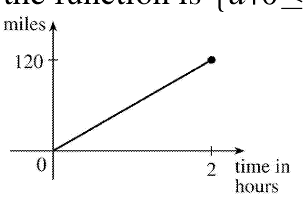
1. **(a)** The point $(-1,-2)$ is on the graph of f , so $f(-1)=-2$.
(b) When $x=2$, y is about 2.8 , so $f(2)\approx 2.8$.
(c) $f(x)=2$ is equivalent to $y=2$. When $y=2$, we have $x=-3$ and $x=1$.
(d) Reasonable estimates for x when $y=0$ are $x=-2.5$ and $x=0.3$.
(e) The domain of f consists of all x –values on the graph of f . For this function, the domain is $-3\leq x\leq 3$, or $[-3,3]$. The range of f consists of all y –values on the graph of f . For this function, the range is $-2\leq y\leq 3$, or $[-2,3]$.
(f) As x increases from -1 to 3 , y increases from -2 to 3 . Thus, f is increasing on the interval $[-1,3]$.

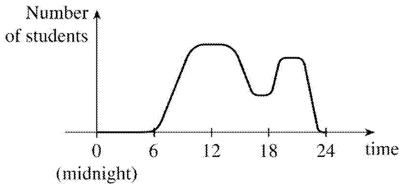
2. **(a)** The point $(-4,-2)$ is on the graph of f , so $f(-4)=-2$. The point $(3,4)$ is on the graph of g , so $g(3)=4$.
(b) We are looking for the values of x for which the y –values are equal. The y –values for f and g are equal at the points $(-2,1)$ and $(2,2)$, so the desired values of x are -2 and 2 .
(c) $f(x)=-1$ is equivalent to $y=-1$. When $y=-1$, we have $x=-3$ and $x=4$.
(d) As x increases from 0 to 4 , y decreases from 3 to -1 . Thus, f is decreasing on the interval $[0,4]$.
(e) The domain of f consists of all x –values on the graph of f . For this function, the domain is $-4\leq x\leq 4$, or $[-4,4]$. The range of f consists of all y –values on the graph of f . For this function, the range is $-2\leq y\leq 3$, or $[-2,3]$.
(f) The domain of g is $[-4,3]$ and the range is $[0.5,4]$.

3. From Figure 1 in the text, the lowest point occurs at about $(t,a)=(12,-85)$. The highest point occurs at about $(17,115)$. Thus, the range of the vertical ground acceleration is $-85\leq a\leq 115$. In Figure 11, the range of the north–south acceleration is approximately $-325\leq a\leq 485$. In Figure 12, the range of the east–west acceleration is approximately $-210\leq a\leq 200$.

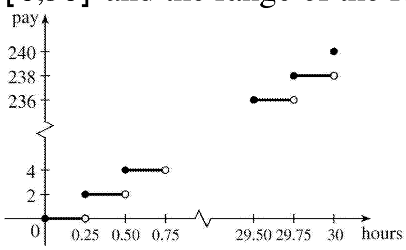
4. *Example 1:* A car is driven at 60 mi / h for 2 hours. The distance d traveled by the car is a function of the time t . The domain of the function is $\{t\mid 0\leq t\leq 2\}$, where t is measured in hours. The range of the function is $\{d\mid 0\leq d\leq 120\}$, where d is measured in miles.



Example 2: At a certain university, the number of students N on campus at any time on a particular day is a function of the time t after midnight. The domain of the function is $\{t\mid 0\leq t\leq 24\}$, where t is measured in hours. The range of the function is $\{N\mid 0\leq N\leq k\}$, where N is an integer and k is the largest number of students on campus at once.



Example 3: A certain employee is paid \$8.00 per hour and works a maximum of 30 hours per week. The number of hours worked is rounded down to the nearest quarter of an hour. This employee's gross weekly pay P is a function of the number of hours worked h . The domain of the function is $[0,30]$ and the range of the function is $\{0,2.00,4.00,\ldots,238.00,240.00\}$.



5. No, the curve is not the graph of a function because a vertical line intersects the curve more than once. Hence, the curve fails the Vertical Line Test.

6. Yes, the curve is the graph of a function because it passes the Vertical Line Test. The domain is $[-2,2]$ and the range is $[-1,2]$.

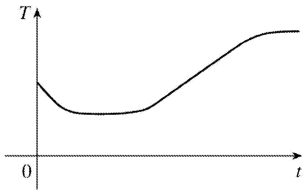
7. Yes, the curve is the graph of a function because it passes the Vertical Line Test. The domain is $[-3,2]$ and the range is $[-3,-2)\cup[-1,3]$.

8. No, the curve is not the graph of a function since for $x=0$, ± 1 , and ± 2 , there are infinitely many points on the curve.

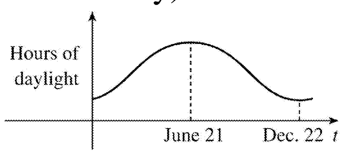
9. The person's weight increased to about 160 pounds at age 20 and stayed fairly steady for 10 years. The person's weight dropped to about 120 pounds for the next 5 years, then increased rapidly to about 170 pounds. The next 30 years saw a gradual increase to 190 pounds. Possible reasons for the drop in weight at 30 years of age: diet, exercise, health problems.

10. The salesman travels away from home from 8 to 9 A.M. and is then stationary until 10 : 00 . The salesman travels farther away from 10 until noon. There is no change in his distance from home until 1 : 00 , at which time the distance from home decreases until 3 : 00 . Then the distance starts increasing again, reaching the maximum distance away from home at 5 : 00 . There is no change from 5 until 6 , and then the distance decreases rapidly until 7 : 00 P.M., at which time the salesman reaches home.

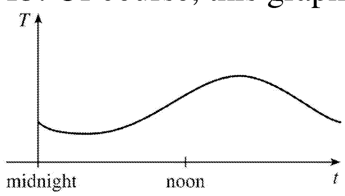
11. The water will cool down almost to freezing as the ice melts. Then, when the ice has melted, the water will slowly warm up to room temperature.



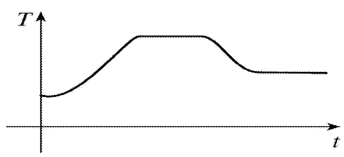
12. The summer solstice (the longest day of the year) is around June 21, and the winter solstice (the shortest day) is around December 22.

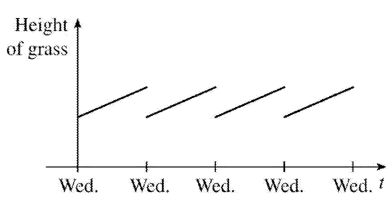


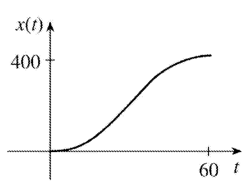
13. Of course, this graph depends strongly on the geographical location!

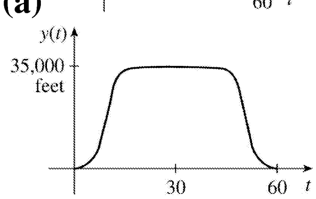


14. The temperature of the pie would increase rapidly, level off to oven temperature, decrease rapidly, and then level off to room temperature.

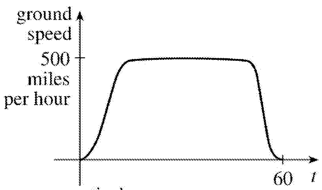


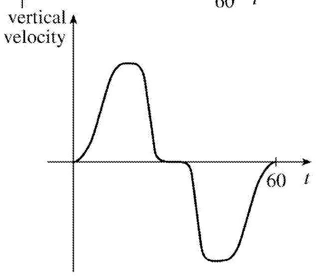
15. 

16. **(a)** 

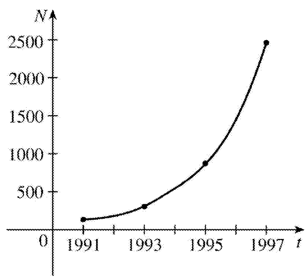
(b) 

(c)



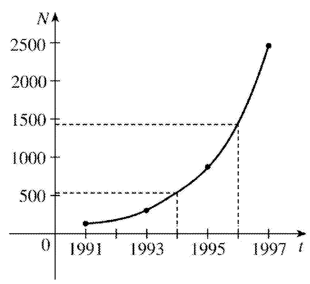


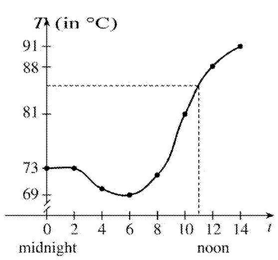
(d)



17. **(a)**

(b) From the graph, we estimate the number of cell–phone subscribers in Malaysia to be about 540 in 1994 and 1450 in 1996.





18. **(a)**

(b) From the graph in part (a), we estimate the temperature at 11:00 A.M. to be about 84.5 $^\circ$ C.

19. $f(x)=3x^2-x+2.$

$f(2)=3(2)^2-2+2=12-2+2=12.$

$f(-2)=3(-2)^2-(-2)+2=12+2+2=16.$

$f(a)=3a^2-a+2.$
$f(-a)=3(-a)^2-(-a)+2=3a^2+a+2.$
$f(a+1)=3(a+1)^2-(a+1)+2=3(a^2+2a+1)-a-1+2=3a^2+6a+3-a+1=3a^2+5a+4.$
$2f(a)=2\cdot f(a)=2(3a^2-a+2)=6a^2-2a+4.$
$f(2a)=3(2a)^2-(2a)+2=3(4a^2)-2a+2=12a^2-2a+2.$
$f(a^2)=3(a^2)^2-(a^2)+2=3(a^4)-a^2+2=3a^4-a^2+2.$

$$\begin{aligned}[f(a)]^2 &=\left[3a^2-a+2\right]^2=\left(3a^2-a+2\right)\left(3a^2-a+2\right)\\ &=9a^4-3a^3+6a^2-3a^3+a^2-2a+6a^2-2a+4=9a^4-6a^3+13a^2-4a+4.\end{aligned}$$

$f(a+h)=3(a+h)^2-(a+h)+2=3(a^2+2ah+h^2)-a-h+2=3a^2+6ah+3h^2-a-h+2.$

20. A spherical balloon with radius $r+1$ has volume $V(r+1)=\frac{4}{3}\pi(r+1)^3=\frac{4}{3}\pi\left(r^3+3r^2+3r+1\right)$. We wish to find the amount of air needed to inflate the balloon from a radius of r to $r+1$. Hence, we need to find the difference $V(r+1)-V(r)=\frac{4}{3}\pi\left(r^3+3r^2+3r+1\right)-\frac{4}{3}\pi r^3=\frac{4}{3}\pi\left(3r^2+3r+1\right)$.

21. $f(x)=x-x^2$, so $f(2+h)=2+h-(2+h)^2=2+h-(4+4h+h^2)=2+h-4-4h-h^2=-\left(h^2+3h+2\right)$,
$f(x+h)=x+h-(x+h)^2=x+h-(x^2+2xh+h^2)=x+h-x^2-2xh-h^2$, and

$$\frac{f(x+h)-f(x)}{h}=\frac{x+h-x^2-2xh-h^2-x+x^2}{h}=\frac{h-2xh-h^2}{h}=\frac{h(1-2x-h)}{h}=1-2x-h.$$

22. $f(x)=\frac{x}{x+1}$, so $f(2+h)=\frac{2+h}{2+h+1}=\frac{2+h}{3+h}$, $f(x+h)=\frac{x+h}{x+h+1}$, and

$$\frac{f(x+h)-f(x)}{h}=\frac{\frac{x+h}{x+h+1}-\frac{x}{x+1}}{h}=\frac{(x+h)(x+1)-x(x+h+1)}{h(x+h+1)(x+1)}=\frac{1}{(x+h+1)(x+1)}.$$

23. $f(x)=x/(3x-1)$ is defined for all x except when $0=3x-1 \Leftrightarrow x=\frac{1}{3}$, so the domain is

$$\left\{x\in R \mid x\neq\frac{1}{3}\right\}=\left(-\infty,\frac{1}{3}\right)\cup\left(\frac{1}{3},\infty\right).$$

24. $f(x)=(5x+4)\big/\left(x^2+3x+2\right)$ is defined for all x except when $0=x^2+3x+2\Leftrightarrow 0=(x+2)(x+1)\Leftrightarrow x=-2$ or -1, so the domain is $\{x\in R \mid x\neq-2,-1\}=(-\infty,-2)\cup(-2,-1)\cup(-1,\infty)$.

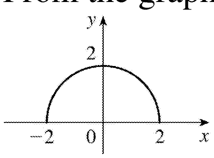
25.

$f(t)=\sqrt{t}+\sqrt[3]{t}$ is defined when $t\geq 0$. These values of t give real number results for $\sqrt{t}$, whereas any value of t gives a real number result for $\sqrt[3]{t}$. The domain is $[0,\infty)$.

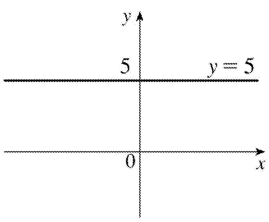
26. $g(u)=\sqrt{u}+\sqrt{4-u}$ is defined when $u\geq 0$ and $4-u\geq 0 \Leftrightarrow u\leq 4$. Thus, the domain is $0\leq u\leq 4=[0,4]$.

27. $h(x)=1\Big/\sqrt[4]{x^2-5x}$ is defined when $x^2-5x>0 \Leftrightarrow x(x-5)>0$. Note that $x^2-5x\neq 0$ since that would result in division by zero. The expression $x(x-5)$ is positive if $x<0$ or $x>5$. (See Appendix A for methods for solving inequalities.) Thus, the domain is $(-\infty,0)\cup(5,\infty)$.

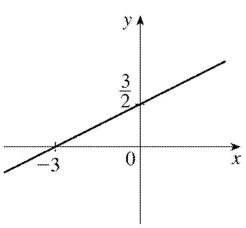
28. $h(x)=\sqrt{4-x^2}$. Now $y=\sqrt{4-x^2} \Rightarrow y^2=4-x^2 \Leftrightarrow x^2+y^2=4$, so the graph is the top half of a circle of radius 2 with center at the origin. The domain is $\left\{ x|4-x^2\geq 0\right\} = \left\{ x|4\geq x^2\right\} =\{x|2\geq |x|\}=[-2,2]$. From the graph, the range is $0\leq y\leq 2$, or $[0,2]$.



29. $f(x)=5$ is defined for all real numbers, so the domain is R , or $(-\infty,\infty)$. The graph of f is a horizontal line with y –intercept 5 .

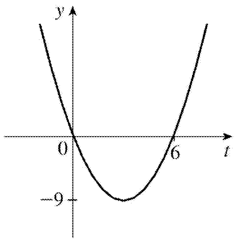


30. $F(x)=\frac{1}{2}(x+3)$ is defined for all real numbers, so the domain is R , or $(-\infty,\infty)$. The graph of F is a line with x –intercept -3 and y –intercept $\frac{3}{2}$.

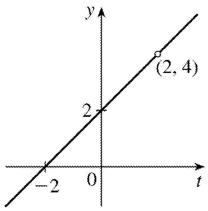


31. $f(t)=t^2-6t$ is defined for all real numbers, so the domain is R , or $(-\infty,\infty)$. The graph of f is a parabola opening upward since the coefficient of t^2 is positive. To find the t –intercepts, let $y=0$ and solve for t . $0=t^2-6t=t(t-6) \Rightarrow t=0$ and $t=6$. The t –coordinate of the vertex is halfway between the t –

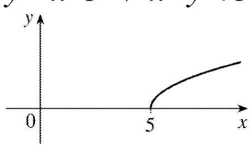
intercepts, that is, at $t=3$. Since $f(3)=3^2-6\cdot 3=-9$, the vertex is $(3,-9)$.



32. $H(t)=\dfrac{4-t^2}{2-t}=\dfrac{(2+t)(2-t)}{2-t}$, so for $t\neq 2$, $H(t)=2+t$. The domain is $\{t\,|\,t\neq 2\}$. So the graph of H is the same as the graph of the function $f(t)=t+2$ (a line) except for the hole at $(2,4)$.



33. $g(x)=\sqrt{x-5}$ is defined when $x-5\geq 0$ or $x\geq 5$, so the domain is $[5,\infty)$. Since $y=\sqrt{x-5}\Rightarrow$ $y^2=x-5\Rightarrow x=y^2+5$, we see that g is the top half of a parabola.

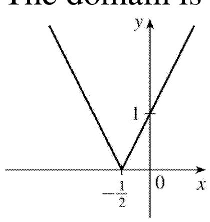


34.

$$F(x)=|2x+1|=\begin{cases} 2x+1 & \text{if } 2x+1\geq 0 \\ -(2x+1) & \text{if } 2x+1<1 \end{cases}$$

$$=\begin{cases} 2x+1 & \text{if } x\geq -\dfrac{1}{2} \\ -2x-1 & \text{if } x<-\dfrac{1}{2} \end{cases}$$

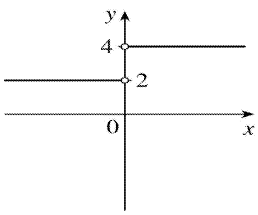
The domain is R , or $(-\infty,\infty)$.



35.

$G(x)=\dfrac{3x+|x|}{x}$. Since $|x|=\begin{cases} x & \text{if } x\geq 0 \\ -x & \text{if } x<0 \end{cases}$, we have

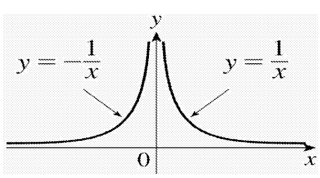
$$G(x)=\begin{cases} \dfrac{3x+x}{x} & \text{if } x>0 \\ \dfrac{3x-x}{x} & \text{if } x<0 \end{cases} = \begin{cases} \dfrac{4x}{x} & \text{if } x>0 \\ \dfrac{2x}{x} & \text{if } x<0 \end{cases} = \begin{cases} 4 & \text{if } x>0 \\ 2 & \text{if } x<0 \end{cases}$$



Note that G is not defined for $x=0$. The domain is $(-\infty,0)\cup(0,\infty)$.

36. $g(x)=\dfrac{|x|}{x^2}$. Since $|x|=\begin{cases} x & \text{if } x\geq 0 \\ -x & \text{if } x<0 \end{cases}$, we have

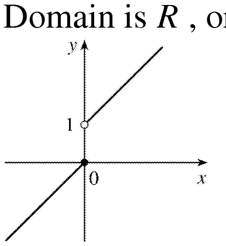
$$g(x)=\begin{cases} \dfrac{x}{x^2} & \text{if } x>0 \\ \dfrac{-x}{x^2} & \text{if } x<0 \end{cases} = \begin{cases} \dfrac{1}{x} & \text{if } x>0 \\ \dfrac{1}{-x} & \text{if } x<0 \end{cases}$$



Note that g is not defined for $x=0$. The domain is $(-\infty,0)\cup(0,\infty)$.

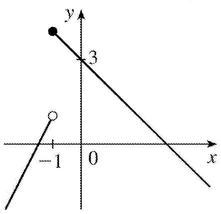
37. $f(x)=\begin{cases} x & \text{if } x\leq 0 \\ x+1 & \text{if } x>0 \end{cases}$

Domain is R , or $(-\infty,\infty)$.



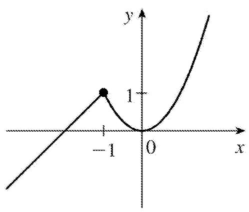
38. $f(x)=\begin{cases} 2x+3 & \text{if } x<-1 \\ 3-x & \text{if } x\geq -1 \end{cases}$

Domain is R , or $(-\infty,\infty)$.



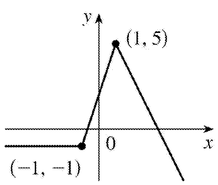
39. $f(x)=\begin{cases} x+2 & \text{if } x\leq -1 \\ x^2 & \text{if } x>-1 \end{cases}$

Note that for $x=-1$, both $x+2$ and x^2 are equal to 1. Domain is R .



40. $f(x)=\begin{cases} -1 & \text{if } x\leq -1 \\ 3x+2 & \text{if } -1<x<1 \\ 7-2x & \text{if } x\geq 1 \end{cases}$

Domain is R .



41. Recall that the slope m of a line between the two points $\left(x_1,y_1\right)$ and $\left(x_2,y_2\right)$ is $m=\dfrac{y_2-y_1}{x_2-x_1}$ and an equation of the line connecting those two points is $y-y_1=m(x-x_1)$. The slope of this line segment is $\dfrac{-6-1}{4-(-2)}=-\dfrac{7}{6}$, so an equation is $y-1=-\dfrac{7}{6}(x+2)$. The function is $f(x)=-\dfrac{7}{6}x-\dfrac{4}{3}$, $-2\leq x\leq 4$.

42. The slope of this line segment is $\dfrac{3-(-2)}{6-(-3)}=\dfrac{5}{9}$, so an equation is $y+2=\dfrac{5}{9}(x+3)$. The function is $f(x)=\dfrac{5}{9}x-\dfrac{1}{3}$, $-3\leq x\leq 6$.

43. We need to solve the given equation for y .

$x+(y-1)^2=0 \Leftrightarrow (y-1)^2=-x \Leftrightarrow y-1=\pm\sqrt{-x} \Leftrightarrow y=1\pm\sqrt{-x}$. The expression with the positive radical represents the top half of the parabola, and the one with the negative radical represents the bottom half. Hence, we want $f(x)=1-\sqrt{-x}$. Note that the domain is $x\leq 0$.

44. $(x-1)^2+y^2=1 \Leftrightarrow y=\pm\sqrt{1-(x-1)^2}=\pm\sqrt{2x-x^2}$. The top half is given by the function $f(x)=\sqrt{2x-x^2}$, $0\leq x\leq 2$.

45. For $-1\leq x\leq 2$, the graph is the line with slope 1 and $y-$ intercept 1 , that is, the line $y=x+1$. For $2<x\leq 4$, the graph is the line with slope $-\frac{3}{2}$ and $x-$ intercept 4 , so $y-0=-\frac{3}{2}(x-4)=-\frac{3}{2}x+6$. So the function is $f(x)=\begin{cases} x+1 & \text{if } -1\leq x\leq 2 \\ -\frac{3}{2}x+6 & \text{if } 2<x\leq 4 \end{cases}$

46. For $x\leq 0$, the graph is the line $y=2$. For $0<x\leq 1$, the graph is the line with slope -2 and $y-$ intercept 2 , that is, the line $y=-2x+2$. For $x>1$, the graph is the line with slope 1 and $x-$ intercept 1 , that is, the line $y=1(x-1)=x-1$. So the function is $f(x)=\begin{cases} 2 & \text{if } x\leq 0 \\ -2x+2 & \text{if } 0<x\leq 1 \\ x-1 & \text{if } 1<x \end{cases}$.

47. Let the length and width of the rectangle be L and W . Then the perimeter is $2L+2W=20$ and the area is $A=LW$. Solving the first equation for W in terms of L gives $W=\frac{20-2L}{2}=10-L$. Thus, $A(L)=L(10-L)=10L-L^2$. Since lengths are positive, the domain of A is $0<L<10$. If we further restrict L to be larger than W , then $5<L<10$ would be the domain.

48. Let the length and width of the rectangle be L and W . Then the area is $LW=16$, so that $W=16/L$. The perimeter is $P=2L+2W$, so $P(L)=2L+2(16/L)=2L+32/L$, and the domain of P is $L>0$, since lengths must be positive quantities. If we further restrict L to be larger than W , then $L>4$ would be the domain.

49. Let the length of a side of the equilateral triangle be x . Then by the Pythagorean Theorem, the height y of the triangle satisfies $y^2+\left(\frac{1}{2}x\right)^2=x^2$, so that $y^2=x^2-\frac{1}{4}x^2=\frac{3}{4}x^2$ and $y=\frac{\sqrt{3}}{2}x$. Using the formula for the area A of a triangle, $A=\frac{1}{2}$ (base) (height) , we obtain $A(x)=\frac{1}{2}(x)\left(\frac{\sqrt{3}}{2}x\right)=\frac{\sqrt{3}}{4}x^2$, with domain $x>0$.

50. Let the volume of the cube be V and the length of an edge be L . Then $V=L^3$ so $L=\sqrt[3]{V}$, and the

surface area is $S(V)=6\left(\sqrt[3]{V}\right)^2=6V^{2/3}$, with domain $V>0$.

51. Let each side of the base of the box have length x , and let the height of the box be h . Since the volume is 2 , we know that $2=hx^2$, so that $h=2/x^2$, and the surface area is $S=x^2+4xh$. Thus, $S(x)=x^2+4x(2/x^2)=x^2+(8/x)$, with domain $x>0$.

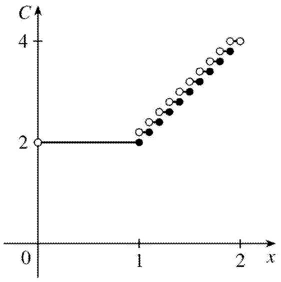
52. The area of the window is $A=xh+\frac{1}{2}\pi\left(\frac{1}{2}x\right)^2=xh+\frac{\pi x^2}{8}$, where h is the height of the rectangular portion of the window. The perimeter is $P=2h+x+\frac{1}{2}\pi x=30\Leftrightarrow 2h=30-x-\frac{1}{2}\pi x\Leftrightarrow$ $h=\frac{1}{4}(60-2x-\pi x)$. Thus,

$$A(x)=x\,\frac{60-2x-\pi x}{4}+\frac{\pi x^2}{8}=15x-\frac{1}{2}x^2-\frac{\pi}{4}x^2+\frac{\pi}{8}x^2=15x-\frac{4}{8}x^2-\frac{\pi}{8}x^2=15x-x^2\left(\frac{\pi+4}{8}\right)$$

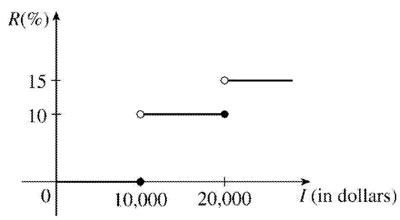
Since the lengths x and h must be positive quantities, we have $x>0$ and $h>0$. For $h>0$, we have $2h>0$ $\Leftrightarrow 30-x-\frac{1}{2}\pi x>0\Leftrightarrow 60>2x+\pi x\Leftrightarrow x<\frac{60}{2+\pi}$. Hence, the domain of A is $0<x<\frac{60}{2+\pi}$.

53. The height of the box is x and the length and width are $L=20-2x$, $W=12-2x$. Then $V=LWx$ and so
$V(x)=(20-2x)(12-2x)(x)=4(10-x)(6-x)(x)=4x\left(60-16x+x^2\right)=4x^3-64x^2+240x$.
The sides L , W , and x must be positive. Thus, $L>0\Leftrightarrow 20-2x>0\Leftrightarrow x<10$; $W>0\Leftrightarrow 12-2x>0\Leftrightarrow x<6$; and $x>0$. Combining these restrictions gives us the domain $0<x<6$.
54.

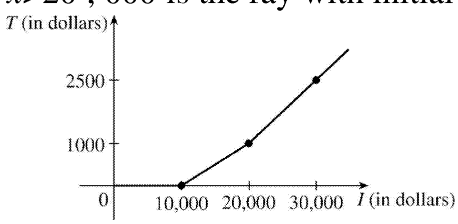
$$C(x)=\begin{cases} \$2.00 & \text{if } 0.0<x\leq 1.0 \\ 2.20 & \text{if } 1.0<x\leq 1.1 \\ 2.40 & \text{if } 1.1<x\leq 1.2 \\ 2.60 & \text{if } 1.2<x\leq 1.3 \\ 2.80 & \text{if } 1.3<x\leq 1.4 \\ 3.00 & \text{if } 1.4<x\leq 1.5 \\ 3.20 & \text{if } 1.5<x\leq 1.6 \\ 3.40 & \text{if } 1.6<x\leq 1.7 \\ 3.60 & \text{if } 1.7<x\leq 1.8 \\ 3.80 & \text{if } 1.8<x\leq 1.9 \\ 4.00 & \text{if } 1.9<x<2.0 \end{cases}$$



55. **(a)**



(b) On \$14 , 000 , tax is assessed on \$4000 , and 10%(\$4000)=\$400 .
On \$26 , 000 , tax is assessed on \$16 , 000 , and 10%(\$10 , 000)+15%(\$6000)=\$1000+\$900=\$1900 .
(c) As in part (b), there is \$1000 tax assessed on \$20,000 of income, so the graph of T is a line segment from (10 , 000,0) to (20 , 000,1000) . The tax on \$30,000 is \$2500, so the graph of T for $x>20$, 000 is the ray with initial point (20 , 000,1000) that passes through (30 , 000,2500) .



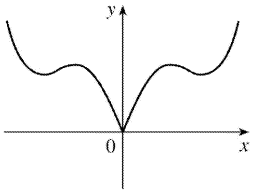
56. One example is the amount paid for cable or telephone system repair in the home, usually measured to the nearest quarter hour. Another example is the amount paid by a student in tuition fees, if the fees vary according to the number of credits for which the student has registered.

57. f is an odd function because its graph is symmetric about the origin. g is an even function because its graph is symmetric with respect to the y –axis.

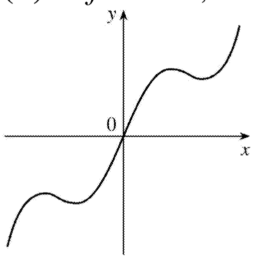
58. f is not an even function since it is not symmetric with respect to the y –axis. f is not an odd function since it is not symmetric about the origin. Hence, f is *neither* even nor odd. g is an even function because its graph is symmetric with respect to the y –axis.

59. **(a)** Because an even function is symmetric with respect to the y –axis, and the point $(5,3)$ is on the graph of this even function, the point $(-5,3)$ must also be on its graph.
(b) Because an odd function is symmetric with respect to the origin, and the point $(5,3)$ is on the graph of this odd function, the point $(-5,-3)$ must also be on its graph.

60. **(a)** If f is even, we get the rest of the graph by reflecting about the y –axis.



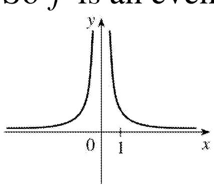
(b) If f is odd, we get the rest of the graph by rotating 180° about the origin.



61. $f(x)=x^{-2}$.

$$f(-x)=(-x)^{-2}=\frac{1}{(-x)^{2}}=\frac{1}{x^{2}}$$
$$=x^{-2}=f(x)$$

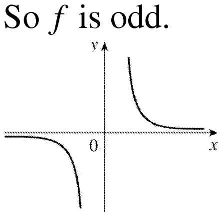
So f is an even function.



62. $f(x)=x^{-3}$.

$$f(-x)=(-x)^{-3}=\frac{1}{(-x)^{3}}=\frac{1}{-x^{3}}$$
$$=-\frac{1}{x^{3}}=-\left(x^{-3}\right)=-f(x)$$

So f is odd.



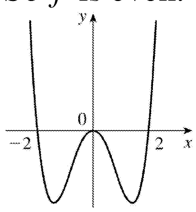
63. $f(x)=x^{2}+x$, so $f(-x)=(-x)^{2}+(-x)=x^{2}-x$. Since this is neither $f(x)$ nor $-f(x)$, the function f is

neither even nor odd.

64. $f(x)=x^4-4x^2$.

$$
\begin{aligned}
f(-x) &= (-x)^4-4(-x)^2 \\
&= x^4-4x^2=f(x)
\end{aligned}
$$

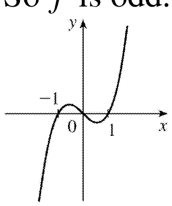
So f is even.



65. $f(x)=x^3-x$.

$$
\begin{aligned}
f(-x) &= (-x)^3-(-x)=-x^3+x \\
&= -\left(x^3-x\right)=-f(x)
\end{aligned}
$$

So f is odd.



66. $f(x)=3x^3+2x^2+1$, so $f(-x)=3(-x)^3+2(-x)^2+1=-3x^3+2x^2+1$. Since this is neither $f(x)$ nor $-f(x)$, the function f is neither even nor odd.

1. **(a)** $f(x)=\sqrt[5]{x}$ is a root function with $n=5$.

(b) $g(x)=\sqrt{1-x^2}$ is an algebraic function because it is a root of a polynomial.

(c) $h(x)=x^9+x^4$ is a polynomial of degree 9 .

(d) $r(x)=\dfrac{x^2+1}{x^3+x}$ is a rational function because it is a ratio of polynomials.

(e) $s(x)=\tan\ 2x$ is a trigonometric function.

(f) $t(x)=\log_{10}\ x$ is a logarithmic function.

2. **(a)** $y=(x-6)/(x+6)$ is a rational function because it is a ratio of polynomials.

(b) $y=x+x^2/\sqrt{x-1}$ is an algebraic function because it involves polynomials and roots of polynomials.

(c) $y=10^x$ is an exponential function (notice that x is the *exponent*).

(d) $y=x^{10}$ is a power function (notice that x is the *base*).

(e) $y=2t^6+t^4-\pi$ is a polynomial of degree 6 .

(f) $y=\cos\ \theta+\sin\ \theta$ is a trigonometric function.

3. We notice from the figure that g and h are even functions (symmetric with respect to the y –axis) and that f is an odd function (symmetric with respect to the origin). So (b) $\left[y=x^5\right]$ must be f . Since g is flatter than h near the origin, we must have (c) $\left[y=x^8\right]$ matched with g and (a) $\left[y=x^2\right]$ matched with h .

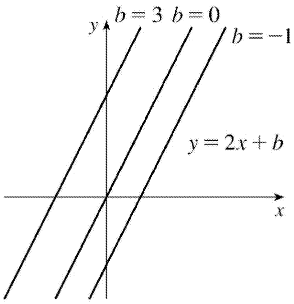
4. **(a)** The graph of $y=3x$ is a line (choice G).

(b) $y=3^x$ is an exponential function (choice f).

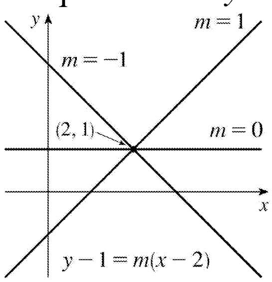
(c) $y=x^3$ is an odd polynomial function or power function (choice F).

(d) $y=\sqrt[3]{x}=x^{1/3}$ is a root function (choice g).

5. **(a)** An equation for the family of linear functions with slope 2 is $y=f(x)=2x+b$, where b is the y –intercept.

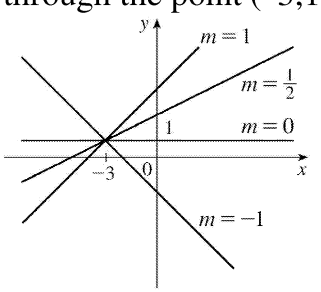


(b) $f(2)=1$ means that the point $(2,1)$ is on the graph of f . We can use the point–slope form of a line to obtain an equation for the family of linear functions through the point $(2,1)$. $y-1=m(x-2)$, which is equivalent to $y=mx+(1-2m)$ in slope–intercept form.

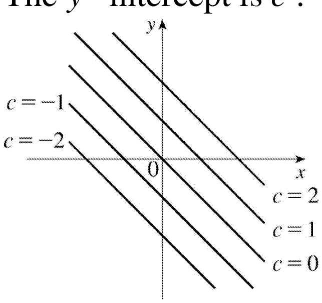


(c) To belong to both families, an equation must have slope $m=2$, so the equation in part (b), $y=mx+(1-2m)$, becomes $y=2x-3$. It is the *only* function that belongs to both families.

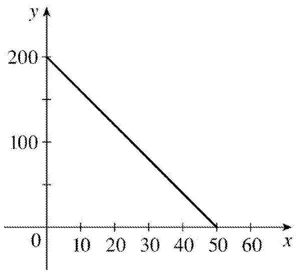
6. All members of the family of linear functions $f(x)=1+m(x+3)$ have graphs that are lines passing through the point $(-3,1)$.



7. All members of the family of linear functions $f(x)=c-x$ have graphs that are lines with slope -1 . The y –intercept is c .

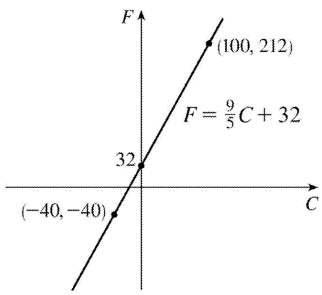


8. **(a)**



(b) The slope of -4 means that for each increase of 1 dollar for a rental space, the number of spaces rented *decreases* by 4 . The y –intercept of 200 is the number of spaces that would be occupied if there were no charge for each space. The x –intercept of 50 is the smallest rental fee that results in no spaces rented.

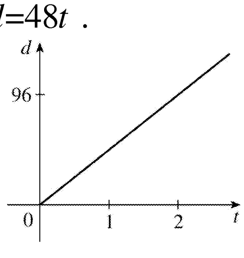
9. **(a)**



(b) The slope of $\frac{9}{5}$ means that F increases $\frac{9}{5}$ degrees for each increase of 1° C. (Equivalently, F increases by 9 when C increases by 5 and F decreases by 9 when C decreases by 5 .) The F –intercept of 32 is the Fahrenheit temperature corresponding to a Celsius temperature of 0 .

10. **(a)** Let d= distance traveled (in miles) and t= time elapsed (in hours). At $t=0$, $d=0$ and at $t=50$ minutes $=50\cdot\frac{1}{60}=\frac{5}{6}$ h, $d=40$. Thus we have two points: $(0,0)$ and $\left(\frac{5}{6},40\right)$, so $m=\frac{40-0}{\frac{5}{6}-0}=48$ and so $d=48t$.

(b)



(c) The slope is 48 and represents the car's speed in mi / h.

11. **(a)** Using N in place of x and T in place of y , we find the slope to be $\frac{T_2-T_1}{N_2-N_1}=\frac{80-70}{173-113}=\frac{10}{60}=\frac{1}{6}$. So a linear equation is $T-80=\frac{1}{6}(N-173) \Leftrightarrow T-80=\frac{1}{6}N-\frac{173}{6} \Leftrightarrow$

$T=\frac{1}{6}N+\frac{307}{6}\left[\frac{307}{6}=51.1\overline{6}\right]$.

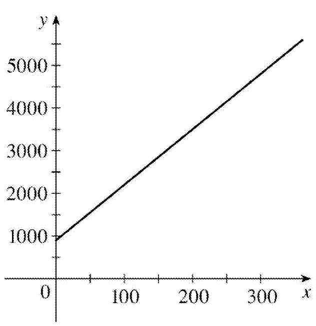
(b) The slope of $\frac{1}{6}$ means that the temperature in Fahrenheit degrees increases one–sixth as rapidly as the number of cricket chirps per minute. Said differently, each increase of 6 cricket chirps per minute corresponds to an increase of 1° F.

(c) When $N=150$, the temperature is given approximately by $T=\frac{1}{6}(150)+\frac{307}{6}=76.1\overline{6}^{\circ}$ F $\approx 76^{\circ}$ F.

12. **(a)** Let x denote the number of chairs produced in one day and y the associated cost. Using the points $(100,2200)$ and $(300,4800)$ we get the slope $\frac{4800-2200}{300-100}=\frac{2600}{200}=13$. So $y-2200=13(x-100)$ $\Leftrightarrow y=13x+900$.

(b) The slope of the line in part (a) is 13 and it represents the cost (in dollars) of producing each additional chair.

(c) The y –intercept is 900 and it represents the fixed daily costs of operating the factory.



13. **(a)** We are given $\frac{\text{change in pressure}}{10\ \text{feet change in depth}}=\frac{4.34}{10}=0.434$. Using P for pressure and d for depth with the point $(d,P)=(0,15)$, we have the slope–intercept form of the line, $P=0.434d+15$.

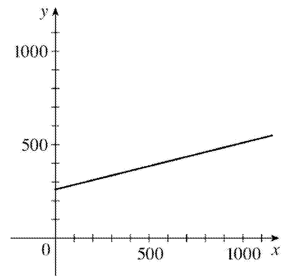
(b) When $P=100$, then $100=0.434d+15 \Leftrightarrow 0.434d=85 \Leftrightarrow d=\frac{85}{0.434}\approx 195.85$ feet. Thus, the pressure is $100\ \text{lb/in}^2$ at a depth of approximately 196 feet.

14. **(a)** Using d in place of x and C in place of y , we find the slope to be

$$\frac{C_2-C_1}{d_2-d_1}=\frac{460-380}{800-480}=\frac{80}{320}=\frac{1}{4}$$

So a linear equation is $C-460=\frac{1}{4}(d-800) \Leftrightarrow C-460=\frac{1}{4}d-200 \Leftrightarrow C=\frac{1}{4}d+260$.

(b) Letting $d=1500$ we get $C=\frac{1}{4}(1500)+260=635$. The cost of driving 1500 miles is \$635.

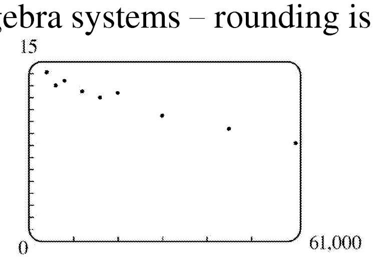
(c) 

The slope of the line represents the cost per mile, \$0.25 .
(d) The y -intercept represents the fixed cost, \$260 .
(e) A linear function gives a suitable model in this situation because you have fixed monthly costs such as insurance and car payments, as well as costs that increase as you drive, such as gasoline, oil, and tires, and the cost of these for each additional mile driven is a constant.

15. **(a)** The data appear to be periodic and a sine or cosine function would make the best model. A model of the form $f(x)=a\cos(bx)+c$ seems appropriate.
(b) The data appear to be decreasing in a linear fashion. A model of the form $f(x)=mx+b$ seems appropriate.

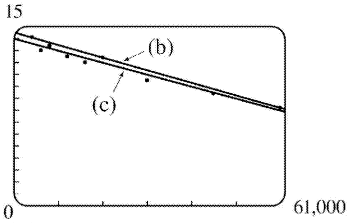
16. **(a)** The data appear to be increasing exponentially. A model of the form $f(x)=a\cdot b^x$ or $f(x)=a\cdot b^x+c$ seems appropriate.
(b) The data appear to be decreasing similarly to the values of the reciprocal function. A model of the form $f(x)=a/x$ seems appropriate.

17. Some values are given to many decimal places. These are the results given by several computer algebra systems – rounding is left to the reader.

(a) 

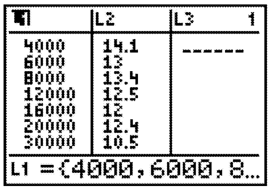
A linear model does seem appropriate.

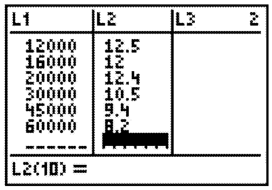
(b) Using the points $(4000,14.1)$ and $(60{,}000,8.2)$, we obtain $y-14.1=\frac{8.2-14.1}{60{,}000-4000}(x-4000)$ or, equivalently, $y\approx-0.000105357x+14.521429$.



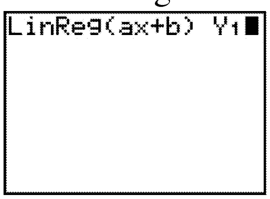
(c) Using a computing device, we obtain the least squares regression line $y=-0.0000997855x+13.950764$.

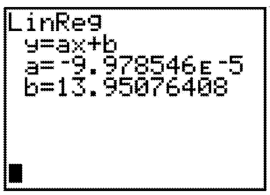
The following commands and screens illustrate how to find the least squares regression line on a TI–83 Plus. Enter the data into list one (L1) and list two (L2). Press STAT 1 to enter the editor.

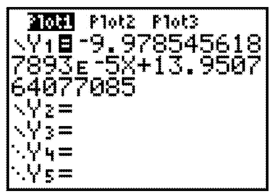




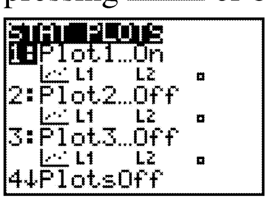
Find the regession line and store it in Y_1 . Press 2nd QUIT STAT ▸ 4 VARS ▸ 1 1 ENTER .

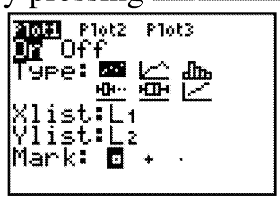




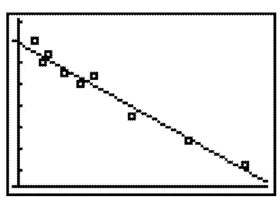


Note from the last figure that the regression line has been stored in Y_1 and that Plot1 has been turned on (Plot1 is highlighted). You can turn on Plot1 from the Y= menu by placing the cursor on Plot1 and pressing ENTER or by pressing 2nd STAT PLOT 1 ENTER .





Now press ZOOM 9 to produce a graph of the data and the regression line. Note that choice 9 of the ZOOM menu automatically selects a window that displays all of the data.

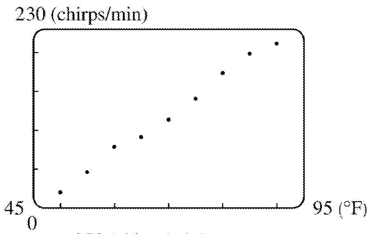


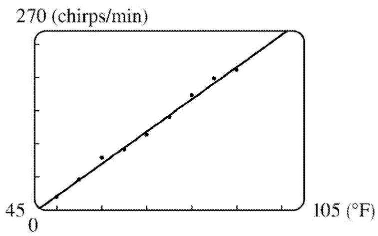
(d) When $x=25,000$, $y\approx 11.456$; or about 11.5 per 100 population.

(e) When $x=80,000$, $y\approx 5.968$; or about a 6% chance.

(f) When $x=200,000$, y is negative, so the model does not apply.

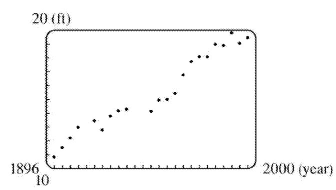
18. **(a)**



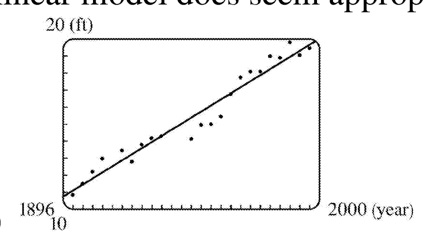


(b)
Using a computing device, we obtain the least squares regression line $y=4.85\overline{6}x-220.9\overline{6}$.

(c) When $x=100^{\circ}$ F, $y=264.7\approx 265$ chirps / min.



19. **(a)**
A linear model does seem appropriate.

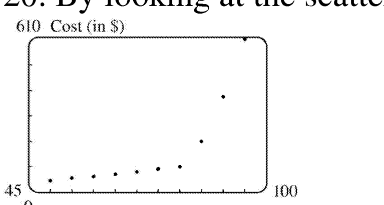


(b)
Using a computing device, we obtain the least squares regression line $y=0.089119747x-158.2403249$, where x is the year and y is the height in feet.

(c) When $x=2000$, the model gives $y\approx 20.00$ ft. Note that the actual winning height for the 2000 Olympics is *less than* the winning height for 1996 – so much for that prediction.

(d) When $x=2100$, $y\approx 28.91$ ft. This would be an increase of 9.49 ft from 1996 to 2100. Even though there was an increase of 8.59 ft from 1900 to 1996, it is unlikely that a similar increase will occur over the next 100 years.

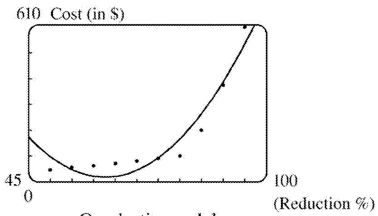
20. By looking at the scatter plot of the data, we rule out the linear and logarithmic models.



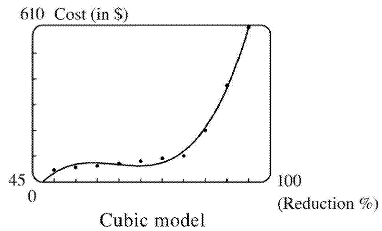
Scatter plot

We try various models:

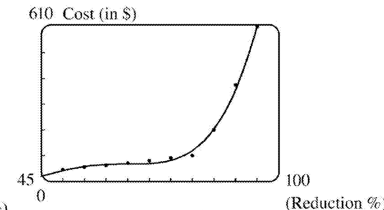
Quadratic:	$y=0.49\overline{6}x^2-62.28\overline{93}x+1970.6\overline{39}$
Cubic:	$y=0.0201243201x^3-3.88037296x^2+247.6754468x-5163.935198$
Quartic:	$y=0.0002951049x^4-0.0654560995x^3+5.27525641x^2-180.2266511x+2203.210956$
Exponential:	$y=2.41422994(1.054516914)^x$
Power:	$y=0.000022854971x^{3.616078251}$



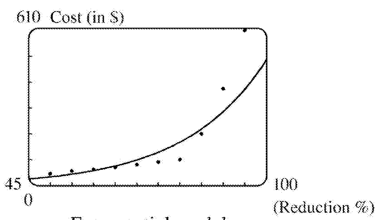
Quadratic model



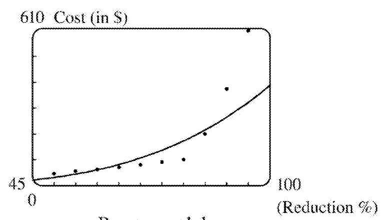
Cubic model



Quartic model

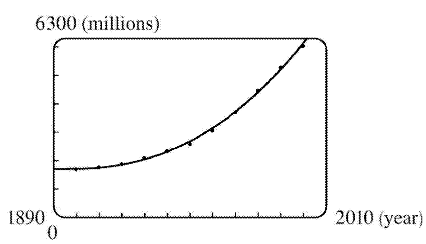


Exponential model



Power model

After examining the graphs of these models, we see that the cubic and quartic models are clearly the best.

21. 

Using a computing device, we obtain the cubic function $y=ax^3+bx^2+cx+d$ with $a=0.0012937$, $b=-7.06142$, $c=12\ ,\ 823$, and $d=-7\ ,\ 743\ ,\ 770$. When $x=1925$, $y\approx 1914$ (million).

22. **(a)** $T=1.000396048d^{1.499661718}$

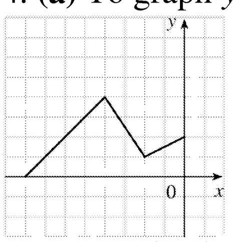
(b) The power model in part (a) is approximately $T=d^{1.5}$. Squaring both sides gives us $T^2=d^3$, so the model matches Kepler's Third Law, $T^2=kd^3$.

1. **(a)** If the graph of f is shifted 3 units upward, its equation becomes $y=f(x)+3$.
(b) If the graph of f is shifted 3 units downward, its equation becomes $y=f(x)-3$.
(c) If the graph of f is shifted 3 units to the right, its equation becomes $y=f(x-3)$.
(d) If the graph of f is shifted 3 units to the left, its equation becomes $y=f(x+3)$.
(e) If the graph of f is reflected about the x –axis, its equation becomes $y=-f(x)$.
(f) If the graph of f is reflected about the y –axis, its equation becomes $y=f(-x)$.
(g) If the graph of f is stretched vertically by a factor of 3 , its equation becomes $y=3f(x)$.
(h) If the graph of f is shrunk vertically by a factor of 3 , its equation becomes $y=\frac{1}{3}f(x)$.

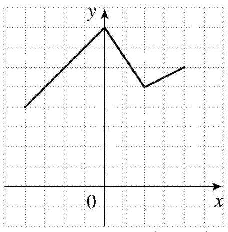
2. **(a)** To obtain the graph of $y=5f(x)$ from the graph of $y=f(x)$, stretch the graph vertically by a factor of 5 .
(b) To obtain the graph of $y=f(x-5)$ from the graph of $y=f(x)$, shift the graph 5 units to the right.
(c) To obtain the graph of $y=-f(x)$ from the graph of $y=f(x)$, reflect the graph about the x –axis.
(d) To obtain the graph of $y=-5f(x)$ from the graph of $y=f(x)$, stretch the graph vertically by a factor of 5 and reflect it about the x –axis.
(e) To obtain the graph of $y=f(5x)$ from the graph of $y=f(x)$, shrink the graph horizontally by a factor of 5 .
(f) To obtain the graph of $y=5f(x)-3$ from the graph of $y=f(x)$, stretch the graph vertically by a factor of 5 and shift it 3 units downward.

3. **(a)** (graph 3) The graph of f is shifted 4 units to the right and has equation $y=f(x-4)$.
(b) (graph 1) The graph of f is shifted 3 units upward and has equation $y=f(x)+3$.
(c) (graph 4) The graph of f is shrunk vertically by a factor of 3 and has equation $y=\frac{1}{3}f(x)$.
(d) (graph 5) The graph of f is shifted 4 units to the left and reflected about the x –axis. Its equation is $y=-f(x+4)$.
(e) (graph 2) The graph of f is shifted 6 units to the left and stretched vertically by a factor of 2 . Its equation is $y=2f(x+6)$.

4. **(a)** To graph $y=f(x+4)$ we shift the graph of f , 4 units to the left.

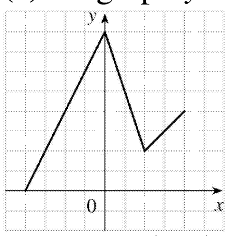


The point $(2,1)$ on the graph of f corresponds to the point $(2-4,1)=(-2,1)$.
(b) To graph $y=f(x)+4$ we shift the graph of f , 4 units upward.



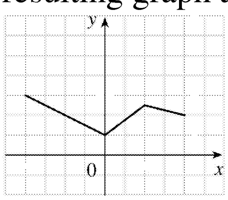
The point $(2,1)$ on the graph of f corresponds to the point $(2,1+4)=(2,5)$.

(c) To graph $y=2f(x)$ we stretch the graph of f vertically by a factor of 2 .



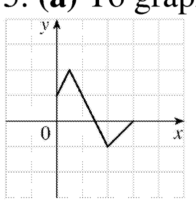
The point $(2,1)$ on the graph of f corresponds to the point $(2,2\cdot 1)=(2,2)$.

(d) To graph $y=-\frac{1}{2}f(x)+3$, we shrink the graph of f vertically by a factor of 2 , then reflect the resulting graph about the x –axis, then shift the resulting graph 3 units upward.



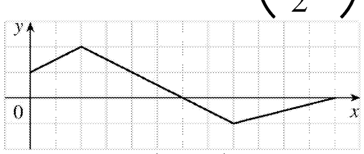
The point $(2,1)$ on the graph of f corresponds to the point $\left(2,-\frac{1}{2}\cdot 1+3\right)=(2,2.5)$.

5. **(a)** To graph $y=f(2x)$ we shrink the graph of f horizontally by a factor of 2 .



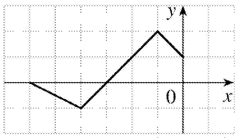
The point $(4,-1)$ on the graph of f corresponds to the point $\left(\frac{1}{2}\cdot 4,-1\right)=(2,-1)$.

(b) To graph $y=f\left(\frac{1}{2}x\right)$ we stretch the graph of f horizontally by a factor of 2 .

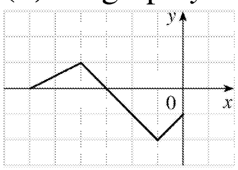


The point $(4,-1)$ on the graph of f corresponds to the point $(2\cdot 4,-1)=(8,-1)$.

(c) To graph $y=f(-x)$ we reflect the graph of f about the y –axis.



The point $(4,-1)$ on the graph of f corresponds to the point $(-1\cdot 4,-1)=(-4,-1)$.
(d) To graph $y=-f(-x)$ we reflect the graph of f about the y –axis, then about the x –axis.



The point $(4,-1)$ on the graph of f corresponds to the point $(-1\cdot 4,-1\cdot -1)=(-4,1)$.

6. The graph of $y=f(x)=\sqrt{3x-x^2}$ has been shifted 2 units to the right and stretched vertically by a factor of 2 . Thus, a function describing the graph is

$$\begin{aligned} y &= 2f(x-2)=2\sqrt{3(x-2)-(x-2)^2} \\ &= 2\sqrt{3x-6-(x2-4x+4)}=2\sqrt{-x2+7x-10} \end{aligned}$$

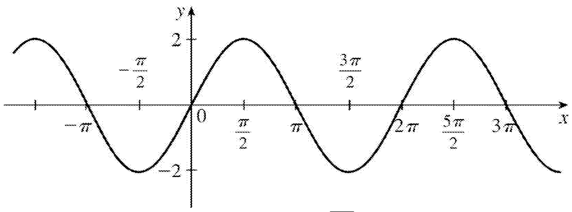
7. The graph of $y=f(x)=\sqrt{3x-x^2}$ has been shifted 4 units to the left, reflected about the x –axis, and shifted downward 1 unit. Thus, a function describing the graph is

$$y=\underbrace{-1\ \cdot}_{\substack{\text{reflect}\\ \text{about}\\ \text{x–axis}}}\ f\,\underbrace{x+4}_{\substack{\text{shift}\\ \text{4units}\\ \text{left}}}\ \underbrace{-1}_{\substack{\text{shift}\\ \text{1unit}\\ \text{down}}}$$

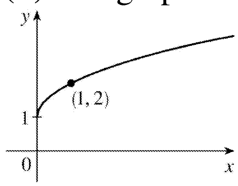
This function can be written as

$$\begin{aligned} y &=-f(x+4)-1=-\sqrt{3(x+4)-(x+4)^2}-1=-\sqrt{3x+12-\left(x^2+8x+16\right)}-1 \\ &=-\sqrt{-x^2-5x-4}-1 \end{aligned}$$

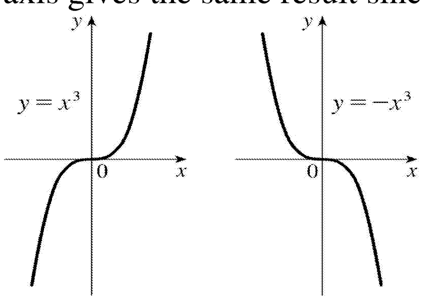
8. **(a)** The graph of $y=2\sin\ x$ can be obtained from the graph of $y=\sin\ x$ by stretching it vertically by a factor of 2 .



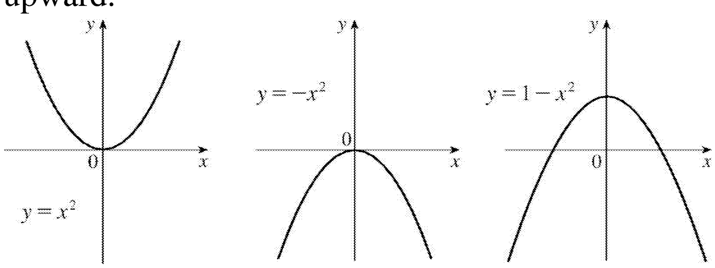
(b) The graph of $y=1+\sqrt{x}$ can be obtained from the graph of $y=\sqrt{x}$ by shifting it upward 1 unit.



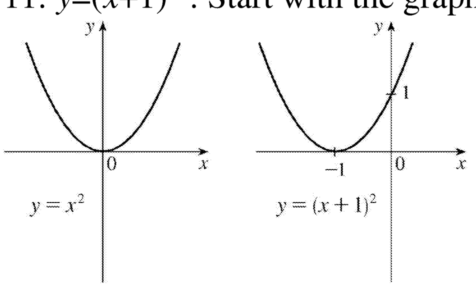
9. $y=-x^3$: Start with the graph of $y=x^3$ and reflect about the x –axis. Note: Reflecting about the y –axis gives the same result since substituting $-x$ for x gives us $y=(-x)^3=-x^3$.



10. $y=1-x^2=-x^2+1$: Start with the graph of $y=x^2$, reflect about the x –axis, and then shift 1 unit upward.

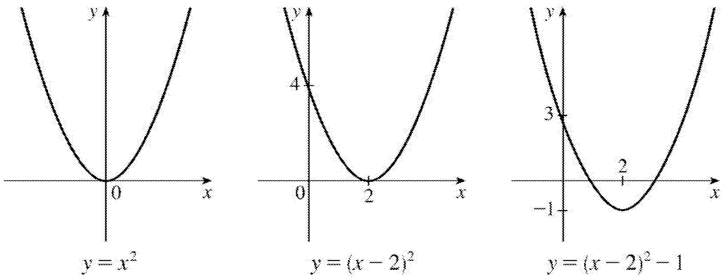


11. $y=(x+1)^2$: Start with the graph of $y=x^2$ and shift 1 unit to the left.

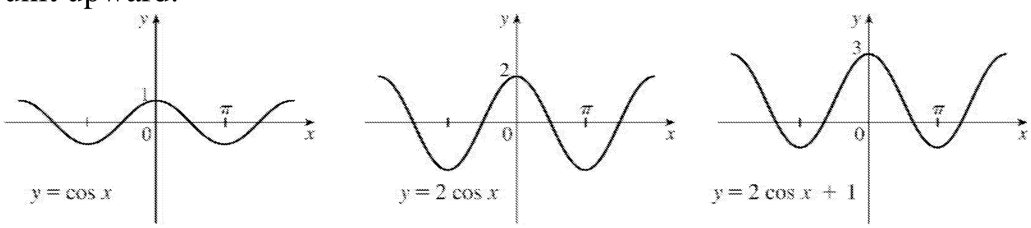


12.

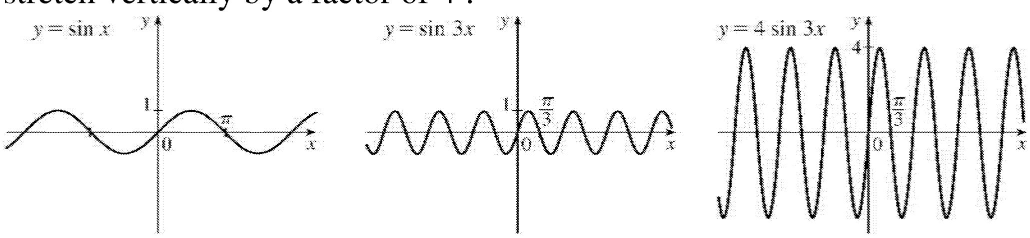
$y=x^2-4x+3=(x^2-4x+4)-1=(x-2)^2-1$: Start with the graph of $y=x^2$, shift 2 units to the right, and then shift 1 unit downward.



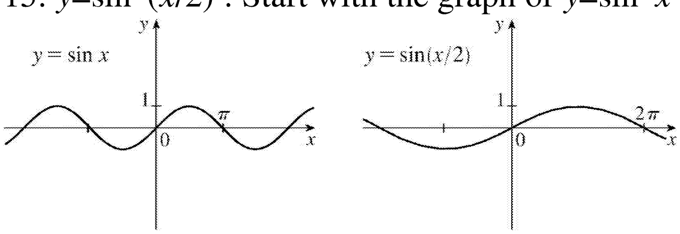
13. $y=1+2\cos\ x$: Start with the graph of $y=\cos\ x$, stretch vertically by a factor of 2 , and then shift 1 unit upward.



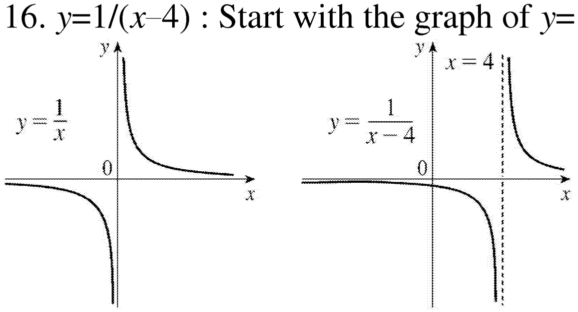
14. $y=4\sin\ 3x$: Start with the graph of $y=\sin\ x$, compress horizontally by a factor of 3 , and then stretch vertically by a factor of 4 .



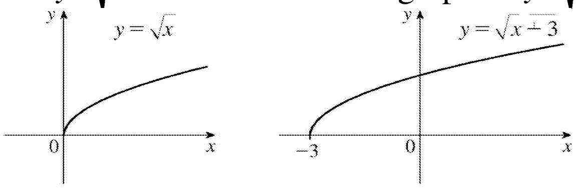
15. $y=\sin\ (x/2)$: Start with the graph of $y=\sin\ x$ and stretch horizontally by a factor of 2 .



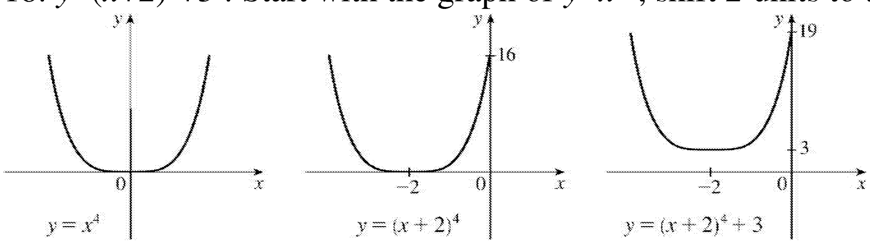
16. $y=1/(x-4)$: Start with the graph of $y=1/x$ and shift 4 units to the right.



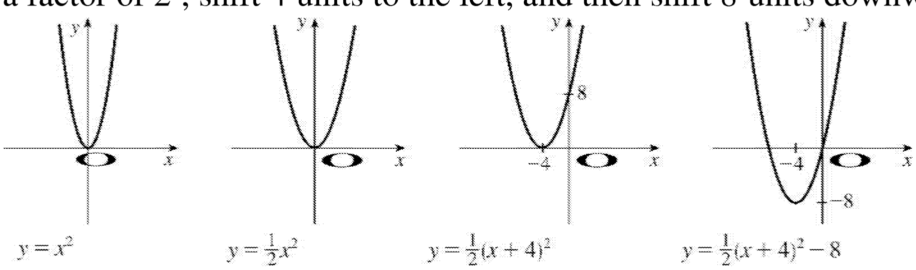
17. $y=\sqrt{x+3}$: Start with the graph of $y=\sqrt{x}$ and shift 3 units to the left.



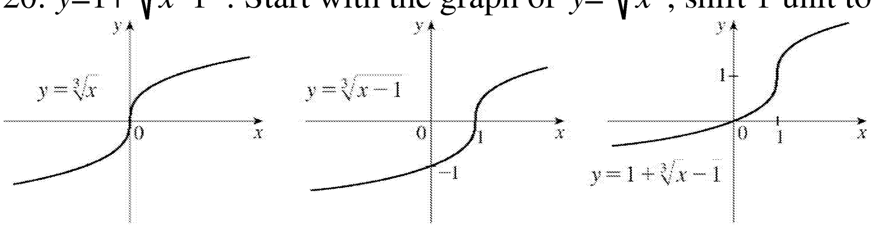
18. $y=(x+2)^4+3$: Start with the graph of $y=x^4$, shift 2 units to the left, and then shift 3 units upward.



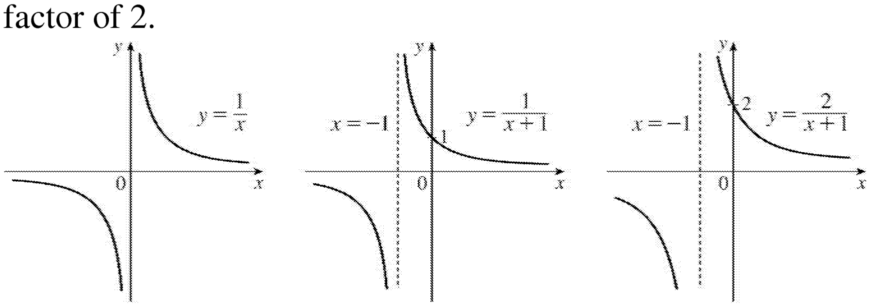
19. $y=\frac{1}{2}(x^2+8x)=\frac{1}{2}(x^2+8x+16)-8=\frac{1}{2}(x+4)^2-8$: Start with the graph of $y=x^2$, compress vertically by a factor of 2 , shift 4 units to the left, and then shift 8 units downward.



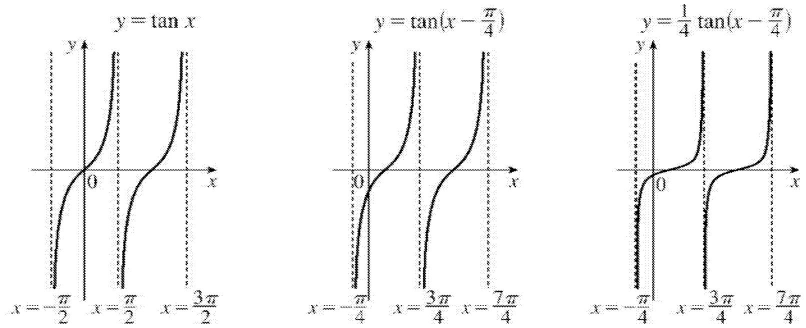
20. $y=1+\sqrt[3]{x-1}$: Start with the graph of $y=\sqrt[3]{x}$, shift 1 unit to the right, and then shift 1 unit upward.



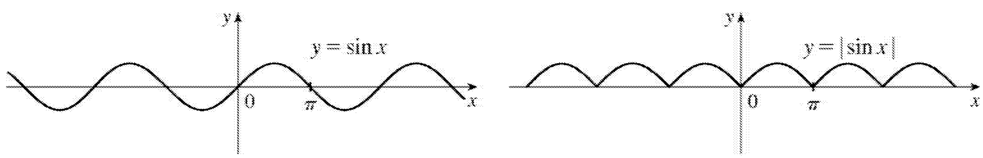
21. $y=2/(x+1)$: Start with the graph of $y=1/x$, shift 1 unit to the left, and then stretch vertically by a factor of 2.



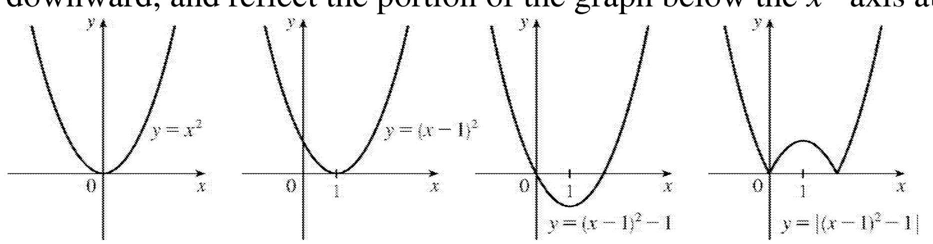
22. $y=\frac{1}{4}\tan\left(x-\frac{\pi}{4}\right)$: Start with the graph of $y=\tan x$, shift $\frac{\pi}{4}$ units to the right, and then compress vertically by a factor of 4 .



23. $y=|\sin x|$: Start with the graph of $y=\sin x$ and reflect all the parts of the graph below the x –axis about the x –axis.



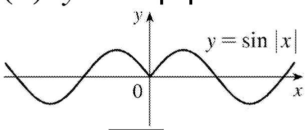
24. $y=\left|x^2-2x\right|=\left|x^2-2x+1-1\right|=\left|(x-1)^2-1\right|$: Start with the graph of $y=x^2$, shift 1 unit right, shift 1 unit downward, and reflect the portion of the graph below the x –axis about the x –axis.



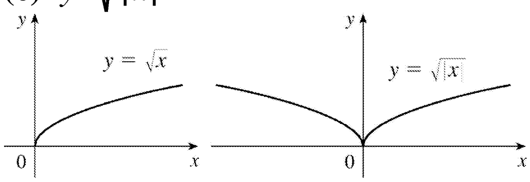
25. This is just like the solution to Example 4 except the amplitude of the curve (the 30 $^\circ$ N curve in Figure 9 on June 21) is $14-12=2$. So the function is $L(t)=12+2\sin\left[\frac{2\pi}{365}(t-80)\right]$. March 31 is the 90 th day of the year, so the model gives $L(90)\approx 12.34$ h. The daylight time (5:51 A.M. to 6:18 P.M.) is 12 hours and 27 minutes, or 12.45 h. The model value differs from the actual value by $\frac{12.45-12.34}{12.45}\approx 0.009$, less than 1% .

26. Using a sine function to model the brightness of Delta Cephei as a function of time, we take its period to be 5.4 days, its amplitude to be 0.35 (on the scale of magnitude), and its average magnitude to be 4.0 . If we take $t=0$ at a time of average brightness, then the magnitude (brightness) as a function of time t in days can be modeled by the formula $M(t)=4.0+0.35\sin\left(\frac{2\pi}{5.4}t\right)$.

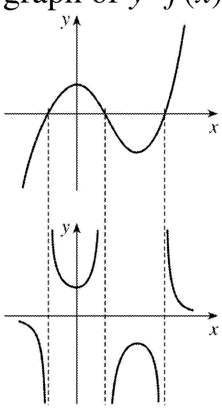
27. **(a)** To obtain $y=f(|x|)$, the portion of the graph of $y=f(x)$ to the right of the y –axis is reflected about the y –axis.
(b) $y=\sin\ |x|$



(c) $y=\sqrt{|x|}$

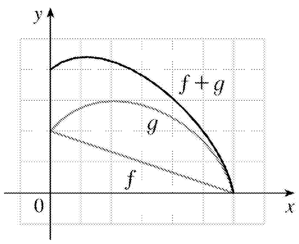


28. The most important features of the given graph are the x –intercepts and the maximum and minimum points. The graph of $y=1/f(x)$ has vertical asymptotes at the x –values where there are x –intercepts on the graph of $y=f(x)$. The maximum of 1 on the graph of $y=f(x)$ corresponds to a minimum of $1/1=1$ on $y=1/f(x)$. Similarly, the minimum on the graph of $y=f(x)$ corresponds to a maximum on the graph of $y=1/f(x)$. As the values of y get large (positively or negatively) on the graph of $y=f(x)$, the values of y get close to zero on the graph of $y=1/f(x)$.



29. Assuming that successive horizontal and vertical gridlines are a unit apart, we can make a table of approximate values as follows.

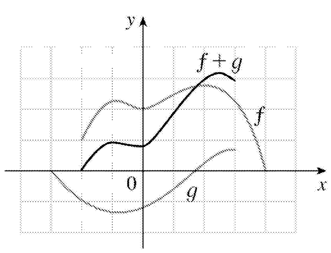
x	0	1	2	3	4	5	6
$f(x)$	2	1.7	1.3	1.0	0.7	0.3	0
$g(x)$	2	2.7	3	2.8	2.4	1.7	0
$g(x)+g(x)$	4	4.4	4.3	3.8	3.1	2.0	0



Connecting the points $(x,f(x)+g(x))$ with a smooth curve gives an approximation to the graph of $f+g$. Extra points can be plotted between those listed above if necessary.

30. First note that the domain of $f+g$ is the intersection of the domains of f and g ; that is, $f+g$ is only defined where both f and g are defined. Taking the horizontal and vertical units of length to be the distances between successive vertical and horizontal gridlines, we can make a table of approximate values as follows:

x	−2	−1	0	1	2	2.5	3
$f(x)$	−1	2.2	2.0	2.4	2.7	2.7	2.3
$g(x)$	1	−1.3	−1.2	−0.6	0.3	0.5	0.7
$f(x)+g(x)$	0	0.9	0.8	1.8	3.0	3.2	3.0



Extra values of x (like the value 2.5 in the table above) can be added as needed.

31. $f(x)=x^3+2x^2$; $g(x)=3x^2-1$. $D=R$ for both f and g .

$(f+g)(x)=(x^3+2x^2)+(3x^2-1)=x^3+5x^2-1$, $D=R$.

$(f-g)(x)=(x^3+2x^2)-(3x^2-1)=x^3-x^2+1$, $D=R$.

$(fg)(x)=(x^3+2x^2)(3x^2-1)=3x^5+6x^4-x^3-2x^2$, $D=R$.

$$\left(\frac{f}{g}\right)(x)=\frac{x^3+2x^2}{3x^2-1} \ , D=\left\{ x|x\neq\pm\frac{1}{\sqrt{3}} \right\} \text{ since } 3x^2-1\neq 0 \ .$$

32. $f(x)=\sqrt{1+x}$, $D=[-1,\infty)$; $g(x)=\sqrt{1-x}$, $D=(-\infty,1]$.

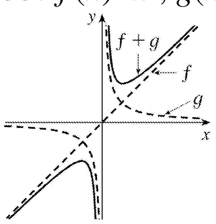
$(f+g)(x)=\sqrt{1+x}+\sqrt{1-x}$, $D=(-\infty,1]\cap[-1,\infty)=[-1,1]$.

$(f-g)(x)=\sqrt{1+x}-\sqrt{1-x}$, $D=[-1,1]$.

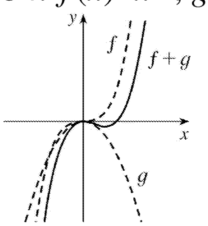
$(fg)(x)=\sqrt{1+x}\cdot\sqrt{1-x}=\sqrt{1-x^2}$, $D=[-1,1]$.

$\left(\frac{f}{g}\right)(x)=\frac{\sqrt{1+x}}{\sqrt{1-x}}$, $D=[-1,1)$. We must exclude $x=1$ since it would make $\frac{f}{g}$ undefined.

33. $f(x)=x$, $g(x)=1/x$



34. $f(x)=x^3$, $g(x)=-x^2$



35. $f(x)=2x^2-x$; $g(x)=3x+2$. $D=R$ for both f and g , and hence for their composites.
$(f\circ g)(x)=f(g(x))=f(3x+2)=2(3x+2)^2-(3x+2)=2(9x^2+12x+4)-3x-2=18x^2+21x+6$.
$(g\circ f)(x)=g(f(x))=g(2x^2-x)=3(2x^2-x)+2=6x^2-3x+2$.
$(f\circ f)(x)=f(f(x))=f(2x^2-x)=2(2x^2-x)^2-(2x^2-x)=2(4x^4-4x^3+x^2)-2x^2+x=8x^4-8x^3+x$.
$(g\circ g)(x)=g(g(x))=g(3x+2)=3(3x+2)+2=9x+6+2=9x+8$.

36. $f(x)=1-x^3$, $D=R$; $g(x)=1/x$, $D=\{x\,|\,x\neq 0\}$.

$(f\circ g)\,(x)=f(g(x))=f\,(1/x)=1-(1/x)^3=1-1/x^3$, $D=\{x\,|\,x\neq 0\}$.

$(g\circ f)\,(x)=g(f(x))=g\left(1-x^3\right)=1/(1-x^3)$, $D=\{x\,|\,1-x^3\neq 0\}=\{x\,|\,x\neq 1\}$.
$(f\circ f)\,(x)=f(f(x))=f(1-x^3)=1-(1-x^3)^3$ $[=x^9-3x^6+3x^3]$, $D=R$.
$(g\circ g)\,(x)=g(g(x))=g\,(1/x)=1/\,(1/x)=x$, $D=\{x\,|\,x\neq 0\}$ because 0 is not in the domain of g .

37. $f(x)=\sin\ x$, $D=R$; $g(x)=1-\sqrt{x}$, $D=[0,\infty)$.
$(f\circ g)\,(x)=f(g(x))=f(1-\sqrt{x})=\sin\ \left(1-\sqrt{x}\right)$, $D=[0,\infty]$.
$(g\circ f)\,(x)=g(f(x))=g\,(\sin\ x)=1-\sqrt{\sin\ x}$. For $\sqrt{\sin\ x}$ to be defined, we must have $\sin\ x\geq 0\Leftrightarrow$
$x\in[0,\pi]\cup[2\pi,3\pi]\cup[-2\pi,-\pi]\cup[4\pi,5\pi]\cup[-4\pi,-3\pi]\cup\ldots$, so
$D=\{x\,|\,x\in[2n\pi,\pi+2n\pi],\text{where } n \text{ is an integer}\}$.

$(f\circ f)(x)=f(f(x))=f(\sin x)=\sin(\sin x)$, $D=R$.
$(g\circ g)(x)=g(g(x))=g\left(1-\sqrt{x}\right)=1-\sqrt{1-\sqrt{x}}$,
$D=\left\{x\geq 0\,|\,1-\sqrt{x}\geq 0\right\}=\left\{x\geq 0\,|\,1\geq\sqrt{x}\right\}=\left\{x\geq 0\,|\,\sqrt{x}\leq 1\right\}=[0,1]$.

38. $f(x)=1-3x$, $D=R$; $g(x)=5x^2+3x+2$, $D=R$.

$$(f\circ g)(x)=f(g(x))=f(5x^2+3x+2)=1-3(5x^2+3x+2)$$
$$=1-15x^2-9x-6=-15x^2-9x-5\ ,\ D=R\ .$$
$$(g\circ f)(x)=g(f(x))=g(1-3x)=5(1-3x)^2+3(1-3x)+2=5(1-6x+9x^2)+3-9x+2$$
$$=5-30x+45x^2-9x+5=45x^2-39x+10\ ,\ D=R\ .$$
$$(f\circ f)(x)=f(f(x))=f(1-3x)=1-3(1-3x)=1-3+9x=9x-2,\ D=R\ .$$
$$(g\circ g)(x)=g(g(x))=g(5x^2+3x+2)=5(5x^2+3x+2)^2+3(5x^2+3x+2)+2$$
$$=5(25x^4+30x^3+29x^2+12x+4)+15x^2+9x+6+2$$
$$=125x^4+150x^3+145x^2+60x+20+15x^2+9x+8$$
$$=125x^4+150x^3+160x^2+69x+28,\ D=R\ .$$

39. $f(x)=x+\dfrac{1}{x}$, $D=\{x\,|\,x\neq 0\}$; $g(x)=\dfrac{x+1}{x+2}$, $D=\{x\,|\,x\neq -2\}$.

$$(f\circ g)(x)=f(g(x))=f\left(\frac{x+1}{x+2}\right)=\frac{x+1}{x+2}+\frac{1}{\dfrac{x+1}{x+2}}=\frac{x+1}{x+2}+\frac{x+2}{x+1}$$
$$=\frac{(x+1)(x+1)+(x+2)(x+2)}{(x+2)(x+1)}=\frac{\left(x^2+2x+1\right)+\left(x^2+4x+4\right)}{(x+2)(x+1)}=\frac{2x^2+6x+5}{(x+2)(x+1)}$$

Since $g(x)$ is not defined for $x=-2$ and $f(g(x))$ is not defined for $x=-2$ and $x=-1$, the domain of ($f\circ g)(x)$ is $D=\{x\,|\,x\neq -2,-1\}$.

$$(g\circ f)(x)=g(f(x))=g\left(x+\frac{1}{x}\right)=\frac{\left(x+\dfrac{1}{x}\right)+1}{\left(x+\dfrac{1}{x}\right)+2}=\frac{\dfrac{x^2+1+x}{x}}{\dfrac{x^2+1+2x}{x}}=\frac{x^2+x+1}{x^2+2x+1}=\frac{x^2+x+1}{(x+1)^2}\ .$$

Since $f(x)$ is not defined for $x=0$ and $g(f(x))$ is not defined for $x=-1$, the domain of $(g\circ f)(x)$ is $D=\{x\,|\,x\neq -1,0\}$.

$$(f\circ f)(x)=f(f(x))=f\left(x+\frac{1}{x}\right)=\left(x+\frac{1}{x}\right)+\frac{1}{x+\frac{1}{x}}=x+\frac{1}{x}+\frac{1}{\frac{x^2+1}{x}}=x+\frac{1}{x}+\frac{x}{x^2+1}$$

$$=\frac{x(x)\left(x^2+1\right)+1\left(x^2+1\right)+x(x)}{x(x^2+1)}=\frac{x^4+x^2+x^2+1+x^2}{x(x^2+1)}$$

$$=\frac{x^4+3x^2+1}{x(x^2+1)}\ ,\ D=\{x\,|\,x\neq 0\}\ .$$

$$(g\circ g)(x)=g(g(x))=g\left(\frac{x+1}{x+2}\right)=\frac{\frac{x+1}{x+2}+1}{\frac{x+1}{x+2}+2}=\frac{\frac{x+1+1(x+2)}{x+2}}{\frac{x+1+2(x+2)}{x+2}}=\frac{x+1+x+2}{x+1+2x+4}=\frac{2x+3}{3x+5}$$. Since $g(x)$ is not defined for $x=-2$ and $g(g(x))$ is not defined for $x=-\frac{5}{3}$, the domain of $(g\circ g)(x)$ is $D=\left\{x\,|\,x\neq-2,-\frac{5}{3}\right\}$.

40. $f(x)=\sqrt{2x+3}$, $D=\left\{x\,|\,x\geq-\frac{3}{2}\right\}$; $g(x)=x^2+1$, $D=R$.

$(f\circ g)(x)=f(x^2+1)=\sqrt{2(x^2+1)+3}=\sqrt{2x^2+5}$, $D=R$.

$(g\circ f)(x)=g\left(\sqrt{2x+3}\right)=\left(\sqrt{2x+3}\right)^2+1=(2x+3)+1=2x+4$, $D=\left\{x\,|\,x\geq-\frac{3}{2}\right\}$.

$(f\circ f)(x)=f\left(\sqrt{2x+3}\right)=\sqrt{2\left(\sqrt{2x+3}\right)+3}=\sqrt{2\sqrt{2x+3}+3}$, $D=\left\{x\,|\,x\geq-\frac{3}{2}\right\}$.

$(g\circ g)(x)=g(x^2+1)=(x^2+1)^2+1=(x^4+2x^2+1)+1=x^4+2x^2+2$, $D=R$.

41.

$$\begin{aligned}(f\circ g\circ h)(x)&=f(g(h(x)))=f(g(x-1))=f(2(x-1))\\&=2(x-1)+1=2x-1\end{aligned}$$

42.

$$\begin{aligned}(f\circ g\circ h)(x)&=f(g(h(x)))=f(g(1-x))=f((1-x)^2)\\&=2(1-x)^2-1=2x^2-4x+1\end{aligned}$$

43.

$$(f\circ g\circ h)(x)=f(g(h(x)))=f(g(x+3))=f((x+3)^2+2)$$
$$=f(x^2+6x+11)=\sqrt{\left(x^2+6x+11\right)-1}=\sqrt{x^2+6x+10}$$

44. $(f\circ g\circ h)(x)=f(g(h(x)))=f\left(g\left(\sqrt{x+3}\right)\right)=f\left(\cos\sqrt{x+3}\right)=\dfrac{2}{\cos\sqrt{x+3}+1}$

45. Let $g(x)=x^2+1$ and $f(x)=x^{10}$. Then $(f\circ g)(x)=f(g(x))=(x^2+1)^{10}=F(x)$.

46. Let $g(x)=\sqrt{x}$ and $f(x)=\sin x$. Then $(f\circ g)(x)=f(g(x))=\sin\left(\sqrt{x}\right)=F(x)$.

47. Let $g(x)=x^2$ and $f(x)=\dfrac{x}{x+4}$. Then $(f\circ g)(x)=f(g(x))=\dfrac{x^2}{x^2+4}=G(x)$.

48. Let $g(x)=x+3$ and $f(x)=1/x$. Then $(f\circ g)(x)=f(g(x))=1/(x+3)=G(x)$.

49. Let $g(t)=\cos t$ and $f(t)=\sqrt{t}$. Then $(f\circ g)(t)=f(g(t))=\sqrt{\cos t}=u(t)$.

50. Let $g(t)=\tan t$ and $f(t)=\dfrac{t}{1+t}$. Then $(f\circ g)(t)=f(g(t))=\dfrac{\tan t}{1+\tan t}=u(t)$.

51. Let $h(x)=x^2$, $g(x)=3^x$, and $f(x)=1-x$. Then $(f\circ g\circ h)(x)=1-3^{x^2}=H(x)$.

52. Let $h(x)=\sqrt{x}$, $g(x)=x-1$, and $f(x)=\sqrt[3]{x}$. Then $(f\circ g\circ h)(x)=\sqrt[3]{\sqrt{x}-1}=H(x)$.

53. Let $h(x)=\sqrt{x}$, $g(x)=\sec x$, and $f(x)=x^4$. Then $(f\circ g\circ h)(x)=\left(\sec\sqrt{x}\right)^4=\sec^4\left(\sqrt{x}\right)=H(x)$.

54. **(a)** $f(g(1))=f(6)=5$
(b) $g(f(1))=g(3)=2$
(c) $f(f(1))=f(3)=4$
(d) $g(g(1))=g(6)=3$
(e) $(g\circ f)(3)=g(f(3))=g(4)=1$
(f) $(f\circ g)(6)=f(g(6))=f(3)=4$

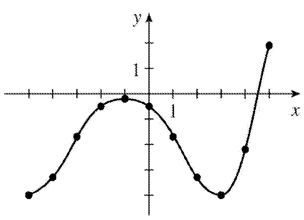
55. **(a)** $g(2)=5$, because the point $(2,5)$ is on the graph of g . Thus, $f(g(2))=f(5)=4$, because the point $(5,4)$ is on the graph of f .
(b) $g(f(0))=g(0)=3$

(c) $(f \circ g)(0)=f(g(0))=f(3)=0$
(d) $(g \circ f)(6)=g(f(6))=g(6)$. This value is not defined, because there is no point on the graph of g that has x –coordinate 6 .
(e) $(g \circ g)(-2)=g(g(-2))=g(1)=4$
(f) $(f \circ f)(4)=f(f(4))=f(2)=-2$

56. To find a particular value of $f(g(x))$, say for $x=0$, we note from the graph that $g(0)\approx 2.8$ and $f(2.8)\approx -0.5$. Thus, $f(g(0))\approx f(2.8)\approx -0.5$. The other values listed in the table were obtained in a similar fashion.

x	$g(x)$	$f(g(x))$
–5	–0.2	–4
–4	1.2	–3.3
–3	2.2	–1.7
–2	2.8	–0.5
–1	3	–0.2

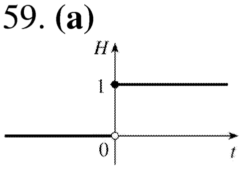
x	$g(x)$	$f(g(x))$
0	2.8	–0.5
1	2.2	–1.7
2	1.2	–3.3
3	–0.2	–4
4	–1.9	–2.2
5	–4.1	1.9



57. **(a)** Using the relationship *distance* = *rate* · *time* with the radius r as the distance, we have $r(t)=60t$.
(b) $A=\pi r^2 \Rightarrow (A\circ r)(t)=A(r(t))=\pi(60t)^2=3600\pi t^2$. This formula gives us the extent of the rippled area (in cm^2) at any time t .

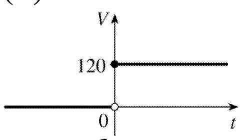
58. **(a)** $d=rt \Rightarrow d(t)=350t$
(b) There is a Pythagorean relationship involving the legs with lengths d and 1 and the hypotenuse with length s : $d^2+1^2=s^2$. Thus, $s(d)=\sqrt{d^2+1}$.
(c) $(s\circ d)(t)=s(d(t))=s(350t)=\sqrt{(350t)^2+1}$

59. **(a)**

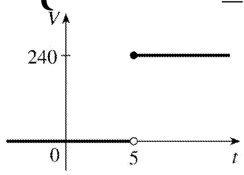


$$H(t)=\begin{cases} 0 & \text{if } t<0 \\ 1 & \text{if } t\geq 0 \end{cases}$$

(b)



$$V(t)=\begin{cases} 0 & \text{if } t<0 \\ 120 & \text{if } t\geq 0 \end{cases} \quad \text{so } V(t)=120H(t)\ .$$

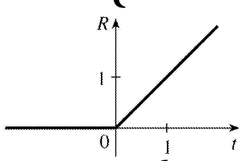


(c)

Starting with the formula in part (b), we replace 120 with 240 to reflect the different voltage. Also, because we are starting 5 units to the right of $t=0$, we replace t with $t-5$ Thus, the formula is $V(t)=240H(t-5)$.

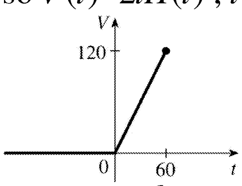
60. **(a)**

$$\begin{aligned} R(t) &= tH(t) \\ &= \begin{cases} 0 & \text{if } t<0 \\ t & \text{if } t\geq 0 \end{cases} \end{aligned}$$

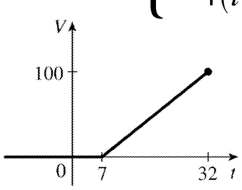


(b) $V(t)=\begin{cases} 0 & \text{if } t<0 \\ 2t & \text{if } 0\leq t\leq 60 \end{cases}$

so $V(t)=2tH(t)$, $t\leq 60$.



(c) $V(t)=\begin{cases} 0 & \text{if } t<7 \\ 4(t-7) & \text{if } 7\leq t\leq 32 \end{cases}$ so $V(t)=4(t-7)H(t-7)$, $t\leq 32$.



61. **(a)** By examining the variable terms in g and h , we deduce that we must square g to get the terms $4x^2$ and $4x$ in h . If we let $f(x)=x^2+c$, then $(f\circ g)(x)=f(g(x))=f(2x+1)=(2x+1)^2+c=4x^2+4x+(1+c)$.

Since $h(x)=4x^2+4x+7$, we must have $1+c=7$. So $c=6$ and $f(x)=x^2+6$.
(b) We need a function g so that $f(g(x))=3(g(x))+5=h(x)$. But $h(x)=3x^2+3x+2=3(x^2+x)+2=3(x^2+x-1)+5$, so we see that $g(x)=x^2+x-1$.

62. We need a function g so that $g(f(x))=g(x+4)=h(x)=4x-1=4(x+4)-17$. So we see that the function g must be $g(x)=4x-17$.

63. We need to examine $h(-x)$.

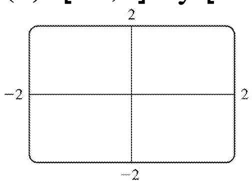
$$h(-x)=(f\circ g)(-x)=f(g(-x))=f(g(x)) \quad \text{[because } g \text{ is even]} \quad =h(x)$$

Because $h(-x)=h(x)$, h is an even function.

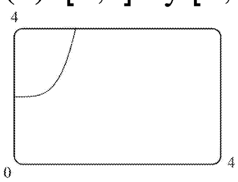
64. $h(-x)=f(g(-x))=f(-g(x))$. At this point, we can't simplify the expression, so we might try to find a counterexample to show that h is not an odd function. Let $g(x)=x$, an odd function, and $f(x)=x^2+x$. Then $h(x)=x^2$+x, which is neither even nor odd.
Now suppose f is an odd function. Then $f(-g(x))=-f(g(x))=-h(x)$. Hence, $h(-x)=-h(x)$, and so h is odd if both f and g are odd.
Now suppose f is an even function. Then $f(-g(x))=f(g(x))=h(x)$. Hence, $h(-x)=h(x)$, and so h is even if g is odd and f is even.

1. $f(x)=x^4+2$

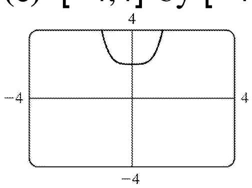
(a) $[-2,2]$ by $[-2,2]$



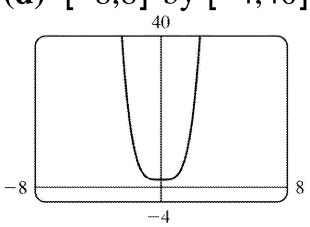
(b) $[0,4]$ by $[0,4]$



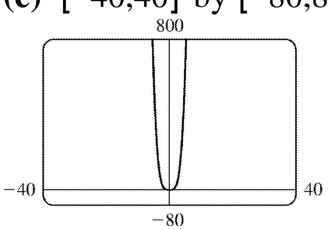
(c) $[-4,4]$ by $[-4,4]$



(d) $[-8,8]$ by $[-4,40]$



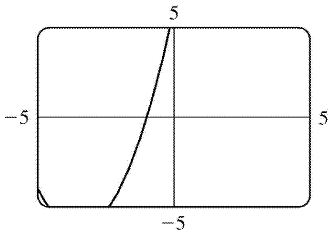
(e) $[-40,40]$ by $[-80,800]$



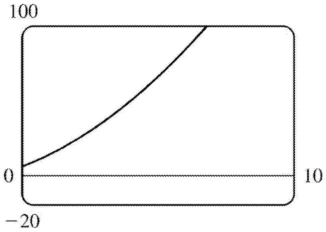
The most appropriate graph is produced in viewing rectangle (d).

2. $f(x)=x^2+7x+6$

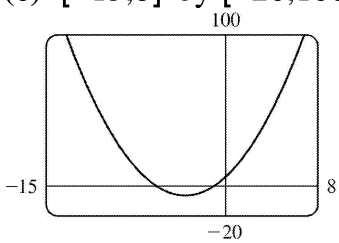
(a) $[-5,5]$ by $[-5,5]$



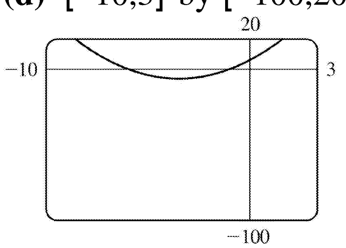
(b) $[0,10]$ by $[-20,100]$



(c) $[-15,8]$ by $[-20,100]$



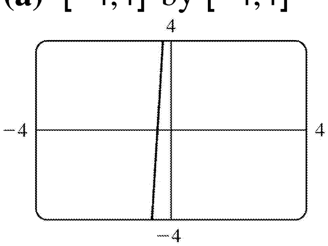
(d) $[-10,3]$ by $[-100,20]$



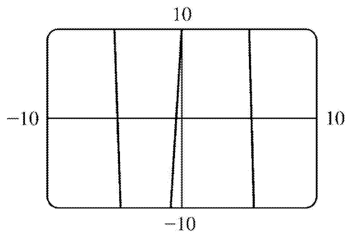
The most appropriate graph is produced in viewing rectangle (c).

3. $f(x)=10+25x-x^3$

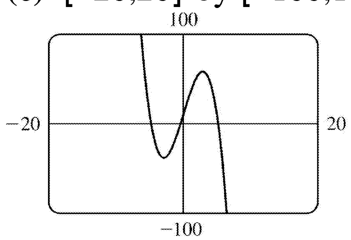
(a) $[-4,4]$ by $[-4,4]$



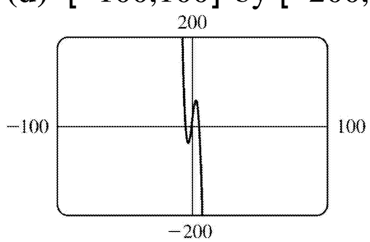
(b) $[-10,10]$ by $[-10,10]$



(c) $[-20,20]$ by $[-100,100]$

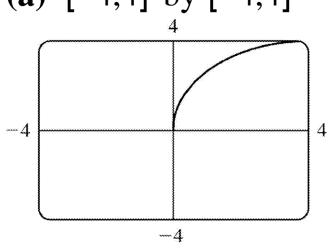


(d) $[-100,100]$ by $[-200,200]$

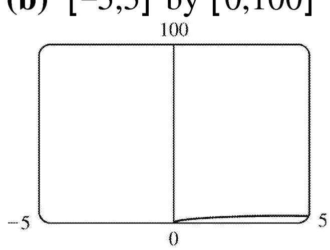


The most appropriate graph is produced in viewing rectangle (c) because the maximum and minimum points are fairly easy to see and estimate.

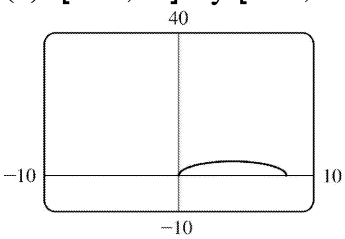
4. $f(x)=\sqrt{8x-x^2}$
(a) $[-4,4]$ by $[-4,4]$



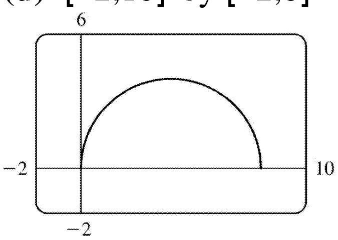
(b) $[-5,5]$ by $[0,100]$



(c) $[-10,10]$ by $[-10,40]$

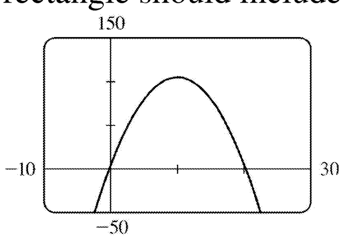


(d) $[-2,10]$ by $[-2,6]$

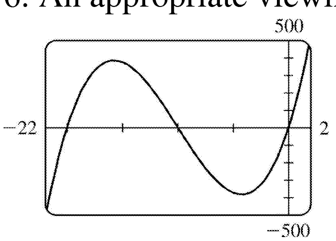


The most appropriate graph is produced in viewing rectangle (d).

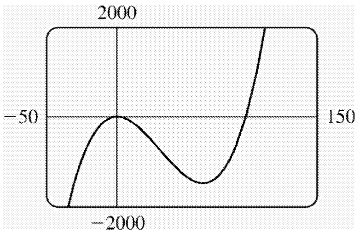
5. Since the graph of $f(x)=5+20x-x^2$ is a parabola opening downward, an appropriate viewing rectangle should include the maximum point.



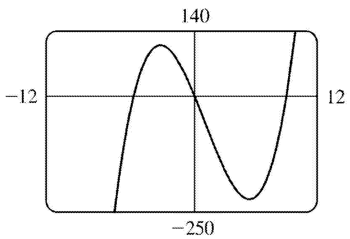
6. An appropriate viewing rectangle for $f(x)=x^3+30x^2+200x$ should include the high and low points.



7. $f(x)=0.01x^3-x^2+5$. Graphing f in a standard viewing rectangle, $[-10,10]$ by $[-10,10]$, shows us what appears to be a parabola. But since this is a cubic polynomial, we know that a larger viewing rectangle will reveal a minimum point as well as the maximum point. After some trial and error, we choose the viewing rectangle $[-50,150]$ by $[-2000,2000]$.

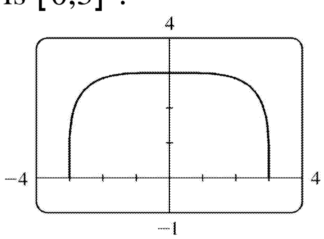


8. $f(x)=x(x+6)(x-9)$

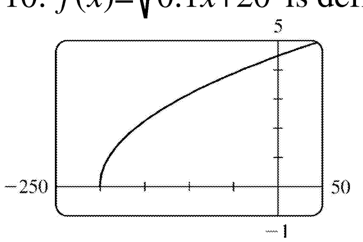


9. $f(x)=\sqrt[4]{81-x^4}$ is defined when

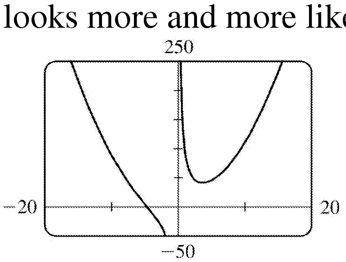
$81-x^4\geq 0\Leftrightarrow x^4\leq 81\Leftrightarrow |x|\leq 3$, so the domain of f is $[-3,3]$. Also $0\leq\sqrt[4]{81-x^4}\leq\sqrt[4]{81}=3$, so the range is $[0,3]$.



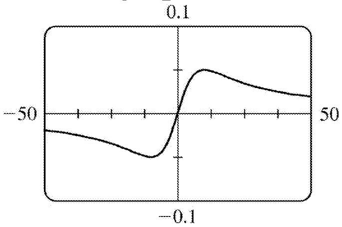
10. $f(x)=\sqrt{0.1x+20}$ is defined when $0.1x+20\geq 0\Leftrightarrow x\geq -200$, so the domain of f is $[-200,\infty)$.



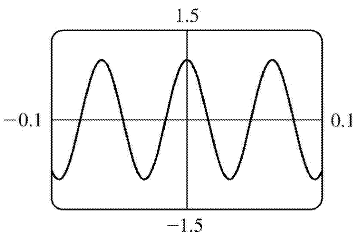
11. The graph of $f(x)=x^2+(100/x)$ has a vertical asymptote of $x=0$. As you zoom out, the graph of f looks more and more like that of $y=x^2$.



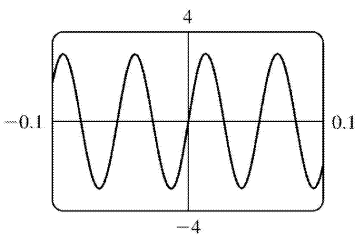
12. The graph of $f(x)=x/(x^2+100)$ is symmetric with respect to the origin.



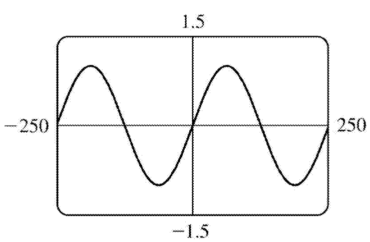
13. $f(x)=\cos(100x)$



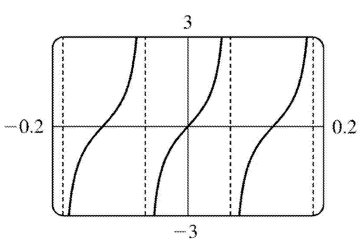
14. $f(x)=3\sin(120x)$



15. $f(x)=\sin(x/40)$

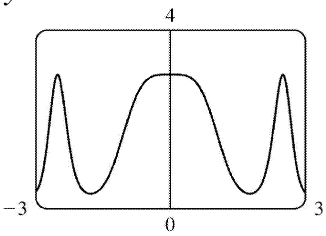


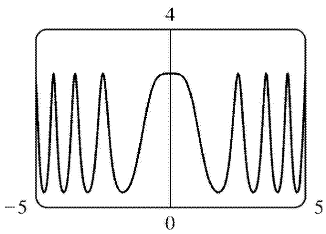
16. $f(x)=\tan(25x)$



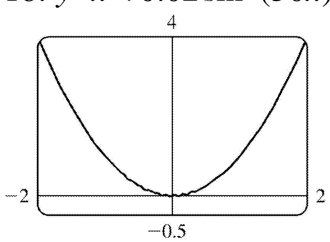
17.

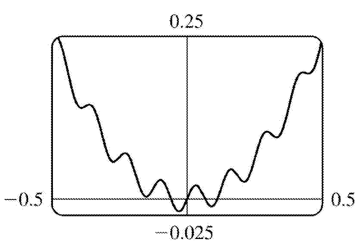
$y=3^{\cos\ (x^2)}$





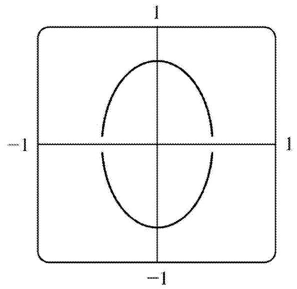
18. $y=x^2+0.02\sin\ (50x)$



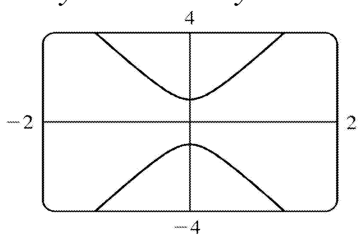


19. We must solve the given equation for y to obtain equations for the upper and lower halves of the ellipse.

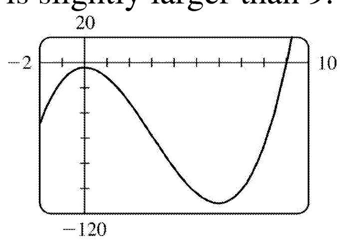
$$4x^2+2y^2=1 \Leftrightarrow 2y^2=1-4x^2 \Leftrightarrow y^2=\frac{1-4x^2}{2} \Leftrightarrow y=\pm\sqrt{\frac{1-4x^2}{2}}$$



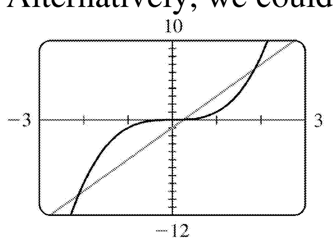
20. $y^2-9x^2=1 \Leftrightarrow y^2=1+9x^2 \Leftrightarrow y=\pm\sqrt{1+9x^2}$



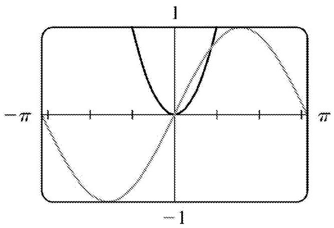
21. From the graph of $f(x)=x^3-9x^2-4$, we see that there is one solution of the equation $f(x)=0$ and it is slightly larger than 9. By zooming in or using a root or zero feature, we obtain $x\approx 9.05$.



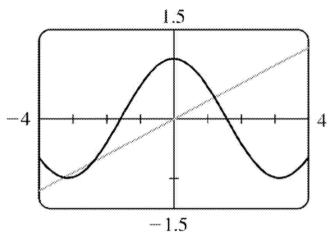
22. We see that the graphs of $f(x)=x^3$ and $g(x)=4x-1$ intersect three times. The x –coordinates of these points (which are the solutions of the equation) are approximately $-2.11, 0.25$, and 1.86 . Alternatively, we could find these values by finding the zeros of $h(x)=x^3-4x+1$.



23. We see that the graphs of $f(x)=x^2$ and $g(x)=\sin\ x$ intersect twice. One solution is $x=0$. The other solution of $f=g$ is the x –coordinate of the point of intersection in the first quadrant. Using an intersect feature or zooming in, we find this value to be approximately 0.88. Alternatively, we could find that value by finding the positive zero of $h(x)=x^2-\sin\ x$.

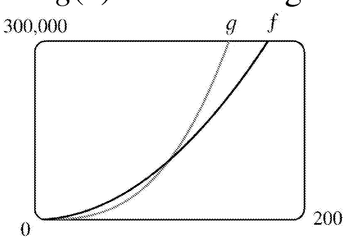


Note : After producing the graph on a TI–83 Plus, we can find the approximate value 0.88 by using the following keystrokes:
2nd CALC 5 ENTER ENTER 1 ENTER . The ''1'' is just a guess for 0.88.

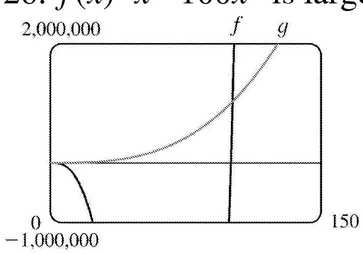


24. **(a)**
The x –coordinates of the three points of intersection are $x \approx -3.29$, -2.36 and 1.20 .
(b) Using trial and error, we find that $m \approx 0.3365$. Note that m could also be negative.

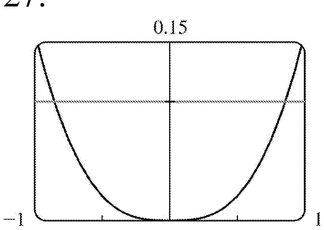
25. $g(x)=x^3/10$ is larger than $f(x)=10x^2$ whenever $x>100$.



26. $f(x)=x^4-100x^3$ is larger than $g(x)=x^3$ whenever $x>101$.

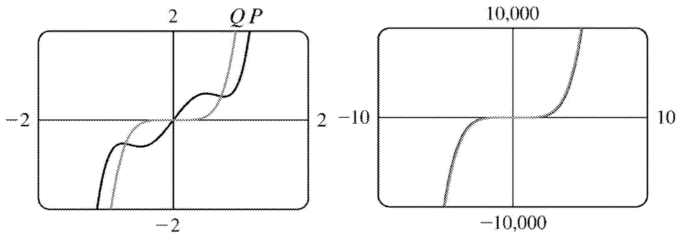


27.



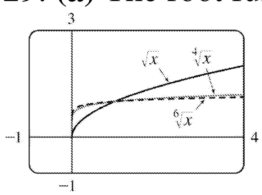
We see from the graphs of $y=|\sin x-x|$ and $y=0.1$ that there are two solutions to the equation

$|\sin x - x| = 0.1$: $x \approx -0.85$ and $x \approx 0.85$. The condition $|\sin x - x| < 0.1$ holds for any x lying between these two values.

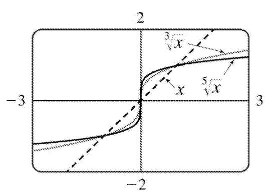
28. 

$P(x)=3x^5-5x^3+2x$, $Q(x)=3x^5$. These graphs are significantly different only in the region close to the origin. The larger a viewing rectangle one chooses, the more similar the two graphs look.

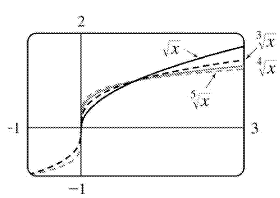
29. **(a)** The root functions $y=\sqrt{x}$, $y=\sqrt[4]{x}$ and $y=\sqrt[6]{x}$



(b) The root functions $y=x$, $y=\sqrt[3]{x}$ and $y=\sqrt[5]{x}$



(c) The root functions $y=\sqrt{x}$, $y=\sqrt[3]{x}$, $y=\sqrt[4]{x}$ and $y=\sqrt[5]{x}$

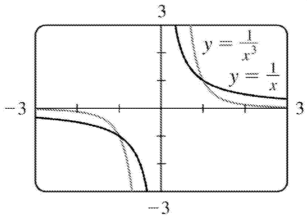


(d)

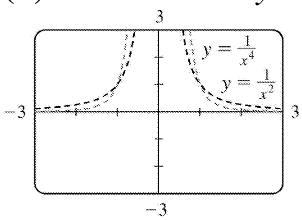
- For any n , the n th root of 0 is 0 and the n th root of 1 is 1 ; that is, all n th root functions pass through the points $(0,0)$ and $(1,1)$.
- For odd n , the domain of the n th root function is R , while for even n , it is $\{x \in R | x \geq 0\}$.
- Graphs of even root functions look similar to that of $\sqrt{x}$, while those of odd root functions resemble that of $\sqrt[3]{x}$.

- As n increases, the graph of $\sqrt[n]{x}$ becomes steeper near 0 and flatter for $x>1$.

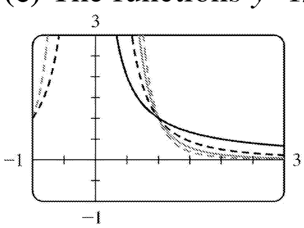
30. **(a)** The functions $y=1/x$ and $y=1/x^3$



(b) The functions $y=1/x^2$ and $y=1/x^4$



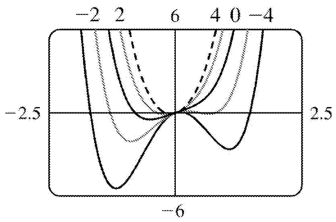
(c) The functions $y=1/x$, $y=1/x^2$, $y=1/x^3$ and $y=1/x^4$



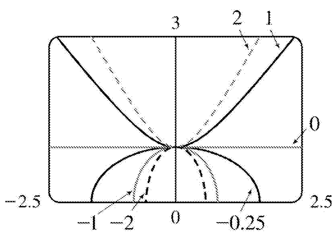
(d)

- The graphs of all functions of the form $y=1/x^n$ pass through the point $(1,1)$.
- If n is even, the graph of the function is entirely above the $x-$ axis. The graphs of $1/x^n$ for n even are similar to one another.
- If n is odd, the function is positive for positive x and negative for negative x . The graphs of $1/x^n$ for n odd are similar to one another.
- As n increases, the graphs of $1/x^n$ approach 0 faster as $x\to\infty$.

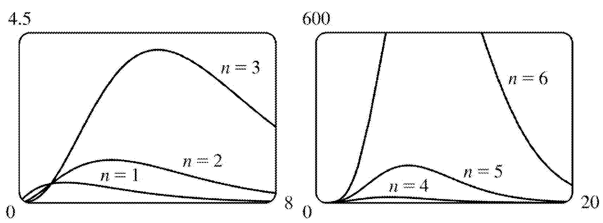
31. $f(x)=x^4+cx^2+x$. If $c<0$, there are three humps: two minimum points and a maximum point. These humps get flatter as c increases, until at $c=0$ two of the humps disappear and there is only one minimum point. This single hump then moves to the right and approaches the origin as c increases.



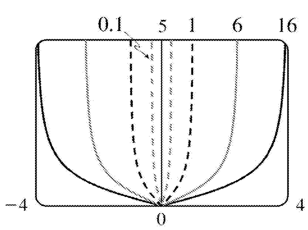
32. $f(x)=\sqrt{1+cx^2}$. If $c<0$, the function is only defined on $\left[-1/\sqrt{-c}\ ,1/\sqrt{-c}\ \right]$, and its graph is the top half of an ellipse. If $c=0$, the graph is the line $y=1$. If $c>0$, the graph is the top half of a hyperbola. As c approaches 0 , these curves become flatter and approach the line $y=1$.



33. $y=x^n 2^{-x}$. As n increases, the maximum of the function moves further from the origin, and gets larger. Note, however, that regardless of n , the function approaches 0 as $x\to\infty$.



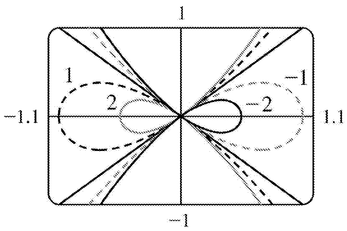
34. $y=\dfrac{|x|}{\sqrt{c-x^2}}$. The ‘‘bullet’’ becomes broader as c increases.



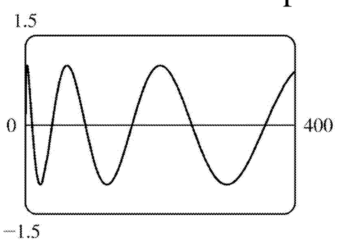
35. $y^2=cx^3+x^2$

If $c<0$, the loop is to the right of the origin, and if c is positive, it is to the left. In both cases, the closer c is to 0 , the larger the loop is.

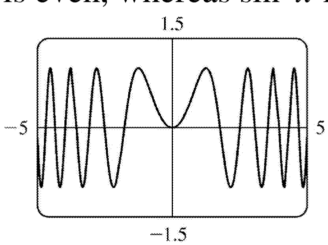
(In the limiting case, $c=0$, the loop is ''infinite'', that is, it doesn't close.) Also, the larger $|c|$ is, the steeper the slope is on the loopless side of the origin.



36. **(a)** $y=\sin\left(\sqrt{x}\right)$
This function is not periodic; it oscillates less frequently as x increases.



(b) $y=\sin\left(x^2\right)$
This function oscillates more frequently as $|x|$ increases. Note also that this function is even, whereas $\sin x$ is odd.



37. The graphing window is 95 pixels wide and we want to start with $x=0$ and end with $x=2\pi$. Since there are 94 ''gaps'' between pixels, the distance between pixels is $\frac{2\pi-0}{94}$. Thus, the x –values that the calculator actually plots are $x=0+\frac{2\pi}{94}\cdot n$, where $n=0, 1, 2, \ldots, 93, 94$. For $y=\sin 2x$, the actual points plotted by the calculator are $\left(\frac{2\pi}{94}\cdot\sin\left(2\cdot\frac{2\pi}{94}\cdot n\right)\right)$ for $n=0, 1, \ldots, 94$. For $y=\sin 96x$, the points plotted are $\left(\frac{2\pi}{94}\cdot\sin\left(96\cdot\frac{2\pi}{94}\cdot n\right)\right)$ for $n=0, 1, \ldots, 94$. But

$$\sin\left(96\cdot\frac{2\pi}{94}\cdot n\right)=\sin\left(94\cdot\frac{2\pi}{94}\cdot n+2\cdot\frac{2\pi}{94}\cdot n\right)=\sin\left(2\pi n+2\cdot\frac{2\pi}{94}\cdot n\right)$$

$$=\sin\left(2\cdot\frac{2\pi}{94}\cdot n\right) \text{ [by periodicity of sine]}, n=0, 1, \ldots, 94$$

So the y –values, and hence the points, plotted for $y=\sin 96x$ are identical to those plotted for $y=\sin 2x$.
Note: Try graphing $y=\sin 94x$. Can you see why all the y –values are zero?

38. As in Exercise 37, we know that the points being plotted for $y=\sin 45x$ are $\left(\frac{2\pi}{94}\cdot\sin\left(45\cdot\frac{2\pi}{94}\cdot n\right)\right)$ for $n=0, 1, \ldots, 94$. But

$$\sin\left(45\cdot\frac{2\pi}{94}\cdot n\right)=\sin\left(47\cdot\frac{2\pi}{94}\cdot n-2\cdot\frac{2\pi}{94}\cdot n\right)=\sin\left(n\pi-2\cdot\frac{2\pi}{94}\cdot n\right)$$

$$=\sin(n\pi)\cos\left(2\cdot\frac{2\pi}{94}\cdot n\right)-\cos(n\pi)\sin\left(2\cdot\frac{2\pi}{94}\cdot n\right) \text{ [Subtraction formula for the sine]}$$

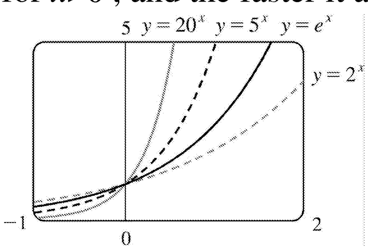
$$=0\cdot\cos\left(2\cdot\frac{2\pi}{94}\cdot n\right)-(\pm 1)\sin\left(2\cdot\frac{2\pi}{94}\cdot n\right)=\pm\sin\left(2\cdot\frac{2\pi}{94}\cdot n\right), n=0, 1, \ldots, 94$$

So the y –values, and hence the points, plotted for $y=\sin 45x$ lie on either $y=\sin 2x$ or $y=-\sin 2x$.

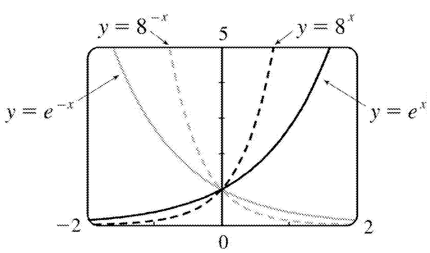
1. **(a)** $f(x)=a^x$, $a>0$
(b) R
(c) $(0,\infty)$
(d) See Figures (c), (b), and (a), respectively.

2. **(a)** The number e is the value of a such that the slope of the tangent line at $x=0$ on the graph of $y=a^x$ is exactly 1 .
(b) $e\approx 2.71828$
(c) $f(x)=e^x$

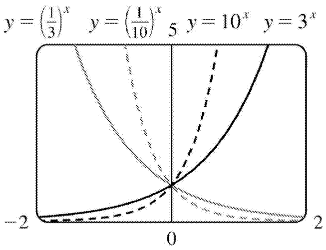
3. All of these graphs approach 0 as $x\to-\infty$, all of them pass through the point $(0,1)$, and all of them are increasing and approach ∞ as $x\to\infty$. The larger the base, the faster the function increases for $x>0$, and the faster it approaches 0 as $x\to-\infty$.



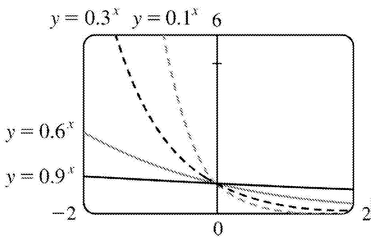
4. The graph of e^{-x} is the reflection of the graph of e^x about the y– axis, and the graph of 8^{-x} is the reflection of that of 8^x about the y –axis. The graph of 8^x increases more quickly than that of e^x for $x>0$, and approaches 0 faster as $x\to-\infty$.



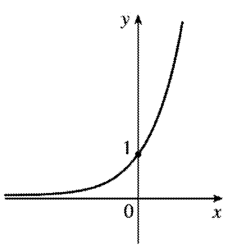
5. The functions with bases greater than 1 (3^x and 10^x) are increasing, while those with bases less than 1 $\left[\left(\frac{1}{3}\right)^x \text{and} \left(\frac{1}{10}\right)^x\right]$ are decreasing. The graph of $\left(\frac{1}{3}\right)^x$ is the reflection of that of 3^x about the y– axis, and the graph of $\left(\frac{1}{10}\right)^x$ is the reflection of that of 10^x about the y– axis. The graph of 10^x increases more quickly than that of 3^x for $x>0$, and approaches 0 faster as $x\to-\infty$.

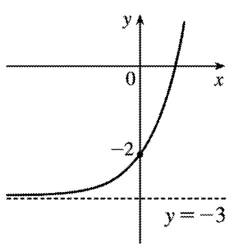


6. Each of the graphs approaches ∞ as $x \to -\infty$, and each approaches 0 as $x \to \infty$. The smaller the base, the faster the function grows as $x \to -\infty$, and the faster it approaches 0 as $x \to \infty$.



7. We start with the graph of $y=4^x$ (Figure 3) and then shift 3 units downward. This shift doesn't affect the domain, but the range of $y=4^x-3$ is $(-3,\infty)$. There is a horizontal asymptote of $y=-3$.

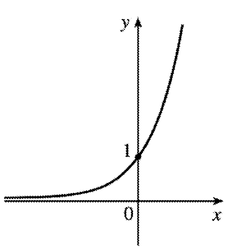


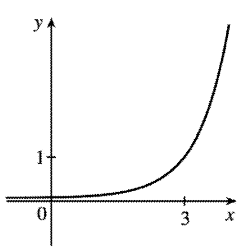


$y=4^x$

$y=4^x-3$

8. We start with the graph of $y=4^x$ (Figure 3) and then shift 3 units to the right. There is a horizontal asymptote of $y=0$.

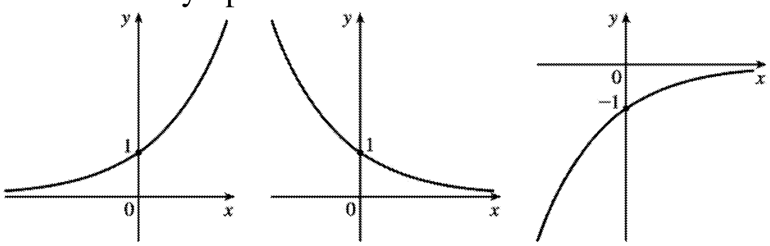




$y=4^x$

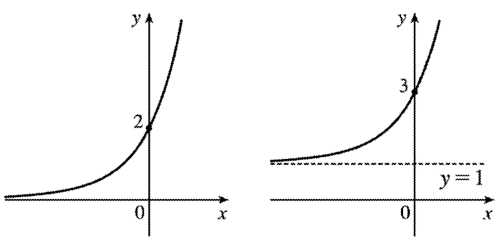
$y=4^{x-3}$

9. We start with the graph of $y=2^x$ (Figure 2), reflect it about the y –axis, and then about the x –axis (or just rotate 180° to handle both reflections) to obtain the graph of $y=-2^{-x}$. In each graph, $y=0$ is the horizontal asymptote.



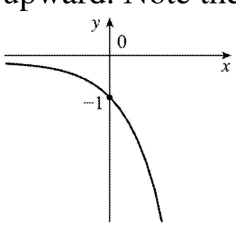
$y=2^x$ $y=2^{-x}$ $y=-2^{-x}$

10. We start with the graph of $y=e^x$ (Figure 13), vertically stretch by a factor of 2, and then shift 1 unit upward. There is a horizontal asymptote of $y=1$.

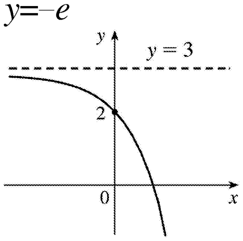


$y=2e^x$ $y=1+2e^x$

11. We start with the graph of $y=e^x$ (Figure 13), reflect it about the x –axis, and then shift 3 units upward. Note the horizontal asymptote of $y=3$.



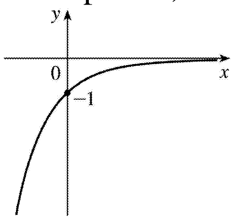
$y=-e^x$



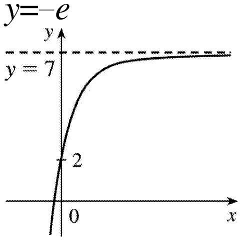
$y=3-e^x$

12. We start with the graph of $y=e^x$ (Figure 13), reflect it about the y –axis, and then about the x –axis

(or just rotate 180° to handle both reflections) to obtain the graph of $y=-e^{-x}$. Now shift this graph 1 unit upward, vertically stretch by a factor of 5 , and then shift 2 units upward.



$y=-e^{-x}$



$y=2+5(1-e^{-x})$

13. **(a)** To find the equation of the graph that results from shifting the graph of $y=e^{x}$ 2 units downward, we subtract 2 from the original function to get $y=e^{x}-2$.
(b) To find the equation of the graph that results from shifting the graph of $y=e^{x}$ 2 units to the right, we replace x with $x-2$ in the original function to get $y=e^{(x-2)}$.
(c) To find the equation of the graph that results from reflecting the graph of $y=e^{x}$ about the x –axis, we multiply the original function by -1 to get $y=-e^{x}$.
(d) To find the equation of the graph that results from reflecting the graph of $y=e^{x}$ about the y –axis, we replace x with $-x$ in the original function to get $y=e^{-x}$.
(e) To find the equation of the graph that results from reflecting the graph of $y=e^{x}$ about the x –axis and then about the y –axis, we first multiply the original function by -1 (to get $y=-e^{x}$) and then replace x with $-x$ in this equation to get $y=-e^{-x}$.

14. **(a)** This reflection consists of first reflecting the graph about the x –axis (giving the graph with equation $y=-e^{x}$) and then shifting this graph $2\cdot 4=8$ units upward. So the equation is $y=-e^{x}+8$.
(b) This reflection consists of first reflecting the graph about the y –axis (giving the graph with equation $y=e^{-x}$) and then shifting this graph $2\cdot 2=4$ units to the right. So the equation is $y=e^{-(x-4)}$.

15. **(a)** The denominator $1+e^{x}$ is never equal to zero because $e^{x}>0$, so the domain of $f(x)=1/(1+e^{x})$ is R .

(b) $1-e^x=0 \Leftrightarrow e^x=1 \Leftrightarrow x=0$, so the domain of $f(x)=1/(1-e^x)$ is $(-\infty,0)\cup(0,\infty)$.

16. **(a)** The sine and exponential functions have domain R , so $g(t)=\sin(e^{-t})$ also has domain R .
(b) The function $g(t)=\sqrt{1-2^t}$ has domain $\{t \mid 1-2^t\geq 0\}=\{t \mid 2^t\leq 1\}=\{t \mid t\leq 0\}=(-\infty,0]$.

17. Use $y=Ca^x$ with the points $(1,6)$ and $(3,24)$. $6=Ca^1$ $\left[C=\frac{6}{a}\right]$ and $24=Ca^3 \Rightarrow 24=\left(\frac{6}{a}\right)a^3 \Rightarrow$ $4=a^2 \Rightarrow a=2$ [since $a>0$] and $C=\frac{6}{2}=3$. The function is $f(x)=3\cdot 2^x$.

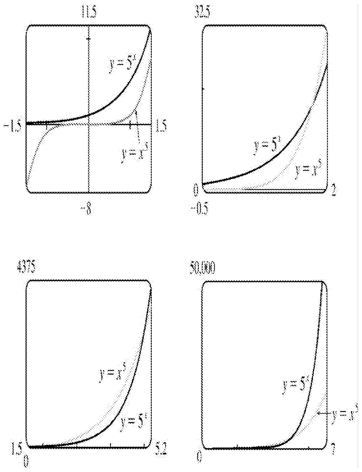
18. Given the y –intercept $(0,2)$, we have $y=Ca^x=2a^x$. Using the point $\left(2,\frac{2}{9}\right)$ gives us $\frac{2}{9}=2a^2 \Rightarrow$ $\frac{1}{9}=a^2 \Rightarrow a=\frac{1}{3}$ [since $a>0$]. The function is $f(x)=2\left(\frac{1}{3}\right)^x$ or $f(x)=2(3)^{-x}$.

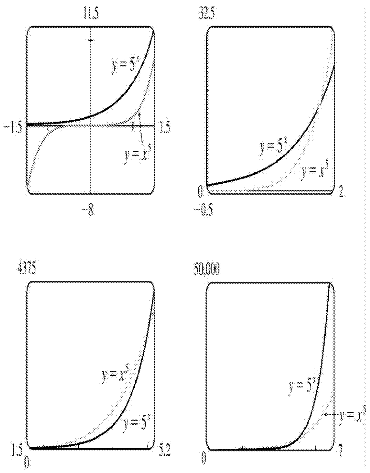
19. If $f(x)=5^x$, then $\frac{f(x+h)-f(x)}{h}=\frac{5^{x+h}-5^x}{h}=\frac{5^x5^h-5^x}{h}=\frac{5^x\left(5^h-1\right)}{h}=5^x\left(\frac{5^h-1}{h}\right)$.

20. Suppose the month is February. Your payment on the 28th day would be $2^{28-1}=2^{27}=134,217,728$ cents, or \$1 , 342 , 177 . 28 . Clearly, the second method of payment results in a larger amount for any month.

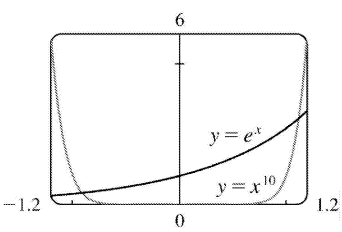
21. 2 ft =24 in, $f(24)=24^2$ in =576 in =48 ft. $g(24)=2^{24}$ in $=2^{24}/(12\cdot 5280)$ mi ≈ 265 mi

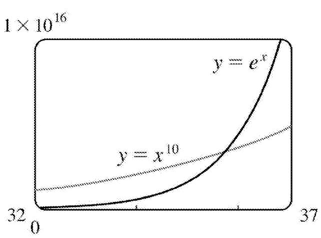
22. We see from the graphs that for x less than about 1.8 , $g(x)=5^x>f(x)=x^5$, and then near the point $(1.8,17.1)$ the curves intersect. Then $f(x)>g(x)$ from $x\approx 1.8$ until $x=5$. At $(5,3125)$ there is another point of intersection, and for $x>5$ we see that $g(x)>f(x)$. In fact, g increases much more rapidly than f beyond that point.





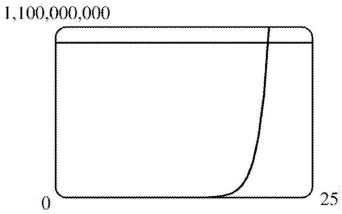
23. The graph of g finally surpasses that of f at $x \approx 35.8$.



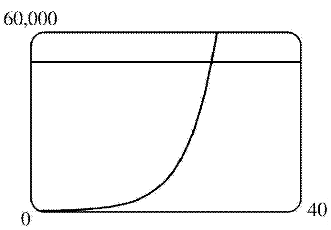


24. We graph $y=e^x$ and $y=1,000,000,000$ and determine where $e^x=1\times 10^9$. This seems to be true at $x\approx 20.723$, so

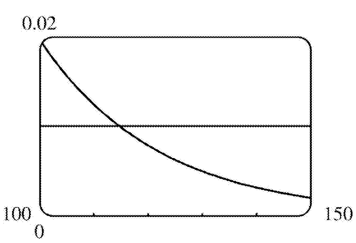
$e^x > 1 \times 10^9$ for $x > 20.723$.



25. **(a)** Fifteen hours represents 5 doubling periods (one doubling period is three hours).
$100 \cdot 2^5 = 3200$
(b) In t hours, there will be $t/3$ doubling periods. The initial population is 100 , so the population y at time t is $y = 100 \cdot 2^{t/3}$.
(c) $t = 20 \Rightarrow y = 100 \cdot 2^{20/3} \approx 10 , 159$
(d) We graph $y_1 = 100 \cdot 2^{x/3}$ and $y_2 = 50 , 000$. The two curves intersect at $x \approx 26.9$, so the population reaches 50 , 000 in about 26.9 hours.

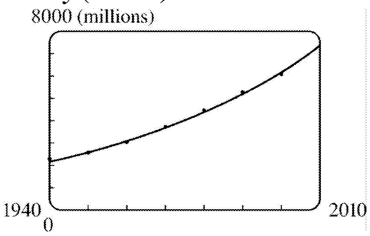


26. **(a)** Sixty hours represents 4 half–life periods. $2 \cdot \left(\frac{1}{2} \right)^4 = \frac{1}{8}$ g
(b) In t hours, there will be $t/15$ half–life periods. The initial mass is 2 g, so the mass y at time t is $y = 2 \cdot \left(\frac{1}{2} \right)^{t/15}$.
(c) 4 days $= 4 \cdot 24 = 96$ hours. $t = 96 \Rightarrow y = 2 \cdot \left(\frac{1}{2} \right)^{96/15} \approx 0.024$ g
(d) $y = 0.01 \Rightarrow t \approx 114.7$ hours

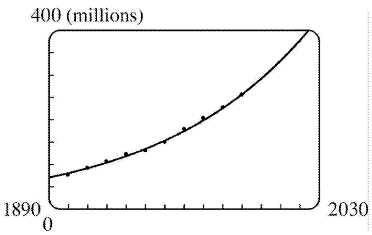


27. An exponential model is

$y = ab^t$, where $a = 3.154832569 \times 10^{-12}$ and $b = 1.017764706$. This model gives $y(1993) \approx 5498$ million and $y(2010) \approx 7417$ million.



28. An exponential model is $y = ab^t$, where $a = 1.9976760197589 \times 10^{-9}$ and $b = 1.0129334321697$. This model gives $y(1925) \approx 111$ million, $y(2010) \approx 330$ million, and $y(2020) \approx 375$ million.



1. **(a)** See Definition 1.
(b) It must pass the Horizontal Line Test.

2. **(a)** $f^{-1}(y)=x \Leftrightarrow f(x)=y$ for any y in B . The domain of f^{-1} is B and the range of f^{-1} is A .
(b) See the steps in (5).
(c) Reflect the graph of f about the line $y=x$.

3. f is not one–to–one because $2\neq 6$, but $f(2)=2.0=f(6)$.

4. f is one–to–one since for any two different domain values, there are different range values.

5. No horizontal line intersects the graph of f more than once. Thus, by the Horizontal Line Test, f is one–to–one.

6. The horizontal line $y=0$ (the x –axis) intersects the graph of f in more than one point. Thus, by the Horizontal Line Test, f is not one–to–one.

7. The horizontal line $y=0$ (the x –axis) intersects the graph of f in more than one point. Thus, by the Horizontal Line Test, f is not one–to–one.

8. No horizontal line intersects the graph of f more than once. Thus, by the Horizontal Line Test, f is one–to–one.

9. The graph of $f(x)=\frac{1}{2}(x+5)$ is a line with slope $\frac{1}{2}$. It passes the Horizontal Line Test, so f is one–to–one.
Algebraic solution : If $x_1\neq x_2$, then $x_1+5\neq x_2+5\Rightarrow \frac{1}{2}\left(x_1+5\right)\neq \frac{1}{2}\left(x_2+5\right)\Rightarrow f\left(x_1\right)\neq f\left(x_2\right)$, so f is one–to–one.

10. The graph of $f(x)=1+4x-x^2$ is a parabola with axis of symmetry $x=-\frac{b}{2a}=-\frac{4}{2(-1)}=2$. Pick any x –values equidistant from 2 to find two equal function values. For example, $f(1)=4$ and $f(3)=4$, so f is not $1-1$.

11. $g(x)=|x|\Rightarrow g(-1)=1=g(1)$, so g is not one–to–one.

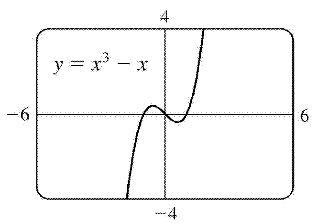
12. $x_1\neq x_2\Rightarrow \sqrt{x_1}\neq\sqrt{x_2}\Rightarrow g\left(x_1\right)\neq g\left(x_2\right)$, so g is $1-1$.

13. A football will attain every height h up to its maximum height twice: once on the way up, and

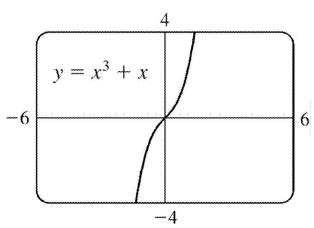
again on the way down. Thus, even if t_1 does not equal t_2 , $f\left(t_1\right)$ may equal $f\left(t_2\right)$, so f is not $1-1$.

14. f is not $1-1$ because eventually we all stop growing and therefore, there are two times at which we have the same height.

15. f does not pass the Horizontal Line Test, so f is not $1-1$.



16. f passes the Horizontal Line Test,
so f is $1-1$.



17. Since $f(2)=9$ and f is $1-1$, we know that $f^{-1}(9)=2$. Remember, if the point $(2,9)$ is on the graph of f , then the point $(9,2)$ is on the graph of f^{-1} .

18. **(a)** First, we must determine x such that $f(x)=3$. By inspection, we see that if $x=0$, then $f(x)=3$. Since f is $1-1$ (f is an increasing function), it has an inverse, and $f^{-1}(3)=0$.
(b) By the second cancellation equation in (4), we have $f\left(f^{-1}(5)\right)=5$.

19. First, we must determine x such that $g(x)=4$. By inspection, we see that if $x=0$, then $g(x)=4$. Since g is $1-1$ (g is an increasing function), it has an inverse, and $g^{-1}(4)=0$.

20. **(a)** f is $1-1$ because it passes the Horizontal Line Test.
(b) Domain of $f=[-3,3]=$ Range of f^{-1} . Range of $f=[-2,2]=$ Domain of f^{-1} .
(c) Since $f(-2)=1$, $f^{-1}(1)=-2$.

21. We solve

$C=\frac{5}{9}(F-32)$ for F : $\frac{9}{5}C=F-32\Rightarrow F=\frac{9}{5}C+32$. This gives us a formula for the inverse function, that is, the Fahrenheit temperature F as a function of the Celsius temperature C . $F\geq -459.67\Rightarrow$ $\frac{9}{5}C+32\geq -459.67\Rightarrow \frac{9}{5}C\geq -491.67\Rightarrow C\geq -273.15$, the domain of the inverse function.

22. $m=\dfrac{m_0}{\sqrt{1-v^2/c^2}}\Rightarrow 1-\dfrac{v^2}{c^2}=\dfrac{m_0^2}{m^2}\Rightarrow \dfrac{v^2}{c^2}=1-\dfrac{m_0^2}{m^2}\Rightarrow v^2=c^2\left(1-\dfrac{m_0^2}{m^2}\right)\Rightarrow v=c\sqrt{1-\dfrac{m_0^2}{m^2}}$. This formula gives us the speed v of the particle in terms of its mass m , that is, $v=f^{-1}(m)$.

23. $f(x)=\sqrt{10-3x}\Rightarrow y=\sqrt{10-3x}\ (y\geq 0)\Rightarrow y^2=10-3x\Rightarrow 3x=10-y^2\Rightarrow x=-\frac{1}{3}y^2+\frac{10}{3}$. Interchange x and y : $y=-\frac{1}{3}x^2+\frac{10}{3}$. So $f^{-1}(x)=-\frac{1}{3}x^2+\frac{10}{3}$. Note that the domain of f^{-1} is $x\geq 0$.

24. $f(x)=\dfrac{4x-1}{2x+3}\Rightarrow y=\dfrac{4x-1}{2x+3}\Rightarrow y(2x+3)=4x-1\Rightarrow 2xy+3y=4x-1\Rightarrow 3y+1=4x-2xy\Rightarrow 3y+1=(4-2y)x\Rightarrow$ $x=\dfrac{3y+1}{4-2y}$. Interchange x and y : $y=\dfrac{3x+1}{4-2x}$.
So $f^{-1}(x)=\dfrac{3x+1}{4-2x}$.

25. $f(x)=e^{x^3}\Rightarrow y=e^{x^3}\Rightarrow \ln y=x^3\Rightarrow x=\sqrt[3]{\ln y}$. Interchange x and y : $y=\sqrt[3]{\ln x}$.
So $f^{-1}(x)=\sqrt[3]{\ln x}$.

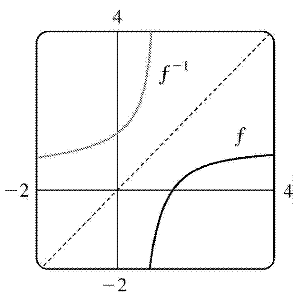
26. $y=f(x)=2x^3+3\Rightarrow y-3=2x^3\Rightarrow \dfrac{y-3}{2}=x^3\Rightarrow x=\sqrt[3]{\dfrac{y-3}{2}}$.
Interchange x and y : $y=\sqrt[3]{\dfrac{x-3}{2}}$. So $f^{-1}(x)=\sqrt[3]{\dfrac{x-3}{2}}$.

27. $y=\ln(x+3)\Rightarrow x+3=e^y\Rightarrow x=e^y-3$. Interchange x and y : $y=e^x-3$. So $f^{-1}(x)=e^x-3$.

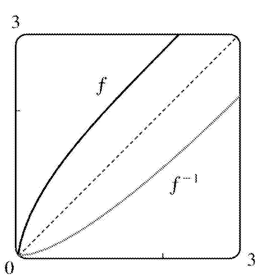
28. $y=\dfrac{1+e^x}{1-e^x}\Rightarrow y-ye^x=1+e^x\Rightarrow y-1=ye^x+e^x\Rightarrow y-1=e^x(y+1)\Rightarrow$

$e^x=\dfrac{y-1}{y+1}\Rightarrow x=\ln\left(\dfrac{y-1}{y+1}\right)$. Interchange x and y : $y=\ln\left(\dfrac{x-1}{x+1}\right)$. So $f^{-1}(x)=\ln\left(\dfrac{x-1}{x+1}\right)$. Note that the domain of f^{-1} is $|x|>1$.

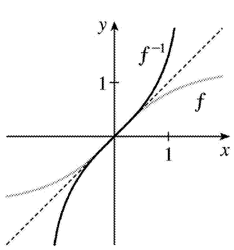
29. $y=f(x)=1-\dfrac{2}{x^2}\Rightarrow 1-y=\dfrac{2}{x^2}\Rightarrow x^2=\dfrac{2}{1-y}\Rightarrow x=\sqrt{\dfrac{2}{1-y}}$, since $x>0$. Interchange x and y : $y=\sqrt{\dfrac{2}{1-x}}$. So $f^{-1}(x)=\sqrt{\dfrac{2}{1-x}}$.



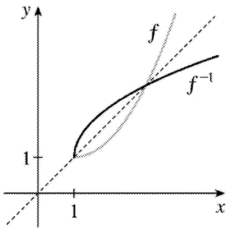
30. $y=f(x)=\sqrt{x^2+2x}$, $x>0\Rightarrow y>0$ and $y^2=x^2+2x\Rightarrow x^2+2x-y^2=0$. Now we use the quadratic formula: $x=\dfrac{-2\pm\sqrt{2^2-4\cdot 1\cdot\left(-y^2\right)}}{2\cdot 1}=-1\pm\sqrt{1+y^2}$. But $x>0$, so the negative root is inadmissible. Interchange x and y : $y=-1+\sqrt{1+x^2}$. So $f^{-1}(x)=-1+\sqrt{1+x^2}$, $x>0$.

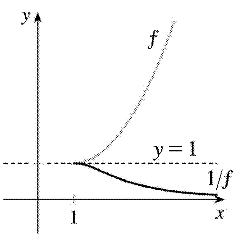


31. The function f is one–to–one, so its inverse exists and the graph of its inverse can be obtained by reflecting the graph of f about the line $y=x$.



32. The function f is one–to–one, so its inverse exists and the graph of its inverse can be obtained by reflecting the graph of f about the line $y=x$. For the graph of $1/f$, the y –coordinates are simply the reciprocals of f . For example, if $f(5)=9$, then $1/f(5)=\frac{1}{9}$. If we draw the horizontal line $y=1$, we see that the only place where the graphs intersect is on that line.





33. **(a)** It is defined as the inverse of the exponential function with base a , that is, $\log_a x=y \Leftrightarrow a^y=x$.
(b) $(0,\infty)$
(c) $\mathbb{R}$
(d) See Figure .

34. **(a)** The natural logarithm is the logarithm with base e , denoted $\ln x$.
(b) The common logarithm is the logarithm with base 10 , denoted $\log x$.
(c) See Figure .

35. **(a)** $\log_2 64=6$ since $2^6=64$.

(b) $\log_6 \frac{1}{36}=-2$ since $6^{-2}=\frac{1}{36}$.

36. **(a)** $\log_8 2=\frac{1}{3}$ since $8^{1/3}=2$.

(b) $\ln e^{\sqrt{2}}=\sqrt{2}$

37. **(a)** $\log_{10} 1.25+\log_{10} 80=\log_{10}(1.25\cdot 80)=\log_{10} 100=\log_{10} 10^2=2$

(b) $\log_5 10+\log_5 20-3\log_5 2=\log_5(10\cdot 20)-\log_5 2^3=\log_5 \frac{200}{8}=\log_5 25=\log_5 5^2=2$

38. **(a)** $2^{\left(\log_2 3+\log_2 5\right)}=2^{\log_2 15}=15$ [*Or:* $2^{\left(\log_2 3+\log_2 5\right)}=2^{\log_2 3}\cdot 2^{\log_2 5}=3\cdot 5=15$]

(b) $e^{3\ln 2}=e^{\ln (2^3)}=e^{\ln 8}=8$ [*Or:* $e^{3\ln 2}=\left(e^{\ln 2}\right)^3=2^3=8$]

39. $2\ln 4-\ln 2=\ln 4^2-\ln 2=\ln 16-\ln 2=\ln \dfrac{16}{2}=\ln 8$

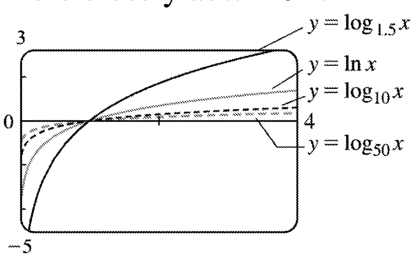
40. $\ln x+a\ln y-b\ln z=\ln x+\ln y^a-\ln z^b=\ln (x\cdot y^a)-\ln z^b=\ln (xy^a/z^b)$

41. $\ln (1+x^2)+\dfrac{1}{2}\ln x-\ln \sin x=\ln (1+x^2)+\ln x^{1/2}-\ln \sin x=\ln [(1+x^2)\sqrt{x}\,]-\ln \sin x=\ln \dfrac{(1+x^2)\sqrt{x}}{\sin x}$

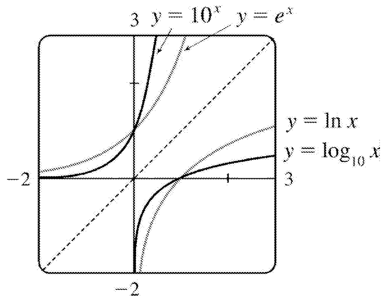
42. **(a)** $\log_{12}10=\dfrac{\ln 10}{\ln 12}\approx 0.926628$

(b) $\log_2 8.4=\dfrac{\ln 8.4}{\ln 2}\approx 3.070389$

43. To graph these functions, we use $\log_{1.5}x=\dfrac{\ln x}{\ln 1.5}$ and $\log_{50}x=\dfrac{\ln x}{\ln 50}$. These graphs all approach $-\infty$ as $x\to 0^+$, and they all pass through the point $(1,0)$. Also, they are all increasing, and all approach ∞ as $x\to\infty$. The functions with larger bases increase extremely slowly, and the ones with smaller bases do so somewhat more quickly. The functions with large bases approach the y– axis more closely as $x\to 0^+$.



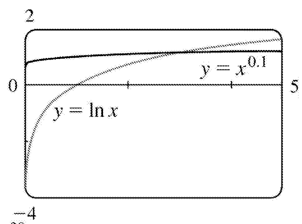
44. We see that the graph of $\ln x$ is the reflection of the graph of e^x about the line $y=x$, and that the graph of $\log_{10}x$ is the reflection of the graph of 10^x about the same line. The graph of 10^x increases more quickly than that of e^x . Also note that $\log_{10}x\to\infty$ as $x\to\infty$ more slowly than $\ln x$.

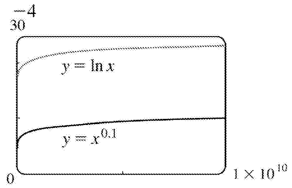


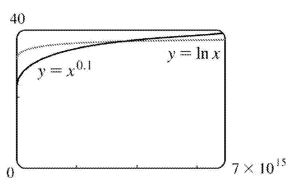
45. 3 ft $=36$ in, so we need x such that $\log_2 x=36 \Leftrightarrow x=2^{36}=68,719,476,736$. In miles, this is

$68,719,476,736 \text{ in} \cdot \frac{1\text{ft}}{12\text{in}} \cdot \frac{1\text{mi}}{5280\text{ft}} \approx 1,084,587.7$ mi.

46.



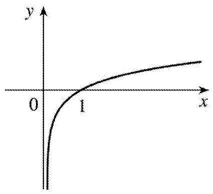




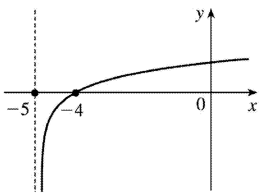
From the graphs, we see that $f(x)=x^{0.1}>g(x)=\ln x$ for approximately $0<x<3.06$, and then $g(x)>f(x)$ for $3.06<x<3.43\times 10^{15}$ (approximately). At that point, the graph of f finally surpasses the graph of g for good.

47. **(a)** Shift the graph of $y=\log_{10} x$ five units to the left to obtain the graph of $y=\log_{10}(x+5)$. Note the vertical asymptote of $x=-5$.

$y=\log_{10} x$

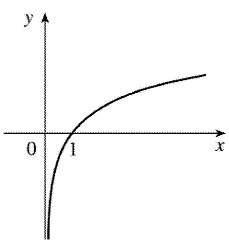


$y = \log_{10}(x+5)$

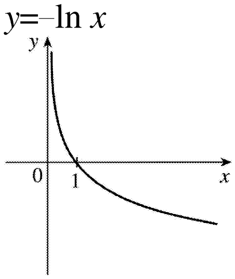


(b) Reflect the graph of $y=\ln x$ about the x –axis to obtain the graph of $y=-\ln x$.

$y=\ln x$

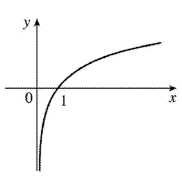


$y=-\ln x$

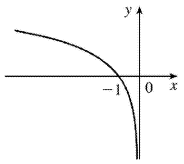


48. **(a)** Reflect the graph of $y=\ln x$ about the y –axis to obtain the graph of $y=\ln(-x)$.

$y=\ln x$

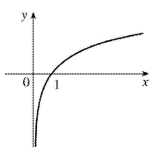


$y=\ln(-x)$

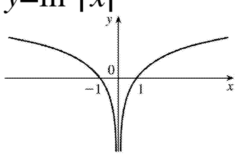


(b) Reflect the portion of the graph of $y = \ln x$ to the right of the y –axis about the y –axis. The graph of $y = \ln |x|$ is that reflection in addition to the original portion.

$y = \ln x$



$y = \ln |x|$



49. **(a)** $2\ln x = 1 \Rightarrow \ln x = \frac{1}{2} \Rightarrow x = e^{1/2} = \sqrt{e}$

(b) $e^{-x} = 5 \Rightarrow -x = \ln 5 \Rightarrow x = -\ln 5$

50. **(a)** $e^{2x+3} - 7 = 0 \Rightarrow e^{2x+3} = 7 \Rightarrow 2x+3 = \ln 7 \Rightarrow 2x = \ln 7 - 3 \Rightarrow x = \frac{1}{2}(\ln 7 - 3)$

(b) $\ln(5-2x) = -3 \Rightarrow 5-2x = e^{-3} \Rightarrow 2x = 5 - e^{-3} \Rightarrow x = \frac{1}{2}(5 - e^{-3})$

51. **(a)** $2^{x-5} = 3 \Leftrightarrow \log_2 3 = x-5 \Leftrightarrow x = 5 + \log_2 3$.

Or: $2^{x-5} = 3 \Leftrightarrow \ln\left(2^{x-5}\right) = \ln 3 \Leftrightarrow (x-5)\ln 2 = \ln 3 \Leftrightarrow x-5 = \frac{\ln 3}{\ln 2} \Leftrightarrow x = 5 + \frac{\ln 3}{\ln 2}$

(b) $\ln x + \ln(x-1) = \ln(x(x-1)) = 1 \Leftrightarrow x(x-1) = e^1 \Leftrightarrow x^2 - x - e = 0$. The quadratic formula (with $a=1$, $b=-1$, and $c=-e$) gives $x = \frac{1}{2}\left(1 \pm \sqrt{1+4e}\right)$, but we reject the negative root since the natural logarithm is not defined for $x<0$. So $x = \frac{1}{2}\left(1 + \sqrt{1+4e}\right)$.

52. **(a)** $\ln(\ln x)=1 \Leftrightarrow e^{\ln(\ln x)}=e^{1} \Leftrightarrow \ln x=e^{1}=e \Leftrightarrow e^{\ln x}=e^{e} \Leftrightarrow x=e^{e}$

(b) $e^{ax}=Ce^{bx} \Leftrightarrow \ln e^{ax}=\ln[C(e^{bx})] \Leftrightarrow ax=\ln C+bx+\ln e^{bx} \Leftrightarrow ax=\ln C+bx \Leftrightarrow ax-bx=\ln C \Leftrightarrow (a-b)x=\ln C \Leftrightarrow$
$x=\dfrac{\ln C}{a-b}$

53. **(a)** $e^{x}<10 \Rightarrow \ln e^{x}<\ln 10 \Rightarrow x<\ln 10 \Rightarrow x\in(-\infty,\ln 10)$

(b) $\ln x>-1 \Rightarrow e^{\ln x}>e^{-1} \Rightarrow x>e^{-1} \Rightarrow x\in(1/\mathrm{e},\infty)$

54. **(a)** $2<\ln x<9 \Rightarrow e^{2}<e^{\ln x}<e^{9} \Rightarrow e^{2}<x<e^{9} \Rightarrow x\in\left(e^{2},\mathrm{e}^{9}\right)$

(b) $e^{2-3x}>4 \Rightarrow \ln e^{2-3x}>\ln 4 \Rightarrow 2-3x>\ln 4 \Rightarrow -3x>\ln 4-2 \Rightarrow x<-\frac{1}{3}(\ln 4-2) \Rightarrow x\in\left(-\infty,\frac{1}{3}(2-\ln 4)\right)$

55. **(a)** For $f(x)=\sqrt{3-e^{2x}}$, we must have $3-e^{2x}\geq 0 \Rightarrow e^{2x}\leq 3 \Rightarrow 2x\leq \ln 3 \Rightarrow x\leq \frac{1}{2}\ln 3$.
Thus, the domain of f is $(-\infty,\frac{1}{2}\ln 3]$.

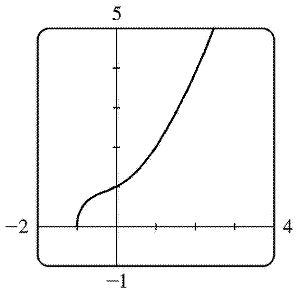
(b) $y=f(x)=\sqrt{3-e^{2x}}$ [note that $y\geq 0$] $\Rightarrow y^{2}=3-e^{2x} \Rightarrow e^{2x}=3-y^{2} \Rightarrow 2x=\ln(3-y^{2}) \Rightarrow x=\frac{1}{2}\ln(3-y^{2})$.
Interchange x and y : $y=\frac{1}{2}\ln(3-x^{2})$. So $f^{-1}(x)=\frac{1}{2}\ln(3-x^{2})$. For the domain of f^{-1} , we must have $3-x^{2}>0 \Rightarrow x^{2}<3 \Rightarrow |x|<\sqrt{3} \Rightarrow -\sqrt{3}<x<\sqrt{3} \Rightarrow 0\leq x<\sqrt{3}$ since $x\geq 0$. Note that the domain of f^{-1} , $[0,\sqrt{3})$, equals the range of f .

56. **(a)** For $f(x)=\ln(2+\ln x)$, we must have $2+\ln x>0 \Rightarrow \ln x>-2 \Rightarrow x>e^{-2}$. Thus, the domain of f is (e^{-2},∞) .

(b) $y=f(x)=\ln(2+\ln x) \Rightarrow e^{y}=2+\ln x \Rightarrow \ln x=e^{y}-2 \Rightarrow x=e^{e^{y}-2}$. Interchange x and y : $y=e^{e^{x}-2}$. So $f^{-1}(x)=e^{e^{x}-2}$. The domain of f^{-1} , as well as the range of f , is $\mathbb{R}$.

57. We see that the graph of $y=f(x)=\sqrt{x^{3}+x^{2}+x+1}$ is increasing, so f is $1-1$. Enter $x=\sqrt{y^{3}+y^{2}+y+1}$ and use your CAS to solve the equation for y . Using Derive, we get two (irrelevant) solutions involving imaginary expressions, as well as one which can be simplified to the following:

$$y=f^{-1}(x)=-\frac{\sqrt[3]{4}}{6}\left(\sqrt[3]{D-27x^{2}+20}-\sqrt[3]{D+27x^{2}-20}+\sqrt[3]{2}\right)$$



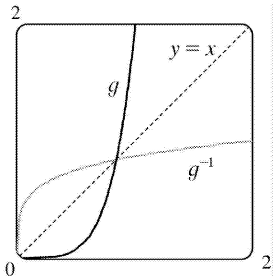
where $D=3\sqrt{3}\sqrt{27x^4-40x^2+16}$. Maple and Mathematica each give two complex expressions and one real expression, and the real expression is equivalent to that given by Derive. For example, Maple's expression simplifies to $\frac{1}{6}\ \frac{M^{2/3}-8-2M^{1/3}}{2M^{1/3}}$, where $M=108x^2+12\sqrt{48-120x^2+81x^4}-80$.

58. **(a)** If we use Derive, then solving $x=y^6+y^4$ for y gives us six solutions of the form $y=\pm\frac{\sqrt{3}}{3}\sqrt{B-1}$, where $B\in\left\{-2\sin\frac{A}{3},2\sin\left(\frac{A}{3}+\frac{\pi}{3}\right),-2\cos\left(\frac{A}{3}+\frac{\pi}{6}\right)\right\}$ and $A=\sin^{-1}\left(\frac{27x-2}{2}\right)$. The inverse for $y=x^6+x^4$ ($x\geq 0$) is $y=\frac{\sqrt{3}}{3}\sqrt{B-1}$ with $B=2\sin\left(\frac{A}{3}+\frac{\pi}{3}\right)$, but because the domain of A is $\left[0,\frac{4}{27}\right]$, this expression is only valid for $x\in\left[0,\frac{4}{27}\right]$.

Happily, Maple gives us the rest of the solution! We solve $x=y^6+y^4$ for y to get the two real solutions $\pm\frac{\sqrt{6}}{6}\ \frac{\sqrt{C^{1/3}\left(C^{2/3}-2C^{1/3}+4\right)}}{C^{1/3}}$, where $C=108x+12\sqrt{3}\sqrt{x(27x-4)}$, and the inverse for $y=x^6+x^4$ ($x\geq 0$) is the positive solution, whose domain is $\left[\frac{4}{27},\infty\right)$.

Mathematica also gives two real solutions, equivalent to those of Maple. The positive one is $\frac{\sqrt{6}}{6}\left(\sqrt[3]{4}\,D^{1/3}+2\sqrt[3]{2}\,D^{-1/3}-2\right)$, where $D=-2+27x+3\sqrt{3}\sqrt{x}\sqrt{27x-4}$. Although this expression also has domain $\left[\frac{4}{27},\infty\right)$, Mathematica is mysteriously able to plot the solution for all $x\geq 0$.

(b)



59. **(a)** $n=100\cdot 2^{t/3}\Rightarrow \frac{n}{100}=2^{t/3}\Rightarrow \log_2\left(\frac{n}{100}\right)=\frac{t}{3}\Rightarrow t=3\log_2\left(\frac{n}{100}\right)$. Using formula ( ), we can write this as $t=3\cdot\frac{\ln(n/100)}{\ln 2}$. This function tells us how long it will take to obtain n bacteria (given the number n).

(b) $n=50{,}000\Rightarrow t=3\log_2\frac{50{,}000}{100}=3\log_2 500=3\left(\frac{\ln 500}{\ln 2}\right)\approx 26.9$ hours

60. **(a)** $Q=Q_0\left(1-e^{-t/a}\right)\Rightarrow \frac{Q}{Q_0}=1-e^{-t/a}\Rightarrow e^{-t/a}=1-\frac{Q}{Q_0}\Rightarrow -\frac{t}{a}=\ln\left(1-\frac{Q}{Q_0}\right)\Rightarrow t=-a\ln(1-Q/Q_0)$. This gives us the time t necessary to obtain a given charge Q .

(b) $Q=0.9Q_0$ and $a=2\Rightarrow t=-2\ln\left(1-0.9\left(Q_0/Q_0\right)\right)=-2\ln 0.1\approx 4.6$ seconds.

61. **(a)** To find the equation of the graph that results from shifting the graph of $y=\ln x$ 3 units upward, we add 3 to the original function to get $y=\ln x+3$.

(b) To find the equation of the graph that results from shifting the graph of $y=\ln x$ 3 units to the left, we replace x with $x+3$ in the original function to get $y=\ln(x+3)$.

(c) To find the equation of the graph that results from reflecting the graph of $y=\ln x$ about the x –axis, we multiply the original equation by -1 to get $y=-\ln x$.

(d) To find the equation of the graph that results from reflecting the graph of $y=\ln x$ about the y –axis, we replace x with $-x$ in the original equation to get $y=\ln(-x)$.

(e) To find the equation of the graph that results from reflecting the graph of $y=\ln x$ about the line $y=x$, we interchange x and y in the original equation to get $x=\ln y\Leftrightarrow y=e^x$.

(f) To find the equation of the graph that results from reflecting the graph of $y=\ln x$ about the x –axis and then about the line $y=x$, we first multiply the original equation by -1 and then interchange x and y in this equation to get $x=-\ln y\Leftrightarrow \ln y=-x\Leftrightarrow y=e^{-x}$.

(g) To find the equation of the graph that results from reflecting the graph of $y=\ln x$ about the y –axis and then about the line $y=x$, we first replace x with $-x$ in the original equation and then interchange x and y to get $x=\ln(-y)\Leftrightarrow -y=e^x\Leftrightarrow y=-e^x$.

(h) To find the equation of the graph that results from shifting the graph of $y=\ln x$ 3 units to the left and then reflecting it about the line $y=x$, we first replace x with $x+3$ in the original equation and then interchange x and y in this equation to get $x = \ln(y+3) \Leftrightarrow y+3 = e^x \Leftrightarrow y = e^x - 3$.

62. **(a)** If the point (x,y) is on the graph of $y=f(x)$, then the point $(x-c,y)$ is that point shifted c units to the left. Since f is $1-1$, the point (y,x) is on the graph of $y=f^{-1}(x)$ and the point corresponding to $(x-c,y)$ on the graph of f is $(y,x-c)$ on the graph of f^{-1} . Thus, the curve's reflection is shifted *down* the same number of units as the curve itself is shifted to the left. So an expression for the inverse function is $g^{-1}(x)=f^{-1}(x)-c$.
(b) If we compress (or stretch) a curve horizontally, the curve's reflection in the line $y=x$ is compressed (or stretched) *vertically* by the same factor. Using this geometric principle, we see that the inverse of $h(x)=f(cx)$ can be expressed as $h^{-1}(x)=(1/c)f^{-1}(x)$.

63. **(a)** $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)=\frac{\pi}{3}$ since $\sin\frac{\pi}{3}=\frac{\sqrt{3}}{2}$ and $\frac{\pi}{3}$ is in $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$.
(b) $\cos^{-1}(-1)=\pi$ since $\cos\pi=-1$ and π is in $[0,\pi]$.

64. **(a)** $\arctan(-1)=-\frac{\pi}{4}$ since $\tan\left(-\frac{\pi}{4}\right)=-1$ and $-\frac{\pi}{4}$ is in $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$.
(b) $\csc^{-1}2=\frac{\pi}{6}$ since $\csc\frac{\pi}{6}=2$ and $\frac{\pi}{6}$ is in $\left(0,\frac{\pi}{2}\right]\cup\left(\pi,\frac{3\pi}{2}\right]$.

65. **(a)** $\tan^{-1}\sqrt{3}=\frac{\pi}{3}$ since $\tan\frac{\pi}{3}=\sqrt{3}$ and $\frac{\pi}{3}$ is in $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$.
(b) $\arcsin\left(-\frac{1}{\sqrt{2}}\right)=-\frac{\pi}{4}$ since $\sin\left(-\frac{\pi}{4}\right)=-\frac{1}{\sqrt{2}}$ and $-\frac{\pi}{4}$ is in $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$.

66. **(a)** $\sec^{-1}\sqrt{2}=\frac{\pi}{4}$ since $\sec\frac{\pi}{4}=\sqrt{2}$ and $\frac{\pi}{4}$ is in $\left[0,\frac{\pi}{2}\right)\cup\left[\pi,\frac{3\pi}{2}\right)$.
(b) $\arcsin 1=\frac{\pi}{2}$ since $\sin\frac{\pi}{2}=1$ and $\frac{\pi}{2}$ is in $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$.

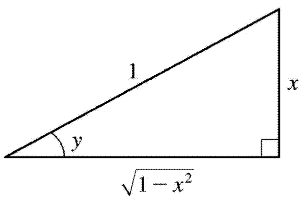
67. **(a)** $\sin(\sin^{-1}0.7)=0.7$ since 0.7 is in $[-1,1]$.
(b) $\tan^{-1}\left(\tan\frac{4\pi}{3}\right)=\tan^{-1}\sqrt{3}=\frac{\pi}{3}$ since $\frac{\pi}{3}$ is in $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$.

68. **(a)** Let $\theta=\arctan 2$, so $\tan\theta=2\Rightarrow\sec^2\theta=1+\tan^2\theta=1+4=5\Rightarrow\sec\theta=\sqrt{5}\Rightarrow\sec(\arctan 2)=\sec\theta=\sqrt{5}$
.
(b) Let

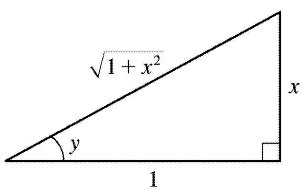
$\theta = \sin^{-1}\frac{5}{13}$. Then $\sin\theta = \frac{5}{13}$, so $\cos\left(2\sin^{-1}\frac{5}{13}\right) = \cos 2\theta = 1 - 2\sin^2\theta = 1 - 2\left(\frac{5}{13}\right)^2 = \frac{119}{169}$.

69. Let $y = \sin^{-1}x$. Then $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2} \Rightarrow \cos y \geq 0$, so $\cos(\sin^{-1}x) = \cos y = \sqrt{1 - \sin^2 y} = \sqrt{1 - x^2}$

70. Let $y = \sin^{-1}x$. Then $\sin y = x$, so from the triangle we see that $\tan(\sin^{-1}x) = \tan y = \dfrac{x}{\sqrt{1-x^2}}$.

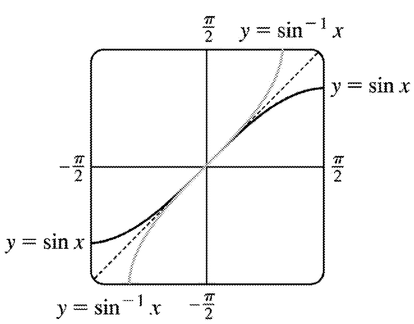


71. Let $y = \tan^{-1}x$. Then $\tan y = x$, so from the triangle we see that $\sin(\tan^{-1}x) = \sin y = \dfrac{x}{\sqrt{1+x^2}}$.



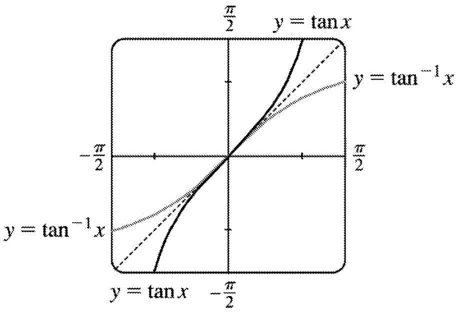
72.

Let $y = \cos^{-1}x$. Then $\cos y = x \Rightarrow \sin y = \sqrt{1-x^2}$ since $0 \leq y \leq \pi$. So
$\sin(2\cos^{-1}x) = \sin 2y = 2\sin y\cos y = 2x\sqrt{1-x^2}$.
73.



The graph of $\sin^{-1}x$ is the reflection of the graph of $\sin x$ about the line $y = x$.

74.



The graph of $\tan^{-1} x$ is the reflection of the graph of $\tan x$ about the line $y=x$.

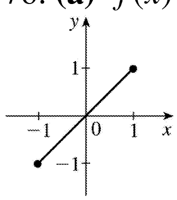
75.

$g(x)=\sin^{-1}(3x+1)$.

Domain $(g)=\{x|-1\leq 3x+1\leq 1\}=\{x|-2\leq 3x\leq 0\}=\left\{ x|-\frac{2}{3}\leq x\leq 0\right\} =\left[-\frac{2}{3},0\right]$.

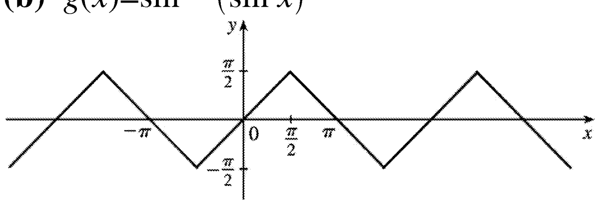
Range $(g)=\left\{ y|-\frac{\pi}{2}\leq y\leq \frac{\pi}{2} \right\} =\left[-\frac{\pi}{2},\frac{\pi}{2} \right]$.

76. **(a)** $f(x)=\sin\left(\sin^{-1} x\right)$



Since one function undoes what the other one does, we get the identity function, $y=x$, on the restricted domain $-1\leq x\leq 1$.

(b) $g(x)=\sin^{-1}(\sin x)$



This is similar to part (a), but with domain R . Equations for g on intervals of the form $\left(-\frac{\pi}{2}+\pi n,\frac{\pi}{2}+\pi n \right)$, for any integer n , can be found using $g(x)=(-1)^n x+(-1)^{n+1} n\pi$. The sine function is monotonic on each of these intervals, and hence, so is g (but in a linear fashion).

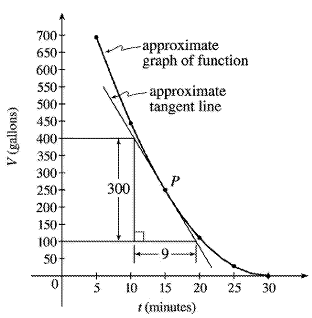
1. **(a)** Using $P(15{,}250)$, we construct the following table:

t	Q	slope=m_{PQ}
5	$(5{,}694)$	$\dfrac{694-250}{5-15}=-\dfrac{444}{10}=-44.4$
10	$(10{,}444)$	$\dfrac{444-250}{10-15}=-\dfrac{194}{5}=-38.8$
20	$(20{,}111)$	$\dfrac{111-250}{20-15}=-\dfrac{139}{5}=-27.8$
25	$(25{,}28)$	$\dfrac{28-250}{25-15}=-\dfrac{222}{10}=-22.2$
30	$(30{,}0)$	$\dfrac{0-250}{30-15}=-\dfrac{250}{15}=-16.\overline{6}$

(b) Using the values of t that correspond to the points closest to P ($t=10$ and $t=20$), we have

$$\frac{-38.8+(-27.8)}{2}=-33.3$$

(c) From the graph, we can estimate the slope of the tangent line at P to be $\dfrac{-300}{9}=-33.\overline{3}$.



2.

(a) $\text{Slope}=\dfrac{2948-2530}{42-36}=\dfrac{418}{6}\approx 69.67$ (b) $\text{Slope}=\dfrac{2948-2661}{42-38}=\dfrac{287}{4}=71.75$

(c) $\text{Slope}=\dfrac{2948-2806}{42-40}=\dfrac{142}{2}=71$ (d) $\text{Slope}=\dfrac{3080-2948}{44-42}=\dfrac{132}{2}=66$

From the data, we see that the patient's heart rate is decreasing from 71 to 66 heartbeats / minute after 42 minutes. After being stable for a while, the patient's heart rate is dropping.

3. **(a)** For the curve $y=x/(1+x)$ and the point $P\left(1,\frac{1}{2}\right)$

	x	Q	m_{PQ}
(i)	0.5	(0.5,0.333333)	0.333333
(ii)	0.9	(0.9,0.473684)	0.263158
(iii)	0.99	(0.99,0.497487)	0.251256
(iv)	0.999	(0.999,0.499750)	0.250125
(v)	1.1	(1.5,06)	0.2
(vi)	1.5	(1.1,0.523810)	0.238095
(vii)	1.01	(1.01,0.502488)	0.248756
(viii)	1.001	(1.001,0.500250)	0.249875

(b) The slope appears to be $\frac{1}{4}$.

(c) $y-\frac{1}{2}=\frac{1}{4}(x-1)$ or $y=\frac{1}{4}x+\frac{1}{4}$.

4. For the curve $y=\ln x$ and the point $P(2,\ln 2)$:

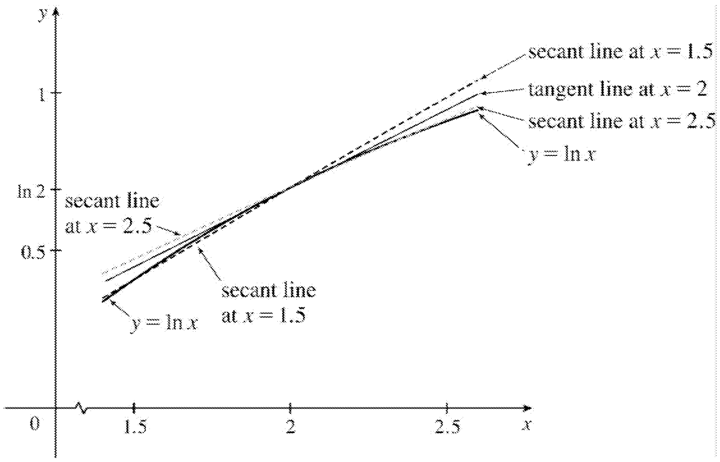
(a)

	x	Q	m_{PQ}
(i)	1.5	(1.5,0.405465)	0.575364
(ii)	1.9	(1.9,0.641854)	0.512933
(iii)	1.99	(1.99,0.688135)	0.501254
(iv)	1.999	(1.999,0.692647)	0.500125
(v)	2.5	(2.5,0.916291)	0.446287
(vi)	2.1	(2.1,0.741937)	0.487902
(vii)	2.01	(2.01,0.698135)	0.498754
(viii)	2.001	(2.001,0.693647)	0.499875

(b) The slope appears to be $\frac{1}{2}$.

(c) $y-\ln 2=\frac{1}{2}(x-2)$ or $y=\frac{1}{2}x-1+\ln 2$

(d)



5. **(a)** $y=y(t)=40t-16t^2$. At $t=2$, $y=40(2)-16(2)^2=16$. The average velocity between times 2 and $2+h$ is $v_{ave}=\dfrac{y(2+h)-y(2)}{(2+h)-2}=\dfrac{\left[40(2+h)-16(2+h)^2\right]-16}{h}=\dfrac{-24h-16h^2}{h}=-24-16h$, if $h\neq 0$.

(i) $[2,2.5]$: $h=0.5$, $v_{ave}=-32$ ft / s (ii) $[2,2.1]$: $h=0.1$, $v_{ave}=-25.6$ ft / s

(iii) $[2,2.05]$: $h=0.05$, $v_{ave}=-24.8$ ft / s (iv) $[2,2.01]$: $h=0.01$, $v_{ave}=-24.16$ ft / s

(b) The instantaneous velocity when $t=2$ (h approaches 0) is -24 ft / s.

6. The average velocity between t and $t+h$ seconds is

$$\frac{58(t+h)-0.83(t+h)^2-\left(58t-0.83t^2\right)}{h}=\frac{58h-1.66th-0.83h^2}{h}=58-1.66t-0.83h \text{ if } h\neq 0 .$$

(a) Here $t=1$, so the average velocity is $58-1.66-0.83h=56.34-0.83h$.

(i) $[1,2]$: $h=1, 55.51$ m / s (ii) $[1,1.5]$: $h=0.5$, 55.925 m / s

(iii) $[1,1.1]$: $h=0.1$, 56.257 m / s (iv) $[1,1.01]$: $h=0.01$, 56.3317 m / s

(v) $[1,1.001]$: $h=0.001$, 56.33917 m / s

(b) The instantaneous velocity after 1 second is 56.34 m / s.

7. **(a)**

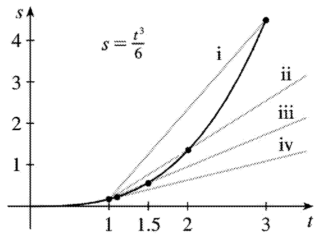
(i) $[1,3]$: $h=2$, $v_{ave}=\dfrac{13}{6}$ ft / s (ii) $[1,2]$: $h=1$, $v_{ave}=\dfrac{7}{6}$ ft / s

(iii) $[1,1.5]$: $h=0.5$, $v_{ave}=\dfrac{19}{24}$ ft / s (iv) $[1,1.1]$: $h=0.1$, $v_{ave}=\dfrac{331}{600}$ ft / s

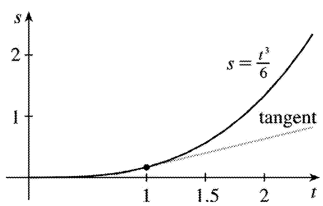
(b) As h approaches 0 , the velocity approaches

$\frac{3}{6} = \frac{1}{2}$ ft / s.

(c)



(d)



8. Average velocity between times t=2 and t=2+h is given by $\frac{s(2+h)-s(2)}{h}$.

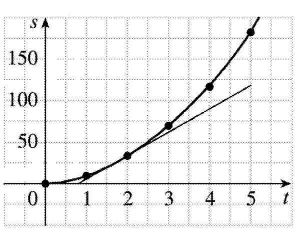
(a)

(i) $h=3 \Rightarrow v_{av} = \frac{s(5)-s(2)}{5-2} = \frac{178-32}{3} = \frac{146}{3} \approx 48.7$ ft / s

(ii) $h=2 \Rightarrow v_{av} = \frac{s(4)-s(2)}{4-2} = \frac{119-32}{2} = \frac{87}{2} = 43.5$ ft / s

(iii) $h=1 \Rightarrow v_{av} = \frac{s(3)-s(2)}{3-2} = \frac{70-32}{1} = 38$ ft / s

(b) Using the points $(0.8,0)$ and $(5,118)$ from the approximate tangent line, the instantaneous velocity at t=2 is about $\frac{118-0}{5-0.8} \approx 28$ ft / s.



9. For the curve $y=\sin(10\pi/x)$ and the point $P(1,0)$:

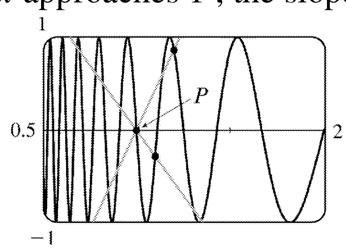
(a)

x	Q	m_{PQ}
2	$(2,0)$	0

1.5	$(1.5,0.8660)$	1.7321
1.4	$(1.4,-0.4339)$	-1.0847
1.3	$(1.3,-0.8230)$	-2.7433
1.2	$(1.2,0.8660)$	4.3301
1.1	$(1.1,-0.2817)$	-2.8173

x	Q	m_{PQ}
0.5	$(0.5,0)$	0
0.6	$(0.6,0.8660)$	-2.1651
0.7	$(0.7,0.7818)$	-2.6061
0.8	$(0.8,1)$	-5
0.9	$(0.9,-0.3420)$	3.4202

As x approaches 1 , the slopes do not appear to be approaching any particular value.



(b)
We see that problems with estimation are caused by the frequent oscillations of the graph. The tangent is so steep at P that we need to take x –values much closer to 1 in order to get accurate estimates of its slope.

(c) If we choose $x=1.001$, then the point Q is $(1.001,-0.0314)$ and $m_{PQ}\approx-31.3794$. If $x=0.999$, then Q is $(0.999,0.0314)$ and $m_{PQ}=-31.4422$. The average of these slopes is -31.4108 . So we estimate that the slope of the tangent line at P is about -31.4 .

1. As x approaches 2 , $f(x)$ approaches 5 . [Or, the values of $f(x)$ can be made as close to 5 as we like by taking x sufficiently close to 2 (but $x \neq 2$).] Yes, the graph could have a hole at $(2,5)$ and be defined such that $f(2)=3$.

2. As x approaches 1 from the left, $f(x)$ approaches 3 ; and as x approaches 1 from the right, $f(x)$ approaches 7 . No, the limit does not exist because the left– and right–hand limits are different.

3. **(a)** $\lim_{x \to -3} f(x)=\infty$ means that the values of $f(x)$ can be made arbitrarily large (as large as we please) by taking x sufficiently close to -3 (but not equal to -3).
(b) $\lim_{x \to 4^+} f(x)=-\infty$ means that the values of $f(x)$ can be made arbitrarily large negative by taking x sufficiently close to 4 through values larger than 4 .

4. **(a)** $\lim_{x \to 0} f(x)=3$
(b) $\lim_{x \to 3^-} f(x)=4$
(c) $\lim_{x \to 3^+} f(x)=2$
(d) $\lim_{x \to 3} f(x)$ does not exist because the limits in part (b) and part (c) are not equal.
(e) $f(3)=3$

5. **(a)** $f(x)$ approaches 2 as x approaches 1 from the left, so $\lim_{x \to 1^-} f(x)=2$.
(b) $f(x)$ approaches 3 as x approaches 1 from the right, so $\lim_{x \to 1^+} f(x)=3$.
(c) $\lim_{x \to 1} f(x)$ does not exist because the limits in part (a) and part (b) are not equal.
(d) $f(x)$ approaches 4 as x approaches 5 from the left and from the right, so $\lim_{x \to 5} f(x)=4$.
(e) $f(5)$ is not defined, so it doesn't exist.

6. **(a)** $\lim_{x \to -2^-} g(x)=-1$
(b) $\lim_{x \to -2^+} g(x)=1$
(c) $\lim_{x \to -2} g(x)$ doesn't exist
(d) $g(-2)=1$
(e) $\lim_{x \to 2^-} g(x)=1$

(f) $\lim_{x \to 2^+} g(x) = 2$

(g) $\lim_{x \to 2} g(x)$ doesn't exist

(h) $g(2) = 2$

(i) $\lim_{x \to 4^+} g(x)$ doesn't exist

(j) $\lim_{x \to 4^-} g(x) = 2$

(k) $g(0)$ doesn't exist

(l) $\lim_{x \to 0} g(x) = 0$

7. **(a)** $\lim_{t \to 0^-} g(t) = -1$

(b) $\lim_{t \to 0^+} g(t) = -2$

(c) $\lim_{t \to 0} g(t)$ does not exist because the limits in part (a) and part (b) are not equal.

(d) $\lim_{t \to 2^-} g(t) = 2$

(e) $\lim_{t \to 2^+} g(t) = 0$

(f) $\lim_{t \to 2} g(t)$ does not exist because the limits in part (d) and part (e) are not equal.

(g) $g(2) = 1$

(h) $\lim_{t \to 4} g(t) = 3$

8. **(a)** $\lim_{x \to 2} R(x) = -\infty$

(b) $\lim_{x \to 5} R(x) = \infty$

(c) $\lim_{x \to -3^-} R(x) = -\infty$

(d) $\lim_{x \to -3^+} R(x) = \infty$

(e) The equations of the vertical asymptotes are $x = -3$, $x = 2$, and $x = 5$.

9. **(a)** $\lim_{x \to -7} f(x) = -\infty$

(b) $\lim_{x \to -3} f(x) = \infty$

(c)

$\lim_{x \to 0} f(x) = \infty$

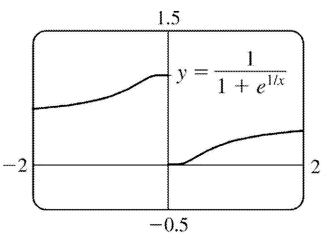
(d) $\lim_{x \to 6^-} f(x) = -\infty$

(e) $\lim_{x \to 6^+} f(x) = \infty$

(f) The equations of the vertical asymptotes are $x=-7$, $x=-3$, $x=0$, and $x=6$.

10. $\lim_{t \to 12^-} f(t) = 150$ mg and $\lim_{t \to 12^+} f(t) = 300$ mg. These limits show that there is an abrupt change in the amount of drug in the patient's bloodstream at $t=12$ h. The left–hand limit represents the amount of the drug just before the fourth injection. The right–hand limit represents the amount of the drug just after the fourth injection.

11.

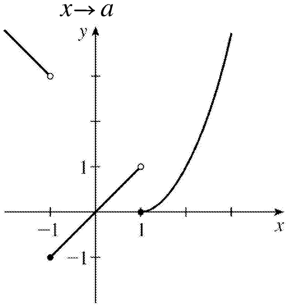


(a) $\lim_{x \to 0^-} f(x) = 1$

(b) $\lim_{x \to 0^+} f(x) = 0$

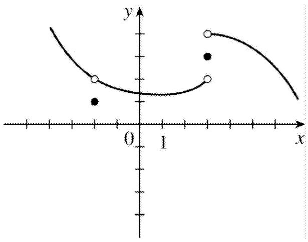
(c) $\lim_{x \to 0} f(x)$ does not exist because the limits in part (a) and part (b) are not equal.

12. $\lim_{x \to a} f(x)$ exists for all a except $a = \pm 1$.

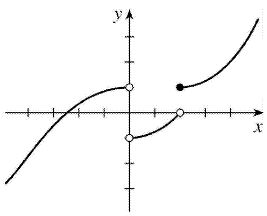


13. $\lim_{x \to 3^+} f(x) = 4$, $\lim_{x \to 3^-} f(x) = 2$,

$\lim_{x\to -2} f(x)=2$, $f(3)=3$, $f(-2)=1$



14. $\lim_{x\to 0^-} f(x)=1$, $\lim_{x\to 0^+} f(x)=-1$, $\lim_{x\to 2^-} f(x)=0$, $\lim_{x\to 2^+} f(x)=1$, $f(2)=1$, $f(0)$ is undefined



15. For $f(x)=\dfrac{x^2-2x}{x^2-x-2}$:

x	$f(x)$
2.5	0.714286
2.1	0.677419
2.05	0.672131
2.01	0.667774
2.005	0.667221
2.001	0.666778

x	$f(x)$
1.9	0.655172
1.95	0.661017
1.99	0.665552
1.995	0.666110
1.999	0.666556

It appears that $\lim_{x\to 2} \dfrac{x^2-2x}{x^2-x-2}=0.\overline{6}=\dfrac{2}{3}$.

16. For $f(x)=\dfrac{x^2-2x}{x^2-x-2}$:

x	$f(x)$
0	0
−0.5	−1
−0.9	−9
−0.95	−19
−0.99	−99
−0.999	−999

x	$f(x)$
−2	2
−1.5	3
−1.1	11
−1.01	101
−1.001	1001

It appears that $\lim\limits_{x\to -1}\dfrac{x^2-2x}{x^2-x-2}$ *does not exist* since $f(x)\to-\infty$ as $x\to-1^-$ and $f(x)\to\infty$ as $x\to-1^+$.

17. For $f(x)=\dfrac{e^x-1-x}{x^2}$:

x	$f(x)$
1	0.718282
0.5	0.594885
0.1	0.517092
0.05	0.508439
0.01	0.501671

x	$f(x)$
−1	0.367879

–0.5	0.426123
–0.1	0.483742
–0.05	0.491770
–0.01	0.498337

It appears that $\lim_{x\to 0} \frac{e^x-1-x}{x^2}=0.5=\frac{1}{2}$.

18. For $f(x)=x\ln\left(x+x^2\right)$:

x	$f(x)$
1	0.693147
0.5	–0.143841
0.1	–0.220727
0.05	–0.147347
0.01	–0.045952
0.005	–0.026467
0.001	–0.006907

It appears that $\lim_{x\to 0^+} x\ln\left(x+x^2\right)=0$.

19. For $f(x)=\frac{\sqrt{x+4}-2}{x}$:

x	$f(x)$
1	0.236068
0.5	0.242641
0.1	0.248457
0.05	0.249224
0.01	0.249844

x	$f(x)$
–1	0.267949

–0.5	0.258343
–0.1	0.251582
–0.05	0.250786
–0.01	0.250156

It appears that $\lim_{x\to 0} \frac{\sqrt{x+4}-2}{x} = 0.25 = \frac{1}{4}$.

20. For $f(x)=\frac{\tan 3x}{\tan 5x}$:

x	$f(x)$
± 0.2	0.439279
± 0.1	0.566236
± 0.05	0.591893
± 0.01	0.599680
± 0.001	0.599997

It appears that $\lim_{x\to 0} \frac{\tan 3x}{\tan 5x} = 0.6 = \frac{3}{5}$.

21. For $f(x)=\frac{x^6-1}{x^{10}-1}$:

x	$f(x)$
0.5	0.985337
0.9	0.719397
0.95	0.660186
0.99	0.612018
0.999	0.601200

x	$f(x)$
1.5	0.183369
1.1	0.484119
1.05	0.540783
1.01	0.588022

1.001	0.598800

It appears that $\lim_{x\to 1} \frac{x^6-1}{x^{10}-1} = 0.6 = \frac{3}{5}$.

22. For $f(x) = \frac{9^x - 5^x}{x}$:

x	$f(x)$
0.5	1.527864
0.1	0.711120
0.05	0.646496
0.01	0.599082
0.001	0.588906

x	$f(x)$
–0.5	0.227761
–0.1	0.485984
–0.05	0.534447
–0.01	0.576706
–0.001	0.586669

It appears that $\lim_{x\to 0} \frac{9^x - 5^x}{x} = 0.59$. Later we will be able to show that the exact value is ln (9/5) .

23. $\lim_{x\to 5^+} \frac{6}{x-5} = \infty$ since $(x-5)\to 0$ as $x\to 5^+$ and $\frac{6}{x-5} > 0$ for $x > 5$.

24. $\lim_{x\to 5^-} \frac{6}{x-5} = -\infty$ since $(x-5)\to 0$ as $x\to 5^-$ and $\frac{6}{x-5} < 0$ for $x < 5$.

25. $\lim_{x\to 1} \frac{2-x}{(x-1)^2} = \infty$ since the numerator is positive and the denominator approaches 0 through positive values as $x\to 1$.

26.

$\lim\limits_{x\to 0} \dfrac{x-1}{x^2(x+2)} = -\infty$ since $x^2 \to 0$ as $x \to 0$ and $\dfrac{x-1}{x^2(x+2)} < 0$ for $0<x<1$ and for $-2<x<0$.

27. $\lim\limits_{x\to -2^+} \dfrac{x-1}{x^2(x+2)} = -\infty$ since $(x+2)\to 0$ as $x\to -2^+$ and $\dfrac{x-1}{x^2(x+2)} < 0$ for $-2<x<0$.

28. $\lim\limits_{x\to \pi^-} \csc x = \lim\limits_{x\to \pi^-} (1/\sin x) = \infty$ since $\sin x \to 0$ as $x\to \pi^-$ and $\sin x > 0$ for $0<x<\pi$.

29. $\lim\limits_{x\to (-\pi/2)^-} \sec x = \lim\limits_{x\to (-\pi/2)^-} (1/\cos x) = -\infty$ since $\cos x \to 0$ as $x \to (-\pi/2)^-$ and $\cos x < 0$ for $-\pi < x < -\pi/2$.

30. $\lim\limits_{x\to 5^+} \ln(x-5) = -\infty$ since $x-5 \to 0^+$ as $x\to 5^+$.

31. **(a)** $f(x) = 1/(x^3-1)$

x	$f(x)$
0.5	–1.14
0.9	–3.69
0.99	–33.7
0.999	–333.7
0.9999	–3333.7
0.99999	–33 , 333.7

x	$f(x)$
1.5	0.42
1.1	3.02
1.01	33.0
1.001	333.0
1.0001	3333.0
1.00001	33 , 333.3

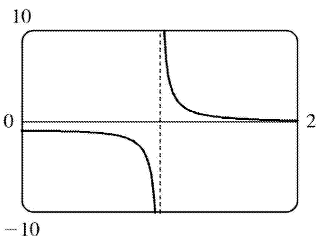
From these calculations, it seems that $\lim\limits_{x\to 1^-} f(x) = -\infty$ and $\lim\limits_{x\to 1^+} f(x) = \infty$.

(b) If x is slightly smaller than 1 , then

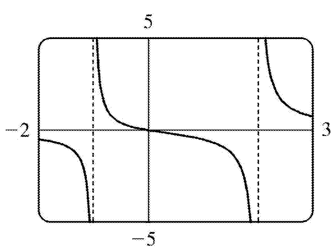
x^3-1 will be a negative number close to 0 , and the reciprocal of x^3-1 , that is, $f(x)$, will be a negative number with large absolute value. So $\lim_{x\to 1^-} f(x)=-\infty$.

If x is slightly larger than 1 , then x^3-1 will be a small positive number, and its reciprocal, $f(x)$, will be a large positive number. So $\lim_{x\to 1^+} f(x)=\infty$.

(c) It appears from the graph of f that $\lim_{x\to 1^-} f(x)=-\infty$ and $\lim_{x\to 1^+} f(x)=\infty$.



32. **(a)** $y=\dfrac{x}{x^2-x-2}=\dfrac{x}{(x-2)(x+1)}$. Therefore, as $x\to -1^+$ or $x\to 2^+$, the denominator approaches 0 , and $y>0$ for $x<-1$ and for $x>2$, so $\lim_{x\to -1^+} y=\lim_{x\to 2^+} y=\infty$. Also, as $x\to -1^-$ or $x\to 2^-$, the denominator approaches 0 and $y<0$ for $-1<x<2$, so $\lim_{x\to -1^-} y=\lim_{x\to 2^-} y=-\infty$.

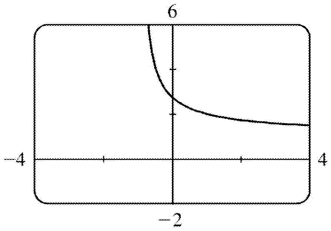
(b) 

33. **(a)** Let $h(x)=(1+x)^{1/x}$.

x	$h(x)$
−0.001	2.71964
−0.0001	2.71842
−0.00001	2.71830
−0.000001	2.71828
0.000001	2.71828
0.00001	2.71827

0.0001	2.71815
0.001	2.71692

It appears that $\lim_{x \to 0} (1+x)^{1/x} \approx 2.71828$, which is approximately e. In Section 7.4 we will see that the value of the limit is exactly e.

(b) 

34. For the curve $y=2^x$ and the points $P(0,1)$ and $Q\left(x,2^x\right)$:

x	Q	m_{PQ}
0.1	(0.1,1.0717735)	0.71773
0.01	(0.01,1.0069556)	0.69556
0.001	(0.001,1.0006934)	0.69339
0.0001	(0.0001,1.0000693)	0.69317

The slope appears to be about 0.693 .

35. **(a)**

x	$f(x)$
1	0.998000
0.8	0.638259
0.6	0.358484
0.4	0.158680
0.2	0.038851
0.1	0.008928
0.05	0.001465

It appears that $\lim_{x \to 0} f(x)=0$.

(b)

x	$f(x)$
0.04	0.000572

0.02	–0.000614
0.01	–0.000907
0.005	–0.000978
0.003	–0.000993
0.001	–0.001000

It appears that $\lim_{x \to 0} f(x) = -0.001$.

36. $h(x) = \dfrac{\tan x - x}{x^3}$

(a)

x	$h(x)$
1.0	0.55740773
0.5	0.37041992
0.1	0.33467209
0.05	0.33366700
0.01	0.33334667
0.005	0.33333667

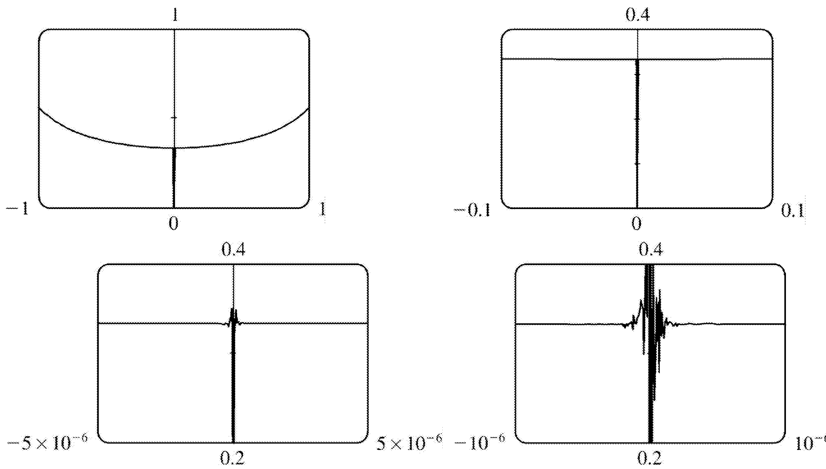
(b) It seems that $\lim_{x \to 0} h(x) = \dfrac{1}{3}$.

(c)

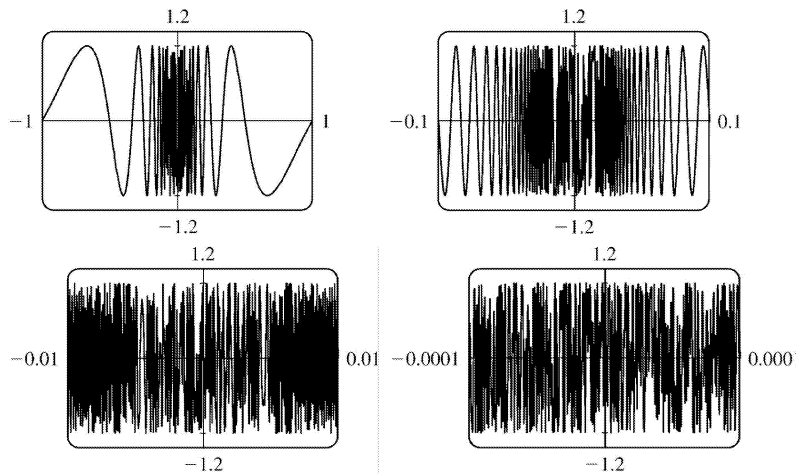
x	$h(x)$
0.001	0.33333350
0.0005	0.33333344
0.0001	0.33333000
0.00005	0.33333600
0.00001	0.33300000
0.000001	0.00000000

Here the values will vary from one calculator to another. Every calculator will eventually give *false values* .

(d) As in part (c), when we take a small enough viewing rectangle we get incorrect output.

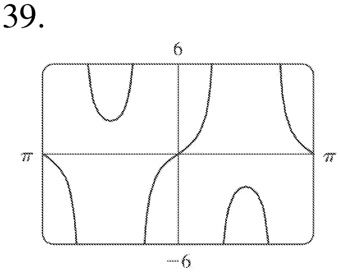


37. No matter how many times we zoom in toward the origin, the graphs of $f(x)=\sin(\pi/x)$ appear to consist of almost–vertical lines. This indicates more and more frequent oscillations as $x\rightarrow 0$.



38. $\lim\limits_{v\rightarrow c^-} m=\lim\limits_{v\rightarrow c^-} \dfrac{m_0}{\sqrt{1-v^2/c^2}}$. As $v\rightarrow c^-$, $\sqrt{1-v^2/c^2}\rightarrow 0^+$, and $m\rightarrow\infty$.

39.

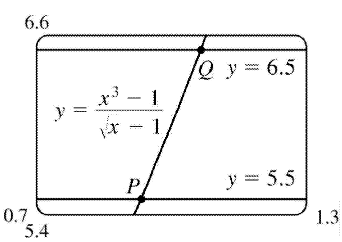


There appear to be vertical asymptotes of the curve $y=\tan(2\sin x)$ at $x\approx\pm 0.90$ and $x\approx\pm 2.24$. To find the exact equations of these asymptotes, we note that the graph of the tangent function has vertical asymptotes at $x=\frac{\pi}{2}+\pi n$. Thus, we must have $2\sin x=\frac{\pi}{2}+\pi n$, or equivalently, $\sin x=\frac{\pi}{4}+\frac{\pi}{2}n$. Since $-1\leq\sin x\leq 1$, we must have $\sin x=\pm\frac{\pi}{4}$ and so $x=\pm\sin^{-1}\frac{\pi}{4}$ (corresponding to $x\approx\pm 0.90$).

Just as 150° is the reference angle for 30° , $\pi-\sin^{-1}\frac{\pi}{4}$ is the reference angle for $\sin^{-1}\frac{\pi}{4}$. So $x=\pm\left(\pi-\sin^{-1}\frac{\pi}{4}\right)$ are also equations of the vertical asymptotes (corresponding to $x\approx\pm 2.24$).

40. **(a)** Let $y=\left(x^{3}-1\right)/\left(\sqrt{x}-1\right)$.

x	y
0.99	5.92531
0.999	5.99250
0.9999	5.99925
1.01	6.07531
1.001	6.00750
1.0001	6.00075



From the table and the graph, we guess that the limit of y as x approaches 1 is 6 .

(b) We need to have $5.5<\frac{x^{3}-1}{\sqrt{x}-1}<6.5$. From the graph we obtain the approximate points of intersection $P\,(0.9313853,5.5)$ and $Q\,(1.0649004,6.5)$. Now $1-0.9313853\approx 0.0686$ and $1.0649004-1\approx 0.0649$, so by requiring that x be within 0.0649 of 1 , we ensure that y is within 0.5 of 6 .

1. **(a)**
$$\lim_{x\to a}\left[f(x)+h(x)\right]=\lim_{x\to a}f(x)+\lim_{x\to a}h(x)$$
$$=-3+8=5$$

(b) $\lim\limits_{x\to a}\left[f(x)\right]^2=\left[\lim\limits_{x\to a}f(x)\right]^2=(-3)^2=9$

(c) $\lim\limits_{x\to a}\sqrt[3]{h(x)}=\sqrt[3]{\lim\limits_{x\to a}h(x)}=\sqrt[3]{8}=2$

(d) $\lim\limits_{x\to a}\dfrac{1}{f(x)}=\dfrac{1}{\lim\limits_{x\to a}f(x)}=\dfrac{1}{-3}=-\dfrac{1}{3}$

(e) $\lim\limits_{x\to a}\dfrac{f(x)}{h(x)}=\dfrac{\lim\limits_{x\to a}f(x)}{\lim\limits_{x\to a}h(x)}=\dfrac{-3}{8}=-\dfrac{3}{8}$

(f) $\lim\limits_{x\to a}\dfrac{g(x)}{f(x)}=\dfrac{\lim\limits_{x\to a}g(x)}{\lim\limits_{x\to a}f(x)}=\dfrac{0}{-3}=0$

(g) The limit does not exist, since $\lim\limits_{x\to a}g(x)=0$ but $\lim\limits_{x\to a}f(x)\neq 0$.

(h) $\lim\limits_{x\to a}\dfrac{2f(x)}{h(x)-f(x)}=\dfrac{2\lim\limits_{x\to a}f(x)}{\lim\limits_{x\to a}h(x)-\lim\limits_{x\to a}f(x)}=\dfrac{2(-3)}{8-(-3)}=-\dfrac{6}{11}$

2. **(a)** $\lim\limits_{x\to 2}\left[f(x)+g(x)\right]=\lim\limits_{x\to 2}f(x)+\lim\limits_{x\to 2}g(x)=2+0=2$

(b) $\lim\limits_{x\to 1}g(x)$ does not exist since its left– and right–hand limits are not equal, so the given limit does not exist.

(c) $\lim\limits_{x\to 0}\left[f(x)g(x)\right]=\lim\limits_{x\to 0}f(x)\cdot\lim\limits_{x\to 0}g(x)=0\cdot 1.3=0$

(d) Since $\lim\limits_{x\to -1}g(x)=0$ and g is in the denominator, but $\lim\limits_{x\to -1}f(x)=-1\neq 0$, the given limit does not exist.

(e) $\lim\limits_{x\to 2}x^3f(x)=\left[\lim\limits_{x\to 2}x^3\right]\left[\lim\limits_{x\to 2}f(x)\right]=2^3\cdot 2=16$

(f) $\lim\limits_{x\to 1}\sqrt{3+f(x)}=\sqrt{3+\lim\limits_{x\to 1}f(x)}=\sqrt{3+1}=2$

3.

$$\lim_{x\to -2}(3x^4+2x^2-x+1) = \lim_{x\to -2}3x^4+\lim_{x\to -2}2x^2-\lim_{x\to -2}x+\lim_{x\to -2}1 \quad \text{[Limit Laws 1 and 2]}$$

$$=3\lim_{x\to -2}x^4+2\lim_{x\to -2}x^2-\lim_{x\to -2}x+\lim_{x\to -2}1 \quad [3]$$

$$=3(-2)^4+2(-2)^2-(-2)+(1) \quad \text{[9, 8, and 7]}$$

$$=48+8+2+1=59$$

4.

$$\lim_{x\to 2}\frac{2x^2+1}{x^2+6x-4} = \frac{\lim_{x\to 2}\left(2x^2+1\right)}{\lim_{x\to 2}\left(x^2+6x-4\right)} \quad \text{[Limit Law 5]}$$

$$=\frac{2\lim_{x\to 2}x^2+\lim_{x\to 2}1}{\lim_{x\to 2}x^2+6\lim_{x\to 2}x-\lim_{x\to 2}4} \quad \text{[2, 1, and 3]}$$

$$=\frac{2(2)^2+1}{(2)^2+6(2)-4}=\frac{9}{12}=\frac{3}{4} \quad \text{[9, 7, and 8]}$$

5.

$$\lim_{x\to 3}(x^2-4)(x^3+5x-1) = \lim_{x\to 3}(x^2-4)\cdot\lim_{x\to 3}(x^3+5x-1) \quad \text{[Limit Law 4]}$$

$$=\left(\lim_{x\to 3}x^2-\lim_{x\to 3}4\right)\cdot\left(\lim_{x\to 3}x^3+5\lim_{x\to 3}x-\lim_{x\to 3}1\right) \quad \text{[2, 1, and 3]}$$

$$=(3^2-4)\cdot(3^3+5\cdot 3-1) \quad \text{[7, 8, and 9]}$$

$$=5\cdot 41=205$$

6.

$$\lim_{t\to -1}(t^2+1)^3(t+3)^5 = \lim_{t\to -1}(t^2+1)^3\cdot\lim_{t\to -1}(t+3)^5 \quad \text{[Limit Law 4]}$$

$$=\left[\lim_{t\to -1}(t^2+1)\right]^3\cdot\left[\lim_{t\to -1}(t+3)\right]^5 \quad [6]$$

$$=\left[\lim_{t\to -1} t^2+\lim_{t\to -1} 1\right]^3\cdot\left[\lim_{t\to -1} t+\lim_{t\to -1} 3\right]^5 \quad [1]$$

$$=\left[(-1)^2+1\right]^3\cdot\left[-1+3\right]^5=8\cdot 32=256 \qquad [9, 7, \text{and } 8]$$

7.

$$\lim_{x\to 1}\left(\frac{1+3x}{1+4x^2+3x^4}\right)^3=\left(\lim_{x\to 1}\frac{1+3x}{1+4x^2+3x^4}\right)^3 \qquad [6]$$

$$=\left[\frac{\lim_{x\to 1}(1+3x)}{\lim_{x\to 1}(1+4x^2+3x^4)}\right]^3 \qquad [5]$$

$$=\left[\frac{\lim_{x\to 1} 1+3\lim_{x\to 1} x}{\lim_{x\to 1} 1+4\lim_{x\to 1} x^2+3\lim_{x\to 1} x^4}\right]^3 \qquad [2, 1, \text{and } 3]$$

$$=\left[\frac{1+3(1)}{1+4(1)^2+3(1)^4}\right]^3=\left[\frac{4}{8}\right]^3=\left(\frac{1}{2}\right)^3=\frac{1}{8} \qquad [7, 8, \text{and } 9]$$

8.

$$\lim_{u\to -2}\sqrt{u^4+3u+6}=\sqrt{\lim_{u\to -2}\left(u^4+3u+6\right)} \qquad [11]$$

$$=\sqrt{\lim_{u\to -2} u^4+3\lim_{u\to -2} u+\lim_{u\to -2} 6} \qquad [1, 2, \text{and } 3]$$

$$=\sqrt{(-2)^4+3(-2)+6} \qquad [9, 8, \text{and } 7]$$

$$=\sqrt{16-6+6}=\sqrt{16}=4$$

9.

$$\lim_{x\to 4^-}\sqrt{16-x^2}=\sqrt{\lim_{x\to 4^-}\left(16-x^2\right)} \qquad [11]$$

$$=\sqrt{\lim_{x\to 4^-} 16-\lim_{x\to 4^-} x^2} \qquad [2]$$

$$=\sqrt{16-(4)^2}=0 \qquad [7 \text{ and } 9]$$

10. **(a)** The left–hand side of the equation is not defined for $x=2$, but the right–hand side is.
(b) Since the equation holds for all $x\neq 2$, it follows that both sides of the equation approach the same limit as $x\to 2$, just as in Example 3. Remember that in finding $\lim\limits_{x\to a} f(x)$, we never consider $x=a$.

11. $\lim\limits_{x\to 2}\dfrac{x^2+x-6}{x-2}=\lim\limits_{x\to 2}\dfrac{(x+3)(x-2)}{x-2}=\lim\limits_{x\to 2}(x+3)=2+3=5$

12. $\lim\limits_{x\to -4}\dfrac{x^2+5x+4}{x^2+3x-4}=\lim\limits_{x\to -4}\dfrac{(x+4)(x+1)}{(x+4)(x-1)}=\lim\limits_{x\to -4}\dfrac{x+1}{x-1}=\dfrac{-4+1}{-4-1}=\dfrac{-3}{-5}=\dfrac{3}{5}$

13. $\lim\limits_{x\to 2}\dfrac{x^2-x+6}{x-2}$ does not exist since $x-2\to 0$ but $x^2-x+6\to 8$ as $x\to 2$.

14. $\lim\limits_{x\to 4}\dfrac{x^2-4x}{x^2-3x-4}=\lim\limits_{x\to 4}\dfrac{x(x-4)}{(x-4)(x+1)}=\lim\limits_{x\to 4}\dfrac{x}{x+1}=\dfrac{4}{4+1}=\dfrac{4}{5}$

15. $\lim\limits_{t\to -3}\dfrac{t^2-9}{2t^2+7t+3}=\lim\limits_{t\to -3}\dfrac{(t+3)(t-3)}{(2t+1)(t+3)}=\lim\limits_{t\to -3}\dfrac{t-3}{2t+1}=\dfrac{-3-3}{2(-3)+1}=\dfrac{-6}{-5}=\dfrac{6}{5}$

16. $\lim\limits_{x\to -1}\dfrac{x^2-4x}{x^2-3x-4}$ does not exist since $x^2-3x-4\to 0$ but $x^2-4x\to 5$ as $x\to -1$.

17. $\lim\limits_{h\to 0}\dfrac{(4+h)^2-16}{h}=\lim\limits_{h\to 0}\dfrac{(16+8h+h^2)-16}{h}=\lim\limits_{h\to 0}\dfrac{8h+h^2}{h}=\lim\limits_{h\to 0}\dfrac{h(8+h)}{h}=\lim\limits_{h\to 0}(8+h)=8+0=8$

18. $\lim\limits_{x\to 1}\dfrac{x^3-1}{x^2-1}=\lim\limits_{x\to 1}\dfrac{(x-1)\left(x^2+x+1\right)}{(x-1)(x+1)}=\lim\limits_{x\to 1}\dfrac{x^2+x+1}{x+1}=\dfrac{1^2+1+1}{1+1}=\dfrac{3}{2}$

19.

$$\lim_{h\to 0}\frac{(1+h)^4-1}{h}=\lim_{h\to 0}\frac{\left(1+4h+6h^2+4h^3+h^4\right)-1}{h}=\lim_{h\to 0}\frac{4h+6h^2+4h^3+h^4}{h}$$
$$=\lim_{h\to 0}\frac{h(4+6h+4h^2+h^3)}{h}=\lim_{h\to 0}\left(4+6h+4h^2+h^3\right)=4+0+0+0=4$$

20.

$$\lim_{h\to 0}\frac{(2+h)^3-8}{h}=\lim_{h\to 0}\frac{\left(8+12h+6h^2+h^3\right)-8}{h}=\lim_{h\to 0}\frac{12h+6h^2+h^3}{h}$$
$$=\lim_{h\to 0}\left(12+6h+h^2\right)=12+0+0=12$$

21. $$\lim_{t\to 9}\frac{9-t}{3-\sqrt{t}}=\lim_{t\to 9}\frac{\left(3+\sqrt{t}\,\right)\left(3-\sqrt{t}\,\right)}{3-\sqrt{t}}=\lim_{t\to 9}\left(3+\sqrt{t}\,\right)=3+\sqrt{9}=6$$

22.

$$\lim_{h\to 0}\frac{\sqrt{1+h}-1}{h}=\lim_{h\to 0}\frac{\sqrt{1+h}-1}{h}\cdot\frac{\sqrt{1+h}+1}{\sqrt{1+h}+1}=\lim_{h\to 0}\frac{(1+h)-1}{h\left(\sqrt{1+h}+1\right)}=\lim_{h\to 0}\frac{h}{h\left(\sqrt{1+h}+1\right)}$$
$$=\lim_{h\to 0}\frac{1}{\sqrt{1+h}+1}=\frac{1}{\sqrt{1}+1}=\frac{1}{2}$$

23.

$$\lim_{x\to 7}\frac{\sqrt{x+2}-3}{x-7}=\lim_{x\to 7}\frac{\sqrt{x+2}-3}{x-7}\cdot\frac{\sqrt{x+2}+3}{\sqrt{x+2}+3}=\lim_{x\to 7}\frac{(x+2)-9}{(x-7)\left(\sqrt{x+2}+3\right)}$$
$$=\lim_{x\to 7}\frac{x-7}{(x-7)\left(\sqrt{x+2}+3\right)}=\lim_{x\to 7}\frac{1}{\sqrt{x+2}+3}=\frac{1}{\sqrt{9}+3}=\frac{1}{6}$$

24.

$$\lim_{x\to 2}\frac{x^4-16}{x-2}=\lim_{x\to 2}\frac{(x+2)(x-2)\left(x^2+4\right)}{x-2}=\lim_{x\to 2}(x+2)\left(x^2+4\right)=\lim_{x\to 2}(x+2)\lim_{x\to 2}\left(x^2+4\right)$$
$$=(2+2)\left(2^2+4\right)=32$$

25.

$$\lim_{x\to -4} \frac{\frac{1}{4}+\frac{1}{x}}{4+x} = \lim_{x\to -4} \frac{\frac{x+4}{4x}}{4+x} = \lim_{x\to -4} \frac{x+4}{4x(4+x)} = \lim_{x\to -4} \frac{1}{4x} = \frac{1}{4(-4)} = -\frac{1}{16}$$

26. $$\lim_{t\to 0}\left(\frac{1}{t}-\frac{1}{t^2+t}\right) = \lim_{t\to 0}\frac{\left(t^2+t\right)-t}{t\left(t^2+t\right)} = \lim_{t\to 0}\frac{t^2}{t\cdot t(t+1)} = \lim_{t\to 0}\frac{1}{t+1} = \frac{1}{0+1} = 1$$

27.

$$\lim_{x\to 9}\frac{x^2-81}{\sqrt{x}-3} = \lim_{x\to 9}\frac{(x-9)(x+9)}{\sqrt{x}-3} = \lim_{x\to 9}\frac{(\sqrt{x}-3)(\sqrt{x}+3)(x+9)}{\sqrt{x}-3} \quad \left[\begin{array}{c}\text{factor } x-9 \text{ as a}\\ \text{difference of squares}\end{array}\right]$$
$$= \lim_{x\to 9}\left[(\sqrt{x}+3)(x+9)\right] = \left(\sqrt{9}+3\right)(9+9) = 6\cdot 18 = 108$$

28.

$$\lim_{h\to 0}\frac{(3+h)^{-1}-3^{-1}}{h} = \lim_{h\to 0}\frac{\frac{1}{3+h}-\frac{1}{3}}{h} = \lim_{h\to 0}\frac{3-(3+h)}{h(3+h)3} = \lim_{h\to 0}\frac{-h}{h(3+h)3}$$
$$= \lim_{h\to 0}\left[-\frac{1}{3(3+h)}\right] = -\frac{1}{\lim_{h\to 0}\left[3(3+h)\right]} = -\frac{1}{3(3+0)} = -\frac{1}{9}$$

29.

$$\lim_{t\to 0}\left(\frac{1}{t\sqrt{1+t}}-\frac{1}{t}\right) = \lim_{t\to 0}\frac{1-\sqrt{1+t}}{t\sqrt{1+t}} = \lim_{t\to 0}\frac{\left(1-\sqrt{1+t}\right)\left(1+\sqrt{1+t}\right)}{t\sqrt{t+1}\left(1+\sqrt{1+t}\right)} = \lim_{t\to 0}\frac{-t}{t\sqrt{1+t}\left(1+\sqrt{1+t}\right)}$$
$$= \lim_{t\to 0}\frac{-1}{\sqrt{1+t}\left(1+\sqrt{1+t}\right)} = \frac{-1}{\sqrt{1+0}\left(1+\sqrt{1+0}\right)} = -\frac{1}{2}$$

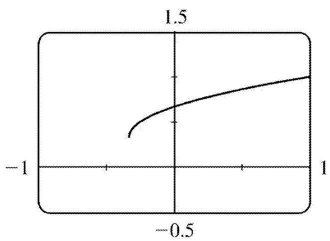
30.

$$\lim_{x\to 1}\frac{\sqrt{x}-x^2}{1-\sqrt{x}} = \lim_{x\to 1}\frac{\sqrt{x}\left(1-x^{3/2}\right)}{1-\sqrt{x}} = \lim_{x\to 1}\frac{\sqrt{x}\left(1-\sqrt{x}\right)\left(1+\sqrt{x}+x\right)}{1-\sqrt{x}} \quad \text{[difference of cubes]}$$
$$= \lim_{x\to 1}\left[\sqrt{x}\left(1+\sqrt{x}+x\right)\right] = \lim_{x\to 1}\left[1(1+1+1)\right] = 3$$

Another method: We ''add and subtract'' 1 in the numerator, and then split up the fraction:

$$\lim_{x\to 1}\frac{\sqrt{x}-x^2}{1-\sqrt{x}}=\lim_{x\to 1}\frac{\left(\sqrt{x}-1\right)+\left(1-x^2\right)}{1-\sqrt{x}}=\lim_{x\to 1}\left[-1+\frac{(1-x)(1+x)}{1-\sqrt{x}}\right]$$

$$=\lim_{x\to 1}\left[-1+\frac{\left(1-\sqrt{x}\right)\left(1+\sqrt{x}\right)(1+x)}{1-\sqrt{x}}\right]=-1+\left(1+\sqrt{1}\right)(1+1)=3$$



31. **(a)**

$$\lim_{x\to 0}\frac{x}{\sqrt{1+3x}-1}\approx\frac{2}{3}$$

(b)

x	$f(x)$
−0.001	0.6661663
−0.0001	0.6666167
−0.00001	0.6666617
−0.000001	0.6666662
0.000001	0.6666672
0.00001	0.6666717
0.0001	0.6667167
0.001	0.6671663

The limit appears to be $\frac{2}{3}$.

(c)

$$\lim_{x\to 0}\left(\frac{x}{\sqrt{1+3x}-1}\cdot\frac{\sqrt{1+3x}+1}{\sqrt{1+3x}+1}\right)=\lim_{x\to 0}\frac{x\left(\sqrt{1+3x}+1\right)}{(1+3x)-1}=\lim_{x\to 0}\frac{x\left(\sqrt{1+3x}+1\right)}{3x}$$

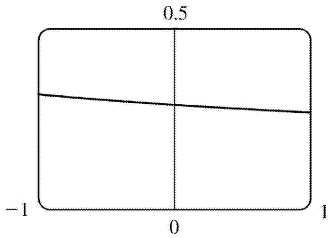
$$=\frac{1}{3}\lim_{x\to 0}\left(\sqrt{1+3x}+1\right) \qquad \text{[Limit Law 3]}$$

$$=\frac{1}{3}\left[\sqrt{\lim_{x\to 0}(1+3x)}+\lim_{x\to 0}1\right] \qquad \text{[1 and 11]}$$

$$=\frac{1}{3}\left(\sqrt{\lim_{x\to 0}1+3\lim_{x\to 0}x}+1\right) \qquad \text{[1, 3, and 7]}$$

$$= \frac{1}{3}\left(\sqrt{1+3\cdot 0}+1\right) \qquad \text{[7 and 8]}$$

$$= \frac{1}{3}(1+1) = \frac{2}{3}$$



32. **(a)**

$$\lim_{x\to 0} \frac{\sqrt{3+x}-\sqrt{3}}{x} \approx 0.29$$

(b)

x	$f(x)$
−0.001	0.2886992
−0.0001	0.2886775
−0.00001	0.2886754
−0.000001	0.2886752
0.000001	0.2886751
0.00001	0.2886749
0.0001	0.2886727
0.001	0.2886511

The limit appears to be approximately 0.2887 .

(c)

$$\lim_{x\to 0}\left(\frac{\sqrt{3+x}-\sqrt{3}}{x}\cdot\frac{\sqrt{3+x}+\sqrt{3}}{\sqrt{3+x}+\sqrt{3}}\right) = \lim_{x\to 0}\frac{(3+x)-3}{x\left(\sqrt{3+x}+\sqrt{3}\right)} = \lim_{x\to 0}\frac{1}{\sqrt{3+x}+\sqrt{3}}$$

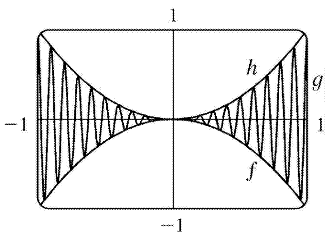
$$= \frac{\lim_{x\to 0} 1}{\lim_{x\to 0}\sqrt{3+x}+\lim_{x\to 0}\sqrt{3}} \qquad \text{[Limit Laws 5 and 1]}$$

$$= \frac{1}{\sqrt{\lim_{x\to 0}(3+x)}+\sqrt{3}} \qquad \text{[7 and 11]}$$

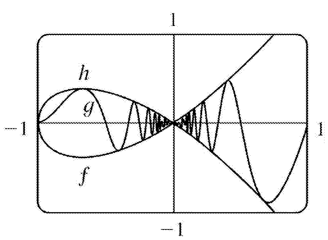
$$= \frac{1}{\sqrt{3+0}+\sqrt{3}} \qquad \text{[1, 7, and 8]}$$

$$=\frac{1}{2\sqrt{3}}$$

33. Let $f(x)=-x^2$, $g(x)=x^2\cos 20\pi x$ and $h(x)=x^2$. Then $-1\leq \cos 20\pi x\leq 1\Rightarrow -x^2\leq x^2\cos 20\pi x\leq x^2\Rightarrow$ $f(x)\leq g(x)\leq h(x)$. So since $\lim_{x\to 0} f(x)=\lim_{x\to 0} h(x)=0$, by the Squeeze Theorem we have $\lim_{x\to 0} g(x)=0$.



34. Let $f(x)=-\sqrt{x^3+x^2}$, $g(x)=\sqrt{x^3+x^2}\sin(\pi/x)$, and $h(x)=\sqrt{x^3+x^2}$. Then $-1\leq \sin(\pi/x)\leq 1\ \Rightarrow$ $-\sqrt{x^3+x^2}\leq \sqrt{x^3+x^2}\sin(\pi/x)\leq \sqrt{x^3+x^2}\ \Rightarrow f(x)\leq g(x)\leq h(x)$. So since $\lim_{x\to 0} f(x)=\lim_{x\to 0} h(x)=0$, by the Squeeze Theorem we have $\lim_{x\to 0} g(x)=0$.



35. $1\leq f(x)\leq x^2+2x+2$ for all x . Now $\lim_{x\to -1} 1=1$ and $\lim_{x\to -1}\left(x^2+2x+2\right)=\lim_{x\to -1} x^2+2\lim_{x\to -1} x+\lim_{x\to -1} 2=(-1)^2+2(-1)+2=1$. Therefore, by the Squeeze Theorem, $\lim_{x\to -1} f(x)=1$.

36. $3x\leq f(x)\leq x^3+2$ for $0\leq x\leq 2$. Now $\lim_{x\to 1} 3x=3$ and $\lim_{x\to 1}\left(x^3+2\right)=\lim_{x\to 1} x^3+\lim_{x\to 1} 2=1^3+2=3$. Therefore, by the Squeeze Theorem, $\lim_{x\to 1} f(x)=3$.

37. $-1\leq \cos(2/x)\leq 1\Rightarrow -x^4\leq x^4\cos(2/x)\leq x^4$. Since $\lim_{x\to 0}\left(-x^4\right)=0$ and $\lim_{x\to 0} x^4=0$, we have

$\lim_{x\to 0}\left[x^4\cos(2/x)\right]=0$ by the Squeeze Theorem.

38. $-1\le\sin(\pi/x)\le 1\Rightarrow e^{-1}\le e^{\sin(\pi/x)}\le e^{1}\Rightarrow\sqrt{x}/e\le\sqrt{x}\,e^{\sin(\pi/x)}\le\sqrt{x}\,e$. Since $\lim_{x\to 0^+}\left(\sqrt{x}/e\right)=0$ and $\lim_{x\to 0^+}\left(\sqrt{x}\,e\right)=0$, we have $\lim_{x\to 0^+}\left[\sqrt{x}\,e^{\sin(\pi/x)}\right]=0$ by the Squeeze Theorem.

39. If $x>-4$, then $|x+4|=x+4$, so $\lim_{x\to -4^+}|x+4|=\lim_{x\to -4^+}(x+4)=-4+4=0$.
If $x<-4$, then $|x+4|=-(x+4)$, so $\lim_{x\to -4^-}|x+4|=\lim_{x\to -4^-}-(x+4)=-(-4+4)=0$.
Since the right and left limits are equal, $\lim_{x\to -4}|x+4|=0$.

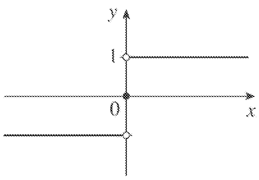
40. If $x<-4$, then $|x+4|=-(x+4)$, so $\lim_{x\to -4^-}\dfrac{|x+4|}{x+4}=\lim_{x\to -4^-}\dfrac{-(x+4)}{x+4}=\lim_{x\to -4^-}(-1)=-1$.

41. If $x>2$, then $|x-2|=x-2$, so $\lim_{x\to 2^+}\dfrac{|x-2|}{x-2}=\lim_{x\to 2^+}\dfrac{x-2}{x-2}=\lim_{x\to 2^+}1=1$. If $x<2$, then $|x-2|=-(x-2)$, so $\lim_{x\to 2^-}\dfrac{|x-2|}{x-2}=\lim_{x\to 2^-}\dfrac{-(x-2)}{x-2}=\lim_{x\to 2^-}-1=-1$. The right and left limits are different, so $\lim_{x\to 2}\dfrac{|x-2|}{x-2}$ does not exist.

42. If $x>\dfrac{3}{2}$, then $|2x-3|=2x-3$, so $\lim_{x\to 1.5^+}\dfrac{2x^2-3x}{|2x-3|}=\lim_{x\to 1.5^+}\dfrac{2x^2-3x}{2x-3}=\lim_{x\to 1.5^+}\dfrac{x(2x-3)}{2x-3}=\lim_{x\to 1.5^+}x=1.5$. If $x<\dfrac{3}{2}$, then $|2x-3|=3-2x$, so $\lim_{x\to 1.5^-}\dfrac{2x^2-3x}{|2x-3|}=\lim_{x\to 1.5^-}\dfrac{2x^2-3x}{-(2x-3)}=\lim_{x\to 1.5^-}\dfrac{x(2x-3)}{-(2x-3)}=\lim_{x\to 1.5^-}-x=-1.5$. The right and left limits are different, so $\lim_{x\to 1.5}\dfrac{2x^2-3x}{|2x-3|}$ does not exist.

43. Since $|x|=-x$ for $x<0$, we have $\lim_{x\to 0^-}\left(\dfrac{1}{x}-\dfrac{1}{|x|}\right)=\lim_{x\to 0^-}\left(\dfrac{1}{x}-\dfrac{1}{-x}\right)=\lim_{x\to 0^-}\dfrac{2}{x}$, which does not exist since the denominator approaches 0 and the numerator does not.

44. Since $|x|=x$ for $x>0$, we have $\lim\limits_{x\to 0^+}\left(\frac{1}{x}-\frac{1}{|x|}\right)=\lim\limits_{x\to 0^+}\left(\frac{1}{x}-\frac{1}{x}\right)=\lim\limits_{x\to 0^+}0=0$.

45. **(a)** 

(b)

(i) Since $\operatorname{sgn}x=1$ for $x>0$, $\lim\limits_{x\to 0^+}\operatorname{sgn}x=\lim\limits_{x\to 0^+}1=1$.

(ii) Since $\operatorname{sgn}x=-1$ for $x<0$, $\lim\limits_{x\to 0^-}\operatorname{sgn}x=\lim\limits_{x\to 0^-}-1=-1$.

(iii) Since $\lim\limits_{x\to 0^-}\operatorname{sgn}x\neq\lim\limits_{x\to 0^+}\operatorname{sgn}x$, $\lim\limits_{x\to 0}\operatorname{sgn}x$ does not exist.

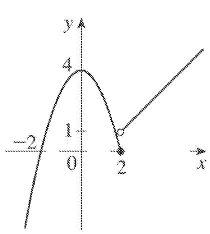
(iv) Since $|\operatorname{sgn}x|=1$ for $x\neq 0$, $\lim\limits_{x\to 0}|\operatorname{sgn}x|=\lim\limits_{x\to 0}1=1$.

46. **(a)**

$$\lim_{x\to 2^-}f(x)=\lim_{x\to 2^-}\left(4-x^2\right)=\lim_{x\to 2^-}4-\lim_{x\to 2^-}x^2$$
$$=4-4=0$$

$$\lim_{x\to 2^+}f(x)=\lim_{x\to 2^+}(x-1)=\lim_{x\to 2^+}x-\lim_{x\to 2^+}1$$
$$=2-1=1$$

(b) No, $\lim\limits_{x\to 2}f(x)$ does not exist since $\lim\limits_{x\to 2^-}f(x)\neq\lim\limits_{x\to 2^+}f(x)$.

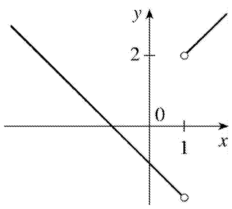
(c) 

47. **(a)**

(i) $\lim_{x\to 1^+} \frac{x^2-1}{|x-1|} = \lim_{x\to 1^+} \frac{x^2-1}{x-1} = \lim_{x\to 1^+} (x+1) = 2$

(ii) $\lim_{x\to 1^-} \frac{x^2-1}{|x-1|} = \lim_{x\to 1^-} \frac{x^2-1}{-(x-1)} = \lim_{x\to 1^-} -(x+1) = -2$

(b) No, $\lim_{x\to 1} F(x)$ does not exist since $\lim_{x\to 1^+} F(x) \neq \lim_{x\to 1^-} F(x)$.

(c) 

48. **(a)**

(i) $\lim_{x\to 0^+} h(x) = \lim_{x\to 0^+} x^2 = 0^2 = 0$

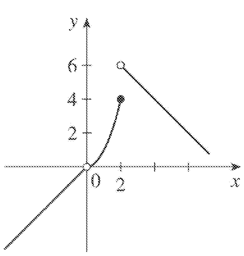
(ii) $\lim_{x\to 0^-} h(x) = \lim_{x\to 0^-} x = 0$, so $\lim_{x\to 0} h(x) = 0$.

(iii) $\lim_{x\to 1} h(x) = \lim_{x\to 1} x^2 = 1^2 = 1$

(iv) $\lim_{x\to 2^-} h(x) = \lim_{x\to 2^-} x^2 = 2^2 = 4$

(v) $\lim_{x\to 2^+} h(x) = \lim_{x\to 2^+} (8-x) = 8-2 = 6$

(vi) Since $\lim_{x\to 2^-} h(x) \neq \lim_{x\to 2^+} h(x)$, $\lim_{x\to 2} h(x)$ does not exist.



(b)

49. **(a)**

(i) $[x]=-2$ for $-2\le x<-1$, so $\lim_{x\to -2^+}[x]=\lim_{x\to -2^+}(-2)=-2$

(ii) $[x]=-3$ for $-3\le x<-2$, so $\lim_{x\to -2^-}[x]=\lim_{x\to -2^-}(-3)=-3$.

The right and left limits are different, so $\lim_{x\to -2}[x]$ does not exist.

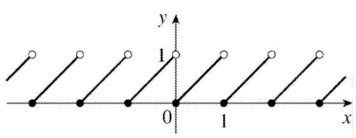
(iii) $[x]=-3$ for $-3\le x<-2$, so $\lim_{x\to -2.4}[x]=\lim_{x\to -2.4}(-3)=-3$.

(b)

(i) $[x]=n-1$ for $n-1\le x<n$, so $\lim_{x\to n^-}[x]=\lim_{x\to n^-}(n-1)=n-1$.

(ii) $[x]=n$ for $n\le x<n+1$, so $\lim_{x\to n^+}[x]=\lim_{x\to n^+}n=n$.

(c) $\lim_{x\to a}[x]$ exists $\Leftrightarrow a$ is not an integer.

50. **(a)** 

(b)

(i) $\lim_{x\to n^-}f(x)=\lim_{x\to n^-}(x-[x])=\lim_{x\to n^-}\left[x-(n-1)\right]=n-(n-1)=1$

(ii) $\lim_{x\to n^+}f(x)=\lim_{x\to n^+}(x-[x])=\lim_{x\to n^+}(x-n)=n-n=0$

(c) $\lim_{x\to a}f(x)$ exists $\Leftrightarrow a$ is not an integer.

51. The graph of $f(x)=[x]+[-x]$ is the same as the graph of $g(x)=-1$ with holes at each integer, since $f(a)=0$ for any integer a . Thus, $\lim_{x\to 2^-}f(x)=-1$ and $\lim_{x\to 2^+}f(x)=-1$, so $\lim_{x\to 2}f(x)=-1$. However, $f(2)=[2]+[-2]=2+(-2)=0$, so $\lim_{x\to 2}f(x)\neq f(2)$.

52. $\lim_{v\to c^-}\left(L_0\sqrt{1-\frac{v^2}{c^2}}\right)=L_0\sqrt{1-1}=0$. As the velocity approaches the speed of light, the length

approaches 0 .
A left–hand limit is necessary since L is not defined for $v>c$.

53. Since $p(x)$ is a polynomial, $p(x)=a_0+a_1x+a_2x^2+\cdots+a_nx^n$. Thus, by the Limit Laws,

$$\begin{aligned}\lim_{x\to a} p(x) &= \lim_{x\to a}\left(a_0+a_1x+a_2x^2+\cdots+a_nx^n\right)\\ &= a_0+a_1\lim_{x\to a} x+a_2\lim_{x\to a} x^2+\cdots+a_n\lim_{x\to a} x^n\\ &= a_0+a_1a+a_2a^2+\cdots+a_na^n=p(a)\end{aligned}$$

Thus, for any polynomial p , $\lim_{x\to a} p(x)=p(a)$.

54. Let $r(x)=\dfrac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are any polynomials, and suppose that $q(a)\neq 0$. Thus,

$$\lim_{x\to a} r(x)=\lim_{x\to a}\frac{p(x)}{q(x)}=\frac{\lim_{x\to a} p(x)}{\lim_{x\to a} q(x)}\ \text{[Limit Law 5]}=\frac{p(a)}{q(a)}\ \text{[Exercise 53]}=r(a) .$$

55. Observe that $0\leq f(x)\leq x^2$ for all x , and $\lim_{x\to 0} 0=0=\lim_{x\to 0} x^2$. So, by the Squeeze Theorem, $\lim_{x\to 0} f(x)=0$.

56. Let $f(x)=[x]$ and $g(x)=-[x]$. Then $\lim_{x\to 3} f(x)$ and $\lim_{x\to 3} g(x)$ do not exist (Example 10) but $\lim_{x\to 3}\left[f(x)+g(x)\right]=\lim_{x\to 3}([x]-[x])=\lim_{x\to 3} 0=0$.

57. Let $f(x)=H(x)$ and $g(x)=1-H(x)$, where H is the Heaviside function defined in Exercise 1.3.59. Thus, either f or g is 0 for any value of x . Then $\lim_{x\to 0} f(x)$ and $\lim_{x\to 0} g(x)$ do not exist, but $\lim_{x\to 0}\left[f(x)g(x)\right]=\lim_{x\to 0} 0=0$.

58.

$$\lim_{x\to 2}\frac{\sqrt{6-x}-2}{\sqrt{3-x}-1}=\lim_{x\to 2}\left(\frac{\sqrt{6-x}-2}{\sqrt{3-x}-1}\cdot\frac{\sqrt{6-x}+2}{\sqrt{6-x}+2}\cdot\frac{\sqrt{3-x}+1}{\sqrt{3-x}+1}\right)$$

$$=\lim_{x\to 2}\left[\frac{\left(\sqrt{6-x}\,\right)^2-2^2}{\left(\sqrt{3-x}\,\right)^2-1^2}\cdot\frac{\sqrt{3-x}+1}{\sqrt{6-x}+2}\right]=\lim_{x\to 2}\left(\frac{6-x-4}{3-x-1}\cdot\frac{\sqrt{3-x}+1}{\sqrt{6-x}+2}\right)$$

$$=\lim_{x\to 2}\frac{(2-x)\left(\sqrt{3-x}+1\right)}{(2-x)\left(\sqrt{6-x}+2\right)}=\lim_{x\to 2}\frac{\sqrt{3-x}+1}{\sqrt{6-x}+2}=\frac{1}{2}$$

59. Since the denominator approaches 0 as $x\to -2$, the limit will exist only if the numerator also approaches 0 as $x\to -2$. In order for this to happen, we need $\lim\limits_{x\to -2}\left(3x^2+ax+a+3\right)=0\Leftrightarrow$ $3(-2)^2+a(-2)+a+3=0\Leftrightarrow 12-2a+a+3=0\Leftrightarrow a=15$. With $a=15$, the limit becomes

$$\lim_{x\to -2}\frac{3x^2+15x+18}{x^2+x-2}=\lim_{x\to -2}\frac{3(x+2)(x+3)}{(x-1)(x+2)}=\lim_{x\to -2}\frac{3(x+3)}{x-1}=\frac{3(-2+3)}{-2-1}=\frac{3}{-3}=-1 .$$

60. *Solution 1:* First, we find the coordinates of P and Q as functions of r . Then we can find the equation of the line determined by these two points, and thus find the $x-$ intercept (the point R), and take the limit as $r\to 0$.

The coordinates of P are $(0,r)$. The point Q is the point of intersection of the two circles $x^2+y^2=r^2$ and $(x-1)^2+y^2=1$. Eliminating y from these equations, we get $r^2-x^2=1-(x-1)^2\Leftrightarrow r^2=1+2x-1\Leftrightarrow x=\frac{1}{2}r^2$. Substituting back into the equation of the shrinking circle to find the $y-$ coordinate, we get $\left(\frac{1}{2}r^2\right)^2+y^2=r^2\Leftrightarrow y^2=r^2\left(1-\frac{1}{4}r^2\right)\Leftrightarrow y=r\sqrt{1-\frac{1}{4}r^2}$ (the positive $y-$ value). So the coordinates of Q are $\left(\frac{1}{2}r^2,r\sqrt{1-\frac{1}{4}r^2}\right)$. The equation of the line joining P and Q is thus

$$y-r=\frac{r\sqrt{1-\frac{1}{4}r^2}-r}{\frac{1}{2}r^2-0}(x-0)$$. We set $y=0$ in order to find the $x-$ intercept, and get

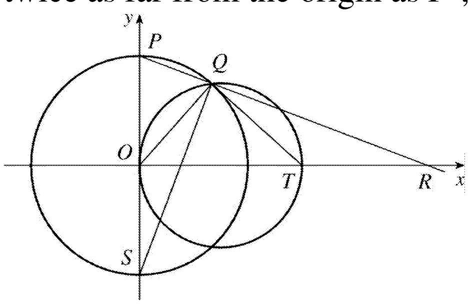
$$x=-r\frac{\frac{1}{2}r^2}{r\left(\sqrt{1-\frac{1}{4}r^2}-1\right)}=\frac{-\frac{1}{2}r^2\left(\sqrt{1-\frac{1}{4}r^2}+1\right)}{1-\frac{1}{4}r^2-1}=2\left(\sqrt{1-\frac{1}{4}r^2}+1\right) .$$

Now we take the limit as $r\to 0^+$: $\lim\limits_{r\to 0^+}x=\lim\limits_{r\to 0^+}2\left(\sqrt{1-\frac{1}{4}r^2}+1\right)=\lim\limits_{r\to 0^+}2\left(\sqrt{1}+1\right)=4$.

So the limiting position of R is the point $(4,0)$.

Solution 2: We add a few lines to the diagram, as shown. Note that $\angle PQS = 90^\circ$ (subtended by diameter PS).

So $\angle SQR = 90^\circ = \angle OQT$ (subtended by diameter OT). It follows that $\angle OQS = \angle TQR$. Also $\angle PSQ = 90^\circ - \angle SPQ = \angle ORP$. Since $\triangle QOS$ is isosceles, so is $\triangle QTR$, implying that $QT = TR$. As the circle C_2 shrinks, the point Q plainly approaches the origin, so the point R must approach a point twice as far from the origin as T, that is, the point $(4,0)$, as above.



1. **(a)** To have $5x+3$ within a distance of 0.1 of 13, we must have $12.9\le 5x+3\le 13.1\Rightarrow 9.9\le 5x\le 10.1\Rightarrow 1.98\le x\le 2.02$. Thus, x must be within 0.02 units of 2 so that $5x+3$ is within 0.1 of 13.
(b) Use 0.01 in place of 0.1 in part (a) to obtain 0.002.

2. **(a)** To have $6x-1$ within a distance of 0.01 of 29, we must have $28.99\le 6x-1\le 29.01\Rightarrow 29.99\le 6x\le 30.01\Rightarrow 4.998\overline{3}\le x\le 5.001\overline{6}$. Thus, x must be within $0.001\overline{6}$ units of 5 so that $6x-1$ is within 0.01 of 29.
(b) As in part (a) with 0.001 in place of 0.01, we obtain $0.0001\overline{6}$.
(c) As in part (a) with 0.0001 in place of 0.01, we obtain $0.00001\overline{6}$.

3. On the left side of $x=2$, we need $|x-2|<\left|\frac{10}{7}-2\right|=\frac{4}{7}$. On the right side, we need $|x-2|<\left|\frac{10}{3}-2\right|=\frac{4}{3}$. For both of these conditions to be satisfied at once, we need the more restrictive of the two to hold, that is, $|x-2|<\frac{4}{7}$. So we can choose $\delta=\frac{4}{7}$, or any smaller positive number.

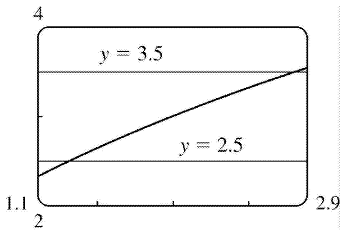
4. On the left side, we need $|x-5|<|4-5|=1$. On the right side, we need $|x-5|<|5.7-5|=0.7$. For both conditions to be satisfied at once, we need the more restrictive condition to hold; that is, $|x-5|<0.7$. So we can choose $\delta=0.7$, or any smaller positive number.

5. The leftmost question mark is the solution of $\sqrt{x}=1.6$ and the rightmost, $\sqrt{x}=2.4$. So the values are $1.6^2=2.56$ and $2.4^2=5.76$. On the left side, we need $|x-4|<|2.56-4|=1.44$. On the right side, we need $|x-4|<|5.76-4|=1.76$. To satisfy both conditions, we need the more restrictive condition to hold — namely, $|x-4|<1.44$. Thus, we can choose $\delta=1.44$, or any smaller positive number.

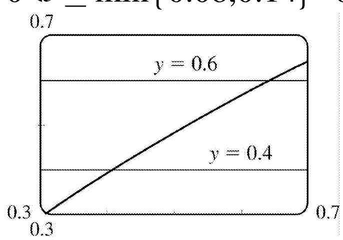
6. The left–hand question mark is the positive solution of $x^2=\frac{1}{2}$, that is, $x=\frac{1}{\sqrt{2}}$, and the right–hand question mark is the positive solution of $x^2=\frac{3}{2}$, that is, $x=\sqrt{\frac{3}{2}}$. On the left side, we need $|x-1|<\left|\frac{1}{\sqrt{2}}-1\right|\approx 0.292$ (rounding down to be safe). On the right side, we need $|x-1|<\left|\sqrt{\frac{3}{2}}-1\right|\approx 0.224$. The more restrictive of these two conditions must apply, so we choose $\delta=0.224$ (or any smaller positive number).

7. $\left|\sqrt{4x+1}-3\right|<0.5\Leftrightarrow 2.5<\sqrt{4x+1}<3.5$. We plot the three parts of this inequality on the same screen

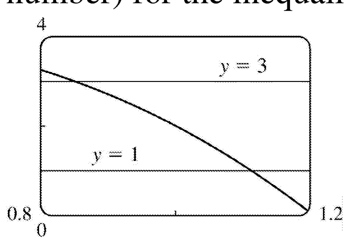
and identify the x –coordinates of the points of intersection using the cursor. It appears that the inequality holds for $1.3125 \le x \le 2.8125$. Since $|2-1.3125|=0.6875$ and $|2-2.8125|=0.8125$, we choose $0<\delta<\min\{0.6875,0.8125\}=0.6875$.

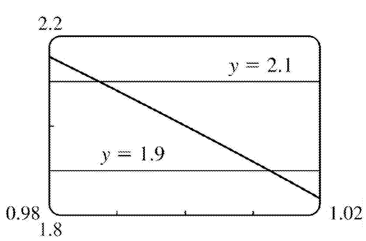


8. $\left|\sin x-\dfrac{1}{2}\right|<0.1 \Leftrightarrow 0.4<\sin x<0.6$. From the graph, we see that for this inequality to hold, we need $0.42 \le x \le 0.64$. So since $|0.5-0.42|=0.08$ and $|0.5-0.64|=0.14$, we choose $0<\delta \le \min\{0.08,0.14\}=0.08$.

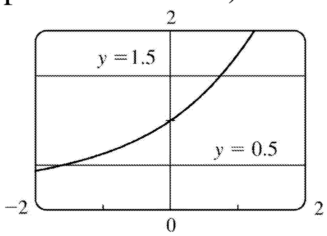


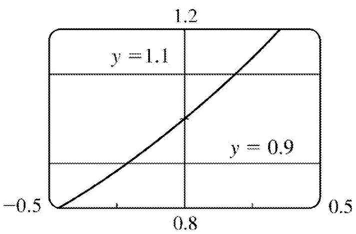
9. For $\varepsilon=1$, the definition of a limit requires that we find δ such that $\left|\left(4+x-3x^3\right)-2\right|<1 \Leftrightarrow 1<4+x-3x^3<3$ whenever $0<|x-1|<\delta$. If we plot the graphs of $y=1$, $y=4+x-3x^3$ and $y=3$ on the same screen, we see that we need $0.86 \le x \le 1.11$. So since $|1-0.86|=0.14$ and $|1-1.11|=0.11$, we choose $\delta=0.11$ (or any smaller positive number). For $\varepsilon=0.1$, we must find δ such that $\left|\left(4+x-3x^3\right)-2\right|<0.1$ $\Leftrightarrow 1.9<4+x-3x^3<2.1$ whenever $0<|x-1|<\delta$. From the graph, we see that we need $0.988 \le x \le 1.012$. So since $|1-0.988|=0.012$ and $|1-1.012|=0.012$, we choose $\delta=0.012$ (or any smaller positive number) for the inequality to hold.



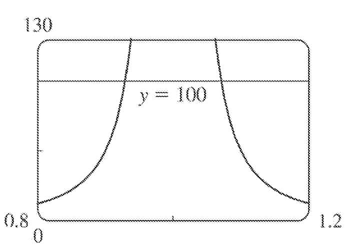


10. For $\varepsilon=0.5$, the definition of a limit requires that we find δ such that $\left|\frac{e^x-1}{x}-1\right|<0.5 \Leftrightarrow$ $0.5<\frac{e^x-1}{x}<1.5$ whenever $0<|x-0|<\delta$. If we plot the graphs of $y=0.5$, $y=\frac{e^x-1}{x}$, and $y=1.5$ on the same screen, we see that we need $-1.59\le x\le 0.76$. So since $|0-(-1.59)|=1.59$ and $|0-0.76|=0.76$, we choose $\delta=0.76$ (or any smaller positive number). For $\varepsilon=0.1$, we must find δ such that $\left|\frac{e^x-1}{x}-1\right|<0.1 \Leftrightarrow 0.9<\frac{e^x-1}{x}<1.1$ whenever $0<|x-0|<\delta$. From the graph, we see that we need $-0.21\le x\le 0.18$. So since $|0-(-0.21)|=0.21$ and $|0-0.18|=0.18$, we choose $\delta=0.18$ (or any smaller positive number) for the inequality to hold.

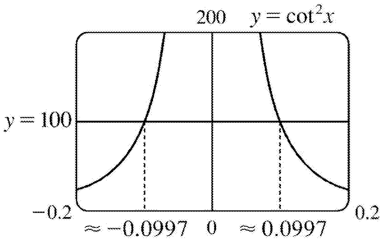




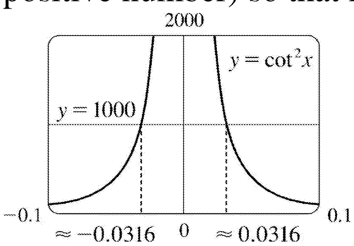
11. From the graph, we see that $\frac{x}{\left(x^2+1\right)(x-1)^2}>100$ whenever $0.93\le x\le 1.07$. So since $|1-0.93|=0.07$ and $|1-1.07|=0.07$, we can take $\delta=0.07$ (or any smaller positive number).



12. For $M=100$, we need $-0.0997<x<0$ or $0<x<0.0997$. Thus, we choose $\delta=0.0997$ (or any smaller positive number) so that if $0<|x|<\delta$, then $\cot^2 x>100$.



For $M=1000$, we need $-0.0316<x<0$ or $0<x<0.0316$. Thus, we choose $\delta=0.0316$ (or any smaller positive number) so that if $0<|x|<\delta$, then $\cot^2 x>1000$.



13. **(a)** $A=\pi r^2$ and $A=1000\text{ cm}^2 \Rightarrow \pi r^2=1000 \Rightarrow r^2=\dfrac{1000}{\pi} \Rightarrow$

$r=\sqrt{\dfrac{1000}{\pi}}\ [r>0] \approx 17.8412$ cm.

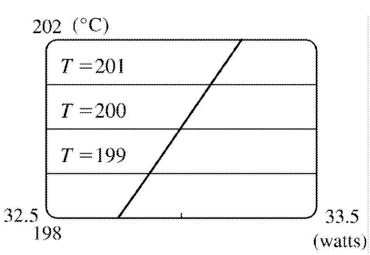
(b) $|A-1000|\leq 5 \Rightarrow -5\leq \pi r^2-1000\leq 5 \Rightarrow 1000-5\leq \pi r^2\leq 1000+5 \Rightarrow$

$\sqrt{\dfrac{995}{\pi}}\leq r\leq \sqrt{\dfrac{1005}{\pi}} \Rightarrow 17.7966\leq r\leq 17.8858$. $\sqrt{\dfrac{1000}{\pi}}-\sqrt{\dfrac{995}{\pi}}\approx 0.04466$ and

$\sqrt{\dfrac{1005}{\pi}}-\sqrt{\dfrac{1000}{\pi}}\approx 0.04455$. So if the machinist gets the radius within 0.0445 cm of 17.8412 , the area will be within 5 cm^2 of 1000 .

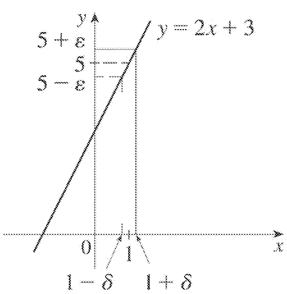
(c) x is the radius, $f(x)$ is the area, a is the target radius given in part (a), L is the target area (1000) , ε is the tolerance in the area (5), and δ is the tolerance in the radius given in part (b).

14. **(a)** $T=0.1w^2+2.155w+20$ and $T=200 \Rightarrow 0.1w^2+2.155w+20=200 \Rightarrow$ [by the quadratic formula or from the graph] $w\approx 33.0$ watts ($w>0$)

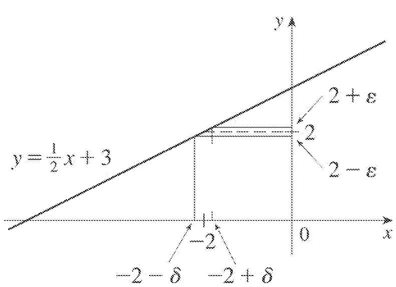


(b) From the graph, $199 \leq T \leq 201 \Rightarrow 32.89 < w < 33.11$.
(c) x is the input power, $f(x)$ is the temperature, a is the target input power given in part (a), L is the target temperature (200), ε is the tolerance in the temperature (1), and δ is the tolerance in the power input in watts indicated in part (b) (0.11 watts).

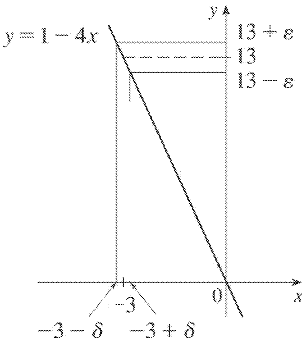
15. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x-1| < \delta$, then $|(2x+3)-5| < \varepsilon$. But $|(2x+3)-5| < \varepsilon \Leftrightarrow |2x-2| < \varepsilon \Leftrightarrow 2|x-1| < \varepsilon \Leftrightarrow |x-1| < \varepsilon/2$. So if we choose $\delta = \varepsilon/2$, then $0 < |x-1| < \delta \Rightarrow |(2x+3)-5| < \varepsilon$. Thus, $\lim_{x \to 1} (2x+3) = 5$ by the definition of a limit.



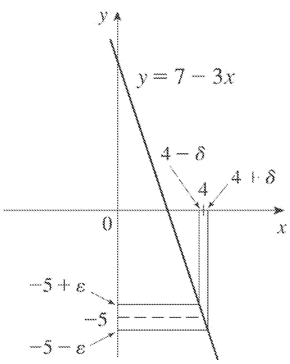
16. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x-(-2)| < \delta$, then $\left|\left(\frac{1}{2}x+3\right)-2\right| < \varepsilon$. But $\left|\left(\frac{1}{2}x+3\right)-2\right| < \varepsilon \Leftrightarrow \left|\frac{1}{2}x+1\right| < \varepsilon \Leftrightarrow \frac{1}{2}|x+2| < \varepsilon \Leftrightarrow |x-(-2)| < 2\varepsilon$. So if we choose $\delta = 2\varepsilon$, then $0 < |x-(-2)| < \delta \Rightarrow \left|\left(\frac{1}{2}x+3\right)-2\right| < \varepsilon$. Thus, $\lim_{x \to -2} \left(\frac{1}{2}x+3\right) = 2$ by the definition of a limit.



17. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x-(-3)| < \delta$, then $|(1-4x)-13| < \varepsilon$. But $|(1-4x)-13| < \varepsilon \Leftrightarrow |-4x-12| < \varepsilon \Leftrightarrow |-4|\ |x+3| < \varepsilon \Leftrightarrow |x-(-3)| < \varepsilon/4$. So if we choose $\delta = \varepsilon/4$, then $0 < |x-(-3)| < \delta \Rightarrow |(1-4x)-13| < \varepsilon$. Thus, $\lim_{x \to -3} (1-4x) = 13$ by the definition of a limit.



18. Given $\varepsilon>0$, we need $\delta>0$ such that if $0<|x-4|<\delta$, then $|(7-3x)-(-5)|<\varepsilon$. But $|(7-3x)-(-5)|<\varepsilon \Leftrightarrow |-3x+12|<\varepsilon \Leftrightarrow |-3|\,|x-4|<\varepsilon \Leftrightarrow |x-4|<\varepsilon/3$. So if we choose $\delta=\varepsilon/3$, then $0<|x-4|<\delta \Rightarrow |(7-3x)-(-5)|<\varepsilon$. Thus, $\lim_{x\to 4}(7-3x)=-5$ by the definition of a limit.



19. Given $\varepsilon>0$, we need $\delta>0$ such that if $0<|x-3|<\delta$, then $\left|\frac{x}{5}-\frac{3}{5}\right|<\varepsilon \Leftrightarrow \frac{1}{5}|x-3|<\varepsilon \Leftrightarrow |x-3|<5\varepsilon$. So choose $\delta=5\varepsilon$. Then $0<|x-3|<\delta \Rightarrow |x-3|<5\varepsilon \Rightarrow \frac{|x-3|}{5}<\varepsilon \Rightarrow \left|\frac{x}{5}-\frac{3}{5}\right|<\varepsilon$. By the definition of a limit, $\lim_{x\to 3}\frac{x}{5}=\frac{3}{5}$.

20. Given $\varepsilon>0$, we need $\delta>0$ such that if $0<|x-6|<\delta$, then $\left|\left(\frac{x}{4}+3\right)-\frac{9}{2}\right|<\varepsilon \Leftrightarrow \left|\frac{x}{4}-\frac{3}{2}\right|<\varepsilon \Leftrightarrow \frac{1}{4}|x-6|<\varepsilon \Leftrightarrow |x-6|<4\varepsilon$. So choose $\delta=4\varepsilon$. Then $0<|x-6|<\delta \Rightarrow |x-6|<4\varepsilon \Rightarrow \frac{|x-6|}{4}<\varepsilon \Rightarrow \left|\frac{x}{4}-\frac{6}{4}\right|<\varepsilon \Rightarrow \left|\left(\frac{x}{4}+3\right)-\frac{9}{2}\right|<\varepsilon$. By the definition of a limit, $\lim_{x\to 6}\left(\frac{x}{4}+3\right)=\frac{9}{2}$.

21. Given $\varepsilon>0$, we need $\delta>0$ such that if $0<|x-(-5)|<\delta$, then $\left|\left(4-\frac{3}{5}x\right)-7\right|<\varepsilon \Leftrightarrow$

$\left|-\frac{3}{5}x-3\right|<\varepsilon \Leftrightarrow \frac{3}{5}|x+5|<\varepsilon \Leftrightarrow |x-(-5)|<\frac{5}{3}\varepsilon$. So choose $\delta=\frac{5}{3}\varepsilon$. Then $|x-(-5)|<\delta \Rightarrow$ $\left|\left(4-\frac{3}{5}x\right)-7\right|<\varepsilon$. Thus, $\lim_{x\to -5}\left(4-\frac{3}{5}x\right)=7$ by the definition of a limit.

22. Given $\varepsilon>0$, we need $\delta>0$ such that if $0<|x-3|<\delta$, then $\left|\frac{x^2+x-12}{x-3}-7\right|<\varepsilon$. Notice that if $0<|x-3|$, then $x\neq 3$, so $\frac{x^2+x-12}{x-3}=\frac{(x+4)(x-3)}{x-3}=x+4$. Thus, when $0<|x-3|$, we have $\left|\frac{x^2+x-12}{x-3}-7\right|<\varepsilon \Leftrightarrow |(x+4)-7|<\varepsilon \Leftrightarrow |x-3|<\varepsilon$. We take $\delta=\varepsilon$ and see that $0<|x-3|<\delta \Rightarrow$ $\left|\frac{x^2+x-12}{x-3}-7\right|<\varepsilon$. By the definition of a limit, $\lim_{x\to 3}\frac{x^2+x-12}{x-3}=7$.

23. Given $\varepsilon>0$, we need $\delta>0$ such that if $0<|x-a|<\delta$, then $|x-a|<\varepsilon$. So $\delta=\varepsilon$ will work.

24. Given $\varepsilon>0$, we need $\delta>0$ such that if $0<|x-a|<\delta$, then $|c-c|<\varepsilon$. But $|c-c|=0$, so this will be true no matter what δ we pick.

25. Given $\varepsilon>0$, we need $\delta>0$ such that if $0<|x-0|<\delta$, then $\left|x^2-0\right|<\varepsilon \Leftrightarrow x^2<\varepsilon \Leftrightarrow |x|<\sqrt{\varepsilon}$. Take $\delta=\sqrt{\varepsilon}$. Then $0<|x-0|<\delta \Rightarrow \left|x^2-0\right|<\varepsilon$. Thus, $\lim_{x\to 0}x^2=0$ by the definition of a limit.

26. Given $\varepsilon>0$, we need $\delta>0$ such that if $0<|x-0|<\delta$, then $\left|x^3-0\right|<\varepsilon \Leftrightarrow |x|^3<\varepsilon \Leftrightarrow |x|<\sqrt[3]{\varepsilon}$. Take $\delta=\sqrt[3]{\varepsilon}$. Then $0<|x-0|<\delta \Rightarrow \left|x^3-0\right|<\delta^3=\varepsilon$. Thus, $\lim_{x\to 0}x^3=0$ by the definition of a limit.

27. Given $\varepsilon>0$, we need $\delta>0$ such that if $0<|x-0|<\delta$, then $\big||x|-0\big|<\varepsilon$. But $\big||x|\big|=|x|$. So this is true if we pick $\delta=\varepsilon$. Thus, $\lim_{x\to 0}|x|=0$ by the definition of a limit.

28. Given $\varepsilon>0$, we need $\delta>0$ such that if $9-\delta<x<9$, then $\left|\sqrt[4]{9-x}-0\right|<\varepsilon \Leftrightarrow \sqrt[4]{9-x}<\varepsilon \Leftrightarrow 9-x<\varepsilon^4 \Leftrightarrow$ $9-\varepsilon^4<x<9$. So take $\delta=\varepsilon^4$. Then $9-\delta<x<9 \Rightarrow \left|\sqrt[4]{9-x}-0\right|<\varepsilon$. Thus, $\lim_{x\to 9^-}\sqrt[4]{9-x}=0$ by the definition of a limit.

29. Given $\varepsilon>0$, we need $\delta>0$ such that if $0<|x-2|<\delta$, then $\left|\left(x^2-4x+5\right)-1\right|<\varepsilon \Leftrightarrow \left|x^2-4x+4\right|<\varepsilon \Leftrightarrow$

$\left|(x-2)^2\right|<\varepsilon$. So take $\delta=\sqrt{\varepsilon}$. Then $0<|x-2|<\delta \Leftrightarrow |x-2|<\sqrt{\varepsilon} \Leftrightarrow \left|(x-2)^2\right|<\varepsilon$. Thus, $\lim\limits_{x\to 2}\left(x^2-4x+5\right)=1$ by the definition of a limit.

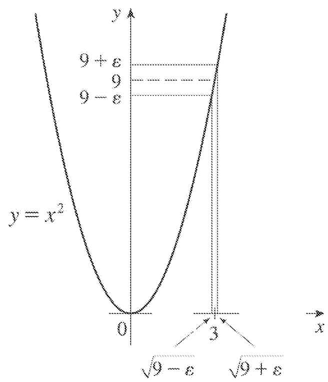
30. Given $\varepsilon>0$, we need $\delta>0$ such that if $0<|x-3|<\delta$, then $\left|\left(x^2+x-4\right)-8\right|<\varepsilon \Leftrightarrow \left|x^2+x-12\right|<\varepsilon \Leftrightarrow$ $|(x-3)(x+4)|<\varepsilon$. Notice that if $|x-3|<1$, then $-1<x-3<1 \Rightarrow 6<x+4<8 \Rightarrow |x+4|<8$. So take $\delta=\min\{1,\varepsilon/8\}$. Then $0<|x-3|<\delta \Leftrightarrow |(x-3)(x+4)| \le |8(x-3)|=8\cdot|x-3|<8\delta \le \varepsilon$. Thus, $\lim\limits_{x\to 3}\left(x^2+x-4\right)=8$ by the definition of a limit.

31. Given $\varepsilon>0$, we need $\delta>0$ such that if $0<|x-(-2)|<\delta$, then $\left|\left(x^2-1\right)-3\right|<\varepsilon$ or upon simplifying we need $\left|x^2-4\right|<\varepsilon$ whenever $0<|x+2|<\delta$. Notice that if $|x+2|<1$, then $-1<x+2<1 \Rightarrow -5<x-2<-3 \Rightarrow$ $|x-2|<5$. So take $\delta=\min\{\varepsilon/5,1\}$. Then $0<|x+2|<\delta \Rightarrow |x-2|<5$ and $|x+2|<\varepsilon/5$, so $\left|\left(x^2-1\right)-3\right|=|(x+2)(x-2)|=|x+2|\,|x-2|<(\varepsilon/5)(5)=\varepsilon$. Thus, by the definition of a limit, $\lim\limits_{x\to -2}\left(x^2-1\right)=3$.

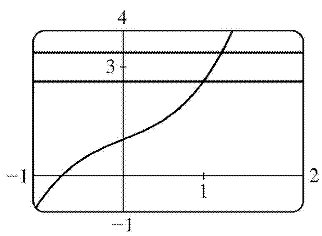
32. Given $\varepsilon>0$, we need $\delta>0$ such that if $0<|x-2|<\delta$, then $\left|x^3-8\right|<\varepsilon$. Now $\left|x^3-8\right|=\left|(x-2)\left(x^2+2x+4\right)\right|$. If $|x-2|<1$, that is, $1<x<3$, then $x^2+2x+4<3^2+2(3)+4=19$ and so $\left|x^3-8\right|=|x-2|\left(x^2+2x+4\right)<19|x-2|$. So if we take $\delta=\min\left\{1,\frac{\varepsilon}{19}\right\}$, then $0<|x-2|<\delta \Rightarrow$ $\left|x^3-8\right|=|x-2|\left(x^2+2x+4\right)<\frac{\varepsilon}{19}\cdot 19=\varepsilon$. Thus, by the definition of a limit, $\lim\limits_{x\to 2} x^3=8$.

33. Given $\varepsilon>0$, we let $\delta=\min\left\{2,\frac{\varepsilon}{8}\right\}$. If $0<|x-3|<\delta$, then $|x-3|<2 \Rightarrow -2<x-3<2 \Rightarrow 4<x+3<8 \Rightarrow$ $|x+3|<8$. Also $|x-3|<\frac{\varepsilon}{8}$, so $\left|x^2-9\right|=|x+3|\,|x-3|<8\cdot\frac{\varepsilon}{8}=\varepsilon$. Thus, $\lim\limits_{x\to 3} x^2=9$.

34. From the figure, our choices for δ are $\delta_1=3-\sqrt{9-\varepsilon}$ and $\delta_2=\sqrt{9+\varepsilon}-3$. The *largest* possible choice for δ is the minimum value of $\{\delta_1,\delta_2\}$; that is, $\delta=\min\{\delta_1,\delta_2\}=\delta_2=\sqrt{9+\varepsilon}-3$.



35. **(a)** The points of intersection in the graph are $(x_1, 2.6)$ and $(x_2, 3.4)$ with $x_1 \approx 0.891$ and $x_2 \approx 1.093$. Thus, we can take δ to be the smaller of $1 - x_1$ and $x_2 - 1$. So $\delta = x_2 - 1 \approx 0.093$.



(b) Solving $x^3 + x + 1 = 3 + \varepsilon$ gives us two nonreal complex roots and one real root, which is
$x(\varepsilon) = \dfrac{\left(216 + 108\varepsilon + 12\sqrt{336 + 324\varepsilon + 81\varepsilon^2}\right)^{2/3} - 12}{6\left(216 + 108\varepsilon + 12\sqrt{336 + 324\varepsilon + 81\varepsilon^2}\right)^{1/3}}$. Thus, $\delta = x(\varepsilon) - 1$.

(c) If $\varepsilon = 0.4$, then $x(\varepsilon) \approx 1.093\,272\,342$ and $\delta = x(\varepsilon) - 1 \approx 0.093$, which agrees with our answer in part (a).

36. *1. Guessing a value for* δ Let $\varepsilon > 0$ be given. We have to find a number $\delta > 0$ such that $\left| \frac{1}{x} - \frac{1}{2} \right| < \varepsilon$ whenever $0 < |x - 2| < \delta$. But $\left| \frac{1}{x} - \frac{1}{2} \right| = \left| \frac{2 - x}{2x} \right| = \frac{|x - 2|}{|2x|} < \varepsilon$. We find a positive constant C such that $\frac{1}{|2x|} < C \Rightarrow \frac{|x - 2|}{|2x|} < C|x - 2|$ and we can make $C|x - 2| < \varepsilon$ by taking $|x - 2| < \frac{\varepsilon}{C} = \delta$. We restrict x to lie in the interval $|x - 2| < 1 \Rightarrow 1 < x < 3$ so $1 > \frac{1}{x} > \frac{1}{3} \Rightarrow \frac{1}{6} < \frac{1}{2x} < \frac{1}{2} \Rightarrow \frac{1}{|2x|} < \frac{1}{2}$. So $C = \frac{1}{2}$ is suitable. Thus, we should choose $\delta = \min\{1, 2\varepsilon\}$.

2. Showing that δ *works* Given $\varepsilon > 0$ we let $\delta = \min\{1, 2\varepsilon\}$. If $0 < |x - 2| < \delta$, then $|x - 2| < 1 \Rightarrow 1 < x < 3 \Rightarrow \frac{1}{|2x|} < \frac{1}{2}$ (as in part 1). Also $|x - 2| < 2\varepsilon$, so $\left| \frac{1}{x} - \frac{1}{2} \right| = \frac{|x - 2|}{|2x|} < \frac{1}{2} \cdot 2\varepsilon = \varepsilon$. This shows that

$\lim_{x\to 2}(1/x)=\frac{1}{2}$.

37. *1. Guessing a value for* δ Given $\varepsilon>0$, we must find $\delta>0$ such that $|\sqrt{x}-\sqrt{a}|<\varepsilon$ whenever $0<|x-a|<\delta$. But $|\sqrt{x}-\sqrt{a}|=\frac{|x-a|}{\sqrt{x}+\sqrt{a}}<\varepsilon$ (from the hint). Now if we can find a positive constant C such that $\sqrt{x}+\sqrt{a}>C$ then $\frac{|x-a|}{\sqrt{x}+\sqrt{a}}<\frac{|x-a|}{C}<\varepsilon$, and we take $|x-a|<C\varepsilon$. We can find this number by restricting x to lie in some interval centered at a . If $|x-a|<\frac{1}{2}a$, then $-\frac{1}{2}a<x-a<\frac{1}{2}a\Rightarrow\frac{1}{2}a<x<\frac{3}{2}a$ $\Rightarrow\sqrt{x}+\sqrt{a}>\sqrt{\frac{1}{2}a}+\sqrt{a}$, and so $C=\sqrt{\frac{1}{2}a}+\sqrt{a}$ is a suitable choice for the constant. So $|x-a|<\left(\sqrt{\frac{1}{2}a}+\sqrt{a}\right)\varepsilon$. This suggests that we let $\delta=\min\left\{\frac{1}{2}a,\left(\sqrt{\frac{1}{2}a}+\sqrt{a}\right)\varepsilon\right\}$.

2. Showing that δ *works* Given $\varepsilon>0$, we let $\delta=\min\left\{\frac{1}{2}a,\left(\sqrt{\frac{1}{2}a}+\sqrt{a}\right)\varepsilon\right\}$. If $0<|x-a|<\delta$, then $|x-a|<\frac{1}{2}a\Rightarrow\sqrt{x}+\sqrt{a}>\sqrt{\frac{1}{2}a}+\sqrt{a}$ (as in part 1). Also $|x-a|<\left(\sqrt{\frac{1}{2}a}+\sqrt{a}\right)\varepsilon$, so $|\sqrt{x}-\sqrt{a}|=\frac{|x-a|}{\sqrt{x}+\sqrt{a}}<\frac{\left(\sqrt{a/2}+\sqrt{a}\right)\varepsilon}{\left(\sqrt{a/2}+\sqrt{a}\right)}=\varepsilon$. Therefore, $\lim_{x\to a}\sqrt{x}=\sqrt{a}$ by the definition of a limit.

38. Suppose that $\lim_{t\to 0}H(t)=L$. Given $\varepsilon=\frac{1}{2}$, there exists $\delta>0$ such that $0<|t|<\delta\Rightarrow|H(t)-L|<\frac{1}{2}\Leftrightarrow$ $L-\frac{1}{2}<H(t)<L+\frac{1}{2}$. For $0<t<\delta$, $H(t)=1$, so $1<L+\frac{1}{2}\Rightarrow L>\frac{1}{2}$. For $-\delta<t<0$, $H(t)=0$, so $L-\frac{1}{2}<0\Rightarrow$ $L<\frac{1}{2}$. This contradicts $L>\frac{1}{2}$. Therefore, $\lim_{t\to 0}H(t)$ does not exist.

39. Suppose that $\lim_{x\to 0}f(x)=L$. Given $\varepsilon=\frac{1}{2}$, there exists $\delta>0$ such that $0<|x|<\delta\Rightarrow|f(x)-L|<\frac{1}{2}$. Take any rational number r with $0<|r|<\delta$. Then $f(r)=0$, so $|0-L|<\frac{1}{2}$, so $L\le|L|<\frac{1}{2}$. Now take any irrational number s with $0<|s|<\delta$. Then $f(s)=1$, so $|1-L|<\frac{1}{2}$. Hence, $1-L<\frac{1}{2}$, so $L>\frac{1}{2}$. This contradicts $L<\frac{1}{2}$, so $\lim_{x\to 0}f(x)$ does not exist.

40. First suppose that $\lim_{x\to a}f(x)=L$. Then, given $\varepsilon>0$ there exists $\delta>0$ so that $0<|x-a|<\delta\Rightarrow$ $|f(x)-L|<\varepsilon$. Then $a-\delta<x<a\Rightarrow 0<|x-a|<\delta$ so $|f(x)-L|<\varepsilon$. Thus,

$\lim_{x\to a^-} f(x)=L$. Also $a<x<a+\delta \Rightarrow 0<|x-a|<\delta$ so $|f(x)-L|<\varepsilon$. Hence, $\lim_{x\to a^+} f(x)=L$.
Now suppose $\lim_{x\to a^-} f(x)=L=\lim_{x\to a^+} f(x)$. Let $\varepsilon>0$ be given. Since $\lim_{x\to a^-} f(x)=L$, there exists $\delta_1>0$ so that $a-\delta_1<x<a \Rightarrow |f(x)-L|<\varepsilon$. Since $\lim_{x\to a^+} f(x)=L$, there exists $\delta_2>0$ so that $a<x<a+\delta_2 \Rightarrow |f(x)-L|<\varepsilon$. Let δ be the smaller of δ_1 and δ_2 . Then $0<|x-a|<\delta \Rightarrow a-\delta_1<x<a$ or $a<x<a+\delta_2$ so $|f(x)-L|<\varepsilon$. Hence, $\lim_{x\to a} f(x)=L$. So we have proved that $\lim_{x\to a} f(x)=L \Leftrightarrow \lim_{x\to a^-} f(x)=L=\lim_{x\to a^+} f(x)$.

41. $\dfrac{1}{(x+3)^4}>10{,}000 \Leftrightarrow (x+3)^4<\dfrac{1}{10{,}000} \Leftrightarrow |x+3|<\dfrac{1}{\sqrt[4]{10{,}000}} \Leftrightarrow |x-(-3)|<\dfrac{1}{10}$

42. Given $M>0$, we need $\delta>0$ such that $0<|x+3|<\delta \Rightarrow 1/(x+3)^4>M$. Now $\dfrac{1}{(x+3)^4}>M \Leftrightarrow (x+3)^4<\dfrac{1}{M} \Leftrightarrow |x+3|<\dfrac{1}{\sqrt[4]{M}}$. So take $\delta=\dfrac{1}{\sqrt[4]{M}}$. Then $0<|x+3|<\delta=\dfrac{1}{\sqrt[4]{M}} \Rightarrow \dfrac{1}{(x+3)^4}>M$, so $\lim_{x\to -3} \dfrac{1}{(x+3)^4}=\infty$.

43. Given $M<0$ we need $\delta>0$ so that $\ln x<M$ whenever $0<x<\delta$; that is, $x=e^{\ln x}<e^M$ whenever $0<x<\delta$. This suggests that we take $\delta=e^M$. If $0<x<e^M$, then $\ln x<\ln e^M=M$. By the definition of a limit, $\lim_{x\to 0^+} \ln x=-\infty$.

44. **(a)** Let M be given. Since $\lim_{x\to a} f(x)=\infty$, there exists $\delta_1>0$ such that $0<|x-a|<\delta_1 \Rightarrow f(x)>M+1-c$. Since $\lim_{x\to a} g(x)=c$, there exists $\delta_2>0$ such that $0<|x-a|<\delta_2 \Rightarrow |g(x)-c|<1 \Rightarrow g(x)>c-1$. Let δ be the smaller of δ_1 and δ_2 . Then $0<|x-a|<\delta \Rightarrow f(x)+g(x)>(M+1-c)+(c-1)=M$. Thus, $\lim_{x\to a} [f(x)+g(x)]=\infty$.
(b) Let $M>0$ be given. Since $\lim_{x\to a} g(x)=c>0$, there exists $\delta_1>0$ such that $0<|x-a|<\delta_1 \Rightarrow |g(x)-c|<c/2 \Rightarrow g(x)>c/2$. Since $\lim_{x\to a} f(x)=\infty$, there exists $\delta_2>0$ such that $0<|x-a|<\delta_2 \Rightarrow f(x)>2M/c$. Let $\delta=\min\left\{\delta_1,\delta_2\right\}$. Then $0<|x-a|<\delta \Rightarrow f(x)\,g(x)>\dfrac{2M}{c}\,\dfrac{c}{2}=M$, so $\lim_{x\to a} f(x)\,g(x)=\infty$.
(c) Let $N<0$ be given. Since $\lim_{x\to a} g(x)=c<0$, there exists $\delta_1>0$ such that $0<|x-a|<\delta_1 \Rightarrow |g(x)-c|<-c/2 \Rightarrow g(x)<c/2$. Since $\lim_{x\to a} f(x)=\infty$, there exists $\delta_2>0$ such that $0<|x-a|<\delta_2 \Rightarrow f(x)>2N/c$. (Note that

$c<0$ and $N<0 \Rightarrow 2N/c>0$.) Let $\delta=\min\left\{\delta_1,\delta_2\right\}$. Then $0<|x-a|<\delta \Rightarrow f(x)>2N/c \Rightarrow$ $f(x)\,g(x)<\frac{2N}{c}\cdot\frac{c}{2}=N$, so $\lim_{x\to a} f(x)\,g(x)=-\infty$.

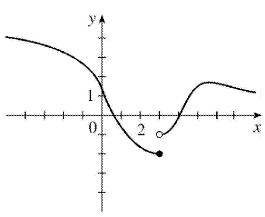
1. From Definition 1, $\lim_{x \to 4} f(x) = f(4)$.

2. The graph of f has no hole, jump, or vertical asymptote.

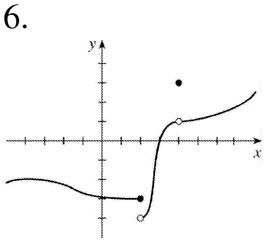
3. **(a)** The following are the numbers at which f is discontinuous and the type of discontinuity at that number: -4 (removable), -2 (jump), 2 (jump), 4 (infinite).
(b) f is continuous from the left at -2 since $\lim_{x \to -2^-} f(x) = f(-2)$. f is continuous from the right at 2 and 4 since $\lim_{x \to 2^+} f(x) = f(2)$ and $\lim_{x \to 4^+} f(x) = f(4)$. It is continuous from neither side at -4 since $f(-4)$ is undefined.

4. g is continuous on $[-4,-2)$, $(-2,2)$, $[2,4)$, $(4,6)$, and $(6,8)$.

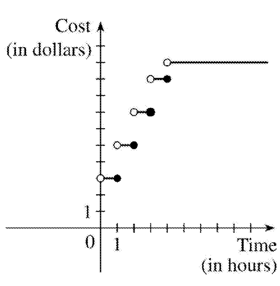
5. The graph of $y=f(x)$ must have a discontinuity at $x=3$ and must show that $\lim_{x \to 3^-} f(x) = f(3)$.



6.



7. **(a)**



(b) There are discontinuities at times $t=1$, 2 , 3 , and 4 . A person parking in the lot would want to keep in mind that the charge will jump at the beginning of each hour.

8. **(a)** Continuous; at the location in question, the temperature changes smoothly as time passes, without any instantaneous jumps from one temperature to another.

(b) Continuous; the temperature at a specific time changes smoothly as the distance due west from New York City increases, without any instantaneous jumps.
(c) Discontinuous; as the distance due west from New York City increases, the altitude above sea level may jump from one height to another without going through all of the intermediate values — at a cliff, for example.
(d) Discontinuous; as the distance traveled increases, the cost of the ride jumps in small increments.
(e) Discontinuous; when the lights are switched on (or off), the current suddenly changes between 0 and some nonzero value, without passing through all of the intermediate values. This is debatable, though, depending on your definition of current.

9. Since f and g are continuous functions,

$$\begin{aligned}\lim_{x\to 3}\left[2f(x)-g(x)\right] &= 2\lim_{x\to 3} f(x)-\lim_{x\to 3} g(x) && \text{[by Limit Laws 2 and 3]}\\ &= 2f(3)-g(3) && \text{[by continuity of } f \text{ and } g \text{ at } x=3\text{]}\\ &= 2\cdot 5-g(3)=10-g(3)\end{aligned}$$

Since it is given that $\lim_{x\to 3}\left[2f(x)-g(x)\right]=4$, we have $10-g(3)=4$, so $g(3)=6$.

10. $\lim_{x\to 4} f(x)=\lim_{x\to 4}\left(x^2+\sqrt{7-x}\right)=\lim_{x\to 4} x^2+\sqrt{\lim_{x\to 4} 7-\lim_{x\to 4} x}=4^2+\sqrt{7-4}=16+\sqrt{3}=f(4)$.
By the definition of continuity, f is continuous at $a=4$.

11. $\lim_{x\to -1} f(x)=\lim_{x\to -1}\left(x+2x^3\right)^4=\left(\lim_{x\to -1} x+2\lim_{x\to -1} x^3\right)^4=\left[-1+2(-1)^3\right]^4=(-3)^4=81=f(-1)$.
By the definition of continuity, f is continuous at $a=-1$.

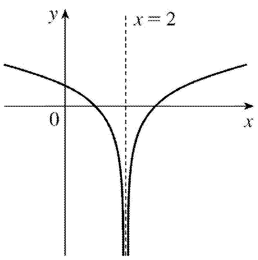
12. $\lim_{x\to 4} g(x)=\lim_{x\to 4}\dfrac{x+1}{2x^2-1}=\dfrac{\lim_{x\to 4} x+\lim_{x\to 4} 1}{2\lim_{x\to 4} x^2-\lim_{x\to 4} 1}=\dfrac{4+1}{2(4)^2-1}=\dfrac{5}{31}=g(4)$. So g is continuous at 4 .

13. For $a>2$, we have $\lim_{x\to a} f(x)=\lim_{x\to a}\dfrac{2x+3}{x-2}=\dfrac{\lim_{x\to a}(2x+3)}{\lim_{x\to a}(x-2)}$ [Limit Law 5] $=\dfrac{2\lim_{x\to a} x+\lim_{x\to a} 3}{\lim_{x\to a} x-\lim_{x\to a} 2}$ [1, 2, and 3] $=\dfrac{2a+3}{a-2}$ [7 and 8] $=f(a)$. Thus, f is continuous at $x=a$ for every a in $(2,\infty)$; that is, f is continuous on $(2,\infty)$.

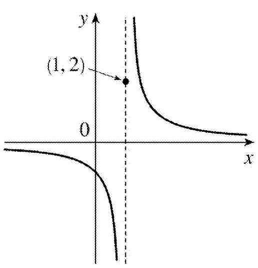
14. For $a<3$, we have $\lim_{x\to a} g(x)=\lim_{x\to a} 2\sqrt{3-x}=2\lim_{x\to a}\sqrt{3-x}$ [Limit Law 3] $=2\sqrt{\lim_{x\to a}(3-x)}$ [11]

$=2\sqrt{\lim\limits_{x\to a} 3-\lim\limits_{x\to a} x}$ [2] $=2\sqrt{3-a}$ [7 and 8] $=g(a)$, so g is continuous at $x=a$ for every a in $(-\infty ,3)$. Also, $\lim\limits_{x\to 3^-} g(x)=0=g(3)$, so g is continuous from the left at 3 . Thus, g is continuous on $(-\infty ,3]$.

15. $f(x)=\ln |x-2|$ is discontinuous at 2 since $f(2)=\ln 0$ is not defined.

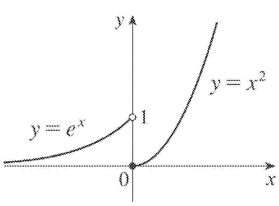


16. $f(x)=\begin{cases} 1/(x-1) & \text{if } x\neq 1 \\ 2 & \text{if } x=1 \end{cases}$ is discontinuous at 1 because $\lim\limits_{x\to 1} f(x)$ does not exist.



17. $f(x)=\begin{cases} e^x & \text{if} x<0 \\ x^2 & \text{if} x\geq 0 \end{cases}$

The left–hand limit of f at $a=0$ is $\lim\limits_{x\to 0^-} f(x)=\lim\limits_{x\to 0^-} e^x=1$. The right–hand limit of f at $a=0$ is $\lim\limits_{x\to 0^+} f(x)=\lim\limits_{x\to 0^+} x^2=0$. Since these limits are not equal, $\lim\limits_{x\to 0} f(x)$ does not exist and f is discontinuous at 0.



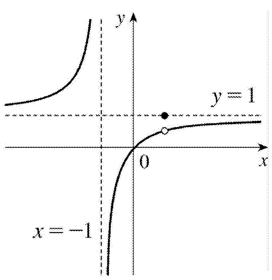
18.

$$f(x)=\begin{cases} \dfrac{x^2-x}{x^2-1} & \text{if } x\neq 1 \\ 1 & \text{if } x=1 \end{cases}$$

$$\lim_{x\to 1} f(x) = \lim_{x\to 1} \frac{x^2-x}{x^2-1} = \lim_{x\to 1} \frac{x(x-1)}{(x+1)(x-1)}$$

$$= \lim_{x\to 1} \frac{x}{x+1} = \frac{1}{2} ,$$

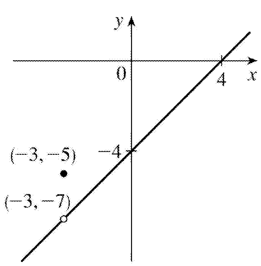
but $f(1)=1$, so f is discontinous at 1.



19. $f(x)=\begin{cases} \dfrac{x^2-x-12}{x+3} & \text{if } x\neq -3 \\ -5 & \text{if } x=-3 \end{cases} = \begin{cases} x-4 & \text{if } x\neq -3 \\ -5 & \text{if } x=-3 \end{cases}$

So $\lim_{x\to -3} f(x)=\lim_{x\to -3} (x-4)=-7$ and $f(-3)=-5$.

Since $\lim_{x\to -3} f(x)\neq f(-3)$, f is discontinuous at -3 .

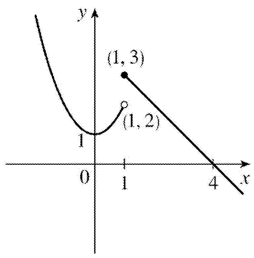


20. $f(x)=\begin{cases} 1+x^2 & \text{if } x<1 \\ 4-x & \text{if } x\geq 1 \end{cases}$

$\lim_{x\to 1^-} f(x)=\lim_{x\to 1^-} (1+x^2)=1+1^2=2$ and

$\lim_{x\to 1^+} f(x)=\lim_{x\to 1^+} (4-x)=4-1=3$.

Thus, f is discontinuous at 1 because $\lim_{x\to 1} f(x)$ does not exist.



21. $F(x)=\dfrac{x}{x^2+5x+6}$ is a rational function. So by Theorem 5 (or Theorem 7), F is continuous at every number in its domain, $\left\{x\mid x^2+5x+6\neq 0\right\}=\{x\mid (x+3)(x+2)\neq 0\}=\{x\mid x\neq -3,-2\}$ or $(-\infty,-3)\cup(-3,-2)\cup(-2,\infty)$.

22. By Theorem 7, the root function $\sqrt[3]{x}$ and the polynomial function $1+x^3$ are continuous on R . By part 4 of Theorem 4, the product $G(x)=\sqrt[3]{x}\left(1+x^3\right)$ is continuous on its domain, R .

23. By Theorem 5, the polynomials x^2 and $2x-1$ are continuous on $(-\infty,\infty)$. By Theorem 7, the root function $\sqrt{x}$ is continuous on $[0,\infty)]$. By Theorem 9, the composite function $\sqrt{2x-1}$ is continuous on its domain, $[\frac{1}{2},\infty)]$. By part 1 of Theorem 4, the sum $R(x)=x^2+\sqrt{2x-1}$ is continuous on $[\frac{1}{2},\infty)]$.

24. By Theorem 7, the trigonometric function $\sin x$ and the polynomial function $x+1$ are continuous on R . By part 5 of Theorem 4, $h(x)=\dfrac{\sin x}{x+1}$ is continuous on its domain, $\{x\mid x\neq -1\}$.

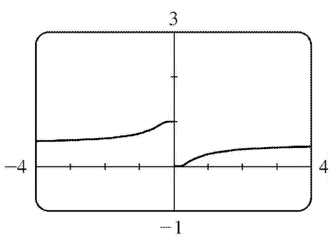
25. By Theorem 5, the polynomial $5x$ is continuous on $(-\infty,\infty)$. By Theorems 9 and 7, $\sin 5x$ is continuous on $(-\infty,\infty)$. By Theorem 7, e^x is continuous on $(-\infty,\infty)$. By part 4 of Theorem 4, the product of e^x and $\sin 5x$ is continuous at all numbers which are in both of their domains, that is, on $(-\infty,\infty)$.

26. By Theorem 5, the polynomial x^2-1 is continuous on $(-\infty,\infty)$. By Theorem 7, $\sin^{-1}$ is continuous on its domain, $[-1,1]$. By Theorem 9, $\sin^{-1}\left(x^2-1\right)$ is continuous on its domain, which is $\left\{x\mid -1\leq x^2-1\leq 1\right\}=\left\{x\mid 0\leq x^2\leq 2\right\}=\left\{x\mid |x|\leq\sqrt{2}\right\}=\left[-\sqrt{2},\sqrt{2}\right]$.

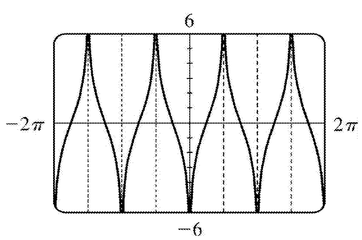
27. By Theorem 5, the polynomial t^4-1 is continuous on $(-\infty,\infty)$. By Theorem 7, ln x is continuous on its domain, $(0,\infty)$. By Theorem 9, ln $\left(t^4-1\right)$ is continuous on its domain, which is $\left\{t\,|\,t^4-1>0\right\}=\left\{t\,|\,t^4>1\right\}=\{t\,|\,|t|>1\}=(-\infty,-1)\cup(1,\infty)$.

28. By Theorem 7, $\sqrt{x}$ is continuous on $[0,\infty)$. By Theorems 7 and 9, $e^{\sqrt{x}}$ is continuous on $[0,\infty)$. Also by Theorems 7 and 9, cos $\left(e^{\sqrt{x}}\right)$ is continuous on $[0,\infty)$.

29. The function $y=\dfrac{1}{1+e^{1/x}}$ is discontinuous at $x=0$ because the left– and right–hand limits at $x=0$ are different.



30. The function $y=\tan^2 x$ is discontinuous at $x=\dfrac{\pi}{2}+\pi k$, where k is any integer. The function $y=\ln\left(\tan^2 x\right)$ is also discontinuous where $\tan^2 x$ is 0 , that is, at $x=\pi k$. So $y=\ln\left(\tan^2 x\right)$ is discontinuous at $x=\dfrac{\pi}{2}n$, n any integer.



31. Because we are dealing with root functions, $5+\sqrt{x}$ is continuous on $[0,\infty)$, $\sqrt{x+5}$ is continuous on $[-5,\infty)$, so the quotient $f(x)=\dfrac{5+\sqrt{x}}{\sqrt{5+x}}$ is continuous on $[0,\infty)$. Since f is continuous at $x=4$, $\lim\limits_{x\to 4} f(x)=f(4)=\dfrac{7}{3}$.

32. Because x is continuous on R , sin x is continuous on R , and $x+\sin x$ is continuous on R , the

composite function $f(x)=\sin(x+\sin x)$ is continuous on R, so $\lim_{x\to\pi} f(x)=f(\pi)=\sin(\pi+\sin\pi)=\sin\pi=0$.

33. Because x^2-x is continuous on R, the composite function $f(x)=e^{x^2-x}$ is continuous on R, so $\lim_{x\to 1} f(x)=f(1)=e^{1-1}=e^0=1$.

34. Because arctan is a continuous function, we can apply Theorem 8.

$$\lim_{x\to 2}\arctan\left(\frac{x^2-4}{3x^2-6x}\right)=\arctan\left(\lim_{x\to 2}\frac{(x+2)(x-2)}{3x(x-2)}\right)=\arctan\left(\lim_{x\to 2}\frac{x+2}{3x}\right)=\arctan\frac{2}{3}\approx 0.588$$

35. $f(x)=\begin{cases} x^2 & \text{if } x<1 \\ \sqrt{x} & \text{if } x\geq 1 \end{cases}$

By Theorem 5, since $f(x)$ equals the polynomial x^2 on $(-\infty,1)$, f is continuous on $(-\infty,1)$. By Theorem 7, since $f(x)$ equals the root function $\sqrt{x}$ on $(1,\infty)$, f is continuous on $(1,\infty)$. At $x=1$, $\lim_{x\to 1^-} f(x)=\lim_{x\to 1^-} x^2=1$ and $\lim_{x\to 1^+} f(x)=\lim_{x\to 1^+}\sqrt{x}=1$. Thus, $\lim_{x\to 1} f(x)$ exists and equals 1. Also, $f(1)=\sqrt{1}=1$. Thus, f is continuous at $x=1$. We conclude that f is continuous on $(-\infty,\infty)$.

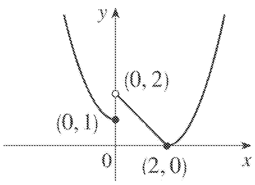
36. $f(x)=\begin{cases} \sin x & \text{if } x<\pi/4 \\ \cos x & \text{if } x\geq\pi/4 \end{cases}$

By Theorem 7, the trigonometric functions are continuous. Since $f(x)=\sin x$ on $(-\infty,\pi/4)$ and $f(x)=\cos x$ on $(\pi/4,\infty)$, f is continuous on $(-\infty,\pi/4)\cup(\pi/4,\infty)$.
$\lim_{x\to(\pi/4)^-} f(x)=\lim_{x\to(\pi/4)^-}\sin x=\sin\dfrac{\pi}{4}=1/\sqrt{2}$ since the sine function is continuous at $\pi/4$. Similarly, $\lim_{x\to(\pi/4)^+} f(x)=\lim_{x\to(\pi/4)^+}\cos x=1/\sqrt{2}$ by continuity of the cosine function at $\pi/4$. Thus, $\lim_{x\to(\pi/4)} f(x)$ exists and equals $1/\sqrt{2}$, which agrees with the value $f(\pi/4)$. Therefore, f is continuous at $\pi/4$, so f is continuous on $(-\infty,\infty)$.

37. $f(x)=\begin{cases} 1+x^2 & \text{if } x\leq 0 \\ 2-x & \text{if } 0<x\leq 2 \\ (x-2)^2 & \text{if } x>2 \end{cases}$

f is continuous on $(-\infty,0)$, $(0,2)$, and $(2,\infty)$ since it is a polynomial on each of these intervals. Now

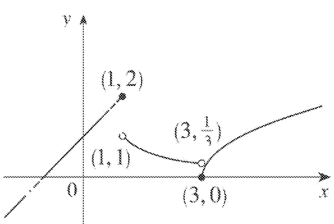
$\lim_{x\to 0^-} f(x)=\lim_{x\to 0^-}(1+x^2)=1$ and



$\lim_{x\to 0^+} f(x)=\lim_{x\to 0^+}(2-x)=2$, so f is discontinuous at 0 . Since $f(0)=1$, f is continuous from the left at 0. Also, $\lim_{x\to 2^-} f(x)=\lim_{x\to 2^-}(2-x)=0, \lim_{x\to 2^+} f(x)=\lim_{x\to 2^+}(x-2)^2=0$, and $f(2)=0$, so f is continuous at 2. The only number at which f is discontinuous is 0 .

38. $f(x)=\begin{cases} x+1 & \text{if } x\le 1 \\ 1/x & \text{if } 1<x<3 \\ \sqrt{x-3} & \text{if } x\ge 3 \end{cases}$

f is continuous on $(-\infty,1)$, $(1,3)$, and $(3,\infty)$, where it is a polynomial, a rational function, and a composite of a root function with a polynomial, respectively. Now $\lim_{x\to 1^-} f(x)=\lim_{x\to 1^-}(x+1)=2$ and $\lim_{x\to 1^+} f(x)=\lim_{x\to 1^+}(1/x)=1$, so f is discontinuous at 1 .

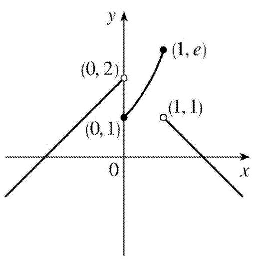


Since $f(1)=2$, f is continuous from the left at 1 . Also, $\lim_{x\to 3^-} f(x)=\lim_{x\to 3^-}(1/x)=1/3$, and $\lim_{x\to 3^+} f(x)=\lim_{x\to 3^+}\sqrt{x-3}=0=f(3)$, so f is discontinuous at 3 , but it is continuous from the right at 3.

39. $f(x)=\begin{cases} x+2 & \text{if } x<0 \\ e^x & \text{if } 0\le x\le 1 \\ 2-x & \text{if } x>1 \end{cases}$

f is continuous on $(-\infty,0)$ and $(1,\infty)$ since on each of these intervals it is a polynomial; it is continuous on $(0,1)$ since it is an exponential. Now

$\lim_{x\to 0^-} f(x)=\lim_{x\to 0^-}(x+2)=2$ and $\lim_{x\to 0^+} f(x)=\lim_{x\to 0^+} e^x=1$, so f is discontinuous at 0 . Since $f(0)=1$, f is continuous from the right at 0 . Also $\lim_{x\to 1^-} f(x)=\lim_{x\to 1^-} e^x=e$ and $\lim_{x\to 1^+} f(x)=\lim_{x\to 1^+}(2-x)=1$, so f is discontinuous at 1 . Since $f(1)=e$, f is continuous from the left at 1 .



40. By Theorem 5, each piece of F is continuous on its domain. We need to check for continuity at $r=R$.
$\lim_{r\to R^-} F(r)=\lim_{r\to R^-}\frac{GMr}{R^3}=\frac{GM}{R^2}$ and $\lim_{r\to R^+} F(r)=\lim_{r\to R^+}\frac{GM}{r^2}=\frac{GM}{R^2}$, so $\lim_{r\to R} F(r)=\frac{GM}{R^2}$. Since $F(R)=\frac{GM}{R^2}$, F is continuous at R . Therefore, F is a continuous function of r .

41. f is continuous on $(-\infty,3)$ and $(3,\infty)$. Now $\lim_{x\to 3^-} f(x)=\lim_{x\to 3^-}(cx+1)=3c+1$ and $\lim_{x\to 3^+} f(x)=\lim_{x\to 3^+}\left(cx^2-1\right)=9c-1$. So f is continuous $\Leftrightarrow 3c+1=9c-1\Leftrightarrow 6c=2\Leftrightarrow c=\frac{1}{3}$. Thus, for f to be continuous on $(-\infty,\infty)$, $c=\frac{1}{3}$.

42. The functions x^2-c^2 and $cx+20$, considered on the intervals $(-\infty,4)$ and $[4,\infty)$ respectively, are continuous for any value of c . So the only possible discontinuity is at $x=4$. For the function to be continuous at $x=4$, the left–hand and right–hand limits must be the same. Now
$\lim_{x\to 4^-} g(x)=\lim_{x\to 4^-}\left(x^2-c^2\right)=16-c^2$ and $\lim_{x\to 4^+} g(x)=\lim_{x\to 4^+}(cx+20)=4c+20=g(4)$. Thus, $16-c^2=4c+20\Leftrightarrow$
$c^2+4c+4=0\Leftrightarrow c=-2$.

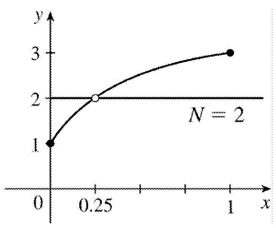
43. **(a)** $f(x)=\frac{x^2-2x-8}{x+2}=\frac{(x-4)(x+2)}{x+2}$ has a removable discontinuity at -2 because $g(x)=x-4$ is continuous on R and $f(x)=g(x)$ for $x\neq -2$. [The discontinuity is removed by defining $f(-2)=-6$.]
(b)

$f(x)=\dfrac{x-7}{|x-7|} \Rightarrow \lim_{x\to 7^-} f(x)=-1$ and $\lim_{x\to 7^+} f(x)=1$. Thus, $\lim_{x\to 7} f(x)$ does not exist, so the discontinuity is not removable. (It is a jump discontinuity.)

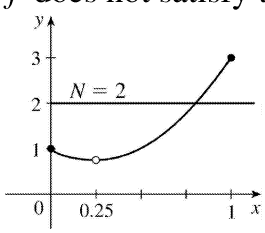
(c) $f(x)=\dfrac{x^3+64}{x+4}=\dfrac{(x+4)(x^2-4x+16)}{x+4}$ has a removable discontinuity at -4 because $g(x)=x^2-4x+16$ is continuous on R and $f(x)=g(x)$ for $x\neq -4$.[The discontinuity is removed by defining $f(-4)=48$.]

(d) $f(x)=\dfrac{3-\sqrt{x}}{9-x}=\dfrac{3-\sqrt{x}}{\left(3-\sqrt{x}\right)\left(3+\sqrt{x}\right)}$ has a removable discontinuity at 9 because $g(x)=\dfrac{1}{3+\sqrt{x}}$ is continuous on R and $f(x)=g(x)$ for $x\neq 9$. [The discontinuity is removed by defining $f(9)=\dfrac{1}{6}$.]

44.



f does not satisfy the conclusion of the Intermediate Value Theorem.



f does satisfy the conclusion of the Intermediate Value Theorem.

45. $f(x)=x^3-x^2+x$ is continuous on the interval $[2,3]$, $f(2)=6$, and $f(3)=21$. Since $6<10<21$, there is a number c in $(2,3)$ such that $f(c)=10$ by the Intermediate Value Theorem.

46. $f(x)=x^2$ is continuous on the interval $[1,2]$, $f(1)=1$, and $f(2)=4$. Since $1<2<4$, there is a number c in $(1,2)$ such that $f(c)=c^2=2$ by the Intermediate Value Theorem.

47. $f(x)=x^4+x-3$ is continuous on the interval $[1,2], f(1)=-1$, and $f(2)=15$. Since $-1<0<15$, there is a number c in $(1,2)$ such that $f(c)=0$ by the Intermediate Value Theorem. Thus, there is a root of the equation $x^4+x-3=0$ in the interval $(1,2)$.

48. $f(x)=\sqrt[3]{x}+x-1$ is continuous on the interval $[0,1], f(0)=-1$, and $f(1)=1$. Since $-1<0<1$, there is a number c in $(0,1)$ such that $f(c)=0$ by the Intermediate Value Theorem. Thus, there is a root of the

equation $\sqrt[3]{x}+x-1=0$, or $\sqrt[3]{x}=1-x$, in the interval (0,1).

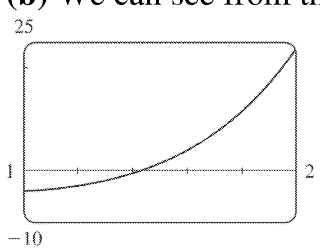
49. $f(x)=\cos x-x$ is continuous on the interval $[0,1]$, $f(0)=1$, and $f(1)=\cos 1-1\approx-0.46$. Since $-0.46<0<1$, there is a number c in $(0,1)$ such that $f(c)=0$ by the Intermediate Value Theorem. Thus, there is a root of the equation $\cos x-x=0$, or $\cos x=x$, in the interval $(0,1)$.

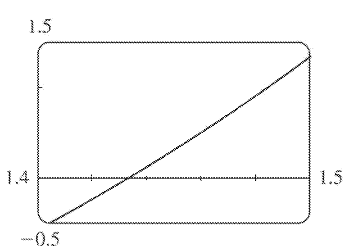
50. $f(x)=\ln x-e^{-x}$ is continuous on the interval $[1,2]$, $f(1)=-e^{-1}\approx-0.37$, and $f(2)=\ln 2-e^{-2}\approx 0.56$. Since $-0.37<0<0.56$, there is a number c in $(1,2)$ such that $f(c)=0$ by the Intermediate Value Theorem. Thus, there is a root of the equation $\ln x-e^{-x}=0$, or $\ln x=e^{-x}$, in the interval $(1,2)$.

51. **(a)** $f(x)=e^{x}+x-2$ is continuous on the interval $[0,1]$, $f(0)=-1<0$, and $f(1)=e-1\approx 1.72>0$. Since $-1<0<1.72$, there is a number c in $(0,1)$ such that $f(c)=0$ by the Intermediate Value Theorem. Thus, there is a root of the equation $e^{x}+x-2=0$, or $e^{x}=2-x$, in the interval $(0,1)$.
(b) $f(0.44)\approx-0.007<0$ and $f(0.45)\approx 0.018>0$, so there is a root between 0.44 and 0.45 .

52. **(a)** $f(x)=\sin x-2+x$ is continuous on $[0,2]$, $f(0)=-2$, and $f(2)=\sin 2\approx 0.91$. Since $-2<0<0.91$, there is a number c in $(0,2)$ such that $f(c)=0$ by the Intermediate Value Theorem. Thus, there is a root of the equation $\sin x-2+x=0$, or $\sin x=2-x$, in the interval $(0,2)$.
(b) $f(1.10)\approx-0.009<0$ and $f(1.11)\approx 0.006>0$, so there is a root between 1.10 and 1.11 .

53. **(a)** Let $f(x)=x^{5}-x^{2}-4$. Then $f(1)=1^{5}-1^{2}-4=-4<0$ and $f(2)=2^{5}-2^{2}-4=24>0$. So by the Intermediate Value Theorem, there is a number c in $(1,2)$ such that $f(c)=c^{5}-c^{2}-4=0$.
(b) We can see from the graphs that, correct to three decimal places, the root is $x\approx 1.434$.

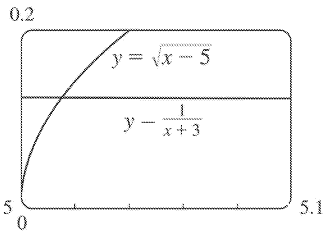




54. **(a)** Let $f(x)=\sqrt{x-5}-\dfrac{1}{x+3}$. Then $f(5)=-\dfrac{1}{8}<0$ and $f(6)=\dfrac{8}{9}>0$, and f is continuous on $[5,\infty)$. So

by the Intermediate Value Theorem, there is a number c in $(5,6)$ such that $f(c)=0$. This implies that $\frac{1}{c+3}=\sqrt{c-5}$.

(b) Using the intersect feature of the graphing device, we find that the root of the equation is $x=5.016$, correct to three decimal places.



55. ($\Rightarrow$) If f is continuous at a , then by Theorem 8 with $g(h)=a+h$, we have $\lim_{h\to 0} f(a+h)=f\left(\lim_{h\to 0}(a+h)\right)=f(a)$.

($\Leftarrow$) Let $\varepsilon>0$. Since $\lim_{h\to 0} f(a+h)=f(a)$, there exists $\delta>0$ such that $0<|h|<\delta \Rightarrow |f(a+h)-f(a)|<\varepsilon$. So if $0<|x-a|<\delta$, then $|f(x)-f(a)|=|f(a+(x-a))-f(a)|<\varepsilon$. Thus, $\lim_{x\to a} f(x)=f(a)$ and so f is continuous at a .

56.

$$\begin{aligned}\lim_{h\to 0}\sin(a+h) &= \lim_{h\to 0}(\sin a\cos h+\cos a\sin h)=\lim_{h\to 0}(\sin a\cos h)+\lim_{h\to 0}(\cos a\sin h)\\ &= \left(\lim_{h\to 0}\sin a\right)\left(\lim_{h\to 0}\cos h\right)+\left(\lim_{h\to 0}\cos a\right)\left(\lim_{h\to 0}\sin h\right)\\ &= (\sin a)(1)+(\cos a)(0)=\sin a\end{aligned}$$

57. As in the previous exercise, we must show that $\lim_{h\to 0}\cos(a+h)=\cos a$ to prove that the cosine function is continuous.

$$\begin{aligned}\lim_{h\to 0}\cos(a+h) &= \lim_{h\to 0}(\cos a\cos h-\sin a\sin h)\\ &= \lim_{h\to 0}(\cos a\cos h)-\lim_{h\to 0}(\sin a\sin h)\\ &= \left(\lim_{h\to 0}\cos a\right)\left(\lim_{h\to 0}\cos h\right)-\left(\lim_{h\to 0}\sin a\right)\left(\lim_{h\to 0}\sin h\right)\\ &= (\cos a)(1)-(\sin a)(0)=\cos a\end{aligned}$$

58. **(a)** Since f is continuous at a , $\lim_{x\to a} f(x)=f(a)$ .Thus, using the Constant Multiple Law of Limits,

we have $\lim_{x\to a}(cf)(x)=\lim_{x\to a} cf(x)=c\lim_{x\to a} f(x)=cf(a)=(cf)(a)$. Therefore, cf is continuous at a .
(b) Since f and g are continuous at a , $\lim_{x\to a} f(x)=f(a)$ and $\lim_{x\to a} g(x)=g(a)$. Since $g(a)\neq 0$, we can use the Quotient Law of Limits: $\lim_{x\to a}\left(\frac{f}{g}\right)(x)=\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}=\frac{f(a)}{g(a)}=\left(\frac{f}{g}\right)(a)$. Thus, $\frac{f}{g}$ is continuous at a .

59. $f(x)=\begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational}\end{cases}$ is continuous nowhere. For, given any number a and any $\delta>0$, the interval $(a-\delta, a+\delta)$ contains both infinitely many rational and infinitely many irrational numbers. Since $f(a)=0$ or 1 , there are infinitely many numbers x with $0<|x-a|<\delta$ and $|f(x)-f(a)|=1$. Thus, $\lim_{x\to a} f(x)\neq f(a)$. [In fact $\lim_{x\to a} f(x)$ does not even exist.]

60. $g(x)=\begin{cases} 0 & \text{if } x \text{ is rational} \\ x & \text{if } x \text{ is irrational}\end{cases}$ is continuous at 0 . To see why, note that $-|x|\leq g(x)\leq |x|$, so by the Squeeze Theorem $\lim_{x\to 0} g(x)=0=g(0)$. But g is continuous nowhere else. For if $a\neq 0$ and $\delta>0$, the interval $(a-\delta, a+\delta)$ contains both infinitely many rational and infinitely many irrational numbers. Since $g(a)=0$ or a , there are infinitely many numbers x with $0<|x-a|<\delta$ and $|g(x)-g(a)|>|a|/2$. Thus, $\lim_{x\to a} g(x)\neq g(a)$.

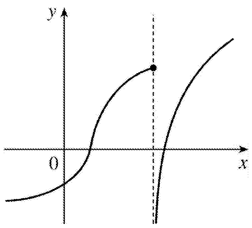
61. If there is such a number, it satisfies the equation $x^3+1=x \Leftrightarrow x^3-x+1=0$. Let the left–hand side of this equation be called $f(x)$. Now $f(-2)=-5<0$, and $f(-1)=1>0$. Note also that $f(x)$ is a polynomial, and thus continuous. So by the Intermediate Value Theorem, there is a number c between -2 and -1 such that $f(c)=0$, so that $c=c^3+1$.

62. **(a)** $\lim_{x\to 0^+} F(x)=0$ and $\lim_{x\to 0^-} F(x)=0$, so $\lim_{x\to 0} F(x)=0$, which is $F(0)$, and hence F is continuous at $x=a$ if $a=0$. For $a>0$, $\lim_{x\to a} F(x)=\lim_{x\to a} x=a=F(a)$. For $a<0$, $\lim_{x\to a} F(x)=\lim_{x\to a}(-x)=-a=F(a)$. Thus, F is continuous at $x=a$; that is, continuous everywhere.
(b) Assume that f is continuous on the interval I . Then for $a\in I$, $\lim_{x\to a}|f(x)|=\left|\lim_{x\to a} f(x)\right|=|f(a)|$ by Theorem 8. (If a is an endpoint of I , use the appropriate one–sided limit.) So $|f|$ is continuous on I .
(c) No, the converse is false. For example, the function $f(x)=\begin{cases} 1 & \text{if } x\geq 0 \\ -1 & \text{if } x<0\end{cases}$ is not continuous at $x=0$, but $|f(x)|=1$ is continuous on R .

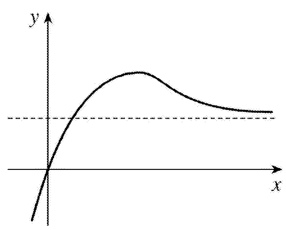
63. Define $u(t)$ to be the monk's distance from the monastery, as a function of time, on the first day, and define $d(t)$ to be his distance from the monastery, as a function of time, on the second day. Let D be the distance from the monastery to the top of the mountain. From the given information we know that $u(0)=0$, $u(12)=D$, $d(0)=D$ and $d(12)=0$. Now consider the function $u-d$, which is clearly continuous. We calculate that $(u-d)(0)=-D$ and $(u-d)(12)=D$. So by the Intermediate Value Theorem, there must be some time t_0 between 0 and 12 such that $(u-d)(t_0)=0 \Leftrightarrow u(t_0)=d(t_0)$. So at time t_0 after 7:00 A.M., the monk will be at the same place on both days.

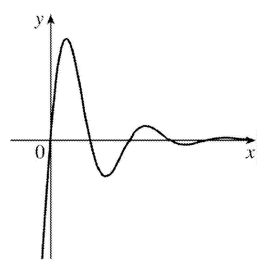
1. **(a)** As x becomes large, the values of $f(x)$ approach 5 .
(b) As x becomes large negative, the values of $f(x)$ approach 3 .

2. **(a)** The graph of a function can intersect a vertical asymptote in the sense that it can meet but not cross it.

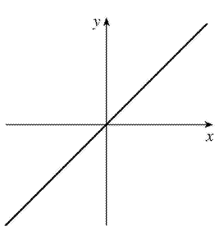


The graph of a function can intersect a horizontal asymptote. It can even intersect its horizontal asymptote an infinite number of times.

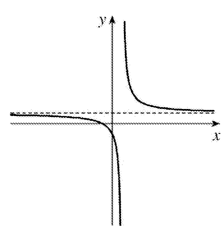




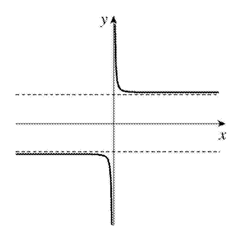
(b) The graph of a function can have 0 , 1 , or 2 horizontal asymptotes. Representative examples are shown.



No horizontal asymptote



One horizontal asymptote



Two horizontal asymptotes

3. **(a)** $\lim_{x \to 2} f(x) = \infty$

(b) $\lim_{x \to -1^-} f(x) = \infty$

(c) $\lim_{x \to -1^+} f(x) = -\infty$

(d) $\lim_{x \to \infty} f(x) = 1$

(e) $\lim_{x \to -\infty} f(x) = 2$

(f) Vertical: $x = -1$, $x = 2$; Horizontal: $y = 1$, $y = 2$

4. **(a)** $\lim_{x \to \infty} g(x) = 2$

(b) $\lim_{x \to -\infty} g(x) = -2$

(c) $\lim_{x \to 3} g(x) = \infty$

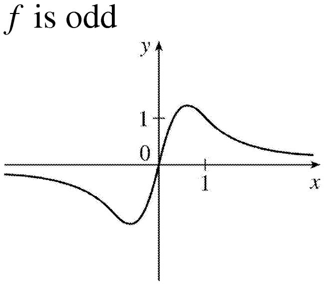
(d) $\lim_{x \to 0} g(x) = -\infty$

(e) $\lim_{x \to -2^+} g(x) = -\infty$

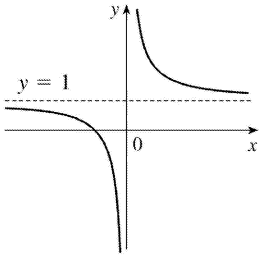
(f) Vertical: $x = -2$, $x = 0$, $x = 3$; Horizontal: $y = -2$, $y = 2$

5. $f(0) = 0$, $f(1) = 1$, $\lim_{x \to \infty} f(x) = 0$,

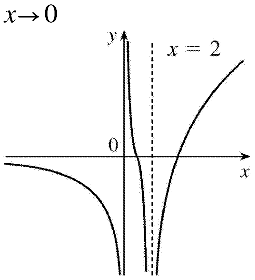
f is odd



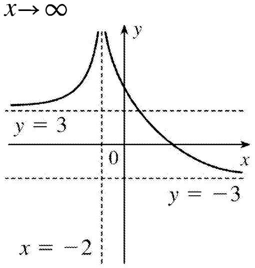
6. $\lim_{x\to 0^+} f(x)=\infty$, $\lim_{x\to 0^-} f(x)=-\infty$,
$\lim_{x\to \infty} f(x)=1$, $\lim_{x\to -\infty} f(x)=1$



7. $\lim_{x\to 2} f(x)=-\infty$, $\lim_{x\to \infty} f(x)=\infty$,
$\lim_{x\to -\infty} f(x)=0$, $\lim_{x\to 0^+} f(x)=\infty$,
$\lim_{x\to 0^-} f(x)=-\infty$



8. $\lim_{x\to -2} f(x)=\infty$, $\lim_{x\to -\infty} f(x)=3$,
$\lim_{x\to \infty} f(x)=-3$



9. If $f(x)=x^2/2^x$, then a calculator gives $f(0)=0$, $f(1)=0.5$, $f(2)=1$, $f(3)=1.125$, $f(4)=1$, $f(5)=0.78125$, $f(6)=0.5625$, $f(7)=0.3828125$, $f(8)=0.25$, $f(9)=0.158203125$, $f(10)=0.09765625$, $f(20)\approx 0.00038147$, $f(50)\approx 2.2204\times 10^{-12}$, $f(100)\approx 7.8886\times 10^{-27}$.
It appears that $\lim_{x\to \infty} \left(x^2/2^x\right)=0$.

10. **(a)** From a graph of $f(x)=(1-2/x)^x$ in a window of $[0,10{,}000]$ by $[0,0.2]$, we estimate that $\lim_{x\to\infty} f(x)=0.14$ (to two decimal places.)

(b)

x	$f(x)$
10,000	0.135308
100,000	0.135333
1,000,000	0.135335

From the table, we estimate that $\lim_{x\to\infty} f(x)=0.1353$ (to four decimal places.)

11.

$$\lim_{x\to\infty}\frac{3x^2-x+4}{2x^2+5x-8}=\lim_{x\to\infty}\frac{(3x^2-x+4)/x^2}{(2x^2+5x-8)/x^2}$$ [divide both the numerator and denominator by x^2 (the highest power of x that appears in the denominator)]

$$=\frac{\lim_{x\to\infty}(3-1/x+4/x^2)}{\lim_{x\to\infty}(2+5/x-8/x^2)}$$ [Limit Law 5]

$$=\frac{\lim_{x\to\infty}3-\lim_{x\to\infty}(1/x)+\lim_{x\to\infty}(4/x^2)}{\lim_{x\to\infty}2+\lim_{x\to\infty}(5/x)-\lim_{x\to\infty}(8/x^2)}$$ [Limit Laws 1 and 2]

$$=\frac{3-\lim_{x\to\infty}(1/x)+4\lim_{x\to\infty}(1/x^2)}{2+5\lim_{x\to\infty}(1/x)-8\lim_{x\to\infty}(1/x^2)}$$ [Limit Laws 7 and 3]

$$=\frac{3-0+4(0)}{2+5(0)-8(0)}$$ [Theorem 5 of Section 2.5]

$$=\frac{3}{2}$$

12.

$$\lim_{x\to\infty}\sqrt{\frac{12x^3-5x+2}{1+4x^2+3x^3}}=\sqrt{\lim_{x\to\infty}\frac{12x^3-5x+2}{1+4x^2+3x^3}}$$ [Limit Law 11]

$$=\sqrt{\lim_{x\to\infty}\frac{12-5/x^2+2/x^3}{1/x^3+4/x+3}}$$ [divide by x^3]

$$=\sqrt{\frac{\lim_{x\to\infty}(12-5/x^2+2/x^3)}{\lim_{x\to\infty}(1/x^3+4/x+3)}}$$ [Limit Law 5]

$$=\sqrt{\frac{\lim_{x\to\infty}12-\lim_{x\to\infty}(5/x^2)+\lim_{x\to\infty}(2/x^3)}{\lim_{x\to\infty}(1/x^3)+\lim_{x\to\infty}(4/x)+\lim_{x\to\infty}3}}$$ [Limit Laws 1 and 2]

$$=\sqrt{\frac{12-5\lim_{x\to\infty}(1/x^2)+2\lim_{x\to\infty}(1/x^3)}{\lim_{x\to\infty}(1/x^3)+4\lim_{x\to\infty}(1/x)+3}}$$ [Limit Laws 7 and 3]

$$=\sqrt{\frac{12-5(0)+2(0)}{0+4(0)+3}}$$ [Theorem 5 of Section 2.5]

$$=\sqrt{\frac{12}{3}}=\sqrt{4}=2$$

13. $$\lim_{x\to\infty}\frac{1}{2x+3}=\lim_{x\to\infty}\frac{1/x}{(2x+3)/x}=\frac{\lim_{x\to\infty}(1/x)}{\lim_{x\to\infty}(2+3/x)}=\frac{\lim_{x\to\infty}(1/x)}{\lim_{x\to\infty}2+3\lim_{x\to\infty}(1/x)}=\frac{0}{2+3(0)}=\frac{0}{2}=0$$

14. $$\lim_{x\to\infty}\frac{3x+5}{x-4}=\lim_{x\to\infty}\frac{(3x+5)/x}{(x-4)/x}=\lim_{x\to\infty}\frac{3+5/x}{1-4/x}=\frac{\lim_{x\to\infty}3+5\lim_{x\to\infty}\frac{1}{x}}{\lim_{x\to\infty}1-4\lim_{x\to\infty}\frac{1}{x}}=\frac{3+5(0)}{1-4(0)}=3$$

15.

$$\lim_{x\to-\infty}\frac{1-x-x^2}{2x^2-7}=\lim_{x\to-\infty}\frac{(1-x-x^2)/x^2}{(2x^2-7)/x^2}=\frac{\lim\limits_{x\to-\infty}(1/x^2-1/x-1)}{\lim\limits_{x\to-\infty}(2-7/x^2)}$$

$$=\frac{\lim\limits_{x\to-\infty}(1/x^2)-\lim\limits_{x\to-\infty}(1/x)-\lim\limits_{x\to-\infty}1}{\lim\limits_{x\to-\infty}2-7\lim\limits_{x\to-\infty}(1/x^2)}=\frac{0-0-1}{2-7(0)}=-\frac{1}{2}$$

16. $$\lim_{y\to\infty}\frac{2-3y^2}{5y^2+4y}=\lim_{y\to\infty}\frac{(2-3y^2)/y^2}{(5y^2+4y)/y^2}=\frac{\lim\limits_{y\to\infty}(2/y^2-3)}{\lim\limits_{y\to\infty}(5+4/y)}=\frac{2\lim\limits_{y\to\infty}(1/y^2)-\lim\limits_{y\to\infty}3}{\lim\limits_{y\to\infty}5+4\lim\limits_{y\to\infty}(1/y)}=\frac{2(0)-3}{5+4(0)}=-\frac{3}{5}$$

17. Divide both the numerator and denominator by x^3 (the highest power of x that occurs in the denominator).

$$\lim_{x\to\infty}\frac{x^3+5x}{2x^3-x^2+4}=\lim_{x\to\infty}\frac{\dfrac{x^3+5x}{x^3}}{\dfrac{2x^3-x^2+4}{x^3}}=\lim_{x\to\infty}\frac{1+\dfrac{5}{x^2}}{2-\dfrac{1}{x}+\dfrac{4}{x^3}}=\frac{\lim\limits_{x\to\infty}\left(1+\dfrac{5}{x^2}\right)}{\lim\limits_{x\to\infty}\left(2-\dfrac{1}{x}+\dfrac{4}{x^3}\right)}$$

$$=\frac{\lim\limits_{x\to\infty}1+5\lim\limits_{x\to\infty}\dfrac{1}{x^2}}{\lim\limits_{x\to\infty}2-\lim\limits_{x\to\infty}\dfrac{1}{x}+4\lim\limits_{x\to\infty}\dfrac{1}{x^3}}=\frac{1+5(0)}{2-0+4(0)}=\frac{1}{2}$$

18. $$\lim_{t\to-\infty}\frac{t^2+2}{t^3+t^2-1}=\lim_{t\to-\infty}\frac{\left(t^2+2\right)/t^3}{\left(t^3+t^2-1\right)/t^3}=\lim_{t\to-\infty}\frac{1/t+2/t^3}{1+1/t-1/t^3}=\frac{0+0}{1+0-0}=0$$

19. First, multiply the factors in the denominator. Then divide both the numerator and denominator by u^4.

$$\lim_{u\to\infty} \frac{4u^4+5}{(u^2-2)(2u^2-1)} = \lim_{u\to\infty} \frac{4u^4+5}{2u^4-5u^2+2} = \lim_{u\to\infty} \frac{\dfrac{4u^4+5}{u^4}}{\dfrac{2u^4-5u^2+2}{u^4}} = \lim_{u\to\infty} \frac{4+\dfrac{5}{u^4}}{2-\dfrac{5}{u^2}+\dfrac{2}{u^4}}$$

$$= \frac{\lim\limits_{u\to\infty}\left(4+\dfrac{5}{u^4}\right)}{\lim\limits_{u\to\infty}\left(2-\dfrac{5}{u^2}+\dfrac{2}{u^4}\right)} = \frac{\lim\limits_{u\to\infty} 4+5\lim\limits_{u\to\infty}\dfrac{1}{u^4}}{\lim\limits_{u\to\infty} 2-5\lim\limits_{u\to\infty}\dfrac{1}{u^2}+2\lim\limits_{u\to\infty}\dfrac{1}{u^4}} = \frac{4+5(0)}{2-5(0)+2(0)}$$

$$= \frac{4}{2} = 2$$

20. $$\lim_{x\to\infty} \frac{x+2}{\sqrt{9x^2+1}} = \lim_{x\to\infty} \frac{(x+2)/x}{\sqrt{9x^2+1}/\sqrt{x^2}} = \lim_{x\to\infty} \frac{1+2/x}{\sqrt{9+1/x^2}} = \frac{1+0}{\sqrt{9+0}} = \frac{1}{3}$$

21.

$$\lim_{x\to\infty} \frac{\sqrt{9x^6-x}}{x^3+1} = \lim_{x\to\infty} \frac{\sqrt{9x^6-x}/x^3}{(x^3+1)/x^3} = \frac{\lim\limits_{x\to\infty}\sqrt{(9x^6-x)/x^6}}{\lim\limits_{x\to\infty}(1+1/x^3)} \qquad [\text{ since } x^3=\sqrt{x^6} \text{ for } x>0 \text{ }]$$

$$= \frac{\lim\limits_{x\to\infty}\sqrt{9-1/x^5}}{\lim\limits_{x\to\infty} 1+\lim\limits_{x\to\infty}(1/x^3)} = \frac{\sqrt{\lim\limits_{x\to\infty} 9-\lim\limits_{x\to\infty}(1/x^5)}}{1+0}$$

$$= \sqrt{9-0} = 3$$

22.

$$\lim_{x\to-\infty} \frac{\sqrt{9x^6-x}}{x^3+1} = \lim_{x\to-\infty} \frac{\sqrt{9x^6-x}/x^3}{(x^3+1)/x^3} = \frac{\lim\limits_{x\to-\infty} -\sqrt{(9x^6-x)/x^6}}{\lim\limits_{x\to-\infty}(1+1/x^3)} \qquad [\text{ since } x^3=-\sqrt{x^6} \text{ for } x<0 \text{ }]$$

$$=\frac{\lim\limits_{x\to-\infty} -\sqrt{9-1/x^5}}{\lim\limits_{x\to-\infty} 1+\lim\limits_{x\to-\infty} (1/x^3)}=\frac{-\sqrt{\lim\limits_{x\to-\infty} 9-\lim\limits_{x\to-\infty} (1/x^5)}}{1+0}$$

$$=-\sqrt{9-0}=-3$$

23.

$$\lim_{x\to\infty}\left(\sqrt{9x^2+x}-3x\right)=\lim_{x\to\infty}\frac{\left(\sqrt{9x^2+x}-3x\right)\left(\sqrt{9x^2+x}+3x\right)}{\sqrt{9x^2+x}+3x}=\lim_{x\to\infty}\frac{\left(\sqrt{9x^2+x}\right)^2-(3x)^2}{\sqrt{9x^2+x}+3x}$$

$$=\lim_{x\to\infty}\frac{\left(9x^2+x\right)-9x^2}{\sqrt{9x^2+x}+3x}=\lim_{x\to\infty}\frac{x}{\sqrt{9x^2+x}+3x}\cdot\frac{1/x}{1/x}$$

$$=\lim_{x\to\infty}\frac{x/x}{\sqrt{9x^2/x^2+x/x^2}+3x/x}=\lim_{x\to\infty}\frac{1}{\sqrt{9+1/x}+3}=\frac{1}{\sqrt{9}+3}=\frac{1}{3+3}=\frac{1}{6}$$

24.

$$\lim_{x\to-\infty}\left(x+\sqrt{x^2+2x}\right)=\lim_{x\to-\infty}\left(x+\sqrt{x^2+2x}\right)\left[\frac{x-\sqrt{x^2+2x}}{x-\sqrt{x^2+2x}}\right]=\lim_{x\to-\infty}\frac{x^2-\left(x^2+2x\right)}{x-\sqrt{x^2+2x}}$$

$$=\lim_{x\to-\infty}\frac{-2x}{x-\sqrt{x^2+2x}}=\lim_{x\to-\infty}\frac{-2}{1+\sqrt{1+2/x}}=\frac{-2}{1+\sqrt{1+2(0)}}=-1$$

Note: In dividing numerator and denominator by x , we used the fact that for $x<0$, $x=-\sqrt{x^2}$.

25.

$$\lim_{x\to\infty}\left(\sqrt{x^2+ax}-\sqrt{x^2+bx}\right)=\lim_{x\to\infty}\frac{\left(\sqrt{x^2+ax}-\sqrt{x^2+bx}\right)\left(\sqrt{x^2+ax}+\sqrt{x^2+bx}\right)}{\sqrt{x^2+ax}+\sqrt{x^2+bx}}$$

$$=\lim_{x\to\infty}\frac{(x^2+ax)-(x^2+bx)}{\sqrt{x^2+ax}+\sqrt{x^2+bx}}=\lim_{x\to\infty}\frac{[(a-b)x]/x}{\left(\sqrt{x^2+ax}+\sqrt{x^2+bx}\right)/\sqrt{x^2}}$$

$$=\lim_{x\to\infty}\frac{a-b}{\sqrt{1+a/x}+\sqrt{1+b/x}}=\frac{a-b}{\sqrt{1+0}+\sqrt{1+0}}=\frac{a-b}{2}$$

26. $\lim_{x\to\infty}\cos x$ does not exist because as x increases $\cos x$ does not approach any one value, but oscillates between 1 and -1 .

27. $\sqrt{x}$ is large when x is large, so $\lim_{x\to\infty}\sqrt{x}=\infty$.

28. $\sqrt[3]{x}$ is large negative when x is large negative, so $\lim_{x\to-\infty}\sqrt[3]{x}=-\infty$.

29. $\lim_{x\to\infty}\left(x-\sqrt{x}\right)=\lim_{x\to\infty}\sqrt{x}\left(\sqrt{x}-1\right)=\infty$ since $\sqrt{x}\to\infty$ and $\sqrt{x}-1\to\infty$ as $x\to\infty$.

30. $\lim_{x\to\infty}\frac{x^3-2x+3}{5-2x^2}=\lim_{x\to\infty}\frac{(x^3-2x+3)/x^2}{(5-2x^2)/x^2}$ [divide by the highest power of x in the denominator]

$=\lim_{x\to\infty}\frac{x-2/x+3/x^2}{5/x^2-2}=-\infty$ because $x-2/x+3/x^2\to\infty$ and $5/x^2-2\to-2$ as $x\to\infty$.

31. $\lim_{x\to-\infty}(x^4+x^5)=\lim_{x\to-\infty}x^5(\frac{1}{x}+1)=-\infty$ because $x^5\to-\infty$ and $1/x+1\to1$ as $x\to-\infty$.

32. $\lim_{x\to\infty}\tan^{-1}\left(x^2-x^4\right)=\lim_{x\to\infty}\tan^{-1}\left(x^2\left(1-x^2\right)\right)$. If we let $t=x^2\left(1-x^2\right)$, we know that $t\to-\infty$ as $x\to\infty$, since $x^2\to\infty$ and $1-x^2\to-\infty$. So $\lim_{x\to\infty}\tan^{-1}\left(x^2\left(1-x^2\right)\right)=\lim_{t\to-\infty}\tan^{-1}t=-\frac{\pi}{2}$.

33.

$\lim_{x\to\infty}\frac{x+x^3+x^5}{1-x^2+x^4}=\lim_{x\to\infty}\frac{(x+x^3+x^5)/x^4}{(1-x^2+x^4)/x^4}$ [divide by the highest power of x in the denominator]

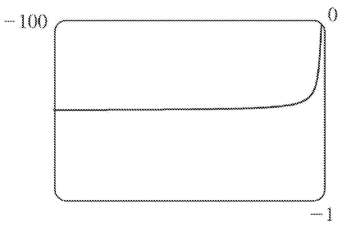
$$=\lim_{x\to\infty}\frac{1/x^3+1/x+x}{1/x^4-1/x^2+1}=\infty$$

because $(1/x^3+1/x+x)\to\infty$ and $(1/x^4-1/x^2+1)\to1$ as $x\to\infty$.

34. If we let $t=\tan x$, then as $x\to(\pi/2)^+$, $t\to-\infty$. Thus,

$$\lim_{x \to (\pi/2)^+} e^{\tan x} = \lim_{t \to -\infty} e^t = 0 .$$

35. **(a)**



From the graph of $f(x) = \sqrt{x^2+x+1} + x$, we estimate the value of $\lim_{x \to -\infty} f(x)$ to be -0.5 .

(b)

x	$f(x)$
−10,000	−0.4999625
−100,000	−0.4999962
−1,000,000	−0.4999996

From the table, we estimate the limit to be -0.5 .

(c)

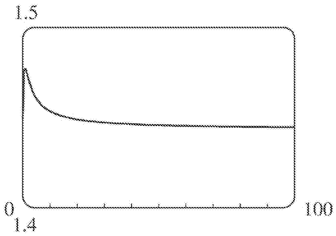
$$\lim_{x \to -\infty} \left(\sqrt{x^2+x+1} + x \right) = \lim_{x \to -\infty} \left(\sqrt{x^2+x+1} + x \right) \left[\frac{\sqrt{x^2+x+1} - x}{\sqrt{x^2+x+1} - x} \right] = \lim_{x \to -\infty} \frac{\left(x^2+x+1 \right) - x^2}{\sqrt{x^2+x+1} - x}$$

$$= \lim_{x \to -\infty} \frac{(x+1)(1/x)}{\left(\sqrt{x^2+x+1} - x \right)(1/x)} = \lim_{x \to -\infty} \frac{1+(1/x)}{-\sqrt{1+(1/x)+\left(1/x^2\right)} - 1}$$

$$= \frac{1+0}{-\sqrt{1+0+0} - 1} = -\frac{1}{2}$$

Note that for $x<0$, we have $\sqrt{x^2} = |x| = -x$, so when we divide the radical by x , with $x<0$, we get $\frac{1}{x}\sqrt{x^2+x+1} = -\frac{1}{\sqrt{x^2}}\sqrt{x^2+x+1} = -\sqrt{1+(1/x)+\left(1/x^2\right)}$.

36. **(a)**



From the graph of $f(x)=\sqrt{3x^2+8x+6}-\sqrt{3x^2+3x+1}$, we estimate (to one decimal place) the value of $\lim_{x\to\infty} f(x)$ to be 1.4 .

(b)

x	$f(x)$
10,000	1.44339
100,000	1.44338
1,000,000	1.44338

From the table, we estimate (to four decimal places) the limit to be 1.4434 .

(c)

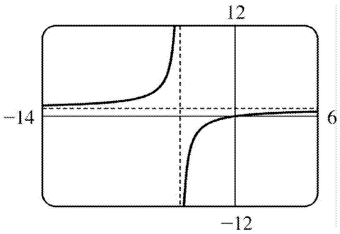
$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{\left(\sqrt{3x^2+8x+6}-\sqrt{3x^2+3x+1}\right)\left(\sqrt{3x^2+8x+6}+\sqrt{3x^2+3x+1}\right)}{\sqrt{3x^2+8x+6}+\sqrt{3x^2+3x+1}}$$

$$=\lim_{x\to\infty} \frac{\left(3x^2+8x+6\right)-\left(3x^2+3x+1\right)}{\sqrt{3x^2+8x+6}+\sqrt{3x^2+3x+1}}$$

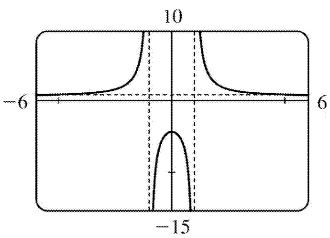
$$=\lim_{x\to\infty} \frac{(5x+5)(1/x)}{\left(\sqrt{3x^2+8x+6}+\sqrt{3x^2+3x+1}\right)(1/x)}$$

$$=\lim_{x\to\infty} \frac{5+5/x}{\sqrt{3+8/x+6/x^2}+\sqrt{3+3/x+1/x^2}} = \frac{5}{\sqrt{3}+\sqrt{3}} = \frac{5}{2\sqrt{3}} = \frac{5\sqrt{3}}{6} \approx 1.443376$$

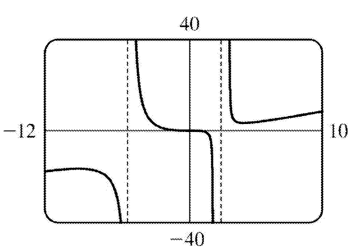
37. $\lim_{x\to\pm\infty} \frac{x}{x+4} = \lim_{x\to\pm\infty} \frac{1}{1+4/x} = \frac{1}{1+0} = 1$, so $y=1$ is a horizontal asymptote. $\lim_{x\to -4^-} \frac{x}{x+4} = \infty$ and $\lim_{x\to -4^+} \frac{x}{x+4} = -\infty$, so $x=-4$ is a vertical asymptote. The graph confirms these calculations.



38. Since $x^2-1\to 0$ as $x\to \pm 1$ and $y<0$ for $-1<x<1$ and $y>0$ for $x<-1$ and $x>1$, we have $\lim\limits_{x\to 1^-} \frac{x^2+4}{x^2-1} = -\infty$, $\lim\limits_{x\to 1^+} \frac{x^2+4}{x^2-1} = \infty$, $\lim\limits_{x\to -1^-} \frac{x^2+4}{x^2-1} = \infty$, and $\lim\limits_{x\to -1^+} \frac{x^2+4}{x^2-1} = -\infty$, so $x=1$ and $x=-1$ are vertical asymptotes. Also $\lim\limits_{x\to \pm\infty} \frac{x^2+4}{x^2-1} = \lim\limits_{x\to \pm\infty} \frac{1+4/x^2}{1-1/x^2} = \frac{1+0}{1-0} = 1$, so $y=1$ is a horizontal asymptote. The graph confirms these calculations.

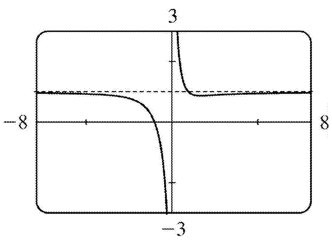


39. $\lim\limits_{x\to \pm\infty} \frac{x^3}{x^2+3x-10} = \lim\limits_{x\to \pm\infty} \frac{x}{1+(3/x)-\left(10/x^2\right)} = \pm\infty$, so there is no horizontal asymptote. $\lim\limits_{x\to 2^+} \frac{x^3}{x^2+3x-10} = \lim\limits_{x\to 2^+} \frac{x^3}{(x+5)(x-2)} = \infty$, since $\frac{x^3}{(x+5)(x-2)} > 0$ for $x>2$. Similarly, $\lim\limits_{x\to 2^-} \frac{x^3}{x^2+3x-10} = -\infty$, $\lim\limits_{x\to -5^-} \frac{x^3}{x^2+3x-10} = -\infty$, and $\lim\limits_{x\to -5^+} \frac{x^3}{x^2+3x-10} = \infty$, so $x=2$ and $x=-5$ are vertical asymptotes. The graph confirms these calculations.



40.

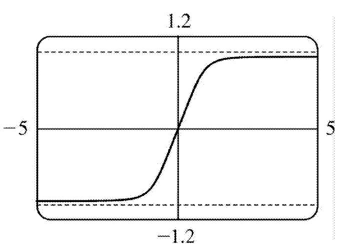
$\lim\limits_{x\to\pm\infty} \dfrac{x^3+1}{x^3+x} = \lim\limits_{x\to\pm\infty} \dfrac{1+1/x^3}{1+1/x^2} = 1$, so $y=1$ is a horizontal asymptote. Since $y=\dfrac{x^3+1}{x^3+x} = \dfrac{x^3+1}{x\left(x^2+1\right)} > 0$ for $x>0$ and $y<0$ for $-1<x<0$, $\lim\limits_{x\to 0^+} \dfrac{x^3+1}{x^3+x} = \infty$ and $\lim\limits_{x\to 0^-} \dfrac{x^3+1}{x^3+x} = -\infty$, so $x=0$ is a vertical asymptote.



41. $\lim\limits_{x\to\infty} \dfrac{x}{\sqrt[4]{x^4+1}} \cdot \dfrac{1/x}{1/\sqrt[4]{x^4}} = \lim\limits_{x\to\infty} \dfrac{1}{\sqrt[4]{1+\dfrac{1}{x^4}}} = \dfrac{1}{\sqrt[4]{1+0}} = 1$ and

$\lim\limits_{x\to-\infty} \dfrac{x}{\sqrt[4]{x^4+1}} \cdot \dfrac{1/x}{-1/\sqrt[4]{x^4}} = \lim\limits_{x\to-\infty} \dfrac{1}{-\sqrt[4]{1+\dfrac{1}{x^4}}} = \dfrac{1}{-\sqrt[4]{1+0}} = -1$, so $y=\pm 1$ are horizontal asymptotes.

There is no vertical asymptote.

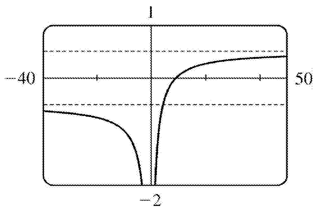


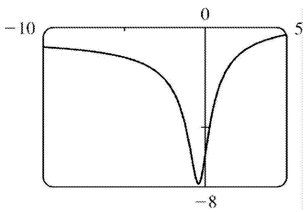
42. $\lim\limits_{x\to\infty} \dfrac{x-9}{\sqrt{4x^2+3x+2}} = \lim\limits_{x\to\infty} \dfrac{1-9/x}{\sqrt{4+(3/x)+\left(2/x^2\right)}} = \dfrac{1-0}{\sqrt{4+0+0}} = \dfrac{1}{2}$.

Using the fact that $\sqrt{x^2} = |x| = -x$ for $x<0$, we divide the numerator by $-x$ and the denominator by $\sqrt{x^2}$.

Thus, $\lim\limits_{x\to-\infty} \dfrac{x-9}{\sqrt{4x^2+3x+2}} = \lim\limits_{x\to-\infty} \dfrac{-1+9/x}{\sqrt{4+(3/x)+\left(2/x^2\right)}} = \dfrac{-1+0}{\sqrt{4+0+0}} = -\dfrac{1}{2}$.

The horizontal asymptotes are $y=\pm\dfrac{1}{2}$. The polynomial $4x^2+3x+2$ is positive for all x , so the denominator never approaches zero, and thus there is no vertical asymptote.





43. Let's look for a rational function.

(1) $\lim_{x \to \pm\infty} f(x)=0 \Rightarrow$ degree of numerator < degree of denominator

(2) $\lim_{x \to 0} f(x)=-\infty \Rightarrow$ there is a factor of x^2 in the denominator (not just x , since that would produce a sign change at $x=0$), and the function is negative near $x=0$.

(3) $\lim_{x \to 3^-} f(x)=\infty$ and $\lim_{x \to 3^+} f(x)=-\infty \Rightarrow$ vertical asymptote at $x=3$; there is a factor of $(x-3)$ in the denominator.

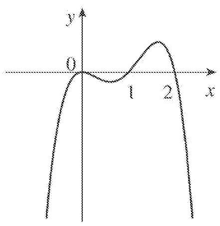
(4) $f(2)=0 \Rightarrow$ 2 is an x –intercept; there is at least one factor of $(x-2)$ in the numerator.

Combining all of this information and putting in a negative sign to give us the desired left– and right–hand limits gives us $f(x)=\dfrac{2-x}{x^2(x-3)}$ as one possibility.

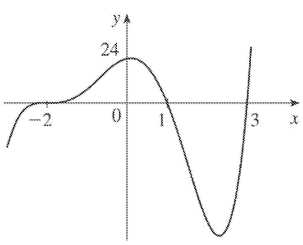
44. Since the function has vertical asymptotes $x=1$ and $x=3$, the denominator of the rational function we are looking for must have factors $(x-1)$ and $(x-3)$. Because the horizontal asymptote is $y=1$, the degree of the numerator must equal the degree of the denominator, and the ratio of the leading coefficients must be 1 . One possibility is $f(x)=\dfrac{x^2}{(x-1)(x-3)}$.

45. $y=f(x)=x^2(x-2)(1-x)$. The y– intercept is $f(0)=0$, and the x– intercepts occur when $y=0 \Rightarrow x=0$, 1 , and 2 . Notice that, since x^2 is always positive, the graph does not cross the x– axis at 0 , but does cross the x– axis at 1 and 2 . $\lim_{x \to \infty} x^2(x-2)(1-x)=-\infty$, since the first two factors are large positive and the third large negative when x is large positive.

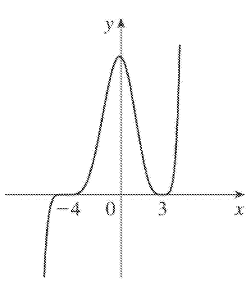
$\lim\limits_{x\to-\infty} x^2(x-2)(1-x)=-\infty$ because the first and third factors are large positive and the second large negative as $x\to-\infty$.



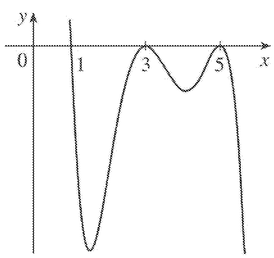
46. $y=(2+x)^3(1-x)(3-x)$. As $x\to\infty$, the first factor is large positive, and the second and third factors are large negative. Therefore, $\lim\limits_{x\to\infty} f(x)=\infty$. As $x\to-\infty$, the first factor is large negative, and the second and third factors are large positive. Therefore, $\lim\limits_{x\to-\infty} f(x)=-\infty$. Now the $y-$ intercept is $f(0)=(2)^3(1)(3)=24$ and the $x-$ intercepts are the solutions to $f(x)=0\Rightarrow x=-2$, 1 and 3 , and the graph crosses the $x-$ axis at all of these points.



47. $y=f(x)=(x+4)^5(x-3)^4$. The $y-$ intercept is $f(0)=4^5(-3)^4=82 , 944$. The $x-$ intercepts occur when $y=0\Rightarrow x=-4$, 3 . Notice that the graph does not cross the $x-$ axis at 3 because $(x-3)^4$ is always positive, but does cross the $x-$ axis at -4 . $\lim\limits_{x\to\infty} (x+4)^5(x-3)^4=\infty$ since both factors are large positive when x is large positive. $\lim\limits_{x\to-\infty} (x+4)^5(x-3)^4=-\infty$ since the first factor is large negative and the second factor is large positive when x is large negative.

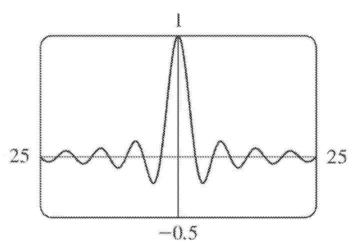


48. $y=(1-x)(x-3)^2(x-5)^2$. As $x\to\infty$, the first factor approaches $-\infty$ while the second and third factors approach ∞ . Therefore, $\lim_{x\to\infty}(x)=-\infty$. As $x\to-\infty$, the factors all approach ∞ . Therefore, $\lim_{x\to-\infty}(x)=\infty$. Now the $y-$ intercept is $f(0)=(1)(-3)^2(-5)^2=225$ and the $x-$ intercepts are the solutions to $f(x)=0\Rightarrow x=1$, 3 , and 5 . Notice that $f(x)$ does not change sign at $x=3$ or $x=5$ because the factors $(x-3)^2$ and $(x-5)^2$ are always positive, so the graph does not cross the $x-$ axis at $x=3$ or $x=5$, but does cross the $x-$ axis at $x=1$.

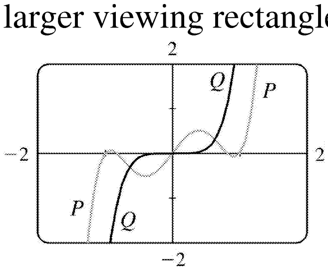


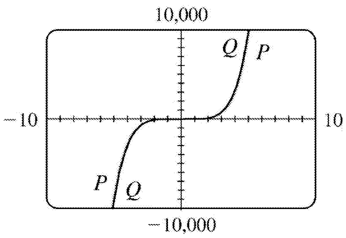
49. **(a)** Since $-1\le\sin x\le 1$ for all x,$-\dfrac{1}{x}\le\dfrac{\sin x}{x}\le\dfrac{1}{x}$ for $x>0$. As $x\to\infty$, $-1/x\to 0$ and $1/x\to 0$, so by the Squeeze Theorem, $(\sin x)/x\to 0$. Thus, $\lim_{x\to\infty}\dfrac{\sin x}{x}=0$.

(b) From part (a), the horizontal asymptote is $y=0$. The function $y=(\sin x)/x$ crosses the horizontal asymptote whenever $\sin x=0$; that is, at $x=\pi n$ for every integer n . Thus, the graph crosses the asymptote *an infinite number of times.*



50. **(a)** In both viewing rectangles, $\lim_{x\to\infty}P(x)=\lim_{x\to\infty}Q(x)=\infty$ and $\lim_{x\to-\infty}P(x)=\lim_{x\to-\infty}Q(x)=-\infty$. In the larger viewing rectangle, P and Q become less distinguishable.

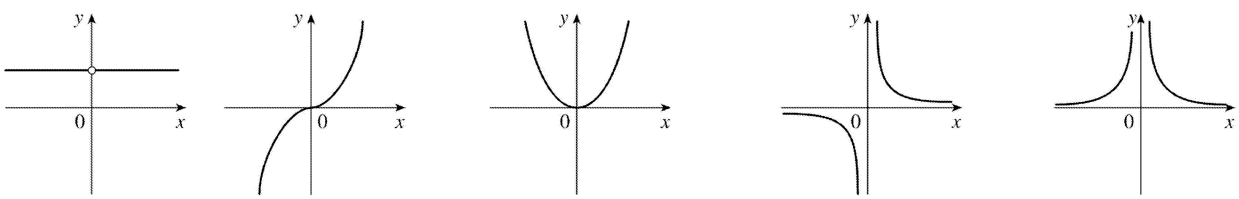




(b) $\lim_{x\to\infty}\dfrac{P(x)}{Q(x)}=\lim_{x\to\infty}\dfrac{3x^5-5x^3+2x}{3x^5}=\lim_{x\to\infty}\left(1-\dfrac{5}{3}\cdot\dfrac{1}{x^2}+\dfrac{2}{3}\cdot\dfrac{1}{x^4}\right)=1-\dfrac{5}{3}(0)+\dfrac{2}{3}(0)=1\Rightarrow$
P and Q have the same end behavior.

51. Divide the numerator and the denominator by the highest power of x in $Q(x)$.
(a) If deg $P<$deg Q, then the numerator $\to 0$ but the denominator doesn't. So $\lim_{x\to\infty}[P(x)/Q(x)]=0$.
(b) If deg $P>$deg Q, then the numerator $\to\pm\infty$ but the denominator doesn't, so $\lim_{x\to\infty}[P(x)/Q(x)]=\pm\infty$ (depending on the ratio of the leading coefficients of P and Q).

52.



(i) $n=0$ (ii) $n>0$ (n odd) (iii) $n>0$ (n even) (iv) $n<0$ (n odd) (v) $n<0$ (n even)

(a) $\lim_{x\to 0^+} x^n=\begin{cases}1 & \text{if } n=0\\ 0 & \text{if } n>0\\ \infty & \text{if } n<0\end{cases}$

(b) $\lim_{x\to 0^-} x^n=\begin{cases}1 & \text{if } n=0\\ 0 & \text{if } n>0\\ -\infty & \text{if } n<0,\quad n \text{ odd}\\ \infty & \text{if } n<0,\quad n \text{ even}\end{cases}$

(c) $\lim_{x\to\infty} x^n=\begin{cases}1 & \text{if } n=0\\ \infty & \text{if } n>0\\ 0 & \text{if } n<0\end{cases}$

(d) $\lim_{x\to-\infty} x^n=\begin{cases}1 & \text{if } n=0\\ -\infty & \text{if } n>0,\quad n \text{ odd}\\ \infty & \text{if } n>0,\quad n \text{ even}\\ 0 & \text{if } n<0\end{cases}$

53.

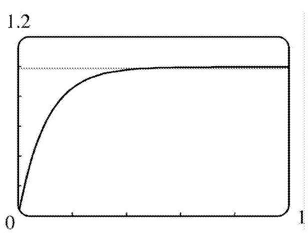
$\lim_{x\to\infty} \frac{4x-1}{x} = \lim_{x\to\infty} \left(4-\frac{1}{x}\right) = 4$, and $\lim_{x\to\infty} \frac{4x^2+3x}{x^2} = \lim_{x\to\infty} \left(4+\frac{3}{x}\right) = 4$. Therefore, by the Squeeze Theorem, $\lim_{x\to\infty} f(x) = 4$.

54. **(a)** After t minutes, $25t$ liters of brine with 30 g of salt per liter has been pumped into the tank, so it contains $(5000+25t)$ liters of water and $25t\cdot 30 = 750t$ grams of salt. Therefore, the salt concentration at time t will be $C(t) = \frac{750t}{5000+25t} = \frac{30t}{200+t} \frac{\text{g}}{\text{L}}$.

(b) $\lim_{t\to\infty} C(t) = \lim_{t\to\infty} \frac{30t}{200+t} = \lim_{t\to\infty} \frac{30t/t}{200/t+t/t} = \frac{30}{0+1} = 30$. So the salt concentration approaches that of the brine being pumped into the tank.

55. **(a)** $\lim_{t\to\infty} v(t) = \lim_{t\to\infty} v^*\left(1-e^{-gt/v^*}\right) = v^*(1-0) = v^*$

(b) We graph $v(t) = 1-e^{-9.8t}$ and $v(t) = 0.99v^*$, or in this case, $v(t) = 0.99$. Using an intersect feature or zooming in on the point of intersection, we find that $t \approx 0.47$ s.

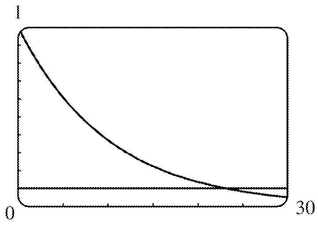


56. **(a)** $y = e^{-x/10}$ and $y = 0.1$ intersect at $x_1 \approx 23.03$.

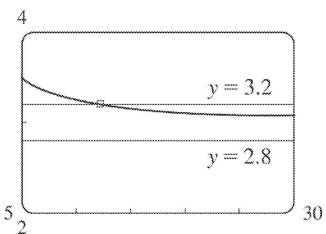
If $x > x_1$, then $e^{-x/10} < 0.1$.

(b) $e^{-x/10} < 0.1 \Rightarrow -x/10 < \ln 0.1 \Rightarrow$

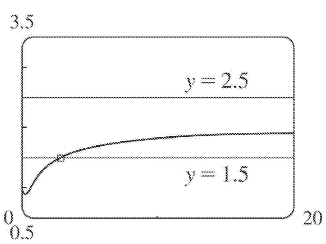
$x > -10\ln \frac{1}{10} = -10\ln 10^{-1} = 10\ln 10$

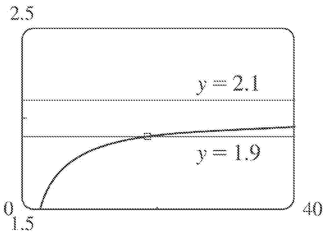


57. $\left|\frac{6x^2+5x-3}{2x^2-1}-3\right|<0.2 \Leftrightarrow 2.8<\frac{6x^2+5x-3}{2x^2-1}<3.2$. So we graph the three parts of this inequality on the same screen, and find that the curve $y=\frac{6x^2+5x-3}{2x^2-1}$ seems to lie between the lines $y=2.8$ and $y=3.2$ whenever $x>12.8$. So we can choose $N=13$ (or any larger number) so that the inequality holds whenever $x\geq N$.



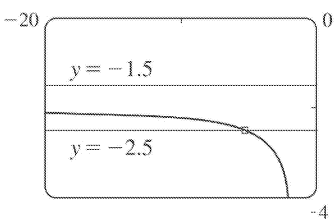
58. For $\varepsilon=0.5$, we must find N such that whenever $x\geq N$, we have $\left|\frac{\sqrt{4x^2+1}}{x+1}-2\right|<0.5 \Leftrightarrow$ $1.5<\frac{\sqrt{4x^2+1}}{x+1}<2.5$. We graph the three parts of this inequality on the same screen, and find that it holds whenever $x\geq 3$. So we choose $N=3$ (or any larger number). For $\varepsilon=0.1$, we must have $1.9<\frac{\sqrt{4x^2+1}}{x+1}<2.1$, and the graphs show that this holds whenever $x\geq 19$. So we choose $N=19$ (or any larger number).

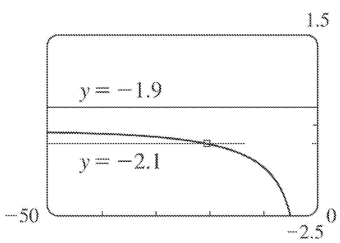




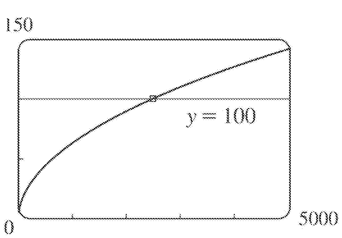
59. For $\varepsilon = 0.5$, we need to find N such that $\left| \frac{\sqrt{4x^2+1}}{x+1} - (-2) \right| < 0.5 \Leftrightarrow -2.5 < \frac{\sqrt{4x^2+1}}{x+1} < -1.5$ whenever $x \leq N$. We graph the three parts of this inequality on the same screen, and see that the inequality holds for $x \leq -6$. So we choose $N = -6$ (or any smaller number).

For $\varepsilon = 0.1$, we need $-2.1 < \frac{\sqrt{4x^2+1}}{x+1} < -1.9$ whenever $x \leq N$. From the graph, it seems that this inequality holds for $x \leq -22$. So we choose $N = -22$ (or any smaller number).





60. We need N such that $\frac{2x+1}{\sqrt{x+1}} > 100$ whenever $x \geq N$. From the graph, we see that this inequality holds for $x \geq 2500$. So we choose $N = 2500$ (or any larger number).



61. **(a)** $1/x^2 < 0.0001 \Leftrightarrow x^2 > 1/0.0001 = 10,000 \Leftrightarrow x > 100$ ($x > 0$)

(b) If $\varepsilon > 0$ is given, then $1/x^2 < \varepsilon \Leftrightarrow x^2 > 1/\varepsilon \Leftrightarrow x > 1/\sqrt{\varepsilon}$. Let $N = 1/\sqrt{\varepsilon}$.
Then $x > N \Rightarrow x > \dfrac{1}{\sqrt{\varepsilon}} \Rightarrow \left| \dfrac{1}{x^2} - 0 \right| = \dfrac{1}{x^2} < \varepsilon$, so $\displaystyle\lim_{x \to \infty} \frac{1}{x^2} = 0$.

62. **(a)** $1/\sqrt{x} < 0.0001 \Leftrightarrow \sqrt{x} > 1/0.0001 = 10^4 \Leftrightarrow x > 10^8$
(b) If $\varepsilon > 0$ is given, then $1/\sqrt{x} < \varepsilon \Leftrightarrow \sqrt{x} > 1/\varepsilon \Leftrightarrow x > 1/\varepsilon^2$. Let $N = 1/\varepsilon^2$.
Then $x > N \Rightarrow x > \dfrac{1}{\varepsilon^2} \Rightarrow \left| \dfrac{1}{\sqrt{x}} - 0 \right| = \dfrac{1}{\sqrt{x}} < \varepsilon$, so $\displaystyle\lim_{x \to \infty} \frac{1}{\sqrt{x}} = 0$.

63. For $x < 0$, $|1/x - 0| = -1/x$. If $\varepsilon > 0$ is given, then $-1/x < \varepsilon \Leftrightarrow x < -1/\varepsilon$.
Take $N = -1/\varepsilon$. Then $x < N \Rightarrow x < -1/\varepsilon \Rightarrow |(1/x) - 0| = -1/x < \varepsilon$, so $\displaystyle\lim_{x \to -\infty} (1/x) = 0$.

64. Given $M > 0$, we need $N > 0$ such that $x > N \Rightarrow x^3 > M$. Now $x^3 > M \Leftrightarrow x > \sqrt[3]{M}$, so take $N = \sqrt[3]{M}$. Then $x > N = \sqrt[3]{M} \Rightarrow x^3 > M$, so $\displaystyle\lim_{x \to \infty} x^3 = \infty$.

65. Given $M > 0$, we need $N > 0$ such that $x > N \Rightarrow e^x > M$. Now $e^x > M \Leftrightarrow x > \ln M$, so take $N = \max(1, \ln M)$. (This ensures that $N > 0$.) Then $x > N = \max(1, \ln M) \Rightarrow e^x > \max(e, M) \geq M$, so $\displaystyle\lim_{x \to \infty} e^x = \infty$.

66. **Definition** Let f be a function defined on some interval $(-\infty, a)$. Then $\displaystyle\lim_{x \to -\infty} f(x) = -\infty$ means that for every negative number M there is a corresponding negative number N such that $f(x) < M$ whenever $x < N$. Now we use the definition to prove that $\displaystyle\lim_{x \to -\infty} \left(1 + x^3\right) = -\infty$. Given a negative number M , we need a negative number N such that $x < N \Rightarrow 1 + x^3 < M$. Now $1 + x^3 < M \Leftrightarrow x^3 < M - 1 \Leftrightarrow x < \sqrt[3]{M-1}$. Thus, we take $N = \sqrt[3]{M-1}$ and find that $x < N \Rightarrow 1 + x^3 < M$. This proves that $\displaystyle\lim_{x \to -\infty} \left(1 + x^3\right) = -\infty$.

67. Suppose that $\displaystyle\lim_{x \to \infty} f(x) = L$. Then for every $\varepsilon > 0$ there is a corresponding positive number N such that $|f(x) - L| < \varepsilon$ whenever $x > N$. If $t = 1/x$, then $x > N \Leftrightarrow 0 < 1/x < 1/N \Leftrightarrow 0 < t < 1/N$. Thus, for every $\varepsilon > 0$ there is a corresponding $\delta > 0$ (namely $1/N$) such that $|f(1/t) - L| < \varepsilon$ whenever $0 < t < \delta$. This proves that

$\lim_{t \to 0^+} f(1/t) = L = \lim_{x \to \infty} f(x)$.

Now suppose that $\lim_{x \to -\infty} f(x) = L$. Then for every $\varepsilon > 0$ there is a corresponding negative number N such that $|f(x) - L| < \varepsilon$ whenever $x < N$. If $t = 1/x$, then $x < N \Leftrightarrow 1/N < 1/x < 0 \Leftrightarrow 1/N < t < 0$. Thus, for every $\varepsilon > 0$ there is a corresponding $\delta > 0$ (namely $-1/N$) such that $|f(1/t) - L| < \varepsilon$ whenever $-\delta < t < 0$. This proves that $\lim_{t \to 0^-} f(1/t) = L = \lim_{x \to -\infty} f(x)$.

1. **(a)** This is just the slope of the line through two points: $m_{PQ}=\dfrac{\Delta y}{\Delta x}=\dfrac{f(x)-f(3)}{x-3}$.

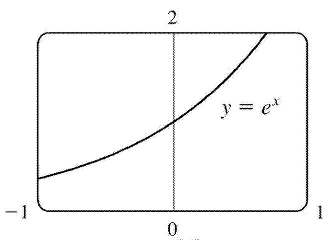
(b) This is the limit of the slope of the secant line PQ as Q approaches P : $m=\lim\limits_{x\to 3}\dfrac{f(x)-f(3)}{x-3}$.

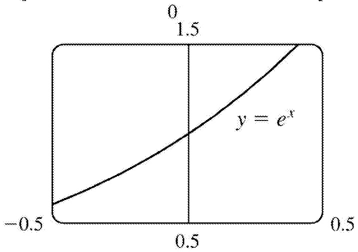
2. **(a)** Average velocity = $\dfrac{\Delta s}{\Delta t}=\dfrac{f(a+h)-f(a)}{(a+h)-a}=\dfrac{f(a+h)-f(a)}{h}$

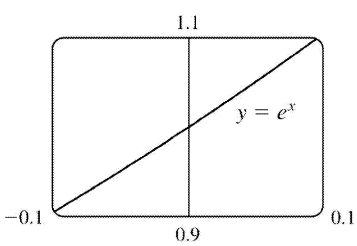
(b) Instantaneous velocity $=\lim\limits_{h\to 0}\dfrac{f(a+h)-f(a)}{h}$

3. The slope at D is the largest positive slope, followed by the positive slope at E . The slope at C is zero. The slope at B is steeper than at A (both are negative). In decreasing order, we have the slopes at: D , E , C , A , and B .

4. The curve looks more like a line as the viewing rectangle gets smaller.







5. **(a)**

(i) Using Definition 1,

$$m=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}\ \lim_{x\to -3}\frac{f(x)-f(-3)}{x-(-3)}=\lim_{x\to -3}\frac{\left(x^{2}+2x\right)-(3)}{x-(-3)}=\lim_{x\to -3}\frac{(x+3)(x-1)}{x+3}$$
$$=\lim_{x\to -3}(x-1)=-4$$

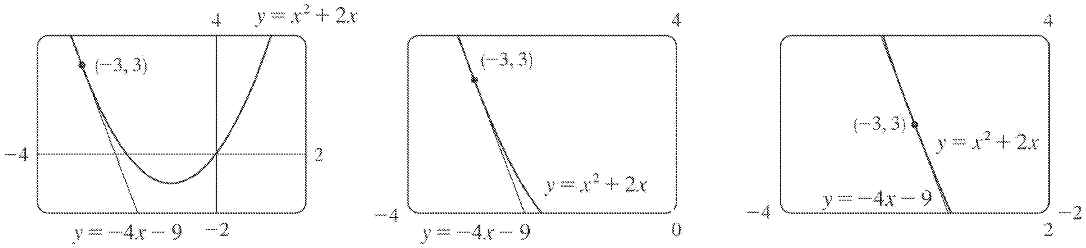
(ii) Using Equation 2,

$$m=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}=\lim_{h\to 0}\frac{f(-3+h)-f(-3)}{h}=\lim_{h\to 0}\frac{\left[(-3+h)^2+2(-3+h)\right]-(3)}{h}$$

$$=\lim_{h\to 0}\frac{9-6h+h^2-6+2h-3}{h}=\lim_{h\to 0}\frac{h(h-4)}{h}=\lim_{h\to 0}(h-4)=-4$$

(b) Using the point–slope form of the equation of a line, an equation of the tangent line is $y-3=-4(x+3)$. Solving for y gives us $y=-4x-9$, which is the slope–intercept form of the equation of the tangent line.

(c)



6. **(a)**

(i)

$$m=\lim_{x\to -1}\frac{f(x)-f(-1)}{x-(-1)}=\lim_{x\to -1}\frac{x^3-(-1)}{x+1}=\lim_{x\to -1}\frac{(x+1)\left(x^2-x+1\right)}{x+1}$$

$$=\lim_{x\to -1}\left(x^2-x+1\right)=3$$

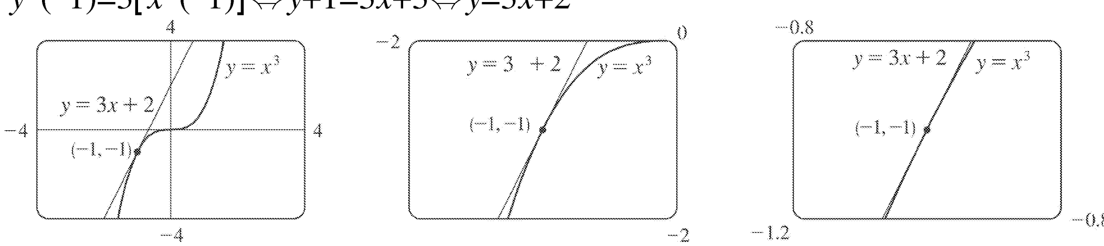
(ii)

$$m=\lim_{h\to 0}\frac{f(-1+h)-f(-1)}{h}=\lim_{h\to 0}\frac{(-1+h)^3-(-1)}{h}=\lim_{h\to 0}\frac{h^3-3h^2+3h-1+1}{h}$$

$$=\lim_{h\to 0}\left(h^2-3h+3\right)=3$$

(b) $y-(-1)=3[x-(-1)]\Leftrightarrow y+1=3x+3\Leftrightarrow y=3x+2$

(c)



7. Using (2) with $f(x)=1+2x-x^3$ and $P(1,2)$,

$$m=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}=\lim_{h\to 0}\frac{f(1+h)-f(1)}{h}=\lim_{h\to 0}\frac{\left[1+2(1+h)-(1+h)^3\right]-2}{h}$$

$$=\lim_{h\to 0}\frac{1+2+2h-(1+3h+3h^2+h^3)-2}{h}=\lim_{h\to 0}\frac{-h^3-3h^2-h}{h}$$

$$=\lim_{h\to 0}\frac{h(-h^2-3h-1)}{h}=\lim_{h\to 0}(-h^2-3h-1)=-1$$

Tangent line: $y-2=-1(x-1)\Leftrightarrow y-2=-x+1\Leftrightarrow y=-x+3$

8. Using (1),

$$m=\lim_{x\to 4}\frac{\sqrt{2x+1}-\sqrt{2(4)+1}}{x-4}=\lim_{x\to 4}\frac{\sqrt{2x+1}-3}{x-4}\cdot\frac{\sqrt{2x+1}+3}{\sqrt{2x+1}+3}$$

$$=\lim_{x\to 4}\frac{(2x+1)-3^2}{(x-4)(\sqrt{2x+1}+3)}=\lim_{x\to 4}\frac{2(x-4)}{(x-4)(\sqrt{2x+1}+3)}$$

$$=\lim_{x\to 4}\frac{2}{(\sqrt{2x+1}+3)}=\frac{2}{3+3}=\frac{1}{3}\ .$$

Tangent line: $y-3=\frac{1}{3}(x-4)\Leftrightarrow y-3=\frac{1}{3}x-\frac{4}{3}\Leftrightarrow y=\frac{1}{3}x+\frac{5}{3}$

9. Using (1) with $f(x)=\frac{x-1}{x-2}$ and $P(3,2)$,

$$m=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}=\lim_{x\to 3}\frac{\frac{x-1}{x-2}-2}{x-3}=\lim_{x\to 3}\frac{\frac{x-1-2(x-2)}{x-2}}{x-3}=\lim_{x\to 3}\frac{3-x}{(x-2)(x-3)}$$

$$=\lim_{x\to 3}\frac{-1}{x-2}=\frac{-1}{1}=-1\ .$$

Tangent line: $y-2=-1(x-3)\Leftrightarrow y-2=-x+3\Leftrightarrow y=-x+5$

10. Using (1), $m=\lim_{x\to 0}\frac{\frac{2x}{(x+1)^2}-0}{x-0}=\lim_{x\to 0}\frac{2x}{x(x+1)^2}=\lim_{x\to 0}\frac{2}{(x+1)^2}=\frac{2}{1^2}=2$.

Tangent line: $y-0=2(x-0)\Leftrightarrow y=2x$

11. **(a)**

$$m=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}=\lim_{x\to a}\frac{2/(x+3)-2/(a+3)}{x-a}=\lim_{x\to a}\frac{2(a+3)-2(x+3)}{(x-a)(x+3)(a+3)}$$

$$=\lim_{x\to a}\frac{2(a-x)}{(x-a)(x+3)(a+3)}=\lim_{x\to a}\frac{-2}{(x+3)(a+3)}=\frac{-2}{(a+3)^2}$$

(b)

(i) $a=-1\Rightarrow m=\dfrac{-2}{(-1+3)^2}=-\dfrac{1}{2}$

(ii) $a=0\Rightarrow m=\dfrac{-2}{(0+3)^2}=-\dfrac{2}{9}$

(iii) $a=1\Rightarrow m=\dfrac{-2}{(1+3)^2}=-\dfrac{1}{8}$

12. **(a)** Using (1),

$$m=\lim_{x\to a}\frac{\left(1+x+x^2\right)-\left(1+a+a^2\right)}{x-a}=\lim_{x\to a}\frac{x+x^2-a-a^2}{x-a}=\lim_{x\to a}\frac{x-a+(x-a)(x+a)}{x-a}$$

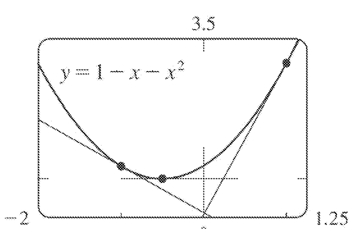
$$=\lim_{x\to a}\frac{(x-a)(1+x+a)}{x-a}=\lim_{x\to a}(1+x+a)=1+2a$$

(b)

(i) $x=-1\Rightarrow m=1+2(-1)=-1$

(ii) $x=-\dfrac{1}{2}\Rightarrow m=1+2\left(-\dfrac{1}{2}\right)=0$

(iii) $x=1\Rightarrow m=1+2(1)=3$

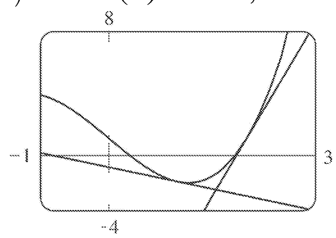
(c) 

13. **(a)** Using (1),

$$m=\lim_{x\to a}\frac{\left(x^3-4x+1\right)-\left(a^3-4a+1\right)}{x-a}=\lim_{x\to a}\frac{\left(x^3-a^3\right)-4(x-a)}{x-a}$$

$$=\lim_{x\to a}\frac{(x-a)\left(x^2+ax+a^2\right)-4(x-a)}{x-a}=\lim_{x\to a}\left(x^2+ax+a^2-4\right)=3a^2-4$$

(b) At $(1,-2)$: $m=3(1)^2-4=-1$, so an equation of the tangent line is $y-(-2)=-1(x-1)\Leftrightarrow y=-x-1$. At $(2,1)$: $m=3(2)^2-4=8$, so an equation of the tangent line is $y-1=8(x-2)\Leftrightarrow y=8x-15$.

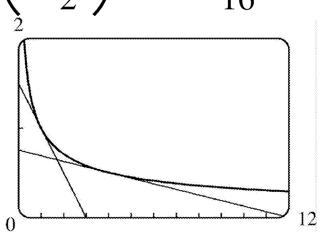
(c) 

14. **(a)** Using (1),

$$m=\lim_{x\to a}\frac{\frac{1}{\sqrt{x}}-\frac{1}{\sqrt{a}}}{x-a}=\lim_{x\to a}\frac{\frac{\sqrt{a}-\sqrt{x}}{\sqrt{ax}}}{x-a}=\lim_{x\to a}\frac{\left(\sqrt{a}-\sqrt{x}\right)\left(\sqrt{a}+\sqrt{x}\right)}{\sqrt{ax}\,(x-a)\left(\sqrt{a}+\sqrt{x}\right)}$$

$$=\lim_{x\to a}\frac{a-x}{\sqrt{ax}\,(x-a)\left(\sqrt{a}+\sqrt{x}\right)}=\lim_{x\to a}\frac{-1}{\sqrt{ax}\left(\sqrt{a}+\sqrt{x}\right)}=\frac{-1}{\sqrt{a^2}\left(2\sqrt{a}\right)}=-\frac{1}{2a^{3/2}}\text{ or }-\frac{1}{2}a^{-3/2}$$

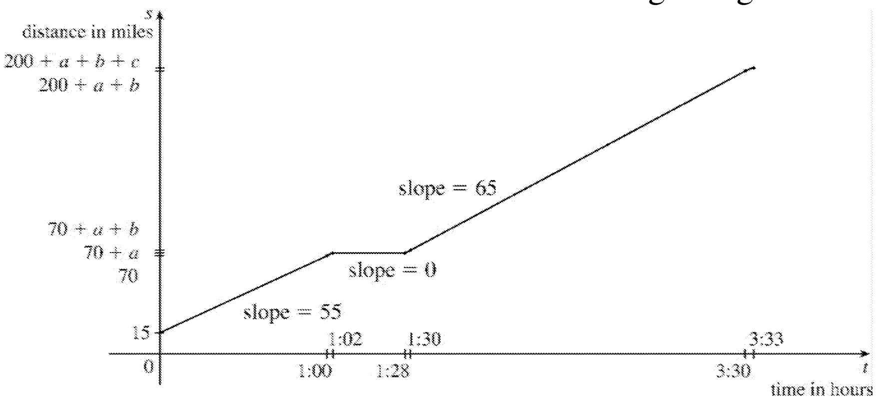
(b) At $(1,1)$: $m=-\frac{1}{2}$, so an equation of the tangent line is $y-1=-\frac{1}{2}(x-1)\Leftrightarrow y=-\frac{1}{2}x+\frac{3}{2}$.
At $\left(4,\frac{1}{2}\right)$: $m=-\frac{1}{16}$, so an equation of the tangent line is $y-\frac{1}{2}=-\frac{1}{16}(x-4)\Leftrightarrow y=-\frac{1}{16}x+\frac{3}{4}$.

(c) 

15. **(a)** Since the slope of the tangent at $t=0$ is 0 , the car's initial velocity was 0 .
(b) The slope of the tangent is greater at C than at B , so the car was going faster at C .
(c) Near A , the tangent lines are becoming steeper as x increases, so the velocity was increasing, so the car was speeding up. Near B , the tangent lines are becoming less steep, so the car was slowing down. The steepest tangent near C is the one at C , so at C the car had just finished speeding up, and was about to start slowing down.

(d) Between D and E , the slope of the tangent is 0 , so the car did not move during that time.

16. Let a denote the distance traveled from 1:00 to 1:02 ,b from 1:28 to 1:30, and c from 3:30 to 3:33, where all the times are relative to t=0 at the beginning of the trip.



17. Let $s(t)=40t-16t^2$.

$$v(2)=\lim_{t\to 2}\frac{s(t)-s(2)}{t-2}=\lim_{t\to 2}\frac{\left(40t-16t^2\right)-16}{t-2}=\lim_{t\to 2}\frac{-16t^2+40t-16}{t-2}=\lim_{t\to 2}\frac{-8\left(2t^2-5t+2\right)}{t-2}$$

$$=\lim_{t\to 2}\frac{-8(t-2)(2t-1)}{t-2}=-8\lim_{t\to 2}(2t-1)=-8(3)=-24$$

Thus, the instantaneous velocity when t=2 is -24 ft / s.

18. **(a)**

$$v(1)=\lim_{h\to 0}\frac{H(1+h)-H(1)}{h}$$

$$=\lim_{h\to 0}\frac{\left(58+58h-0.83-1.66h-0.83h^2\right)-57.17}{h}=\lim_{h\to 0}(56.34-0.83h)=56.34 \text{ m / s}$$

(b)

$$v(a)=\lim_{h\to 0}\frac{H(a+h)-H(a)}{h}$$

$$=\lim_{h\to 0}\frac{\left(58a+58h-0.83a^2-1.66ah-0.83h^2\right)-\left(58a-0.83a^2\right)}{h}$$

$$=\lim_{h\to 0}(58-1.66a-0.83h)=58-1.66a \text{ m / s}$$

(c) The arrow strikes the moon when the height is 0 , that is,

$58t-0.83t^2=0 \Leftrightarrow t(58-0.83t)=0 \Leftrightarrow t=\dfrac{58}{0.83} \approx 69.9$ s (since t can't be 0).

(d) Using the time from part (c), $v\left(\dfrac{58}{0.83}\right)=58-1.66\left(\dfrac{58}{0.83}\right)=-58$ m / s. Thus, the arrow will have a velocity of -58 m / s.

19.

$$v(a)=\lim_{h\to 0}\frac{s(a+h)-s(a)}{h}=\lim_{h\to 0}\frac{4(a+h)^3+6(a+h)+2-\left(4a^3+6a+2\right)}{h}$$

$$=\lim_{h\to 0}\frac{4a^3+12a^2h+12ah^2+4h^3+6a+6h+2-4a^3-6a-2}{h}$$

$$=\lim_{h\to 0}\frac{12a^2h+12ah^2+4h^3+6h}{h}=\lim_{h\to 0}\left(12a^2+12ah+4h^2+6\right)=\left(12a^2+6\right)\text{ m / s}$$

So $v(1)=12(1)^2+6=18$ m / s, $v(2)=12(2)^2+6=54$ m / s, and $v(3)=12(3)^2+6=114$ m / s.

20. **(a)** The average velocity between times t and $t+h$ is

$$\frac{s(t+h)-s(t)}{(t+h)-t}=\frac{(t+h)^2-8(t+h)+18-\left(t^2-8t+18\right)}{h}$$

$$=\frac{t^2+2th+h^2-8t-8h+18-t^2+8t-18}{h}=\frac{2th+h^2-8h}{h}$$

$$=(2t+h-8)\ \text{ m/s}$$

(i) $[3,4]$: $t=3$, $h=4-3=1$, so the average velocity is $2(3)+1-8=-1$ m / s.

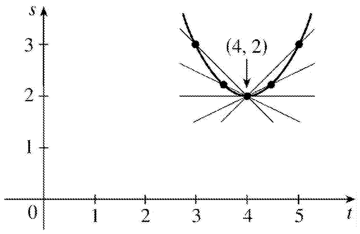
(ii) $[3.5,4]$: $t=3.5$, $h=0.5$, so the average velocity is $2(3.5)+0.5-8=-0.5$ m / s.

(iii) $[4,5]$: $t=4$, $h=1$, so the average velocity is $2(4)+1-8=1$ m / s.

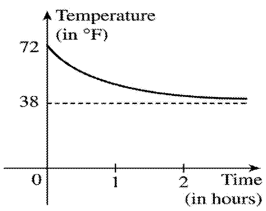
(iv) $[4,4.5]$: $t=4$, $h=0.5$, so the average velocity is $2(4)+0.5-8=0.5$ m / s.

(b) $v(t)=\lim_{h\to 0}\dfrac{s(t+h)-s(t)}{h}=\lim_{h\to 0}(2t+h-8)=2t-8$, so $v(4)=0$.

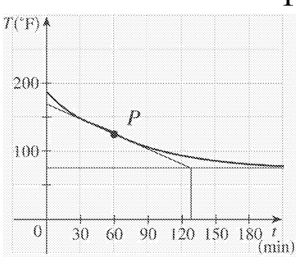
(c)



21. The sketch shows the graph for a room temperature of 72° and a refrigerator temperature of 38° . The initial rate of change is greater in magnitude than the rate of change after an hour.



22. The slope of the tangent (that is, the rate of change of temperature with respect to time) at t=1 h seems to be about $\dfrac{75-168}{132-0}\approx -0.7^{\circ}$ F / min.

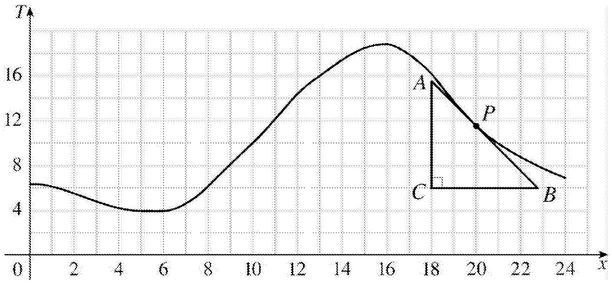


23. **(a)** (i) $[20,23]$: $\dfrac{7.9-11.5}{23-20}=-1.2^{\circ}$ C / h

(ii) $[20,22]$: $\dfrac{9.0-11.5}{22-20}=-1.25^{\circ}$ C / h

(iii) $[20,21]$: $\dfrac{10.2-11.5}{21-20}=-1.3^{\circ}$ C / h

(b) In the figure, we estimate A to be $(18,15.5)$ and B as $(23,6)$. So the slope is $\dfrac{6-15.5}{23-18}=-1.9^{\circ}$ C / h at 8:00 P.M.



24. **(a)**

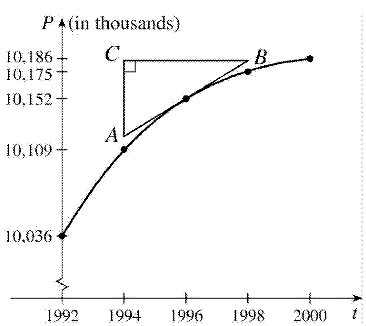
(i) $[1992,1996]$: $\dfrac{P(1996)-P(1992)}{1996-1992}=\dfrac{10{,}152-10{,}036}{4}=\dfrac{116}{4}=29$ thousand / year

(ii) $[1994,1996]$: $\dfrac{P(1996)-P(1994)}{1996-1994}=\dfrac{10{,}152-10{,}109}{2}=\dfrac{43}{2}=21.5$ thousand / year

(iii) $[1996,1998]$: $\dfrac{P(1998)-P(1996)}{1998-1996}=\dfrac{10{,}175-10{,}152}{2}=\dfrac{23}{2}=11.5$ thousand / year

(b) Using the values from (ii) and (iii), we have $\dfrac{21.5+11.5}{2}=16.5$ thousand / year.

(c) Estimating A as (1994,10 , 125) and B as $(1998{,}10{,}182)$, the slope at 1996 is $\dfrac{10{,}182-10{,}125}{1998-1994}=\dfrac{57}{4}=14.25$ thousand / year.



25. **(a)**

(i) $[1995,1997]$: $\dfrac{N(1997)-N(1995)}{1997-1995}=\dfrac{2461-873}{2}=\dfrac{1588}{2}=794$ thousand / year

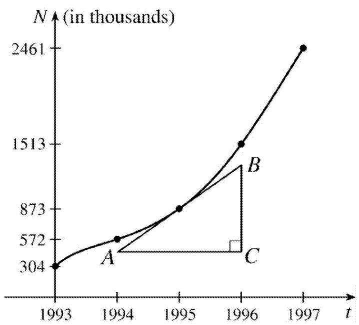
(ii) $[1995,1996]$: $\dfrac{N(1996)-N(1995)}{1996-1995}=\dfrac{1513-873}{1}=640$ thousand / year

(iii) $[1994,1995]$: $\dfrac{N(1995)-N(1994)}{1995-1994}=\dfrac{873-572}{1}=301$ thousand / year

(b) Using the values from (ii) and (iii), we have

$\frac{640+301}{2}=\frac{941}{2}$ =470.5 thousand / year.

(c) A as $(1994,420)$ and B as $(1996,1275)$, the slope at 1995 is $\frac{1275-420}{1996-1994}=\frac{855}{2}$ =427.5 thousand / year



26. **(a)**

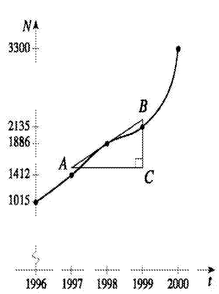
(i) $[1996,1998]$: $\frac{N(1998)-N(1996)}{1998-1996}=\frac{1886-1015}{2}=\frac{871}{2}$ =435.5 locations / year

(ii) $[1997,1998]$: $\frac{N(1998)-N(1997)}{1998-1997}=\frac{1886-1412}{1}$ =474 locations / year

(iii) $[1998,1999]$: $\frac{N(1999)-N(1998)}{1999-1998}=\frac{2135-1886}{1}$ =249 locations / year

(b) Using the values from (ii) and (iii), we have $\frac{474+249}{2}=\frac{723}{2}$ =361.5≈362 locations / year.

(c) Estimating A as $(1997,1525)$ and B as $(1999,2250)$, the slope at 1998 is $\frac{2250-1525}{1999-1997}=\frac{725}{2}$ =362.5 locations / year.



27. **(a)**

(i) $\dfrac{\Delta C}{\Delta x} = \dfrac{C(105)-C(100)}{105-100} = \dfrac{6601.25-6500}{5}$ =\$20.25/ unit.

(ii) $\dfrac{\Delta C}{\Delta x} = \dfrac{C(101)-C(100)}{101-100} = \dfrac{6520.05-6500}{1}$ =\$20.05/ unit.

(b)

$$\frac{C(100+h)-C(100)}{h} = \frac{\left[5000+10(100+h)+0.05(100+h)^2\right]-6500}{h} = \frac{20h+0.05h^2}{h}$$

$$=20+0.05h\ ,\ h\neq 0$$

So the instantaneous rate of change is $\lim_{h\to 0} \dfrac{C(100+h)-C(100)}{h} = \lim_{h\to 0} (20+0.05h)=\$20/$ unit.

28.

$$\Delta V = V(t+h)-V(t)=100\,,000\left(1-\frac{t+h}{60}\right)^2-100\,,000\left(1-\frac{t}{60}\right)^2$$

$$=100\,,000\left[\left(1-\frac{t+h}{30}+\frac{(t+h)^2}{3600}\right)-\left(1-\frac{t}{30}+\frac{t^2}{3600}\right)\right]=100\,,000\left(-\frac{h}{30}+\frac{2th}{3600}+\frac{h^2}{3600}\right)$$

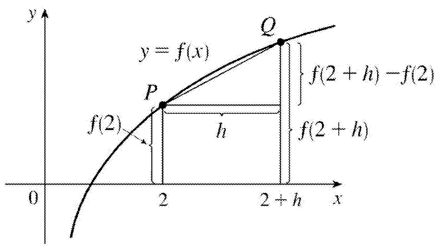
$$=\frac{100{,}000}{3600}h(-120+2t+h)=\frac{250}{9}h(-120+2t+h)$$

Dividing ΔV by h and then letting $h\to 0$, we see that the instantaneous rate of change is $\dfrac{500}{9}(t-60)$ gal / min.

t	*Flow rate (gal/min)*	*Water remaining $V(t)$ (gal)*
0	$-3333.\overline{3}$	100,000
10	$-2777.\overline{7}$	$69{,}444.\overline{4}$
20	$-2222.\overline{2}$	$44{,}444.\overline{4}$
30	$-1666.\overline{6}$	25,000
40	$-1111.\overline{1}$	$11{,}111.\overline{1}$
50	$-555.\overline{5}$	$2{,}777.\overline{7}$
60	0	0

The magnitude of the flow rate is greatest at the beginning and gradually decreases to 0.

1.



The line from $P(2,f(2))$ to $Q(2+h,f(2+h))$ is the line that has slope $\dfrac{f(2+h)-f(2)}{h}$

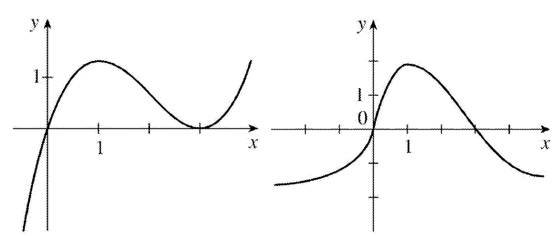
2. As h decreases, the line PQ becomes steeper, so its slope increases. So $0<\dfrac{f(4)-f(2)}{4-2}<\dfrac{f(3)-f(2)}{3-2}<\lim\limits_{x\to 2}\dfrac{f(x)-f(2)}{x-2}$. Thus, $0<\frac{1}{2}[f(4)-f(2)]<f(3)-f(2)<f'(2)$.

3. $g'(0)$ is the only negative value. The slope at $x=4$ is smaller than the slope at $x=2$ and both are smaller than the slope at $x=-2$. Thus, $g'(0)<0<g'(4)<g'(2)<g'(-2)$.

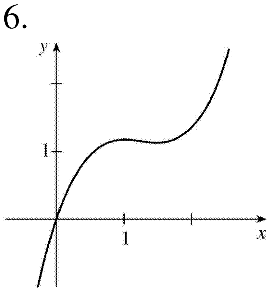
4. Since $(4,3)$ is on $y=f(x)$, $f(4)=3$. The slope of the tangent line between $(0,2)$ and $(4,3)$ is $\dfrac{1}{4}$, so $f'(4)=\dfrac{1}{4}$.

5.

We begin by drawing a curve through the origin at a slope of 3 to satisfy $f(0)=0$ and $f'(0)=3$. Since $f'(1)=0$, we will round off our figure so that there is a horizontal tangent directly over $x=1$. Lastly, we make sure that the curve has a slope of -1 as we pass over $x=2$. Two of the many possibilities are shown.



6.



7. Using Definition 2 with

$f(x)=3x^2-5x$ and the point $(2,2)$, we have

$$f'(2)=\lim_{h\to 0}\frac{f(2+h)-f(2)}{h}=\lim_{h\to 0}\frac{[3(2+h)^2-5(2+h)]-2}{h}$$

$$=\lim_{h\to 0}\frac{(12+12h+3h^2-10-5h)-2}{h}=\lim_{h\to 0}\frac{3h^2+7h}{h}=\lim_{h\to 0}(3h+7)=7 .$$

So an equation of the tangent line at $(2,2)$ is $y-2=7(x-2)$ or $y=7x-12$.

8. Using Definition 2 with $g(x)=1-x^3$ and the point $(0,1)$, we have

$$g'(0)=\lim_{h\to 0}\frac{g(0+h)-g(0)}{h}=\lim_{h\to 0}\frac{[1-(0+h)^3]-1}{h}=\lim_{h\to 0}\frac{(1-h^3)-1}{h}=\lim_{h\to 0}(-h^2)=0 .$$

So an equation of the tangent line is $y-1=0(x-0)$ or $y=1$.

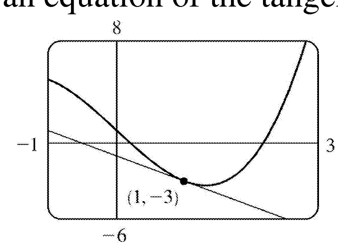
9. **(a)** Using Definition 2 with $F(x)=x^3-5x+1$ and the point $(1,-3)$, we have

$$F'(1)=\lim_{h\to 0}\frac{F(1+h)-F(1)}{h}=\lim_{h\to 0}\frac{[(1+h)^3-5(1+h)+1]-(-3)}{h}$$

$$=\lim_{h\to 0}\frac{(1+3h+3h^2+h^3-5-5h+1)+3}{h}=\lim_{h\to 0}\frac{h^3+3h^2-2h}{h}$$

$$=\lim_{h\to 0}\frac{h(h^2+3h-2)}{h}=\lim_{h\to 0}(h^2+3h-2)=-2$$

So an equation of the tangent line at $(1,-3)$ is $y-(-3)=-2(x-1)\Leftrightarrow y=-2x-1$.

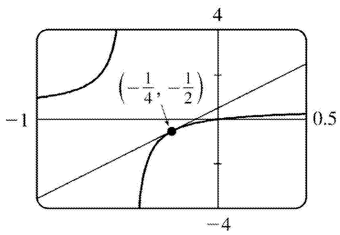
(b) 

10. **(a)**

$$G'(a)=\lim_{h\to 0}\frac{G(a+h)-G(a)}{h}=\lim_{h\to 0}\frac{\dfrac{a+h}{1+2(a+h)}-\dfrac{a}{1+2a}}{h}$$

$$=\lim_{h\to 0}\frac{a+2a^2+h+2ah-a-2a^2-2ah}{h(1+2a+2h)(1+2a)}=\lim_{h\to 0}\frac{1}{(1+2a+2h)(1+2a)}=(1+2a)^{-2}$$

So the slope of the tangent at the point $(-\frac{1}{4},-\frac{1}{2})$ is $m=\left[1+2(-\frac{1}{4})\right]^{-2}=4$, and thus an equation is $y+\frac{1}{2}=4(x+\frac{1}{4})$ or $y=4x+\frac{1}{2}$.

(b) 

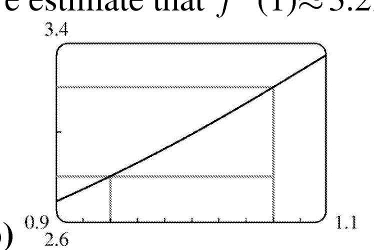
11. **(a)** $f'(1)=\lim_{h\to 0}\frac{f(1+h)-f(1)}{h}=\lim_{h\to 0}\frac{3^{1+h}-3^1}{h}$.

So let $F(h)=\frac{3^{1+h}-3}{h}$. We calculate:

h	$F(h)$
0.1	3.484
0.01	3.314
0.001	3.298
0.0001	3.296

h	$F(h)$
–0.1	3.121
–0.01	3.278
–0.001	3.294
–0.0001	3.296

We estimate that $f'(1)\approx 3.296$.

(b) 

From the graph, we estimate that the slope of the tangent is about $\frac{3.2-2.8}{1.06-0.94}=\frac{0.4}{0.12}\approx 3.3$.

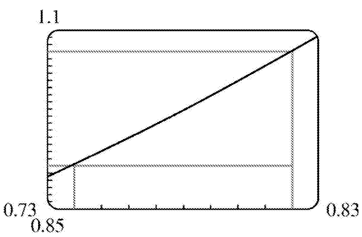
12. **(a)**

$$g'\left(\frac{\pi}{4}\right)=\lim_{h\to 0}\frac{g\left(\frac{\pi}{4}+h\right)-g\left(\frac{\pi}{4}\right)}{h}=\lim_{h\to 0}\frac{\tan\left(\frac{\pi}{4}+h\right)-\tan\left(\frac{\pi}{4}\right)}{h}.$$

So let $G(h)=\dfrac{\tan\left(\frac{\pi}{4}+h\right)-1}{h}$. We calculate:

h	$G(h)$
0.1	2.2305
0.01	2.0203
0.001	2.0020
0.0001	2.0002

h	$G(h)$
–0.1	1.8237
–0.01	1.9803
–0.001	1.9980
–0.0001	1.9998

(b) 

From the graph, we estimate that the slope of the tangent is about $\dfrac{1.07-0.91}{0.82-0.74}=\dfrac{0.16}{0.08}=2$.

13. Use Definition 2 with $f(x)=3-2x+4x^2$.

$$f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}=\lim_{h\to 0}\frac{[3-2(a+h)+4(a+h)^2]-(3-2a+4a^2)}{h}$$

$$=\lim_{h\to 0}\frac{[3-2a-2h+4a^2+8ah+4h^2]-(3-2a+4a^2)}{h}=\lim_{h\to 0}\frac{-2h+8ah+4h^2}{h}$$

$$=\lim_{h\to 0}\frac{h(-2+8a+4h)}{h}=\lim_{h\to 0}(-2+8a+4h)=-2+8a$$

14.

$$f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}=\lim_{h\to 0}\frac{[(a+h)^4-5(a+h)]-(a^4-5a)}{h}$$

$$=\lim_{h\to 0}\frac{(a^4+4a^3h+6a^2h^2+4ah^3+h^4-5a-5h)-(a^4-5a)}{h}$$

$$=\lim_{h\to 0}\frac{4a^3h+6a^2h^2+4ah^3+h^4-5h}{h}=\lim_{h\to 0}\frac{h(4a^3+6a^2h+4ah^2+h^3-5)}{h}$$

$$=\lim_{h\to 0}(4a^3+6a^2h+4ah^2+h^3-5)=4a^3-5$$

15. Use Definition 2 with $f(t)=(2t+1)/(t+3)$.

$$f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}=\lim_{h\to 0}\frac{\frac{2(a+h)+1}{(a+h)+3}-\frac{2a+1}{a+3}}{h}$$

$$=\lim_{h\to 0}\frac{(2a+2h+1)(a+3)-(2a+1)(a+h+3)}{h(a+h+3)(a+3)}$$

$$=\lim_{h\to 0}\frac{(2a^2+6a+2ah+6h+a+3)-(2a^2+2ah+6a+a+h+3)}{h(a+h+3)(a+3)}$$

$$=\lim_{h\to 0}\frac{5h}{h(a+h+3)(a+3)}=\lim_{h\to 0}\frac{5}{(a+h+3)(a+3)}=\frac{5}{(a+3)^2}$$

16.

$$f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}=\lim_{h\to 0}\frac{\frac{(a+h)^2+1}{(a+h)-2}-\frac{a^2+1}{a-2}}{h}$$

$$=\lim_{h\to 0}\frac{(a^2+2ah+h^2+1))(a-2)-(a^2+1)(a+h-2)}{h(a+h-2)(a-2)}$$

$$=\lim_{h\to 0}\frac{(a^3-2a^2+2a^2h-4ah+ah^2-2h^2+a-2)-(a^3+a^2h-2a^2+a+h-2)}{h(a+h-2)(a-2)}$$

$$=\lim_{h\to 0}\frac{a^2h-4ah+ah^2-2h^2-h}{h(a+h-2)(a-2)}=\lim_{h\to 0}\frac{h(a^2-4a+ah-2h-1)}{h(a+h-2)(a-2)}$$

$$=\lim_{h\to 0}\frac{a^2-4a+ah-2h-1}{(a+h-2)(a-2)}=\frac{a^2-4a-1}{(a-2)^2}$$

17. Use Definition 2 with $f(x)=1/\sqrt{x+2}$.

$$f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}=\lim_{h\to 0}\frac{\frac{1}{\sqrt{(a+h)+2}}-\frac{1}{\sqrt{a+2}}}{h}$$

$$=\lim_{h\to 0}\frac{\frac{\sqrt{a+2}-\sqrt{a+h+2}}{\sqrt{a+h+2}\,\sqrt{a+2}}}{h}=\lim_{h\to 0}\left[\frac{\sqrt{a+2}-\sqrt{a+h+2}}{h\sqrt{a+h+2}\,\sqrt{a+2}}\cdot\frac{\sqrt{a+2}+\sqrt{a+h+2}}{\sqrt{a+2}+\sqrt{a+h+2}}\right]$$

$$=\lim_{h\to 0}\frac{(a+2)-(a+h+2)}{h\sqrt{a+h+2}\,\sqrt{a+2}(\sqrt{a+2}+\sqrt{a+h+2})}$$

$$=\lim_{h\to 0}\frac{-h}{h\sqrt{a+h+2}\,\sqrt{a+2}(\sqrt{a+2}+\sqrt{a+h+2})}$$

$$=\lim_{h\to 0}\frac{-1}{\sqrt{a+h+2}\,\sqrt{a+2}(\sqrt{a+2}+\sqrt{a+h+2})}$$

$$=\frac{-1}{(\sqrt{a+2})^2(2\sqrt{a+2})}=-\frac{1}{2(a+2)^{3/2}}$$

18.

$$f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}=\lim_{h\to 0}\frac{\sqrt{3(a+h)+1}-\sqrt{3a+1}}{h}$$

$$=\lim_{h\to 0}\frac{(\sqrt{3a+3h+1}-\sqrt{3a+1})(\sqrt{3a+3h+1}+\sqrt{3a+1})}{h(\sqrt{3a+3h+1}+\sqrt{3a+1})}$$

$$=\lim_{h\to 0}\frac{(3a+3h+1)-(3a+1)}{h(\sqrt{3a+3h+1}+\sqrt{3a+1})}=\lim_{h\to 0}\frac{3h}{h(\sqrt{3a+3h+1}+\sqrt{3a+1})}$$

$$=\lim_{h\to 0}\frac{3}{\sqrt{3a+3h+1}+\sqrt{3a+1}}=\frac{3}{2\sqrt{3a+1}}$$

19. By Definition 2, $\lim_{h\to 0}\frac{(1+h)^{10}-1}{h}=f'(1)$, where $f(x)=x^{10}$ and $a=1$. *Or:* By Definition 2, $\lim_{h\to 0}\frac{(1+h)^{10}-1}{h}=f'(0)$, where $f(x)=(1+x)^{10}$ and $a=0$.

20. By Definition 2,

$\lim\limits_{h\to 0}\dfrac{\sqrt[4]{16+h}-2}{h}=f'(16)$, where $f(x)=\sqrt[4]{x}$ and $a=16$. *Or* : By Definition 2, $\lim\limits_{h\to 0}\dfrac{\sqrt[4]{16+h}-2}{h}=f'(0)$, where $f(x)=\sqrt[4]{16+x}$ and $a=0$.

21. By Equation 3, $\lim\limits_{x\to 5}\dfrac{2^x-32}{x-5}=f'(5)$, where $f(x)=2^x$ and $a=5$.

22. By Equation 3, $\lim\limits_{x\to \pi/4}\dfrac{\tan x-1}{x-\pi/4}=f'(\pi/4)$, where $f(x)=\tan x$ and $a=\pi/4$.

23. By Definition 2, $\lim\limits_{h\to 0}\dfrac{\cos(\pi+h)+1}{h}=f'(\pi)$, where $f(x)=\cos x$ and $a=\pi$. *Or* : By Definition 2, $\lim\limits_{h\to 0}\dfrac{\cos(\pi+h)+1}{h}=f'(0)$, where $f(x)=\cos(\pi+x)$ and $a=0$.

24. By Equation 3, $\lim\limits_{t\to 1}\dfrac{t^4+t-2}{t-1}=f'(1)$, where $f(t)=t^4+t$ and $a=1$.

25.

$$v(2)=f'(2)=\lim_{h\to 0}\frac{f(2+h)-f(2)}{h}=\lim_{h\to 0}\frac{[(2+h)^2-6(2+h)-5]-[2^2-6(2)-5]}{h}$$

$$=\lim_{h\to 0}\frac{(4+4h+h^2-12-6h-5)-(-13)}{h}=\lim_{h\to 0}\frac{h^2-2h}{h}=\lim_{h\to 0}(h-2)=-2\,m/s$$

26.

$$v(2)=f'(2)=\lim_{h\to 0}\frac{f(2+h)-f(2)}{h}$$

$$=\lim_{h\to 0}\frac{[2(2+h)^3-(2+h)+1]-[2(2)^3-2+1]}{h}$$

$$=\lim_{h\to 0}\frac{(2h^3+12h^2+24h+16-2-h+1)-15}{h}$$

$$=\lim_{h\to 0}\frac{2h^3+12h^2+23h}{h}=\lim_{h\to 0}(2h^2+12h+23)=23m/s$$

27. **(a)** $f'(x)$ is the rate of change of the production cost with respect to the number of ounces of gold produced. Its units are dollars per ounce.
(b) After 800 ounces of gold have been produced, the rate at which the production cost is increasing is \$17/ ounce. So the cost of producing the 800 th (or 801 st) ounce is about \$17 .
(c) In the short term, the values of $f'(x)$ will decrease because more efficient use is made of start–up costs as x increases. But eventually $f'(x)$ might increase due to large–scale operations.

28. **(a)** $f'(5)$ is the rate of growth of the bacteria population when $t=5$ hours. Its units are bacteria per hour.
(b) With unlimited space and nutrients, f' should increase as t increases; so $f'(5)<f'(10)$. If the supply of nutrients is limited, the growth rate slows down at some point in time, and the opposite may be true.

29. **(a)** $f'(v)$ is the rate at which the fuel consumption is changing with respect to the speed. Its units are $(gal\ /\ h)\ /\ (mi\ /\ h)$.
(b) The fuel consumption is decreasing by $0.05(gal\ /\ \mathrm{h})\ /\ (mi\ /\ h)$ as the car's speed reaches 20 $mi\ /\ h$. So if you increase your speed to 21 $mi\ /\ h$, you could expect to decrease your fuel consumption by about $0.05(gal\ /\ \mathrm{h})\ /\ (mi\ /\ h)$.

30. **(a)** $f'(8)$ is the rate of change of the quantity of coffee sold with respect to the price per pound when the price is \$8 per pound. The units for $f'(8)$ are pounds /(dollars / pound) .
(b) $f'(8)$ is negative since the quantity of coffee sold will decrease as the price charged for it increases. People are generally less willing to buy a product when its price increases.

31. $T'(10)$ is the rate at which the temperature is changing at 10:00 A.M. To estimate the value of $T'(10)$, we will average the difference quotients obtained using the times $t=8$ and $t=12$. Let $A=\frac{T(8)-T(10)}{8-10}=\frac{72-81}{-2}=4.5$ and $B=\frac{T(12)-T(10)}{12-10}=\frac{88-81}{2}=3.5$. Then
$T'(10)=\lim_{t\to 10}\frac{T(t)-T(10)}{t-10}\approx\frac{A+B}{2}=\frac{4.5+3.5}{2}=4^{\circ}F/h$.

32. **For 1910:** We will average the difference quotients obtained using the years 1900 and 1920.

Let $A=\dfrac{E(1900)-E(1910)}{1900-1910}=\dfrac{48.3-51.1}{-10}=0.28$ and
$B=\dfrac{E(1920)-E(1910)}{1920-1910}=\dfrac{55.2-51.1}{10}=0.41$. Then
$E'(1910)=\lim\limits_{t\to 1910}\dfrac{E(t)-E(1910)}{t-1910}\approx\dfrac{A+B}{2}=0.345$. This means that life expectancy at birth was increasing at about 0.345 year / year in 1910.
For 1950: Using data for 1940 and 1960 in a similar fashion, we obtain
$E'(1950)\approx[0.31+0.10]/2=0.205$. So life expectancy at birth was increasing at about 0.205 year / year in 1950.

33. **(a)** $S'(T)$ is the rate at which the oxygen solubility changes with respect to the water temperature. Its units are (mg / L)/° C .
(b) For $T=16°$ C , it appears that the tangent line to the curve goes through the points (0,14) and (32,6) . So $S'(16)\approx\dfrac{6-14}{32-0}=-\dfrac{8}{32}=-0.25$(mg / L)/° C . This means that as the temperature increases past 16° C , the oxygen solubility is decreasing at a rate of 0.25(mg / L)/° C .

34. **(a)** $S'(T)$ is the rate of change of the maximum sustainable speed of Coho salmon with respect to the temperature. Its units are (cm / s)/° C .
(b) For $T=15°$ C, it appears the tangent line to the curve goes through the points (10,25) and (20,32) . So $S'(15)\approx\dfrac{32-25}{20-10}=0.7$(cm / s)/° C . This tells us that at $T=15°$ C, the maximum sustainable speed of Coho salmon is changing at a rate of 0.7 (cm / s)/° C . In a similar fashion for $T=25°$ C, we can use the points (20,35) and (25,25) to obtain $S'(25)\approx\dfrac{25-35}{25-20}=-2$ (cm / s) /° C . As it gets warmer than 20° C, the maximum sustainable speed decreases rapidly.

35. Since $f(x)=x\sin(1/x)$ when $x\neq 0$ and $f(0)=0$, we have
$f'(0)=\lim\limits_{h\to 0}\dfrac{f(0+h)-f(0)}{h}=\lim\limits_{h\to 0}\dfrac{h\sin(1/h)-0}{h}=\lim\limits_{h\to 0}(\sin(1/h))$. This limit does not exist since $\sin(1/h)$ takes the values -1 and 1 on any interval containing 0 . (Compare with Example 4 in Section 2.2.)

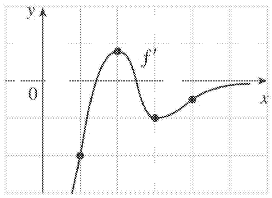
36. Since $f(x)=x^2\sin(1/x)$ when $x\neq 0$ and $f(0)=0$, we have
$f'(0)=\lim\limits_{h\to 0}\dfrac{f(0+h)-f(0)}{h}=\lim\limits_{h\to 0}\dfrac{h^2\sin(1/h)-0}{h}=\lim\limits_{h\to 0}(h\sin(1/h))$. Since $-1\leq\sin\dfrac{1}{h}\leq 1$, we have

$-|h| \leq |h| \, sin \, \frac{1}{h} \leq |h| \Rightarrow -|h| \leq hsin \, \frac{1}{h} \leq |h|$. Because $\lim_{h \to 0}(-|h|)=0$ and $\lim_{h \to 0}|h|=0$, we know that $\lim_{h \to 0}(h \, sin \, \frac{1}{h})=0$ by the Squeeze Theorem. Thus, $f^{/}(0)=0$.

1. *Note:* Your answers may vary depending on your estimates. By estimating the slopes of tangent lines on the graph of f , it appears that

(a) $f'(1)\approx-2$ (b) $f'(2)\approx 0.8$

(c) $f'(3)\approx-1$ (d) $f'(4)\approx-0.5$

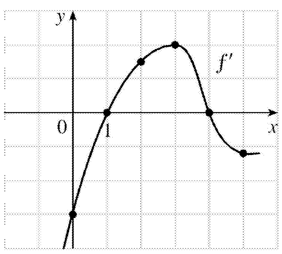


2. *Note:* Your answers may vary depending on your estimates. By estimating the slopes of tangent lines on the graph of f , it appears that

(a) $f'(0)\approx-3$ (b) $f'(1)\approx 0$

(c) $f'(2)\approx 1.5$ (d) $f'(3)\approx 2$

(e) $f'(4)\approx 0$ (f) $f'(5)\approx-1.2$



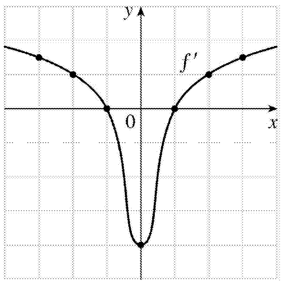
3. It appears that f is an odd function, so f' will be an even function ??? that is, $f'(-a)=f'(a)$

(a) $f'(-3)\approx 1.5$ (b) $f'(-2)\approx 1$

(c) $f'(-1)\approx-0$ (d) $f'(0)\approx-4$

(e) $f'(1)\approx-0$ (f) $f'(2)\approx-1$

(g) $f'(3)\approx 1.5$

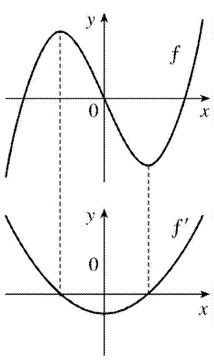


4. **(a)** $(a)'$ = II, since from left to right, the slopes of the tangents to graph (a) start out negative, become 0 , then positive, then 0 , then negative again. The actual function values in graph II follow the same pattern.

(b) $(b)'$ = IV, since from left to right, the slopes of the tangents to graph (b) start out at a fixed positive quantity, then suddenly become negative, then positive again. The discontinuities in graph IV indicate sudden changes in the slopes of the tangents.

(c) $(c)'$ = I, since the slopes of the tangents to graph (c) are negative for $x<0$ and positive for $x>0$, as are the function values of graph I.

(d) $(d)'$ = III, since from left to right, the slopes of the tangents to graph (d) are positive, then 0 , then negative, then 0 , then positive, then 0 , then negative again, and the function values in graph III follow the same pattern.



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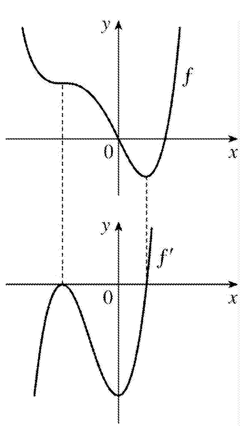

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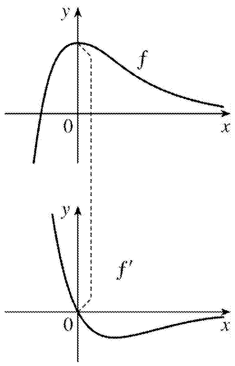
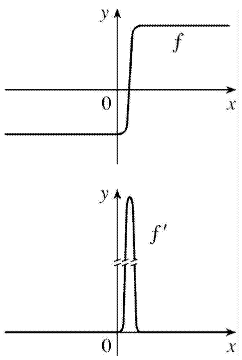
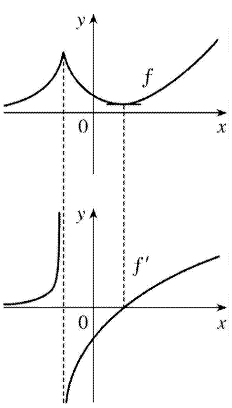
7. 

8. 

9. 

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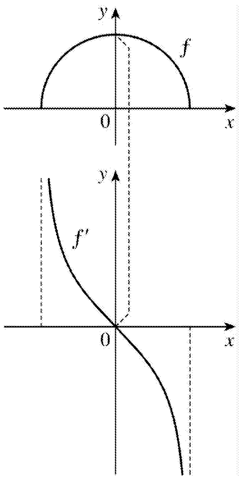

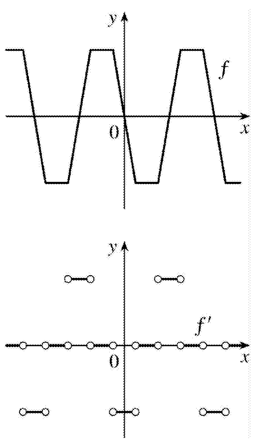

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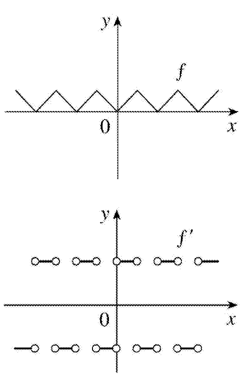
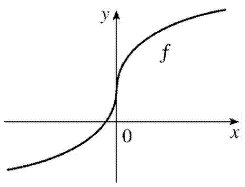
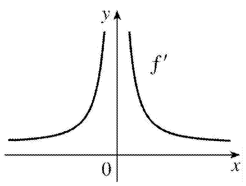


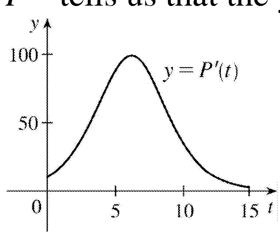
12.

13.

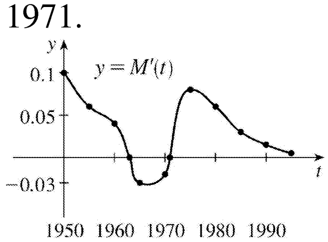




14. The slopes of the tangent lines on the graph of $y=P(t)$ are always positive, so the y –values of $y=P'(t)$ are always positive. These values start out relatively small and keep increasing, reaching a maximum at about $t=6$. Then the y –values of $y=P'(t)$ decrease and get close to zero. The graph of P' tells us that the yeast culture grows most rapidly after 6 hours and then the growth rate declines.

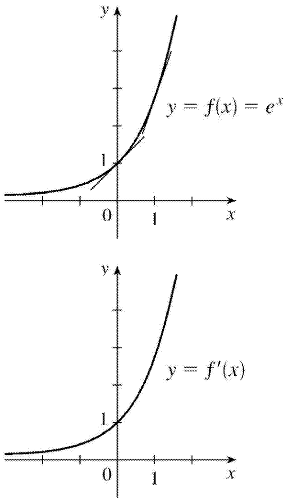


15. It appears that there are horizontal tangents on the graph of M for $t=1963$ and $t=1971$. Thus, there are zeros for those values of t on the graph of M' . The derivative is negative for the years 1963 to 1971.



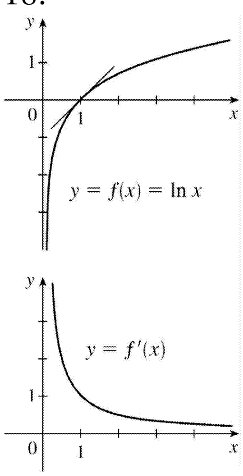
16. See Figure 1 in Section 3.4.

17.



The slope at 0 appears to be 1 and the slope at 1 appears to be 2.7 . As x decreases, the slope gets closer to 0 . Since the graphs are so similar, we might guess that $f'(x)=e^x$.

18.



As x increases toward 1 , $f'(x)$ decreases from very large numbers to 1 . As x becomes large, $f'(x)$ gets closer to 0 . As a guess, $f'(x)=1/x^2$ or $f'(x)=1/x$ make sense.

19. **(a)** By zooming in, we estimate that $f'(0)=0$, $f'\left(\frac{1}{2}\right)=1$, $f'(1)=2$, and $f'(2)=4$.

(b) By symmetry, $f'(-x)=-f'(x)$. So $f'\left(-\frac{1}{2}\right)=-1$,

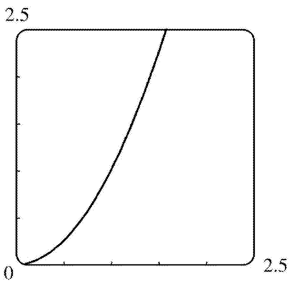
$f'(-1)=-2$, and $f'(-2)=-4$.

(c) It appears that $f'(x)$ is twice the value of x , so we guess that $f'(x)=2x$.

(d)

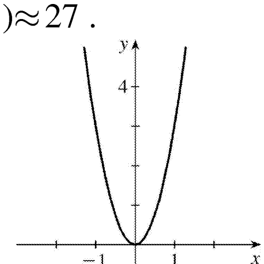
$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{(x+h)^2-x^2}{h}$$

$$=\lim_{h\to 0}\frac{(x^2+2hx+h^2)-x^2}{h}=\lim_{h\to 0}\frac{2hx+h^2}{h}=\lim_{h\to 0}\frac{h(2x+h)}{h}=\lim_{h\to 0}(2x+h)=2x$$



20. **(a)** By zooming in, we estimate that $f'(0)=0$, $f'\left(\frac{1}{2}\right)\approx 0.75$, $f'(1)\approx 3$, $f'(2)\approx 12$, and $f'(3)\approx 27$.

(b) By symmetry, $f'(-x)=f'(x)$. So $f'\left(-\frac{1}{2}\right)\approx 0.75$, $f'(-1)\approx 3$, $f'(-2)\approx 12$, and $f'(-3)\approx 27$.

(c) 

(d) Since $f'(0)=0$, it appears that f' may have the form $f'(x)=ax^2$. Using $f'(1)=3$, we have $a=3$, so $f'(x)=3x^2$.

(e) $$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{(x+h)^3-x^3}{h}=\lim_{h\to 0}\frac{(x^3+3x^2h+3xh^2+h^3)-x^3}{h}$$

$$=\lim_{h\to 0}\frac{3x^2h+3xh^2+h^3}{h}=\lim_{h\to 0}\frac{h(3x^2+3xh+h^2)}{h}=\lim_{h\to 0}(3x^2+3xh+h^2)=3x^2$$

21. $$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{37-37}{h}=\lim_{h\to 0}\frac{0}{h}=\lim_{h\to 0}0=0$$

Domain of f= domain of $f'=R$.

22.

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{[12+7(x+h)]-(12+7x)}{h}$$
$$=\lim_{h\to 0}\frac{12+7x+7h-12-7x}{h}=\lim_{h\to 0}\frac{7h}{h}=\lim_{h\to 0}7=7$$

Domain of f= domain of $f'=R$.

23.

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{[1-3(x+h)^2]-(1-3x^2)}{h}$$
$$=\lim_{h\to 0}\frac{[1-3(x^2+2xh+h^2)]-(1-3x^2)}{h}=\lim_{h\to 0}\frac{1-3x^2-6xh-3h^2-1+3x^2}{h}$$
$$=\lim_{h\to 0}\frac{-6xh-3h^2}{h}=\lim_{h\to 0}\frac{h(-6x-3h)}{h}=\lim_{h\to 0}(-6x-3h)=-6x$$

Domain of f= domain of $f'=R$.

24.

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{[5(x+h)^2+3(x+h)-2]-(5x^2+3x-2)}{h}$$
$$=\lim_{h\to 0}\frac{5x^2+10xh+5h^2+3x+3h-2-5x^2-3x+2}{h}=\lim_{h\to 0}\frac{10xh+5h^2+3h}{h}$$
$$=\lim_{h\to 0}\frac{h(10x+5h+3)}{h}=\lim_{h\to 0}(10x+5h+3)=10x+3$$

Domain of f= domain of $f'=R$.

25.

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{[(x+h)^3-3(x+h)+5]-(x^3-3x+5)}{h}$$

$$=\lim_{h\to 0}\frac{(x^3+3x^2h+3xh^2+h^3-3x-3h+5)-(x^3-3x+5)}{h}$$

$$=\lim_{h\to 0}\frac{3x^2h+3xh^2+h^3-3h}{h}=\lim_{h\to 0}\frac{h(3x^2+3xh+h^2-3)}{h}$$

$$=\lim_{h\to 0}(3x^2+3xh+h^2-3)=3x^2-3$$

Domain of f= domain of $f'=R$.

26.

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{(x+h+\sqrt{x+h})-(x+\sqrt{x})}{h}$$

$$=\lim_{h\to 0}\left(\frac{h}{h}+\frac{\sqrt{x+h}-\sqrt{x}}{h}\cdot\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}\right)=\lim_{h\to 0}\left[1+\frac{(x+h)-x}{h(\sqrt{x+h}+\sqrt{x})}\right]$$

$$=\lim_{h\to 0}\left(1+\frac{1}{\sqrt{x+h}+\sqrt{x}}\right)=1+\frac{1}{\sqrt{x}+\sqrt{x}}=1+\frac{1}{2\sqrt{x}}$$

Domain of $f=[0,\infty)$, domain of $f'=(0,\infty)$.

27.

$$g'(x)=\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}=\lim_{h\to 0}\frac{\sqrt{1+2(x+h)}-\sqrt{1+2x}}{h}\left[\frac{\sqrt{1+2(x+h)}+\sqrt{1+2x}}{\sqrt{1+2(x+h)}+\sqrt{1+2x}}\right]$$

$$=\lim_{h\to 0}\frac{(1+2x+2h)-(1+2x)}{h[\sqrt{1+2(x+h)}+\sqrt{1+2x}]}=\lim_{h\to 0}\frac{2}{\sqrt{1+2x+2h}+\sqrt{1+2x}}=\frac{2}{2\sqrt{1+2x}}=\frac{1}{\sqrt{1+2x}}$$

Domain of $g=\left[-\frac{1}{2},\infty\right)$, domain of $g'=\left(-\frac{1}{2},\infty\right)$.

28.

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\dfrac{3+(x+h)}{1-3(x+h)}-\dfrac{3+x}{1-3x}}{h}$$

$$=\lim_{h\to 0}\frac{(3+x+h)(1-3x)-(3+x)(1-3x-3h)}{h(1-3x-3h)(1-3x)}$$

$$=\lim_{h\to 0}\frac{(3-9x+x-3x^2+h-3hx)-(3-9x-9h+x-3x^2-3hx)}{h(1-3x-3h)(1-3x)}$$

$$=\lim_{h\to 0}\frac{10h}{h(1-3x-3h)(1-3x)}=\lim_{h\to 0}\frac{10}{(1-3x-3h)(1-3x)}=\frac{10}{(1-3x)^2}$$

Domain of f= domain of $f'=\left(-\infty,\frac{1}{3}\right)\cup\left(\frac{1}{3},\infty\right)$.

29.

$$G'(t)=\lim_{h\to 0}\frac{G(t+h)-G(t)}{h}=\lim_{h\to 0}\frac{\frac{4(t+h)}{(t+h)+1}-\frac{4t}{t+1}}{h}=\lim_{h\to 0}\frac{\frac{4(t+h)(t+1)-4t(t+h+1)}{(t+h+1)(t+1)}}{h}$$

$$=\lim_{h\to 0}\frac{(4t^2+4ht+4t+4h)-(4t^2+4ht+4t)}{h(t+h+1)(t+1)}$$

$$=\lim_{h\to 0}\frac{4h}{h(t+h+1)(t+1)}=\lim_{h\to 0}\frac{4}{(t+h+1)(t+1)}=\frac{4}{(t+1)^2}$$

Domain of G= domain of $G'=(-\infty,-1)\cup(-1,\infty)$.

30.

$$g'(x)=\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}=\lim_{h\to 0}\frac{\frac{1}{(x+h)^2}-\frac{1}{x^2}}{h}=\lim_{h\to 0}\frac{x^2-(x+h)^2}{h(x+h)^2x^2}$$

$$=\lim_{h\to 0}\frac{x^2-(x^2+2xh+h^2)}{h}=\lim_{h\to 0}\frac{-2xh-h^2}{h(x+h)^2x^2}=\lim_{h\to 0}\frac{-2x-h}{(x+h)^2x^2}=\frac{-2x}{x^4}=-2x^{-3}$$

Domain of g= domain of $g'=\{x\,|\,x\neq 0\}$.

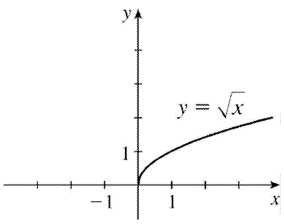
31.

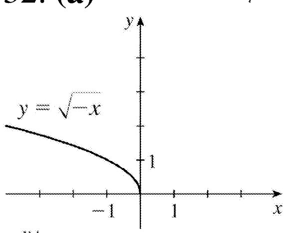
$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{(x+h)^4-x^4}{h}=\lim_{h\to 0}\frac{(x^4+4x^3h+6x^2h^2+4xh^3+h^4)-x^4}{h}$$

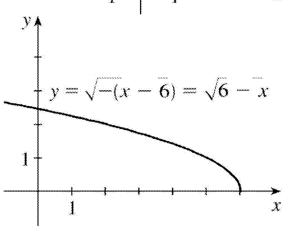
$$=\lim_{h\to 0}\frac{4x^3h+6x^2h^2+4xh^3+h^4}{h}=\lim_{h\to 0}(4x^3+6x^2h+4xh^2+h^3)=4x^3$$

Domain of f= domain of $f'=R$.

32. **(a)**







(b) Note that the third graph in part (a) has small negative values for its slope, f' ; but as $x\to 6^-$, $f'\to -\infty$.
See the graph in part (d).
(c)

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\sqrt{6-(x+h)}-\sqrt{6-x}}{h}\left[\frac{\sqrt{6-(x+h)}+\sqrt{6-x}}{\sqrt{6-(x+h)}+\sqrt{6-x}}\right]$$

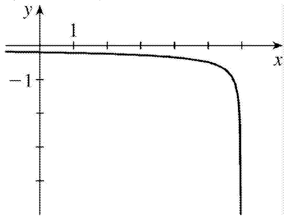
$$=\lim_{h\to 0}\frac{[6-(x+h)]-(6-x)}{h[\sqrt{6-(x+h)}+\sqrt{6-x}]}=\lim_{h\to 0}\frac{-h}{h(\sqrt{6-x-h}+\sqrt{6-x})}$$

$$=\lim_{h\to 0}\frac{-1}{\sqrt{6-x-h}+\sqrt{6-x}}=\frac{-1}{2\sqrt{6-x}}$$

Domain of $f=(-\infty,6]$, domain of

$f' = (-\infty, 6)$.

(d)



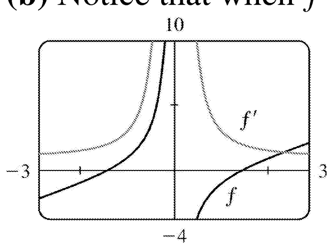
33. **(a)**

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\left[x+h-\left(\frac{2}{x+h}\right)\right]-\left[x-\left(\frac{2}{x}\right)\right]}{h}$$

$$= \lim_{h\to 0}\left[\frac{h-\frac{2}{(x+h)}+\frac{2}{x}}{h}\right] = \lim_{h\to 0}\left[1+\frac{-2x+2(x+h)}{hx(x+h)}\right] = \lim_{h\to 0}\left[1+\frac{2h}{hx(x+h)}\right]$$

$$= \lim_{h\to 0}\left[1+\frac{2}{x(x+h)}\right] = 1+\frac{2}{x^2}$$

(b) Notice that when f has steep tangent lines, $f'(x)$ is very large. When f is flatter, $f'(x)$ is smaller.

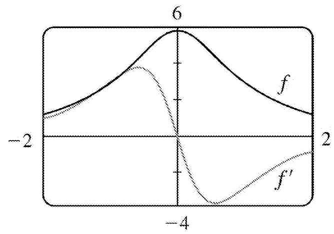


34. **(a)**

$$f'(t) = \lim_{h\to 0}\frac{f(t+h)-f(t)}{h} = \lim_{h\to 0}\frac{\frac{6}{1+(t+h)^2}-\frac{6}{1+t^2}}{h} = \lim_{h\to 0}\frac{6+6t^2-6-6(t+h)^2}{h[1+(t+h)^2](1+t^2)}$$

$$= \lim_{h\to 0}\frac{-12th-6h^2}{h[1+(t+h)^2](1+t^2)} = \lim_{h\to 0}\frac{-12t-6h}{[1+(t+h)^2](1+t^2)} = \frac{-12t}{(1+t^2)^2}$$

(b) Notice that f has a horizontal tangent when $t=0$. This corresponds to $f'(0)=0$. f' is positive when f is increasing and negative when f is decreasing.



35. **(a)** $U'(t)$ is the rate at which the unemployment rate is changing with respect to time. Its units are percent per year.

(b) To find $U'(t)$, we use $\lim_{h\to 0}\frac{U(t+h)-U(t)}{h} \approx \frac{U(t+h)-U(t)}{h}$ for small values of h.

For 1991: $U'(1991)=\frac{U(1992)-U(1991)}{1992-1991}=\frac{7.5-6.8}{1}=0.70$

For 1992: We estimate $U'(1992)$ by using $h=-1$ and $h=1$, and then average the two results to obtain a final estimate.

$h=-1 \Rightarrow U'(1992)\approx\frac{U(1991)-U(1992)}{1991-1992}=\frac{6.8-7.5}{-1}=0.70$;

$h=1 \Rightarrow U'(1992)\approx\frac{U(1993)-U(1992)}{1993-1992}=\frac{6.9-7.5}{1}=-0.60$.

So we estimate that $U'(1992)\approx\frac{1}{2}[0.70+(-0.60)]=0.05$.

t	1991	1992	1993	1994	1995	1996	1997	1998	1999	2000
$U'(t)$	0.70	0.05	–0.70	–0.65	–0.35	–0.35	–0.45	–0.35	–0.25	–0.20

36. **(a)** $P'(t)$ is the rate at which the percentage of Americans under the age of 18 is changing with respect to time. Its units are percent per year (%/yr).

(b) To find $P'(t)$, we use $\lim_{h\to 0}\frac{P(t+h)-P(t)}{h} \approx \frac{P(t+h)-P(t)}{h}$ for small values of h.

For 1950: $P'(1950)=\frac{P(1960)-P(1950)}{1960-1950}=\frac{35.7-31.1}{10}=0.46$

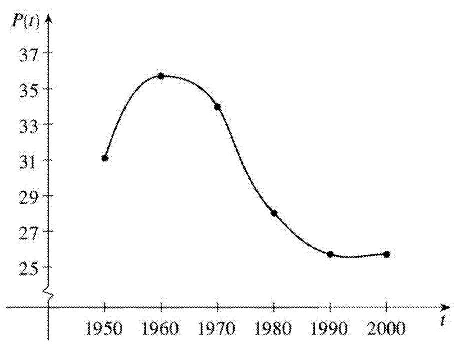
For 1960: We estimate $P'(1960)$ by using $h=-10$ and $h=10$, and then average the two results to obtain a final estimate.

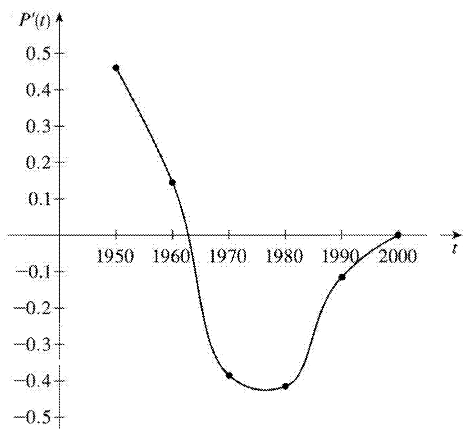
$h=-10 \Rightarrow P'(1960)\approx\frac{P(1950)-P(1960)}{1950-1960}=\frac{31.1-35.7}{-10}=0.46$

$h=10 \Rightarrow P'(1960) \approx \frac{P(1970)-P(1960)}{1970-1960} = \frac{34.0-35.7}{10} = -0.17$

So we estimate that $P'(1960) \approx \frac{1}{2}[0.46+(-0.17)]=0.145$.

t	1950	1960	1970	1980	1990	2000
$P'(t)$	0.460	0.145	–0.385	–0.415	–0.115	0.000





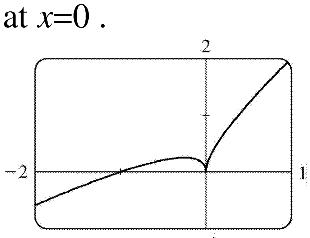
(c)

(d) We could get more accurate values for $P'(t)$ by obtaining data for the mid–decade years 1955, 1965, 1975, 1985, and 1995.

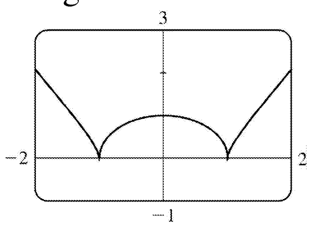
37. f is not differentiable at $x=-1$ or at $x=11$ because the graph has vertical tangents at those points; at $x=4$, because there is a discontinuity there; and at $x=8$, because the graph has a corner there.

38. **(a)** g is discontinuous at $x=-2$ (a removable discontinuity), at $x=0$ (g is not defined there), and at $x=5$ (a jump discontinuity).
(b) g is not differentiable at the points mentioned in part (a) (by Theorem 4), nor is it differentiable at $x=-1$ (corner), $x=2$ (vertical tangent), or $x=4$ (vertical tangent).

39. As we zoom in toward $(-1,0)$, the curve appears more and more like a straight line, so $f(x)=x+\sqrt{|x|}$ is differentiable at $x=-1$. But no matter how much we zoom in toward the origin, the curve doesn't straighten out—we can't eliminate the sharp point (a cusp). So f is not differentiable at $x=0$.



40. As we zoom in toward (0,1) , the curve appears more and more like a straight line, so f is differentiable at $x=0$. But no matter how much we zoom in toward (1,0) or (−1,0) , the curve doesn't straighten out—we can't eliminate the sharp point (a cusp). So f is not differentiable at $x=\pm 1$.



41. **(a)** Note that we have factored $x-a$ as the difference of two cubes in the third step.

$$f'(a)=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}=\lim_{x\to a}\frac{x^{1/3}-a^{1/3}}{x-a}=\lim_{x\to a}\frac{x^{1/3}-a^{1/3}}{(x^{1/3}-a^{1/3})(x^{2/3}+x^{1/3}a^{1/3}+a^{2/3})}$$

$$=\lim_{x\to a}\frac{1}{x^{2/3}+x^{1/3}a^{1/3}+a^{2/3}}=\frac{1}{3a^{2/3}}\text{ or }\frac{1}{3}a^{-2/3}$$

(b) $f'(0)=\lim\limits_{h\to 0}\dfrac{f(0+h)-f(0)}{h}=\lim\limits_{h\to 0}\dfrac{\sqrt[3]{h}-0}{h}=\lim\limits_{h\to 0}\dfrac{1}{h^{2/3}}$. This function increases without bound, so the limit does not exist, and therefore $f'(0)$ does not exist.

(c) $\lim\limits_{x\to 0}\left|f'(x)\right|=\lim\limits_{x\to 0}\dfrac{1}{3x^{2/3}}=\infty$ and f is continuous at $x=0$ (root function), so f has a vertical tangent at $x=0$.

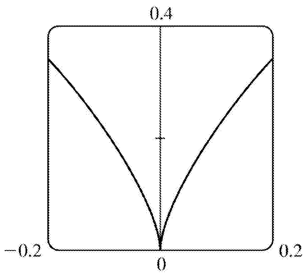
42. **(a)** $g'(0)=\lim\limits_{x\to 0}\dfrac{g(x)-g(0)}{x-0}=\lim\limits_{x\to 0}\dfrac{x^{2/3}-0}{x}=\lim\limits_{x\to 0}\dfrac{1}{x^{1/3}}$, which does not exist.

(b)

$$g'(a)=\lim_{x\to a}\frac{g(x)-g(a)}{x-a}=\lim_{x\to a}\frac{x^{2/3}-a^{2/3}}{x-a}=\lim_{x\to a}\frac{(x^{1/3}-a^{1/3})(x^{1/3}+a^{1/3})}{(x^{1/3}-a^{1/3})(x^{2/3}+x^{1/3}a^{1/3}+a^{2/3})}$$

$$=\lim_{x\to a}\frac{x^{1/3}+a^{1/3}}{x^{2/3}+x^{1/3}a^{1/3}+a^{2/3}}=\frac{2a^{1/3}}{3a^{2/3}}=\frac{2}{3a^{1/3}}\text{ or }\frac{2}{3}a^{-1/3}$$

(c) $g(x)=x^{2/3}$ is continuous at $x=0$ and $\lim\limits_{x\to 0}\left|g'(x)\right|=\lim\limits_{x\to 0}\dfrac{2}{3|x|^{1/3}}=\infty$. This shows that g has a vertical tangent line at $x=0$.

(d) 

43. $f(x)=|x-6| = \begin{cases} -(x-6) & \text{if } x<6 \\ x-6 & \text{if } x\geq 6 \end{cases} = \begin{cases} 6-x & \text{if } x<6 \\ x-6 & \text{if } x\geq 6 \end{cases}$

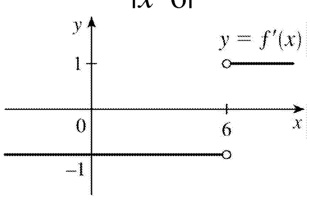
$$\lim_{x\to 6^+} \frac{f(x)-f(6)}{x-6} = \lim_{x\to 6^+} \frac{|x-6|-0}{x-6} = \lim_{x\to 6^+} \frac{x-6}{x-6} = \lim_{x\to 6^+} 1=1 .$$

But

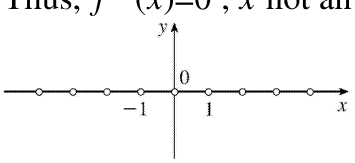
$$\lim_{x\to 6^-} \frac{f(x)-f(6)}{x-6} = \lim_{x\to 6^-} \frac{|x-6|-0}{x-6} = \lim_{x\to 6^-} \frac{6-x}{x-6}$$

$$= \lim_{x\to 6^-} (-1)=-1$$

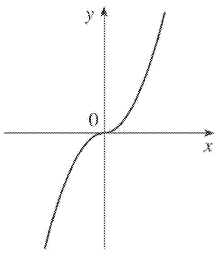
So, $f'(6)$ does not exist. However, $f'(x)=\begin{cases} -1 & \text{if } x<6 \\ 1 & \text{if } x\geq 6 \end{cases}$ Another way of writing the answer is $f'(x) = \dfrac{x-6}{|x-6|}$.



44. $f(x)=[[x]]$ is not continuous at any integer n , so f is not differentiable at n by the contrapositive of Theorem 4. If a is not an integer, then f is constant on an open interval containing a , so $f'(a)=0$. Thus, $f'(x)=0$, x not an integer.



45. **(a)** $f(x)=x|x|=\begin{cases} x^2 & \text{if } x\geq 0 \\ -x^2 & \text{if } x<0 \end{cases}$



(b) Since $f(x)=x^2$ for $x\geq 0$, we have $f'(x)=2x$ for $x>0$. [See Exercise 2.9.19(d).] Similarly, since $f(x)=-x^2$ for $x<0$, we have $f'(x)=-2x$ for $x<0$. At $x=0$, we have

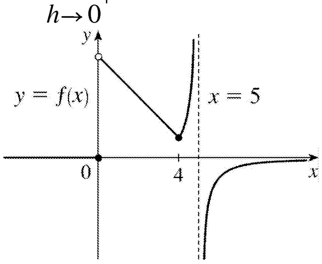
$$f'(0)=\lim_{x\to 0}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0}\frac{x|x|}{x}=\lim_{x\to 0}|x|=0.$$

So f is differentiable at 0 . Thus, f is differentiable for all x .

(c) From part (b), we have $f'(x)=\left\{\begin{array}{ll} 2x & \text{if } x\geq 0 \\ -2x & \text{if } x<0 \end{array}\right\}=2|x|$.

46. **(a)** $f'_-(4)=\lim_{h\to 0^-}\dfrac{f(4+h)-f(4)}{h}=\lim_{h\to 0^-}\dfrac{5-(4+h)-1}{h}=\lim_{h\to 0^-}\dfrac{-h}{h}=-1$ and

$$f'_+(4)=\lim_{h\to 0^+}\frac{f(4+h)-f(4)}{h}=\lim_{h\to 0^+}\frac{\dfrac{1}{5-(4+h)}-1}{h}=\lim_{h\to 0^+}\frac{1-(1-h)}{h(1-h)}=\lim_{h\to 0^+}\frac{1}{1-h}=1\ .$$

(b) 

(c) $f(x)=\left\{\begin{array}{ll} 0 & \text{if } x\leq 0 \\ 5-x & \text{if } 0<x<4 \\ 1/(5-x) & \text{if } x\geq 4 \end{array}\right.$ These expressions show that f is continuous on the intervals $(-\infty,0)$, $(0,4)$, $(4,5)$ and $(5,\infty)$. Since $\lim_{x\to 0^+}f(x)=\lim_{x\to 0^+}(5-x)=5\neq 0=\lim_{x\to 0^-}f(x)$, $\lim_{x\to 0}f(x)$ does not exist, so f is discontinuous (and therefore not differentiable) at 0 .

At 4 we have $\lim_{x\to 4^-}f(x)=\lim_{x\to 4^-}(5-x)=1$ and $\lim_{x\to 4^+}f(x)=\lim_{x\to 4^+}\dfrac{1}{5-x}=1$, so $\lim_{x\to 4}f(x)=1=f(4)$ and f is continuous at 4 . Since $f(5)$ is not defined, f is discontinuous at 5 .

(d) From (a), f is not differentiable at 4 since $f'_-(4)\neq f'_+(4)$, and from (c), f is not differentiable at 0 or 5 .

47. **(a)** If f is even, then

$$f'(-x) = \lim_{h\to 0}\frac{f(-x+h)-f(-x)}{h} = \lim_{h\to 0}\frac{f[-(x-h)]-f(-x)}{h} = \lim_{h\to 0}\frac{f(x-h)-f(x)}{h}$$

$$= -\lim_{h\to 0}\frac{f(x-h)-f(x)}{-h}\ [\text{let}\,\Delta x=-h] = -\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x} = -f'(x)$$

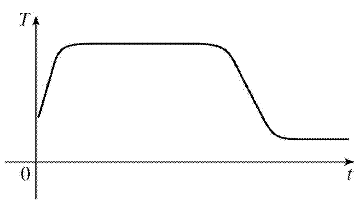
Therefore, f' is odd.

(b) If f is odd, then

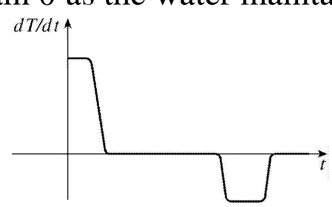
$$f'(-x) = \lim_{h\to 0}\frac{f(-x+h)-f(-x)}{h} = \lim_{h\to 0}\frac{f[-(x-h)]-f(-x)}{h} = \lim_{h\to 0}\frac{-f(x-h)+f(x)}{h}$$

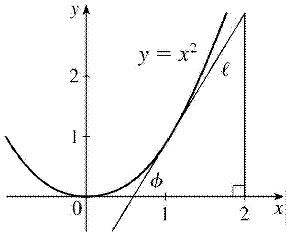
$$= \lim_{h\to 0}\frac{f(x-h)-f(x)}{-h}\ [\text{let}\,\Delta x=-h] = \lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x} = f'(x)$$

Therefore, f' is even.

48. **(a)** 

(b) The initial temperature of the water is close to room temperature because of the water that was in the pipes. When the water from the hot water tank starts coming out, dT/dt is large and positive as T increases to the temperature of the water in the tank. In the next phase, $dT/dt=0$ as the water comes out at a constant, high temperature. After some time, dT/dt becomes small and negative as the contents of the hot water tank are exhausted. Finally, when the hot water has run out, dT/dt is once again 0 as the water maintains its (cold) temperature.

(c) 

49. 

In the right triangle in the diagram, let Δy be the side opposite angle ϕ and Δx the side adjacent angle ϕ. Then the slope of the tangent line ℓ is $m=\Delta y/\Delta x=\tan\phi$. Note that

$0<\phi<\frac{\pi}{2}$. We know (see Exercise 19) that the derivative of $f(x)=x^2$ is $f'(x)=2x$. So the slope of the tangent to the curve at the point (1,1) is 2 . Thus, ϕ is the angle between 0 and $\frac{\pi}{2}$ whose tangent is 2 ; that is, $\phi=\tan^{-1}2\approx 63^{\circ}$.

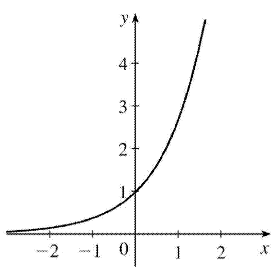
1. **(a)** e is the number such that $\lim_{h\to 0} \frac{e^h - 1}{h} = 1$.

(b)

x	$(2.7^x - 1)/x$
-0.001	0.9928
-0.0001	0.9932
0.001	0.9937
0.0001	0.9933

x	$(2.8^x - 1)/x$
-0.001	1.0291
-0.0001	1.0296
0.001	1.0301
0.0001	1.0297

From the tables (to two decimal places), $\lim_{h\to 0} \frac{2.7^h - 1}{h} = 0.99$ and $\lim_{h\to 0} \frac{2.8^h - 1}{h} = 1.03$. Since $0.99 < 1 < 1.03$, $2.7 < e < 2.8$.

2. **(a)** 

The function value at $x=0$ is 1 and the slope at $x=0$ is 1 .

(b) $f(x)=e^x$ is an exponential function and $g(x)=x^e$ is a power function. $\frac{d}{dx}(e^x)=e^x$ and $\frac{d}{dx}(x^e)=ex^{e-1}$.

(c) $f(x)=e^x$ grows more rapidly than $g(x)=x^e$ when x is large.

3. $f(x)=186.5$ is a constant function, so its derivative is 0 , that is, $f'(x)=0$.

4. $f(x)=\sqrt{30}$ is a constant function, so its derivative is 0 , that is, $f'(x)=0$.

5. $f(x)=5x-1 \Rightarrow f'(x)=5-0=5$

6. $F(x)=-4x^{10} \Rightarrow F'(x)=-4(10x^{10-1})=-40x^9$

7. $f(x)=x^2+3x-4 \Rightarrow f'(x)=2x^{2-1}+3-0=2x+3$

8. $g(x)=5x^8-2x^5+6 \Rightarrow g'(x)=5(8x^{8-1})-2(5x^{5-1})+0=40x^7-10x^4$

9. $g(x)=5x^8-2x^5+6 \Rightarrow g'(x)=5(8x^{8-1})-2(5x^{5-1})+0=40x^7-10x^4$

10. $f(t)=\frac{1}{2}t^6-3t^4+t \Rightarrow f'(t)=\frac{1}{2}(6t^5)-3(4t^3)+1=3t^5-12t^3+1$

11. $y=x^{-2/5} \Rightarrow y'=-\frac{2}{5}x^{(-2/5)-1}=-\frac{2}{5}x^{-7/5}=-\frac{2}{5x^{7/5}}$

12. $y=5e^x+3 \Rightarrow y'=5(e^x)+0=5e^x$

13. $V(r)=\frac{4}{3}\pi r^3 \Rightarrow V'(r)=\frac{4}{3}\pi(3r^2)=4\pi r^2$

14. $R(t)=5t^{-3/5} \Rightarrow R'(t)=5\left[-\frac{3}{5}t^{(-3/5)-1}\right]=-3t^{-8/5}$

15. $Y(t)=6t^{-9} \Rightarrow Y'(t)=6(-9)t^{-10}=-54t^{-10}$

16. $R(x)=\frac{\sqrt{10}}{x^7}=\sqrt{10}\,x^{-7} \Rightarrow R'(x)=-7\sqrt{10}\,x^{-8}=-\frac{7\sqrt{10}}{x^8}$

17. $G(x)=\sqrt{x}-2e^x=x^{1/2}-2e^x \Rightarrow G'(x)=\frac{1}{2}x^{-1/2}-2e^x=\frac{1}{2\sqrt{x}}-2e^x$

18. $y=\sqrt[3]{x}=x^{1/3} \Rightarrow y'=\frac{1}{3}x^{-2/3}=\frac{1}{3x^{2/3}}$

19. $F(x)=\left(\frac{1}{2}x\right)^5=\left(\frac{1}{2}\right)^5x^5=\frac{1}{32}x^5 \Rightarrow F'(x)=\frac{1}{32}(5x^4)=\frac{5}{32}x^4$

20. $f(t)=\sqrt{t}-\frac{1}{\sqrt{t}}=t^{1/2}-t^{-1/2} \Rightarrow f'(t)=\frac{1}{2}t^{-1/2}-\left(-\frac{1}{2}t^{-3/2}\right)=\frac{1}{2\sqrt{t}}+\frac{1}{2t\sqrt{t}}$

21. $g(x)=x^2+\frac{1}{x^2}=x^2+x^{-2} \Rightarrow g'(x)=2x+(-2)x^{-3}=2x-\frac{2}{x^3}$

22. $y=\sqrt{x}\,(x-1)=x^{3/2}-x^{1/2}\Rightarrow y'=\frac{3}{2}x^{1/2}-\frac{1}{2}x^{-1/2}=\frac{1}{2}x^{-1/2}(3x-1)$ [factor out $\frac{1}{2}x^{-1/2}$] or $y'=\frac{3x-1}{2\sqrt{x}}$.

23. $y=\frac{x^2+4x+3}{\sqrt{x}}=x^{3/2}+4x^{1/2}+3x^{-1/2}\Rightarrow$
$y'=\frac{3}{2}x^{1/2}+4\left(\frac{1}{2}\right)x^{-1/2}+3\left(-\frac{1}{2}\right)x^{-3/2}=\frac{3}{2}\sqrt{x}+\frac{2}{\sqrt{x}}-\frac{3}{2x\sqrt{x}}$
$\left[\text{note that } x^{3/2}=x^{2/2}\cdot x^{1/2}=x\sqrt{x}\right]$

24. $y=\frac{x^2-2\sqrt{x}}{x}=x-2x^{-1/2}\Rightarrow y'=1-2\left(-\frac{1}{2}\right)x^{-3/2}=1+1/(x\sqrt{x})$

25. $y=4\pi^2\Rightarrow y'=0$ since $4\pi^2$ is a constant.

26. $g(u)=\sqrt{2}\,u+\sqrt{3u}=\sqrt{2}\,u+\sqrt{3}\,\sqrt{u}\Rightarrow g'(u)=\sqrt{2}\,(1)+\sqrt{3}\left(\frac{1}{2}u^{-1/2}\right)=\sqrt{2}+\sqrt{3}/(2\sqrt{u})$

27. $y=ax^2+bx+c\Rightarrow y'=2ax+b$

28. $y=ae^v+\frac{b}{v}+\frac{c}{v^2}=ae^v+bv^{-1}+cv^{-2}\Rightarrow y'=ae^v-bv^{-2}-2cv^{-3}=ae^v-\frac{b}{v^2}-\frac{2c}{v^3}$

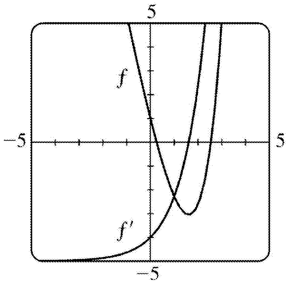
29. $v=t^2-\frac{1}{\sqrt[4]{t^3}}=t^2-t^{-3/4}\Rightarrow v'=2t-\left(-\frac{3}{4}\right)t^{-7/4}=2t+\frac{3}{4t^{7/4}}=2t+\frac{3}{4t\sqrt[4]{t^3}}$

30. $u=\sqrt[3]{t^2}+2\sqrt{t^3}=t^{2/3}+2t^{3/2}\Rightarrow u'=\frac{2}{3}t^{-1/3}+2\left(\frac{3}{2}\right)t^{1/2}=\frac{2}{3\sqrt[3]{t}}+3\sqrt{t}$

31. $z=\frac{A}{y^{10}}+Be^y=Ay^{-10}+Be^y\Rightarrow z'=-10Ay^{-11}+Be^y=-\frac{10A}{y^{11}}+Be^y$

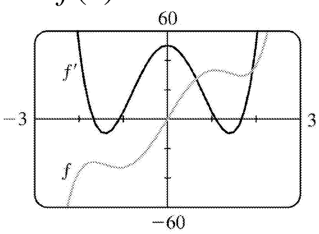
32. $y=e^{x+1}+1=e^x e^1+1=e\cdot e^x+1\Rightarrow y'=e\cdot e^x=e^{x+1}$

33. $f(x)=e^x-5x\Rightarrow f'(x)=e^x-5$.



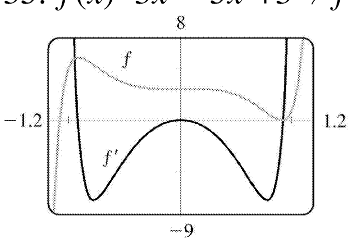
Notice that $f'(x)=0$ when f has a horizontal tangent, f' is positive when f is increasing, and f' is negative when f is decreasing.

34. $f(x)=3x^5-20x^3+50x \Rightarrow f'(x)=15x^4-60x^2+50$.



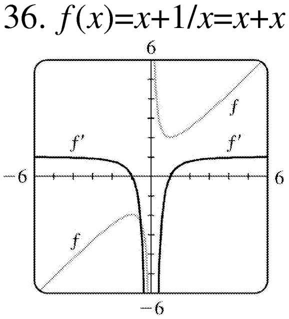
Notice that $f'(x)=0$ when f has a horizontal tangent and that f' is an even function while f is an odd function.

35. $f(x)=3x^{15}-5x^3+3 \Rightarrow f'(x)=45x^{14}-15x^2$.



Notice that $f'(x)=0$ when f has a horizontal tangent, f' is positive when f is increasing, and f' is negative when f is decreasing.

36. $f(x)=x+1/x=x+x^{-1} \Rightarrow f'(x)=1-x^{-2}=1-1/x^2$.

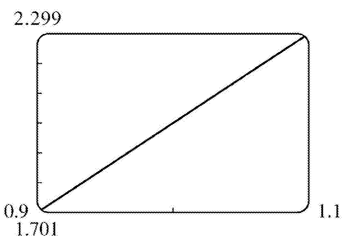


Notice that $f'(x)=0$ when f has a horizontal tangent, f' is positive when f is increasing, and f' is negative when f is decreasing.

37. To graphically estimate the value of $f'(1)$ for $f(x)=3x^2-x^3$, we'll graph f in the viewing rectangle $[1-0.1, 1+0.1]$ by $[f(0.9), f(1.1)]$, as shown in the figure. If we have sufficiently zoomed in on the graph of f, we should obtain a graph that looks like a diagonal line; if not, graph again with $1-0.01$ and $1+0.01$, etc.

Estimated value: $f'(1)\approx \dfrac{2.299-1.701}{1.1-0.9}=\dfrac{0.589}{0.2}=2.99$.

Exact value: $f(x)=3x^2-x^3 \Rightarrow f'(x)=6x-3x^2$, so $f'(1)=6-3=3$.



38. See the previous exercise. Since f is a decreasing function, assign $Y_1(3.9)$ to Y□ and $Y_1(4.1)$ to Y_{min} .

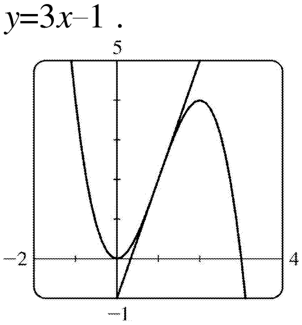
Estimated value: $f'(4)\approx \dfrac{0.49386-0.50637}{4.1-3.9}=\dfrac{-0.01251}{0.2}=-0.06255$.

Exact value: $f(x)=x^{-1/2} \Rightarrow f'(x)=-\dfrac{1}{2}x^{-3/2}$, so $f'(4)=-\dfrac{1}{2}(4^{-3/2})=-\dfrac{1}{2}\left(\dfrac{1}{8}\right)=-\dfrac{1}{16}=-0.0625$.

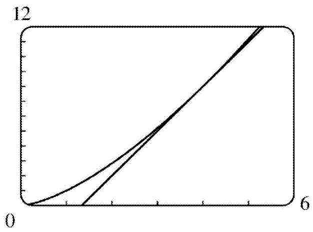
39. $y=x^4+2e^x \Rightarrow y'=4x^3+2e^x$. At $(0, 2)$, $y'=2$ and an equation of the tangent line is $y-2=2(x-0)$ or $y=2x+2$.

40. $y=(1+2x)^2=1+4x+4x^2 \Rightarrow y'=4+8x$. At $(1, 9)$, $y'=12$ and an equation of the tangent line is $y-9=12(x-1)$ or $y=12x-3$.

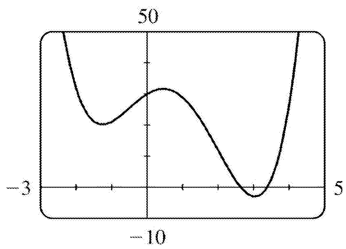
41. $y=3x^2-x^3 \Rightarrow y'=6x-3x^2$. At $(1, 2)$, $y'=6-3=3$, so an equation of the tangent line is $y-2=3(x-1)$, or $y=3x-1$.



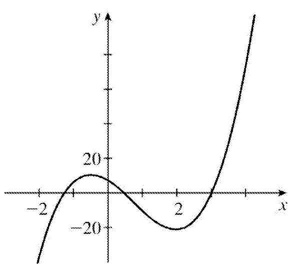
42. $y=x\sqrt{x}=x^{3/2}\Rightarrow y'=\frac{3}{2}x^{1/2}$. At (4, 8) , $y'=\frac{3}{2}(2)=3$, so an equation of the tangent line is $y-8=3(x-4)$, or $y=3x-4$.



43. **(a)**

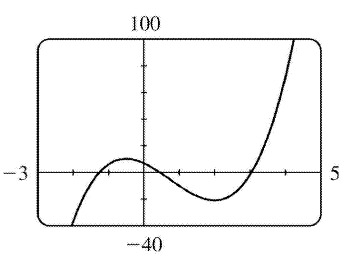


(b)

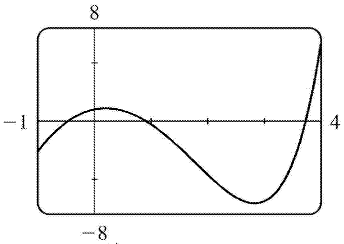


From the graph in part (a), it appears that f' is zero at $x_1\approx -1.25$, $x_2\approx 0.5$, and $x_3\approx 3$. The slopes are negative (so f' is negative) on $(-\infty, x_1)$ and (x_2, x_3). The slopes are positive (so f' is positive) on (x_1, x_2) and (x_3, ∞).

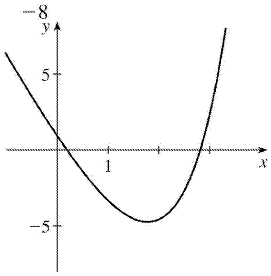
(c) $f(x)=x^4-3x^3-6x^2+7x+30\Rightarrow f'(x)=4x^3-9x^2-12x+7$



44. **(a)**

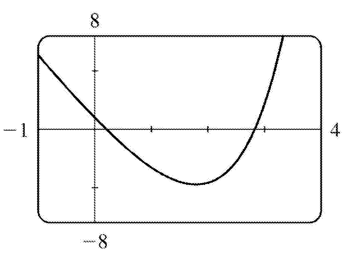


(b)



From the graph in part (a), it appears that f' is zero at $x_1 \approx 0.2$ and $x_2 \approx 2.8$. The slopes are positive (so f' is positive) on $(-\infty, x_1)$ and (x_2, ∞) . The slopes are negative (so f' is negative) on (x_1, x_2) .

(c) $g(x)=e^x-3x^2 \Rightarrow g'(x)=e^x-6x$

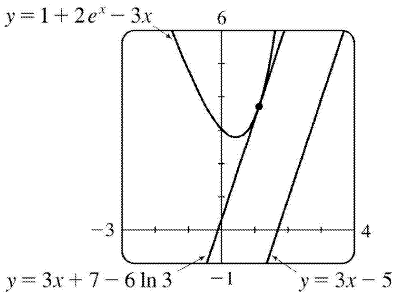


45. The curve $y=2x^3+3x^2-12x+1$ has a horizontal tangent when $y'=6x^2+6x-12=0 \Leftrightarrow$
$6(x^2+x-2)=0 \Leftrightarrow 6(x+2)(x-1)=0 \Leftrightarrow x=-2$ or $x=1$. The points on the curve are $(-2, 21)$ and $(1, -6)$.

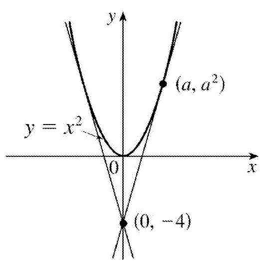
46. $f(x)=x^3+3x^2+x+3$ has a horizontal tangent when $f'(x)=3x^2+6x+1=0 \Leftrightarrow$
$x=\dfrac{-6\pm\sqrt{36-12}}{6}=-1\pm\dfrac{1}{3}\sqrt{6}$.

47. $y=6x^3+5x-3 \Rightarrow m=y'=18x^2+5$, but $x^2 \geq 0$ for all x , so $m \geq 5$ for all x .

48. The slope of $y=1+2e^x-3x$ is given by $m=y'=2e^x-3$.
The slope of $3x-y=5 \Leftrightarrow y=3x-5$ is 3 .
$m=3 \Rightarrow 2e^x-3=3 \Rightarrow e^x=3 \Rightarrow x=\ln 3$. This occurs at the point $(\ln 3, 7-3\ln 3) \approx (1.1, 3.7)$.



49.

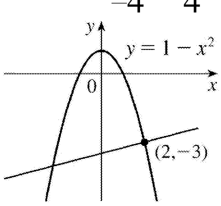


Let (a, a^2) be a point on the parabola at which the tangent line passes through the point $(0, -4)$. The tangent line has slope $2a$ and equation $y-(-4)=2a(x-0) \Leftrightarrow y=2ax-4$. Since (a, a^2) also lies on the line, $a^2=2a(a)-4$, or $a^2=4$. So $a=\pm 2$ and the points are $(2, 4)$ and $(-2, 4)$.

50. If $y=x^2+x$, then $y'=2x+1$. If the point at which a tangent meets the parabola is (a, a^2+a), then the slope of the tangent is $2a+1$. But since it passes through $(2, -3)$, the slope must also be $\dfrac{\Delta y}{\Delta x}=\dfrac{a^2+a+3}{a-2}$.

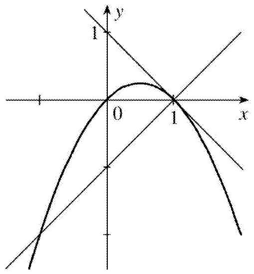
Therefore, $2a+1=\dfrac{a^2+a+3}{a-2}$. Solving this equation for a we get $a^2+a+3=2a^2-3a-2 \Leftrightarrow$ $a^2-4a-5=(a-5)(a+1)=0 \Leftrightarrow a=5$ or -1 . If $a=-1$, the point is $(-1, 0)$ and the slope is -1, so the equation is $y-0=(-1)(x+1)$ or $y=-x-1$. If $a=5$, the point is $(5, 30)$ and the slope is 11, so the equation is $y-30=11(x-5)$ or $y=11x-25$.

51. $y=f(x)=1-x^2 \Rightarrow f'(x)=-2x$, so the tangent line at $(2, -3)$ has slope $f'(2)=-4$. The normal line has slope $-\dfrac{1}{-4}=\dfrac{1}{4}$ and equation $y+3=\dfrac{1}{4}(x-2)$ or $y=\dfrac{1}{4}x-\dfrac{7}{2}$.



52. $y=f(x)=x-x^2 \Rightarrow f'(x)=1-2x$. So $f'(1)=-1$, and the slope of the normal line is the negative

reciprocal of that of the tangent line, that is, $-1/(-1)=1$. So the equation of the normal line at $(1, 0)$ is $y-0=1(x-1) \Leftrightarrow y=x-1$. Substituting this into the equation of the parabola, we obtain $x-1=x-x^2 \Leftrightarrow x=\pm 1$. The solution $x=-1$ is the one we require. Substituting $x=-1$ into the equation of the parabola to find the y-coordinate, we have $y=-2$. So the point of intersection is $(-1, -2)$, as shown in the sketch.



53.

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\frac{1}{x+h}-\frac{1}{x}}{h}=\lim_{h\to 0}\frac{x-(x+h)}{hx(x+h)}$$

$$=\lim_{h\to 0}\frac{-h}{hx(x+h)}=\lim_{h\to 0}\frac{-1}{x(x+h)}=-\frac{1}{x^2}$$

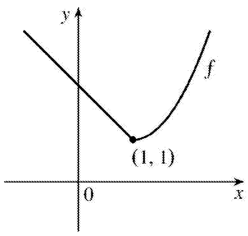
54. Substituting $x=1$ and $y=1$ into $y=ax^2+bx$ gives us $a+b=1$ **(1)**. The slope of the tangent line $y=3x-2$ is 3 and the slope of the tangent to the parabola at (x, y) is $y'=2ax+b$. At $x=1$, $y'=3 \Rightarrow 3=2a+b$ **(2)**. Subtracting **(1)** from **(2)** gives us $2=a$ and it follows that $b=-1$. The parabola has equation $y=2x^2-x$.

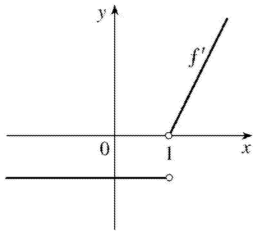
55. $f(x)=2-x$ if $x\leq 1$ and $f(x)=x^2-2x+2$ if $x>1$. Now we compute the right– and left–hand derivatives defined in Exercise :

$$f'_-(1)=\lim_{h\to 0^-}\frac{f(1+h)-f(1)}{h}=\lim_{h\to 0^-}\frac{2-(1+h)-1}{h}=\lim_{h\to 0^-}\frac{-h}{h}=\lim_{h\to 0^-}-1=-1 \text{ and}$$

$$f'_+(1)=\lim_{h\to 0^+}\frac{f(1+h)-f(1)}{h}=\lim_{h\to 0^+}\frac{(1+h)^2-2(1+h)+2-1}{h}=\lim_{h\to 0^+}\frac{h^2}{h}=\lim_{h\to 0^+}h=0 .$$

Thus, $f'(1)$ does not exist since $f'_-(1)\neq f'_+(1)$, so f is not differentiable at 1. But $f'(x)=-1$ for $x<1$ and $f'(x)=2x-2$ if $x>1$.





56. $g(x)=\begin{cases} -1-2x & \text{if } x<-1 \\ x^2 & \text{if } -1\leq x\leq 1 \\ x & \text{if } x>1 \end{cases}$

$$\lim_{h\to 0^-}\frac{g(-1+h)-g(-1)}{h}=\lim_{h\to 0^-}\frac{[-1-2(-1+h)]-1}{h}=\lim_{h\to 0^-}\frac{-2h}{h}=\lim_{h\to 0^-}(-2)=-2 \text{ and}$$

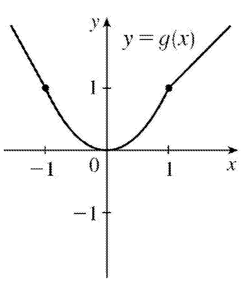
$$\lim_{h\to 0^+}\frac{g(-1+h)-g(-1)}{h}=\lim_{h\to 0^+}\frac{(-1+h)^2-1}{h}=\lim_{h\to 0^+}\frac{-2h+h^2}{h}=\lim_{h\to 0^+}(-2+h)=-2,$$

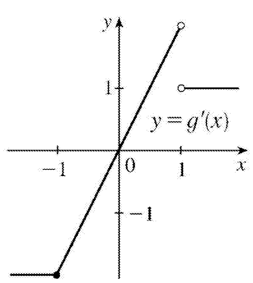
so g is differentiable at -1 and $g'(-1)=-2$.

$$\lim_{h\to 0^-}\frac{g(1)-g(1)}{h}=\lim_{h\to 0^-}\frac{(1)^2-1}{h}=\lim_{h\to 0^-}\frac{2h+h^2}{h}=\lim_{h\to 0^-}(2+h)=2 \text{ and}$$

$$\lim_{h\to 0^+}\frac{g(1)-g(1)}{h}=\lim_{h\to 0^+}\frac{(1)-1}{h}=\lim_{h\to 0^+}\frac{h}{h}=\lim_{h\to 0^+}1=1 \text{ , so } g'(1) \text{ does not exist.}$$

Thus, g is differentiable except when $x=1$, and $g'(x)=\begin{cases} -2 & \text{if } x<-1 \\ 2x & \text{if } -1\leq x<1 \\ 1 & \text{if } x>1 \end{cases}$





57. **(a)** Note that $x^2-9<0$ for $x^2<9\Leftrightarrow|x|<3\Leftrightarrow-3<x<3$. So

$$f(x)=\begin{cases} x^2-9 & \text{if } x\leq -3 \\ -x^2+9 & \text{if } -3<x<3 \\ x^2-9 & \text{if} x\geq 3 \end{cases} \Rightarrow f'(x)=\begin{cases} 2x & \text{if } x<-3 \\ -2x & \text{if } -3<x<3 \\ 2x & \text{if } x>3 \end{cases} = \begin{cases} 2x & \text{if } |x|>3 \\ -2x & \text{if } |x|<3 \end{cases}$$

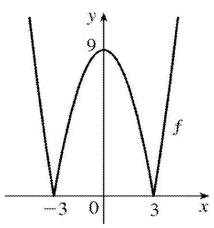
To show that

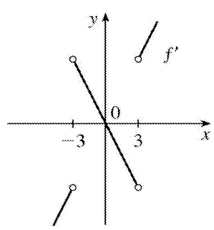
$f'(3)$ does not exist we investigate $\lim_{h\to 0}\frac{f(3+h)-f(3)}{h}$ by computing the left– and right–hand derivatives defined in Exercise .

$f'_{-}(3)=\lim_{h\to 0^-}\frac{f(3+h)-f(3)}{h}=\lim_{h\to 0^-}\frac{[-(3+h)^2+9]-0}{h}=\lim_{h\to 0^-}(-6-h)=-6$ and

$f'_{+}(3)=\lim_{h\to 0^+}\frac{f(3+h)-f(3)}{h}=\lim_{h\to 0^+}\frac{\left[(3+h)^2-9\right]-0}{h}=\lim_{h\to 0^+}\frac{6h+h^2}{h}=\lim_{h\to 0^+}(6+h)=6$.

Since the left and right limits are different, $\lim_{h\to 0}\frac{f(3+h)-f(3)}{h}$ does not exist, that is, $f'(3)$ does not exist. Similarly, $f'(-3)$ does not exist. Therefore, f is not differentiable at 3 or at -3 .



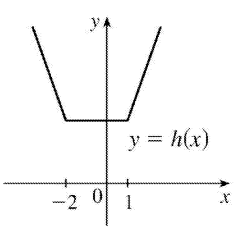


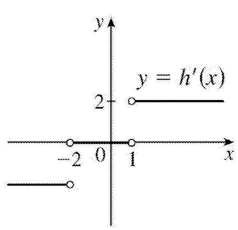
(b)

58. If $x\geq 1$, then $h(x)=|x-1|+|x+2|=x-1+x+2=2x+1$.
If $-2<x<1$, then $h(x)=-(x-1)+x+2=3$.
If $x\leq -2$, then $h(x)=-(x-1)-(x+2)=-2x-1$. Therefore,

$$h(x)=\begin{cases}-2x-1 & \text{if } x\leq -2\\ 3 & \text{if } -2<x<1\\ 2x+1 & \text{if } x\geq 1\end{cases}\quad\Rightarrow h'(x)=\begin{cases}-2 & \text{if } x<-2\\ 0 & \text{if } -2<x<1\\ 2 & \text{if } x>1\end{cases}$$

To see that $h'(1)=\lim_{x\to 1}\frac{h(x)-h(1)}{x-1}$ does not exist, observe that $\lim_{x\to 1^-}\frac{h(x)-h(1)}{x-1}=\lim_{x\to 1^-}\frac{3-3}{3-1}=0$ but $\lim_{x\to 1^+}\frac{h(x)-h(1)}{x-1}=\lim_{x\to 1^+}\frac{2x-2}{x-1}=2$. Similarly, $h'(-2)$ does not exist.





59. $y=f(x)=ax^2 \Rightarrow f'(x)=2ax$. So the slope of the tangent to the parabola at $x=2$ is $m=2a(2)=4a$. The slope of the given line, $2x+y=b \Leftrightarrow y=-2x+b$, is seen to be -2, so we must have $4a=-2 \Leftrightarrow a=-\frac{1}{2}$. So when $x=2$, the point in question has y-coordinate $-\frac{1}{2}\cdot 2^2=-2$. Now we simply require that the given line, whose equation is $2x+y=b$, pass through the point $(2, -2)$: $2(2)+(-2)=b \Leftrightarrow b=2$. So we must have $a=-\frac{1}{2}$ and $b=2$.

60. f is clearly differentiable for $x<2$ and for $x>2$. For $x<2$, $f'(x)=2x$, so $f'_-(2)=4$. For $x>2$, $f'(x)=m$, so $f'_+(2)=m$. For f to be differentiable at $x=2$, we need $4=f'_-(2)=f'_+(2)=m$. So $f(x)=4x+b$. We must also have continuity at $x=2$, so $4=f(2)=\lim_{x\to 2^+} f(x)=\lim_{x\to 2^+}(4x+b)=8+b$. Hence, $b=-4$.

61. $y=f(x)=ax^3+bx^2+cx+d \Rightarrow f'(x)=3ax^2+2bx+c$. The point $(-2, 6)$ is on f, so $f(-2)=6 \Rightarrow$ $-8a+4b-2c+d=6$ **(1)** . The point $(2, 0)$ is on f, so $f(2)=0 \Rightarrow 8a+4b+2c+d=0$ **(2)** . Since there are horizontal tangents at $(-2, 6)$ and $(2, 0)$, $f'(\pm 2)=0$. $f'(-2)=0 \Rightarrow 12a-4b+c=0$ **(3)** and $f'(2)=0 \Rightarrow 12a+4b+c=0$ **(4)** . Subtracting equation **(3)** from **(4)** gives $8b=0 \Rightarrow b=0$. Adding **(1)** and **(2)** gives $8b+2d=6$, so $d=3$ since $b=0$. From **(3)** we have $c=-12a$, so **(2)** becomes $8a+4(0)+2(-12a)+3=0$ $\Rightarrow 3=16a \Rightarrow a=\frac{3}{16}$. Now $c=-12a=-12\left(\frac{3}{16}\right)=-\frac{9}{4}$ and the desired cubic function is $y=\frac{3}{16}x^3-\frac{9}{4}x+3$.

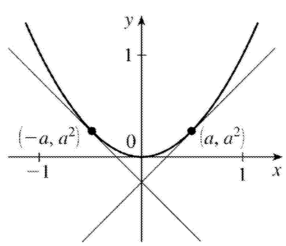
62. **(a)** $xy=c \Rightarrow y=\frac{c}{x}$. Let $P=\left(a, \frac{c}{a}\right)$. The slope of the tangent line at $x=a$ is $y'(a)=-\frac{c}{a^2}$. Its equation is $y-\frac{c}{a}=-\frac{c}{a^2}(x-a)$ or $y=-\frac{c}{a^2}x+\frac{2c}{a}$. so its y-intercept is $\frac{2c}{a}$. Setting $y=0$ gives $x=2a$, so the x-intercept is $2a$. The midpoint of the line segment joining $\left(0, \frac{2c}{a}\right)$ and $(2a, 0)$ is $\left(a, \frac{c}{a}\right)=P$.

(b) We know the x- and y-intercepts of the tangent line from part (a), so the area of the triangle bounded by the axes and the tangent is $\frac{1}{2}$(base)(height)$=\frac{1}{2}xy=\frac{1}{2}(2a)(2c/a)=2c$, a constant.

63. *Solution 1:* Let $f(x)=x^{1000}$. Then, by the definition of a derivative,
$f'(1)=\lim_{x\to 1}\frac{f(x)-f(1)}{x-1}=\lim_{x\to 1}\frac{x^{1000}-1}{x-1}$. But this is just the limit we want to find, and we know (from the Power Rule) that $f'(x)=1000x^{999}$, so $f'(1)=1000(1)^{999}=1000$. So $\lim_{x\to 1}\frac{x^{1000}-1}{x-1}=1000$.

Solution 2: Note that $(x^{1000}-1)=(x-1)(x^{999}+x^{998}+x^{997}+\cdots+x^2+x+1)$. So

$$\begin{aligned}\lim_{x\to 1}\frac{x^{1000}-1}{x-1} &= \lim_{x\to 1}\frac{(x-1)(x^{999}+x^{998}+x^{997}+\cdots+x^2+x+1)}{x-1}\\ &= \lim_{x\to 1}(x^{999}+x^{998}+x^{997}+\cdots+x^2+x+1) = \underbrace{1+1+1+\ldots+1+1+1}_{\text{1000 ones}}\\ &= 1000, \text{ as above.}\end{aligned}$$



64.
In order for the two tangents to intersect on the y –axis, the points of tangency must be at equal distances from the y –axis, since the parabola $y=x^2$ is symmetric about the y –axis. Say the points of tangency are (a, a^2) and $(-a, a^2)$, for some $a>0$. Then since the derivative of $y=x^2$ is $dy/dx=2x$, the left–hand tangent has slope $-2a$ and equation $y-a^2=-2a(x+a)$, or $y=-2ax-a^2$, and similarly the right–hand tangent line has equation
$y-a^2=2a(x-a)$, or $y=2ax-a^2$. So the two lines intersect at $(0, -a^2)$. Now if the lines are perpendicular, then the product of their slopes is -1, so $(-2a)(2a)=-1\Leftrightarrow a^2=\frac{1}{4}\Leftrightarrow a=\frac{1}{2}$. So the lines intersect at $\left(0, -\frac{1}{4}\right)$.

1. $V=x^3 \Rightarrow \dfrac{dV}{dt}=\dfrac{dV}{dx}\dfrac{dx}{dt}=3x^2\dfrac{dx}{dt}$

2. **(a)** $A=\pi r^2 \Rightarrow \dfrac{dA}{dt}=\dfrac{dA}{dr}\dfrac{dr}{dt}=2\pi r\dfrac{dr}{dt}$

(b) $\dfrac{dA}{dt}=2\pi r\dfrac{dr}{dt}=2\pi(30\text{m})(1\text{m/s})=60\pi\ \text{m}^2/\text{s}$

3. $y=x^3+2x \Rightarrow \dfrac{dy}{dt}=\dfrac{dy}{dx}\dfrac{dx}{dt}=(3x^2+2)(5)=5(3x^2+2)$. When $x=2$, $\dfrac{dy}{dt}=5(14)=70$.

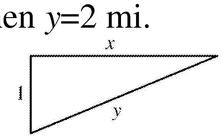
4. $x^2+y^2=25 \Rightarrow 2x\dfrac{dx}{dt}+2y\dfrac{dy}{dt}=0 \Rightarrow x\dfrac{dx}{dt}=-y\dfrac{dy}{dt} \Rightarrow \dfrac{dx}{dt}=-\dfrac{y}{x}\dfrac{dy}{dt}$.

When $y=4$, $x^2+4^2=25 \Rightarrow x=\pm 3$. For $\dfrac{dy}{dt}=6$, $\dfrac{dx}{dt}=-\dfrac{4}{\pm 3}(6)=\mp 8$.

5. $z^2=x^2+y^2 \Rightarrow 2z\dfrac{dz}{dt}=2x\dfrac{dx}{dt}+2y\dfrac{dy}{dt} \Rightarrow \dfrac{dz}{dt}=\dfrac{1}{z}\left(x\dfrac{dx}{dt}+y\dfrac{dy}{dt}\right)$. When $x=5$ and $y=12$, $z^2=5^2+12^2 \Rightarrow$ $z^2=169 \Rightarrow z=\pm 13$. For $\dfrac{dx}{dt}=2$ and $\dfrac{dy}{dt}=3$, $\dfrac{dz}{dt}=\dfrac{1}{\pm 13}(5\cdot 2+12\cdot 3)=\pm\dfrac{46}{13}$.

6. $y=\sqrt{1+x^3} \Rightarrow \dfrac{dy}{dt}=\dfrac{dy}{dx}\dfrac{dx}{dt}=\dfrac{1}{2}(1+x^3)^{-1/2}(3x^2)\dfrac{dx}{dt}=\dfrac{3x^2}{2\sqrt{1+x^3}}\dfrac{dx}{dt}$. With $\dfrac{dy}{dt}=4$ when $x=2$ and $y=3$, we have $4=\dfrac{3(4)}{2(3)}\dfrac{dx}{dt} \Rightarrow \dfrac{dx}{dt}=2$ cm / s.

7. **(a)** Given: a plane flying horizontally at an altitude of 1 mi and a speed of 500 mi / h passes directly over a radar station. If we let t be time (in hours) and x be the horizontal distance traveled by the plane (in mi), then we are given that $dx/dt=500$ mi / h.
(b) Unknown: the rate at which the distance from the plane to the station is increasing when it is 2 mi from the station. If we let y be the distance from the plane to the station, then we want to find dy/dt when $y=2$ mi.

(c) 

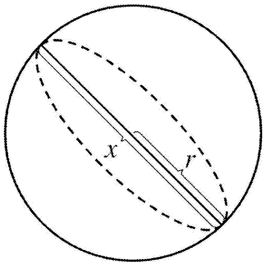
(d) By the Pythagorean Theorem, $y^2=x^2+1 \Rightarrow 2y(dy/dt)=2x(dx/dt)$.

(e) $\dfrac{dy}{dt}=\dfrac{x}{y}\dfrac{dx}{dt}=\dfrac{x}{y}(500)$. Since $y^2=x^2+1$, when $y=2$, $x=\sqrt{3}$, so $\dfrac{dy}{dt}=\dfrac{\sqrt{3}}{2}(500)=250\sqrt{3}\approx 433$ mi / h.

8. **(a)** Given: the rate of decrease of the surface area is 1 cm

^{2}/ min. If we let t be time (in minutes) and S be the surface area (in cm^2), then we are given that $dS/dt = -1$ cm^2/ s.

(b) Unknown: the rate of decrease of the diameter when the diameter is 10 cm. If we let x be the diameter, then we want to find dx/dt when $x=10$ cm.



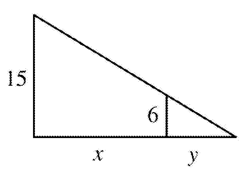
(c)

(d) If the radius is r and the diameter $x=2r$, then $r=\frac{1}{2}x$ and $S=4\pi r^2=4\pi\left(\frac{1}{2}x\right)^2=\pi x^2 \Rightarrow$
$\frac{dS}{dt}=\frac{dS}{dx}\frac{dx}{dt}=2\pi x\frac{dx}{dt}$.

(e) $-1=\frac{dS}{dt}=2\pi x\frac{dx}{dt} \Rightarrow \frac{dx}{dt}=-\frac{1}{2\pi x}$. When $x=10$, $\frac{dx}{dt}=-\frac{1}{20\pi}$. So the rate of decrease is $\frac{1}{20\pi}$ cm / min.

9. **(a)** Given: a man 6 ft tall walks away from a street light mounted on a 15 –ft–tall pole at a rate of 5 ft / s. If we let t be time (in s) and x be the distance from the pole to the man (in ft), then we are given that $dx/dt=5$ ft / s.

(b) Unknown: the rate at which the tip of his shadow is moving when he is 40 ft from the pole. If we let y be the distance from the man to the tip of
his shadow (in ft), then we want to find $\frac{d}{dt}(x+y)$ when $x=40$ ft.



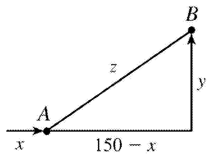
(c)

(d) By similar triangles, $\frac{15}{6}=\frac{x+y}{y} \Rightarrow 15y=6x+6y \Rightarrow 9y=6x \Rightarrow y=\frac{2}{3}x$.

(e) The tip of the shadow moves at a rate of $\frac{d}{dt}(x+y)=\frac{d}{dt}\left(x+\frac{2}{3}x\right)=\frac{5}{3}\frac{dx}{dt}=\frac{5}{3}(5)=\frac{25}{3}$ ft / s.

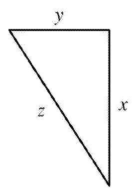
10. **(a)** Given: at noon, ship A is 150 km west of ship B; ship A is sailing east at 35 km / h, and ship B is sailing north at 25 km / h. If we let t be time (in hours), x be the distance traveled by ship A (in km), and y be the distance traveled by ship B (in km), then we are given that $dx/dt=35$ km / h and $dy/dt=25$ km / h.

(b) Unknown: the rate at which the distance between the ships is changing at 4:00 P.M. If we let z be the distance between the ships, then we want to find dz/dt when $t=4$ h.

(c) 

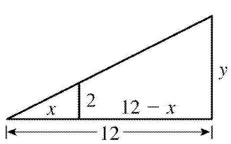
(d) $z^2=(150-x)^2+y^2 \Rightarrow 2z\dfrac{dz}{dt}=2(150-x)\left(-\dfrac{dx}{dt}\right)+2y\dfrac{dy}{dt}$

(e) At 4:00 P.M., $x=4(35)=140$ and $y=4(25)=100 \Rightarrow z=\sqrt{(150-140)^2+100^2}=\sqrt{10{,}100}$. So $\dfrac{dz}{dt}=\dfrac{1}{z}\left[(x-150)\dfrac{dx}{dt}+y\dfrac{dy}{dt}\right]=\dfrac{-10(35)+100(25)}{\sqrt{10{,}100}}=\dfrac{215}{\sqrt{101}}\approx 21.4$ km / h.

11. 

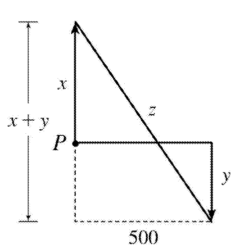
We are given that $\dfrac{dx}{dt}=60$ mi / h and $\dfrac{dy}{dt}=25$ mi / h. $z^2=x^2+y^2 \Rightarrow 2z\dfrac{dz}{dt}=2x\dfrac{dx}{dt}+2y\dfrac{dy}{dt} \Rightarrow$ $z\dfrac{dz}{dt}=x\dfrac{dx}{dt}+y\dfrac{dy}{dt} \Rightarrow \dfrac{dz}{dt}=\dfrac{1}{z}\left(x\dfrac{dx}{dt}+y\dfrac{dy}{dt}\right)$.

After 2 hours, $x=2(60)=120$ and $y=2(25)=50 \Rightarrow z=\sqrt{120^2+50^2}=130$, so $\dfrac{dz}{dt}=\dfrac{1}{z}\left(x\dfrac{dx}{dt}+y\dfrac{dy}{dt}\right)=\dfrac{120(60)+50(25)}{130}=65$ mi / h.

12. 

We are given that $\dfrac{dx}{dt}=1.6$ m / s. By similar triangles, $\dfrac{y}{12}=\dfrac{2}{x} \Rightarrow y=\dfrac{24}{x} \Rightarrow \dfrac{dy}{dt}=-\dfrac{24}{x^2}\dfrac{dx}{dt}=-\dfrac{24}{x^2}(1.6)$.

When $x=8$, $\dfrac{dy}{dt}=-\dfrac{24(1.6)}{64}=-0.6$ m / s, so the shadow is decreasing at a rate of 0.6 m / s.

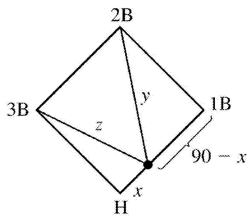
13. 

We are given that $\dfrac{dx}{dt}=4$ ft / s and $\dfrac{dy}{dt}=5$ ft / s. $z^2=(x+y)^2+500^2 \Rightarrow 2z\dfrac{dz}{dt}=2(x+y)\left(\dfrac{dx}{dt}+\dfrac{dy}{dt}\right)$. 15

minutes after the woman starts, we have $x = (4\text{ft/s})(20\text{min})(60\text{s/min}) = 4800$ ft and $y = 5 \cdot 15 \cdot 60 = 4500 \Rightarrow$ $z = \sqrt{(4800+4500)^2 + 500^2} = \sqrt{86{,}740{,}000}$, so

$$\frac{dz}{dt} = \frac{x+y}{z}\left(\frac{dx}{dt} + \frac{dy}{dt}\right) = \frac{4800+4500}{\sqrt{86{,}740{,}000}}(4+5) = \frac{837}{\sqrt{8674}} \approx 8.99 \text{ ft / s}.$$

14. We are given that $\frac{dx}{dt} = 24$ ft / s.



(a)

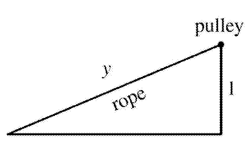
$$y^2 = (90-x)^2 + 90^2 \Rightarrow 2y\frac{dy}{dt} = 2(90-x)\left(-\frac{dx}{dt}\right).$$

When $x = 45$, $y = \sqrt{45^2 + 90^2} = 45\sqrt{5}$, so $\frac{dy}{dt} = \frac{90-x}{y}\left(-\frac{dx}{dt}\right) = \frac{45}{45\sqrt{5}}(-24) = -\frac{24}{\sqrt{5}}$,

so the distance from second base is decreasing at a rate of $\frac{24}{\sqrt{5}} \approx 10.7$ ft / s.

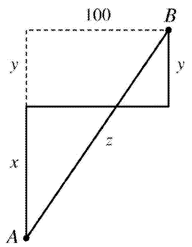
(b) Due to the symmetric nature of the problem in part (a), we expect to get the same answer and we do. $z^2 = x^2 + 90^2 \Rightarrow 2z\frac{dz}{dt} = 2x\frac{dx}{dt}$. When $x = 45$, $z = 45\sqrt{5}$, so $\frac{dz}{dt} = \frac{45}{45\sqrt{5}}(24) = \frac{24}{\sqrt{5}} \approx 10.7$ ft / s.

15. $A = \frac{1}{2}bh$, where b is the base and h is the altitude. We are given that $\frac{dh}{dt} = 1$ cm / min and $\frac{dA}{dt} = 2$ cm ^{2}/ min. Using the Product Rule, we have $\frac{dA}{dt} = \frac{1}{2}\left(b\frac{dh}{dt} + h\frac{db}{dt}\right)$. When $h = 10$ and $A = 100$, we have $100 = \frac{1}{2}b(10) \Rightarrow \frac{1}{2}b = 10 \Rightarrow b = 20$, so $2 = \frac{1}{2}\left(20 \cdot 1 + 10\frac{db}{dt}\right) \Rightarrow 4 = 20 + 10\frac{db}{dt} \Rightarrow$
$\frac{db}{dt} = \frac{4-20}{10} = -1.6$ cm / min.

16. 

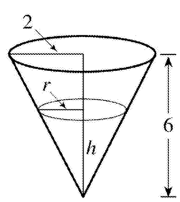
Given $\frac{dy}{dt} = -1$ m / s, find $\frac{dx}{dt}$ when $x = 8$ m. $y^2 = x^2 + 1 \Rightarrow 2y\frac{dy}{dt} = 2x\frac{dx}{dt} \Rightarrow \frac{dx}{dt} = \frac{y}{x}\frac{dy}{dt} = -\frac{y}{x}$. When $x = 8$, $y = \sqrt{65}$, so

$\frac{dx}{dt} = -\frac{\sqrt{65}}{8}$. Thus, the boat approaches the dock at $\frac{\sqrt{65}}{8} \approx 1.01$ m / s.

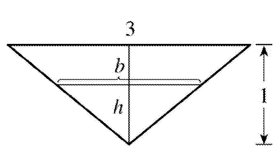
17. 

We are given that $\frac{dx}{dt} = 35$ km / h and $\frac{dy}{dt} = 25$ km / h. $z^2 = (x+y)^2 + 100^2 \Rightarrow 2z\frac{dz}{dt} = 2(x+y)\left(\frac{dx}{dt} + \frac{dy}{dt}\right)$.
At 4:00 P.M., $x=4(35)=140$ and $y=4(25)=100 \Rightarrow z=\sqrt{(140+100)^2+100^2} = \sqrt{67{,}600} = 260$, so
$\frac{dz}{dt} = \frac{x+y}{z}\left(\frac{dx}{dt} + \frac{dy}{dt}\right) = \frac{140+100}{260}(35+25) = \frac{720}{13} \approx 55.4$ km / h.

18. Let D denote the distance from the origin (0,0) to the point on the curve $y=\sqrt{x}$.
$D=\sqrt{(x-0)^2+(y-0)^2} = \sqrt{x^2+(\sqrt{x})^2} = \sqrt{x^2+x} \Rightarrow \frac{dD}{dt} = \frac{1}{2}(x^2+x)^{-1/2}(2x+1)\frac{dx}{dt} = \frac{2x+1}{2\sqrt{x^2+x}}\frac{dx}{dt}$. With
$\frac{dx}{dt} = 3$ when $x=4$, $\frac{dD}{dt} = \frac{9}{2\sqrt{20}}(3) = \frac{27}{4\sqrt{5}} \approx 3.02$ cm / s.

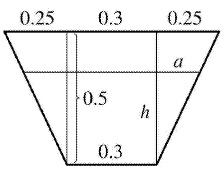
19. 

If C= the rate at which water is pumped in, then $\frac{dV}{dt} = C - 10{,}000$, where $V=\frac{1}{3}\pi r^2 h$ is the volume at time t . By similar triangles, $\frac{r}{2} = \frac{h}{6} \Rightarrow r = \frac{1}{3}h \Rightarrow V = \frac{1}{3}\pi\left(\frac{1}{3}h\right)^2 h = \frac{\pi}{27}h^3 \Rightarrow \frac{dV}{dt} = \frac{\pi}{9}h^2\frac{dh}{dt}$.
When $h=200$cm , $\frac{dh}{dt} = 20$cm/min , so $C - 10{,}000 = \frac{\pi}{9}(200)^2(20) \Rightarrow C = 10{,}000 + \frac{800{,}000}{9}\pi \approx 289{,}253$ cm^3/ min.

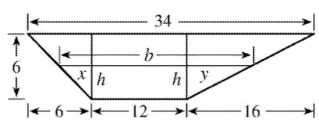
20. 

By similar triangles,

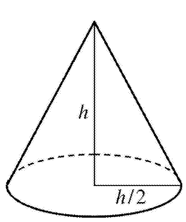
$\frac{3}{1}=\frac{b}{h}$, so $b=3h$. The trough has volume $V=\frac{1}{2}bh(10)=5(3h)h=15h^2 \Rightarrow 12=\frac{dV}{dt}=30h\frac{dh}{dt} \Rightarrow \frac{dh}{dt}=\frac{2}{5h}$. When $h=\frac{1}{2}$, $\frac{dh}{dt}=\frac{2}{5\cdot\frac{1}{2}}=\frac{4}{5}$ ft / min.

21. 

The figure is labeled in meters. The area A of a trapezoid is $\frac{1}{2}(\text{ base }_1+\text{ base }_2)(\text{ height })$, and the volume V of the 10–meter–long trough is $10A$. Thus, the volume of the trapezoid with height h is $V=(10)\frac{1}{2}[0.3+(0.3+2a)]h$. By similar triangles, $\frac{a}{h}=\frac{0.25}{0.5}=\frac{1}{2}$, so $2a=h \Rightarrow V=5(0.6+h)h=3h+5h^2$. Now $\frac{dV}{dt}=\frac{dV}{dh}\frac{dh}{dt} \Rightarrow 0.2=(3+10h)\frac{dh}{dt} \Rightarrow \frac{dh}{dt}=\frac{0.2}{3+10h}$. When $h=0.3$, $\frac{dh}{dt}=\frac{0.2}{3+10(0.3)}=\frac{0.2}{6}$ m / min $=\frac{1}{30}$ m / min or $\frac{10}{3}$ cm / min.

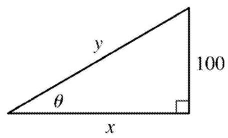
22. 

The figure is drawn without the top 3 feet. $V=\frac{1}{2}(b+12)h(20)=10(b+12)h$ and, from similar triangles, $\frac{x}{h}=\frac{6}{6}$ and $\frac{y}{h}=\frac{16}{6}=\frac{8}{3}$, so $b=x+12+y=h+12+\frac{8h}{3}=12+\frac{11h}{3}$. Thus, $V=10\left(24+\frac{11h}{3}\right)h=240h+\frac{110h^2}{3}$ and so $0.8=\frac{dV}{dt}=\left(240+\frac{220}{3}h\right)\frac{dh}{dt}$. When $h=5$, $\frac{dh}{dt}=\frac{0.8}{240+5(220/3)}=\frac{3}{2275}\approx 0.00132$ ft / min.

23. 

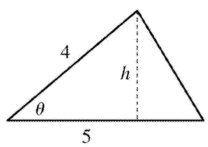
We are given that $\frac{dV}{dt}=30$ ft ^{3}/ min. $V=\frac{1}{3}\pi r^2 h=\frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h=\frac{\pi h^3}{12} \Rightarrow$

$\frac{dV}{dt}=\frac{dV}{dh}\frac{dh}{dt}\Rightarrow 30=\frac{\pi h^2}{4}\frac{dh}{dt}\Rightarrow\frac{dh}{dt}=\frac{120}{\pi h^2}$. When h=10 ft, $\frac{dh}{dt}=\frac{120}{10^2\pi}=\frac{6}{5\pi}\approx 0.38$ ft / min.

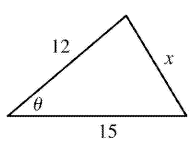
24. 

We are given dx/dt=8 ft / s. $\cot\theta=\frac{x}{100}\Rightarrow x=100\cot\theta\Rightarrow\frac{dx}{dt}=-100\csc^2\theta\frac{d\theta}{dt}\Rightarrow\frac{d\theta}{dt}=-\frac{\sin^2\theta}{100}\cdot 8$.

When $y=200$, $\sin\theta=\frac{100}{200}=\frac{1}{2}\Rightarrow\frac{d\theta}{dt}=-\frac{(1/2)^2}{100}\cdot 8=-\frac{1}{50}$ rad / s. The angle is decreasing at a rate of $\frac{1}{50}$ rad / s.

25. 

$A=\frac{1}{2}bh$, but b=5 m and $\sin\theta=\frac{h}{4}\Rightarrow h=4\sin\theta$, so $A=\frac{1}{2}(5)(4\sin\theta)=10\sin\theta$. We are given $\frac{d\theta}{dt}=0.06$ rad / s, so $\frac{dA}{dt}=\frac{dA}{d\theta}\frac{d\theta}{dt}=(10\cos\theta)(0.06)=0.6\cos\theta$. When $\theta=\frac{\pi}{3}$,
$\frac{dA}{dt}=0.6\left(\cos\frac{\pi}{3}\right)=(0.6)\left(\frac{1}{2}\right)=0.3$ m^2/ s.

26. 

We are given $d\theta/dt=2^\circ$ / min $=\frac{\pi}{90}$ rad / min. By the Law of Cosines,
$x^2=12^2+15^2-2(12)(15)\cos\theta=369-360\cos\theta\Rightarrow 2x\frac{dx}{dt}=360\sin\theta\frac{d\theta}{dt}\Rightarrow\frac{dx}{dt}=\frac{180\sin\theta}{x}\frac{d\theta}{dt}$. When $\theta=60^\circ$, $x=\sqrt{369-360\cos 60^\circ}=\sqrt{189}=3\sqrt{21}$, so $\frac{dx}{dt}=\frac{180\sin 60^\circ}{3\sqrt{21}}\frac{\pi}{90}=\frac{\pi\sqrt{3}}{3\sqrt{21}}=\frac{\sqrt{7}\pi}{21}\approx 0.396$ m / min.

27. Differentiating both sides of $PV=C$ with respect to t and using the Product Rule gives us $P\frac{dV}{dt}+V\frac{dP}{dt}=0\Rightarrow\frac{dV}{dt}=-\frac{V}{P}\frac{dP}{dt}$. When $V=600$, $P=150$ and $\frac{dP}{dt}=20$, so we have

$\frac{dV}{dt}=-\frac{600}{150}(20)=-80$. Thus, the volume is decreasing at a rate of 80 cm ^{3}/ min.

28. $PV^{1.4}=C\Rightarrow P\cdot 1.4V^{0.4}\frac{dV}{dt}+V^{1.4}\frac{dP}{dt}=0\Rightarrow \frac{dV}{dt}=-\frac{V^{1.4}}{P\cdot 1.4V^{0.4}}\frac{dP}{dt}=-\frac{V}{1.4P}\frac{dP}{dt}$. When $V=400$, $P=80$ and $\frac{dP}{dt}=-10$, so we have $\frac{dV}{dt}=-\frac{400}{1.4(80)}(-10)=\frac{250}{7}$. Thus, the volume is increasing at a rate of $\frac{250}{7}\approx 36$ cm ^{3}/ min.

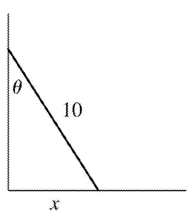
29. With $R_1=80$ and $R_2=100$, $\frac{1}{R}=\frac{1}{R_1}+\frac{1}{R_2}=\frac{1}{80}+\frac{1}{100}=\frac{180}{8000}=\frac{9}{400}$, so $R=\frac{400}{9}$. Differentiating $\frac{1}{R}=\frac{1}{R_1}+\frac{1}{R_2}$ with respect to t , we have $-\frac{1}{R^2}\frac{dR}{dt}=-\frac{1}{R_1^2}\frac{dR_1}{dt}-\frac{1}{R_2^2}\frac{dR_2}{dt}\Rightarrow$

$\frac{dR}{dt}=R^2\left(\frac{1}{R_1^2}\frac{dR_1}{dt}+\frac{1}{R_2^2}\frac{dR_2}{dt}\right)$. When $R_1=80$ and $R_2=100$,

$\frac{dR}{dt}=\frac{400^2}{9^2}\left[\frac{1}{80^2}(0.3)+\frac{1}{100^2}(0.2)\right]=\frac{107}{810}\approx 0.132\Omega/\text{ s.}$

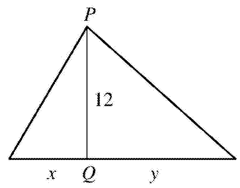
30. We want to find $\frac{dB}{dt}$ when $L=18$ using $B=0.007W^{2/3}$ and $W=0.12L^{2.53}$.

$$\frac{dB}{dt}=\frac{dB}{dW}\frac{dW}{dL}\frac{dL}{dt}=\left(0.007\cdot\frac{2}{3}W^{-1/3}\right)(0.12\cdot 2.53\cdot L^{1.53})\left(\frac{20-15}{10{,}000{,}000}\right)$$
$$=\left[0.007\cdot\frac{2}{3}(0.12\cdot 18^{2.53})^{-1/3}\right](0.12\cdot 2.53\cdot 18^{1.53})\left(\frac{5}{10^7}\right)\approx 1.045\times 10^{-8}\text{g/yr}$$

31. 

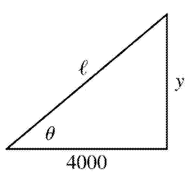
We are given that $\frac{dx}{dt}=2$ ft / s. $\sin\theta=\frac{x}{10}\Rightarrow x=10\sin\theta\Rightarrow\frac{dx}{dt}=10\cos\theta\frac{d\theta}{dt}$. When $\theta=\frac{\pi}{4}$,

$2=10\cos\frac{\pi}{4}\frac{d\theta}{dt}\Rightarrow\frac{d\theta}{dt}=\frac{2}{10(1/\sqrt{2})}=\frac{\sqrt{2}}{5}$ rad / s.

32. 

Using Q for the origin, we are given $\frac{dx}{dt}=-2$ ft / s and need to find $\frac{dy}{dt}$ when $x=-5$. Using the Pythagorean Theorem twice, we have $\sqrt{x^2+12^2}+\sqrt{y^2+12^2}=39$, the total length of the rope. Differentiating with respect to t , we get $\frac{x}{\sqrt{x^2+12^2}}\frac{dx}{dt}+\frac{y}{\sqrt{y^2+12^2}}\frac{dy}{dt}=0$, so

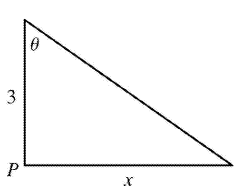
$\frac{dy}{dt}=-\frac{x\sqrt{y^2+12^2}}{y\sqrt{x^2+12^2}}\frac{dx}{dt}$. Now when $x=-5$, $39=\sqrt{(-5)^2+12^2}+\sqrt{y^2+12^2}=13+\sqrt{y^2+12^2}\Leftrightarrow\sqrt{y^2+12^2}=26$, and $y=\sqrt{26^2-12^2}=\sqrt{532}$. So when $x=-5$, $\frac{dy}{dt}=-\frac{(-5)(26)}{\sqrt{532}(13)}(-2)=-\frac{10}{\sqrt{133}}\approx-0.87$ ft / s. So cart B is moving towards Q at about 0.87 ft / s.

33. **(a)** 

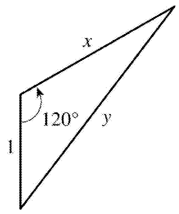
By the Pythagorean Theorem, $4000^2+y^2=\ell^2$. Differentiating with respect to t , we obtain $2y\frac{dy}{dt}=2\ell\frac{d\ell}{dt}$. We know that $\frac{dy}{dt}=600$ ft / s, so when $y=3000$ ft,
$\ell=\sqrt{4000^2+3000^2}=\sqrt{25{,}000{,}000}=5000$ ft and $\frac{d\ell}{dt}=\frac{y}{\ell}\frac{dy}{dt}=\frac{3000}{5000}(600)=\frac{1800}{5}=360$ ft / s.

(b) Here $\tan\theta=\frac{y}{4000}\Rightarrow\frac{d}{dt}(\tan\theta)=\frac{d}{dt}\left(\frac{y}{4000}\right)\Rightarrow\sec^2\theta\frac{d\theta}{dt}=\frac{1}{4000}\frac{dy}{dt}\Rightarrow\frac{d\theta}{dt}=\frac{\cos^2\theta}{4000}\frac{dy}{dt}$.

When $y=3000$ ft, $\frac{dy}{dt}=600$ ft / s, $\ell=5000$ and $\cos\theta=\frac{4000}{\ell}=\frac{4000}{5000}=\frac{4}{5}$, so $\frac{d\theta}{dt}=\frac{(4/5)^2}{4000}(600)=0.096$ rad / s.

34. 

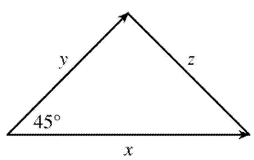
We are given that $\frac{d\theta}{dt}=4(2\pi)=8\pi$ rad / min. $x=3\tan\theta \Rightarrow \frac{dx}{dt}=3\sec^2\theta\,\frac{d\theta}{dt}$. When $x=1$, $\tan\theta=\frac{1}{3}$, so $\sec^2\theta=1+\left(\frac{1}{3}\right)^2=\frac{10}{9}$ and $\frac{dx}{dt}=3\left(\frac{10}{9}\right)(8\pi)=\frac{80\pi}{3}\approx 83.8$ km / min.

35. 

We are given that $\frac{dx}{dt}=300$ km / h. By the Law of Cosines,

$y^2=x^2+1^2-2(1)(x)\cos 120^\circ=x^2+1-2x\left(-\frac{1}{2}\right)=x^2+x+1$, so $2y\frac{dy}{dt}=2x\frac{dx}{dt}+\frac{dx}{dt} \Rightarrow \frac{dy}{dt}=\frac{2x+1}{2y}\,\frac{dx}{dt}$. After 1 minute,

$x=\frac{300}{60}=5$ km $\Rightarrow y=\sqrt{5^2+5+1}=\sqrt{31}$ km $\Rightarrow \frac{dy}{dt}=\frac{2(5)+1}{2\sqrt{31}}(300)=\frac{1650}{\sqrt{31}}\approx 296$ km / h.

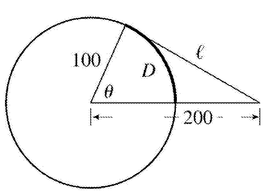
36. 

We are given that $\frac{dx}{dt}=3$ mi / h and $\frac{dy}{dt}=2$ mi / h. By the Law of Cosines,

$z^2=x^2+y^2-2xy\cos 45^\circ=x^2+y^2-\sqrt{2}\,xy \Rightarrow 2z\frac{dz}{dt}=2x\frac{dx}{dt}+2y\frac{dy}{dt}-\sqrt{2}\,x\frac{dy}{dt}-\sqrt{2}\,y\frac{dx}{dt}$. After 15 minutes $\left[=\frac{1}{4}\text{ h}\right]$,

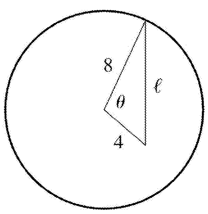
we have $x=\frac{3}{4}$ and $y=\frac{2}{4}=\frac{1}{2} \Rightarrow z^2=\left(\frac{3}{4}\right)^2+\left(\frac{2}{4}\right)^2-\sqrt{2}\left(\frac{3}{4}\right)\left(\frac{2}{4}\right) \Rightarrow z=\frac{\sqrt{13-6\sqrt{2}}}{4}$ and

$\frac{dz}{dt}=\frac{2}{\sqrt{13-6\sqrt{2}}}\left[2\left(\frac{3}{4}\right)3+2\left(\frac{1}{2}\right)2-\sqrt{2}\left(\frac{3}{4}\right)2-\sqrt{2}\left(\frac{1}{2}\right)3\right]=\frac{2}{\sqrt{13-6\sqrt{2}}}\,\frac{13-6\sqrt{2}}{2}=\sqrt{13-6\sqrt{2}}$ ≈ 2.125mi/h.

37. 

Let the distance between the runner and the friend be ℓ . Then by the Law of Cosines,

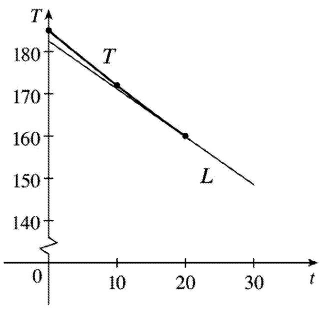
$\ell^2 = 200^2 + 100^2 - 2 \cdot 200 \cdot 100 \cdot \cos\theta = 50{,}000 - 40{,}000\cos\theta$ (*). Differentiating implicitly with respect to t, we obtain $2\ell \frac{d\ell}{dt} = -40{,}000(-\sin\theta)\frac{d\theta}{dt}$. Now if D is the distance run when the angle is θ radians, then by the formula for the length of an arc on a circle, $s = r\theta$, we have $D = 100\theta$, so $\theta = \frac{1}{100} D \Rightarrow \frac{d\theta}{dt} = \frac{1}{100}\frac{dD}{dt} = \frac{7}{100}$. To substitute into the expression for $\frac{d\ell}{dt}$, we must know $\sin\theta$ at the time when $\ell = 200$, which we find from (*): $200^2 = 50{,}000 - 40{,}000\cos\theta \Leftrightarrow \cos\theta = \frac{1}{4}$ $\Rightarrow \sin\theta = \sqrt{1 - \left(\frac{1}{4}\right)^2} = \frac{\sqrt{15}}{4}$. Substituting, we get $2(200)\frac{d\ell}{dt} = 40{,}000\frac{\sqrt{15}}{4}\left(\frac{7}{100}\right) \Rightarrow$ $d\ell/dt = \frac{7\sqrt{15}}{4} \approx 6.78$ m / s. Whether the distance between them is increasing or decreasing depends on the direction in which the runner is running.

38. 

The hour hand of a clock goes around once every 12 hours or, in radians per hour, $\frac{2\pi}{12} = \frac{\pi}{6}$ rad / h. The minute hand goes around once an hour, or at the rate of 2π rad / h. So the angle θ between them (measuring clockwise from the minute hand to the hour hand) is changing at the rate of $d\theta/dt = \frac{\pi}{6} - 2\pi = -\frac{11\pi}{6}$ rad / h. Now, to relate θ to ℓ, we use the Law of Cosines:
$\ell^2 = 4^2 + 8^2 - 2 \cdot 4 \cdot 8 \cdot \cos\theta = 80 - 64\cos\theta$ (*).
Differentiating implicitly with respect to t, we get $2\ell\frac{d\ell}{dt} = -64(-\sin\theta)\frac{d\theta}{dt}$. At 1:00, the angle between the two hands is one–twelfth of the circle, that is, $\frac{2\pi}{12} = \frac{\pi}{6}$ radians. We use (*) to find ℓ at 1:00: $\ell = \sqrt{80 - 64\cos\frac{\pi}{6}} = \sqrt{80 - 32\sqrt{3}}$. Substituting, we get $2\ell\frac{d\ell}{dt} = 64\sin\frac{\pi}{6}\left(-\frac{11\pi}{6}\right) \Rightarrow$ $\frac{d\ell}{dt} = \frac{64\left(\frac{1}{2}\right)\left(-\frac{11\pi}{6}\right)}{2\sqrt{80 - 32\sqrt{3}}} = -\frac{88\pi}{3\sqrt{80 - 32\sqrt{3}}} \approx -18.6$. So at 1:00, the distance between the tips of the hands is decreasing at a rate of 18.6 mm / h ≈ 0.005 mm / s.

1. As in Example 1, $T(0)=185$, $T(10)=172$, $T(20)=160$, and $T'(20)\approx \frac{T(10)-T(20)}{10-20}=\frac{172-160}{-10}=-1.2^\circ$ F / min. $T(30)\approx T(20)+T'(20)(30-20)\approx 160-1.2(10)=148^\circ$ F.

We would expect the temperature of the turkey to get closer to 75° F as time increases. Since the temperature decreased 13° F in the first 10 minutes and 12° F in the second 10 minutes, we can assume that the slopes of the tangent line are increasing through negative values: $-1.3,-1.2,\ldots$. Hence, the tangent lines are under the curve and 148° F



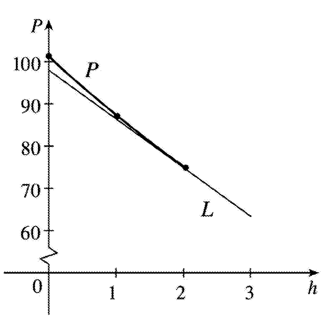
is an underestimate. From the figure, we estimate the slope of the tangent line at $t=20$ to be $\frac{184-147}{0-30}=-\frac{37}{30}$.

Then the linear approximation becomes $T(30)\approx T(20)+T'(20)\cdot 10\approx 160-\frac{37}{30}(10)=147\frac{2}{3}\approx 147.7$.

2. $P'(2)\approx \frac{P(1)-P(2)}{1-2}=\frac{87.1-74.9}{-1}=-12.2$ kilopascals / km.

$P(3)\approx P(2)+P'(2)(3-2)\approx 74.9-12.2(1)=62.7$ kPa.

From the figure, we estimate the slope of the tangent line at $h=2$ to be $\frac{98-63}{0-3}=-\frac{35}{3}$. Then the linear approximation becomes $P(3)\approx P(2)+P'(2)\cdot 1\approx 74.9-\frac{35}{3}\approx 63.23$ kPa.



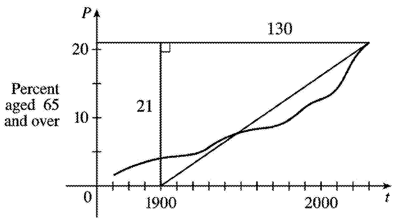
3. Extend the tangent line at the point (2030,21) to the t –axis. Answers will vary based on this approximation—we'll use $t=1900$ as our t –intercept. The linearization is then

$P(t)\quad \approx P(2030)+P'(2030)(t-2030)$

$$\approx 21+\frac{21}{130}(t-2030)$$

$P(2040)=21+\dfrac{21}{130}(2040-2030)\approx 22.6\%$

$P(2050)=21+\dfrac{21}{130}(2050-2030)\approx 24.2\%$



These predictions are probably too high since the tangent line lies above the graph at $t=2030$.

4. Let $A=\dfrac{N(1980)-N(1985)}{1980-1985}=\dfrac{15.0-17.0}{-5}=0.4$ and $B=\dfrac{N(1990)-N(1985)}{1990-1985}=\dfrac{19.3-17.0}{5}=0.46$. Then $N'(1985)=\lim\limits_{t\to 1985}\dfrac{N(t)-N(1985)}{t-1985}\approx\dfrac{A+B}{2}=0.43$ million / year. So

$N(1984)\approx N(1985)+N'(1985)(1984-1985)\approx 17.0+0.43(-1)=16.57$ million.

$N'(2000)\approx\dfrac{N(1995)-N(2000)}{1995-2000}=\dfrac{22.0-24.9}{-5}=0.58$ million / year.

$N(2006)\approx N(2000)+N'(2000)(2006-2000)\approx 24.9+0.58(6)=28.38$ million.

5. $f(x)=x^3\Rightarrow f'(x)=3x^2$, so $f(1)=1$ and $f'(1)=3$. With $a=1$, $L(x)=f(a)+f'(a)(x-a)$ becomes $L(x)=f(1)+f'(1)(x-1)=1+3(x-1)=3x-2$.

6. $f(x)=\ln x\Rightarrow f'(x)=1/x$, so $f(1)=0$ and $f'(1)=1$. Thus, $L(x)=f(1)+f'(1)(x-1)=0+1(x-1)=x-1$.

7. $f(x)=\cos x\Rightarrow f'(x)=-\sin x$, so $f\left(\dfrac{\pi}{2}\right)=0$ and $f'\left(\dfrac{\pi}{2}\right)=-1$. Thus,

$L(x)=f\left(\dfrac{\pi}{2}\right)+f'\left(\dfrac{\pi}{2}\right)\left(x-\dfrac{\pi}{2}\right)=0-1\left(x-\dfrac{\pi}{2}\right)=-x+\dfrac{\pi}{2}$.

8. $f(x)=\sqrt[3]{x}=x^{1/3}\Rightarrow f'(x)=\dfrac{1}{3}x^{-2/3}$, so $f(-8)=-2$ and $f'(-8)=\dfrac{1}{12}$. Thus,

$L(x)=f(-8)+f'(-8)(x+8)=-2+\dfrac{1}{12}(x+8)=\dfrac{1}{12}x-\dfrac{4}{3}$.

9. $f(x)=\sqrt{1-x}\Rightarrow$

$f'(x)=\dfrac{-1}{2\sqrt{1-x}}$, so $f(0)=1$ and $f'(0)=-\dfrac{1}{2}$. Therefore,

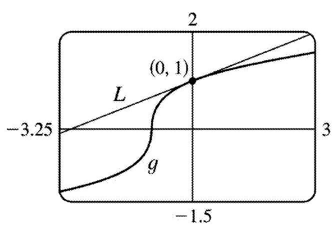
$$\begin{aligned}\sqrt{1-x}=f(x)&\approx f(0)+f'(0)(x-0)\\ &= 1+\left(-\frac{1}{2}\right)(x-0)=1-\frac{1}{2}x\end{aligned}$$

So $\sqrt{0.9}=\sqrt{1-0.1}\approx 1-\frac{1}{2}(0.1)=0.95$ and $\sqrt{0.99}=\sqrt{1-0.01}\approx 1-\frac{1}{2}(0.01)=0.995$.

10. $g(x)=\sqrt[3]{1+x}=(1+x)^{1/3}\Rightarrow g'(x)=\frac{1}{3}(1+x)^{-2/3}$,

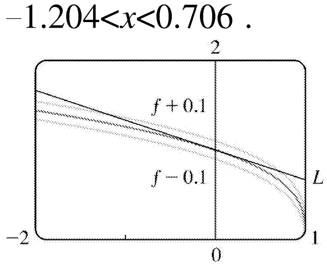
so $g(0)=1$ and $g'(0)=\frac{1}{3}$. Therefore, $\sqrt[3]{1+x}=g(x)\approx g(0)+g'(0)(x-0)=1+\frac{1}{3}x$. So

$\sqrt[3]{0.95}=\sqrt[3]{1+(-0.05)}\approx 1+\frac{1}{3}(-0.05)=0.98\overline{3}$, and $\sqrt[3]{1.1}=\sqrt[3]{1+0.1}\approx 1+\frac{1}{3}(0.1)=1.0\overline{3}$.

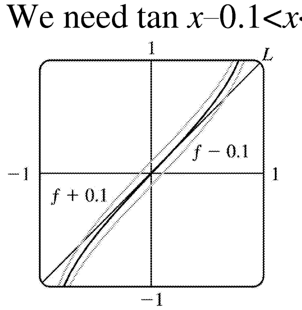


11. $f(x)=\sqrt[3]{1-x}=(1-x)^{1/3}\Rightarrow f'(x)=-\frac{1}{3}(1-x)^{-2/3}$, so $f(0)=1$ and $f'(0)=-\frac{1}{3}$. Thus,

$f(x)\approx f(0)+f'(0)(x-0)=1-\frac{1}{3}x$. We need $\sqrt[3]{1-x}-0.1<1-\frac{1}{3}x<\sqrt[3]{1-x}+0.1$, which is true when $-1.204<x<0.706$.

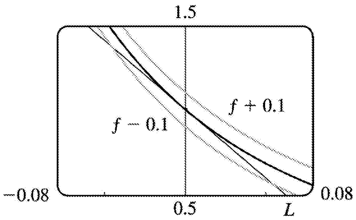


12. $f(x)=\tan x\Rightarrow f'(x)=\sec^2 x$, so $f(0)=0$ and $f'(0)=1$. Thus, $f(x)\approx f(0)+f'(0)(x-0)=0+1(x-0)=x$. We need $\tan x-0.1<x<\tan x+0.1$, which is true when $-0.63<x<0.63$.



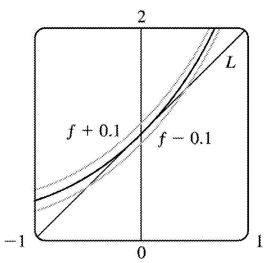
13. $f(x)=\dfrac{1}{(1+2x)^4}=(1+2x)^{-4}\Rightarrow f'(x)=-4(1+2x)^{-5}(2)=\dfrac{-8}{(1+2x)^5}$, so $f(0)=1$ and $f'(0)=-8$. Thus, $f(x)\approx f(0)+f'(0)(x-0)=1+(-8)(x-0)=1-8x$.

We need $1/(1+2x)^4-0.1<1-8x<1/(1+2x)^4+0.1$, which is true when $-0.045<x<0.055$.



14. $f(x)=e^x\Rightarrow f'(x)=e^x$, so $f(0)=1$ and $f'(0)=1$. Thus, $f(x)\approx f(0)+f'(0)(x-0)=1+1(x-0)=1+x$.

We need $e^x-0.1<1+x<e^x+0.1$, which is true when $-0.483<x<0.416$.



15. If $y=f(x)$, then the differential dy is equal to $f'(x)dx$. $y=x^4+5x\Rightarrow dy=(4x^3+5)dx$.

16. $y=\cos\pi x\Rightarrow dy=-\sin\pi x\cdot\pi\,dx=-\pi\sin\pi x\,dx$

17. $y=x\ln x\Rightarrow dy=\left(x\cdot\dfrac{1}{x}+\ln x\cdot 1\right)dx=(1+\ln x)dx$

18. $y=\sqrt{1+t^2}\Rightarrow dy=\dfrac{1}{2}(1+t^2)^{-1/2}(2t)dt=\dfrac{t}{\sqrt{1+t^2}}dt$

19. $y=\dfrac{u+1}{u-1}\Rightarrow dy=\dfrac{(u-1)(1)-(u+1)(1)}{(u-1)^2}du=\dfrac{-2}{(u-1)^2}du$

20. $y=(1+2r)^{-4}\Rightarrow dy=-4(1+2r)^{-5}\cdot 2dr=-8(1+2r)^{-5}dr$

21. **(a)** $y=x^2+2x\Rightarrow dy=(2x+2)dx$

(b) When $x=3$ and

$dx = \frac{1}{2}$, $dy = [2(3)+2]\left(\frac{1}{2}\right) = 4$.

22. **(a)** $y = e^{x/4} \Rightarrow dy = \frac{1}{4} e^{x/4} dx$

(b) When $x = 0$ and $dx = 0.1$, $dy = \left(\frac{1}{4} e^{0}\right)(0.1) = 0.025$.

23. **(a)** $y = \sqrt{4+5x} \Rightarrow dy = \frac{1}{2}(4+5x)^{-1/2} \cdot 5\,dx = \frac{5}{2\sqrt{4+5x}}\,dx$

(b) When $x = 0$ and $dx = 0.04$, $dy = \frac{5}{2\sqrt{4}}(0.04) = \frac{5}{4} \cdot \frac{1}{25} = \frac{1}{20} = 0.05$.

24. **(a)** $y = 1/(x+1) \Rightarrow dy = -\frac{1}{(x+1)^2}\,dx$

(b) When $x = 1$ and $dx = -0.01$, $dy = -\frac{1}{2^2}(-0.01) = \frac{1}{4} \cdot \frac{1}{100} = \frac{1}{400} = 0.0025$.

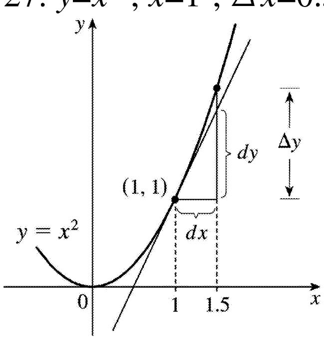
25. **(a)** $y = \tan x \Rightarrow dy = \sec^2 x\,dx$

(b) When $x = \pi/4$ and $dx = -0.1$, $dy = [\sec(\pi/4)]^2(-0.1) = (\sqrt{2})^2(-0.1) = -0.2$.

26. **(a)** $y = \cos x \Rightarrow dy = -\sin x\,dx$

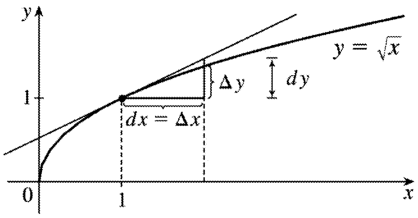
(b) When $x = \pi/3$ and $dx = 0.05$, $dy = -\sin(\pi/3)(0.05) = -0.5\sqrt{3}(0.05) = -0.025\sqrt{3} \approx -0.043$.

27. $y = x^2$, $x = 1$, $\Delta x = 0.5 \Rightarrow \Delta y = (1.5)^2 - 1^2 = 1.25$. $dy = 2x\,dx = 2(1)(0.5) = 1$

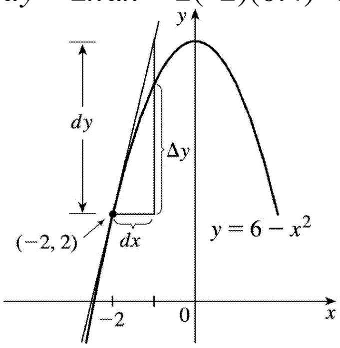


28. $y = \sqrt{x}$, $x = 1$, $\Delta x = 1 \Rightarrow \Delta y = \sqrt{2} - \sqrt{1} = \sqrt{2} - 1 \approx 0.414$

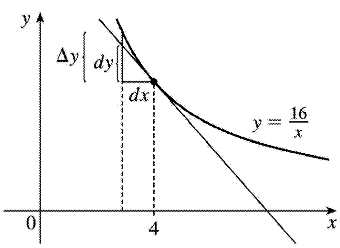
$dy = \frac{1}{2\sqrt{x}}\,dx = \frac{1}{2}(1) = 0.5$



29. $y=6-x^2$, $x=-2$, $\Delta x=0.4 \Rightarrow \Delta y=(6-(-1.6)^2)-(6-(-2)^2)=1.44$
$dy=-2x\,dx=-2(-2)(0.4)=1.6$



30. $y=\dfrac{16}{x}$, $x=4$, $\Delta x=-1 \Rightarrow \Delta y=\dfrac{16}{3}-\dfrac{16}{4}=\dfrac{4}{3}$. $dy=-\left(\dfrac{16}{x^2}\right)dx=-\left(\dfrac{16}{4^2}\right)(-1)=1$



31. $y=f(x)=x^5 \Rightarrow dy=5x^4\,dx$. When $x=2$ and $dx=0.001$, $dy=5(2)^4(0.001)=0.08$, so
$(2.001)^5=f(2.001)\approx f(2)+dy=32+0.08=32.08$.

32. $y=f(x)=\sqrt{x} \Rightarrow dy=\dfrac{1}{2\sqrt{x}}\,dx$. When $x=100$ and $dx=-0.2$, $dy=\dfrac{1}{2\sqrt{100}}(-0.2)=-0.01$, so
$\sqrt{99.8}=f(99.8)\approx f(100)+dy=10-0.01=9.99$.

33. $y=f(x)=x^{2/3} \Rightarrow dy=\dfrac{2}{3\sqrt[3]{x}}\,dx$. When $x=8$ and $dx=0.06$, $dy=\dfrac{2}{3\sqrt[3]{8}}(0.06)=0.02$, so
$(8.06)^{2/3}=f(8.06)\approx f(8)+dy=4+0.02=4.02$.

34. $y=f(x)=1/x \Rightarrow dy=(-1/x^2)dx$. When $x=1000$ and $dx=2, dy=[-1/(1000)^2](2)=-0.000002$, so
$1/1002=f(1002)\approx f(1000)+dy=1/1000-0.000002=0.000998$

35. $y=f(x)=\tan x \Rightarrow dy=\sec^2 x\,dx$. When $x=45^\circ$ and $dx=-1^\circ$, $dy=\sec^2 45^\circ(-\pi/180)=(\sqrt{2})^2(-\pi/180)=-\pi/90$, so $\tan 44^\circ=f(44^\circ)\approx f(45^\circ)+dy=1-\pi/90\approx 0.965$.

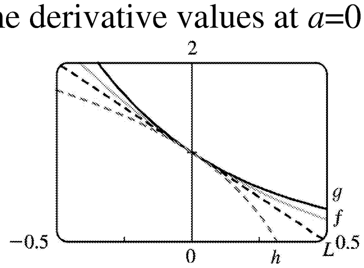
36. $y=f(x)=\ln x \Rightarrow dy=\frac{1}{x}\,dx$. When $x=1$ and $dx=0.07$, $dy=\frac{1}{1}(0.07)=0.07$, so $\ln 1.07=f(1.07)\approx f(1)+dy=0+0.07=0.07$.

37. $y=f(x)=\sec x \Rightarrow f'(x)=\sec x\tan x$, so $f(0)=1$ and $f'(0)=1\cdot 0=0$. The linear approximation of f at 0 is $f(0)+f'(0)(x-0)=1+0(x)=1$. Since 0.08 is close to 0 , approximating $\sec 0.08$ with 1 is reasonable.

38. If $y=x^6$, $y'=6x^5$ and the tangent line approximation at $(1,1)$ has slope 6 . If the change in x is 0.01 , the change in y on the tangent line is 0.06 , and approximating $(1.01)^6$ with 1.06 is reasonable.

39. $y=f(x)=\ln x \Rightarrow f'(x)=1/x$, so $f(1)=0$ and $f'(1)=1$. The linear approximation of f at 1 is $f(1)+f'(1)(x-1)=0+1(x-1)=x-1$. Now $f(1.05)=\ln 1.05\approx 1.05-1=0.05$, so the approximation is reasonable.

40. **(a)** $f(x)=(x-1)^2 \Rightarrow f'(x)=2(x-1)$, so $f(0)=1$ and $f'(0)=-2$.
Thus, $f(x)\approx L_f(x)=f(0)+f'(0)(x-0)=1-2x$.
$g(x)=e^{-2x} \Rightarrow g'(x)=-2e^{-2x}$, so $g(0)=1$ and $g'(0)=-2$.
Thus, $g(x)\approx L_g(x)=g(0)+g'(0)(x-0)=1-2x$.
$h(x)=1+\ln(1-2x) \Rightarrow h'(x)=\frac{-2}{1-2x}$, so $h(0)=1$ and $h'(0)=-2$.
Thus, $h(x)\approx L_h(x)=h(0)+h'(0)(x-0)=1-2x$.
Notice that $L_f=L_g=L_h$. This happens because f , g , and h have the same function values and the same derivative values at $a=0$.

(b) 

The linear approximation appears to be the best for the function f since it is closer to f for a larger domain than it is to g and h . The approximation looks worst for h since h moves away from L faster than f and g do.

41. **(a)** If x is the edge length, then $V=x^3 \Rightarrow dV=3x^2\,dx$. When $x=30$ and $dx=0.1$, $dV=3(30)^2(0.1)=270$, so the maximum possible error in computing the volume of the cube is about 270 cm^3 . The relative error is calculated by dividing the change in V , ΔV , by V . We approximate ΔV with dV .

Relative error = $\dfrac{\Delta V}{V} \approx \dfrac{dV}{V} = \dfrac{3x^2dx}{x^3} = 3\dfrac{dx}{x} = 3\left(\dfrac{0.1}{30}\right) = 0.01$.

Percentage error = relative error$\times$ 100%=0.01$\times$ 100%=1% .

(b) $S=6x^2 \Rightarrow dS=12x\,dx$. When $x=30$ and $dx=0.1$, $dS=12(30)(0.1)=36$, so the maximum possible error in computing the surface area of the cube is about 36 cm^2 .

Relative error = $\dfrac{\Delta S}{S} \approx \dfrac{dS}{S} = \dfrac{12x\,dx}{6x^2} = 2\dfrac{dx}{x} = 2\left(\dfrac{0.1}{30}\right) = 0.00\overline{6}$.

Percentage error = relative error$\times$ 100%=$0.00\overline{6}\times$ 100%=$0.\overline{6}$% .

42. **(a)** $A=\pi r^2 \Rightarrow dA=2\pi r\,dr$. When $r=24$ and $dr=0.2$, $dA=2\pi(24)(0.2)=9.6\pi$, so the maximum possible error in the calculated area of the disk is about $9.6\pi \approx 30$ cm^2 .

(b) Relative error = $\dfrac{\Delta A}{A} \approx \dfrac{dA}{A} = \dfrac{2\pi r\,dr}{\pi r^2} = \dfrac{2dr}{r} = \dfrac{2(0.2)}{24} = \dfrac{0.2}{12} = \dfrac{1}{60} = 0.01\overline{6}$.

Percentage error = relative error$\times$ 100%=$0.01\overline{6}\times$ 100%=$1.\overline{6}$% .

43. **(a)** For a sphere of radius r , the circumference is $C=2\pi r$ and the surface area is $S=4\pi r^2$, so $r=C/(2\pi) \Rightarrow S=4\pi(C/2\pi)^2=C^2/\pi \Rightarrow dS=(2/\pi)C\,dC$. When $C=84$ and $dC=0.5$, $dS=\dfrac{2}{\pi}(84)(0.5)=\dfrac{84}{\pi}$, so the maximum error is about $\dfrac{84}{\pi} \approx 27$ cm^2 . Relative error $\approx \dfrac{dS}{S} = \dfrac{84/\pi}{84^2/\pi} = \dfrac{1}{84} \approx 0.012$

(b) $V=\dfrac{4}{3}\pi r^3=\dfrac{4}{3}\pi\left(\dfrac{C}{2\pi}\right)^3=\dfrac{C^3}{6\pi^2} \Rightarrow dV=\dfrac{1}{2\pi^2}C^2\,dC$. When $C=84$ and $dC=0.5$, $dV=\dfrac{1}{2\pi^2}(84)^2(0.5)=\dfrac{1764}{\pi^2}$, so the maximum error is about $\dfrac{1764}{\pi^2} \approx 179$ cm^3 . The relative error is approximately $\dfrac{dV}{V} = \dfrac{1764/\pi^2}{(84)^3/(6\pi^2)} = \dfrac{1}{56} \approx 0.018$.

44. For a hemispherical dome,

$V=\frac{2}{3}\pi r^3\Rightarrow dV=2\pi r^2 dr$. When $r=\frac{1}{2}(50)=25$ m and $dr=0.05$ cm $=0.0005$ m, $dV=2\pi(25)^2(0.0005)=\frac{5\pi}{8}$, so the amount of paint needed is about $\frac{5\pi}{8}\approx 2$ m^3 .

45. **(a)** $V=\pi r^2 h\Rightarrow \Delta V\approx dV=2\pi rh\,dr=2\pi rh\,\Delta r$

(b) The error is

$$\begin{aligned}\Delta V-dV &=[\pi(r+\Delta r)^2 h-\pi r^2 h]-2\pi rh\,\Delta r=\pi r^2 h+2\pi rh\,\Delta r+\pi(\Delta r)^2 h-\pi r^2 h-2\pi rh\Delta r\\ &=\pi(\Delta r)^2 h\end{aligned}$$

46. $F=kR^4\Rightarrow dF=4kR^3 dR\Rightarrow \frac{dF}{F}=\frac{4kR^3 dR}{kR^4}=4\left(\frac{dR}{R}\right)$. Thus, the relative change in F is about 4 times the relative change in R . So a 5% increase in the radius corresponds to a 20% increase in blood flow.

47. **(a)** $dc=\frac{dc}{dx}dx=0\,dx=0$

(b) $d(cu)=\frac{d}{dx}(cu)dx=c\frac{du}{dx}dx=c\,du$

(c) $d(u+v)=\frac{d}{dx}(u+v)dx=\left(\frac{du}{dx}+\frac{dv}{dx}\right)dx=\frac{du}{dx}dx+\frac{dv}{dx}dx=du+dv$

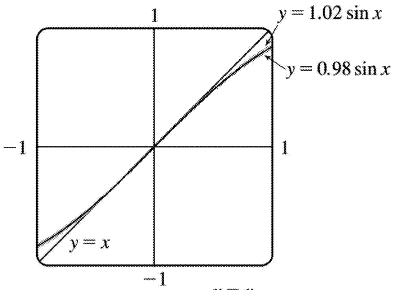
(d) $d(uv)=\frac{d}{dx}(uv)dx=\left(u\frac{dv}{dx}+v\frac{du}{dx}\right)dx=u\frac{dv}{dx}dx+v\frac{du}{dx}dx=u\,dv+v\,du$

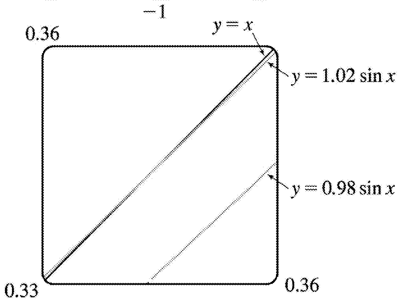
(e) $d\left(\frac{u}{v}\right)=\frac{d}{dx}\left(\frac{u}{v}\right)dx=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}dx=\frac{v\frac{du}{dx}dx-u\frac{dv}{dx}dx}{v^2}=\frac{v\,du-u\,dv}{v^2}$

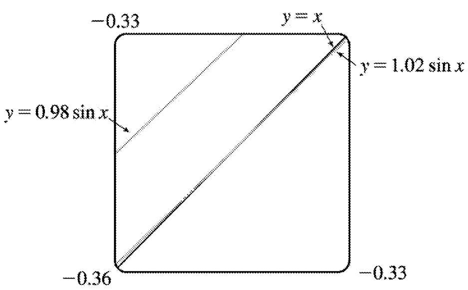
(f) $d(x^n)=\frac{d}{dx}(x^n)dx=nx^{n-1}dx$

48. **(a)** $f(x)=\sin x\Rightarrow f'(x)=\cos x$, so $f(0)=0$ and $f'(0)=1$. Thus, $f(x)\approx f(0)+f'(0)(x-0)=0+1(x-0)=x$.

(b)







We want to know the values of x for which $y=x$ approximates $y=\sin x$ with less than a 2% difference; that is, the values of x for which

$$\left|\frac{x-\sin x}{\sin x}\right|<0.02 \Leftrightarrow -0.02<\frac{x-\sin x}{\sin x}<0.02 \Leftrightarrow$$

$$\begin{cases} -0.02\sin x<x-\sin x<0.02\sin x & \text{if } \sin x>0 \\ -0.02\sin x>x-\sin x>0.02\sin x & \text{if } \sin x<0 \end{cases} \Leftrightarrow \begin{cases} 0.98\sin x<x<1.02\sin x & \text{if } \sin x>0 \\ 1.02\sin x<x<0.98\sin x & \text{if } \sin x<0 \end{cases}$$

In the first figure, we see that the graphs are very close to each other near $x=0$. Changing the viewing rectangle and using an intersect feature (see the second figure) we find that $y=x$ intersects $y=1.02\sin x$ at $x\approx 0.344$. By symmetry, they also intersect at $x\approx -0.344$ (see the third figure.). Converting 0.344 radians to degrees, we get $0.344\left(\frac{180^\circ}{\pi}\right)\approx 19.7^\circ\approx 20^\circ$, which verifies the statement.

49. **(a)** The graph shows that $f'(1)=2$, so $L(x)=f(1)+f'(1)(x-1)=5+2(x-1)=2x+3$. $f(0.9)\approx L(0.9)=4.8$ and $f(1.1)\approx L(1.1)=5.2$.

(b) From the graph, we see that $f'(x)$ is positive and decreasing. This means that the slopes of the tangent lines are positive, but the tangents are becoming less steep. So the tangent lines lie *above* the curve. Thus, the estimates in part (a) are too large.

50. **(a)** $g'(x)=\sqrt{x^2+5} \Rightarrow g'(2)=\sqrt{9}=3$. $g(1.95)\approx g(2)+g'(2)(1.95-2)=-4+3(-0.05)=-4.15$.
$g(2.05)\approx g(2)+g'(2)(2.05-2)=-4+3(0.05)=-3.85$.

(b) The formula $g'(x)=\sqrt{x^2+5}$ shows that $g'(x)$ is positive and increasing. This means that the slopes of the tangent lines are positive and the tangents are getting steeper. So the tangent lines lie *below* the graph of g . Hence, the estimates in part (a) are too small.

1. Product Rule:
$y=(x^2+1)(x^3+1)\Rightarrow$

$y'=(x^2+1)(3x^2)+(x^3+1)(2x)=3x^4+3x^2+2x^4+2x=5x^4+3x^2+2x$.
Multiplying first: $y=(x^2+1)(x^3+1)=x^5+x^3+x^2+1\Rightarrow y'=5x^4+3x^2+2x$ (equivalent).

2. Quotient Rule: $F(x)=\dfrac{x-3x\sqrt{x}}{\sqrt{x}}=\dfrac{x-3x^{3/2}}{x^{1/2}}\Rightarrow$

$$F'(x)=\frac{x^{1/2}\left(1-\frac{9}{2}x^{1/2}\right)-\left(x-3x^{3/2}\right)\left(\frac{1}{2}x^{-1/2}\right)}{\left(x^{1/2}\right)^2}$$

$$=\frac{x^{1/2}-\frac{9}{2}x-\frac{1}{2}x^{1/2}+\frac{3}{2}x}{x}=\frac{\frac{1}{2}x^{1/2}-3x}{x}=\frac{1}{2}x^{-1/2}-3$$

Simplifying first: $F(x)=\dfrac{x-3x\sqrt{x}}{\sqrt{x}}=\sqrt{x}-3x=x^{1/2}-3x\Rightarrow F'(x)=\dfrac{1}{2}x^{-1/2}-3$ (equivalent).
For this problem, simplifying first seems to be the better method.

3. By the Product Rule, $f(x)=x^2e^x\Rightarrow f'(x)=x^2\dfrac{d}{dx}(e^x)+e^x\dfrac{d}{dx}(x^2)=x^2e^x+e^x(2x)=xe^x(x+2)$.

4. By the Product Rule, $g(x)=\sqrt{x}\,e^x=x^{1/2}e^x\Rightarrow g'(x)=x^{1/2}(e^x)+e^x\left(\dfrac{1}{2}x^{-1/2}\right)=\dfrac{1}{2}x^{-1/2}e^x(2x+1)$.

5. By the Quotient Rule, $y=\dfrac{e^x}{x^2}\Rightarrow y'=\dfrac{x^2\frac{d}{dx}(e^x)-e^x\frac{d}{dx}(x^2)}{(x^2)^2}=\dfrac{x^2(e^x)-e^x(2x)}{x^4}=\dfrac{xe^x(x-2)}{x^4}=\dfrac{e^x(x-2)}{x^3}$

6. By the Quotient Rule, $y=\dfrac{e^x}{1+x}\Rightarrow y'=\dfrac{(1+x)e^x-e^x(1)}{(1+x)^2}=\dfrac{e^x+xe^x-e^x}{(x+1)^2}=\dfrac{xe^x}{(x+1)^2}$.

7. $g(x)=\dfrac{3x-1}{2x+1}\overset{\text{QR}}{\Rightarrow}g'(x)=\dfrac{(2x+1)(3)-(3x-1)(2)}{(2x+1)^2}=\dfrac{6x+3-6x+2}{(2x+1)^2}=\dfrac{5}{(2x+1)^2}$

8. $f(t)=\dfrac{2t}{4+t^2}\overset{\text{QR}}{\Rightarrow}f'(t)=\dfrac{(4+t^2)(2)-(2t)(2t)}{(4+t^2)^2}=\dfrac{8+2t^2-4t^2}{(4+t^2)^2}=\dfrac{8-2t^2}{(4+t^2)^2}$

9. $V(x)=(2x^3+3)(x^4-2x) \overset{\text{PR}}{\Rightarrow}$

$V'(x)=(2x^3+3)(4x^3-2)+(x^4-2x)(6x^2)=(8x^6+8x^3-6)+(6x^6-12x^3)=14x^6-4x^3-6$

10. $Y(u)=(u^{-2}+u^{-3})(u^5-2u^2)\overset{\text{PR}}{\Rightarrow}$

$Y'(u)=(u^{-2}+u^{-3})(5u^4-4u)+(u^5-2u^2)(-2u^{-3}-3u^{-4})$

$=(5u^2-4u^{-1}+5u-4u^{-2})+(-2u^2-3u+4u^{-1}+6u^{-2})=3u^2+2u+2u^{-2}$

11. $F(y)=\left(\dfrac{1}{y^2}-\dfrac{3}{y^4}\right)(y+5y^3)=\left(y^{-2}-3y^{-4}\right)\left(y+5y^3\right)\overset{\text{PR}}{\Rightarrow}$

$F'(y)=\left(y^{-2}-3y^{-4}\right)\left(1+15y^2\right)+\left(y+5y^3\right)\left(-2y^{-3}+12y^{-5}\right)$

$=\left(y^{-2}+15-3y^{-4}-45y^{-2}\right)+(-2y^{-2}+12y^{-4}-10+60y^{-2})$

$=5+14y^{-2}+9y^{-4}$ or $5+14/y^2+9/y^4$

12. $R(t)=(t+e^t)(3-\sqrt{t})=$

$R'(t)=(t+e^t)(-\dfrac{1}{2}t^{-1/2})+(3-\sqrt{t})(1+e^t)$

$=\left(-\dfrac{1}{2}t^{1/2}-\dfrac{1}{2}t^{-1/2}e^t\right)+(3+3e^t-\sqrt{t}-\sqrt{t}\,e^t)=3+3e^t-\dfrac{3}{2}\sqrt{t}-\sqrt{t}\,e^t-e^t/(2\sqrt{t})$

13. $y=\dfrac{t^2}{3t^2-2t+1}\overset{\text{QR}}{\Rightarrow}$

$y'=\dfrac{(3t^2-2t+1)(2t)-t^2(6t-2)}{(3t^2-2t+1)^2}=\dfrac{2t[3t^2-2t+1-t(3t-1)]}{(3t^2-2t+1)^2}$

$=\dfrac{2t(3t^2-2t+1-3t^2+t)}{(3t^2-2t+1)^2}=\dfrac{2t(1-t)}{(3t^2-2t+1)^2}$

14. $y=\dfrac{t^3+t}{t^4-2}\overset{\text{QR}}{\Rightarrow}y'=\dfrac{(t^4-2)(3t^2+1)-(t^3+t)(4t^3)}{(t^4-2)^2}=\dfrac{(3t^6+t^4-6t^2-2)-(4t^6+4t^4)}{(t^4-2)^2}$

$=\dfrac{-t^6-3t^4-6t^2-2}{(t^4-2)^2}=-\dfrac{t^6+3t^4+6t^2+2}{(t^4-2)^2}$

15. $y=(r^2-2r)e^r \Rightarrow y'=(r^2-2r)(e^r)+e^r(2r-2)=e^r(r^2-2r+2r-2)=e^r(r^2-2)$

16. $y=\dfrac{1}{s+ke^s} \Rightarrow y'=\dfrac{(s+ke^s)(0)-(1)(1+ke^s)}{(s+ke^s)^2}=-\dfrac{1+ke^s}{(s+ke^s)^2}$

17. $y=\dfrac{v^3-2v\sqrt{v}}{v}=v^2-2\sqrt{v}=v^2-2v^{1/2}\Rightarrow y'=2v-2\left(\dfrac{1}{2}\right)v^{-1/2}=2v-v^{-1/2}$.

We can change the form of the answer as follows: $2v-v^{-1/2}=2v-\dfrac{1}{\sqrt{v}}=\dfrac{2v\sqrt{v}-1}{\sqrt{v}}=\dfrac{2v^{3/2}-1}{\sqrt{v}}$

18. $z=w^{3/2}(w+ce^w)=w^{5/2}+cw^{3/2}e^w \Rightarrow z'=\dfrac{5}{2}w^{3/2}+c\left(w^{3/2}\cdot e^w+e^w\cdot\dfrac{3}{2}w^{1/2}\right)=\dfrac{5}{2}w^{3/2}+\dfrac{1}{2}cw^{1/2}e^w(2w+3)$

19. $y=\dfrac{1}{x^4+x^2+1} \Rightarrow y'=\dfrac{(x^4+x^2+1)(0)-1(4x^3+2x)}{(x^4+x^2+1)^2}=-\dfrac{2x(2x^2+1)}{(x^4+x^2+1)^2}$

20. $y=\dfrac{\sqrt{x}-1}{\sqrt{x}+1} \Rightarrow y'=\dfrac{(\sqrt{x}+1)\left(\dfrac{1}{2\sqrt{x}}\right)-(\sqrt{x}-1)\left(\dfrac{1}{2\sqrt{x}}\right)}{(\sqrt{x}+1)^2}=\dfrac{\dfrac{1}{2}+\dfrac{1}{2\sqrt{x}}-\dfrac{1}{2}+\dfrac{1}{2\sqrt{x}}}{(\sqrt{x}+1)^2}=\dfrac{1}{\sqrt{x}(\sqrt{x}+1)^2}$

21. $f(x)=\dfrac{x}{x+c/x} \Rightarrow f'(x)=\dfrac{(x+c/x)(1)-x(1-c/x^2)}{\left(x+\dfrac{c}{x}\right)^2}=\dfrac{x+c/x-x+c/x}{\left(\dfrac{x^2+c}{x}\right)^2}=\dfrac{2c/x}{\dfrac{(x^2+c)^2}{x^2}}\cdot\dfrac{x^2}{x^2}=\dfrac{2cx}{(x^2+c)^2}$

22. $f(x)=\dfrac{ax+b}{cx+d} \Rightarrow f'(x)=\dfrac{(cx+d)(a)-(ax+b)(c)}{(cx+d)^2}=\dfrac{acx+ad-acx-bc}{(cx+d)^2}=\dfrac{ad-bc}{(cx+d)^2}$

23. $y=\dfrac{2x}{x+1} \Rightarrow y'=\dfrac{(x+1)(2)-(2x)(1)}{(x+1)^2}=\dfrac{2}{(x+1)^2}$. At $(1,1)$, $y'=\dfrac{1}{2}$, and an equation of the tangent line is $y-1=\dfrac{1}{2}(x-1)$, or $y=\dfrac{1}{2}x+\dfrac{1}{2}$.

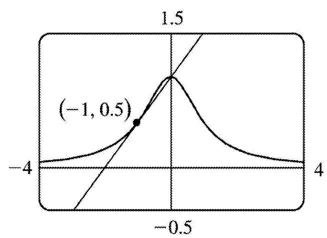
24. $y=\dfrac{\sqrt{x}}{x+1} \Rightarrow$

$y' = \dfrac{(x+1)\left(\dfrac{1}{2\sqrt{x}}\right) - \sqrt{x}\,(1)}{(x+1)^2} = \dfrac{(x+1)-(2x)}{2\sqrt{x}\,(x+1)^2} = \dfrac{1-x}{2\sqrt{x}\,(x+1)^2}$. At $(4,0.4)$, $y' = \dfrac{-3}{100} = -0.03$, and an equation of the tangent line is $y-0.4=-0.03(x-4)$, or $y=-0.03x+0.52$.

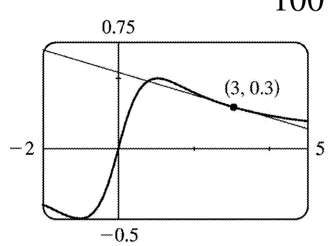
25. $y=2xe^x \Rightarrow y' = 2(x\cdot e^x + e^x \cdot 1) = 2e^x(x+1)$. At $(0, 0)$, $y' = 2e^0(0+1) = 2\cdot 1\cdot 1 = 2$, and an equation of the tangent line is $y-0=2(x-0)$, or $y=2x$.

26. $y=\dfrac{e^x}{x} \Rightarrow y' = \dfrac{x\cdot e^x - e^x \cdot 1}{x^2} = \dfrac{e^x(x-1)}{x^2}$. At $(1, e)$, $y'=0$, and an equation of the tangent line is $y-e=0(x-1)$, or $y=e$.

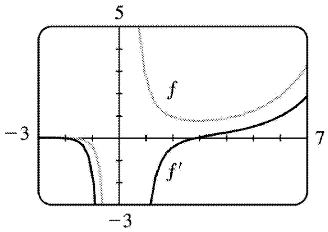
27. **(a)** $y=f(x)=\dfrac{1}{1+x^2} \Rightarrow f'(x)=\dfrac{(1+x^2)(0)-1(2x)}{(1+x^2)^2} = \dfrac{-2x}{(1+x^2)^2}$. So the slope of the tangent line at the point $\left(-1, \dfrac{1}{2}\right)$ is $f'(-1)=\dfrac{2}{2^2}=\dfrac{1}{2}$ and its equation is $y-\dfrac{1}{2}=\dfrac{1}{2}(x+1)$ or $y=\dfrac{1}{2}x+1$.

(b) 

28. **(a)** $y=f(x)=\dfrac{x}{1+x^2} \Rightarrow f'(x)=\dfrac{(1+x^2)1-x(2x)}{(1+x^2)^2} = \dfrac{1-x^2}{(1+x^2)^2}$. So the slope of the tangent line at the point $(3,0.3)$ is $f'(3)=\dfrac{-8}{100}$ and its equation is $y-0.3=-0.08(x-3)$ or $y=-0.08x+0.54$.

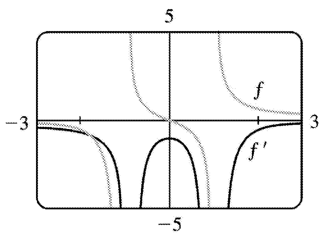
(b) 

29. **(a)** $f(x)=\dfrac{e^x}{x^3} \Rightarrow f'(x)=\dfrac{x^3(e^x)-e^x(3x^2)}{(x^3)^2} = \dfrac{x^2e^x(x-3)}{x^6} = \dfrac{e^x(x-3)}{x^4}$

(b) 

$f'=0$ when f has a horizontal tangent line, f' is negative when f is decreasing, and f' is positive when f is increasing.

30. $f(x)=\dfrac{x}{x^2-1} \Rightarrow f'(x)=\dfrac{(x^2-1)1-x(2x)}{(x^2-1)^2}=\dfrac{-x^2-1}{(x^2-1)^2}=-\dfrac{x^2+1}{(x^2-1)^2}$ Notice that the slopes of all tangents to f are negative and $f'(x)<0$ always.



31. We are given that $f(5)=1$, $f'(5)=6$, $g(5)=-3$, and $g'(5)=2$.

(a) $(fg)'(5)=f(5)g'(5)+g(5)f'(5)=(1)(2)+(-3)(6)=2-18=-16$

(b) $\left(\dfrac{f}{g}\right)'(5)=\dfrac{g(5)f'(5)-f(5)g'(5)}{[g(5)]^2}=\dfrac{(-3)(6)-(1)(2)}{(-3)^2}=-\dfrac{20}{9}$

(c) $\left(\dfrac{g}{f}\right)'(5)=\dfrac{f(5)g'(5)-g(5)f'(5)}{[f(5)]^2}=\dfrac{(1)(2)-(-3)(6)}{(1)^2}=20$

32. We are given that $f(3)=4$, $g(3)=2$, $f'(3)=-6$, and $g'(3)=5$.

(a) $(f+g)'(3)=f'(3)+g'(3)=-6+5=-1$

(b) $(fg)'(3)=f(3)g'(3)+g(3)f'(3)=(4)(5)+(2)(-6)=20-12=8$

(c) $\left(\dfrac{f}{g}\right)'(3)=\dfrac{g(3)f'(3)-f(3)g'(3)}{[g(3)]^2}=\dfrac{(2)(-6)-(4)(5)}{(2)^2}=\dfrac{-32}{4}=-8$

(d)

$$\left(\frac{f}{f-g}\right)'(3)=\frac{[f(3)-g(3)]f'(3)-f(3)[f'(3)-g'(3)]}{[f(3)-g(3)]^2}$$
$$=\frac{(4-2)(-6)-4(-6-5)}{(4-2)^2}=\frac{-12+44}{2^2}=8$$

33. $f(x)=e^x g(x)\Rightarrow f'(x)=e^x g'(x)+g(x)e^x=e^x\left[g'(x)+g(x)\right]$.
$f'(0)=e^0\left[g'(0)+g(0)\right]=1(5+2)=7$

34. $\frac{d}{dx}\left[\frac{h(x)}{x}\right]=\frac{xh'(x)-h(x)\cdot 1}{x^2}\Rightarrow\frac{d}{dx}\left[\frac{h(x)}{x}\right]_{x=2}=\frac{2h'(2)-h(2)}{2^2}=\frac{2(-3)-(4)}{4}=\frac{-10}{4}=-2.5$

35. **(a)** From the graphs of f and g, we obtain the following values: $f(1)=2$ since the point $(1,2)$ is on the graph of f; $g(1)=1$ since the point $(1,1)$ is on the graph of g; $f'(1)=2$ since the slope of the line segment between $(0,0)$ and $(2,4)$ is $\frac{4-0}{2-0}=2$; $g'(1)=-1$ since the slope of the line segment between $(-2,4)$ and $(2,0)$ is $\frac{0-4}{2-(-2)}=-1$. Now $u(x)=f(x)g(x)$, so $u'(1)=f(1)g'(1)+g(1)f'(1)=2\cdot(-1)+1\cdot 2=0$.

(b) $v(x)=f(x)/g(x)$, so $v'(5)=\frac{g(5)f'(5)-f(5)g'(5)}{[g(5)]^2}=\frac{2\left(-\frac{1}{3}\right)-3\cdot\frac{2}{3}}{2^2}=\frac{-\frac{8}{3}}{4}=-\frac{2}{3}$

36. **(a)** $P(x)=F(x)G(x)$, so $P'(2)=F(2)G'(2)+G(2)F'(2)=3\cdot\frac{2}{4}+2\cdot 0=\frac{3}{2}$.

(b) $Q(x)=F(x)/G(x)$, so $Q'(7)=\frac{G(7)F'(7)-F(7)G'(7)}{[G(7)]^2}=\frac{1\cdot\frac{1}{4}-5\cdot\left(-\frac{2}{3}\right)}{1^2}=\frac{1}{4}+\frac{10}{3}=\frac{43}{12}$

37. **(a)** $y=xg(x)\Rightarrow y'=xg'(x)+g(x)\cdot 1=xg'(x)+g(x)$

(b) $y=\frac{x}{g(x)}\Rightarrow y'=\frac{g(x)\cdot 1-xg'(x)}{[g(x)]^2}=\frac{g(x)-xg'(x)}{[g(x)]^2}$

(c) $y=\frac{g(x)}{x}\Rightarrow y'=\frac{xg'(x)-g(x)\cdot 1}{(x)^2}=\frac{xg'(x)-g(x)}{x^2}$

38. **(a)**

$y=x^2 f(x) \Rightarrow y' = x^2 f'(x)+f(x)(2x)$

(b) $y=\dfrac{f(x)}{x^2} \Rightarrow y' = \dfrac{x^2 f'(x)-f(x)(2x)}{(x^2)^2} = \dfrac{xf'(x)-2f(x)}{x^3}$

(c) $y=\dfrac{x^2}{f(x)} \Rightarrow y' = \dfrac{f(x)(2x)-x^2 f'(x)}{[f(x)]^2}$

(d) $y=\dfrac{1+xf(x)}{\sqrt{x}} \Rightarrow$

$$y' = \frac{\sqrt{x}[xf'(x)+f(x)]-[1+xf(x)]\dfrac{1}{2\sqrt{x}}}{(\sqrt{x})^2}$$

$$= \frac{x^{3/2}f'(x)+x^{1/2}f(x)-\frac{1}{2}x^{-1/2}-\frac{1}{2}x^{1/2}f(x)}{x} \cdot \frac{2x^{1/2}}{2x^{1/2}} = \frac{xf(x)+2x^2 f'(x)-1}{2x^{3/2}}$$

39. If $P(t)$ denotes the population at time t and $A(t)$ the average annual income, then $T(t)=P(t)A(t)$ is the total personal income. The rate at which $T(t)$ is rising is given by $T'(t)=P(t)A'(t)+A(t)P'(t) \Rightarrow$

$$T'(1999) = P(1999)A'(1999)+A(1999)P'(1999) = (961{,}400)(\$1400/\text{yr})+(\$30{,}593)(9200/\text{yr})$$
$$= \$1{,}345{,}960{,}000/\text{yr} + \$281{,}455{,}600/\text{yr} = \$1{,}627{,}415{,}600/\text{yr}$$

So the total personal income was rising by about \$ 1.627 billion per year in 1999.
The term $P(t)A'(t) \approx \$\,1.346$ billion represents the portion of the rate of change of total income due to the existing population's increasing income. The term $A(t)P'(t) \approx \$\,281$ million represents the portion of the rate of change of total income due to increasing population.

40. **(a)** $f(20)=10{,}000$ means that when the price of the fabric is \$20/ yard, 10 , 000 yards will be sold.
$f'(20)=-350$ means that as the price of the fabric increases past \$20/ yard, the amount of fabric which will be sold is decreasing at a rate of 350 yards per (dollar per yard).

(b) $R(p)=pf(p) \Rightarrow R'(p)=pf'(p)+f(p)\cdot 1 \Rightarrow R'(20)=20f'(20)+f(20)\cdot 1=20(-350)+10{,}000=3000$.
This means that as the price of the fabric increases past \$20/ yard, the total revenue is increasing at \$3000/(\$/yard). Note that the Product Rule indicates that we will lose \$7000/(\$/yard) due to selling less fabric, but that that loss is more than made up for by the additional revenue due to the increase in price.

41. If $y=f(x)=\frac{x}{x+1}$, then $f'(x)=\frac{(x+1)(1)-x(1)}{(x+1)^2}=\frac{1}{(x+1)^2}$. When $x=a$, the equation of the tangent line is $y-\frac{a}{a+1}=\frac{1}{(a+1)^2}(x-a)$. This line passes through (1,2) when $2-\frac{a}{a+1}=\frac{1}{(a+1)^2}(1-a)\Leftrightarrow$

$2(a+1)^2-a(a+1)=1-a\Leftrightarrow 2a^2+4a+2-a^2-a-1+a=0\Leftrightarrow a^2+4a+1=0$.

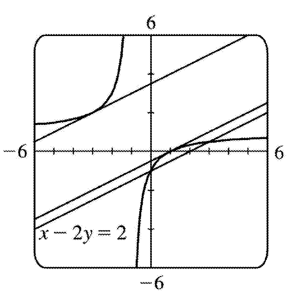
The quadratic formula gives the roots of this equation as $a=\frac{-4\pm\sqrt{4^2-4(1)(1)}}{2(1)}=\frac{-4\pm\sqrt{12}}{2}=-2\pm\sqrt{3}$, so there are two such tangent lines. Since

$$f(-2\pm\sqrt{3})=\frac{-2\pm\sqrt{3}}{-2\pm\sqrt{3}+1}=\frac{-2\pm\sqrt{3}}{-1\pm\sqrt{3}}\cdot\frac{-1\mp\sqrt{3}}{-1\mp\sqrt{3}}$$
$$=\frac{2\pm 2\sqrt{3}\mp\sqrt{3}-3}{1-3}=\frac{-1\pm\sqrt{3}}{-2}=\frac{1\mp\sqrt{3}}{2},$$

the lines touch the curve at $A\left(-2+\sqrt{3},\frac{1-\sqrt{3}}{2}\right)\approx(-0.27,-0.37)$ and $B\left(-2-\sqrt{3},\frac{1+\sqrt{3}}{2}\right)\approx(-3.73,1.37)$.

42. $y=\frac{x-1}{x+1}\Rightarrow y'=\frac{(x+1)(1)-(x-1)(1)}{(x+1)^2}=\frac{2}{(x+1)^2}$. If the tangent intersects the curve when $x=a$, then its slope is $2/(a+1)^2$. But if the tangent is parallel to $x-2y=2$, that is, $y=\frac{1}{2}x-1$, then its slope is $\frac{1}{2}$. Thus, $\frac{2}{(a+1)^2}=\frac{1}{2}\Rightarrow(a+1)^2=4\Rightarrow a+1=\pm 2\Rightarrow a=1$ or -3 . When $a=1$, $y=0$ and the equation of the tangent is $y-0=\frac{1}{2}(x-1)$ or $y=\frac{1}{2}x-\frac{1}{2}$.

When $a=-3$, $y=2$ and the equation of the tangent is $y-2=\frac{1}{2}(x+3)$ or $y=\frac{1}{2}x+\frac{7}{2}$.



43. **(a)** $(fgh)'=[(fg)h]'=(fg)'h+(fg)h'=(f'g+fg')h+(fg)h'=f'gh+fg'h+fgh'$

(b) Putting $f=g=h$ in part (a) , we have

$$\frac{d}{dx}[f(x)]^3=(fff)'=f'ff+ff'f+fff'=3fff'=3[f(x)]^2f'(x).$$

(c) $\frac{d}{dx}(e^{3x})=\frac{d}{dx}(e^x)^3=3(e^x)^2e^x=3e^{2x}e^x=3e^{3x}$

44. **(a)**

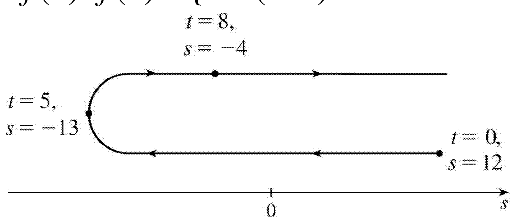
$$\frac{d}{dx}\left(\frac{1}{g(x)}\right)=\frac{g(x)\cdot\frac{d}{dx}(1)-1\cdot\frac{d}{dx}[g(x)]}{[g(x)]^2}\quad\text{[Quotient Rule]}$$

$$=\frac{g(x)\cdot 0-1\cdot g'(x)}{[g(x)]^2}=\frac{0-g'(x)}{[g(x)]^2}=-\frac{g'(x)}{[g(x)]^2}$$

(b) $y=\frac{1}{x^4+x^2+1}\Rightarrow y'=-\frac{4x^3+2x}{(x^4+x^2+1)^2}$ or $\frac{-2x(2x^2+1)}{(x^4+x^2+1)^2}$

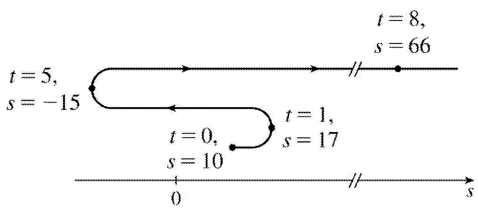
(c) $\frac{d}{dx}(x^{-n})=\frac{d}{dx}\left(\frac{1}{x^n}\right)=-\frac{(x^n)'}{(x^n)^2}$ [by the Reciprocal Rule] $=-\frac{nx^{n-1}}{x^{2n}}=-nx^{n-1-2n}=-nx^{-n-1}$

1. **(a)** $s=f(t)=t^2-10t+12 \Rightarrow v(t)=f'(t)=2t-10$
(b) $v(3)=2(3)-10=-4$ ft / s
(c) The particle is at rest when $v(t)=0 \Leftrightarrow 2t-10=0 \Leftrightarrow t=5$ s.
(d) The particle is moving in the positive direction when $v(t)>0 \Leftrightarrow 2t-10>0 \Leftrightarrow 2t>10 \Leftrightarrow t>5$.
(e) Since the particle is moving in the positive direction and in the negative direction, we need to calculate the distance traveled in the intervals [0,5] and [5,8] separately. $|f(5)-f(0)|=|-13-12|=25$ ft and $|f(8)-f(5)|=|\{-4-(-13)|=9$ ft. The total distance traveled during the first 8 s is $25+9=34$ ft.

(f) 

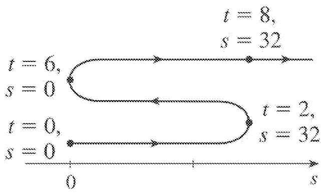
2. **(a)** $s=f(t)=t^3-9t^2+15t+10 \Rightarrow v(t)=f'(t)=3t^2-18t+15=3(t-1)(t-5)$

(b) $v(3)=3(2)(-2)=-12$ ft / s
(c) $v(t)=0 \Leftrightarrow t=1$ s or 5 s
(d) $v(t)>0 \Leftrightarrow 0 \le t<1$ or $t>5$
(e) $|f(1)-f(0)|=|17-10|=7$, $|f(5)-f(1)|=|-15-17|=32$, and $|f(8)-f(5)|=|66-(-15)|=81$. Total distance $=7+32+81=120$ ft.

(f) 

3. **(a)** $s=f(t)=t^3-12t^2+36t \Rightarrow v(t)=f'(t)=3t^2-24t+36$
(b) $v(3)=27-72+36=-9$ ft / s

(c) The particle is at rest when $v(t)=0$. $3t^2-24t+36=0 \Rightarrow 3(t-2)(t-6)=0 \Rightarrow t=2$ s or 6 s.
(d) The particle is moving in the positive direction when $v(t)>0$. $3(t-2)(t-6)>0 \Leftrightarrow 0 \le t<2$ or $t>6$.
(e) Since the particle is moving in the positive direction and in the negative direction, we need to calculate the distance traveled in the intervals (0,2) , (2,6) , and [6,8] separately.
$|f(2)-f(0)|=|32-0|=32$.
$|f(6)-f(2)|=|0-32|=32$.
$|f(8)-f(6)|=|32-0|=32$.
The total distance is $32+32+32=96$ ft.
(f)



4. **(a)** $s=f(t)=t^4-4t+1 \Rightarrow v(t)=f'(t)=4t^3-4$

(b) $v(3)=4(3)^3-4=104$ ft / s

(c) It is at rest when $v(t)=4(t^3-1)=4(t-1)(t^2+t+1)=0 \Leftrightarrow t=1$ s.

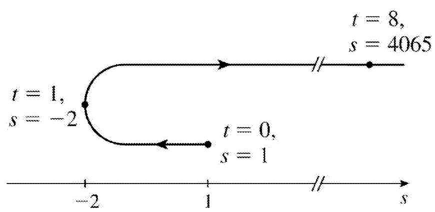
(d) It moves in the positive direction when $4(t^3-1)>0 \Leftrightarrow t>1$.

(e) Distance in positive direction $=|f(8)-f(1)|=|4065-(-2)|=4067$ ft

Distance in negative direction $=|f(1)-f(0)|=|-2-1|=3$ ft

Total distance traveled $=4067+3=4070$ ft

(f)



5. **(a)** $s=\dfrac{t}{t^2+1} \Rightarrow v(t)=s'(t)=\dfrac{(t^2+1)(1)-t(2t)}{(t^2+1)^2}=\dfrac{1-t^2}{(t^2+1)^2}$

(b) $v(3)=\dfrac{1-(3)^2}{(3^2+1)^2}=\dfrac{1-9}{10^2}=\dfrac{-8}{100}=-\dfrac{2}{25}$ ft / s

(c) It is at rest when $v=0 \Leftrightarrow 1-t^2=0 \Leftrightarrow t=1$ s [$t\neq -1$ since $t\geq 0$].

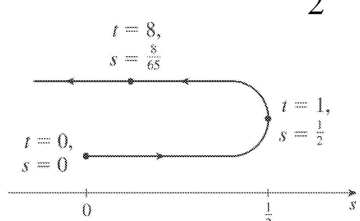
(d) It moves in the positive direction when $v>0 \Leftrightarrow 1-t^2>0 \Leftrightarrow t^2<1 \Leftrightarrow 0\leq t<1$.

(e) Distance in positive direction $=|s(1)-s(0)|=\left|\dfrac{1}{2}-0\right|=\dfrac{1}{2}$ ft

Distance in negative direction $=|s(8)-s(1)|=\left|\dfrac{8}{65}-\dfrac{1}{2}\right|=\dfrac{49}{130}$ ft

Total distance traveled $=\dfrac{1}{2}+\dfrac{49}{130}=\dfrac{57}{65}$ ft

(f)



6. **(a)** $s=\sqrt{t}(3t^2-35t+90)=3t^{5/2}-35t^{3/2}+90t^{1/2}\Rightarrow$

$v(t)=s'(t)=\frac{15}{2}t^{3/2}-\frac{105}{2}t^{1/2}+45t^{-1/2}=\frac{15}{2}t^{-1/2}(t^2-7t+6)=\frac{15}{2\sqrt{t}}(t-1)(t-6)$

(b) $v(3)=\frac{15}{2\sqrt{3}}(2)(-3)=-15\sqrt{3}$ ft / s

(c) It is at rest when $v=0\Leftrightarrow t=1$ s or 6 s.

(d) It moves in the positive direction when $v>0\Leftrightarrow(t-1)(t-6)>0\Leftrightarrow 0\leq t<1$ or $t>6$.

(e)

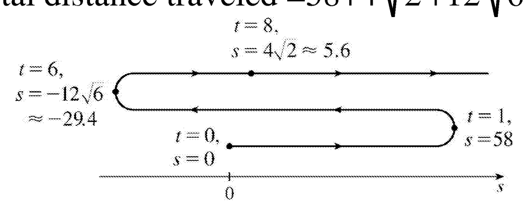
Distance in positive direction $=|s(1)-s(0)|+|s(8)-s(6)|=|58-0|+|4\sqrt{2}-(-12\sqrt{6})$

$=58+4\sqrt{2}+12\sqrt{6}\approx 93.05$ ft

Distance in negative direction $=|s(6)-s(1)|=|-12\sqrt{6}-58|=58+12\sqrt{6}\approx 87.39$ ft

Total distance traveled $=58+4\sqrt{2}+12\sqrt{6}+58+12\sqrt{6}=116+4\sqrt{2}+24\sqrt{6}\approx 180.44$ ft

(f)



7. $s(t)=t^3-4.5t^2-7t\Rightarrow v(t)=s'(t)=3t^2-9t-7=5\Leftrightarrow 3t^2-9t-12=0\Leftrightarrow 3(t-4)(t+1)=0\Leftrightarrow t=4$ or -1 . Since $t\geq 0$, the particle reaches a velocity of 5 m / s at $t=4$ s.

8. **(a)** $s=5t+3t^2\Rightarrow v(t)=\frac{ds}{dt}=5+6t$, so $v(2)=5+6(2)=17$ m / s.

(b) $v(t)=35\Rightarrow 5+6t=35\Rightarrow 6t=30\Rightarrow t=5$ s.

9. **(a)** $h=10t-0.83t^2\Rightarrow v(t)=\frac{dh}{dt}=10-1.66t$, so $v(3)=10-1.66(3)=5.02$ m / s.

(b) $h=25\Rightarrow 10t-0.83t^2=25\Rightarrow 0.83t^2-10t+25=0\Rightarrow t=\frac{10\pm\sqrt{17}}{1.66}\approx 3.54$ or 8.51 .

The value $t_1=(10-\sqrt{17})/1.66$ corresponds to the time it takes for the stone to rise 25 m and $t_2=(10+\sqrt{17})/1.66$ corresponds to the time when the stone is 25 m high on the way down. Thus, $v(t_1)=10-1.66[(10-\sqrt{17})/1.66]=\sqrt{17}\approx 4.12$ m / s.

10. **(a)** At maximum height the velocity of the ball is 0 ft / s. $v(t)=s'(t)=80-32t=0\Leftrightarrow 32t=80\Leftrightarrow t=\frac{5}{2}$.

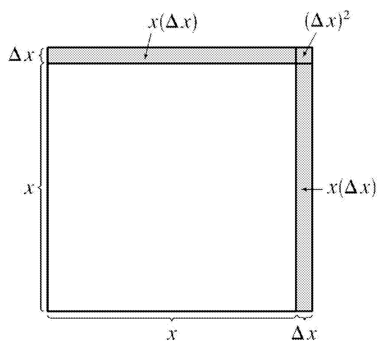
So the maximum height is

$s\left(\frac{5}{2}\right)=80\left(\frac{5}{2}\right)-16\left(\frac{5}{2}\right)^2=200-100=100$ ft.

(b) $s(t)=80t-16t^2=96 \Leftrightarrow 16t^2-80t+96=0 \Leftrightarrow 16(t^2-5t+6)=0 \Leftrightarrow 16(t-3)(t-2)=0$.
So the ball has a height of 96 ft on the way up at $t=2$ and on the way down at $t=3$. At these times the velocities are $v(2)=80-32(2)=16$ ft / s and $v(3)=80-32(3)=-16$ ft / s, respectively.

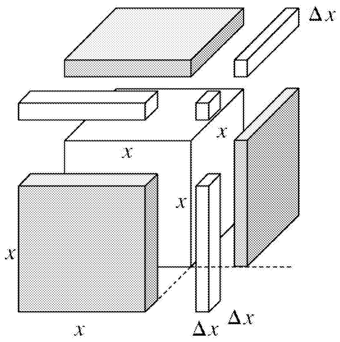
11. **(a)** $A(x)=x^2 \Rightarrow A'(x)=2x$. $A'(15)=30$ mm ^{2}/ mm is the rate at which the area is increasing with respect to the side length as x reaches 15 mm.

(b) The perimeter is $P(x)=4x$, so $A'(x)=2x=\frac{1}{2}(4x)=\frac{1}{2}P(x)$. The figure suggests that if Δx is small, then the change in the area of the square is approximately half of its perimeter (2 of the 4 sides) times Δx . From the figure, $\Delta A=2x(\Delta x)+(\Delta x)^2$. If Δx is small, then $\Delta A\approx 2x(\Delta x)$ and so $\Delta A/\Delta x\approx 2x$.



12. **(a)** $V(x)=x^3 \Rightarrow \frac{dV}{dx}=3x^2$. $\left.\frac{dV}{dx}\right|_{x=3}=3(3)^2=27$ mm ^{3}/ mm is the rate at which the volume is increasing as x increases past 3 mm.

(b) The surface area is $S(x)=6x^2$, so $V'(x)=3x^2=\frac{1}{2}(6x^2)=\frac{1}{2}S(x)$. The figure suggests that if Δx is small, then the change in the volume of the cube is approximately half of its surface area (the area of 3 of the 6 faces) times Δx . From the figure, $\Delta V=3x^2(\Delta x)+3x(\Delta x)^2+(\Delta x)^3$. If Δx is small, then $\Delta V\approx 3x^2(\Delta x)$ and so $\Delta V/\Delta x\approx 3x^2$.



13. **(a)**

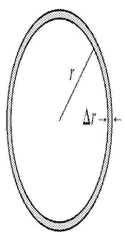
(i) $\dfrac{A(3)-A(2)}{3-2}=\dfrac{9\pi-4\pi}{1}=5\pi$

(ii) $\dfrac{A(2.5)-A(2)}{2.5-2}=\dfrac{6.25\pi-4\pi}{0.5}=4.5\pi$

(iii) $\dfrac{A(2.1)-A(2)}{2.1-2}=\dfrac{4.41\pi-4\pi}{0.1}=4.1\pi$

(b) $A(r)=\pi r^2 \Rightarrow A'(r)=2\pi r$, so $A'(2)=4\pi$.

(c) The circumference is $C(r)=2\pi r=A'(r)$. The figure suggests that if Δr is small, then the change in the area of the circle (a ring around the outside) is approximately equal to its circumference times Δr . Straightening out this ring gives us a shape that is approximately rectangular with length $2\pi r$ and width Δr , so $\Delta A\approx 2\pi r(\Delta r)$. Algebraically, $\Delta A=A(r+\Delta r)-A(r)=\pi(r+\Delta r)^2-\pi r^2=2\pi r(\Delta r)+\pi(\Delta r)^2$. So we see that if Δr is small, then $\Delta A\approx 2\pi r(\Delta r)$ and therefore, $\Delta A/\Delta r\approx 2\pi r$.



14. **(a)** $A'(1)=7200\pi\ \text{cm}^2/\text{s}$

(b) $A'(3)=21,600\pi\ \text{cm}^2/\text{s}$

(c) $A'(5)=36,000\pi\ \text{cm}^2/\text{s}$

15. **(a)** $S'(1)=8\pi\ \text{ft}^2/\text{ft}$

(b) $S'(2)=16\pi\ \text{ft}^2/\text{ft}$

(c) $S'(3)=24\pi\ \text{ft}^2/\text{ft}$

16. **(a)**

(a)
$$\frac{V(8)-V(5)}{8-5}=\frac{\frac{4}{3}\pi(512)-\frac{4}{3}\pi(125)}{3}=172\pi\ \mu\text{ m}^3/\mu\text{ m}$$

(b)
$$\frac{V(8)-V(5)}{8-5}=\frac{\frac{4}{3}\pi(512)-\frac{4}{3}\pi(125)}{3}=172\pi\ \mu\text{ m}^3/\mu\text{ m}$$

(c)
$$\frac{V(6)-V(5)}{6-5}=\frac{\frac{4}{3}\pi(216)-\frac{4}{3}\pi(125)}{1}=121.\overline{3}\pi\ \mu\text{ m}^3/\mu\text{ m}$$

(d)
$$\frac{V(6)-V(5)}{6-5}=\frac{\frac{4}{3}\pi(216)-\frac{4}{3}\pi(125)}{1}=121.\overline{3}\pi\ \mu\text{ m}^3/\mu\text{ m}$$

(e)
$$\frac{V(5.1)-V(5)}{5.1-5}=\frac{\frac{4}{3}\pi(5.1)^3-\frac{4}{3}\pi(5)^3}{0.1}=102.01\overline{3}\pi\ \mu\text{ m}^3/\mu\text{ m}$$

(f)
$$\frac{V(5.1)-V(5)}{5.1-5}=\frac{\frac{4}{3}\pi(5.1)^3-\frac{4}{3}\pi(5)^3}{0.1}=102.01\overline{3}\pi\ \mu\text{ m}^3/\mu\text{ m}$$

(b) $V'(r)=4\pi r^2$, so $V'(5)=100\pi\ \mu\text{ m}^3/\mu\text{ m}$.

(c) $V(r)=\frac{4}{3}\pi r^3\Rightarrow V'(r)=4\pi r^2=S(r)$. By analogy with Exercise 13(c) , we can say that the change in the volume of the spherical shell, ΔV , is approximately equal to its thickness, Δr , times the surface area of the inner sphere. Thus, $\Delta V\approx 4\pi r^2(\Delta r)$ and so $\Delta V/\Delta r\approx 4\pi r^2$.

17. **(a)** $\rho(1)=6$ kg / m

(b) $\rho(2)=12$ kg / m

(c) $\rho(3)=18$ kg / m

18. **(a)** $V'(5)=-250\left(1-\frac{5}{40}\right)=-218.75$ gal / min

(b) $V'(10)=-250\left(1-\frac{10}{40}\right)=-187.5$ gal / min

(c) $V'(20)=-250\left(1-\frac{20}{40}\right)=-125$ gal / min

(d)

$V'(40)=-250\left(1-\dfrac{40}{40}\right)=0$ gal / min

19. **(a)** $Q'(0.5)=3(0.5)^2-4(0.5)+6=4.75$ A
(b) $Q'(1)=3(1)^2-4(1)+6=5$ A

20. **(a)** $F=\dfrac{GmM}{r^2}=(GmM)r^{-2}\Rightarrow \dfrac{dF}{dr}=-2(GmM)r^{-3}=-\dfrac{2GmM}{r^3}$, which is the rate of change of the force with respect to the distance between the bodies. The minus sign indicates that as the distance r between the bodies increases, the magnitude of the force F exerted by the body of mass m on the body of mass M is decreasing.
(b) Given $F'(20,000)=-2$, find $F'(10,000)$. $-2=-\dfrac{2GmM}{20{,}000^3}\Rightarrow GmM=20,000^3$.

$F'(10{,}000)=-\dfrac{2\left(20{,}000^3\right)}{10{,}000^3}=-2\cdot 2^3=-16$ N / km

21. **(a)** To find the rate of change of volume with respect to pressure, we first solve for V in terms of P .
$PV=C\Rightarrow V=\dfrac{C}{P}\Rightarrow \dfrac{dV}{dP}=-\dfrac{C}{P^2}$.
(b) From the formula for dV/dP in part (a), we see that as P increases, the absolute value of dV/dP decreases. Thus, the volume is decreasing more rapidly at the beginning.
(c) $\beta=-\dfrac{1}{V}\dfrac{dV}{dP}=-\dfrac{1}{V}\left(-\dfrac{C}{P^2}\right)=\dfrac{C}{(PV)P}=\dfrac{C}{CP}=\dfrac{1}{P}$

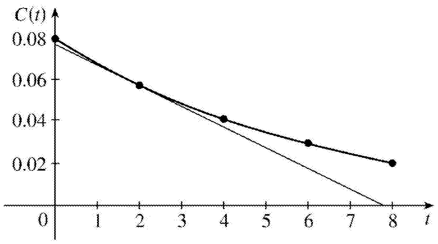
22. **(a)**

(i)
$$\frac{C(6)-C(2)}{6-2}=\frac{0.0295-0.0570}{4}=-0.006875\,\text{(moles/L)}/\text{ min}$$
$$\frac{C(4)-C(2)}{4-2}=\frac{0.0408-0.0570}{2}=-0.008\,\text{(moles/L)}/\text{ min}$$

(iii) $\dfrac{C(2)-C(0)}{2-0}=\dfrac{0.0570-0.0800}{2}$

$=-0.0115\,(\text{moles/L})/\text{ min}$

(b) Slope $=\dfrac{\Delta C}{\Delta t}\approx-\dfrac{0.077}{7.8}\approx-0.01\,(\text{moles/L})/\text{min}$



23. **(a)** **1920:** $m_1=\dfrac{1860-1750}{1920-1910}=\dfrac{110}{10}=11$, $m_2=\dfrac{2070-1860}{1930-1920}=\dfrac{210}{10}=21$,
$(m_1+m_2)2=(11+21)/2=16$ million / year

1980: $m_1=\dfrac{4450-3710}{1980-1970}=\dfrac{740}{10}=74$, $m_2=\dfrac{5280-4450}{1990-1980}=\dfrac{830}{10}=83$,
$(m_1+m_2)2=(74+83)/2=78.5$ million / year

(b) $P(t)=at^3+bt^2+ct+d$ (in millions of people), where $a\approx0.0012937063$, $b\approx-7.061421911$, $c\approx12{,}822.97902$, and $d\approx-7{,}743{,}770.396$.

(c) $P(t)=at^3+bt^2+ct+d\Rightarrow P'(t)=3at^2+2bt+c$ (in millions of people per year)

(d)

$$P'(1920)=3(0.0012937063)(1920)^2+2(-7.061421911)(1920)+12{,}822.97902$$
$$\approx14.48\text{ million / year}$$

$$P'(1980)\approx75.29\text{ million / year (smaller, but close)}$$

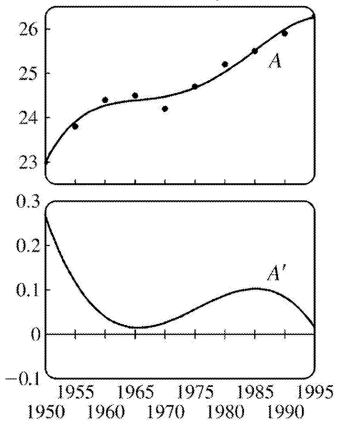
(e) $P'(1985)\approx81.62$ million / year, so the rate of growth in 1985 was about 81.62 million / year.

24. **(a)** $A(t)=at^4+bt^3+ct^2+dt+e$, where $a=-5.8275058275396\times10^{-6}$, $b=0.0460458430461$, $c=-136.43277039706$, $d=179\text{ , }661.02676871$, and $e=-88\text{ , }717\text{ , }597.060767$.

(b) $A(t)=at^4+bt^3+ct^2+dt+e\Rightarrow A'(t)=4at^3+3bt^2+2ct+d$

(c)

$A'(1990) \approx 0.0833$ years of age per year

(d) 

25. **(a)** $[\mathrm{C}]=\dfrac{a^2kt}{akt+1} \Rightarrow$ rate of reaction

$$=\frac{d[\mathrm{C}]}{dt}=\frac{(akt+1)(a^2k)-(a^2kt)(ak)}{(akt+1)^2}=\frac{a^2k(akt+1-akt)}{(akt+1)^2}=\frac{a^2k}{(akt+1)^2}$$

(b) If $x=[\mathrm{C}]$, then $a-x=a-\dfrac{a^2kt}{akt+1}=\dfrac{a^2kt+a-a^2kt}{akt+1}=\dfrac{a}{akt+1}$.

So $k(a-x)^2=k\left(\dfrac{a}{akt+1}\right)^2=\dfrac{a^2k}{(akt+1)^2}=\dfrac{d[\mathrm{C}]}{dt}=\dfrac{dx}{dt}$.

26. **(a)** After an hour the population is $n(1)=3\cdot 500$; after two hours it is $n(2)=3(3\cdot 500)=3^2\cdot 500$; after three hours, $n(3)=3(3^2\cdot 500)=3^3\cdot 500$; after four hours, $n(4)=3^4\cdot 500$. From this pattern, we see that the population after t hours is $n(t)=3^t\cdot 500=500\cdot 3^t$.

(b) From (5) in Section 3.1, we have $\dfrac{d}{dx}(3^x)\approx(1.10)3^x$. Thus, for $n(t)=500\cdot 3^t$,

$$\frac{dn}{dt}=500\frac{d}{dt}(3^t)\approx 500(1.10)3^t \Rightarrow \left.\frac{dn}{dt}\right|_{t=6}\approx 500(1.10)3^6\approx 400\text{ , }950\text{ bacteria / hour.}$$

27. **(a)** Using $v=\dfrac{P}{4\eta l}(R^2-r^2$ with $R=0.01$, $l=3$, $P=3000$, and $\eta=0.027$, we have v as a function of r :

$v(r)=\dfrac{3000}{4(0.027)3}(0.01^2-r^2)$. $v(0)=0.\overline{925}$ cm / s, $v(0.005)=0.69\overline{4}$ cm / s, $v(0.01)=0$.

(b) $v(r)=\dfrac{P}{4\eta l}(R^2-r^2)\Rightarrow v'(r)=\dfrac{P}{4\eta l}(-2r)=-\dfrac{Pr}{2\eta l}$. When $l=3$, $P=3000$, and $\eta=0.027$, we have

$v'(r)=-\dfrac{3000r}{2(0.027)3}$. $v'(0)=0$, $v'(0.005)=-92.\overline{592}$(cm/s)/ cm, and $v'(0.01)=-185.\overline{185}$(cm/s)/ cm.

(c) The velocity is greatest where $r=0$ (at the center) and the velocity is changing most where $r=R=0.01$ cm (at the edge).

28. **(a)**

(a) $$f=\frac{1}{2L}\sqrt{\frac{T}{\rho}}=\left(\frac{1}{2}\sqrt{\frac{T}{\rho}}\right)L^{-1}\Rightarrow\frac{df}{dL}=-\left(\frac{1}{2}\sqrt{\frac{T}{\rho}}\right)L^{-2}=-\frac{1}{2L^2}\sqrt{\frac{T}{\rho}}$$

(b) $$f=\frac{1}{2L}\sqrt{\frac{T}{\rho}}=\left(\frac{1}{2}\sqrt{\frac{T}{\rho}}\right)L^{-1}\Rightarrow\frac{df}{dL}=-\left(\frac{1}{2}\sqrt{\frac{T}{\rho}}\right)L^{-2}=-\frac{1}{2L^2}\sqrt{\frac{T}{\rho}}$$

(c) $$f=\frac{1}{2L}\sqrt{\frac{T}{\rho}}=\left(\frac{1}{2L\sqrt{\rho}}\right)T^{1/2}\Rightarrow\frac{df}{dT}=\frac{1}{2}\left(\frac{1}{2L\sqrt{\rho}}\right)T^{-1/2}=\frac{1}{4L\sqrt{T\rho}}$$

(d) $$f=\frac{1}{2L}\sqrt{\frac{T}{\rho}}=\left(\frac{1}{2L\sqrt{\rho}}\right)T^{1/2}\Rightarrow\frac{df}{dT}=\frac{1}{2}\left(\frac{1}{2L\sqrt{\rho}}\right)T^{-1/2}=\frac{1}{4L\sqrt{T\rho}}$$

(e) $$f=\frac{1}{2L}\sqrt{\frac{T}{\rho}}=\left(\frac{\sqrt{T}}{2L}\right)\rho^{-1/2}\Rightarrow\frac{df}{d\rho}=-\frac{1}{2}\left(\frac{\sqrt{T}}{2L}\right)\rho^{-3/2}=-\frac{\sqrt{T}}{4L\rho^{3/2}}$$

(f) $$f=\frac{1}{2L}\sqrt{\frac{T}{\rho}}=\left(\frac{\sqrt{T}}{2L}\right)\rho^{-1/2}\Rightarrow\frac{df}{d\rho}=-\frac{1}{2}\left(\frac{\sqrt{T}}{2L}\right)\rho^{-3/2}=-\frac{\sqrt{T}}{4L\rho^{3/2}}$$

(b)

(i) $\dfrac{df}{dL}<0$ and L is decreasing $\Rightarrow f$ is increasing $\Rightarrow$ higher note

(ii) $\dfrac{df}{dT}>0$ and T is increasing $\Rightarrow f$ is increasing $\Rightarrow$ higher note

(iii) $\dfrac{df}{d\rho}<0$ and ρ is increasing $\Rightarrow f$ is decreasing $\Rightarrow$ lower note

29. **(a)** $C(x)=2000+3x+0.01x^2+0.0002x^3\Rightarrow C'(x)=3+0.02x+0.0006x^2$

(b) $C'(100)=3+0.02(100)+0.0006(10,000)=3+2+6=\$11$/ pair. $C'(100)$ is the rate at which the cost is increasing as the 100 th pair of jeans is produced. It predicts the cost of the 101 st pair.

(c) The cost of manufacturing the 101 st pair of jeans is

$$C(101)-C(100) = (2000+303+102.01+206.0602)-(2000+300+100+200)$$
$$= 11.0702\approx \$11.07$$

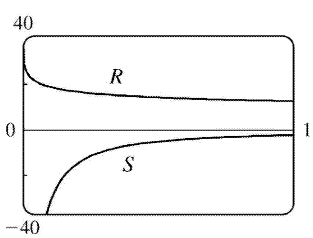
30. **(a)** $C(x)=84+0.16x-0.0006x^2+0.000003x^3\Rightarrow C'(x)=0.16-0.0012x+0.000009x^2\Rightarrow C'(100)=0.13$. This is the rate at which the cost is increasing as the 100 th item is produced.
(b) $C(101)-C(100)=97.13030299-97\approx \$0.13$.

31. **(a)** $A(x)=\dfrac{p(x)}{x}\Rightarrow A'(x)=\dfrac{xp'(x)-p(x)\cdot 1}{x^2}=\dfrac{xp'(x)-p(x)}{x^2}$. $A'(x)>0\Rightarrow A(x)$ is increasing; that is, the average productivity increases as the size of the workforce increases.
(b) $p'(x)$ is greater than the average productivity $\Rightarrow p'(x)>A(x)\Rightarrow p'(x)>\dfrac{p(x)}{x}\Rightarrow$

$xp'(x)>p(x)\Rightarrow xp'(x)-p(x)>0\Rightarrow \dfrac{xp'(x)-p(x)}{x^2}>0\Rightarrow A'(x)>0$.

32. **(a)**

$$S=\frac{dR}{dx}=\frac{\left(1+4x^{0.4}\right)\left(9.6x^{-0.6}\right)-\left(40+24x^{0.4}\right)\left(1.6x^{-0.6}\right)}{\left(1+4x^{0.4}\right)^2}$$
$$=\frac{9.6x^{-0.6}+38.4x^{-0.2}-64x^{-0.6}-38.4x^{-0.2}}{\left(1+4x^{0.4}\right)^2}=-\frac{54.4x^{-0.6}}{\left(1+4x^{0.4}\right)^2}$$



(b)
At low levels of brightness, R is quite large and is quickly decreasing, that is, S is negative with large absolute value. This is to be expected: at low levels of brightness, the eye is more sensitive to slight changes than it is at higher levels of brightness.

33. $PV=nRT\Rightarrow T=\dfrac{PV}{nR}=\dfrac{PV}{(10)(0.0821)}=\dfrac{1}{0.821}(PV)$. Using the Product Rule, we have
$\dfrac{dT}{dt}=\dfrac{1}{0.821}[P(t)V'(t)+V(t)P'(t)]=\dfrac{1}{0.821}[(8)(-0.15)+(10)(0.10)]\approx -0.2436$ K / min.

34. **(a)** If $dP/dt=0$, the population is stable (it is constant).

(b) $\dfrac{dP}{dt}=0 \Rightarrow \beta P=r_0\left(1-\dfrac{P}{P_c}\right)P \Rightarrow \dfrac{\beta}{r_0}=1-\dfrac{P}{P_c} \Rightarrow \dfrac{P}{P_c}=1-\dfrac{\beta}{r_0} \Rightarrow P=P_c\left(1-\dfrac{\beta}{r_0}\right)$.

If $P_c=10{,}000$, $r_0=5\%=0.05$, and $\beta=4\%=0.04$, then $P=10{,}000\left(1-\dfrac{4}{5}\right)=2000$.

(c) If $\beta=0.05$, then $P=10{,}000\left(1-\dfrac{5}{5}\right)=0$. There is no stable population.

35. **(a)** If the populations are stable, then the growth rates are neither positive nor negative; that is, $\dfrac{dC}{dt}=0$ and $\dfrac{dW}{dt}=0$.

(b) ''The caribou go extinct'' means that the population is zero, or mathematically, $C=0$.

(c) We have the equations $\dfrac{dC}{dt}=aC-bCW$ and $\dfrac{dW}{dt}=-cW+dCW$. Let $dC/dt=dW/dt=0$, $a=0.05$, $b=0.001$, $c=0.05$, and $d=0.0001$ to obtain $0.05C-0.001CW=0$ **(1)** and $-0.05W+0.0001CW=0$ **(2)** . Adding 10 times **(2)** to **(1)** eliminates the CW –terms and gives us $0.05C-0.5W=0 \Rightarrow C=10W$. Substituting $C=10W$ into **(1)** results in

$0.05(10W)-0.001(10W)W=0 \Leftrightarrow 0.5W-0.01W^2=0 \Leftrightarrow 50W-W^2=0 \Leftrightarrow W(50-W)=0 \Leftrightarrow W=0$ or 50 . Since $C=10W$, $C=0$ or 500 . Thus, the population pairs (C,W) that lead to stable populations are (0,0) and (500,50) . So it is possible for the two species to live in harmony.

1. $f(x)=x-3\sin x \Rightarrow f'(x)=1-3\cos x$

2. $f(x)=x\sin x \Rightarrow f'(x)=x\cdot\cos x+(\sin x)\cdot 1=x\cos x+\sin x$

3. $y=\sin x+10\tan x \Rightarrow y'=\cos x+10\sec^2 x$

4. $y=2\ \ x+5\cos x \Rightarrow y'=-2\ \ x\cot x-5\sin x$

5. $g(t)=t^3\cos t \Rightarrow g'(t)=t^3(-\sin t)+(\cos t)\cdot 3t^2=3t^2\cos t-t^3\sin t$ or $t^2(3\cos t-t\sin t)$

6. $g(t)=4\sec t+\tan t \Rightarrow g'(t)=4\sec t\tan t+\sec^2 t$

7. $h(\theta)=\csc\theta+e^{\theta}\cot\theta \Rightarrow$
$h'(\theta)=-\csc\theta\cot\theta+e^{\theta}(-\csc^2\theta)+(\cot\theta)e^{\theta}=-\csc\theta\cot\theta+e^{\theta}(\cot\theta-\csc^2\theta)$

8. $y=e^u(\cos u+cu) \Rightarrow y'=e^u(-\sin u+c)+(\cos u+cu)e^u=e^u(\cos u-\sin u+cu+c)$

9. $y=\dfrac{x}{\cos x} \Rightarrow y'=\dfrac{(\cos x)(1)-(x)(-\sin x)}{(\cos x)^2}=\dfrac{\cos x+x\sin x}{\cos^2 x}$

10. $y=\dfrac{1+\sin x}{x+\cos x} \Rightarrow$

$$y'=\frac{(x+\cos x)(\cos x)-(1+\sin x)(1-\sin x)}{(x+\cos x)^2}=\frac{x\cos x+\cos^2 x-(1-\sin^2 x)}{(x+\cos x)^2}$$
$$=\frac{x\cos x+\cos^2 x-(\cos^2 x)}{(x+\cos x)^2}=\frac{x\cos x}{(x+\cos x)^2}$$

11. $f(\theta)=\dfrac{\sec\theta}{1+\sec\theta} \Rightarrow$

$$f'(\theta)=\frac{(1+\sec\theta)(\sec\theta\tan\theta)-(\sec\theta)(\sec\theta\tan\theta)}{(1+\sec\theta)^2}=\frac{(\sec\theta\tan\theta)[(1+\sec\theta)-\sec\theta]}{(1+\sec\theta)^2}=\frac{\sec\theta\tan\theta}{(1+\sec\theta)^2}$$

12. $y=\dfrac{\tan x-1}{\sec x} \Rightarrow$

$$\frac{dy}{dx}=\frac{\sec x\sec^2 x-(\tan x-1)\sec x\tan x}{\sec^2 x}=\frac{\sec x\left(\sec^2 x-\tan^2 x+\tan x\right)}{\sec^2 x}=\frac{1+\tan x}{\sec x}$$

Another method: Simplify y first: $y=\sin x-\cos x\Rightarrow y'=\cos x+\sin x$.

13. $y=\dfrac{\sin x}{x^2}\Rightarrow y'=\dfrac{x^2\cos x-(\sin x)(2x)}{\left(x^2\right)^2}=\dfrac{x(x\cos x-2\sin x)}{x^4}=\dfrac{x\cos x-2\sin x}{x^3}$

14. $y=\theta(\theta+\cot\theta)\Rightarrow$

$$\begin{aligned} y' &=\theta(1-\csc^2\theta)+(\theta+\cot\theta)(-\ \theta\cot\theta)=\theta(1-\csc^2\theta-\theta\cot\theta-\cot^2\theta) \\ &=\theta(-\cot^2\theta-\theta\cot\theta-\cot^2\theta) \qquad \{1+\cot^2\theta=\csc^2\theta\} \\ &=\theta(-\theta\cot\theta-2\cot^2\theta)=-\theta\cot\theta(\theta+2\cot\theta) \end{aligned}$$

15. $y=\sec\theta\tan\theta\Rightarrow y'=\sec\theta(\sec^2\theta)+\tan\theta(\sec\theta\tan\theta)=\sec\theta(\sec^2\theta+\tan^2\theta)$

Using the identity $1+\tan^2\theta=\sec^2\theta$, we can write alternative forms of the answer as $\sec\theta(1+2\tan^2\theta)$ or $\sec\theta(2\sec^2\theta-1)$

16. Recall that if $y=fgh$, then $y'=f'gh+fg'h+fgh'$. $y=x\sin x\cos x\Rightarrow$

$$\frac{dy}{dx}=\sin x\cos x+x\cos x\cos x+x\sin x(-\sin x)=\sin x\cos x+x\cos^2 x-x\sin^2 x$$

17. $\dfrac{d}{dx}(\csc(x))=\dfrac{d}{dx}\left(\dfrac{1}{\sin x}\right)=\dfrac{(\sin x)(0)-1(\cos x)}{\sin^2 x}=\dfrac{-\cos x}{\sin^2 x}=-\dfrac{1}{\sin x}\cdot\dfrac{\cos x}{\sin x}=-\ x\cot x$

18. $\dfrac{d}{dx}(\sec x)=\dfrac{d}{dx}\left(\dfrac{1}{\cos x}\right)=\dfrac{(\cos x)(0)-1(-\sin x)}{\cos^2 x}=\dfrac{\sin x}{\cos^2 x}=\dfrac{1}{\cos x}\cdot\dfrac{\sin x}{\cos x}=\sec x\tan x$

19. $\dfrac{d}{dx}(\cot x)=\dfrac{d}{dx}\left(\dfrac{\cos x}{\sin x}\right)=\dfrac{(\sin x)(-\sin x)-(\cos x)(\cos x)}{\sin^2 x}=-\dfrac{\sin^2 x+\cos^2 x}{\sin^2 x}=-\dfrac{1}{\sin^2 x}=-\ {}^2 x$

20. $f(x)=\cos x\Rightarrow$

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\cos(x+h)-\cos x}{h}=\lim_{h\to 0}\frac{\cos x\cos h-\sin x\sin h-\cos x}{h}$$

$$=\lim_{h\to 0}\left(\cos x\,\frac{\cos h-1}{h}-\sin x\,\frac{\sin h}{h}\right)=\cos x\lim_{h\to 0}\frac{\cos h-1}{h}-\sin x\lim_{h\to 0}\frac{\sin h}{h}$$
$$=(\cos x)(0)-(\sin x)(1)=-\sin x$$

21. $y=\tan x\Rightarrow y'=\sec^2 x\Rightarrow$ the slope of the tangent line at $\left(\frac{\pi}{4},1\right)$ is $\sec^2\frac{\pi}{4}=(\sqrt{2})^2=2$ and an equation of the tangent line is $y-1=2\left(x-\frac{\pi}{4}\right)$ or $y=2x+1-\frac{\pi}{2}$.

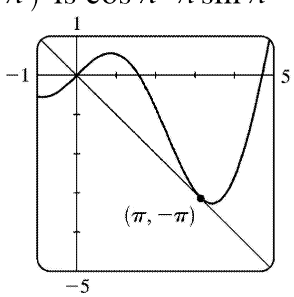
22. $y=e^x\cos x\Rightarrow y'=e^x(-\sin x)+(\cos x)e^x=e^x(\cos x-\sin x)\Rightarrow$ the slope of the tangent line at (0, 1) is $e^0(\cos 0-\sin 0)=1(1-0)=1$ and an equation is $y-1=1(x-0)$ or $y=x+1$.

23. $y=x+\cos x\Rightarrow y'=1-\sin x$. At $(0,1)$, $y'=1$, and an equation of the tangent line is $y-1=1(x-0)$, or $y=x+1$.

24. $y=\dfrac{1}{\sin x+\cos x}\Rightarrow y'=-\dfrac{\cos x-\sin x}{(\sin x+\cos x)^2}$ [Reciprocal Rule]. At $(0,1)$, $y'=-\dfrac{1-0}{(0+1)^2}=-1$, and an equation of the tangent line is $y-1=-1(x-0)$, or $y=-x+1$.

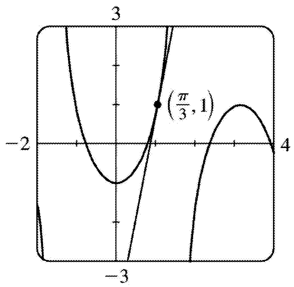
25. **(a)** $y=x\cos x\Rightarrow y'=x(-\sin x)+\cos x(1)=\cos x-x\sin x$. So the slope of the tangent at the point $(\pi,-\pi)$ is $\cos\pi-\pi\sin\pi=-1-\pi(0)=-1$, and an equation is $y+\pi=-(x-\pi)$ or $y=-x$.

(b)

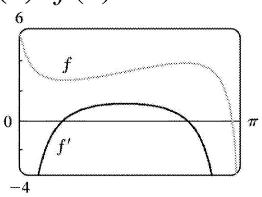


26. **(a)** $y=\sec x-2\cos x\Rightarrow y'=\sec x\tan x+2\sin x\Rightarrow$
the slope of the tangent line at $\left(\frac{\pi}{3},1\right)$ is $\sec\frac{\pi}{3}\tan\frac{\pi}{3}+2\sin\frac{\pi}{3}=2\cdot\sqrt{3}+2\cdot\frac{\sqrt{3}}{2}=3\sqrt{3}$ and an equation is $y-1=3\sqrt{3}\left(x-\frac{\pi}{3}\right)$ or $y=3\sqrt{3}x+1-\pi\sqrt{3}$.

(b)



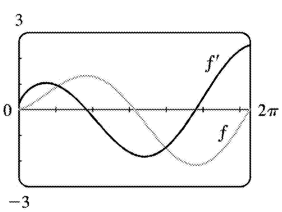
27. **(a)** $f(x)=2x+\cot x \Rightarrow f'(x)=2-\csc^2 x$

(b) 

Notice that $f'(x)=0$ when f has a horizontal tangent.

f' is positive when f is increasing and f' is negative when f is decreasing. Also, $f'(x)$ is large negative when the graph of f is steep.

28. **(a)** $f(x)=\sqrt{x}\sin x \Rightarrow f'(x)=\sqrt{x}\cos x+(\sin x)\left(\frac{1}{2}x^{-1/2}\right)=\sqrt{x}\cos x+\frac{\sin x}{2\sqrt{x}}$

(b) 

Notice that $f'(x)=0$ when f has a horizontal tangent.

f' is positive when f is increasing and f' is negative when f is decreasing.

29. $f(x)=x+2\sin x$ has a horizontal tangent when $f'(x)=0 \Leftrightarrow 1+2\cos x=0 \Leftrightarrow \cos x=-\frac{1}{2} \Leftrightarrow$ $x=\frac{2\pi}{3}+2\pi n$ or $\frac{4\pi}{3}+2\pi n$, where n is an integer. Note that $\frac{4\pi}{3}$ and $\frac{2\pi}{3}$ are $\pm\frac{\pi}{3}$ units from π. This allows us to write the solutions in the more compact equivalent form $(2n+1)\pi\pm\frac{\pi}{3}$, n an integer.

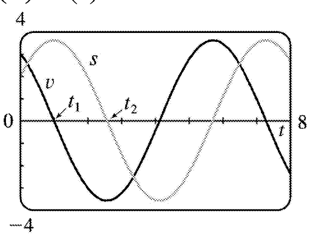
30. $y=\frac{\cos x}{2+\sin x} \Rightarrow y'=\frac{(2+\sin x)(-\sin x)-\cos x\cos x}{(2+\sin x)^2}=\frac{-2\sin x-\sin^2 x-\cos^2 x}{(2+\sin x)^2}=\frac{-2\sin x-1}{(2+\sin x)^2}=0$ when

$-2\sin x-1=0 \Leftrightarrow \sin x=-\frac{1}{2} \Leftrightarrow x=\frac{11\pi}{6}+2\pi n$ or $x=\frac{7\pi}{6}+2\pi n$, n an integer. So $y=\frac{1}{\sqrt{3}}$ or $y=-\frac{1}{\sqrt{3}}$ and the points on the curve with horizontal tangents are: $\left(\frac{11\pi}{6}+2\pi n, \frac{1}{\sqrt{3}}\right)$, $\left(\frac{7\pi}{6}+2\pi n, -\frac{1}{\sqrt{3}}\right)$, n an integer.

31. **(a)** $x(t)=8\sin t \Rightarrow v(t)=x'(t)=8\cos t$

(b) The mass at time $t=\frac{2\pi}{3}$ has position $x\left(\frac{2\pi}{3}\right)=8\sin\frac{2\pi}{3}=8\left(\frac{\sqrt{3}}{2}\right)=4\sqrt{3}$ and velocity $v\left(\frac{2\pi}{3}\right)=8\cos\frac{2\pi}{3}=8\left(-\frac{1}{2}\right)=-4$. Since $v\left(\frac{2\pi}{3}\right)<0$, the particle is moving to the left.

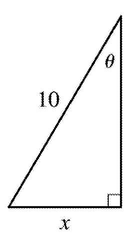
32. **(a)** $s(t)=2\cos t+3\sin t \Rightarrow v(t)=-2\sin t+3\cos t$

(b) 

(c) $s=0 \Rightarrow t_2 \approx 2.55$. So the mass passes through the equilibrium position for the first time when $t \approx 2.55$ s.

(d) $v=0 \Rightarrow t_1 \approx 0.98$, $s(t_1) \approx 3.61$ cm. So the mass travels a maximum of about 3.6 cm (upward and downward) from its equilibrium position.

(e) The speed $|v|$ is greatest when $s=0$; that is, when $t=t_2+n\pi$, n a positive integer.

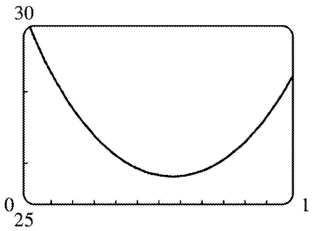
33. 

From the diagram we can see that $\sin\theta=x/10 \Leftrightarrow x=10\sin\theta$. We want to find the rate of change of x with respect to θ ; that is, $dx/d\theta$. Taking the derivative of the above expression, $dx/d\theta=10(\cos\theta)$. So when $\theta=\frac{\pi}{3}$, $\frac{dx}{d\theta}=10\cos\left(\frac{\pi}{3}\right)=10\cdot\left(\frac{1}{2}\right)=5$ ft/rad.

34. **(a)** $F=\frac{\mu W}{\mu\sin\theta+\cos\theta} \Rightarrow \frac{dF}{d\theta}=\frac{(\mu\sin\theta+\cos\theta)(0)-\mu W(\mu\cos\theta-\sin\theta)}{(\mu\sin\theta+\cos\theta)^2}=\frac{\mu W(\sin\theta-\mu\cos\theta)}{(\mu\sin\theta+\cos\theta)^2}$

(b)

$$\frac{dF}{d\theta}=0 \Rightarrow \mu W(\sin\theta-\mu\cos\theta)=0 \Rightarrow \sin\theta=\mu\cos\theta \Rightarrow \tan\theta=\mu \Rightarrow \theta=\tan^{-1}\mu$$



(c)

From the graph of $F=\dfrac{0.6(50)}{0.6\sin\theta+\cos\theta}$ for $0\le\theta\le 1$, we see that $\dfrac{dF}{d\theta}=0 \Rightarrow \theta\approx 0.54$. Checking this with part (b) and $\mu=0.6$, we calculate $\theta=\tan^{-1}0.6\approx 0.54$. So the value from the graph is consistent with the value in part (b).

35.

$$\begin{aligned}\lim_{x\to 0}\frac{\sin 3x}{x} &= \lim_{x\to 0}\frac{3\sin 3x}{3x} && \text{[multiply numerator and denominator by 3]}\\ &=3\lim_{3x\to 0}\frac{\sin 3x}{3x} && \text{[as } x\to 0\text{ , } 3x\to 0\text{]}\\ &=3\lim_{\theta\to 0}\frac{\sin\theta}{\theta} && \text{[let } \theta=3x\text{]}\\ &=3(1) && \text{[Equation 2]}\\ &=3\end{aligned}$$

36.

$$\begin{aligned}\lim_{x\to 0}\frac{\sin 4x}{\sin 6x} &= \lim_{x\to 0}\left(\frac{\sin 4x}{x}\cdot\frac{x}{\sin 6x}\right)=\lim_{x\to 0}\frac{4\sin 4x}{4x}\cdot\lim_{x\to 0}\frac{6x}{6\sin 6x}\\ &=4\lim_{x\to 0}\frac{\sin 4x}{4x}\cdot\frac{1}{6}\lim_{x\to 0}\frac{6x}{\sin 6x}=4(1)\cdot\frac{1}{6}(1)=\frac{2}{3}\end{aligned}$$

37.

$$\begin{aligned}\lim_{t\to 0}\frac{\tan 6t}{\sin 2t} &= \lim_{t\to 0}\left(\frac{\sin 6t}{t}\cdot\frac{1}{\cos 6t}\cdot\frac{t}{\sin 2t}\right)=\lim_{t\to 0}\frac{6\sin 6t}{6t}\cdot\lim_{t\to 0}\frac{1}{\cos 6t}\cdot\lim_{t\to 0}\frac{2t}{2\sin 2t}\\ &=6\lim_{t\to 0}\frac{\sin 6t}{6t}\cdot\lim_{t\to 0}\frac{1}{\cos 6t}\cdot\frac{1}{2}\lim_{t\to 0}\frac{2t}{\sin 2t}=6(1)\cdot\frac{1}{1}\cdot\frac{1}{2}(1)=3\end{aligned}$$

38.

$$\lim_{\theta\to 0}\frac{\cos\theta-1}{\sin\theta}=\lim_{\theta\to 0}\frac{\dfrac{\cos\theta-1}{\theta}}{\dfrac{\sin\theta}{\theta}}=\frac{\displaystyle\lim_{\theta\to 0}\frac{\cos\theta-1}{\theta}}{\displaystyle\lim_{\theta\to 0}\frac{\sin\theta}{\theta}}=\frac{0}{1}=0$$

39. $\displaystyle\lim_{\theta\to 0}\frac{\sin(\cos\theta)}{\sec\theta}=\frac{\sin\left(\displaystyle\lim_{\theta\to 0}(\cos\theta)\right)}{\displaystyle\lim_{\theta\to 0}(\sec\theta)}=\frac{\sin 1}{1}=\sin 1$

40.

$$\lim_{t\to 0}\frac{\sin^2 3t}{t^2}=\lim_{t\to 0}\left(\frac{\sin 3t}{t}\cdot\frac{\sin 3t}{t}\right)=\lim_{t\to 0}\frac{\sin 3t}{t}\cdot\lim_{t\to 0}\frac{\sin 3t}{t}$$

$$=\left(\lim_{t\to 0}\frac{\sin 3t}{t}\right)^2=\left(3\lim_{t\to 0}\frac{\sin 3t}{3t}\right)^2=(3\cdot 1)^2=9$$

41.

$$\lim_{x\to 0}\frac{\cot 2x}{x}=\lim_{x\to 0}\frac{\cos 2x\cdot\sin x}{\sin 2x}=\lim_{x\to 0}\left(\cos 2x\left[\frac{(\sin x)/x}{(\sin 2x)/x}\right]\right)=\lim_{x\to 0}\left(\cos 2x\left[\frac{\displaystyle\lim_{x\to 0}[(\sin x)/x]}{2\displaystyle\lim_{x\to 0}[(\sin 2x)/2x]}\right]\right)$$

$$=1\cdot\frac{1}{2\cdot 1}=\frac{1}{2}$$

42.

$$\lim_{x\to\pi/4}\frac{\sin x-\cos x}{\cos 2x}=\lim_{x\to\pi/4}\frac{\sin x-\cos x}{\cos^2 x-\sin^2 x}=\lim_{x\to\pi/4}\frac{\sin x-\cos x}{(\cos x+\sin x)(\cos x-\sin x)}$$

$$=\lim_{x\to\pi/4}\frac{-1}{\cos x+\sin x}=\frac{-1}{\cos\dfrac{\pi}{4}+\sin\dfrac{\pi}{4}}=\frac{-1}{\sqrt{2}}$$

43. Divide numerator and denominator by θ. ($\sin(\theta)$ also works.)

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta+\tan\theta}=\lim_{\theta\to 0}\frac{\dfrac{\sin\theta}{\theta}}{1+\dfrac{\sin\theta}{\theta}\cdot\dfrac{1}{\cos\theta}}=\frac{\displaystyle\lim_{\theta\to 0}\frac{\sin\theta}{\theta}}{1+\displaystyle\lim_{\theta\to 0}\frac{\sin\theta}{\theta}\ \lim_{\theta\to 0}\frac{1}{\cos\theta}}=\frac{1}{1+1\cdot 1}=\frac{1}{2}$$

44.

$$\lim_{x\to 1}\frac{\sin(x-1)}{x^2+x-2}=\lim_{x\to 1}\frac{\sin(x-1)}{(x+2)(x-1)}=\lim_{x\to 1}\frac{1}{x+2}\lim_{x\to 1}\frac{\sin(x-1)}{x-1}=\frac{1}{3}\cdot 1=\frac{1}{3}$$

45. **(a)** $\frac{d}{dx}\tan x=\frac{d}{dx}\frac{\sin x}{\cos x}\Rightarrow \sec^2 x=\frac{\cos x\cos x-\sin x(-\sin x)}{\cos^2 x}=\frac{\cos^2 x+\sin^2 x}{\cos^2 x}$. So $\sec^2 x=\frac{1}{\cos^2 x}$.

(b) $\frac{d}{dx}\sec x=\frac{d}{dx}\frac{1}{\cos x}\Rightarrow \sec x\tan x=\frac{(\cos x)(0)-1(-\sin x)}{\cos^2 x}$. So $\sec x\tan x=\frac{\sin x}{\cos^2 x}$.

(c) $\frac{d}{dx}(\sin x+\cos x)=\frac{d}{dx}\frac{1+\cot x}{x}\Rightarrow$

$$\cos x-\sin x=\frac{x(-\csc^2 x)-(1+\cot x)(-\ x\cot x)}{\csc^2 x}=\frac{x[-\csc^2 x+(1+\cot x)\cot x]}{\csc^2 x}$$

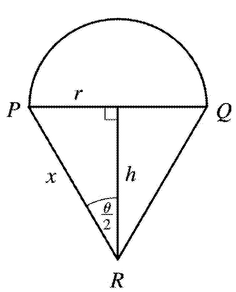
$$=\frac{-\csc^2 x+\cot^2 x+\cot x}{x}=\frac{-1+\cot x}{x}$$

So $\cos x-\sin x=\frac{\cot x-1}{x}$.

46. Let $|PR|=x$. Then we get the following formulas for r and h in terms of θ and x : $\sin\frac{\theta}{2}=\frac{r}{x}\Rightarrow r=x\sin\frac{\theta}{2}$ and $\cos\frac{\theta}{2}=\frac{h}{x}\Rightarrow h=x\cos\frac{\theta}{2}$. Now $A(\theta)=\frac{1}{2}\pi r^2$ and $B(\theta)=\frac{1}{2}(2r)h=rh$. So

$$\lim_{\theta\to 0^+}\frac{A(\theta)}{B(\theta)}=\lim_{\theta\to 0^+}\frac{\frac{1}{2}\pi r^2}{rh}=\frac{1}{2}\pi\lim_{\theta\to 0^+}\frac{r}{h}=\frac{1}{2}\pi\lim_{\theta\to 0^+}\frac{x\sin(\theta/2)}{x\cos(\theta/2)}$$

$$=\frac{1}{2}\pi\lim_{\theta\to 0^+}\tan(\theta/2)=0 .$$



47. By the definition of radian measure, $s=r\theta$, where r is the radius of the circle.

By drawing the bisector of the angle θ , we can see that $\sin\frac{\theta}{2} = \frac{d/2}{r} \Rightarrow d = 2r\sin\frac{\theta}{2}$.

So $\lim_{\theta\to 0^+}\frac{s}{d} = \lim_{\theta\to 0^+}\frac{r\theta}{2r\sin(\theta/2)} = \lim_{\theta\to 0^+}\frac{2\cdot(\theta/2)}{2\sin(\theta/2)} = \lim_{\theta\to 0}\frac{\theta/2}{\sin(\theta/2)} = 1$.

1. Let $u=g(x)=4x$ and $y=f(u)=\sin u$. Then $\frac{dy}{dx}=\frac{dy}{du}\ \frac{du}{dx}=(\cos u)(4)=4\cos 4x$.

2. Let $u=g(x)=4+3x$ and $y=f(u)=\sqrt{u}=u^{1/2}$. Then $\frac{dy}{dx}=\frac{dy}{du}\ \frac{du}{dx}=\frac{1}{2}\,u^{-1/2}(3)=\frac{3}{2\sqrt{u}}=\frac{3}{2\sqrt{4+3x}}$.

3. Let $u=g(x)=1-x^2$ and $y=f(u)=u^{10}$. Then $\frac{dy}{dx}=\frac{dy}{du}\ \frac{du}{dx}=(10u^9)(-2x)=-20x(1-x^2)^9$.

4. Let $u=g(x)=\sin x$ and $y=f(u)=\tan u$. Then $\frac{dy}{dx}=\frac{dy}{du}\ \frac{du}{dx}=(\sec^2 u)(\cos x)=(\sec^2 \mathrm{u})(\sin x)\cdot\cos x$, or equivalently, $[\sec(\sin x)]^2\cos x$.

5. Let $u=g(x)=\sin x$ and $y=f(u)=\sqrt{u}$. Then $\frac{dy}{dx}=\frac{dy}{du}\ \frac{du}{dx}=\frac{1}{2}\,u^{-1/2}\cos x=\frac{\cos x}{2\sqrt{u}}=\frac{\cos x}{2\sqrt{\sin x}}$.

6. Let $u=g(x)=e^x$ and $y=f(u)=\sin u$. Then $\frac{dy}{dx}=\frac{dy}{du}\ \frac{du}{dx}=(\cos u)(e^x)=e^x\cos e^x$.

7. $F(x)=(x^3+4x)^7\Rightarrow F'(x)=7(x^3+4x)^6(3x^2+4)$ [or $7x^6(x^2+4)^6(3x^2+4)$]

8. $F(x)=(x^2-x+1)^3\Rightarrow F'(x)=3(x^2-x+1)^2(2x-1)$

9. $F(x)=\sqrt[4]{1+2x+x^3}=(1+2x+x^3)^{1/4}\Rightarrow$

$$F'(x)=\frac{1}{4}(1+2x+x^3)^{-3/4}\cdot\frac{d}{dx}(1+2x+x^3)=\frac{1}{4(1+2x+x^3)^{3/4}}\cdot(2+3x^2)$$

$$=\frac{2+3x^2}{4(1+2x+x^3)^{3/4}}=\frac{2+3x^2}{4\sqrt[4]{(1+2x+x^3)^3}}$$

10. $f(x)=(1+x^4)^{2/3}\Rightarrow f'(x)=\frac{2}{3}(1+x^4)^{-1/3}(4x^3)=\frac{8x^3}{3\sqrt[3]{1+x^4}}$

11. $g(t)=\frac{1}{(t^4+1)^3}=(t^4+1)^{-3}\Rightarrow g'(t)=-3(t^4+1)^{-4}(4t^3)=-12t^3(t^4+1)^{-4}=\frac{-12t^3}{(t^4+1)^4}$

12.

$f(t)=\sqrt[3]{1+\tan t}=(1+\tan t)^{1/3}\Rightarrow f'(t)=\frac{1}{3}(1+\tan t)^{-2/3}\sec^2 t=\frac{\sec^2 t}{3\sqrt[3]{(1+\tan t)^2}}$

13. $y=\cos(a^3+x^3)\Rightarrow y'=-\sin(a^3+x^3)\cdot 3x^2$ [a^3 is just a constant] $=-3x^2\sin(a^3+x^3)$

14. $y=a^3+\cos^3 x\Rightarrow y'=3(\cos x)^2(-\sin x)$ [a^3 is just a constant] $=-3\sin x\cos^2 x$

15. $y=e^{-mx}\Rightarrow y'=e^{-mx}\frac{d}{dx}(-mx)=e^{-mx}(-m)=-me^{-mx}$

16. $y=4\sec 5x\Rightarrow y'=4\sec 5x\tan 5x(5)=20\sec 5x\tan 5x$

17. $g(x)=(1+4x)^5(3+x-x^2)^8\Rightarrow$

$$\begin{aligned}g'(x)&=(1+4x)^5\cdot 8(3+x-x^2)^7(1-2x)+(3+x-x^2)^8\cdot 5(1+4x)^4\cdot 4\\&=4(1+4x)^4(3+x-x^2)^7[2(1+4x)(1-2x)+5(3+x-x^2)]\\&=4(1+4x)^4(3+x-x^2)^7[(2+4x-16x^2)+(15+5x-5x^2)]\\&=4(1+4x)^4(3+x-x^2)^7(17+9x-21x^2)\end{aligned}$$

18. $h(t)=(t^4-1)^3(t^3+1)^4\Rightarrow$

$$\begin{aligned}h'(t)&=(t^4-1)^3\cdot 4(t^3+1)^3(3t^2)+(t^3+1)^4\cdot 3(t^4-1)^2(4t^3)\\&=12t^2(t^4-1)^2(t^3+1)^3[(t^4-1)+t(t^3+1)]=12t^2(t^4-1)^2(t^3+1)^3(2t^4+t-1)\end{aligned}$$

19. $y=(2x-5)^4(8x^2-5)^{-3}\Rightarrow$

$$\begin{aligned}y'&=4(2x-5)^3(2)(8x^2-5)^{-3}+(2x-5)^4(-3)(8x^2-5)^{-4}(16x)\\&=8(2x-5)^3(8x^2-5)^{-3}-48x(2x-5)^4(8x^2-5)^{-4}\end{aligned}$$

20. $y=(x^2+1)(x^2+2)^{1/3}\Rightarrow y'=2x(x^2+2)^{1/3}+(x^2+1)\left(\frac{1}{3}\right)(x^2+2)^{-2/3}(2x)=2x(x^2+2)^{1/3}\left[1+\frac{x^2+1}{3(x^2+2)}\right]$

21. $y=xe^{-x^2}\Rightarrow y'=xe^{-x^2}(-2x)+e^{-x^2}\cdot 1=e^{-x^2}(-2x^2+1)=e^{-x^2}(1-2x^2)$

22. $y=e^{-5x}\cos 3x \Rightarrow y'=e^{-5x}(-3\sin 3x)+(\cos 3x)(-5e^{-5x})=-e^{-5x}(3\sin 3x+5\cos 3x)$

23. $y=e^{x\cos x} \Rightarrow y'=e^{x\cos x}\cdot \frac{d}{dx}(x\cos x)=e^{x\cos x}[x(-\sin x)+(\cos x)\cdot 1]=e^{x\cos x}(\cos x-x\sin x)$

24. Using Formula 5 and the Chain Rule, $y=10^{1-x^2} \Rightarrow y'=10^{1-x^2}(\ln 10)\cdot \frac{d}{dx}(1-x^2)=-2x(\ln 10)10^{1-x^2}$.

25. $F(z)=\sqrt{\frac{z-1}{z+1}}=\left(\frac{z-1}{z+1}\right)^{1/2} \Rightarrow$

$$F'(z)=\frac{1}{2}\left(\frac{z-1}{z+1}\right)^{-1/2}\cdot \frac{d}{dz}\left(\frac{z-1}{z+1}\right)=\frac{1}{2}\left(\frac{z+1}{z-1}\right)^{1/2}\cdot \frac{(z+1)(1)-(z-1)(1)}{(z+1)^2}$$

$$=\frac{1}{2}\frac{(z+1)^{1/2}}{(z-1)^{1/2}}\cdot \frac{z+1-z+1}{(z+1)^2}=\frac{1}{2}\frac{(z+1)^{1/2}}{(z-1)^{1/2}}\cdot \frac{2}{(z+1)^2}=\frac{1}{(z-1)^{1/2}(z+1)^{3/2}}$$

26. $G(y)=\frac{(y-1)^4}{(y^2+2y)^5} \Rightarrow$

$$G'(y)=\frac{(y^2+2y)^5\cdot 4(y-1)^3\cdot 1-(y-1)^4\cdot 5(y^2+2y)^4(2y+2)}{[(y^2+2y)^5]^2}$$

$$=\frac{2(y^2+2y)^4(y-1)^3[2(y^2+2y)-5(y-1)(y+1)]}{(y^2+2y)^{10}}$$

$$=\frac{2(y-1)^3[(2y^2+4y)+(-5y^2+5)]}{(y^2+2y)^6}=\frac{2(y-1)^3(-3y^2+4y+5)}{(y^2+2y)^6}$$

27. $y=\frac{r}{\sqrt{r^2+1}} \Rightarrow$

$$y'=\frac{\sqrt{r^2+1}\,(1)-r\cdot \frac{1}{2}(r^2+1)^{-1/2}(2r)}{\left(\sqrt{r^2+1}\right)^2}=\frac{\sqrt{r^2+1}-\frac{r^2}{\sqrt{r^2+1}}}{\left(\sqrt{r^2+1}\right)^2}=\frac{\frac{\sqrt{r^2+1}\,\sqrt{r^2+1}-r^2}{\sqrt{r^2+1}}}{\left(\sqrt{r^2+1}\right)^2}$$

$$=\frac{(r^2+1)-r^2}{\left(\sqrt{r^2+1}\right)^3}=\frac{1}{(r^2+1)^{3/2}} \text{ or } (r^2+1)^{-3/2}$$

Another solution: Write y as a product and make use of the Product Rule. $y=r\,(r^2+1)^{-1/2}\Rightarrow$

$$y'=r\cdot -\frac{1}{2}\,(r^2+1)^{-3/2}(2r)+r^2+1)^{-1/2}\cdot 1$$

$$=(r^2+1)^{-3/2}\,[-r^2+(r^2+1)^1]=(r^2+1)^{-3/2}(1)=(r^2+1)^{-3/2}$$

The step that students usually have trouble with is factoring out $(r^2+1)^{-3/2}$. But this is no different than factoring out x^2 from x^2+x^5; that is, we are just factoring out a factor with the *smallest* exponent that appears on it. In this case, $-\frac{3}{2}$ is smaller than $-\frac{1}{2}$.

28. $y=\dfrac{e^{2u}}{e^u+e^{-u}}\Rightarrow$

$$y'=\frac{(e^u+e^{-u})(e^{2u}\cdot 2)-e^{2u}(e^u-e^{-u})}{(e^u+e^{-u})^2}=\frac{e^{2u}(2e^u+2e^{-u}-e^u+e^{-u})}{(e^u+e^{-u})^2}=\frac{e^{2u}(e^u+3e^{-u})}{(e^u+e^{-u})^2}$$

Another solution: Eliminate negative exponents by first changing the form of y.

$$y=\frac{e^{2u}}{e^u+e^{-u}}\cdot\frac{e^u}{e^u}=\frac{e^{3u}}{e^{2u}+1}\Rightarrow$$

$$y'=\frac{(e^{2u}+1)(3e^{3u})-e^{3u}(2e^{2u})}{(e^{2u}+1)^2}=\frac{e^{3u}(3e^{2u}+3-2e^{2u})}{(e^{2u}+1)^2}=\frac{e^{3u}(e^{2u}+3)}{(e^{2u}+1)^2}$$

29. $y=\tan(\cos x)\Rightarrow y'=\sec^2(\cos x)\cdot(-\sin x)=-\sin x\cdot\sec^2(\cos x)$

30. $y=\dfrac{\sin^2 x}{\cos x}\Rightarrow$

$$y'=\frac{\cos x(2\sin x\cdot\cos x)-\sin^2 x(-\sin x)}{\cos^2 x}=\frac{\sin x(2\cos^2 x+\sin^2 x)}{\cos^2 x}=\frac{\sin x(1+\cos^2 x)}{\cos^2 x}$$

$$=\sin x\,(1+\sec^2 x)$$

Another method: $y=\tan x\cdot\sin x\Rightarrow$

$y'=\sec^2 x\cdot \sin x+\tan x\cdot \cos x=\sec^2 x\cdot \sin x+\sin x$

31. Using Formula 5 and the Chain Rule, $y=2^{\sin \pi x}\Rightarrow$
$y'=2^{\sin \pi x}(\ln 2)\cdot \dfrac{d}{dx}(\sin \pi x)=2^{\sin \pi x}(\ln 2)\cdot \cos \pi x\cdot \pi=2^{\sin \pi x}(\pi \ln 2)\cos \pi x$

32. $y=\tan^2(3\theta)=(\tan 3\theta)^2\Rightarrow y'=2(\tan 3\theta)\cdot \dfrac{d}{d\theta}(\tan 3\theta)=2\tan 3\theta\cdot \sec^2 3\theta\cdot 3=6\tan 3\theta \sec^2 3\theta$

33. $y=(1+\cos^2 x)^6\Rightarrow\ y'=6(1+\cos^2 x)^5\cdot 2\cos x(-\sin x)=-12\cos x\sin x(1+\cos^2 x)^5$

34. $y=x\sin \dfrac{1}{x}\Rightarrow\ y'=\sin \dfrac{1}{x}+x\cos \dfrac{1}{x}\left(-\dfrac{1}{x^2}\right)=\sin \dfrac{1}{x}-\dfrac{1}{x}\cos \dfrac{1}{x}$

35. $y=\sec^2 x+\tan^2 x=(\sec x)^2+(\tan x)^2\Rightarrow$
$y'=2(\sec x)\cdot (\sec x\tan x)+2(\tan x)(\sec^2 x)=2\sec^2 x\cdot \tan x+2\sec^2 x\cdot \tan x=4\sec^2 x\cdot \tan x$

36. $y=e^{k\tan \sqrt{x}}\Rightarrow\ y'=e^{k\tan \sqrt{x}}\cdot \dfrac{d}{dx}(k\tan \sqrt{x})=e^{k\tan \sqrt{x}}\left(k\sec^2\sqrt{x}\cdot \dfrac{1}{2}x^{-1/2}\right)=\dfrac{k\sec^2\sqrt{x}}{2\sqrt{x}}e^{k\tan \sqrt{x}}$

37. $y=\cot^2(\sin \theta)=[\cot (\sin \theta)]^2\Rightarrow$
$y'=2[\cot (\sin \theta)]\cdot \dfrac{d}{d\theta}[\cot (\sin \theta)]=2\cot (\sin \theta)\cdot [-\csc^2(\sin \theta)\cdot \cos \theta]=-2\cos \theta\cdot \cot (\sin \theta)\cdot \csc^2(\sin \theta)$

38. $y=\sin (\sin (\sin x))\Rightarrow y'=\cos (\sin (\sin x))\dfrac{d}{dx}(\sin (\sin x))=\cos (\sin (\sin x))\cdot \cos (\sin x)\cdot \cos x$

39. $y=\sqrt{x+\sqrt{x}}\Rightarrow\ y'=\dfrac{1}{2}(x+\sqrt{x})^{-1/2}\left(1+\dfrac{1}{2}x^{-1/2}\right)=\dfrac{1}{2\sqrt{x+\sqrt{x}}}\left(1+\dfrac{1}{2\sqrt{x}}\right)$

40. $y=\sqrt{x+\sqrt{x+\sqrt{x}}}\Rightarrow\ y'=\dfrac{1}{2}\left(x+\sqrt{x+\sqrt{x}}\right)^{-1/2}\left[1+\dfrac{1}{2}(x+\sqrt{x})^{-1/2}\left(1+\dfrac{1}{2}x^{-1/2}\right)\right]$

41. $y=\sin (\tan \sqrt{\sin x})\Rightarrow$
$y'=\cos (\tan \sqrt{\sin x})\cdot \dfrac{d}{dx}(\tan \sqrt{\sin x})=\cos (\tan \sqrt{\sin x})\cdot \sec^2\sqrt{\sin x}\cdot \dfrac{d}{dx}(\sin x)^{1/2}$

$=\cos (\tan \sqrt{\sin x})\sec^2\sqrt{\sin x}\cdot \dfrac{1}{2}(\sin x)^{-1/2}\cdot \cos x$

$=\cos(\tan\sqrt{\sin x})\left(\sec^2\sqrt{\sin x}\right)\left(\dfrac{1}{2\sqrt{\sin x}}\right)(\cos x)$

42. $y=2^{3^{x^2}} \Rightarrow y'=2^{3^{x^2}}(\ln 2)\dfrac{d}{dx}\left(3^{x^2}\right)=2^{3^{x^2}}(\ln 2)3^{x^2}(\ln 3)(2x)$

43. $y=(1+2x)^{10} \Rightarrow y'=10(1+2x)^9\cdot 2=20(1+2x)^9$. At $(0,1)$, $y'=20(1+0)^9=20$, and an equation of the tangent line is $y-1=20(x-0)$, or $y=20x+1$.

44. $y=\sin x+\sin^2 x \Rightarrow y'=\cos x+2\sin x\cos x$. At $(0,0)$, $y'=1$, and an equation of the tangent line is $y-0=1(x-0)$, or $y=x$.

45. $y=\sin(\sin x) \Rightarrow y'=\cos(\sin x)\cdot\cos x$. At $(\pi,0)$, $y'=\cos(\sin\pi)\cdot\cos\pi=\cos(0)\cdot(-1)=1(-1)=-1$, and an equation of the tangent line is $y-0=-1(x-\pi)$, or $y=-x+\pi$.

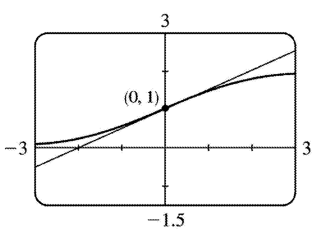
46. $y=x^2e^{-x} \Rightarrow y'=x^2(-e^{-x})+e^{-x}(2x)=2xe^{-x}-x^2e^{-x}$. At $\left(1, \dfrac{1}{e}\right)$, $y'=2e^{-1}-e^{-1}=\dfrac{1}{e}$. So an equation of the tangent line is $y-\dfrac{1}{e}=\dfrac{1}{e}(x-1)$ or $y=\dfrac{1}{e}x$.

47. **(a)** $y=\dfrac{2}{1+e^{-x}} \Rightarrow y'=\dfrac{(1+e^{-x})(0)-2(-e^{-x})}{(1+e^{-x})^2}=\dfrac{2e^{-x}}{(1+e^{-x})^2}$.

At $(0, 1)$, $y'=\dfrac{2e^0}{(1+e^0)^2}=\dfrac{2(1)}{(1+1)^2}=\dfrac{2}{2^2}=\dfrac{1}{2}$.

So an equation of the tangent line is $y-1=\dfrac{1}{2}(x-0)$ or $y=\dfrac{1}{2}x+1$.

(b)

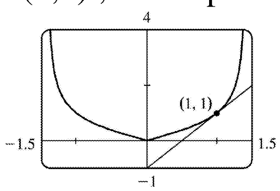


48. **(a)** For $x>0$, $|x|=x$, and $y=f(x)=\dfrac{x}{\sqrt{2-x^2}} \Rightarrow$

$$f'(x) = \frac{\sqrt{2-x^2}\,(1)-x\left(\frac{1}{2}\right)(2-x^2)^{-1/2}(-2x)}{\left(\sqrt{2-x^2}\right)^2}\cdot\frac{(2-x^2)^{1/2}}{(2-x^2)^{1/2}}$$

$$=\frac{(2-x^2)+x^2}{(2-x^2)^{3/2}}=\frac{2}{(2-x^2)^{3/2}}$$

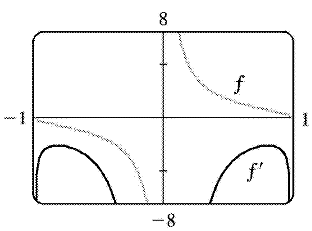
So at $(1,1)$, the slope of the tangent line is $f'(1)=2$ and its equation is $y-1=2(x-1)$ or $y=2x-1$.

(b) 

49. **(a)** $f(x)=\dfrac{\sqrt{1-x^2}}{x}\Rightarrow$

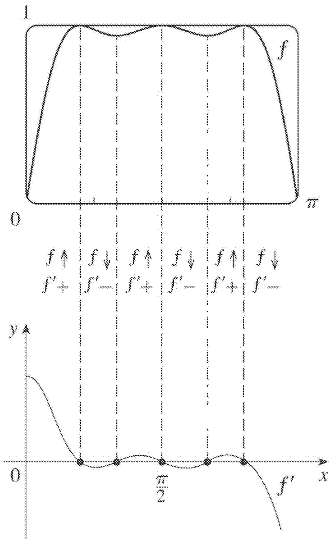
$$f'(x) = \frac{x\cdot\frac{1}{2}(1-x^{2})^{-1/2}(-2x)-\sqrt{1-x^2}\,(1)}{x^2}\cdot\frac{\sqrt{1-x^2}}{\sqrt{1-x^2}}$$

$$=\frac{-x^2-(1-x^2)}{x^2\sqrt{1-x^2}}=\frac{-1}{x^2\sqrt{1-x^2}}$$

(b) 

Notice that all tangents to the graph of f have negative slopes and $f'(x)<0$ always.

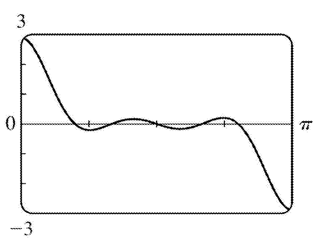
50. **(a)**



From the graph of f, we see that there are 5 horizontal tangents, so there must be 5 zeros on the graph of f'. From the symmetry of the graph of f, we must have the graph of f' as high at $x=0$ as it is low at $x=\pi$. The intervals of increase and decrease as well as the signs of f' are indicated in the figure.

(b)

$$\begin{aligned} f(x) &= \sin(x+\sin 2x) \Rightarrow \\ f'(x) &= \cos(x+\sin 2x)\cdot \frac{d}{dx}(x+\sin 2x) \\ &= \cos(x+\sin 2x)(1+2\cos 2x) \end{aligned}$$



51. For the tangent line to be horizontal, $f'(x)=0$. $f(x)=2\sin x+\sin^2 x \Rightarrow f'(x)=2\cos x+2\sin x\cos x=0$ $\Leftrightarrow 2\cos x(1+\sin x)=0 \Leftrightarrow \cos x=0$ or $\sin x=-1$, so $x=\frac{\pi}{2}+2n\pi$ or $\frac{3\pi}{2}+2n\pi$, where n is any integer. Now $f\left(\frac{\pi}{2}\right)=3$ and $f\left(\frac{3\pi}{2}\right)=-1$, so the points on the curve with a horizontal tangent are $\left(\frac{\pi}{2}+2n\pi, 3\right)$ and $\left(\frac{3\pi}{2}+2n\pi, -1\right)$, where n is any integer.

52. $f(x)=\sin 2x-2\sin x \Rightarrow$

$f'(x)=2\cos 2x-2\cos x=4\cos^2 x-2\cos x-2$, and $4\cos^2 x-2\cos x-2=0\Leftrightarrow(\cos x-1)(4\cos x+2)=0\Leftrightarrow$ $\cos x=1$ or $\cos x=-\frac{1}{2}$. So $x=2n\pi$ or $(2n+1)\pi\pm\frac{\pi}{3}$, n any integer.

53. $F(x)=f(g(x))\Rightarrow F'(x)=f'(g(x))\cdot g'(x)$,
so $F'(3)=f'(g(3))\cdot g'(3)=f'(6)\cdot g'(3)=7\cdot 4=28$. Notice that we did not use $f'(3)=2$.

54. $w=u\circ v\Rightarrow w(x)=u(v(x))\Rightarrow w'(x)=u'(v(x))\cdot v'(x)$, so
$w'(0)=u'(v(0))\cdot v'(0)=u'(2)\cdot v'(0)=4\cdot 5=20$. The other pieces of information, $u(0)=1$, $u'(0)=3$, and $v'(2)=6$, were not needed.

55. **(a)** $h(x)=f(g(x))\Rightarrow h'(x)=f'(g(x))\cdot g'(x)$, so $h'(1)=f'(g(1))\cdot g'(1)=f'(2)\cdot 6=5\cdot 6=30$.
(b) $H(x)=g(f(x))\Rightarrow H'(x)=g'(f(x))\cdot f'(x)$, so $H'(1)=g'(f(1))\cdot f'(1)=g'(3)\cdot 4=9\cdot 4=36$.

56. **(a)** $F(x)=f(f(x))\Rightarrow F'(x)=f'(f(x))\cdot f'(x)$, so $F'(2)=f'(f(2))\cdot f'(2)=f'(1)\cdot 5=4\cdot 5=20$.
(b) $G(x)=g(g(x))\Rightarrow G'(x)=g'(g(x))\cdot g'(x)$, so $G'(3)=g'(g(3))\cdot g'(3)=g'(2)\cdot 9=7\cdot 9=63$.

57. **(a)** $u(x)=f(g(x))\Rightarrow u'(x)=f'(g(x))g'(x)$. So $u'(1)=f'(g(1))g'(1)=f'(3)g'(1)$. To find $f'(3)$, note that f is linear from $(2,4)$ to $(6,3)$, so its slope is $\frac{3-4}{6-2}=-\frac{1}{4}$. To find $g'(1)$, note that g is linear from $(0,6)$ to $(2,0)$, so its slope is $\frac{0-6}{2-0}=-3$. Thus, $f'(3)g'(1)=\left(-\frac{1}{4}\right)(-3)=\frac{3}{4}$.
(b) $v(x)=g(f(x))\Rightarrow v'(x)=g'(f(x))f'(x)$. So $v'(1)=g'(f(1))f'(1)=g'(2)f'(1)$, which does not exist since $g'(2)$ does not exist.
(c) $w(x)=g(g(x))\Rightarrow w'(x)=g'(g(x))g'(x)$. So $w'(1)=g'(g(1))g'(1)=g'(3)g'(1)$. To find $g'(3)$, note that g is linear from $(2,0)$ to $(5,2)$, so its slope is $\frac{2-0}{5-2}=\frac{2}{3}$. Thus, $g'(3)\cdot g'(1)=\left(\frac{2}{3}\right)(-3)=-2$.

58. **(a)** $h(x)=f(f(x))\Rightarrow h'(x)=f'(f(x))f'(x)$.
So $h'(2)=f'(f(2))f'(2)=f'(1)f'(2)\approx(-1)(-1)=1$.
(b) $g(x)=f(x^2)\Rightarrow g'(x)=f'(x^2)\cdot\frac{d}{dx}(x^2)=f'(x^2)(2x)$. So $g'(2)=f'(2^2)(2\cdot 2)=4f'(4)\approx 4(1.5)=6$.

59. $h(x)=f(g(x)) \Rightarrow h'(x)=f'(g(x))g'(x)$. So $h'(0.5)=f'(g(0.5))g'(0.5)=f'(0.1)g'(0.5)$. We can estimate the derivatives by taking the average of two secant slopes.

For $f'(0.1)$: $m_1=\dfrac{14.8-12.6}{0.1-0}=22$, $m_2=\dfrac{18.4-14.8}{0.2-0.1}=36$. So $f'(0.1)\approx\dfrac{m_1+m_2}{2}=\dfrac{22+36}{2}=29$.

For $g'(0.5)$: $m_1=\dfrac{0.10-0.17}{0.5-0.4}=-0.7$, $m_2=\dfrac{0.05-0.10}{0.6-0.5}=-0.5$. So $g'(0.5)\approx\dfrac{m_1+m_2}{2}=-0.6$.

Hence, $h'(0.5)=f'(0.1)g'(0.5)\approx(29)(-0.6)=-17.4$.

60. $g(x)=f(f(x)) \Rightarrow g'(x)=f'(f(x))f'(x)$. So $g'(1)=f'(f(1))f'(1)=f'(2)f'(1)$.

For $f'(2)$: $m_1=\dfrac{3.1-2.4}{2.0-1.5}=1.4$, $m_2=\dfrac{4.4-3.1}{2.5-2.0}=2.6$. So $f'(2)\approx\dfrac{m_1+m_2}{2}=2$.

For $f'(1)$: $m_1=\dfrac{2.0-1.8}{1.0-0.5}=0.4$, $m_2=\dfrac{2.4-2.0}{1.5-1.0}=0.8$. So $f'(1)\approx\dfrac{m_1+m_2}{2}=0.6$.

Hence, $g'(1)=f'(2)f'(1)\approx(2)(0.6)=1.2$.

61. **(a)** $F(x)=f(e^x) \Rightarrow F'(x)=f'(e^x)\dfrac{d}{dx}(e^x)=f'(e^x)e^x$

(b) $G(x)=e^{f(x)} \Rightarrow G'(x)=e^{f(x)}\dfrac{d}{dx}f(x)=e^{f(x)}f'(x)$

62. **(a)** $F(x)=f(x^\alpha) \Rightarrow F'(x)=f'(x^\alpha)\dfrac{d}{dx}(x^\alpha)=f'(x^\alpha)\alpha x^{\alpha-1}$

(b) $G(x)=[f(x)]^\alpha \Rightarrow G'(x)=\alpha[f(x)]^{\alpha-1}f'(x)$

63. **(a)** $f(x)=L(x^4) \Rightarrow f'(x)=L'(x^4)\cdot 4x^3=(1/x^4)\cdot 4x^3=4/x$ for $x>0$.

(b) $g(x)=L(4x) \Rightarrow g'(x)=L'(4x)\cdot 4=(1/(4x))\cdot 4=1/x$ for $x>0$.

(c) $F(x)=[L(x)]^4 \Rightarrow F'(x)=4[L(x)]^3\cdot L'(x)=4[L(x)]^3\cdot(1/x)=4[L(x)]^3/x$

(d) $G(x)=L(1/x) \Rightarrow G'(x)=L'(1/x)\cdot(-1/x^2)=(1/(1/x))\cdot(-1/x^2)=x\cdot(-1/x^2)=-1/x$ for $x>0$.

64. $r(x)=f(g(h(x))) \Rightarrow r'(x)=f'(g(h(x)))\cdot g'(h(x))\cdot h'(x)$, so
$r'(1)=f'(g(h(1)))\cdot g'(h(1))\cdot h'(1)=f'(g(2))\cdot g'(2)\cdot 4=f'(3)\cdot 5\cdot 4=6\cdot 5\cdot 4=120$

65. $s(t)=10+\dfrac{1}{4}\sin(10\pi t) \Rightarrow$ the velocity after t seconds is

$v(t)=s'(t)=\frac{1}{4}\cos(10\pi t)(10\pi)=\frac{5\pi}{2}\cos(10\pi t)$ cm / s.

66. **(a)** $s=A\cos(\omega t+\delta)\Rightarrow$ velocity $=s'=-\omega A\sin(\omega t+\delta)$.

(b) If $A\neq 0$ and $\omega\neq 0$, then $s'=0\Leftrightarrow\sin(\omega t+\delta)=0\Leftrightarrow\omega t+\delta=n\pi\Leftrightarrow t=\frac{n\pi-\delta}{\omega}$, n an integer.

67. **(a)** $B(t)=4.0+0.35\sin\frac{2\pi t}{5.4}\Rightarrow\frac{dB}{dt}=\left(0.35\cos\frac{2\pi t}{5.4}\right)\left(\frac{2\pi}{5.4}\right)=\frac{0.7\pi}{5.4}\cos\frac{2\pi t}{5.4}=\frac{7\pi}{54}\cos\frac{2\pi t}{5.4}$

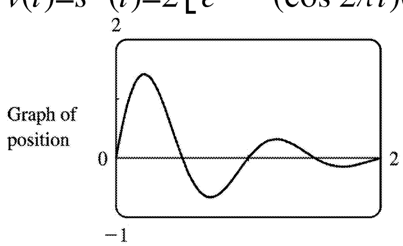
(b) At $t=1$, $\frac{dB}{dt}=\frac{7\pi}{54}\cos\frac{2\pi}{5.4}\approx 0.16$.

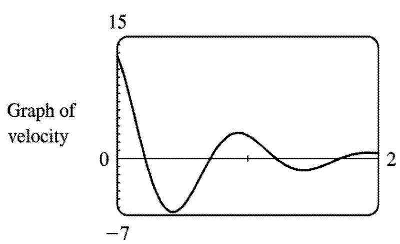
68. $L(t)=12+2.8\sin\left(\frac{2\pi}{365}(t-80)\right)\Rightarrow L'(t)=2.8\cos\left(\frac{2\pi}{365}(t-80)\right)\left(\frac{2\pi}{365}\right)$.

On March 21, $t=80$, and $L'(80)\approx 0.0482$ hours per day. On May 21, $t=141$, and $L'(141)\approx 0.02398$, which is approximately one–half of $L'(80)$.

69. $s(t)=2e^{-1.5t}\sin 2\pi t\Rightarrow$

$v(t)=s'(t)=2\left[e^{-1.5t}(\cos 2\pi t)(2\pi)+(\sin 2\pi t)e^{-1.5t}(-1.5)\right]=2e^{-1.5t}(2\pi\cos 2\pi t-1.5\sin 2\pi t)$

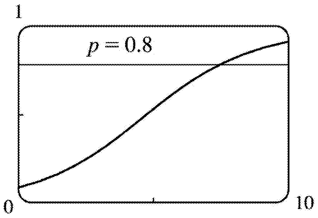




70. **(a)** $\lim_{t\to\infty}p(t)=\lim_{t\to\infty}\frac{1}{1+ae^{-kt}}=\frac{1}{1+a\cdot 0}=1$, since $k>0\Rightarrow -kt\to-\infty\Rightarrow e^{-kt}\to 0$.

(b) $p(t)=(1+ae^{-kt})^{-1}\Rightarrow\frac{dp}{dt}=-(1+ae^{-kt})^{-2}(-kae^{-kt})=\frac{kae^{-kt}}{(1+ae^{-kt})^2}$

(c)

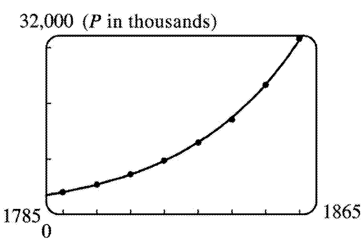


From the graph of $p(t)=(1+10e^{-0.5t})^{-1}$, it seems that $p(t)=0.8$ (indicating that 80% of the population has heard the rumor) when $t\approx 7.4$ hours.

71. **(a)** Using a calculator or CAS, we obtain the model $Q=ab^t$ with $a=100.0124369$ and $b=0.000045145933$. We can change this model to one with base e and exponent $\ln b$ [$b^t=e^{t\ln b}$ from precalculus mathematics or from Section 7.3]: $Q=ae^{t\ln b}=100.012437e^{-10.005531t}$.

(b) Use $Q'(t)=ab^t\ln b$ or the calculator command nDeriv(Y_1, X, .04) with $Y_1=ab^x$ to get $Q'(0.04)\approx -670.63\mu A$. The result of Example 2 in Section 2.1 was $-670\mu A$.

72. **(a)** $P=ab^t$ with $a=4.502714\times 10^{-20}$ and $b=1.029953851$,
where P is measured in thousands of people. The fit appears to be very good.



(b) **For 1800:** $m_1=\dfrac{5308-3929}{1800-1790}=137.9$, $m_2=\dfrac{7240-5308}{1810-1800}=193.2$.

So $P'(1800)\approx(m_1+m_2)/2=165.55$ thousand people / year.

For 1850: $m_1=\dfrac{23{,}192-17{,}063}{1850-1840}=612.9$, $m_2=\dfrac{31{,}443-23{,}192}{1860-1850}=825.1$.

So $P'(1850)\approx(m_1+m_2)/2=719$ thousand people / year.

(c) Use the calculator command nDeriv(Y_1, X, .04) with $Y_1=ab^x$ to get

$P'(1800)\approx 156.85$ and $P'(1850)\approx 686.07$. These estimates are somewhat less than the ones in part (b).

(d) $P(1870)\approx 41{,}946.56$. The difference of 3.4 million people is most likely due to the Civil War (1861—1865).

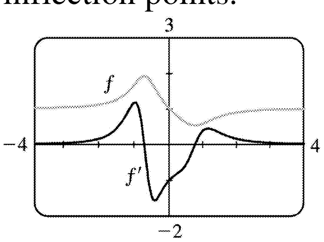
73. **(a)** Derive gives $g'(t)=\dfrac{45(t-2)^8}{(2t+1)^{10}}$ without simplifying. With either Maple or Mathematica, we first get $g'(t)=9\dfrac{(t-2)^8}{(2t+1)^9}-18\dfrac{(t-2)^9}{(2t+1)^{10}}$, and the simplification command results in the above expression.

(b) Derive gives $y'=2(x^3-x+1)^3(2x+1)^4(17x^3+6x^2-9x+3)$ without simplifying.
With either Maple or Mathematica, we first get $y'=10(2x+1)^4(x^3-x+1)^4+4(2x+1)^5(x^3-x+1)^3(3x^2-1)$.
If we use Mathematica's **Factor** or **Simplify**, or Maple's **factor**, we get the above expression, but Maple's **simplify** gives the polynomial expansion instead. For locating horizontal tangents, the factored form is the most helpful.

74. **(a)** $f(x)=\left(\dfrac{x^4-x+1}{x^4+x+1}\right)^{1/2}$. Derive gives $f'(x)=\dfrac{(3x^4-1)\sqrt{\dfrac{x^4-x+1}{x^4+x+1}}}{(x^4+x+1)(x^4-x+1)}$ whereas either Maple or Mathematica give $f'(x)=\dfrac{3x^4-1}{\sqrt{\dfrac{x^4-x+1}{x^4+x+1}}\,(x^4+x+1)^2}$ after simplification.

(b) $f'(x)=0\Leftrightarrow 3x^4-1=0\Leftrightarrow x=\pm\sqrt[4]{\dfrac{1}{3}}\approx\pm 0.7598$.

(c) $f'(x)=0$ where f has horizontal tangents. f' has two maxima and one minimum where f has inflection points.



75. **(a)** If f is even, then $f(x)=f(-x)$. Using the Chain Rule to differentiate this equation, we get $f'(x)=f'(-x)\dfrac{d}{dx}(-x)=-f'(-x)$. Thus, $f'(-x)=-f'(x)$, so f' is odd.

(b) If f is odd, then $f(x)=-f(-x)$. Differentiating this equation, we get $f'(x)=-f'(-x)(-1)=f'(-x)$, so f' is even.

76.

$$\left[\frac{f(x)}{g(x)}\right]' = \left\{f(x)[g(x)]^{-1}\right\}' = f'(x)[g(x)]^{-1} + (-1)[g(x)]^{-2}g'(x)f(x)$$

$$= \frac{f'(x)}{g(x)} - \frac{f(x)g'(x)}{[g(x)]^2} = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

77. **(a)**

$$\frac{d}{dx}\left(\sin^n x\cos nx\right) = n\sin^{n-1}x\cos x\cos nx + \sin^n x(-n\sin nx) \quad \text{[Product Rule]}$$

$$= n\sin^{n-1}x(\cos nx\cos x - \sin nx\sin x) \quad \text{[factor out } n\sin^{n-1}x\text{]}$$

$$= n\sin^{n-1}x\cos(nx+x) \quad \text{[Addition Formula for cosine]}$$

$$= n\sin^{n-1}x\cos[(n+1)x] \quad \text{[factor out } x\text{]}$$

(b)

$$\frac{d}{dx}\left(\cos^n x\cos nx\right) = n\cos^{n-1}x(-\sin x)\cos nx + \cos^n x(-n\sin nx) \quad \text{[Product Rule]}$$

$$= -n\cos^{n-1}x(\cos nx\sin x + \sin nx\cos x) \quad \text{[factor out } n\cos^{n-1}x\text{]}$$

$$= -n\cos^{n-1}x\sin(nx+x) \quad \text{[Addition Formula for sine]}$$

$$= -n\cos^{n-1}x\sin[(n+1)x] \quad \text{[factor out } x\text{]}$$

78. ''The rate of change of y^5 with respect to x is eighty times the rate of change of y with respect to x '' $\Leftrightarrow \frac{d}{dx}y^5 = 80\frac{dy}{dx} \Leftrightarrow 5y^4\frac{dy}{dx} = 80\frac{dy}{dx} \Leftrightarrow 5y^4 = 80$ (Note that $\frac{dy}{dx} \neq 0$ since the curve never has a horizontal tangent) $\Leftrightarrow y^4 = 16 \Leftrightarrow y = 2$ (since $y>0$ for all x)

79. Since $\theta^\circ = \left(\frac{\pi}{180}\right)\theta$ rad, we have

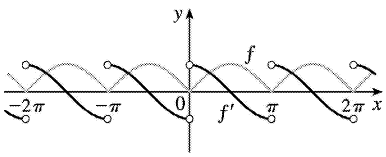
$$\frac{d}{d\theta}\left(\sin\theta^\circ\right) = \frac{d}{d\theta}\left(\sin\frac{\pi}{180}\theta\right) = \frac{\pi}{180}\cos\frac{\pi}{180}\theta = \frac{\pi}{180}\cos\theta^\circ .$$

80. **(a)** $f(x) = |x| = \sqrt{x^2} = (x^2)^{1/2} \Rightarrow f'(x) = \frac{1}{2}(x^2)^{-1/2}(2x) = \frac{x}{\sqrt{x^2}} = \frac{x}{|x|}$ for $x \neq 0$.

f is not differentiable at $x=0$.

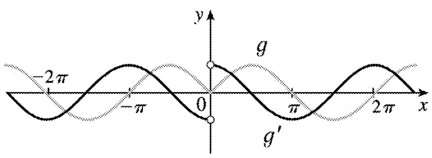
(b) $f(x)=|\sin x|=\sqrt{\sin^2 x} \Rightarrow$

$$f'(x)=\frac{1}{2}(\sin^2 x)^{-1/2}\cdot 2\sin x\cdot\cos x=\frac{\sin x}{|\sin x|}\cos x=\begin{cases}\cos x & \text{if } \sin x>0\\ -\cos x & \text{if } \sin x<0\end{cases}$$



f is not differentiable when $x=n\pi$, n an integer.

(c) $g(x)=\sin|x|=\sin\sqrt{x^2} \Rightarrow g'(x)=\cos|x|\cdot\frac{x}{|x|}=\frac{x}{|x|}\cos x=\begin{cases}\cos x & \text{if } x>0\\ -\cos x & \text{if } x<0\end{cases}$



g is not differentiable at 0 .

81. First note that products and differences of polynomials are polynomials and that the derivative of a polynomial is also a polynomial. When $n=1$,

$$f^{(1)}(x)=\left(\frac{P(x)}{Q(x)}\right)'=\frac{Q(x)P'(x)-P(x)Q'(x)}{[Q(x)]^2}=\frac{A_1(x)}{[Q(x)]^{1+1}}\text{, where } A_1(x)=Q(x)P'(x)-P(x)Q'(x).$$

Suppose the result is true for $n=k$, where $k\geq 1$. Then $f^{(k)}(x)=\dfrac{A_k(x)}{[Q(x)]^{k+1}}$, so

$$f^{(k+1)}(x)=\left(\frac{A_k(x)}{[Q(x)]^{k+1}}\right)'=\frac{[Q(x)]^{k+1}A_k'(x)-A_k(x)\cdot(k+1)[Q(x)]^k\cdot Q'(x)}{\{[Q(x)]^{k+1}\}^2}$$

$$=\frac{[Q(x)]^{k+1}A_k'(x)-(k+1)A_k(x)[Q(x)]^kQ'(x)}{[Q(x)]^{2k+2}}$$

$$=\frac{[Q(x)]^{k}\{^{1}A_k'(x)-(k+1)A_k(x)Q'(x)\}}{[Q(x)]^k[Q(x)]^{k+2}}=\frac{Q(x)A_k'(x)-(k+1)A_k(x)Q'(x)}{[Q(x)]^{k+2}}$$

$$=A_{k+1}(x)/[Q(x)]^{k+2},\text{ where } A_{k+1}(x)=Q(x)A_k'(x)-(k+1)A_k(x)Q'(x)\ .$$

We have shown that the formula holds for $n=1$, and that when it holds for $n=k$ it also holds for $n=k+1$. Thus, by mathematical induction, the formula holds for all positive integers n.

1. **(a)** $\frac{d}{dx}(xy+2x+3x^2)=\frac{d}{dx}(4)\Rightarrow(x\cdot y'+y\cdot 1)+2+6x=0\Rightarrow xy'=-y-2-6x\Rightarrow y'=\frac{-y-2-6x}{x}$ or $y'=-6-\frac{y+2}{x}$.

(b) $xy+2x+3x^2=4\Rightarrow xy=4-2x-3x^2\Rightarrow y=\frac{4-2x-3x^2}{x}=\frac{4}{x}-2-3x$, so $y'=-\frac{4}{x^2}-3$.

(c) From part (a), $y'=\frac{-y-2-6x}{x}=\frac{-(4/x-2-3x)-2-6x}{x}=\frac{-4/x-3x}{x}=-\frac{4}{x^2}-3$.

2. **(a)** $\frac{d}{dx}(4x^2+9y^2)=\frac{d}{dx}(36)\Rightarrow 8x+18y\cdot y'=0\Rightarrow y'=-\frac{8x}{18y}=-\frac{4x}{9y}$

(b) $4x^2+9y^2=36\Rightarrow 9y^2=36-4x^2\Rightarrow y^2=\frac{4}{9}(9-x^2)\Rightarrow y=\pm\frac{2}{3}\sqrt{9-x^2}$, so

$y'=\pm\frac{2}{3}\cdot\frac{1}{2}(9-x^2)^{-1/2}(-2x)=\mp\frac{2x}{3\sqrt{9-x^2}}$

(c) From part (a), $y'=-\frac{4x}{9y}=-\frac{4x}{9\left(\pm\frac{2}{3}\sqrt{9-x^2}\right)}=\mp\frac{2x}{3\sqrt{9-x^2}}$.

3. **(a)** $\frac{d}{dx}\left(\frac{1}{x}+\frac{1}{y}\right)=\frac{d}{dx}(1)\Rightarrow-\frac{1}{x^2}-\frac{1}{y^2}y'=0\Rightarrow-\frac{1}{y^2}y'=\frac{1}{x^2}\Rightarrow y'=-\frac{y^2}{x^2}$

(b) $\frac{1}{x}+\frac{1}{y}=1\Rightarrow\frac{1}{y}=1-\frac{1}{x}=\frac{x-1}{x}\Rightarrow y=\frac{x}{x-1}$, so $y'=\frac{(x-1)(1)-(x)(1)}{(x-1)^2}=\frac{-1}{(x-1)^2}$.

(c) $y'=-\frac{y^2}{x^2}=-\frac{[x/(x-1)]^2}{x^2}=-\frac{x^2}{x^2(x-1)^2}=-\frac{1}{(x-1)^2}$

4. **(a)** $\frac{d}{dx}(\sqrt{x}+\sqrt{y})=\frac{d}{dx}(4)\Rightarrow\frac{1}{2\sqrt{x}}+\frac{1}{2\sqrt{y}}y'=0\Rightarrow y'=-\frac{\sqrt{y}}{\sqrt{x}}$

(b) $\sqrt{y}=4-\sqrt{x}\Rightarrow y=(4-\sqrt{x})^2=16-8\sqrt{x}+x\Rightarrow y'=-\frac{4}{\sqrt{x}}+1$

(c) $y'=-\frac{\sqrt{y}}{\sqrt{x}}=-\frac{4-\sqrt{x}}{\sqrt{x}}=-\frac{4}{\sqrt{x}}+1$

5. $\frac{d}{dx}(x^2+y^2)=\frac{d}{dx}(1)\Rightarrow 2x+2yy'=0\Rightarrow 2yy'=-2x\Rightarrow y'=-\frac{x}{y}$

6. $\frac{d}{dx}(x^2-y^2)=\frac{d}{dx}(1)\Rightarrow 2x-2yy'=0\Rightarrow 2x=2yy'\Rightarrow\ y'=\frac{x}{y}$

7. $\frac{d}{dx}(x^3+x^2y+4y^2)=\frac{d}{dx}(6)\Rightarrow 3x^2+(x^2y'+y\cdot 2x)+8yy'=0\Rightarrow x^2y'+8yy'=-3x^2-2xy\Rightarrow$

$(x^2+8y)y'=-3x^2-2xy\Rightarrow\ y'=-\frac{3x^2+2xy}{x^2+8y}=-\frac{x(3x+2y)}{x^2+8y}$

8. $\frac{d}{dx}(x^2-2xy+y^3)=\frac{d}{dx}(c)\Rightarrow 2x-2(xy'+y\cdot 1)+3y^2y'=0\Rightarrow 2x-2y=2xy'-3y^2y'\Rightarrow 2x-2y=y'(2x-3y^2)\Rightarrow$
$y'=\frac{2x-2y}{2x-3y^2}$

9. $\frac{d}{dx}(x^2y+xy^2)=\frac{d}{dx}(3x)\Rightarrow(x^2y'+y\cdot 2x)+(x\cdot 2yy'+y^2\cdot 1)=3\Rightarrow x^2y'+2xyy'=3-2xy-y^2\Rightarrow$

$y'(x^2+2xy)=3-2xy-y^2\Rightarrow\ y'=\frac{3-2xy-y^2}{x^2+2xy}$

10. $\frac{d}{dx}(y^5+x^2y^3)=\frac{d}{dx}(1+x^4y)\Rightarrow 5y^4y'+x^2\cdot 3y^2y'+y^3\cdot 2x=0+x^4y'+y\cdot 4x^3\Rightarrow$

$y'(5y^4+3x^2y^2-x^4)=4x^3y-2xy^3\Rightarrow\ y'=\frac{4x^3y-2xy^3}{5y^4+3x^2y^2-x^4}$

11. $\frac{d}{dx}(x^2y^2+x\sin y)=\frac{d}{dx}(4)\Rightarrow\ x^2\cdot 2yy'+y^2\cdot 2x+x\cos y\cdot y'+\sin y\cdot 1=0\Rightarrow$

$2x^2yy'+x\cos y\cdot y'=-2xy^2-\sin y\Rightarrow(2x^2y+x\cos y)y'=-2xy^2-\sin y\Rightarrow\ y'=\frac{-2xy^2-\sin y}{2x^2y+x\cos y}$

12. $\frac{d}{dx}(1+x)=\frac{d}{dx}[\sin(xy^2)]\Rightarrow 1=[\cos(xy^2)](x\cdot 2yy'+y^2\cdot 1)\Rightarrow 1=2xy\cos(xy^2)y'+y^2\cos(xy^2)\Rightarrow$

$1-y^2\cos(xy^2)=2xy\cos(xy^2)y'\Rightarrow\ y'=\frac{1-y^2\cos(xy^2)}{2xy\cos(xy^2)}$

13. $\frac{d}{dx}(4\cos x\sin y)=\frac{d}{dx}(1)\Rightarrow 4[\cos x\cdot\cos y\cdot y'+\sin y\cdot(-\sin x)]=0\Rightarrow$

$y'(4\cos x\cos y)=4\sin x\sin y\Rightarrow y'=\dfrac{4\sin x\sin y}{4\cos x\cos y}=\tan x\tan y$

14. $\dfrac{d}{dx}[y\sin(x^2)]=\dfrac{d}{dx}[x\sin(y^2)]\Rightarrow y\cos(x^2)\cdot 2x+\sin(x^2)\cdot y'=x\cos(y^2)\cdot 2yy'+\sin(y^2)\cdot 1\Rightarrow$

$y'[\sin(x^2)-2xy\cos(y^2)]=\sin(y^2)-2xy\cos(x^2)\Rightarrow y'=\dfrac{\sin(y^2)-2xy\cos(x^2)}{\sin(x^2)-2xy\cos(y^2)}$

15. $\dfrac{d}{dx}\left(e^{x^2y}\right)=\dfrac{d}{dx}(x+y)\Rightarrow e^{x^2y}(x^2y'+y\cdot 2x)=1+y'\Rightarrow x^2e^{x^2y}y'+2xye^{x^2y}=1+y'\Rightarrow$

$x^2e^{x^2y}y'-y'=1-2xye^{x^2y}\Rightarrow y'(x^2e^{x^2y}-1)=1-2xye^{x^2y}\Rightarrow y'=\dfrac{1-2xye^{x^2y}}{x^2e^{x^2y}-1}$

16. $\dfrac{d}{dx}(\sqrt{x+y})=\dfrac{d}{dx}(1+x^2y^2)\Rightarrow \dfrac{1}{2}(x+y)^{-1/2}(1+y')=x^2\cdot 2yy'+y^2\cdot 2x\Rightarrow$

$\dfrac{1}{2\sqrt{x+y}}+\dfrac{y'}{2\sqrt{x+y}}=2x^2yy'+2xy^2\Rightarrow 1+y'=4x^2y\sqrt{x+y}\,y'+4xy^2\sqrt{x+y}\Rightarrow$

$y'-4x^2y\sqrt{x+y}\,y'=4xy^2\sqrt{x+y}-1\Rightarrow y'(1-4x^2y\sqrt{x+y})=4xy^2\sqrt{x+y}-1\Rightarrow y'=\dfrac{4xy^2\sqrt{x+y}-1}{1-4x^2y\sqrt{x+y}}$

17. $\sqrt{xy}=1+x^2y\Rightarrow \dfrac{1}{2}(xy)^{-1/2}(xy'+y\cdot 1)=0+x^2y'+y\cdot 2x\Rightarrow \dfrac{x}{2\sqrt{xy}}y'+\dfrac{y}{2\sqrt{xy}}=x^2y'+2xy\Rightarrow$

$y'\left(\dfrac{x}{2\sqrt{xy}}-x^2\right)=2xy-\dfrac{y}{2\sqrt{xy}}\Rightarrow y'\left(\dfrac{x-2x^2\sqrt{xy}}{2\sqrt{xy}}\right)=\dfrac{4xy\sqrt{xy}-y}{2\sqrt{xy}}\Rightarrow y'=\dfrac{4xy\sqrt{xy}-y}{x-2x^2\sqrt{xy}}$

18. $\tan(x-y)=\dfrac{y}{1+x^2}\Rightarrow(1+x^2)\tan(x-y)=y\Rightarrow(1+x^2)\sec^2(x-y)\cdot(1-y')+\tan(x-y)\cdot 2x=y'\Rightarrow$

$(1+x^2)\sec^2(x-y)-(1+x^2)\sec^2(x-y)\cdot y'+2x\tan(x-y)=y'\Rightarrow$

$(1+x^2)\sec^2(x-y)+2x\tan(x-y)=[1+(1+x^2)]\sec^2(x-y)\cdot y'\Rightarrow y'=\dfrac{(1+x^2)\sec^2(x-y)+2x\tan(x-y)}{1+(1+x^2)\sec^2(x-y)}$

19. $xy=\cot(xy)\Rightarrow y+xy'=-\csc^2(xy)(y+xy')\Rightarrow(y+xy')[1+\csc^2(xy)]=0\Rightarrow y+xy'=0\Rightarrow y'=-y/x$

20. $\sin x+\cos y=\sin x\cos y\Rightarrow$

$\cos x - \sin y \cdot y' = \sin x(-\sin y \cdot y') + \cos y \cos x \Rightarrow (\sin x \sin y - \sin y)\, y' = \cos x \cos y - \cos x \Rightarrow$
$y' = \dfrac{\cos x(\cos y - 1)}{\sin y(\sin x - 1)}$

21. $\dfrac{d}{dx}\{1+f(x)+x^2[f(x)]^3\} = \dfrac{d}{dx}(0) \Rightarrow f'(x)+x^2 \cdot 3[f(x)]^2 \cdot f'(x)+[f(x)]^3 \cdot 2x=0$. If $x=1$, we have $f'(1)+1^2 \cdot 3[f(1)]^2 \cdot f'(1)+[f(1)]^3 \cdot 2(1)=0 \Rightarrow f'(1)+1 \cdot 3 \cdot 2^2 \cdot f'(1)+2^3 \cdot 2=0 \Rightarrow f'(1)+12f'(1)=-16 \Rightarrow$ $13f'(1)=-16 \Rightarrow f'(1)=-\dfrac{16}{13}$.

22. $\dfrac{d}{dx} = \dfrac{d}{dx}(x^2) \Rightarrow g'(x)+x\cos g(x) \cdot g'(x)+\sin g(x) \cdot 1=2x$. If $x=1$, we have $g'(1)+1\cos g(1) \cdot g'(1)+\sin g(1)=2(1) \Rightarrow g'(1)+\cos 0 \cdot g'(1)+\sin 0=2 \Rightarrow g'(1)+g'(1)=2 \Rightarrow 2g'(1)=2 \Rightarrow$ $g'(1)=1$.

23. $y^4+x^2y^2+yx^4=y+1 \Rightarrow 4y^3+\left(x^2 \cdot 2y+y^2 \cdot 2x\dfrac{dx}{dy}\right)+\left(y \cdot 4x^3\dfrac{dx}{dy}+x^4 \cdot 1\right)=1 \Rightarrow 2xy^2$
$\dfrac{dx}{dy}+4x^3y\dfrac{dx}{dy}=1-4y^3-2x^2y-x^4 \Rightarrow \dfrac{dx}{dy}=\dfrac{1-4y^3-2x^2y-x^4}{2xy^2+4x^3y}$

24. $(x^2+y^2)^2=ax^2y \Rightarrow 2(x^2+y^2)\left(2x\dfrac{dx}{dy}+2y\right)=2ayx\dfrac{dx}{dy}+ax^2 \Rightarrow \dfrac{dx}{dy}=\dfrac{ax^2-4y(x^2+y^2)}{4x(x^2+y^2)-2axy}$

25. $x^2+xy+y^2=3 \Rightarrow 2x+xy'+y \cdot 1+2yy'=0 \Rightarrow xy'+2yy'=-2x-y \Rightarrow y'(x+2y)=-2x-y \Rightarrow y'=\dfrac{-2x-y}{x+2y}$. When $x=1$ and $y=1$, we have $y'=\dfrac{-2-1}{1+2 \cdot 1}=\dfrac{-3}{3}=-1$, so an equation of the tangent line is $y-1=-1(x-1)$ or $y=-x+2$.

26. $x^2+2xy-y^2+x=2 \Rightarrow 2x+2(xy'+y \cdot 1)-2yy'+1=0 \Rightarrow 2xy'-2yy'=-2x-2y-1 \Rightarrow y'(2x-2y)=-2x-2y-1 \Rightarrow$ $y'=\dfrac{-2x-2y-1}{2x-2y}$. When $x=1$ and $y=2$, we have $y'=\dfrac{-2-4-1}{2-4}=\dfrac{-7}{-2}=\dfrac{7}{2}$, so an equation of the tangent line is $y-2=\dfrac{7}{2}(x-1)$ or $y=\dfrac{7}{2}x-\dfrac{3}{2}$.

27. $x^2+y^2=(2x^2+2y^2-x)^2 \Rightarrow 2x+2yy'=2(2x^2+2y^2-x)(4x+4yy'-1)$. When $x=0$ and $y=\dfrac{1}{2}$, we have

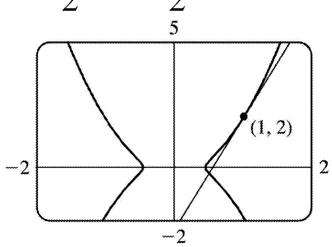
$0+y'=2\left(\frac{1}{2}\right)(2y'-1)\Rightarrow y'=2y'-1\Rightarrow y'=1$, so an equation of the tangent line is $y-\frac{1}{2}=1(x-0)$ or $y=x+\frac{1}{2}$.

28. $x^{2/3}+y^{2/3}=4\Rightarrow \frac{2}{3}x^{-1/3}+\frac{2}{3}y^{-1/3}y'=0\Rightarrow \frac{1}{\sqrt[3]{x}}+\frac{y'}{\sqrt[3]{y}}=0\Rightarrow y'=-\frac{\sqrt[3]{y}}{\sqrt[3]{x}}$. When $x=-3\sqrt{3}$ and $y=1$, we have $y'=-\frac{1}{(-3\sqrt{3})^{1/3}}=-\frac{(-3\sqrt{3})^{2/3}}{-3\sqrt{3}}=\frac{3}{3\sqrt{3}}=\frac{1}{\sqrt{3}}$, so an equation of the tangent line is $y-1=\frac{1}{\sqrt{3}}(x+3\sqrt{3})$ or $y=\frac{1}{\sqrt{3}}x+4$.

29. $2(x^2+y^2)^2=25(x^2-y^2)\Rightarrow 4(x^2+y^2)(2x+2yy')=25(2x-2yy')\Rightarrow$ $4(x+yy')(x^2+y^2)=25(x-yy')\Rightarrow 4yy'(x^2+y^2)+25yy'=25x-4x(x^2+y^2)\Rightarrow y'=\frac{25x-4x(x^2+y^2)}{25y+4y(x^2+y^2)}$. When $x=3$ and $y=1$, we have $y'=\frac{75-120}{25+40}=-\frac{45}{65}=-\frac{9}{13}$, so an equation of the tangent line is $y-1=-\frac{9}{13}(x-3)$ or $y=-\frac{9}{13}x+\frac{40}{13}$.

30. $y^2(y^2-4)=x^2(x^2-5)\Rightarrow y^4-4y^2=x^4-5x^2\Rightarrow 4y^3y'-8yy'=4x^3-10x$. When $x=0$ and $y=-2$, we have $-32y'+16y'=0\Rightarrow -16y'=0\Rightarrow y'=0$, so an equation of the tangent line is $y+2=0(x-0)$ or $y=-2$.

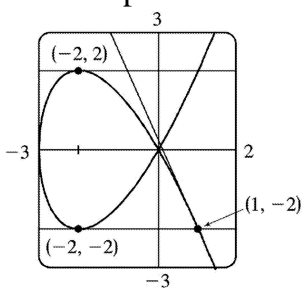
31. **(a)** $y^2=5x^4-x^2\Rightarrow 2yy'=5(4x^3)-2x\Rightarrow y'=\frac{10x^3-x}{y}$. So at the point (1,2) we have $y'=\frac{10(1)^3-1}{2}=\frac{9}{2}$, and an equation of the tangent line is $y-2=\frac{9}{2}(x-1)$ or $y=\frac{9}{2}x-\frac{5}{2}$.

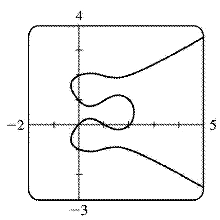
(b) 

32. **(a)** $y^2=x^3+3x^2\Rightarrow 2yy'=3x^2+3(2x)\Rightarrow y'=\frac{3x^2+6x}{2y}$. So at the point (1,–2) we have

$y' = \frac{3(1)^2+6(1)}{2(-2)} = -\frac{9}{4}$, and an equation of the tangent line is $y+2=-\frac{9}{4}(x-1)$ or $y=-\frac{9}{4}x+\frac{1}{4}$.

(b) The curve has a horizontal tangent where $y'=0 \Leftrightarrow 3x^2+6x=0 \Leftrightarrow 3x(x+2)=0 \Leftrightarrow x=0$ or $x=-2$. But note that at $x=0$, $y=0$ also, so the derivative does not exist. At $x=-2$, $y^2=(-2)^3+3(-2)^2=-8+12=4$, so $y=\pm 2$. So the two points at which the curve has a horizontal tangent are $(-2,-2)$ and $(-2,2)$.

(c) 

33. **(a)** 

There are eight points with horizontal tangents: four at $x \approx 1.57735$ and four at $x \approx 0.42265$.

(b) $y' = \frac{3x^2-6x+2}{2(2y^3-3y^2-y+1)} \Rightarrow y'=-1$ at (0, 1) and $y'=\frac{1}{3}$ at (0, 2)

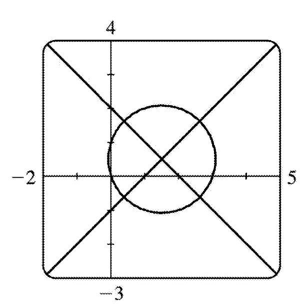
Equations of the tangent lines are $y=-x+1$ and $y=\frac{1}{3}x+2$.

(c) $y'=0 \Rightarrow 3x^2-6x+2=0 \Rightarrow x=1\pm\frac{1}{3}\sqrt{3}$

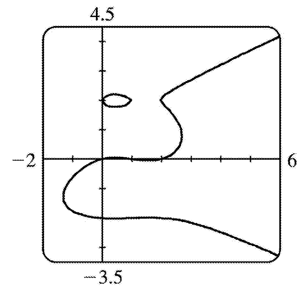
(d)

By multiplying the right side of the equation by $x-3$, we obtain the first graph.

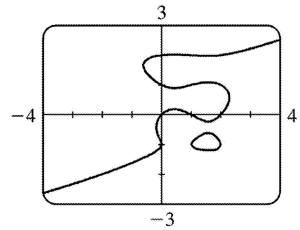
By modifying the equation in other ways, we can generate the other graphs.



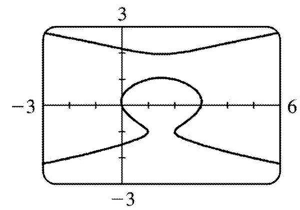
$y(y^2-1)(y-2)=x(x-1)(x-2)(x-3)$



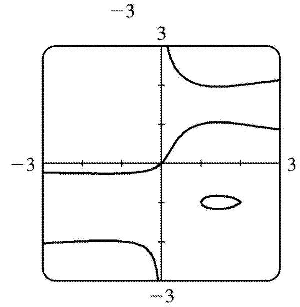
$y(y^2-4)(y-2)=x(x-1)(x-2)$



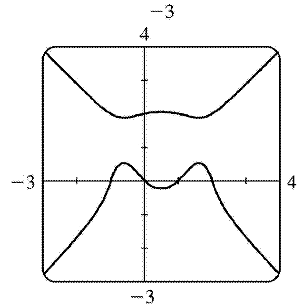
$y(y+1)(y^2-1)(y-2)=(x-1)(x-2)$



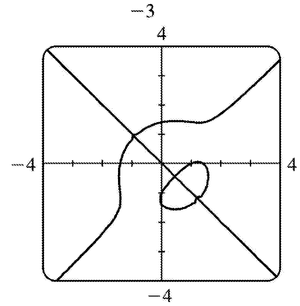
$(y+1)(y^2-1)(y-2)=x(x-1)(x-2)$



$x(y+1)(y^2-1)(y-2)=y(x-1)(x-2)$

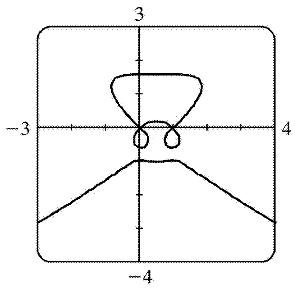


$y(y^2+1)(y-2)=x(x^2-1)(x-2)$



$y(y+1)(y^2-2)=x(x-1)(x^2-2)$

34. **(a)**



(b) There are 9 points with horizontal tangents: 3 at $x=0$, 3 at $x=\frac{1}{2}$, and 3 at $x=1$. The three horizontal tangents along the top of the wagon are hard to find, but by limiting the y –range of the graph (to $[1.6,1.7]$, for example) they are distinguishable.

35. From Exercise 29, a tangent to the lemniscate will be horizontal if $y'=0 \Rightarrow 25x-4x(x^2+y^2)=0 \Rightarrow x[25-4(x^2+y^2)]=0 \Rightarrow x^2+y^2=\frac{25}{4}$ (**1**). (Note that when x is 0 , y is also 0 , and there is no horizontal tangent at the origin.) Substituting $\frac{25}{4}$ for x^2+y^2 in the equation of the lemniscate, $2(x^2+y^2)^2=25(x^2-y^2)$, we get $x^2-y^2=\frac{25}{8}$ (**2**). Solving (**1**) and (**2**), we have $x^2=\frac{75}{16}$ and $y^2=\frac{25}{16}$, so the four points are $\left(\pm\frac{5\sqrt{3}}{4},\pm\frac{5}{4}\right)$.

36. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \Rightarrow \frac{2x}{a^2}+\frac{2yy'}{b^2}=0 \Rightarrow y'=-\frac{b^2x}{a^2y} \Rightarrow$ an equation of the tangent line at (x_0,y_0) is $y-y_0=\frac{-b^2x_0}{a^2y_0}(x-x_0)$. Multiplying both sides by $\frac{y_0}{b^2}$ gives $\frac{y_0y}{b^2}-\frac{y_0^2}{b^2}=-\frac{x_0x}{a^2}+\frac{x_0^2}{a^2}$. Since (x_0,y_0) lies on the ellipse, we have $\frac{x_0x}{a^2}+\frac{y_0y}{b^2}=\frac{x_0^2}{a^2}+\frac{y_0^2}{b^2}=1$.

37. $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \Rightarrow \frac{2x}{a^2}-\frac{2yy'}{b^2}=0 \Rightarrow y'=\frac{b^2x}{a^2y} \Rightarrow$ an equation of the tangent line at (x_0,y_0) is $y-y_0=\frac{b^2x_0}{a^2y_0}(x-x_0)$. Multiplying both sides by $\frac{y_0}{b^2}$ gives $\frac{y_0y}{b^2}-\frac{y_0^2}{b^2}=\frac{x_0x}{a^2}-\frac{x_0^2}{a^2}$. Since (x_0,y_0) lies on the hyperbola, we have

$\frac{x_0 x}{a^2} - \frac{y_0 y}{b^2} = \frac{x_0^2}{a^2} - \frac{y_0^2}{b^2} = 1$.

38. $\sqrt{x}+\sqrt{y}=\sqrt{c} \Rightarrow \frac{1}{2\sqrt{x}} + \frac{y'}{2\sqrt{y}} = 0 \Rightarrow y' = -\frac{\sqrt{y}}{\sqrt{x}} \Rightarrow$ an equation of the tangent line at (x_0,y_0) is $y-y_0 = -\frac{\sqrt{y_0}}{\sqrt{x_0}}(x-x_0)$. Now $x=0 \Rightarrow y=y_0-\frac{\sqrt{y_0}}{\sqrt{x_0}}(-x_0)=y_0+\sqrt{x_0}\sqrt{y_0}$, so the y –intercept is $y_0+\sqrt{x_0}\sqrt{y_0}$. And $y=0 \Rightarrow -y_0 = -\frac{\sqrt{y_0}}{\sqrt{x_0}}(x-x_0) \Rightarrow x-x_0 = \frac{y_0\sqrt{x_0}}{\sqrt{y_0}} \Rightarrow x=x_0+\sqrt{x_0}\sqrt{y_0}$, so the x –intercept is $x_0+\sqrt{x_0}\sqrt{y_0}$. The sum of the intercepts is
$\left(y_0+\sqrt{x_0}\sqrt{y_0}\right)+\left(x_0+\sqrt{x_0}\sqrt{y_0}\right)=x_0+2\sqrt{x_0}\sqrt{y_0}+y_0=\left(\sqrt{x_0}+\sqrt{y_0}\right)^2=(\sqrt{c})^2=c$.

39. If the circle has radius r , its equation is $x^2+y^2=r^2 \Rightarrow 2x+2yy'=0 \Rightarrow y' = -\frac{x}{y}$, so the slope of the tangent line at $P(x_0,y_0)$ is $-\frac{x_0}{y_0}$. The negative reciprocal of that slope is $\frac{-1}{-x_0/y_0} = \frac{y_0}{x_0}$, which is the slope of OP , so the tangent line at P is perpendicular to the radius OP .

40. $y^q=x^p \Rightarrow qy^{q-1}y'=px^{p-1} \Rightarrow y' = \frac{px^{p-1}}{qy^{q-1}} = \frac{px^{p-1}y}{qy^q} = \frac{px^{p-1}x^{p/q}}{qx^p} = \frac{p}{q}x^{(p/q)-1}$

41. $y=\tan^{-1}\sqrt{x} \Rightarrow y' = \frac{1}{1+(\sqrt{x})^2}\cdot\frac{d}{dx}(\sqrt{x}) = \frac{1}{1+x}\left(\frac{1}{2}x^{-1/2}\right) = \frac{1}{2\sqrt{x}(1+x)}$

42. $y=\sqrt{\tan^{-1}x}=(\tan^{-1}x)^{1/2} \Rightarrow$
$y' = \frac{1}{2}(\tan^{-1}x)^{-1/2}\cdot\frac{d}{dx}(\tan^{-1}x) = \frac{1}{2\sqrt{\tan^{-1}x}}\cdot\frac{1}{1+x^2} = \frac{1}{2\sqrt{\tan^{-1}x}(1+x^2)}$

43. $y=\sin^{-1}(2x+1) \Rightarrow$

$$y' = \frac{1}{\sqrt{1-(2x+1)^2}} \cdot \frac{d}{dx}(2x+1) = \frac{1}{\sqrt{1-(4x^2+4x+1)}} \cdot 2 = \frac{2}{\sqrt{-4x^2-4x}} = \frac{1}{\sqrt{-x^2-x}}$$

44. $h(x)=\sqrt{1-x^2}\arcsin x \Rightarrow h'(x)=\sqrt{1-x^2}\cdot\frac{1}{\sqrt{1-x^2}}+\arcsin x\left[\frac{1}{2}(1-x^2)^{-1/2}(-2x)\right]=1-\frac{x\arcsin x}{\sqrt{1-x^2}}$

45. $H(x)=(1+x^2)\arctan x \Rightarrow H'(x)=(1+x^2)\frac{1}{1+x^2}+(\arctan x)(2x)=1+2x\arctan x$

46. $y=\tan^{-1}\left(x-\sqrt{x^2+1}\right)\Rightarrow$

$$y' = \frac{1}{1+\left(x-\sqrt{x^2+1}\right)^2}\left(1-\frac{x}{\sqrt{x^2+1}}\right) = \frac{1}{1+x^2-2x\sqrt{x^2+1}+x^2+1}\left(\frac{\sqrt{x^2+1}-x}{\sqrt{x^2+1}}\right)$$

$$= \frac{\sqrt{x^2+1}-x}{2\left(1+x^2-x\sqrt{x^2+1}\right)\sqrt{x^2+1}} = \frac{\sqrt{x^2+1}-x}{2\left[\sqrt{x^2+1}\,(1+x^2)-x(x^2+1)\right]}$$

$$= \frac{\sqrt{x^2+1}-x}{2\left[(1+x^2)\left(\sqrt{x^2+1}-x\right)\right]} = \frac{1}{2(1+x^2)}$$

47. $h(t)=\cot^{-1}(t)+\cot^{-1}(1/t)\Rightarrow$

$$h'(t) = -\frac{1}{1+t^2}-\frac{1}{1+(1/t)^2}\cdot\frac{d}{dt}\frac{1}{t} = -\frac{1}{1+t^2}-\frac{t^2}{t^2+1}\cdot\left(-\frac{1}{t^2}\right) = -\frac{1}{1+t^2}+\frac{1}{t^2+1} = 0\,.$$

Note that this makes sense because $h(t)=\frac{\pi}{2}$ for $t>0$ and $h(t)=-\frac{\pi}{2}$ for $t<0$.

48. $y=x\cos^{-1}x-\sqrt{1-x^2}\Rightarrow y'=\cos^{-1}x-\frac{x}{\sqrt{1-x^2}}+\frac{x}{\sqrt{1-x^2}}=\cos^{-1}x$

49. $y=\cos^{-1}(e^{2x})\Rightarrow y'=-\frac{1}{\sqrt{1-(e^{2x})^2}}\cdot\frac{d}{dx}(e^{2x})=-\frac{2e^{2x}}{\sqrt{1-e^{4x}}}$

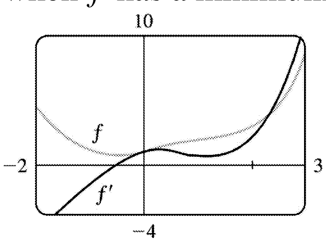
50. $y=\arctan(\cos\theta)\Rightarrow$

$$y'=\frac{1}{1+(\cos\theta)^2}(-\sin\theta)=-\frac{\sin\theta}{1+\cos^2\theta}$$

51. $f(x)=e^x-x^2\arctan x\Rightarrow$

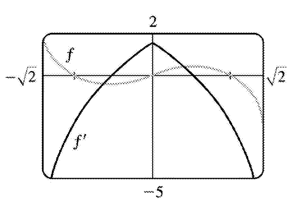
$$f'(x)=e^x-\left[x^2\left(\frac{1}{1+x^2}\right)+(\arctan x)(2x)\right]$$
$$=e^x-\frac{x^2}{1+x^2}-2x\arctan x$$

This is reasonable because the graphs show that f is increasing when f' is positive, and f' is zero when f has a minimum.



52. $f(x)=x\arcsin(1-x^2)\Rightarrow$

$$f'(x)=x\left[\frac{-2x}{\sqrt{1-(1-x^2)^2}}\right]+\arcsin(1-x^2)\cdot 1$$
$$=\arcsin(1-x^2)-\frac{2x^2}{\sqrt{2x^2-x^4}}$$



This is reasonable because the graphs show that f is increasing when f' is positive, and that f has an inflection point when f' changes from increasing to decreasing.

53. Let $y=\cos^{-1}x$. Then $\cos y=x$ and $0\le y\le\pi\Rightarrow -\sin y\,\frac{dy}{dx}=1\Rightarrow$

$$\frac{dy}{dx}=-\frac{1}{\sin y}=-\frac{1}{\sqrt{1-\cos^2 y}}=-\frac{1}{\sqrt{1-x^2}}.$$ (Note that $\sin y\ge 0$ for $0\le y\le\pi$).

54. **(a)** Let $y=\sec^{-1}x$. Then $\sec y=x$ and $y\in\left(0,\ \frac{\pi}{2}\right]\cup\left(\pi,\ \frac{3\pi}{2}\right]$. Differentiate with respect to x: $\sec y\tan y\left(\frac{dy}{dx}\right)=1\Rightarrow\frac{dy}{dx}=\frac{1}{\sec y\tan y}=\frac{1}{\sec y\sqrt{\sec^2 y-1}}=\frac{1}{x\sqrt{x^2-1}}$. Note that $\tan^2 y=\sec^2 y-1\Rightarrow$ $\tan y=\sqrt{\sec^2 y-1}$ since $\tan y>0$ when $0<y<\frac{\pi}{2}$ or $\pi<y<\frac{3\pi}{2}$.

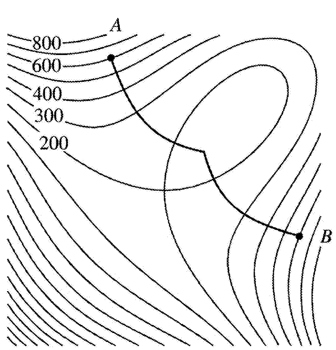
(b) $y=\sec^{-1}x\Rightarrow\sec y=x\Rightarrow\sec y\tan y\frac{dy}{dx}=1\Rightarrow\frac{dy}{dx}=\frac{1}{\sec y\tan y}$. Now $\tan^2 y=\sec^2 y-1=x^2-1$, so $\tan y=\pm\sqrt{x^2-1}$. For $y\in\left[0,\ \frac{\pi}{2}\right)$, $x\geq 1$, so $\sec y=x=|x|$ and $\tan y\geq 0\Rightarrow$

$\frac{dy}{dx}=\frac{1}{x\sqrt{x^2-1}}=\frac{1}{|x|\sqrt{x^2-1}}$. For $y\in\left(\frac{\pi}{2},\pi\right]$, $x\leq -1$, so $|x|=-x$ and $\tan y=-\sqrt{x^2-1}\Rightarrow$

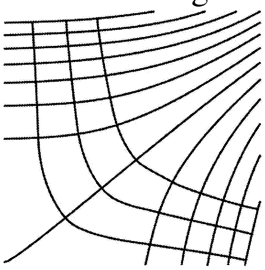
$$\frac{dy}{dx}=\frac{1}{\sec y\tan y}=\frac{1}{x\left(-\sqrt{x^2-1}\right)}=\frac{1}{(-x)\sqrt{x^2-1}}=\frac{1}{|x|\sqrt{x^2-1}}\ .$$

55. $2x^2+y^2=3$ and $x=y^2$ intersect when $2x^2+x-3=0\Leftrightarrow(2x+3)(x-1)=0\Leftrightarrow x=-\frac{3}{2}$ or 1 , but $-\frac{3}{2}$ is extraneous since $x=y^2$ is nonnegative. When $x=1$, $1=y^2\Rightarrow y=\pm 1$, so there are two points of intersection: $(1,\pm 1)$. $2x^2+y^2=3\Rightarrow 4x+2yy'=0\Rightarrow y'=-2x/y$, and $x=y^2\Rightarrow 1=2yy'\Rightarrow y'=1/(2y)$. At $(1,1)$, the slopes are $m_1=-2(1)/1=-2$ and $m_2=1/(2\cdot 1)=\frac{1}{2}$, so the curves are orthogonal (since m_1 and m_2 are negative reciprocals of each other). By symmetry, the curves are also orthogonal at $(1,-1)$.

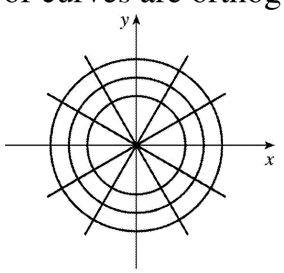
56. $x^2-y^2=5$ and $4x^2+9y^2=72$ intersect when $4x^2+9(x^2-5)=72\Leftrightarrow 13x^2=117\Leftrightarrow x=\pm 3$, so there are four points of intersection: $(\pm 3,\pm 2)$. $x^2-y^2=5\Rightarrow 2x-2yy'=0\Rightarrow y'=x/y$, and $4x^2+9y^2=72\Rightarrow 8x+18yy'=0\Leftrightarrow$ $y'=-4x/9y$. At $(3,2)$, the slopes are $m_1=\frac{3}{2}$ and $m_2=-\frac{2}{3}$, so the curves are orthogonal. By symmetry, the curves are also orthogonal at $(3,-2)$, $(-3,2)$ and $(-3,-2)$.

57. 

58. The orthogonal family represents the direction of the wind.

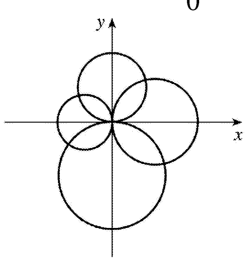


59. $x^2+y^2=r^2$ is a circle with center O and $ax+by=0$ is a line through O . $x^2+y^2=r^2 \Rightarrow 2x+2yy'=0 \Rightarrow y'=-x/y$, so the slope of the tangent line at $P_0(x_0,y_0)$ is $-x_0/y_0$. The slope of the line OP_0 is y_0/x_0 , which is the negative reciprocal of $-x_0/y_0$. Hence, the curves are orthogonal, and the families of curves are orthogonal trajectories of each other.

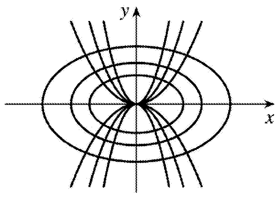


60. The circles $x^2+y^2=ax$ and $x^2+y^2=by$ intersect at the origin where the tangents are vertical and horizontal. If (x_0,y_0) is the other point of intersection, then $x_0^2+y_0^2=ax_0$ (**1**) and $x_0^2+y_0^2=by_0$ (**2**). Now $x^2+y^2=ax \Rightarrow 2x+2yy'=a \Rightarrow y'=\dfrac{a-2x}{2y}$ and $x^2+y^2=by \Rightarrow 2x+2yy'=by' \Rightarrow y'=\dfrac{2x}{b-2y}$. Thus, the curves are orthogonal at

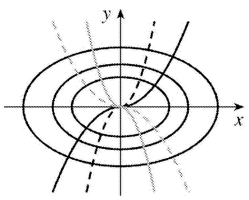
$(x_0,y_0) \Leftrightarrow \dfrac{a-2x_0}{2y_0}=-\dfrac{b-2y_0}{2x_0} \Leftrightarrow 2ax_0-4x_0^2=4y_0^2-2by_0 \Leftrightarrow ax_0+by_0=2(x_0^2+y_0^2)$, which is true by (**1**) and (**2**).



61. $y=cx^2 \Rightarrow y'=2cx$ and $x^2+2y^2=k \Rightarrow 2x+4yy'=0 \Rightarrow 2yy'=-x \Rightarrow y'=-\dfrac{x}{2(y)}=-\dfrac{x}{2(cx^2)}=-\dfrac{1}{2cx}$, so the curves are orthogonal.

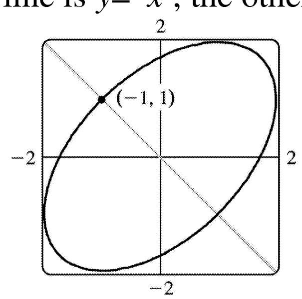


62. $y=ax^3 \Rightarrow y'=3ax^2$ and $x^2+3y^2=b \Rightarrow 2x+6yy'=0 \Rightarrow 3yy'=-x \Rightarrow y'=-\dfrac{x}{3(y)}=-\dfrac{x}{3(ax^3)}=-\dfrac{1}{3ax^2}$, so the curves are orthogonal.



63. To find the points at which the ellipse $x^2-xy+y^2=3$ crosses the x –axis, let $y=0$ and solve for x . $y=0 \Rightarrow x^2-x(0)+0^2=3 \Leftrightarrow x=\pm\sqrt{3}$. So the graph of the ellipse crosses the x –axis at the points $(\pm\sqrt{3},0)$. Using implicit differentiation to find y' , we get $2x-xy'-y+2yy'=0 \Rightarrow y'(2y-x)=y-2x \Leftrightarrow y'=\dfrac{y-2x}{2y-x}$. So y' at $(\sqrt{3},0)$ is $\dfrac{0-2\sqrt{3}}{2(0)-\sqrt{3}}=2$ and y' at $(-\sqrt{3},0)$ is $\dfrac{0+2\sqrt{3}}{2(0)+\sqrt{3}}=2$. Thus, the tangent lines at these points are parallel.

64. **(a)** We use implicit differentiation to find $y'=\dfrac{y-2x}{2y-x}$ as in Exercise 49. The slope of the tangent line at $(-1,1)$ is $m=\dfrac{1-2(-1)}{2(1)-(-1)}=\dfrac{3}{3}=1$, so the slope of the normal line is $-\dfrac{1}{m}=-1$, and its equation is $y-1=-1(x+1) \Leftrightarrow y=-x$. Substituting $-x$ for y in the equation of the ellipse, we get $x^2-x(-x)+(-x)^2=3 \Rightarrow 3x^2=3 \Leftrightarrow x=\pm 1$. So the normal line must intersect the ellipse again at $x=1$, and since the equation of the line is $y=-x$, the other point of intersection must be $(1,-1)$.

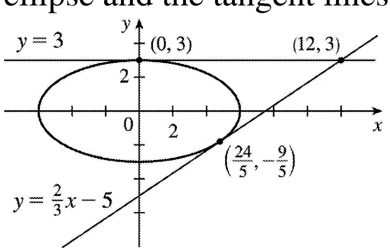
(b) 

65. $x^2y^2+xy=2 \Rightarrow x^2\cdot 2yy'+y^2\cdot 2x+x\cdot y'+y\cdot 1=0 \Leftrightarrow y'(2x^2y+x)=-2xy^2-y \Leftrightarrow$

$y'=-\dfrac{2xy^2+y}{2x^2y+x}$. So $-\dfrac{2xy^2+y}{2x^2y+x}=-1\Leftrightarrow 2xy^2+y=2x^2y+x\Leftrightarrow y(2xy+1)=x(2xy+1)\Leftrightarrow$ $y(2xy+1)-x(2xy+1)=0\Leftrightarrow (2xy+1)(y-x)=0\Leftrightarrow xy=-\dfrac{1}{2}$ or $y=x$. But $xy=-\dfrac{1}{2}\Rightarrow x^2y^2+xy=\dfrac{1}{4}-\dfrac{1}{2}\neq 2$, so we must have $x=y$. Then $x^2y^2+xy=2\Rightarrow x^4+x^2=2\Leftrightarrow x^4+x^2-2=0\Leftrightarrow (x^2+2)(x^2-1)=0$. So $x^2=-2$, which is impossible, or $x^2=1\Leftrightarrow x=\pm 1$. Since $x=y$, the points on the curve where the tangent line has a slope of -1 are $(-1,-1)$ and $(1,1)$.

66. $x^2+4y^2=36\Rightarrow 2x+8yy'=0\Rightarrow y'=-\dfrac{x}{4y}$. Let (a,b) be a point on $x^2+4y^2=36$ whose tangent line passes through $(12,3)$. The tangent line is then $y-3=-\dfrac{a}{4b}(x-12)$, so $b-3=-\dfrac{a}{4b}(a-12)$. Multiplying both sides by $4b$ gives $4b^2-12b=-a^2+12a$, so $4b^2+a^2=12(a+b)$. But $4b^2+a^2=36$, so $36=12(a+b)\Rightarrow$ $a+b=3\Rightarrow b=3-a$. Substituting $3-a$ for b into $a^2+4b^2=36$ gives $a^2+4(3-a)^2=36\Leftrightarrow a^2+36-24a+4a^2=36\Leftrightarrow$ $5a^2-24a=0\Leftrightarrow a(5a-24)=0$, so $a=0$ or $a=\dfrac{24}{5}$. If $a=0$, $b=3-0=3$, and if $a=\dfrac{24}{5}$, $b=3-\dfrac{24}{5}=-\dfrac{9}{5}$. So the two points on the ellipse are $(0,3)$ and $\left(\dfrac{24}{5},-\dfrac{9}{5}\right)$.

Using $y-3=-\dfrac{a}{4b}(x-12)$ with (a,b)=(0,3) gives us the tangent line $y-3=0$ or $y=3$. With (a,b)$=\left(\dfrac{24}{5},-\dfrac{9}{5}\right)$, we have $y-3=-\dfrac{24/5}{4(-9/5)}(x-12)\Leftrightarrow y-3=\dfrac{2}{3}(x-12)\Leftrightarrow y=\dfrac{2}{3}x-5$. A graph of the ellipse and the tangent lines confirms our results.



67. **(a)** If $y=f^{-1}(x)$, then $f(y)=x$. Differentiating implicitly with respect to x and remembering that y is a function of x, we get $f'(y)\dfrac{dy}{dx}=1$, so $\dfrac{dy}{dx}=\dfrac{1}{f'(y)}\Rightarrow (f^{-1})'(x)=\dfrac{1}{f'(f^{-1}(x))}$.

(b) $f(4)=5\Rightarrow f^{-1}(5)=4$. By part (a), $(f^{-1})'(5)=1/f'(f^{-1}(5))=1/f'(4)=1/\left(\dfrac{2}{3}\right)=\dfrac{3}{2}$.

68. **(a)** $f(x)=2x+\cos x\Rightarrow f'(x)=2-\sin x>0$ for all x . Thus, f is increasing for all x and is therefore one–to–one.

(b) Since f is one–to–one, $f^{-1}(1)=k \Leftrightarrow f(k)=1$. By inspection, we see that $f(0)=2(0)+\cos 0=1$, so $k=f^{-1}(1)=0$.

(c) $(f^{-1})'(1)=1/f'(f^{-1}(1))=1/f'(0)=1/(2-\sin 0)=\frac{1}{2}$

69. $x^2+4y^2=5 \Rightarrow 2x+4(2yy')=0 \Rightarrow y'=-\frac{x}{4y}$. Now let h be the height of the lamp, and let (a,b) be the point of tangency of the line passing through the points (3,h) and (–5,0) . This line has slope $\frac{h-0}{3-(-5)}=\frac{1}{8}h$. But the slope of the tangent line through the point (a,b) can be expressed as $y'=-\frac{a}{4b}$, or as $\frac{b-0}{a-(-5)}=\frac{b}{a+5}$, so $-\frac{a}{4b}=\frac{b}{a+5} \Leftrightarrow 4b^2=-a^2-5a \Leftrightarrow a^2+4b^2=-5a$. But $a^2+4b^2=5$, so $5=-5a \Leftrightarrow a=-1$. Then $4b^2=-a^2-5a=-1-5(-1)=4 \Rightarrow b=1$, since the point is on the top half of the ellipse. So $\frac{h}{8}=\frac{b}{a+5}=\frac{1}{-1+5}=\frac{1}{4} \Rightarrow h=2$. So the lamp is located 2 units above the x –axis.

1. $a=f$, $b=f'$, $c=f''$. We can see this because where a has a horizontal tangent, $b=0$, and where b has a horizontal tangent, $c=0$. We can immediately see that c can be neither f nor f' , since at the points where c has a horizontal tangent, neither a nor b is equal to 0 .

2. Where d has horizontal tangents, only c is 0 , so $d'=c$. c has negative tangents for $x<0$ and b is the only graph that is negative for $x<0$, so $c'=b$. b has positive tangents on R (except at $x=0$), and the only graph that is positive on the same domain is a , so $b'=a$. We conclude that $d=f$, $c=f'$, $b=f''$ and $a=f'''$.

3. We can immediately see that a is the graph of the acceleration function, since at the points where a has a horizontal tangent, neither c nor b is equal to 0 . Next, we note that $a=0$ at the point where b has a horizontal tangent, so b must be the graph of the velocity function, and hence, $b'=a$. We conclude that c is the graph of the position function.

4. a must be the jerk since none of the graphs are 0 at its high and low points. a is 0 where b has a maximum, so $b'=a$. b is 0 where c has a maximum, so $c'=b$. We conclude that d is the position function, c is the velocity, b is the acceleration, and a is the jerk.

5. $f(x)=x^5+6x^2-7x \Rightarrow f'(x)=5x^4+12x-7 \Rightarrow f''(x)=20x^3+12$

6. $f(t)=t^8-7t^6+2t^4 \Rightarrow f'(t)=8t^7-42t^5+8t^3 \Rightarrow f''(t)=56t^6-210t^4+24t^2$

7. $y=\cos 2\theta \Rightarrow y'=-2\sin 2\theta \Rightarrow y''=-4\cos 2\theta$

8. $y=\theta\sin\theta \Rightarrow y'=\theta\cos\theta+\sin\theta \Rightarrow y''=\theta(-\sin\theta)+\cos\theta\cdot 1+\cos\theta=2\cos\theta-\theta\sin\theta$

9. $F(t)=(1-7t)^6 \Rightarrow F'(t)=6(1-7t)^5(-7)=-42(1-7t)^5 \Rightarrow F''(t)=-42\cdot 5(1-7t)^4(-7)=1470(1-7t)^4$

10. $g(x)=\dfrac{2x+1}{x-1} \Rightarrow g'(x)=\dfrac{(x-1)(2)-(2x+1)(1)}{(x-1)^2}=\dfrac{2x-2-2x-1}{(x-1)^2}=\dfrac{-3}{(x-1)^2}$ or $-3(x-1)^{-2}$

$\Rightarrow g''(x)=-3(-2)(x-1)^{-3}=6(x-1)^{-3}$ or $\dfrac{6}{(x-1)^3}$

11. $h(u)=\dfrac{1-4u}{1+3u} \Rightarrow h'(u)=\dfrac{(1+3u)(-4)-(1-4u)(3)}{(1+3u)^2}=\dfrac{-4-12u-3+12u}{(1+3u)^2}=\dfrac{-7}{(1+3u)^2}$ or $-7(1+3u)^{-2} \Rightarrow$

$h''(u)=-7(-2)(1+3u)^{-3}(3)=42(1+3u)^{-3}$ or

$$\frac{42}{(1+3u)^3}$$

12.

$$H(s) = a\sqrt{s}+\frac{b}{\sqrt{s}} = as^{1/2}+bs^{-1/2} \Rightarrow$$

$$H'(s) = a\cdot\frac{1}{2}s^{-1/2}+b\left(-\frac{1}{2}s^{-3/2}\right)=\frac{1}{2}as^{-1/2}-\frac{1}{2}bs^{-3/2}\Rightarrow$$

$$H''(s)=\frac{1}{2}a\left(-\frac{1}{2}s^{-3/2}\right)-\frac{1}{2}b\left(-\frac{3}{2}s^{-5/2}\right)=-\frac{1}{4}as^{-3/2}+\frac{3}{4}bs^{-5/2}$$

13. $h(x)=\sqrt{x^2+1} \Rightarrow h'(x)=\frac{1}{2}(x^2+1)^{-1/2}(2x)=\frac{x}{\sqrt{x^2+1}} \Rightarrow$

$$h''(x)=\frac{\sqrt{x^2+1}\cdot 1-x\left[\frac{1}{2}(x^2+1)^{-1/2}(2x)\right]}{\left(\sqrt{x^2+1}\right)^2}=\frac{(x^2+1)^{-1/2}[(x^2+1)-x^2]}{(x^2+1)^1}=\frac{1}{(x^2+1)^{3/2}}$$

14. $y=xe^{cx} \Rightarrow y'=x\cdot e^{cx}\cdot c+e^{cx}\cdot 1=e^{cx}(cx+1)\Rightarrow\ y''=e^{cx}(c)+(cx+1)e^{cx}\cdot c=ce^{cx}(1+cx+1)=ce^{cx}(cx+2)$

15. $y=(x^3+1)^{2/3}\Rightarrow y'=\frac{2}{3}(x^3+1)^{-1/3}(3x^2)=2x^2(x^3+1)^{-1/3}\Rightarrow$

$$y''=2x^2\left(-\frac{1}{3}\right)(x^3+1)^{-4/3}(3x^2)+(x^3+1)^{-1/3}(4x)=4x(x^3+1)^{-1/3}-2x^4(x^3+1)^{-4/3}$$

16.

$$y = \frac{4x}{\sqrt{x+1}} \Rightarrow$$

$$y' = \frac{\sqrt{x+1}\cdot 4-4x\cdot\frac{1}{2}(x+1)^{-1/2}}{(\sqrt{x+1})^2}=\frac{4\sqrt{x+1}-2x/\sqrt{x+1}}{x+1}=\frac{4(x+1)-2x}{(x+1)^{3/2}}=\frac{2x+4}{(x+1)^{3/2}}\Rightarrow$$

$$y'' = \frac{(x+1)^{3/2}\cdot 2-(2x+4)\cdot\frac{3}{2}(x+1)^{1/2}}{[(x+1)^{3/2}]^2}=\frac{(x+1)^{1/2}[2(x+1)-3(x+2)]}{(x+1)^3}$$

$$=\frac{2x+2-3x-6}{(x+1)^{5/2}}=\frac{-x-4}{(x+1)^{5/2}}$$

17. $H(t)=\tan 3t\Rightarrow$

$H^{/}(t)=3\sec^{2}3t\Rightarrow H^{//}(t)=2\cdot 3\sec 3t\,\dfrac{d}{dt}\,(\sec 3t)=6\sec 3t\,(3\sec 3t\cdot\tan 3t)=18\sec^{2}3t\cdot\tan 3t$

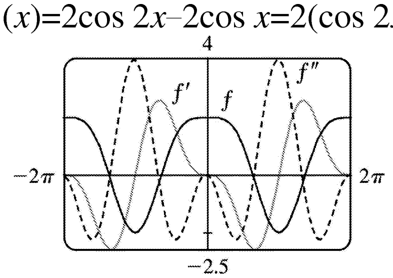
18. $g(s)=s^{2}\cos s\Rightarrow g^{/}(s)=2s\cdot\cos s-s^{2}\sin s\Rightarrow$
$g^{//}(s)=2\cos s-2s\cdot\sin s-2s\cdot\sin s-s^{2}\cos s=(2-s^{2})\cos s-4s\cdot\sin s$

19. $g(t)=t^{3}e^{5t}\Rightarrow g^{/}(t)=t^{3}e^{5t}\cdot 5+e^{5t}\cdot 3t^{2}=t^{2}e^{5t}(5t+3)\Rightarrow$
$g^{//}(t)=(2t)e^{5t}(5t+3)+t^{2}(5e^{5t})(5t+3)+t^{2}e^{5t}(5)$
$\quad=te^{5t}\left[2(5t+3)+5t(5t+3)+5t\right]=te^{5t}(25t^{2}+30t+6)$

20. $h(x)=\tan^{-1}(x^{2})\Rightarrow h^{/}(x)=\dfrac{1}{1+(x^{2})^{2}}\cdot 2x=\dfrac{2x}{1+x^{4}}\Rightarrow h^{//}(x)=\dfrac{(1+x^{4})(2)-(2x)(4x^{3})}{(1+x^{4})^{2}}=\dfrac{2-6x^{4}}{(1+x^{4})^{2}}$

21. **(a)** $f(x)=2\cos x+\sin^{2}x\Rightarrow f^{/}(x)=2(-\sin x)+2\sin x(\cos x)=\sin 2x-2\sin x\Rightarrow$
$f^{//}(x)=2\cos 2x-2\cos x=2(\cos 2x-\cos x)$

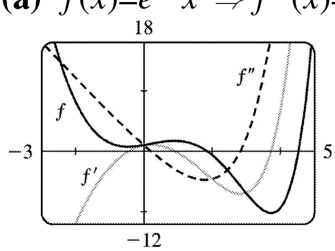
(b)



We can see that our answers are plausible, since f has horizontal tangents where $f^{/}(x)=0$, and $f^{/}$ has horizontal tangents where $f^{//}(x)=0$.

22. **(a)** $f(x)=e^{x}-x^{3}\Rightarrow f^{/}(x)=e^{x}-3x^{2}\Rightarrow f^{//}(x)=e^{x}-6x$

(b)



The graphs seem reasonable since f has horizontal tangents where $f^{/}$ is zero, $f^{/}$ is positive where f is increasing, and $f^{/}$ is negative where f is decreasing; and the same relationships exist between $f^{/}$ and $f^{//}$.

23. $y=\sqrt{2x+3}=(2x+3)^{1/2}\Rightarrow y'=\frac{1}{2}(2x+3)^{-1/2}\cdot 2=(2x+3)^{-1/2}\Rightarrow y''=-\frac{1}{2}(2x+3)^{-3/2}\cdot 2=-(2x+3)^{-3/2}\Rightarrow$
$y'''=\frac{3}{2}(2x+3)^{-5/2}\cdot 2=3(2x+3)^{-5/2}$

24. $y=\frac{x}{2x-1}\Rightarrow y'=\frac{(2x-1)(1)-x(2)}{(2x-1)^2}=\frac{-1}{(2x-1)^2}$ or $-1(2x-1)^{-2}\Rightarrow$
$y''=-1(-2)(2x-1)^{-3}(2)=4(2x-1)^{-3}\Rightarrow$
$y'''=4(-3)(2x-1)^{-4}(2)=-24(2x-1)^{-4}$ or $-24/(2x-1)^4$

25. $f(t)=t\cos t\Rightarrow f'(t)=t(-\sin t)+\cos t\cdot 1\Rightarrow f''(t)=t(-\cos t)-\sin t\cdot 1-\sin t\Rightarrow$
$f'''(t)=t\sin t-\cos t\cdot 1-\cos t-\cos t=t\sin t-3\cos t$, so $f'''(0)=0-3=-3$.

26. $g(x)=\sqrt{5-2x}\Rightarrow g'(x)=\frac{1}{2}(5-2x)^{-1/2}(-2)=-(5-2x)^{-1/2}\Rightarrow g''(x)=\frac{1}{2}(5-2x)^{-3/2}(-2)=-(5-2x)^{-3/2}\Rightarrow$
$g'''(x)=\frac{3}{2}(5-2x)^{-5/2}(-2)=-3(5-2x)^{-5/2}$, so $g'''(2)=-3(1)^{-5/2}=-3$.

27. $f(\theta)=\cot\theta\Rightarrow f'(\theta)=-\csc^2\theta\Rightarrow f''(\theta)=-2\csc\theta\,(-\csc\theta\cdot\cot\theta)=2\csc^2\theta\cdot\cot\theta\Rightarrow$
$f'''(\theta)=2(-2\csc^2\theta\cdot\cot\theta)\cot\theta+2\csc^2\theta\,(-\csc^2\theta)=-2\csc^2\theta\,(2\cot^2\theta+\csc^2\theta)\Rightarrow$
$f'''\left(\frac{\pi}{6}\right)=-2(2)^2[2(\sqrt{3})^2+(2)^2]=-80$

28. $g(x)=\sec x\Rightarrow g'(x)=\sec x\cdot\tan x\Rightarrow$
$g''(x)=\sec x\cdot\sec^2 x+\tan x(\sec x\cdot\tan x)=\sec^3 x+\sec x\cdot\tan^2 x=\sec^3 x+\sec x(\sec^2 x-1)=2\sec^3 x-\sec x\Rightarrow$
$g'''(x)=6\sec^2 x(\sec x\cdot\tan x)-\sec x\cdot\tan x=\sec x\cdot\tan x(6\sec^2 x-1)\Rightarrow\ g'''\left(\frac{\pi}{4}\right)=\sqrt{2}\,(1)(6\cdot 2-1)=11\sqrt{2}$

29. $9x^2+y^2=9\Rightarrow 18x+2yy'=0\Rightarrow 2yy'=-18x\Rightarrow y'=-9x/y\Rightarrow$
$y''=-9\left(\frac{y\cdot 1-x\cdot y'}{y^2}\right)=-9\left(\frac{y-x(-9x/y)}{y^2}\right)=-9\cdot\frac{y^2+9x^2}{y^3}=-9\cdot\frac{9}{y^3}$ [since x and y must satisfy the original equation, $9x^2+y^2=9$] . Thus, $y''=-81/y^3$.

30.

$\sqrt{x}+\sqrt{y}=1 \Rightarrow \dfrac{1}{2\sqrt{x}}+\dfrac{y'}{2\sqrt{y}}=0 \Rightarrow y'=-\dfrac{\sqrt{y}}{\sqrt{x}} \Rightarrow$

$$y''=-\frac{\sqrt{x}\,\frac{1}{(2\sqrt{y})}\,y'-\sqrt{y}\,\frac{1}{(2\sqrt{x})}}{x}=-\frac{\sqrt{x}\left(\frac{1}{\sqrt{y}}\right)\left(-\frac{\sqrt{y}}{\sqrt{x}}\right)-\sqrt{y}\left(\frac{1}{\sqrt{x}}\right)}{2x}=\frac{1+\frac{\sqrt{y}}{\sqrt{x}}}{2x}$$

$=\dfrac{\sqrt{x}+\sqrt{y}}{2x\sqrt{x}}=\dfrac{1}{2x\sqrt{x}}$ since x and y must satisfy the original equation, $\sqrt{x}+\sqrt{y}=1$.

31. $x^3+y^3=1 \Rightarrow 3x^2+3y^2y'=0 \Rightarrow y'=-\dfrac{x^2}{y^2} \Rightarrow$

$$y''=-\frac{y^2(2x)-x^2\cdot 2yy'}{(y^2)^2}=-\frac{2xy^2-2x^2y\left(-\frac{x^2}{y^2}\right)}{y^4}=-\frac{2xy^4+2x^4y}{y^6}=-\frac{2xy(y^3+x^3)}{y^6}=-\frac{2x}{y^5}\ ,$$

since x and y must satisfy the original equation, $x^3+y^3=1$.

32. $x^4+y^4=a^4 \Rightarrow 4x^3+4y^3y'=0 \Rightarrow 4y^3y'=-4x^3 \Rightarrow y'=-x^3/y^3 \Rightarrow$

$$y''=-\left(\frac{y^3\cdot 3x^2-x^3\cdot 3y^2y'}{(y^3)^2}\right)=-3x^2y^2\cdot\frac{y-x\left(-\frac{x^3}{y^3}\right)}{y^6}=-3x^2\cdot\frac{y^4+x^4}{y^4y^3}=-3x^2\cdot\frac{a^4}{y^7}=\frac{-3a^4x^2}{y^7}$$

33. $f(x)=x^n \Rightarrow f'(x)=nx^{n-1} \Rightarrow f''(x)=n(n-1)x^{n-2} \Rightarrow \cdots \Rightarrow f^{(n)}(x)=n(n-1)(n-2)\cdots 2\cdot 1x^{n-n}=\mathrm{n}!$

34. $f(x)=\dfrac{1}{5x-1}=(5x-1)^{-1} \Rightarrow f'(x)=-1(5x-1)^{-2}\cdot 5 \Rightarrow f''(x)=(-1)(-2)(5x-1)^{-3}\cdot 5^2 \Rightarrow$

$f'''(x)=(-1)(-2)(-3)(5x-1)^{-4}\cdot 5^3 \Rightarrow \cdots \Rightarrow f^{(n)}(x)=(-1)^n\mathrm{n}!\,5^n(5x-1)^{-(n+1)}$

35. $f(x)=e^{2x} \Rightarrow f'(x)=2e^{2x} \Rightarrow f''(x)=2\cdot 2e^{2x}=2^2e^{2x} \Rightarrow$

$f'''(x)=2^2\cdot 2e^{2x}=2^3e^{2x} \Rightarrow \cdots \Rightarrow f^{(n)}(x)=2^ne^{2x}$

36. $f(x)=\sqrt{x}=x^{1/2} \Rightarrow f'(x)=\dfrac{1}{2}x^{-1/2} \Rightarrow$

$f''(x)=\dfrac{1}{2}\left(-\dfrac{1}{2}\right)x^{-3/2} \Rightarrow f'''(x)=\dfrac{1}{2}\left(-\dfrac{1}{2}\right)\left(-\dfrac{3}{2}\right)x^{-5/2} \Rightarrow$

$$f^{(4)}(x)=\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)x^{-7/2}=-\frac{1\cdot 3\cdot 5}{2^4}x^{-7/2}\Rightarrow$$

$$f^{(5)}(x)=\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(-\frac{7}{2}\right)x^{-9/2}=\frac{1\cdot 3\cdot 5\cdot 7}{2^5}x^{-9/2}\Rightarrow\cdots\Rightarrow$$

$$f^{(n)}(x)=\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\cdots\left(\frac{1}{2}-n+1\right)x^{-(2n-1)/2}=(-1)^{n-1}\frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-3)}{2^n}x^{-(2n-1)/2}$$

37. $f(x)=1/(3x^3)=\frac{1}{3}x^{-3}\Rightarrow f'(x)=\frac{1}{3}(-3)x^{-4}\Rightarrow f''(x)=\frac{1}{3}(-3)(-4)x^{-5}\Rightarrow$

$f'''(x)=\frac{1}{3}(-3)(-4)(-5)x^{-6}\Rightarrow\cdots\Rightarrow$

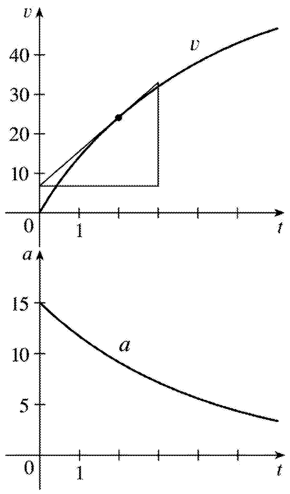
$$f^{(n)}(x)=\frac{1}{3}(-3)(-4)\cdots[-(n+2)]x^{-(n+3)}=\frac{(-1)^n\cdot 3\cdot 4\cdot 5\cdot\cdots\cdot(n+2)}{3x^{n+3}}\cdot\frac{2}{2}=\frac{(-1)^n(n+2)!}{6x^{n+3}}$$

38. $D\sin x=\cos x\Rightarrow D^2\sin x=-\sin x\Rightarrow D^3\sin x=-\cos x\Rightarrow D^4\sin x=\sin x$. The derivatives of $\sin x$ occur in a cycle of four. Since $74=4(18)+2$, we have $D^{74}\sin x=D^2\sin x=-\sin x$.

39. Let $f(x)=\cos x$. Then $Df(2x)=2f'(2x)$, $D^2f(2x)=2^2f''(2x)$, $D^3f(2x)=2^3f'''(2x)$, … , $D^{(n)}f(2x)=2^nf^{(n)}(2x)$. Since the derivatives of $\cos x$ occur in a cycle of four, and since $103=4(25)+3$, we have $f^{(103)}(x)=f^{(3)}(x)=\sin x$ and $D^{103}\cos 2x=2^{103}f^{(103)}(2x)=2^{103}\sin 2x$.

40. $f(x)=xe^{-x}\Rightarrow f'(x)=x(-e^{-x})+e^{-x}=(1-x)e^{-x}\Rightarrow f''(x)=(1-x)(-e^{-x})+e^{-x}(-1)=(x-2)e^{-x}\Rightarrow$
$f'''(x)=(x-2)(-e^{-x})+e^{-x}=(3-x)e^{-x}\Rightarrow f^{(4)}(x)=(3-x)(-e^{-x})+e^{-x}(-1)=(x-4)e^{-x}\Rightarrow\cdots\Rightarrow$
$f^{(n)}(x)=(-1)^n(x-n)e^{-x}$.
So $D^{1000}xe^{-x}=(x-1000)e^{-x}$.

41. By measuring the slope of the graph of $s=f(t)$ at $t=0$, 1 , 2 , 3 , 4 , and 5 , and using the method of Example 1 in Section 3.2, we plot the graph of the velocity function $v=f'(t)$ in the first figure. The acceleration when $t=2$ s is $a=f''(2)$, the slope of the tangent line to the graph of f' when $t=2$. We estimate the slope of this tangent line to be $a(2)=f''(2)=v'(2)\approx\frac{27}{3}=9\text{ ft / s}^2$. Similar measurements enable us to graph the acceleration function in the second figure.



42. **(a)** Since we estimate the velocity to be a maximum at $t=10$, the acceleration is 0 at $t=10$.

(b) Drawing a tangent line at $t=10$ on the graph of a , a appears to decrease by 10 ft/s^2 over a period of 20 s. So at $t=10$ s, the jerk is approximately $-10/20=-0.5\left(\text{ft/s}^2\right)/\text{s}$ or ft/s^3 .

43. **(a)** $s=2t^3-15t^2+36t+2\Rightarrow v(t)=s'(t)=6t^2-30t+36\Rightarrow a(t)=v'(t)=12t-30$

(b) $a(1)=12\cdot 1-30=-18\ \text{m/s}^2$

(c) $v(t)=6(t^2-5t+6)=6(t-2)(t-3)=0$ when $t=2$ or 3 and $a(2)=24-30=-6\ \text{m/s}^2$, $a(3)=36-30=6\ \text{m/s}^2$.

44. **(a)** $s=2t^3-3t^2-12t\Rightarrow v(t)=s'(t)=6t^2-6t-12\Rightarrow a(t)=v'(t)=12t-6$

(b) $a(1)=12\cdot 1-6=6\ \text{m/s}^2$

(c) $v(t)=6(t^2-t-2)=6(t+1)(t-2)=0$ when $t=-1$ or 2 . Since $t\geq 0$, $t\neq -1$ and $a(2)=24-6=18\ \text{m/s}^2$.

45. **(a)** $s=\sin\left(\frac{\pi}{6}t\right)+\cos\left(\frac{\pi}{6}t\right)$, $0\leq t\leq 2$.

$v(t)=s'(t)=\cos\left(\frac{\pi}{6}t\right)\cdot\frac{\pi}{6}-\sin\left(\frac{\pi}{6}t\right)\cdot\frac{\pi}{6}=\frac{\pi}{6}\left[\cos\left(\frac{\pi}{6}t\right)-\sin\left(\frac{\pi}{6}t\right)\right]\Rightarrow$

$a(t)=v'(t)=\frac{\pi}{6}\left[-\sin\left(\frac{\pi}{6}t\right)\cdot\frac{\pi}{6}-\cos\left(\frac{\pi}{6}t\right)\cdot\frac{\pi}{6}\right]=-\frac{\pi^2}{36}\left[\sin\left(\frac{\pi}{6}t\right)+\cos\left(\frac{\pi}{6}t\right)\right]$

(b) $a(1)=-\frac{\pi^2}{36}\left[\sin\left(\frac{\pi}{6}\cdot 1\right)+\cos\left(\frac{\pi}{6}\cdot 1\right)\right]=-\frac{\pi^2}{36}\left[\frac{1}{2}+\frac{\sqrt{3}}{2}\right]=-\frac{\pi^2}{72}(1+\sqrt{3})\approx -0.3745\ \text{m/s}^2$

(c) $v(t)=0$ for $0\leq t\leq 2\Rightarrow\cos\left(\frac{\pi}{6}t\right)=\sin\left(\frac{\pi}{6}t\right)\Rightarrow$

$1=\dfrac{\sin\left(\frac{\pi}{6}t\right)}{\cos\left(\frac{\pi}{6}t\right)} \Rightarrow$

$\tan\left(\frac{\pi}{6}t\right)=1 \Rightarrow \frac{\pi}{6}t=\tan^{-1}1 \Rightarrow t=\frac{6}{\pi}\cdot\frac{\pi}{4}=\frac{3}{2}=1.5$ s. Thus,

$a(\frac{3}{2})=-\frac{\pi^2}{36}\left[\sin\left(\frac{\pi}{6}\cdot\frac{3}{2}\right)+\cos\left(\frac{\pi}{6}\cdot\frac{3}{2}\right)\right]=-\frac{\pi^2}{36}\left[\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}\right]=-\frac{\pi^2}{36}\sqrt{2}\approx-0.3877\ \mathrm{m/s}^2$.

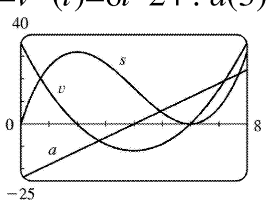
46. **(a)** $s=2t^3-7t^2+4t+1 \Rightarrow v(t)=s'(t)=6t^2-14t+4 \Rightarrow a(t)=v'(t)=12t-14$
(b) $a(1)=12-14=-2\ \mathrm{m/s}^2$
(c) $v(t)=2\left(3t^2-7t+2\right)=2(3t-1)(t-2)=0$ when $t=\frac{1}{3}$ or 2 and $a\left(\frac{1}{3}\right)=12\left(\frac{1}{3}\right)-14=-10\ \mathrm{m/s}^2$,

$a(2)=12(2)-14=10\ \mathrm{m/s}^2$.

47. **(a)** $s(t)=t^4-4t^3+2 \Rightarrow v(t)=s'(t)=4t^3-12t^2 \Rightarrow a(t)=v'(t)=12t^2-24t=12t(t-2)=0$ when $t=0$ or 2 .
(b) $s(0)=2$ m, $v(0)=0$ m / s, $s(2)=-14$ m, $v(2)=-16$ m / s

48. **(a)** $s(t)=2t^3-9t^2 \Rightarrow v(t)=s'(t)=6t^2-18t \Rightarrow a(t)=v'(t)=12t-18=0$ when $t=1.5$.
(b) $s(1.5)=-13.5$ m, $v(1.5)=-13.5$ m / s

49. **(a)** $s=f(t)=t^3-12t^2+36t$, $t\geq 0 \Rightarrow v(t)=f'(t)=3t^2-24t+36$.
$a(t)=v'(t)=6t-24$. $a(3)=6(3)-24=-6\,(\mathrm{m/s})/\mathrm{s}$ or $\mathrm{m/s}^2$.

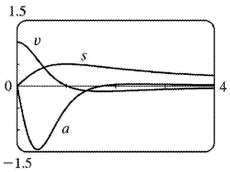
(b) 

(c) The particle is speeding up when v and a have the same sign. This occurs when $2<t<4$ and when $t>6$. It is slowing down when v and a have opposite signs; that is, when $0\leq t<2$ and when $4<t<6$.

50. **(a)** $x(t)=\dfrac{t}{1+t^2} \Rightarrow v(t)=x'(t)=\dfrac{(1+t^2)(1)-t(2t)}{(1+t^2)^2}=\dfrac{1-t^2}{(1+t^2)^2}$.

$a(t)=v'(t)=\dfrac{2t(t^2-3)}{(1+t^2)^3}$. $a(t)=0 \Rightarrow 2t(t^2-3)=0 \Rightarrow t=0$ or $\sqrt{3}$

(b)



(c) v and a have the same sign and the particle is speeding up when $1<t<\sqrt{3}$. The particle is slowing down and v and a have opposite signs when $0<t<1$ and when $t>\sqrt{3}$.

51. **(a)** $y(t)=A\sin\omega t \Rightarrow v(t)=y'(t)=A\omega\cos\omega t \Rightarrow a(t)=v'(t)=-A\omega^2\sin\omega t$

(b) $a(t)=-A\omega^2\sin\omega t=-\omega^2(A\sin\omega t)=-\omega^2 y(t)$, so $a(t)$ is proportional to $y(t)$.

(c) speed $=|v(t)|=A\omega\,|\cos\omega t|$ is a maximum when $\cos\omega t=\pm 1$. But when $\cos\omega t=\pm 1$, we have $\sin\omega t=0$, and $a(t)=-A\omega^2\sin\omega t=-A\omega^2(0)=0$.

52. By the Chain Rule, $a(t)=\dfrac{dv}{dt}=\dfrac{dv}{ds}\,\dfrac{ds}{dt}=\dfrac{dv}{ds}\,v(t)=v(t)\,\dfrac{dv}{ds}$. The derivative dv/dt is the rate of change of the velocity with respect to time (in other words, the acceleration) whereas the derivative dv/ds is the rate of change of the velocity with respect to the displacement.

53. Let $P(x)=ax^2+bx+c$. Then $P'(x)=2ax+b$ and $P''(x)=2a$. $P''(2)=2\Rightarrow 2a=2\Rightarrow a=1$.
$P'(2)=3\Rightarrow 2(1)(2)+b=3\Rightarrow 4+b=3\Rightarrow b=-1$.
$P(2)=5\Rightarrow 1(2)^2+(-1)(2)+c=5\Rightarrow 2+c=5\Rightarrow c=3$. So $P(x)=x^2-x+3$.

54. Let $Q(x)=ax^3+bx^2+cx+d$. Then $Q'(x)=3ax^2+2bx+c$, $Q''(x)=6ax+2b$ and $Q'''(x)=6a$. Thus, $Q(1)=a+b+c+d=1$, $Q'(1)=3a+2b+c=3$, $Q''(1)=6a+2b=6$ and $Q'''(1)=6a=12$. Solving these four equations in four unknowns a , b , c and d we get $a=2$, $b=-3$, $c=3$ and $d=-1$, so $Q(x)=2x^3-3x^2+3x-1$.

55. $y=A\sin x+B\cos x\Rightarrow y'=A\cos x-B\sin x\Rightarrow y''=-A\sin x-B\cos x$. Substituting into $y''+y'-2y=\sin x$ gives us $(-3A-B)\sin x+(A-3B)\cos x=1\sin x$, so we must have $-3A-B=1$ and $A-3B=0$. Solving for A and B , we add the first equation to three times the second to get $B=-\dfrac{1}{10}$ and $A=-\dfrac{3}{10}$.

56. $y=Ax^2+Bx+C\Rightarrow y'=2Ax+B\Rightarrow y''=2A$. We substitute these expressions into the equation $y''+y'-2y=x^2$ to get

$$\begin{aligned}(2A)+(2Ax+B)-2(Ax^2+Bx+C) &= x^2\\ 2A+2Ax+B-2Ax^2-2Bx-2C &= x^2\end{aligned}$$

$(-2A)x^2+(2A-2B)x+(2A+B-2C) = (1)x^2+(0)x+(0)$

The coefficients of x^2 on each side must be equal, so $-2A=1 \Rightarrow A=-\frac{1}{2}$. Similarly, $2A-2B=0 \Rightarrow$ $A=B=-\frac{1}{2}$ and $2A+B-2C=0 \Rightarrow -1-\frac{1}{2}-2C=0 \Rightarrow C=-\frac{3}{4}$.

57. $y=e^{rx} \Rightarrow y'=re^{rx} \Rightarrow y''=r^2e^{rx}$, so $y''+5y'-6y=r^2e^{rx}+5re^{rx}-6e^{rx}=e^{rx}(r^2+5r-6)=e^{rx}(r+6)(r-1)=0 \Rightarrow$ $(r+6)(r-1)=0 \Rightarrow r=1$ or -6.

58. $y=e^{\lambda x} \Rightarrow y'=\lambda e^{\lambda x} \Rightarrow y''=\lambda^2 e^{\lambda x}$. Thus, $y+y'=y'' \Leftrightarrow e^{\lambda x}+\lambda e^{\lambda x}=\lambda^2 e^{\lambda x} \Leftrightarrow e^{\lambda x}(\lambda^2-\lambda-1)=0 \Leftrightarrow$ $\lambda=\frac{1\pm\sqrt{5}}{2}$, since $e^{\lambda x}\neq 0$.

59. $f(x)=xg(x^2) \Rightarrow f'(x)=x\cdot g'(x^2)\cdot 2x+g(x^2)\cdot 1=g(x^2)+2x^2g'(x^2) \Rightarrow$
$f''(x)=g'(x^2)\cdot 2x+2x^2\cdot g''(x^2)\cdot 2x+g'(x^2)\cdot 4x=6xg'(x^2)+4x^3g''(x^2)$

60. $f(x)=\frac{g(x)}{x} \Rightarrow f'(x)=\frac{xg'(x)-g(x)}{x^2} \Rightarrow$

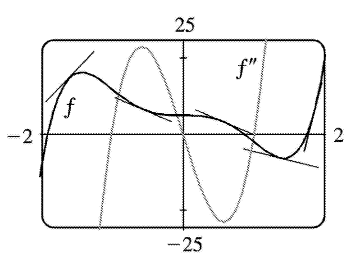
$$f''(x)=\frac{x^2[g'(x)+xg''(x)-g'(x)]-2x[xg'(x)-g(x)]}{x^4}=\frac{x^2g''(x)-2xg'(x)+2g(x)}{x^3}$$

61.

$$f(x) = g(\sqrt{x}) \Rightarrow f'(x)=g'(\sqrt{x})\cdot \frac{1}{2}x^{-1/2}=\frac{g'(\sqrt{x})}{2\sqrt{x}} \Rightarrow$$

$$f''(x)=\frac{2\sqrt{x}\cdot g''(\sqrt{x})\cdot \frac{1}{2}x^{-1/2}-g'(\sqrt{x})\cdot 2\cdot \frac{1}{2}x^{-1/2}}{(2\sqrt{x})^2}=\frac{x^{-1/2}[\sqrt{x}\,g''(\sqrt{x})-g'(\sqrt{x})]}{4x}$$

$$=\frac{\sqrt{x}\,g''(\sqrt{x})-g'(\sqrt{x})}{4x\sqrt{x}}$$

62. 

$f(x)=3x^5-10x^3+5 \Rightarrow f'(x)=15x^4-30x^2 \Rightarrow f''(x)=60x^3-60x=60x(x^2-1)=60x(x+1)(x-1)$

So $f''(x)>0$ when $-1<x<0$ or $x>1$, and on these intervals the graph of f lies above its tangent lines; and $f''(x)<0$ when $x<-1$ or $0<x<1$, and on these intervals the graph of f lies below its tangent lines.

63. **(a)**

$$f(x) = \frac{1}{x^2+x} \Rightarrow f'(x)=\frac{-(2x+1)}{(x^2+x)^2} \Rightarrow$$

$$f''(x) = \frac{(x^2+x)^2(-2)+(2x+1)(2)(x^2+x)(2x+1)}{(x^2+x)^4} = \frac{2(3x^2+3x+1)}{(x^2+x)^3} \Rightarrow$$

$$f'''(x)=\frac{(x^2+x)^3(2)(6x+3)-2(3x^2+3x+1)(3)(x^2+x)^2(2x+1)}{(x^2+x)^6}$$

$$= \frac{-6(4x^3+6x^2+4x+1)}{(x^2+x)^4} \Rightarrow$$

$$f^{(4)}(x) = \frac{(x^2+x)^4(-6)(12x^2+12x+4)+6(4x^3+6x^2+4x+1)(4)(x^2+x)^3(2x+1)}{(x^2+x)^8}$$

$$= \frac{24(5x^4+10x^3+10x^2+5x+1)}{(x^2+x)^5}$$

$f^{(5)}(x) =?$

(b) $f(x)=\dfrac{1}{x(x+1)} = \dfrac{1}{x} - \dfrac{1}{x+1} \Rightarrow f'(x)=-x^{-2}+(x+1)^{-2} \Rightarrow f''(x)=2x^{-3}-2(x+1)^{-3} \Rightarrow$

$f'''(x)=(-3)(2)x^{-4}+(3)(2)(x+1)^{-4} \Rightarrow \cdots \Rightarrow f^{(n)}(x)=(-1)^n n!\left[x^{-(n+1)}-(x+1)^{-(n+1)}\right]$

64. **(a)** For $f(x)=\dfrac{7x+17}{2x^2-7x-4}$, a CAS gives us $f'''(x)=\dfrac{-6(56x^4+544x^3-2184x^2+6184x-6139)}{(2x^2-7x-4)^4}$

(b) Using a CAS we get $f(x)=\dfrac{7x+17}{2x^2-7x-4} = \dfrac{-3}{2x+1} + \dfrac{5}{x-4}$. Now we differentiate three times to obtain

$f'''(x)=\dfrac{144}{(2x+1)^4} - \dfrac{30}{(x-4)^4}$.

65.

For $f(x)=x^2e^x$, $f'(x)=x^2e^x+e^x(2x) =e^x(x^2+2x)$. Similarly, we have

$$f''(x)=e^x(x^2+4x+2)$$
$$f'''(x)=e^x(x^2+6x+6)$$
$$f^{(4)}(x)=e^x(x^2+8x+12)$$
$$f^{(5)}(x)=e^x(x^2+10x+20)$$

It appears that the coefficient of x in the quadratic term increases by 2 with each differentiation. The pattern for the constant terms seems to be $0=1\cdot 0$, $2=2\cdot 1$, $6=3\cdot 2$, $12=4\cdot 3$, $20=5\cdot 4$. So a reasonable guess is that $f^{(n)}(x)=e^x\left[x^2+2nx+n(n-1)\right]$.

Proof: Let S_n be the statement that $f^{(n)}(x)=e^x\left[x^2+2nx+n(n-1)\right]$. 1. S_1 is true because $f'(x)=e^x(x^2+2x)$.

2. Assume that S_k is true; that is, $f^{(k)}(x)=e^x\left[x^2+2kx+k(k-1)\right]$. Then

$$\begin{aligned} f^{(k+1)}(x)&=\frac{d}{dx}\left[f^{(k)}(x)\right]=e^x(2x+2k)+\left[x^2+2kx+k(k-1)\right]e^x \\ &=e^x\left[x^2+(2k+2)x+(k^2+k)\right]=e^x\left[x^2+2(k+1)x+(k+1)k\right] \end{aligned}$$

This shows that S_{k+1} is true.

3. Therefore, by mathematical induction, S_n is true for all n; that is, $f^{(n)}(x)=e^x\left[x^2+2nx+n(n-1)\right]$ for every positive integer n.

66. **(a)** Use the Product Rule repeatedly: $F=fg\Rightarrow F'=f'g+fg'\Rightarrow$
$F''=(f''g+f'g')+(f'g'+fg'')=f''g+2f'g'+fg''$.

(b) $F'''=f'''g+f''g'+2(f''g'+f'g'')+f'g''+fg'''=f'''g+3f''g'+3f'g''+fg'''\Rightarrow$

$$\begin{aligned} F^{(4)}&=f^{(4)}g+f'''g'+3(f'''g'+f''g'')+3(f''g''+f'g''')+f'g'''+fg^{(4)} \\ &=f^{(4)}g+4f'''g'+6f''g''+4f'g'''+fg^{(4)} \end{aligned}$$

(c) By analogy with the Binomial Theorem, we make the guess:

$$F^{(n)}=f^{(n)}g+nf^{(n-1)}g'+\binom{n}{2}f^{(n-2)}g''+\cdot\cdot\cdot+\binom{n}{k}f^{(n-k)}g^{(k)}+\cdot\cdot\cdot+nf'g^{(n-1)}+fg^{(n)}$$

where $\binom{n}{k}=\dfrac{n!}{k!(n-k)!}=\dfrac{n(n-1)(n-2)\cdot\cdot\cdot(n-k+1)}{k!}$.

67. The Chain Rule says that

$\frac{dy}{dx}=\frac{dy}{du}\frac{du}{dx}$, so

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dx}\left(\frac{dy}{du}\frac{du}{dx}\right)=\left[\frac{d}{dx}\left(\frac{dy}{du}\right)\right]\frac{du}{dx}+\frac{dy}{du}\frac{d}{dx}\left(\frac{du}{dx}\right) \text{ [Product Rule]}$$

$$=\left[\frac{d}{du}\left(\frac{dy}{du}\right)\frac{du}{dx}\right]\frac{du}{dx}+\frac{dy}{du}\frac{d^2u}{dx^2}=\frac{d^2y}{du^2}\left(\frac{du}{dx}\right)^2+\frac{dy}{du}\frac{d^2u}{dx^2}$$

68. From Exercise 65, $\frac{d^2y}{dx^2}=\frac{d^2y}{du^2}\left(\frac{du}{dx}\right)^2+\frac{dy}{du}\frac{d^2u}{dx^2}\Rightarrow$

$$\frac{d^3y}{dx^3}=\frac{d}{dx}\frac{d^2y}{dx^2}=\frac{d}{dx}\left[\frac{d^2y}{du^2}\left(\frac{du}{dx}\right)^2\right]+\frac{d}{dx}\left[\frac{dy}{du}\frac{d^2u}{dx^2}\right]$$

$$=\left[\frac{d}{dx}\left(\frac{d^2y}{du^2}\right)\right]\left(\frac{du}{dx}\right)^2+\left[\frac{d}{dx}\left(\frac{du}{dx}\right)^2\right]\frac{d^2y}{du^2}+\left[\frac{d}{dx}\left(\frac{dy}{du}\right)\right]\frac{d^2u}{dx^2}+\left[\frac{d}{dx}\left(\frac{d^2u}{dx^2}\right)\right]\frac{dy}{du}$$

$$=\left[\frac{d}{du}\left(\frac{d^2y}{du^2}\right)\frac{du}{dx}\right]\left(\frac{du}{dx}\right)^2+2\frac{du}{dx}\frac{d^2u}{dx^2}\frac{d^2y}{du^2}+\left[\frac{d}{du}\left(\frac{dy}{du}\right)\frac{du}{dx}\right]\left(\frac{d^2u}{dx^2}\right)+\frac{d^3u}{dx^3}\frac{dy}{du}$$

$$=\frac{d^3y}{du^3}\left(\frac{du}{dx}\right)^3+3\frac{du}{dx}\frac{d^2u}{dx^2}\frac{d^2y}{du^2}+\frac{dy}{du}\frac{d^3u}{dx^3}$$

69. We will show that for each positive integer n , the n th derivative $f^{(n)}$ exists and equals one of f , f' , f'' , f''' , … , $f^{(p-1)}$. Since $f^{(p)}=f$, the first p derivatives of f are f' , f'' , f''' , … , $f^{(p-1)}$, and f. In particular, our statement is true for $n=1$. Suppose that k is an integer, $k\geq 1$, for which f is k –times differentiable with $f^{(k)}$ in the set $S=\{f , f' , f'' , \ldots , f^{(p-1)}\}$. Since f is p –times differentiable, every member of S is differentiable, so $f^{(k+1)}$ exists and equals the derivative of some member of S . Thus, $f^{(k+1)}$ is in the set $\{f' , f''$, f''' , … , $f^{(p)}\}$, which equals S since $f^{(p)}=f$. We have shown that the statement is true for $n=1$ and that its truth for $n=k$ implies its truth for $n=k+1$. By mathematical induction, the statement is true for all positive integers n.

1. The differentiation formula for logarithmic functions, $\frac{d}{dx}(\log_a x)=\frac{1}{x\ln a}$, is simplest when $a=e$ because $\ln e=1$.

2. $f(x)=\ln(x^2+10) \Rightarrow f'(x)=\frac{1}{x^2+10}\frac{d}{dx}(x^2+10)=\frac{2x}{x^2+10}$

3. $f(\theta)=\ln(\cos\theta) \Rightarrow f'(\theta)=\frac{1}{\cos\theta}\frac{d}{d\theta}(\cos\theta)=\frac{-\sin\theta}{\cos\theta}=-\tan\theta$

4. $f(x)=\cos(\ln x) \Rightarrow f'(x)=-\sin(\ln x)\cdot\frac{1}{x}=\frac{-\sin(\ln x)}{x}$

5. $f(x)=\log_2(1-3x) \Rightarrow f'(x)=\frac{1}{(1-3x)\ln 2}\frac{d}{dx}(1-3x)=\frac{-3}{(1-3x)\ln 2}$ or $\frac{3}{(3x-1)\ln 2}$

6. $f(x)=\log_{10}\left(\frac{x}{x-1}\right)=\log_{10}x-\log_{10}(x-1) \Rightarrow f'(x)=\frac{1}{x\ln 10}-\frac{1}{(x-1)\ln 10}$ or $-\frac{1}{x(x-1)\ln 10}$

7. $f(x)=\sqrt[5]{\ln x}=(\ln x)^{1/5} \Rightarrow f'(x)=\frac{1}{5}(\ln x)^{-4/5}\frac{d}{dx}(\ln x)=\frac{1}{5(\ln x)^{4/5}}\cdot\frac{1}{x}=\frac{1}{5x\sqrt[5]{(\ln x)^4}}$

8. $f(x)=\ln\sqrt[5]{x}=\ln x^{1/5}=\frac{1}{5}\ln x \Rightarrow f'(x)=\frac{1}{5}\cdot\frac{1}{x}=\frac{1}{5x}$

9. $f(x)=\sqrt{x}\ln x \Rightarrow f'(x)=\sqrt{x}\left(\frac{1}{x}\right)+(\ln x)\cdot\frac{1}{2\sqrt{x}}=\frac{1}{\sqrt{x}}+\frac{\ln x}{2\sqrt{x}}=\frac{2+\ln x}{2\sqrt{x}}$

10. $f(t)=\frac{1+\ln t}{1-\ln t} \Rightarrow$

$$f'(t)=\frac{(1-\ln t)(1/t)-(1+\ln t)(-1/t)}{(1-\ln t)^2}=\frac{(1/t)\left[(1-\ln t)+(1+\ln t)\right]}{(1-\ln t)^2}=\frac{2}{t(1-\ln t)^2}$$

11. $F(t)=\ln\frac{(2t+1)^3}{(3t-1)^4}=\ln(2t+1)^3-\ln(3t-1)^4=3\ln(2t+1)-4\ln(3t-1) \Rightarrow$

$F'(t)=3\cdot\frac{1}{2t+1}\cdot 2-4\cdot\frac{1}{3t-1}\cdot 3=\frac{6}{2t+1}-\frac{12}{3t-1}$, or combined, $\frac{-6(t+3)}{(2t+1)(3t-1)}$.

12.

$$h(x)=\ln\left(x+\sqrt{x^2-1}\right)\Rightarrow h'(x)=\frac{1}{x+\sqrt{x^2-1}}\left(1+\frac{x}{\sqrt{x^2-1}}\right)=\frac{1}{x+\sqrt{x^2-1}}\cdot\frac{\sqrt{x^2-1}+x}{\sqrt{x^2-1}}=\frac{1}{\sqrt{x^2-1}}$$

13. $g(x)=\ln\dfrac{a-x}{a+x}=\ln(a-x)-\ln(a+x)\Rightarrow$

$$g'(x)=\frac{1}{a-x}(-1)-\frac{1}{a+x}=\frac{-(a+x)-(a-x)}{(a-x)(a+x)}=\frac{-2a}{a^2-x^2}$$

14. $F(y)=y\ln(1+e^y)\Rightarrow F'(y)=y\cdot\dfrac{1}{1+e^y}\cdot e^y+\ln(1+e^y)\cdot 1=\dfrac{ye^y}{1+e^y}+\ln(1+e^y)$

15. $f(u)=\dfrac{\ln u}{1+\ln(2u)}\Rightarrow$

$$f'(u)=\frac{[1+\ln(2u)]\cdot\frac{1}{u}-\ln u\cdot\frac{1}{2u}\cdot 2}{[1+\ln(2u)]^2}=\frac{\frac{1}{u}[1+\ln(2u)-\ln u]}{[1+\ln(2u)]^2}$$

$$=\frac{1+(\ln 2+\ln u)-\ln u}{u[1+\ln(2u)]^2}=\frac{1+\ln 2}{u[1+\ln(2u)]^2}$$

16. $y=\ln(x^4\sin^2 x)=\ln x^4+\ln(\sin x)^2=4\ln x+2\ln\sin x\Rightarrow y'=4\cdot\dfrac{1}{x}+2\cdot\dfrac{1}{\sin x}\cdot\cos x=\dfrac{4}{x}+2\cot x$

17. $y=\ln|2-x-5x^2|\Rightarrow y'=\dfrac{1}{2-x-5x^2}\cdot(-1-10x)=\dfrac{-10x-1}{2-x-5x^2}$ or $\dfrac{10x+1}{5x^2+x-2}$

18. $G(u)=\ln\sqrt{\dfrac{3u+2}{3u-2}}=\dfrac{1}{2}[\ln(3u+2)-\ln(3u-2)]\Rightarrow G'(u)=\dfrac{1}{2}\left(\dfrac{3}{3u+2}-\dfrac{3}{3u-2}\right)=\dfrac{-6}{9u^2-4}$

19. $y=\ln(e^{-x}+xe^{-x})=\ln(e^{-x}(1+x))=\ln(e^{-x})+\ln(1+x)=-x+\ln(1+x)\Rightarrow y'=-1+\dfrac{1}{1+x}=\dfrac{-1-x+1}{1+x}=-\dfrac{x}{1+x}$

20. $y=[\ln(1+e^x)]^2\Rightarrow y'=2[\ln(1+e^x)]\cdot\dfrac{1}{1+e^x}\cdot e^x=\dfrac{2e^x\ln(1+e^x)}{1+e^x}$

21. $y=x\ln x\Rightarrow y'=x(1/x)+(\ln x)\cdot 1=1+\ln x\Rightarrow y''=1/x$

22. $y=\dfrac{\ln x}{x^2}\Rightarrow y'=\dfrac{x^2(1/x)-(\ln x)(2x)}{\left(x^2\right)^2}=\dfrac{x(1-2\ln x)}{x^4}=\dfrac{1-2\ln x}{x^3}\Rightarrow$

$y''=\dfrac{x^3(-2/x)-(1-2\ln x)(3x^2)}{(x^3)^2}=\dfrac{x^2(-2-3+6\ln x)}{x^6}=\dfrac{6\ln x-5}{x^4}$

23. $y=\log_{10}x\Rightarrow y'=\dfrac{1}{x\ln 10}=\dfrac{1}{\ln 10}\left(\dfrac{1}{x}\right)\Rightarrow y''=\dfrac{1}{\ln 10}\left(-\dfrac{1}{x^2}\right)=-\dfrac{1}{x^2\ln 10}$

24. $y=\ln(\sec x+\tan x)\Rightarrow y'=\dfrac{\sec x\tan x+\sec^2 x}{\sec x+\tan x}=\sec x\Rightarrow y''=\sec x\tan x$

25. $f(x)=\dfrac{x}{1-\ln(x-1)}\Rightarrow$

$f'(x)=\dfrac{[1-\ln(x-1)]\cdot 1-x\cdot\dfrac{-1}{x-1}}{[1-\ln(x-1)]^2}=\dfrac{\dfrac{(x-1)[1-\ln(x-1)]+x}{x-1}}{[1-\ln(x-1)]^2}=\dfrac{x-1-(x-1)\ln(x-1)+x}{(x-1)^2}$

$=\dfrac{2x-1-(x-1)\ln(x-1)}{(x-1)[1-\ln(x-1)]^2}$

$\mathrm{Dom}(f)=\{x\,|\,x-1>0 \text{ and } 1-\ln(x-1)\neq 0\}=\{x\,|\,x>1 \text{ and } \ln(x-1)\neq 1\}$

$=\{x\,|\,x>1 \text{ and } x-1\neq e^1\}=\{x\,|\,x>1 \text{ and } x\neq 1+e\}=(1,1+e)\cup(1+e,\infty)$

26. $f(x)=\dfrac{1}{1+\ln x}\Rightarrow f'(x)=-\dfrac{1/x}{(1+\ln x)^2}=-\dfrac{1}{x(1+\ln x)^2}$. $\mathrm{Dom}(f)=\{x\,|\,x>0 \text{ and } \ln x\neq -1\}=$ $\{x\,|\,x>0 \text{ and } x\neq 1/e\}=(0,1/e)\cup(1/e,\infty)$.

27. $f(x)=x^2\ln(1-x^2)\Rightarrow f'(x)=2x\ln(1-x^2)+\dfrac{x^2(-2x)}{1-x^2}=2x\ln(1-x^2)-\dfrac{2x^3}{1-x^2}$.

$\mathrm{Dom}(f)=\{x\,|\,1-x^2>0\}=\{x\,|\,|x|<1\}=(-1,1)$.

28. $f(x)=\ln\ln\ln x\Rightarrow f'(x)=\dfrac{1}{\ln\ln x}\cdot\dfrac{1}{\ln x}\cdot\dfrac{1}{x}$.

$\mathrm{Dom}(f)=\{x\,|\,\ln\ln x>0\}=\{x\,|\,\ln x>1\}=\{x\,|\,x>e\}=(e,\infty)$.

29. $f(x)=$

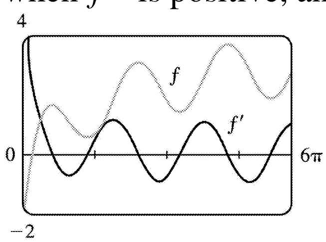
$$\frac{x}{\ln x} \Rightarrow f'(x) = \frac{\ln x - x(1/x)}{(\ln x)^2} = \frac{\ln x - 1}{(\ln x)^2} \Rightarrow f'(e) = \frac{1-1}{1^2} = 0$$

30. $f(x) = x^2 \ln x \Rightarrow f'(x) = 2x\ln x + x^2\left(\frac{1}{x}\right) = 2x\ln x + x \Rightarrow f'(1) = 2\ln 1 + 1 = 1$

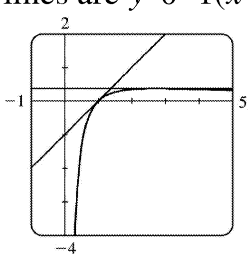
31. $y = f(x) = \ln \ln x \Rightarrow f'(x) = \frac{1}{\ln x}\left(\frac{1}{x}\right) \Rightarrow f'(e) = \frac{1}{e}$, so an equation of the tangent line at $(e, 0)$ is $y - 0 = \frac{1}{e}(x - e)$, or $y = \frac{1}{e}x - 1$, or $x - ey = e$.

32. $y = \ln(x^3 - 7) \Rightarrow y' = \frac{1}{x^3 - 7} \cdot 3x^2 \Rightarrow y'(2) = \frac{12}{8-7} = 12$, so an equation of a tangent line at $(2, 0)$ is $y - 0 = 12(x - 2)$ or $y = 12x - 24$.

33. $f(x) = \sin x + \ln x \Rightarrow f'(x) = \cos x + 1/x$. This is reasonable, because the graph shows that f increases when f' is positive, and $f'(x) = 0$ when f has a horizontal tangent.



34. $y = \frac{\ln x}{x} \Rightarrow y' = \frac{x(1/x) - \ln x}{x^2} = \frac{1 - \ln x}{x^2}$. $y'(1) = \frac{1-0}{1^2} = 1$ and $y'(e) = \frac{1-1}{e^2} = 0 \Rightarrow$ equations of tangent lines are $y - 0 = 1(x - 1)$ or $y = x - 1$ and $y - 1/e = 0(x - e)$ or $y = 1/e$.



35. $y = (2x+1)^5(x^4-3)^6 \Rightarrow \ln y = \ln\left((2x+1)^5(x^4-3)^6\right) \Rightarrow$

$\ln y = 5\ln(2x+1) + 6\ln(x^4 - 3) \Rightarrow \frac{1}{y}y' = 5 \cdot \frac{1}{2x+1} \cdot 2 + 6 \cdot \frac{1}{x^4-3} \cdot 4x^3 \Rightarrow$

$$y' = y\left(\frac{10}{2x+1} + \frac{24x^3}{x^4-3}\right) = (2x+1)^5(x^4-3)^6\left(\frac{10}{2x+1} + \frac{24x^3}{x^4-3}\right).$$

36. $y=\sqrt{x}\,e^{x^2}(x^2+1)^{10}\Rightarrow \ln y=\ln\sqrt{x}+\ln e^{x^2}+\ln(x^2+1)^{10}\Rightarrow \ln y=\frac{1}{2}\ln x+x^2+10\ln(x^2+1)\Rightarrow$

$\frac{1}{y}y'=\frac{1}{2}\cdot\frac{1}{x}+2x+10\cdot\frac{1}{x^2+1}\cdot 2x\Rightarrow y'=\sqrt{x}\,e^{x^2}(x^2+1)^{10}\left(\frac{1}{2x}+2x+\frac{20x}{x^2+1}\right)$

37. $y=\frac{\sin^2 x\tan^4 x}{(x^2+1)^2}\Rightarrow \ln y=\ln(\sin^2 x\tan^4 x)-\ln(x^2+1)^2\Rightarrow$

$\ln y=\ln(\sin x)^2+\ln(\tan x)^4-\ln(x^2+1)^2\Rightarrow \ln y=2\ln|\sin x|+4\ln|\tan x|-2\ln(x^2+1)\Rightarrow$

$\frac{1}{y}y'=2\cdot\frac{1}{\sin x}\cdot\cos x+4\cdot\frac{1}{\tan x}\cdot\sec^2 x-2\cdot\frac{1}{x^2+1}\cdot 2x\Rightarrow$

$y'=\frac{\sin^2 x\tan^4 x}{(x^2+1)^2}\left(2\cot x+\frac{4\sec^2 x}{\tan x}-\frac{4x}{x^2+1}\right)$

38. $y=\sqrt[4]{\frac{x^2+1}{x^2-1}}\Rightarrow \ln y=\frac{1}{4}\ln(x^2+1)-\frac{1}{4}\ln(x^2-1)\Rightarrow \frac{1}{y}y'=\frac{1}{4}\cdot\frac{1}{x^2+1}\cdot 2x-\frac{1}{4}\cdot\frac{1}{x^2-1}\cdot 2x\Rightarrow$

$y'=\sqrt[4]{\frac{x^2+1}{x^2-1}}\cdot\frac{1}{2}\left(\frac{x}{x^2+1}-\frac{x}{x^2-1}\right)=\frac{1}{2}\sqrt[4]{\frac{x^2+1}{x^2-1}}\left(\frac{-2x}{x^4-1}\right)=\frac{x}{1-x^4}\sqrt[4]{\frac{x^2+1}{x^2-1}}$

39. $y=x^x\Rightarrow \ln y=\ln x^x\Rightarrow \ln y=x\ln x\Rightarrow y'/y=x(1/x)+(\ln x)\cdot 1\Rightarrow y'=y(1+\ln x)\Rightarrow y'=x^x(1+\ln x)$

40. $y=x^{1/x}\Rightarrow \ln y=\frac{1}{x}\ln x\Rightarrow \frac{y'}{y}=\frac{1}{x}\left(\frac{1}{x}\right)+(\ln x)\left(-\frac{1}{x^2}\right)\Rightarrow y'=x^{1/x}\frac{1-\ln x}{x^2}$

41. $y=x^{\sin x}\Rightarrow \ln y=\ln x^{\sin x}\Rightarrow \ln y=\sin x\ln x\Rightarrow \frac{y'}{y}=(\sin x)\cdot\frac{1}{x}+(\ln x)(\cos x)\Rightarrow$

$y'=y\left(\frac{\sin x}{x}+\ln x\cos x\right)\Rightarrow y'=x^{\sin x}\left(\frac{\sin x}{x}+\ln x\cos x\right)$

42. $y=(\sin x)^x\Rightarrow \ln y=x\ln(\sin x)\Rightarrow \frac{y'}{y}=x\cdot\frac{1}{\sin x}\cdot\cos x+[\ln(\sin x)]\cdot 1\Rightarrow y'=(\sin x)^x[x\cot x+\ln(\sin x)]$

43. $y=(\ln x)^x\Rightarrow \ln y=\ln(\ln x)^x\Rightarrow \ln y=x\ln\ln x\Rightarrow \frac{y'}{y}=x\cdot\frac{1}{\ln x}\cdot\frac{1}{x}+(\ln\ln x)\cdot 1\Rightarrow$

$y'=y\left(\dfrac{x}{x\ln x}+\ln\ln x\right)\Rightarrow\ y'=(\ln x)^{x}\left(\dfrac{1}{\ln x}+\ln\ln x\right)$

44. $y=x^{\ln x}\Rightarrow\ln y=\ln x\ln x=(\ln x)^{2}\Rightarrow\ \dfrac{y'}{y}=2\ln x\left(\dfrac{1}{x}\right)\Rightarrow\ y'=x^{\ln x}\left(\dfrac{2\ln x}{x}\right)$

45. $y=x^{e^{x}}\Rightarrow\ln y=e^{x}\ln x\Rightarrow\ \dfrac{y'}{y}=e^{x}\cdot\dfrac{1}{x}+(\ln x)\cdot e^{x}\Rightarrow\ y'=x^{e^{x}}e^{x}\left(\ln x+\dfrac{1}{x}\right)$

46. $y=(\ln x)^{\cos x}\Rightarrow\ln y=\cos x\ln(\ln x)\Rightarrow\ \dfrac{y'}{y}=\cos x\cdot\dfrac{1}{\ln x}\cdot\dfrac{1}{x}+(\ln\ln x)(-\sin x)\Rightarrow$
$y'=(\ln x)^{\cos x}\left(\dfrac{\cos x}{x\ln x}-\sin x\ln\ln x\right)$

47. $y=\ln(x^{2}+y^{2})\Rightarrow y'=\dfrac{1}{x^{2}+y^{2}}\ \dfrac{d}{dx}(x^{2}+y^{2})\Rightarrow y'=\dfrac{2x+2yy'}{x^{2}+y^{2}}\Rightarrow x^{2}y'+y^{2}y'=2x+2yy'\Rightarrow$
$x^{2}y'+y^{2}y'-2yy'=2x\Rightarrow(x^{2}+y^{2}-2y)\,y'=2x\Rightarrow\ y'=\dfrac{2x}{x^{2}+y^{2}-2y}$

48. $x^{y}=y^{x}\Rightarrow y\ln x=x\ln y\Rightarrow y\cdot\dfrac{1}{x}+(\ln x)\cdot y'=x\cdot\dfrac{1}{y}\cdot y'+\ln y\Rightarrow y'\ln x-\dfrac{x}{y}y'=\ln y-\dfrac{y}{x}\Rightarrow\ y'=\dfrac{\ln y-y/x}{\ln x-x/y}$

49. $f(x)=\ln(x-1)\Rightarrow f'(x)=1/(x-1)=(x-1)^{-1}\Rightarrow f''(x)=-(x-1)^{-2}\Rightarrow$
$f'''(x)=2(x-1)^{-3}\Rightarrow f^{(4)}(x)=-2\cdot 3(x-1)^{-4}\Rightarrow\cdots\Rightarrow$
$f^{(n)}(x)=(-1)^{n-1}\cdot 2\cdot 3\cdot 4\cdot\ldots\cdot(n-1)(x-1)^{-n}=(-1)^{n-1}\dfrac{(n-1)!}{(x-1)^{n}}$

50. $y=x^{8}\ln x$, so $D^{9}y=D^{8}y'=D^{8}(8x^{7}\ln x+x^{7})$. But the eighth derivative of x^{7} is 0, so we now have
$D^{8}(8x^{7}\ln x)=D^{7}(8\cdot 7x^{6}\ln x+8x^{6})=D^{7}(8\cdot 7x^{6}\ln x)$
$=D^{6}(8\cdot 7\cdot 6x^{5}\ln x)=\ldots=D(8!\,x^{0}\ln x)=8!/\text{x}.$

51. If $f(x)=\ln(1+x)$, then $f'(x)=\dfrac{1}{1+x}$, so $f'(0)=1$. Thus,
$\lim\limits_{x\to 0}\dfrac{\ln(1+x)}{x}=\lim\limits_{x\to 0}\dfrac{f(x)}{x}=\lim\limits_{x\to 0}\dfrac{f(x)-f(0)}{x-0}=f'(0)=1.$

52. Let $m=n/x$. Then $n=xm$, and as $n\to\infty$, $m\to\infty$. Therefore,

$$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^{n}=\lim_{m\to\infty}\left(1+\frac{1}{m}\right)^{mx}=\left[\lim_{m\to\infty}\left(1+\frac{1}{m}\right)^{m}\right]^{x}=e^{x}$$ by Equation 6.

1. **(a)** $\sinh 0=\frac{1}{2}(e^{0}-e^{0})=0$

(b) $\cosh 0=\frac{1}{2}(e^{0}+e^{0})=\frac{1}{2}(1+1)=1$

2. **(a)** $\tanh 0=\frac{(e^{0}-e^{-0})/2}{(e^{0}+e^{-0})/2}=0$

(b) $\tanh 1=\frac{e^{1}-e^{-1}}{e^{1}+e^{-1}}=\frac{e^{2}-1}{e^{2}+1}\approx 0.76159$

3. **(a)** $\sinh(\ln 2)=\frac{e^{\ln 2}-e^{-\ln 2}}{2}=\frac{e^{\ln 2}-(e^{\ln 2})^{-1}}{2}=\frac{2-2^{-1}}{2}=\frac{2-\frac{1}{2}}{2}=\frac{3}{4}$

(b) $\sinh 2=\frac{1}{2}(e^{2}-e^{-2})\approx 3.62686$

4. **(a)** $\cosh 3=\frac{1}{2}(e^{3}+e^{-3})\approx 10.06766$

(b) $\cosh(\ln 3)=\frac{e^{\ln 3}+e^{-\ln 3}}{2}=\frac{3+\frac{1}{3}}{2}=\frac{5}{3}$

5. **(a)** $\operatorname{sech} 0=\frac{1}{\cosh 0}=\frac{1}{1}=1$

(b) $\cosh^{-1} 1=0$ because $\cosh 0=1$.

6. **(a)** $\sinh 1=\frac{1}{2}(e^{1}-e^{-1})\approx 1.17520$

(b) Using Equation 3, we have $\sinh^{-1} 1=\ln\left(1+\sqrt{1^{2}+1}\right)=\ln(1+\sqrt{2})\approx 0.88137$.

7. $\sinh(-x)=\frac{1}{2}\left[e^{-x}-e^{-(-x)}\right]=\frac{1}{2}(e^{-x}-e^{x})=-\frac{1}{2}(e^{x}-e^{-x})=-\sinh x$

8. $\cosh(-x)=\frac{1}{2}\left[e^{-x}+e^{-(-x)}\right]=\frac{1}{2}(e^{-x}+e^{x})=\frac{1}{2}(e^{x}+e^{-x})=\cosh x$

9. $\cosh x+\sinh x=\frac{1}{2}(e^{x}+e^{-x})+\frac{1}{2}(e^{x}-e^{-x})=\frac{1}{2}(2e^{x})=e^{x}$

10. $\cosh x - \sinh x = \frac{1}{2}(e^{x}+e^{-x}) - \frac{1}{2}(e^{x}-e^{-x}) = \frac{1}{2}(2e^{-x}) = e^{-x}$

11. $\sinh x\cosh y + \cosh x\sinh y = \left[\frac{1}{2}(e^{x}-e^{-x})\right]\left[\frac{1}{2}(e^{y}+e^{-y})\right] + \left[\frac{1}{2}(e^{x}+e^{-x})\right]\left[\frac{1}{2}(e^{y}-e^{-y})\right]$

$= \frac{1}{4}\left[(e^{x+y}+e^{x-y}-e^{-x+y}-e^{-x-y})+(e^{x+y}-e^{x-y}+e^{-x+y}-e^{-x-y})\right]$

$= \frac{1}{4}(2e^{x+y}-2e^{-x-y}) = \frac{1}{2}\left[e^{x+y}-e^{-(x+y)}\right] = \sinh(x+y)$

12. $\cosh x\cosh y + \sinh x\sinh y = \left[\frac{1}{2}(e^{x}+e^{-x})\right]\left[\frac{1}{2}(e^{y}+e^{-y})\right] + \left[\frac{1}{2}(e^{x}-e^{-x})\right]\left[\frac{1}{2}(e^{y}-e^{-y})\right]$

$= \frac{1}{4}\left[(e^{x+y}+e^{x-y}+e^{-x+y}+e^{-x-y})+(e^{x+y}-e^{x-y}-e^{-x+y}+e^{-x-y})\right]$

$= \frac{1}{4}(2e^{x+y}+2e^{-x-y}) = \frac{1}{2}\left[e^{x+y}+e^{-(x+y)}\right] = \cosh(x+y)$

13. Divide both sides of the identity $\cosh^{2}x - \sinh^{2}x = 1$ by $\sinh^{2}x$:

$$\frac{\cosh^{2}x}{\sinh^{2}x} - \frac{\sinh^{2}x}{\sinh^{2}x} = \frac{1}{\sinh^{2}x} \Leftrightarrow \coth^{2}x - 1 = \operatorname{csch}^{2}x.$$

14.

$$\tanh(x+y) = \frac{\sinh(x+y)}{\cosh(x+y)} = \frac{\sinh x\cosh y+\cosh x\sinh y}{\cosh x\cosh y+\sinh x\sinh y} = \frac{\dfrac{\sinh x\cosh y}{\cosh x\cosh y}+\dfrac{\cosh x\sinh y}{\cosh x\cosh y}}{\dfrac{\cosh x\cosh y}{\cosh x\cosh y}+\dfrac{\sinh x\sinh y}{\cosh x\cosh y}}$$

$$= \frac{\tanh x+\tanh y}{1+\tanh x\tanh y}$$

15. Putting $y=x$ in the result from Exercise 11, we have
$\sinh 2x = \sinh(x+x) = \sinh x\cosh x + \cosh x\sinh x = 2\sinh x\cosh x$.

16. Putting $y=x$ in the result from Exercise 12, we have
$\cosh 2x = \cosh(x+x) = \cosh x\cosh x + \sinh x\sinh x = \cosh^{2}x + \sinh^{2}x$.

17.

$$\tanh(\ln x) = \frac{\sinh(\ln x)}{\cosh(\ln x)} = \frac{(e^{\ln x}-e^{-\ln x})/2}{(e^{\ln x}+e^{-\ln x})/2} = \frac{x-(e^{\ln x})^{-1}}{x+(e^{\ln x})^{-1}} = \frac{x-x^{-1}}{x+x^{-1}}$$

$$=\frac{x-1/x}{x+1/x}=\frac{(x^2-1)/x}{(x^2+1)/x}=\frac{x^2-1}{x^2+1}$$

18.

$$\frac{1+\tanh x}{1-\tanh x}=\frac{1+(\sinh x)/\cosh x}{1-(\sinh x)/\cosh x}=\frac{\cosh x+\sinh x}{\cosh x-\sinh x}=\frac{\frac{1}{2}(e^x+e^{-x})+\frac{1}{2}(e^x-e^{-x})}{\frac{1}{2}(e^x+e^{-x})-\frac{1}{2}(e^x-e^{-x})}$$

$$=\frac{e^x+e^{-x}+e^x-e^{-x}}{e^x+e^{-x}-e^x+e^{-x}}=\frac{2e^x}{2e^{-x}}=e^{2x}$$

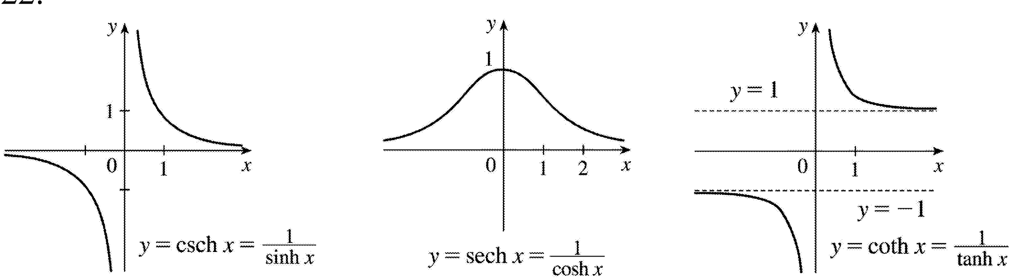
Or: Using the results of Exercises 9 and 10, $\frac{\cosh x+\sinh x}{\cosh x-\sinh x}=\frac{e^x}{e^{-x}}=e^{2x}$

19. By Exercise 9, $(\cosh x+\sinh x)^n=(e^x)^n=e^{nx}=\cosh nx+\sinh nx$.

20. $\sinh x=\frac{3}{4}\Rightarrow \operatorname{csch} x=1/\sinh x=\frac{4}{3}$. $\cosh^2 x=\sinh^2 x+1=\frac{9}{16}+1=\frac{25}{16}\Rightarrow \cosh x=\frac{5}{4}$ (since $\cosh x>0$). $\coth x=1/\cosh x=\frac{4}{5}$, $\tanh x=\sinh x/\cosh x=\frac{3/4}{5/4}=\frac{3}{5}$, and $\coth x=1/\tanh x=\frac{5}{3}$.

21. $\tanh x=\frac{4}{5}>0$, so $x>0$. $\coth x=1/\tanh x=\frac{5}{4}$, $\operatorname{sech}^2 x=1-\tanh^2 x=1-\left(\frac{4}{5}\right)^2=\frac{9}{25}\Rightarrow \operatorname{sech} x=\frac{3}{5}$ (since $\operatorname{sech} x>0$), $\cosh x=1/\operatorname{sech} x=\frac{5}{3}$, $\sinh x=\tanh x\cosh x=\frac{4}{5}\cdot\frac{5}{3}=\frac{4}{3}$, and $\operatorname{csch} x=1/\sinh x=\frac{3}{4}$.

22.



23. **(a)**

$$\lim_{x\to\infty} \tanh x = \lim_{x\to\infty} \frac{e^x - e^{-x}}{e^x + e^{-x}} \cdot \frac{e^{-x}}{e^{-x}} = \lim_{x\to\infty} \frac{1 - e^{-2x}}{1 + e^{-2x}} = \frac{1-0}{1+0} = 1$$

(b) $\lim_{x\to-\infty} \tanh x = \lim_{x\to-\infty} \frac{e^x - e^{-x}}{e^x + e^{-x}} \cdot \frac{e^x}{e^x} = \lim_{x\to-\infty} \frac{e^{2x} - 1}{e^{2x} + 1} = \frac{0-1}{0+1} = -1$

(c) $\lim_{x\to\infty} \sinh x = \lim_{x\to\infty} \frac{e^x - e^{-x}}{2} = \infty$

(d) $\lim_{x\to-\infty} \sinh x = \lim_{x\to-\infty} \frac{e^x - e^{-x}}{2} = -\infty$

(e) $\lim_{x\to\infty} \operatorname{sech} x = \lim_{x\to\infty} \frac{2}{e^x + e^{-x}} = 0$

(f) $\lim_{x\to\infty} \coth x = \lim_{x\to\infty} \frac{e^x + e^{-x}}{e^x - e^{-x}} \cdot \frac{e^{-x}}{e^{-x}} = \lim_{x\to\infty} \frac{1 + e^{-2x}}{1 - e^{-2x}} = \frac{1+0}{1-0} = 1$

(g) $\lim_{x\to 0^+} \coth x = \lim_{x\to 0^+} \frac{\cosh x}{\sinh x} = \infty$, since $\sinh x \to 0$ through positive values and $\cosh x \to 1$.

(h) $\lim_{x\to 0^-} \coth x = \lim_{x\to 0^-} \frac{\cosh x}{\sinh x} = -\infty$, since $\sinh x \to 0$ through negative values and $\cosh x \to 1$.

(i) $\lim_{x\to-\infty} \operatorname{csch} x = \lim_{x\to-\infty} \frac{2}{e^x - e^{-x}} = 0$

24. **(a)** $\frac{d}{dx} \cosh x = \frac{d}{dx} \left[\frac{1}{2} (e^x + e^{-x}) \right] = \frac{1}{2} (e^x - e^{-x}) = \sinh x$

(b) $\frac{d}{dx} \tanh x = \frac{d}{dx} \left[\frac{\sinh x}{\cosh x} \right] = \frac{\cosh x \cosh x - \sinh x \sinh x}{\cosh^2 x} = \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} = \frac{1}{\cosh^2 x} = \operatorname{sech}^2 x$

(c) $\frac{d}{dx} \operatorname{csch} x = \frac{d}{dx} \left[\frac{1}{\sinh x} \right] = -\frac{\cosh x}{\sinh^2 x} = -\frac{1}{\sinh x} \cdot \frac{\cosh x}{\sinh x} = -\operatorname{csch} x \coth x$

(d) $\frac{d}{dx} \operatorname{sech} x = \frac{d}{dx} \left[\frac{1}{\cosh x} \right] = -\frac{\sinh x}{\cosh^2 x} = -\frac{1}{\cosh x} \cdot \frac{\sinh x}{\cosh x} = -\operatorname{sech} x \tanh x$

(e)

$$\frac{d}{dx} \coth x = \frac{d}{dx} \left[\frac{\cosh x}{\sinh x} \right] = \frac{\sinh x \sinh x - \cosh x \cosh x}{\sinh^2 x} = \frac{\sinh^2 x - \cosh^2 x}{\sinh^2 x} = -\frac{1}{\sinh^2 x}$$

$$= -\operatorname{csch}^2 x$$

25. Let $y=\sinh^{-1}x$. Then $\sinh y=x$ and, by Example 1(a), $\cosh^2 y-\sinh^2 y=1\Rightarrow$ [with $\cosh y>0$] $\cosh y=\sqrt{1+\sinh^2 y}=\sqrt{1+x^2}$. So by Exercise 9, $e^y=\sinh y+\cosh y=x+\sqrt{1+x^2}\Rightarrow y=\ln\left(x+\sqrt{1+x^2}\right)$.

26. Let $y=\cosh^{-1}x$. Then $\cosh y=x$ and $y\geq 0$, so $\sinh y=\sqrt{\cosh^2 y-1}=\sqrt{x^2-1}$. So, by Exercise 9, $e^y=\cosh y+\sinh y=x+\sqrt{x^2-1}\Rightarrow y=\ln\left(x+\sqrt{x^2-1}\right)$.
Another method: Write $x=\cosh y=\frac{1}{2}(e^y+e^{-y})$ and solve a quadratic, as in Example 3.

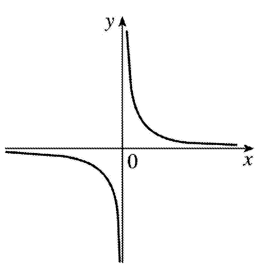
27. **(a)** Let $y=\tanh^{-1}x$. Then $x=\tanh y=\dfrac{\sinh y}{\cosh y}=\dfrac{(e^y-e^{-y})/2}{(e^y+e^{-y})/2}\cdot\dfrac{e^y}{e^y}=\dfrac{e^{2y}-1}{e^{2y}+1}\Rightarrow$
$xe^{2y}+x=e^{2y}-1\Rightarrow 1+x=e^{2y}-xe^{2y}\Rightarrow 1+x=e^{2y}(1-x)\Rightarrow$
$e^{2y}=\dfrac{1+x}{1-x}\Rightarrow 2y=\ln\left(\dfrac{1+x}{1-x}\right)\Rightarrow y=\dfrac{1}{2}\ln\left(\dfrac{1+x}{1-x}\right)$.
(b) Let $y=\tanh^{-1}x$. Then $x=\tanh y$, so from Exercise 18 we have
$e^{2y}=\dfrac{1+\tanh y}{1-\tanh y}=\dfrac{1+x}{1-x}\Rightarrow 2y=\ln\left(\dfrac{1+x}{1-x}\right)\Rightarrow y=\dfrac{1}{2}\ln\left(\dfrac{1+x}{1-x}\right)$.

28. **(a)**

(i) $y=\operatorname{csch}^{-1}x\Leftrightarrow \operatorname{csch} y=x$ ($x\neq 0$)

(ii) We sketch the graph of csch^{-1} by reflecting the graph of csch (see Exercise 22) about the line y=x.

(iii) Let $y=\operatorname{csch}^{-1}x$. Then $x=\operatorname{csch} y=\dfrac{2}{e^y-e^{-y}}\Rightarrow xe^y-xe^{-y}=2\Rightarrow x(e^y)^2-2e^y-x=0\Rightarrow e^y=\dfrac{1\pm\sqrt{x^2+1}}{x}$.



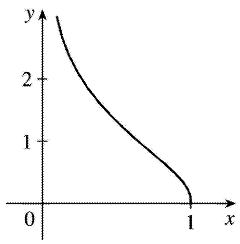
But $e^y>0$, so for $x>0$, $e^y=\dfrac{1+\sqrt{x^2+1}}{x}$ and for $x<0$, $e^y=\dfrac{1-\sqrt{x^2+1}}{x}$. Thus,
$\operatorname{csch}^{-1}x=\ln\left(\dfrac{1}{x}+\dfrac{\sqrt{x^2+1}}{|x|}\right)$.

(b)

(i) $y=\operatorname{sech}^{-1}x \Leftrightarrow \operatorname{sech} y=x$ and $y>0$.

(ii) We sketch the graph of $\coth^{-1}$ by reflecting the graph of coth (see Exercise 22) about the line $y=x$.

(iii) Let $y=\coth^{-1}x$, so $x=\operatorname{sech} y=\dfrac{2}{e^{y}+e^{-y}} \Rightarrow xe^{y}+xe^{-y}=2 \Rightarrow x(e^{y})^{2}-2e^{y}+x=0 \Leftrightarrow e^{y}=\dfrac{1\pm\sqrt{1-x^{2}}}{x}$.



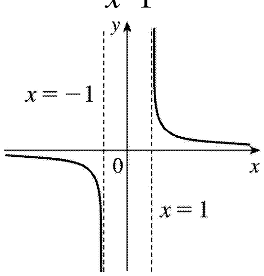
But $y>0 \Rightarrow e^{y}>1$. This rules out the minus sign because $\dfrac{1-\sqrt{1-x^{2}}}{x}>1 \Leftrightarrow 1-\sqrt{1-x^{2}}>x \Leftrightarrow$ $1-x>\sqrt{1-x^{2}} \Leftrightarrow 1-2x+x^{2}>1-x^{2} \Leftrightarrow x^{2}>x \Leftrightarrow x>1$, but $x=\operatorname{sech} y\leq 1$. Thus, $e^{y}=\dfrac{1+\sqrt{1-x^{2}}}{x} \Rightarrow$ $\operatorname{sech}^{-1}x=\ln\left(\dfrac{1+\sqrt{1-x^{2}}}{x}\right)$.

(c)

(i) $y=\coth^{-1}x \Leftrightarrow \coth y=x$

(ii) We sketch the graph of $\coth^{-1}$ by reflecting the graph of coth (see Exercise 22) about the line $y=x$.

(iii) Let $y=\coth^{-1}x$. Then $x=\coth y=\dfrac{e^{y}+e^{-y}}{e^{y}-e^{-y}} \Rightarrow xe^{y}-xe^{-y}=e^{y}+e^{-y} \Rightarrow (x-1)e^{y}=(x+1)e^{-y} \Rightarrow e^{2y}=\dfrac{x+1}{x-1} \Rightarrow$ $2y=\ln\dfrac{x+1}{x-1} \Rightarrow \coth^{-1}x=\dfrac{1}{2}\ln\dfrac{x+1}{x-1}$



29. **(a)** Let $y=\cosh^{-1}x$. Then $\cosh y=x$ and $y\geq 0\Rightarrow \sinh y\,\dfrac{dy}{dx}=1\Rightarrow$

$\dfrac{dy}{dx}=\dfrac{1}{\sinh y}=\dfrac{1}{\sqrt{\cosh^2 y-1}}=\dfrac{1}{\sqrt{x^2-1}}$ (since $\sinh y\geq 0$ for $y\geq 0$). *Or:* Use Formula 4.

(b) Let $y=\tanh^{-1}x$. Then $\tanh y=x\Rightarrow \operatorname{sech}^2 y\,\dfrac{dy}{dx}=1\Rightarrow\ \dfrac{dy}{dx}=\dfrac{1}{\operatorname{sech}^2 y}=\dfrac{1}{1-\tanh^2 y}=\dfrac{1}{1-x^2}$.

Or: Use Formula 5.

(c) Let $y=\operatorname{csch}^{-1}x$. Then $\operatorname{csch} y=x\Rightarrow -\operatorname{csch} y\coth y\,\dfrac{dy}{dx}=1\Rightarrow\ \dfrac{dy}{dx}=-\dfrac{1}{\operatorname{csch} y\coth y}$. By Exercise 13, $\coth y=\pm\sqrt{\operatorname{csch}^2 y+1}=\pm\sqrt{x^2+1}$. If $x>0$, then $\coth y>0$, so $\coth y=\sqrt{x^2+1}$. If $x<0$, then $\coth y<0$, so $\coth y=-\sqrt{x^2+1}$. In either case we have $\dfrac{dy}{dx}=-\dfrac{1}{\operatorname{csch} y\coth y}=-\dfrac{1}{|x|\sqrt{x^2+1}}$.

(d) Let $y=\operatorname{sech}^{-1}x$. Then $\operatorname{sech} y=x\Rightarrow -\operatorname{sech} y\tanh y\,\dfrac{dy}{dx}=1\Rightarrow$

$\dfrac{dy}{dx}=-\dfrac{1}{\operatorname{sech} y\tanh y}=-\dfrac{1}{\operatorname{sech} y\sqrt{1-\operatorname{sech}^2 y}}=-\dfrac{1}{x\sqrt{1-x^2}}$. (Note that $y>0$ and so $\tanh y>0$.)

(e) Let $y=\coth^{-1}x$. Then $\coth y=x\Rightarrow -\operatorname{csch}^2 y\,\dfrac{dy}{dx}=1\Rightarrow\ \dfrac{dy}{dx}=-\dfrac{1}{\operatorname{csch}^2 y}=\dfrac{1}{1-\coth^2 y}=\dfrac{1}{1-x^2}$ by Exercise 13.

30. $f(x)=\tanh 4x\Rightarrow f'(x)=4\operatorname{sech}^2 4x$

31. $f(x)=x\cosh x\Rightarrow f'(x)=x(\cosh x)'+(\cosh x)(x)'=x\sinh x+\cosh x$

32. $g(x)=\sinh^2 x\Rightarrow g'(x)=2\sinh x\cosh x$

33. $h(x)=\sinh (x^2)\Rightarrow h'(x)=\cosh (x^2)\cdot 2x=2x\cosh (x^2)$

34. $F(x)=\sinh x\tanh x\Rightarrow F'(x)=\sinh x\operatorname{sech}^2 x+\tanh x\cosh x$

35. $G(x)=\dfrac{1-\cosh x}{1+\cosh x}\Rightarrow$

$$G'(x)=\frac{(1+\cosh x)(-\sinh x)-(1-\cosh x)(\sinh x)}{(1+\cosh x)^2}$$

$$=\frac{-\sinh x-\sinh x\cosh x-\sinh x+\sinh x\cosh x}{(1+\cosh x)^2}=\frac{-2\sinh x}{(1+\cosh x)^2}$$

36. $f(t)=e^t\operatorname{sech} t\Rightarrow f'(t)=e^t(-\operatorname{sech} t\tanh t)+(\operatorname{sech} t)e^t=e^t\operatorname{sech} t\,(1-\tanh t)$

37. $h(t)=\coth\sqrt{1+t^2}\Rightarrow h'(t)=-\operatorname{csch}^2\sqrt{1+t^2}\cdot\frac{1}{2}(1+t^2)^{-1/2}(2t)=-\frac{t\operatorname{csch}^2\sqrt{1+t^2}}{\sqrt{1+t^2}}$

38. $f(t)=\ln(\sinh t)\Rightarrow f'(t)=\frac{1}{\sinh t}\cosh t=\coth t$

39. $H(t)=\tanh(e^t)\Rightarrow H'(t)=\operatorname{sech}^2(e^t)\cdot e^t=e^t\operatorname{sech}^2(e^t)$

40. $y=\sinh(\cosh x)\Rightarrow y'=\cosh(\cosh x)\cdot\sinh x$

41. $y=e^{\cosh 3x}\Rightarrow y'=e^{\cosh 3x}\cdot\sinh 3x\cdot 3=3e^{\cosh 3x}\sinh 3x$

42. $y=x^2\sinh^{-1}(2x)\Rightarrow y'=x^2\cdot\frac{1}{\sqrt{1+(2x)^2}}\cdot 2+\sinh^{-1}(2x)\cdot 2x=2x\left[\frac{x}{\sqrt{1+4x^2}}+\sinh^{-1}(2x)\right]$

43. $y=\tanh^{-1}\sqrt{x}\Rightarrow y'=\frac{1}{1-(\sqrt{x})^2}\cdot\frac{1}{2}x^{-1/2}=\frac{1}{2\sqrt{x}(1-x)}$

44. $y=x\tanh^{-1}x+\ln\sqrt{1-x^2}=x\tanh^{-1}x+\frac{1}{2}\ln(1-x^2)\Rightarrow$

$y'=\tanh^{-1}x+\frac{x}{1-x^2}+\frac{1}{2}\left(\frac{1}{1-x^2}\right)(-2x)=\tanh^{-1}x$

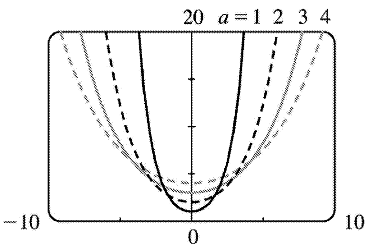
45. $y=x\sinh^{-1}(x/3)-\sqrt{9+x^2}\Rightarrow$

$y'=\sinh^{-1}\left(\frac{x}{3}\right)+x\frac{1/3}{\sqrt{1+(x/3)^2}}-\frac{2x}{2\sqrt{9+x^2}}=\sinh^{-1}\left(\frac{x}{3}\right)+\frac{x}{\sqrt{9+x^2}}-\frac{x}{\sqrt{9+x^2}}=\sinh^{-1}\left(\frac{x}{3}\right)$

46. $y=\operatorname{sech}^{-1}\sqrt{1-x^2}\Rightarrow$

$$y'=-\frac{1}{\sqrt{1-x^2}\sqrt{1-(1-x^2)}}\ \frac{-2x}{2\sqrt{1-x^2}}=\frac{x}{(1-x^2)|x|}$$

47. $y=\coth^{-1}\sqrt{x^2+1}\Rightarrow y'=\frac{1}{1-(x^2+1)}\ \frac{2x}{2\sqrt{x^2+1}}=-\frac{1}{x\sqrt{x^2+1}}$

48. 

For $y=a\cosh(x/a)$ with $a>0$, we have the y –intercept equal to a. As a increases, the graph flattens.

49. **(a)** $y=20\cosh(x/20)-15\Rightarrow y'=20\sinh(x/20)\cdot\frac{1}{20}=\sinh(x/20)$. Since the right pole is positioned at $x=7$, we have $y'(7)=\sinh\frac{7}{20}\approx 0.3572$.

(b) If α is the angle between the tangent line and the x –axis, then $\tan\alpha=$ slope of the line $=\sinh\frac{7}{20}$, so $\alpha=\tan^{-1}\left(\sinh\frac{7}{20}\right)\approx 0.343$ rad $\approx 19.66^\circ$. Thus, the angle between the line and the pole is $\theta=90^\circ-\alpha\approx 70.34^\circ$.

50. We differentiate the function twice, then substitute into the differential equation: $y=\frac{T}{\rho g}\cosh\frac{\rho gx}{T}$

$$\Rightarrow\frac{dy}{dx}=\frac{T}{\rho g}\sinh\left(\frac{\rho gx}{T}\right)\frac{\rho g}{T}=\sinh\frac{\rho gx}{T}\Rightarrow\frac{d^2y}{dx^2}=\cosh\left(\frac{\rho gx}{T}\right)\frac{\rho g}{T}=\frac{\rho g}{T}\cosh\frac{\rho gx}{T}\ .$$

We evaluate the two sides separately: LHS$=\frac{d^2y}{dx^2}=\frac{\rho g}{T}\cosh\frac{\rho gx}{T}$,

RHS$=\frac{\rho g}{T}\sqrt{1+\left(\frac{dy}{dx}\right)^2}=\frac{\rho g}{T}\sqrt{1+\sinh^2\frac{\rho gx}{T}}=\frac{\rho g}{T}\ \frac{\rho gx}{T}$ by the identity proved in Example 1(a).

51. **(a)** $y=A\sinh mx+B\cosh mx\Rightarrow y'=mA\cosh mx+mB\sinh mx\Rightarrow$
$y''=m^2A\sinh mx+m^2B\cosh mx=m^2(A\sinh mx+B\cosh mx)=m^2y$

(b) From part (a), a solution of $y''=9y$ is $y(x)=A\sinh 3x+B\cosh 3x$. So $-4=y(0)=A\sinh 0+B\cosh 0=B$, so $B=-4$. Now $y'(x)=3A\cosh 3x-12\sinh 3x \Rightarrow 6=y'(0)=3A \Rightarrow A=2$, so $y=2\sinh 3x-4\cosh 3x$.

52. $\lim\limits_{x\to\infty} \dfrac{\sinh x}{e^x} = \lim\limits_{x\to\infty} \dfrac{e^x-e^{-x}}{2e^x} = \lim\limits_{x\to\infty} \dfrac{1-e^{-2x}}{2} = \dfrac{1-0}{2} = \dfrac{1}{2}$

53. The tangent to $y=\cosh x$ has slope 1 when $y'=\sinh x=1 \Rightarrow x=\sinh^{-1}1=\ln(1+\sqrt{2})$, by Equation 3. Since $\sinh x=1$ and $y=\cosh x=\sqrt{1+\sinh^2 x}$, we have $\cosh x=\sqrt{2}$. The point is $(\ln(1+\sqrt{2}), \sqrt{2})$.

54.

$$\cosh x = \cosh[\ln(\sec\theta+\tan\theta)] = \frac{1}{2}\left[e^{\ln(\sec\theta+\tan\theta)}+e^{-\ln(\sec\theta+\tan\theta)}\right]$$

$$= \frac{1}{2}\left[\sec\theta+\tan\theta+\frac{1}{\sec\theta+\tan\theta}\right] = \frac{1}{2}\left[\sec\theta+\tan\theta+\frac{\sec\theta-\tan\theta}{(\sec\theta+\tan\theta)(\sec\theta-\tan\theta)}\right]$$

$$= \frac{1}{2}\left[\sec\theta+\tan\theta+\frac{\sec\theta-\tan\theta}{\sec^2\theta-\tan^2\theta}\right] = \frac{1}{2}(\sec\theta+\tan\theta+\sec\theta-\tan\theta)=\sec\theta$$

55. If $ae^x+be^{-x}=\alpha\cosh(x+\beta)$ [or $\alpha\sinh(x+\beta)$], then $ae^x+be^{-x}=\dfrac{\alpha}{2}(e^{x+\beta}\pm e^{-x-\beta})=\dfrac{\alpha}{2}(e^x e^\beta \pm e^{-x}e^{-\beta})=\left(\dfrac{\alpha}{2}e^\beta\right)e^x \pm \left(\dfrac{\alpha}{2}e^{-\beta}\right)e^{-x}$. Comparing coefficients of e^x and e^{-x} , we have $a=\dfrac{\alpha}{2}e^\beta$ **(1)** and $b=\pm\dfrac{\alpha}{2}e^{-\beta}$ **(2)** . We need to find α and β . Dividing equation **(1)** by
equation **(2)** gives us $\dfrac{a}{b}=\pm e^{2\beta} \Rightarrow (*) 2\beta=\ln(\pm\dfrac{a}{b}) \Rightarrow \beta=\dfrac{1}{2}\ln\left(\pm\dfrac{a}{b}\right)$. Solving equations **(1)** and **(2)** for e^β gives us $e^\beta=\dfrac{2a}{\alpha}$ and $e^\beta=\pm\dfrac{\alpha}{2b}$, so $\dfrac{2a}{\alpha}=\pm\dfrac{\alpha}{2b} \Rightarrow \alpha^2=\pm 4ab \Rightarrow \alpha=2\sqrt{\pm ab}$.

(*) If $\dfrac{a}{b}>0$, we use the + sign and obtain a cosh function, whereas if $\dfrac{a}{b}<0$, we use the – sign and obtain a sinh function.

In summary, if a and b have the same sign, we have $ae^x+be^{-x}=2\sqrt{ab}\cosh\left(x+\dfrac{1}{2}\ln\dfrac{a}{b}\right)$, whereas, if a and b have the opposite sign, then $ae^x+be^{-x}=2\sqrt{-ab}\sinh\left(x+\dfrac{1}{2}\ln\left(-\dfrac{a}{b}\right)\right)$.

1. A function f has an **absolute minimum** at $x=c$ if $f(c)$ is the smallest function value on the entire domain of f, whereas f has a **local minimum** at c if $f(c)$ is the smallest function value when x is near c.

2. **(a)** The Extreme Value Theorem
(b) See the Closed Interval Method.

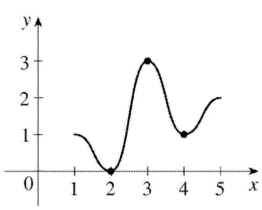
3. Absolute maximum at b; absolute minimum at d; local maxima at b and e; local minima at d and s;
neither a maximum nor a minimum at a, c, r, and t.

4. Absolute maximum at e; absolute minimum at t; local maxima at c, e, and s; local minima at b, c, d, and r;
neither a maximum nor a minimum at a.

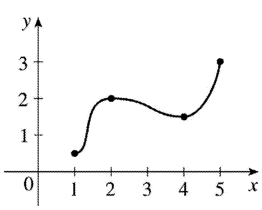
5. Absolute maximum value is $f(4)=4$; absolute minimum value is $f(7)=0$; local maximum values are $f(4)=4$ and $f(6)=3$; local minimum values are $f(2)=1$ and $f(5)=2$.

6. Absolute maximum value is $f(8)=5$; absolute minimum value is $f(2)=0$; local maximum values are $f(1)=2$, $f(4)=4$, and $f(6)=3$; local minimum values are $f(2)=0$, $f(5)=2$, and $f(7)=1$.

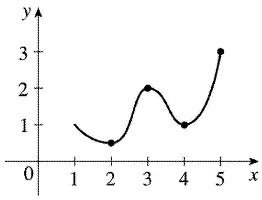
7. Absolute minimum at 2, absolute maximum at 3, local minimum at 4



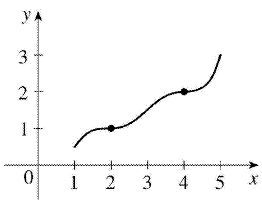
8. Absolute minimum at 1, absolute maximum at 5, local maximum at 2, local minimum at 4

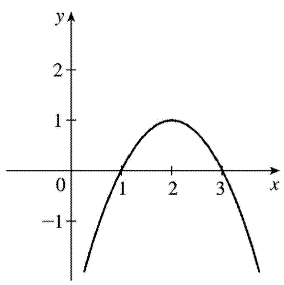


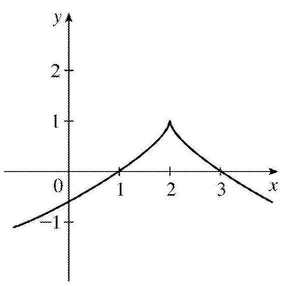
9. Absolute maximum at 5, absolute minimum at 2, local maximum at 3, local minima at 2 and 4

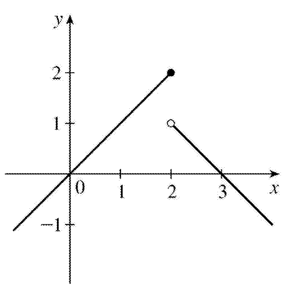


10. f has no local maximum or minimum, but 2 and 4 are critical numbers

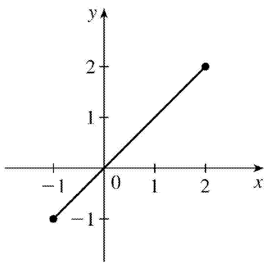


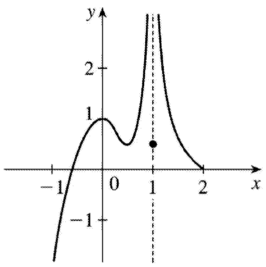
11. **(a)** 

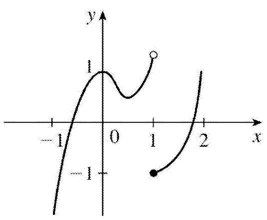
(b) 

(c) 

12. **(a)** Note that a local maximum cannot occur at an endpoint.

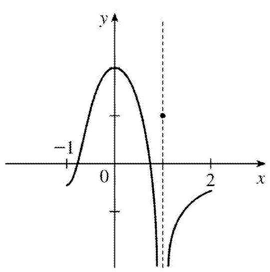


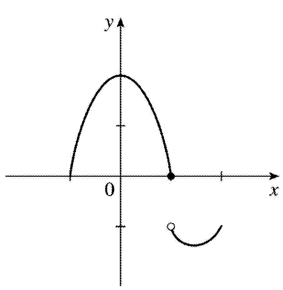
(b) 



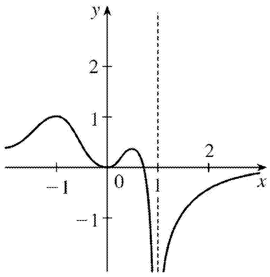
Note: By the Extreme Value Theorem, f must *not* be continuous.

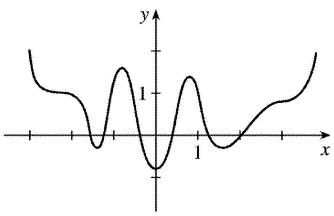
13. **(a)** *Note:* By the Extreme Value Theorem, f must *not* be continuous; because if it were, it would attain an absolute minimum.



(b) 

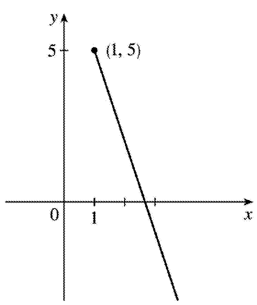
14. **(a)**



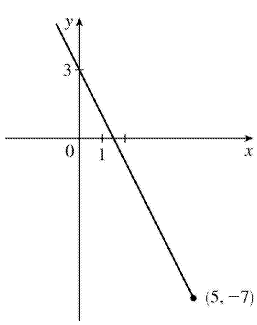


(b)

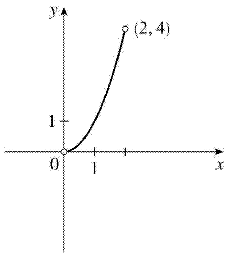
15. $f(x)=8-3x$, $x\geq 1$. Absolute maximum $f(1)=5$; no local maximum. No absolute or local minimum.



16. $f(x)=3-2x$, $x\leq 5$. Absolute minimum $f(5)=-7$; no local minimum. No absolute or local maximum.

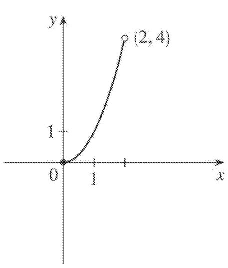


17. $f(x)=x^2$, $0<x<2$. No absolute or local maximum or minimum value.

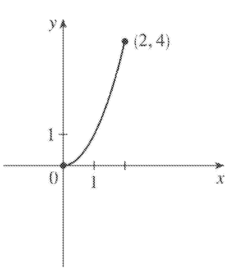


18. $f(x)=x^2$, $0<x\le 2$. Absolute maximum $f(2)=4$; no local maximum. No absolute or local minimum.

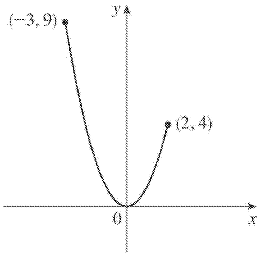
19. $f(x)=x^2$, $0\le x<2$. Absolute minimum $f(0)=0$; no local minimum. No absolute or local maximum.



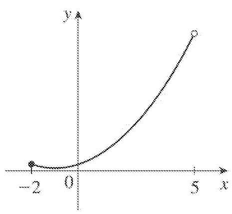
20. $f(x)=x^2$, $0\le x\le 2$. Absolute maximum $f(2)=4$. Absolute minimum $f(0)=0$. No local maximum or minimum.



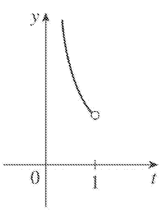
21. $f(x)=x^2$, $-3\le x\le 2$. Absolute maximum $f(-3)=9$. No local maximum. Absolute and local minimum $f(0)=0$.



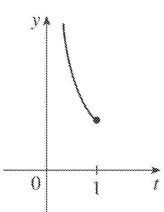
22. $f(x)=1+(x+1)^2$, $-2\le x<5$. No absolute or local maximum. Absolute and local minimum $f(-1)=1$.



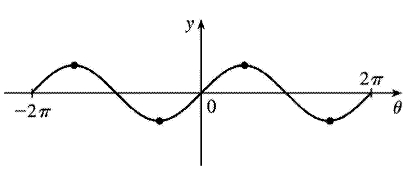
23. $f(t)=1/t$, $0<t<1$. No maximum or minimum.



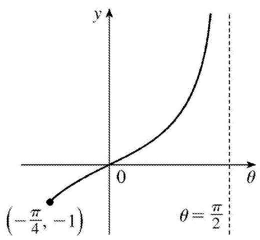
24. $f(t)=1/t$, $0<t\le 1$. Absolute minimum $f(1)=1$; no local minimum. No local or absolute maximum.



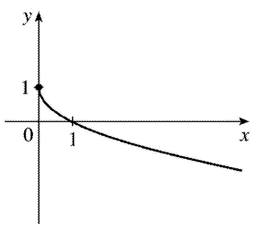
25. $f(\theta)=\sin\theta$, $-2\pi\le\theta\le 2\pi$. Absolute and local maxima $f\left(-\frac{3\pi}{2}\right)=f\left(\frac{\pi}{2}\right)=1$. Absolute and local minima $f\left(-\frac{\pi}{2}\right)=f\left(\frac{3\pi}{2}\right)=-1$.



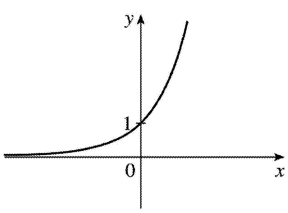
26. $f(\theta)=\tan\theta$, $-\frac{\pi}{4}\le\theta<\frac{\pi}{2}$. Absolute minimum $f\left(-\frac{\pi}{4}\right)=-1$; no local minimum. No absolute or local maximum.



27. $f(x)=1-\sqrt{x}$. Absolute maximum $f(0)=1$; no local maximum. No absolute or local minimum.

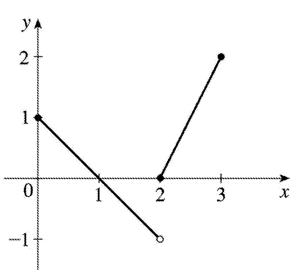


28. $f(x)=e^{x}$. No absolute or local maximum or minimum value.



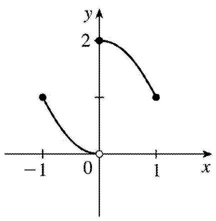
29. $f(x)=\begin{cases} 1-x & \text{if } 0\le x<2 \\ 2x-4 & \text{if } 2\le x\le 3 \end{cases}$

Absolute maximum $f(3)=2$; no local maximum. No absolute or local minimum.



30. $f(x)=\begin{cases} x^2 & \text{if} -1\le x<0 \\ 2-x^2 & \text{if} 0\le x\le 1 \end{cases}$

Absolute and local maximum $f(0)=2$.
No absolute or local minimum.



31. $f(x)=5x^2+4x \Rightarrow f'(x)=10x+4$. $f'(x)=0 \Rightarrow x=-\frac{2}{5}$, so $-\frac{2}{5}$ is the only critical number.

32. $f(x)=x^3+x^2-x \Rightarrow f'(x)=3x^2+2x-1$. $f'(x)=0 \Rightarrow (x+1)(3x-1)=0 \Rightarrow x=-1$, $\frac{1}{3}$. These are the only critical numbers.

33. $f(x)=x^3+3x^2-24x \Rightarrow f'(x)=3x^2+6x-24=3\left(x^2+2x-8\right)$.
$f'(x)=0 \Rightarrow 3(x+4)(x-2)=0 \Rightarrow x=-4$, 2 . These are the only critical numbers.
34. $f(x)=x^3+x^2+x \Rightarrow f'(x)=3x^2+2x+1$. $f'(x)=0 \Rightarrow 3x^2+2x+1=0 \Rightarrow x=\frac{-2\pm\sqrt{4-12}}{6}$. Neither of these is a real number. Thus, there are no critical numbers.

35. $s(t)=3t^4+4t^3-6t^2 \Rightarrow s'(t)=12t^3+12t^2-12t$. $s'(t)=0 \Rightarrow 12t\left(t^2+t-1\right) \Rightarrow t=0$ or $t^2+t-1=0$. Using the quadratic formula to solve the latter equation gives us $t=\frac{-1\pm\sqrt{1^2-4(1)(-1)}}{2(1)}=\frac{-1\pm\sqrt{5}}{2}\approx 0.618$, -1.618 . The three critical numbers are 0 , $\frac{-1\pm\sqrt{5}}{2}$.

36. $f(z)=\frac{z+1}{z^2+z+1} \Rightarrow f'(z)=\frac{\left(z^2+z+1\right)1-(z+1)(2z+1)}{\left(z^2+z+1\right)^2}=\frac{-z^2-2z}{\left(z^2+z+1\right)^2}=0 \Leftrightarrow z(z+2)=0 \Rightarrow z=0$, -2 are the critical numbers. (Note that $z^2+z+1\neq 0$ since the discriminant <0 .)

37. $g(x)=|2x+3|=\begin{cases} 2x+3 & \text{if } 2x+3\geq 0 \\ -(2x+3) & \text{if } 2x+3<0 \end{cases} \Rightarrow g'(x)=\begin{cases} 2 & \text{if } x>-\frac{3}{2} \\ -2 & \text{if } x<-\frac{3}{2} \end{cases}$ $g'(x)$ is never 0 , but $g'(x)$ does not exist for

$x=-\frac{3}{2}$, so $-\frac{3}{2}$ is the only critical number.

38. $g(x)=x^{1/3}-x^{-2/3} \Rightarrow g'(x)=\frac{1}{3}x^{-2/3}+\frac{2}{3}x^{-5/3}=\frac{1}{3}x^{-5/3}(x+2)=\frac{x+2}{3x^{5/3}}$.

$g'(-2)=0$ and $g'(0)$ does not exist, but 0 is not in the domain of g , so the only critical number is -2 .

39. $g(t)=5t^{2/3}+t^{5/3} \Rightarrow g'(t)=\frac{10}{3}t^{-1/3}+\frac{5}{3}t^{2/3}$. $g'(0)$ does not exist, so $t=0$ is a critical number.
$g'(t)=\frac{5}{3}t^{-1/3}(2+t)=0 \Leftrightarrow t=-2$, so $t=-2$ is also a critical number.

40. $g(t)=\sqrt{t}\,(1-t)=t^{1/2}-t^{3/2} \Rightarrow g'(t)=\frac{1}{2\sqrt{t}}-\frac{3}{2}\sqrt{t}$. $g'(0)$ does not exist, so $t=0$ is a critical number.
$0=g'(t)=\frac{1-3t}{2\sqrt{t}} \Rightarrow t=\frac{1}{3}$, so $t=\frac{1}{3}$ is also a critical number.

41. $F(x)=x^{4/5}(x-4)^2 \Rightarrow$

$$F'(x)=x^{4/5}\cdot 2(x-4)+(x-4)^2\cdot\frac{4}{5}x^{-1/5}=\frac{1}{5}x^{-1/5}(x-4)[5\cdot x\cdot 2+(x-4)\cdot 4]$$

$$=\frac{(x-4)(14x-16)}{5x^{1/5}}=\frac{2(x-4)(7x-8)}{5x^{1/5}}=0 \text{ when } x=4,\ \frac{8}{7}\text{ ; and } F'(0) \text{ does not exist.}$$

Critical numbers are 0 , $\frac{8}{7}$, 4 .

42. $G(x)=\sqrt[3]{x^2-x} \Rightarrow G'(x)=\frac{1}{3}\left(x^2-x\right)^{-2/3}(2x-1)$. $G'(x)$ does not exist when $x^2-x=0$, that is, when $x=0$ or 1 . $G'(x)=0 \Leftrightarrow 2x-1=0 \Leftrightarrow x=\frac{1}{2}$. So the critical numbers are $x=0$, $\frac{1}{2}$, 1 .

43. $f(\theta)=2\cos\theta+\sin^2\theta \Rightarrow f'(\theta)=-2\sin\theta+2\sin\theta\cos\theta$. $f'(\theta)=0 \Rightarrow 2\sin\theta(\cos\theta-1)=0 \Rightarrow \sin\theta=0$ or $\cos\theta=1 \Rightarrow \theta=n\pi$ (n an integer) or $\theta=2n\pi$. The solutions $\theta=n\pi$ include the solutions $\theta=2n\pi$, so the critical numbers are $\theta=n\pi$.

44. $g(\theta)=4\theta-\tan\theta \Rightarrow g'(\theta)=4-\sec^2\theta$. $g'(\theta)=0 \Rightarrow \sec^2\theta=4 \Rightarrow \sec\theta=\pm 2 \Rightarrow \cos\theta=\pm\frac{1}{2} \Rightarrow \theta=\frac{\pi}{3}+2n\pi$,

$\frac{5\pi}{3}+2n\pi$, $\frac{2\pi}{3}+2n\pi$, and $\frac{4\pi}{3}+2n\pi$ are critical numbers.

Note: The values of θ that make $g'(\theta)$ undefined are not in the domain of g .

45. $f(x)=x\ln x \Rightarrow f'(x)=x(1/x)+(\ln x)\cdot 1=\ln x+1$. $f'(x)=0 \Leftrightarrow \ln x=-1 \Leftrightarrow x=e^{-1}=1/e$. Therefore, the only critical number is $x=1/e$.

46. $f(x)=xe^{2x} \Rightarrow f'(x)=x(2e^{2x})+e^{2x}=e^{2x}(2x+1)$. Since e^{2x} is never 0 , we have $f'(x)=0$ only when $2x+1=0 \Leftrightarrow x=-\frac{1}{2}$. So $-\frac{1}{2}$ is the only critical number.

47. $f(x)=3x^2-12x+5$, $[0,3]$. $f'(x)=6x-12=0 \Leftrightarrow x=2$. Applying the Closed Interval Method, we find that $f(0)=5$, $f(2)=-7$, and $f(3)=-4$. So $f(0)=5$ is the absolute maximum value and $f(2)=-7$ is the absolute minimum value.

48. $f(x)=x^3-3x+1$, $[0,3]$. $f'(x)=3x^2-3=0 \Leftrightarrow x=\pm 1$, but -1 is not in $[0,3]$. $f(0)=1$, $f(1)=-1$, and $f(3)=19$. So $f(3)=19$ is the absolute maximum value and $f(1)=-1$ is the absolute minimum value.

49. $f(x)=2x^3-3x^2-12x+1$, $[-2,3]$. $f'(x)=6x^2-6x-12=6\left(x^2-x-2\right)=6(x-2)(x+1)=0 \Leftrightarrow x=2,-1$. $f(-2)=-3$, $f(-1)=8$, $f(2)=-19$, and $f(3)=-8$. So $f(-1)=8$ is the absolute maximum value and $f(2)=-19$ is the absolute minimum value.

50. $f(x)=x^3-6x^2+9x+2$, $[-1,4]$. $f'(x)=3x^2-12x+9=3\left(x^2-4x+3\right)=3(x-1)(x-3)=0 \Leftrightarrow x=1,3$. $f(-1)=-14$, $f(1)=6$, $f(3)=2$, and $f(4)=6$. So $f(1)=f(4)=6$ is the absolute maximum value and $f(-1)=-14$ is the absolute minimum value.

51. $f(x)=x^4-2x^2+3$, $[-2,3]$. $f'(x)=4x^3-4x=4x\left(x^2-1\right)=4x(x+1)(x-1)=0 \Leftrightarrow x=-1$, 0 , 1 . $f(-2)=11$, $f(-1)=2$, $f(0)=3$, $f(1)=2$, $f(3)=66$. So $f(3)=66$ is the absolute maximum value and $f(\pm 1)=2$ is the absolute minimum value.

52. $f(x)=\left(x^2-1\right)^3$, $[-1,2]$. $f'(x)=3\left(x^2-1\right)^2(2x)=6x(x+1)^2(x-1)^2=0 \Leftrightarrow x=-1,0,1$. $f(\pm 1)=0$, $f(0)=-1$, and $f(2)=27$. So $f(2)=27$ is the absolute maximum value and $f(0)=-1$ is the absolute minimum value.

53. $f(x)=\frac{x}{x^2+1}$, $[0,2]$. $f'(x)=\frac{\left(x^2+1\right)-x(2x)}{\left(x^2+1\right)^2}=\frac{1-x^2}{\left(x^2+1\right)^2}=0 \Leftrightarrow x=\pm 1$, but -1 is not in $[0,2]$. $f(0)=0$, $f(1)=\frac{1}{2}$, $f(2)=\frac{2}{5}$. So $f(1)=\frac{1}{2}$ is the absolute maximum value and $f(0)=0$ is the absolute

minimum value.

54. $f(x)=\dfrac{x^2-4}{x^2+4}$, $[-4,4]$. $f'(x)=\dfrac{\left(x^2+4\right)(2x)-\left(x^2-4\right)(2x)}{\left(x^2+4\right)^2}=\dfrac{16x}{\left(x^2+4\right)^2}=0\Leftrightarrow x=0$. $f(\pm 4)=\dfrac{12}{20}=\dfrac{3}{5}$ and $f(0)=-1$. So $f(\pm 4)=\dfrac{3}{5}$ is the absolute maximum value and $f(0)=-1$ is the absolute minimum value.

55. $f(t)=t\sqrt{4-t^2}$, $[-1,2]$.
$f'(t)=t\cdot\dfrac{1}{2}\left(4-t^2\right)^{-1/2}(-2t)+\left(4-t^2\right)^{1/2}\cdot 1=\dfrac{-t^2}{\sqrt{4-t^2}}+\sqrt{4-t^2}=\dfrac{-t^2+\left(4-t^2\right)}{\sqrt{4-t^2}}=\dfrac{4-2t^2}{\sqrt{4-t^2}}$. $f'(t)=0\Rightarrow$ $4-2t^2=0\Rightarrow t^2=2\Rightarrow t=\pm\sqrt{2}$, but $t=-\sqrt{2}$ is not in the given interval, $[-1,2]$. $f'(t)$ does not exist if $4-t^2=0\Rightarrow t=\pm 2$, but -2 is not in the given interval. $f(-1)=-\sqrt{3}$, $f\left(\sqrt{2}\right)=2$, and $f(2)=0$. So $f\left(\sqrt{2}\right)=2$ is the absolute maximum value and $f(-1)=-\sqrt{3}$ is the absolute minimum value.

56. $f(t)=\sqrt[3]{t}\,(8-t)$, $[0,8]$. $f(t)=8t^{1/3}-t^{4/3}\Rightarrow f'(t)=\dfrac{8}{3}t^{-2/3}-\dfrac{4}{3}t^{1/3}=\dfrac{4}{3}t^{-2/3}(2-t)=\dfrac{4(2-t)}{3\sqrt[3]{t^2}}$. $f'(t)=0\Rightarrow t=2$. $f'(t)$ does not exist if $t=0$. $f(0)=0$, $f(2)=6\sqrt[3]{2}\approx 7.56$, and $f(8)=0$.
So $f(2)=6\sqrt[3]{2}$ is the absolute maximum value and $f(0)=f(8)=0$ is the absolute minimum value.

57. $f(x)=\sin x+\cos x$, $\left[0,\dfrac{\pi}{3}\right]$. $f'(x)=\cos x-\sin x=0\Leftrightarrow\sin x=\cos x\Rightarrow\dfrac{\sin x}{\cos x}=1\Rightarrow\tan x=1\Rightarrow$ $x=\dfrac{\pi}{4}$. $f(0)=1$, $f\left(\dfrac{\pi}{4}\right)=\sqrt{2}\approx 1.41$, $f\left(\dfrac{\pi}{3}\right)=\dfrac{\sqrt{3}+1}{2}\approx 1.37$. So $f\left(\dfrac{\pi}{4}\right)=\sqrt{2}$ is the absolute maximum value and $f(0)=1$ is the absolute minimum value.

58. $f(x)=x-2\cos x$, $[-\pi,\pi]$. $f'(x)=1+2\sin x=0\Leftrightarrow\sin x=-\dfrac{1}{2}\Leftrightarrow x=-\dfrac{5\pi}{6}$, $-\dfrac{\pi}{6}$. $f(-\pi)=2-\pi\approx -1.14$, $f\left(-\dfrac{5\pi}{6}\right)=\sqrt{3}-\dfrac{5\pi}{6}\approx -0.886$, $f\left(-\dfrac{\pi}{6}\right)=-\dfrac{\pi}{6}-\sqrt{3}\approx -2.26$, $f(\pi)=\pi+2\approx 5.14$. So $f(\pi)=\pi+2$ is the absolute maximum value and $f\left(-\dfrac{\pi}{6}\right)=-\dfrac{\pi}{6}-\sqrt{3}$ is the absolute minimum value.

59. $f(x)=xe^{-x}$, $[0,2]$. $f'(x)=x(-e^{-x})+e^{-x}=e^{-x}(1-x)=0\Leftrightarrow x=1$.
$f(0)=0$, $f(1)=e^{-1}=1/e\approx 0.37$, $f(2)=2/e^2\approx 0.27$. So $f(1)=1/e$ is the absolute maximum value and $f(0)=0$ is the absolute minimum value.

60. $f(x)=\dfrac{\ln x}{x}$, $[1,3]$. $f'(x)=\dfrac{x(1/x)-\ln x}{x^2}=\dfrac{1-\ln x}{x^2}=0 \Leftrightarrow 1-\ln x=0 \Leftrightarrow \ln x=1 \Leftrightarrow x=e$. $f(1)=0/1=0$, $f(e)=1/e\approx 0.368$, $f(3)=(\ln 3)/3\approx 0.366$. So $f(e)=1/e$ is the absolute maximum value and $f(1)=0$ is the absolute minimum value.

61. $f(x)=x-3\ln x$, $[1,4]$. $f'(x)=1-\dfrac{3}{x}=\dfrac{x-3}{x}=0 \Leftrightarrow x=3$. f' does not exist for $x=0$, but 0 is not in the domain of f . $f(1)=1$, $f(3)=3-3\ln 3\approx -0.296$, $f(4)=4-3\ln 4\approx -0.159$. So $f(1)=1$ is the absolute maximum value and $f(3)=3-3\ln 3\approx -0.296$ is the absolute minimum value.

62. $f(x)=e^{-x}-e^{-2x}$, $[0,1]$. $f'(x)=e^{-x}(-1)-e^{-2x}(-2)=\dfrac{2}{e^{2x}}-\dfrac{1}{e^{x}}=\dfrac{2-e^{x}}{e^{2x}}=0 \Leftrightarrow e^{x}=2 \Leftrightarrow x=\ln 2\approx 0.69$. $f(0)=0$, $f(\ln 2)=e^{-\ln 2}-e^{-2\ln 2}=\left(e^{\ln 2}\right)^{-1}-\left(e^{\ln 2}\right)^{-2}=2^{-1}-2^{-2}=\dfrac{1}{2}-\dfrac{1}{4}=\dfrac{1}{4}$, $f(1)=e^{-1}-e^{-2}\approx 0.233$. So $f(\ln 2)=\dfrac{1}{4}$ is the absolute maximum value and $f(0)=0$ is the absolute minimum value.

63. $f(x)=x^{a}(1-x)^{b}$, $0\le x\le 1$, $a>0$, $b>0$.

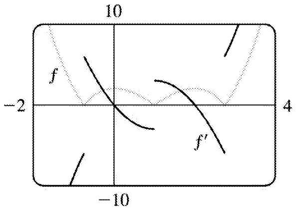
$$\begin{aligned} f'(x) &= x^{a}\cdot b(1-x)^{b-1}(-1)+(1-x)^{b}\cdot ax^{a-1}=x^{a-1}(1-x)^{b-1}[x\cdot b(-1)+(1-x)\cdot a] \\ &= x^{a-1}(1-x)^{b-1}(a-ax-bx) \end{aligned}$$

At the endpoints, we have $f(0)=f(1)=0$ [the minimum value of f]. In the interval $(0,1)$, $f'(x)=0 \Leftrightarrow x=\dfrac{a}{a+b}$.

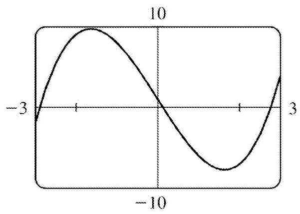
$$f\left(\frac{a}{a+b}\right)=\left(\frac{a}{a+b}\right)^{a}\left(1-\frac{a}{a+b}\right)^{b}=\frac{a^{a}}{(a+b)^{a}}\left(\frac{a+b-a}{a+b}\right)^{b}=\frac{a^{a}}{(a+b)^{a}}\cdot\frac{b^{b}}{(a+b)^{b}}=\frac{a^{a}b^{b}}{(a+b)^{a+b}} .$$

So $f\left(\dfrac{a}{a+b}\right)=\dfrac{a^{a}b^{b}}{(a+b)^{a+b}}$ is the absolute maximum value.

64.



We see that $f'(x)=0$ at about $x=0.0$ and 2.0 , and that $f'(x)$ does not exist at about $x=-0.7$, 1.0 , and 2.7 , so the critical numbers of f are about -0.7 , 0.0 , 1.0 , 2.0 , and 2.7 .

65. **(a)** 

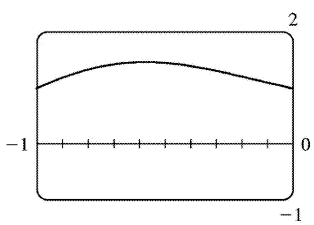
From the graph, it appears that the absolute maximum value is about $f(-1.63)=9.71$, and the absolute minimum value is about $f(1.63)=-7.71$. These values make sense because the graph is symmetric about the point $(0,1)$. ($y=x^3-8x$ is symmetric about the origin.)

(b) $f(x)=x^3-8x+1 \Rightarrow f'(x)=3x^2-8$. So $f'(x)=0 \Rightarrow x=\pm\sqrt{\frac{8}{3}}$.

$$f\left(\pm\sqrt{\frac{8}{3}}\right)=\left(\pm\sqrt{\frac{8}{3}}\right)^3-8\left(\pm\sqrt{\frac{8}{3}}\right)+1=\pm\frac{8}{3}\sqrt{\frac{8}{3}}\mp 8\sqrt{\frac{8}{3}}+1$$

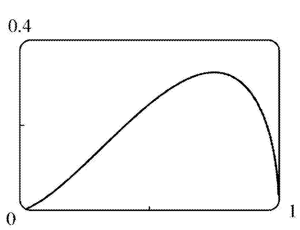
$$=-\frac{16}{3}\sqrt{\frac{8}{3}}+1=1-\frac{32\sqrt{6}}{9} \text{ or } \frac{16}{3}\sqrt{\frac{8}{3}}+1=1+\frac{32\sqrt{6}}{9}$$

(From the graph, we see that the extreme values do not occur at the endpoints.)

66. **(a)** 

From the graph, it appears that the absolute maximum value is about $f(-0.58)=1.47$, and the absolute minimum value is about $f(-1)=f(0)=1.00$; that is, at both endpoints.

(b) $f(x)=e^{x^3-x} \Rightarrow f'(x)=e^{x^3-x}\left(3x^2-1\right)$. So $f'(x)=0$ on $[-1,0] \Rightarrow x=-\sqrt{1/3}$. $f(-1)=f(0)=1$ (minima) and $f\left(-\sqrt{1/3}\right)=e^{-\sqrt{3}/9+\sqrt{3}/3}=e^{2\sqrt{3}/9}$ (maximum).

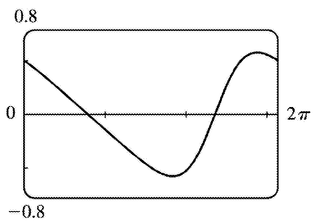
67. **(a)** 

From the graph, it appears that the absolute maximum value is about $f(0.75)=0.32$, and the absolute minimum value is $f(0)=f(1)=0$; that is, at both endpoints.

(b) $f(x)=x\sqrt{x-x^2} \Rightarrow f'(x)=x\cdot \dfrac{1-2x}{2\sqrt{x-x^2}}+\sqrt{x-x^2}=\dfrac{\left(x-2x^2\right)+\left(2x-2x^2\right)}{2\sqrt{x-x^2}}=\dfrac{3x-4x^2}{2\sqrt{x-x^2}}$. So $f'(x)=0\Rightarrow$

$3x-4x^2=0\Rightarrow x(3-4x)=0\Rightarrow x=0$ or $\dfrac{3}{4}$. $f(0)=f(1)=0$,

and $f\left(\dfrac{3}{4}\right)=\dfrac{3}{4}\sqrt{\dfrac{3}{4}-\left(\dfrac{3}{4}\right)^2}=\dfrac{3\sqrt{3}}{16}$.

68. **(a)** 

From the graph, it appears that the absolute maximum value is about $f(5.76)=0.58$, and the absolute minimum value is about $f(3.67)=-0.58$.

(b) $f(x)=\dfrac{\cos x}{2+\sin x}\Rightarrow f'(x)=\dfrac{(2+\sin x)(-\sin x)-(\cos x)(\cos x)}{(2+\sin x)^2}=\dfrac{-1-2\sin x}{(2+\sin x)^2}$.

So $f'(x)=0\Rightarrow \sin x=-\dfrac{1}{2}\Rightarrow x=\dfrac{7\pi}{6}$ or $\dfrac{11\pi}{6}$. Now $f\left(\dfrac{7\pi}{6}\right)=\dfrac{-\sqrt{3}/2}{3/2}=-\dfrac{1}{\sqrt{3}}$,

and $f\left(\dfrac{11\pi}{6}\right)=\dfrac{\sqrt{3}/2}{3/2}=\dfrac{1}{\sqrt{3}}$.

69. The density is defined as $\rho=\dfrac{\text{mass}}{\text{volume}}=\dfrac{1000}{V(T)}$ (in g / cm ^{3}). But a critical point of ρ will also be a critical point of V since [$\dfrac{d\rho}{dT}=-1000V^{-2}\dfrac{dV}{dT}$ and V is never 0], and V is easier to differentiate than ρ .

$V(T)=999.87-0.06426T+0.0085043T^2-0.0000679T^3\Rightarrow$

$V'(T)=-0.06426+0.0170086T-0.0002037T^2$. Setting this equal to 0 and using the quadratic formula to

find T , we get $T=\dfrac{-0.0170086\pm\sqrt{0.0170086^2-4\cdot 0.0002037\cdot 0.06426}}{2(-0.0002037)}\approx 3.9665^\circ$ C or 79.5318° C.

Since we are only interested in the region 0° C $\le T\le 30^\circ$ C, we check the density ρ at the endpoints and at 3.9665° C: $\rho(0)\approx\dfrac{1000}{999.87}\approx 1.00013$; $\rho(30)\approx\dfrac{1000}{1003.7628}\approx 0.99625$;

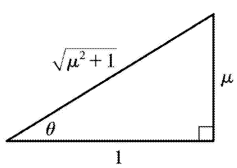
$\rho(3.9665) \approx \dfrac{1000}{999.7447} \approx 1.000255$. So water has its maximum density at about 3.9665° C.

70. $F=\dfrac{\mu W}{\mu \sin\theta+\cos\theta} \Rightarrow \dfrac{dF}{d\theta}=\dfrac{(\mu \sin\theta+\cos\theta)(0)-\mu W(\mu \cos\theta-\sin\theta)}{(\mu \sin\theta+\cos\theta)^2}=\dfrac{-\mu W(\mu \cos\theta-\sin\theta)}{(\mu \sin\theta+\cos\theta)^2}$.

So $\dfrac{dF}{d\theta}=0 \Rightarrow \mu \cos\theta-\sin\theta=0 \Rightarrow \mu=\dfrac{\sin\theta}{\cos\theta}=\tan\theta$. Substituting $\tan\theta$ for μ in F gives us

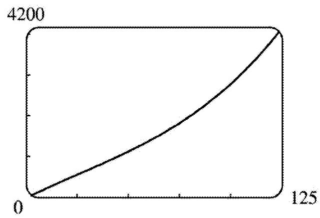
$F=\dfrac{(\tan\theta)W}{(\tan\theta)\sin\theta+\cos\theta}=\dfrac{W\tan\theta}{\dfrac{\sin^2\theta}{\cos\theta}+\cos\theta}=\dfrac{W\tan\theta\cos\theta}{\sin^2\theta+\cos^2\theta}=\dfrac{W\sin\theta}{1}=W\sin\theta$.

If $\tan\theta=\mu$, then $\sin\theta=\dfrac{\mu}{\sqrt{\mu^2+1}}$ (see the figure), so $F=\dfrac{\mu}{\sqrt{\mu^2+1}}W$. We compare this with the value of F at the endpoints: $F(0)=\mu W$ and $F\left(\dfrac{\pi}{2}\right)=W$. Now because $\dfrac{\mu}{\sqrt{\mu^2+1}} \le 1$ and $\dfrac{\mu}{\sqrt{\mu^2+1}} \le \mu$, we have that $\dfrac{\mu}{\sqrt{\mu^2+1}}W$



is less than or equal to each of $F(0)$ and $F\left(\dfrac{\pi}{2}\right)$. Hence, $\dfrac{\mu}{\sqrt{\mu^2+1}}W$ is the absolute minimum value of $F(\theta)$, and it occurs when $\tan\theta=\mu$.

71. We apply the Closed Interval Method to the continuous function $I(t)=0.00009045t^5+0.001438t^4-0.06561t^3+0.4598t^2-0.6270t+99.33$ on $[0,10]$. Its derivative is $I'(t)=0.00045225t^4+0.005752t^3-0.19683t^2+0.9196t-0.6270$. Since I' exists for all t , the only critical numbers of I occur when $I'(t)=0$. We use a root-finder on a computer algebra system (or a graphing device) to find that $I'(t)=0$ when $t \approx -29.7186$, 0.8231 , 5.1309 , or 11.0459 , but only the second and third roots lie in the interval $[0,10]$. The values of I at these critical numbers are $I(0.8231) \approx 99.09$ and $I(5.1309) \approx 100.67$. The values of I at the endpoints of the interval are $I(0)=99.33$ and $I(10) \approx 96.86$. Comparing these four numbers, we see that food was most expensive at $t \approx 5.1309$ (corresponding roughly to August, 1989) and cheapest at $t=10$ (midyear 1994).

72. **(a)** 

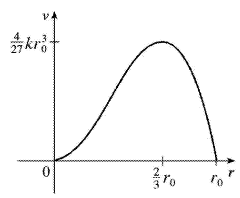
The equation of the graph in the figure is

$v(t)=0.00146t^3-0.11553t^2+24.98169t-21.26872$.

(b) $a(t)=v'(t)=0.00438t^2-0.23106t+24.98169 \Rightarrow a'(t)=0.00876t-0.23106$. $a'(t)=0 \Rightarrow$ $t_1=\dfrac{0.23106}{0.00876}\approx 26.4$. $a(0)\approx 24.98$, $a(t_1)\approx 21.93$, and $a(125)\approx 64.54$. The maximum acceleration is about 64.5 ft / s ^{2} and the minimum acceleration is about 21.93 ft / s ^{2} .

73. **(a)** $v(r)=k(r_0-r)r^2=kr_0r^2-kr^3 \Rightarrow v'(r)=2kr_0r-3kr^2$. $v'(r)=0 \Rightarrow kr(2r_0-3r)=0 \Rightarrow r=0$ or $\frac{2}{3}r_0$ (but 0 is not in the interval). Evaluating v at $\frac{1}{2}r_0$, $\frac{2}{3}r_0$, and r_0 , we get $v\left(\frac{1}{2}r_0\right)=\frac{1}{8}kr_0^3$, $v\left(\frac{2}{3}r_0\right)=\frac{4}{27}kr_0^3$, and $v(r_0)=0$. Since $\frac{4}{27}>\frac{1}{8}$, v attains its maximum value at $r=\frac{2}{3}r_0$. This supports the statement in the text.

(b) From part (a), the maximum value of v is $\frac{4}{27}kr_0^3$.

(c) 

74. $g(x)=2+(x-5)^3 \Rightarrow g'(x)=3(x-5)^2 \Rightarrow g'(5)=0$, so 5 is a critical number. But $g(5)=2$ and g takes on values >2 and values <2 in any open interval containing 5 , so g does not have a local maximum or minimum at 5 .

75. $f(x)=x^{101}+x^{51}+x+1 \Rightarrow f'(x)=101x^{100}+51x^{50}+1\geq 1$ for all x , so $f'(x)=0$ has no solution. Thus, $f(x)$ has no critical number, so $f(x)$ can have no local maximum or minimum.

76. Suppose that f has a minimum value at c , so $f(x)\geq f(c)$ for all x near c . Then $g(x)=-f(x)\leq -f(c)=g(c)$ for all x near c , so $g(x)$ has a maximum value at c .

77. If f has a local minimum at c , then $g(x)=-f(x)$ has a local maximum at c , so $g'(c)=0$ by the case

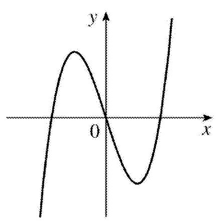
of Fermat's Theorem proved in the text. Thus, $f'(c)=-g'(c)=0$.

78.

(a) $f(x)=ax^3+bx^2+cx+d$, $a\neq 0$. So $f'(x)=3ax^2+2bx+c$ is a quadratic and hence has either 2 , 1 , or 0 real roots, so $f(x)$ has either 2 , 1 or 0 critical numbers.

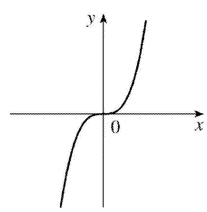
Case (i) (2 critical numbers):

$f(x)=x^3-3x\Rightarrow$
$f'(x)=3x^2-3$, so $x=-1,1$
are critical numbers.



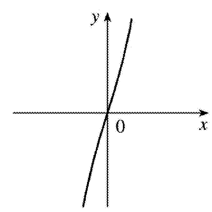
Case (ii) (1 critical number):

$f(x)=x \Rightarrow$
$f'(x)=3x$, *so* $x=0$
is the only critical number.



Case (iii) (no critical number):

$f(x)=x^3+3x\Rightarrow$
$f'(x)=3x^2+3$,
so there are no real roots.



(b) Since there are at most two critical numbers, it can have at most two local extreme values and by (i) this can occur. By (iii) it can have no local extreme value. However, if there is only one critical number, then there is no local extreme value.

1. $f(x)=6x^2-8x+3 \Rightarrow F(x)=6\dfrac{x^{2+1}}{2+1}-8\dfrac{x^{1+1}}{1+1}+3x+C=2x^3-4x^2+3x+C$

Check: $F^{\prime}(x)=2\cdot 3x^2-4\cdot 2x+3+0=6x^2-8x+3=f(x)$

2. $f(x)=4+x^2-5x^3 \Rightarrow F(x)=4x+\dfrac{1}{3}x^3-\dfrac{5}{4}x^4+C$

3. $f(x)=1-x^3+5x^5-3x^7 \Rightarrow F(x)=x-\dfrac{x^{3+1}}{3+1}+5\dfrac{x^{5+1}}{5+1}-3\dfrac{x^{7+1}}{7+1}+C=x-\dfrac{1}{4}x^4+\dfrac{5}{6}x^6-\dfrac{3}{8}x^8+C$

4. $f(x)=x^{20}+4x^{10}+8 \Rightarrow F(x)=\dfrac{1}{21}x^{21}+\dfrac{4}{11}x^{11}+8x+C$

5. $f(x)=5x^{1/4}-7x^{3/4} \Rightarrow F(x)=5\dfrac{x^{1/4+1}}{\frac{1}{4}+1}-7\dfrac{x^{3/4+1}}{\frac{3}{4}+1}+C=5\dfrac{x^{5/4}}{5/4}-7\dfrac{x^{7/4}}{7/4}+C=4x^{5/4}-4x^{7/4}+C$

6. $f(x)=2x+3x^{1.7} \Rightarrow F(x)=x^2+\dfrac{3}{2.7}x^{2.7}+C=x^2+\dfrac{10}{9}x^{2.7}+C$

7. $f(x)=6\sqrt{x}-\sqrt[6]{x}=6x^{1/2}-x^{1/6} \Rightarrow$

$F(x)=6\dfrac{x^{1/2+1}}{\frac{1}{2}+1}-\dfrac{x^{1/6+1}}{\frac{1}{6}+1}+C=6\dfrac{x^{3/2}}{3/2}-\dfrac{x^{7/6}}{7/6}+C=4x^{3/2}-\dfrac{6}{7}x^{7/6}+C$

8. $f(x)=\sqrt[4]{x^3}+\sqrt[3]{x^4}=x^{3/4}+x^{4/3} \Rightarrow F(x)=\dfrac{x^{7/4}}{7/4}+\dfrac{x^{7/3}}{7/3}+C=\dfrac{4}{7}x^{7/4}+\dfrac{3}{7}x^{7/3}+C$

9. $f(x)=\dfrac{10}{x^9}=10x^{-9}$ has domain $(-\infty,0)\cup(0,\infty)$, so

$$F(x)=\begin{cases} \dfrac{10x^{-8}}{-8}+C_1=-\dfrac{5}{4x^8}+C_1 & \text{if } x<0 \\ -\dfrac{5}{4x^8}+C_2 & \text{if } x>0 \end{cases}$$

See Example 1 for a similar problem.

10.

$g(x)=\dfrac{5-4x^3+2x^6}{x^6}=5x^{-6}-4x^{-3}+2$ has domain $(-\infty,0)\cup(0,\infty)$, so

$$G(x)=\begin{cases} 5\dfrac{x^{-5}}{-5}-4\dfrac{x^{-2}}{-2}+2x+C_1=-\dfrac{1}{x^5}+\dfrac{2}{x^2}+2x+C_1 & \text{if } x<0 \\ -\dfrac{1}{x^5}+\dfrac{2}{x^2}+2x+C_2 & \text{if } x>0 \end{cases}$$

11. $f(u)=\dfrac{u^4+3\sqrt{u}}{u^2}=\dfrac{u^4}{u^2}+\dfrac{3u^{1/2}}{u^2}=u^2+3u^{-3/2}\Rightarrow$

$F(u)=\dfrac{u^3}{3}+3\dfrac{u^{-3/2+1}}{-3/2+1}+C=\dfrac{1}{3}u^3+3\dfrac{u^{-1/2}}{-1/2}+C=\dfrac{1}{3}u^3-\dfrac{6}{\sqrt{u}}+C$

12. $f(x)=3e^x+7\sec^2 x\Rightarrow F(x)=3e^x+7\tan x+C_n$ on the interval $\left(n\pi-\dfrac{\pi}{2},n\pi+\dfrac{\pi}{2}\right)$.

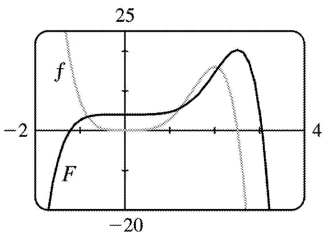
13. $g(\theta)=\cos\theta-5\sin\theta\Rightarrow G(\theta)=\sin\theta-5(-\cos\theta)+C=\sin\theta+5\cos\theta+C$

14. $h(\theta)=\dfrac{\sin\theta}{\cos^2\theta}=\dfrac{1}{\cos\theta}\cdot\dfrac{\sin\theta}{\cos\theta}=\sec\theta\tan\theta\Rightarrow H(\theta)=\sec\theta+C_n$ on the interval $\left(n\pi-\dfrac{\pi}{2},n\pi+\dfrac{\pi}{2}\right)$.

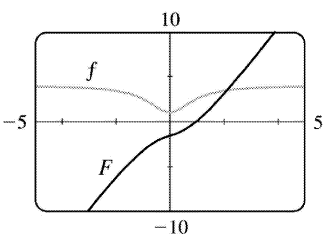
15. $f(x)=2x+5\left(1-x^2\right)^{-1/2}=2x+\dfrac{5}{\sqrt{1-x^2}}\Rightarrow F(x)=x^2+5\sin^{-1}x+C$

16. $f(x)=\dfrac{x^2+x+1}{x}=x+1+\dfrac{1}{x}\Rightarrow F(x)=\begin{cases}\dfrac{1}{2}x^2+x+\ln|x|+C_1 & \text{if } x<0\\ \dfrac{1}{2}x^2+x+\ln|x|+C_2 & \text{if } x>0\end{cases}$

17. $f(x)=5x^4-2x^5\Rightarrow F(x)=5\cdot\dfrac{x^5}{5}-2\cdot\dfrac{x^6}{6}+C=x^5-\dfrac{1}{3}x^6+C$. $F(0)=4\Rightarrow 0^5-\dfrac{1}{3}\cdot 0^6+C=4\Rightarrow C=4$, so $F(x)=x^5-\dfrac{1}{3}x^6+4$. The graph confirms our answer since $f(x)=0$ when F has a local maximum, f is positive when F is increasing, and f is negative when F is decreasing.



18. $f(x)=4-3\left(1+x^2\right)^{-1}=4-\dfrac{3}{1+x^2}\Rightarrow F(x)=4x-3\tan^{-1}x+C$. $F(1)=0\Rightarrow 4-3\left(\dfrac{\pi}{4}\right)+C=0\Rightarrow C=\dfrac{3\pi}{4}-4$, so $F(x)=4x-3\tan^{-1}x+\dfrac{3\pi}{4}-4$. Note that f is positive and F is increasing on R . Also, f has smaller values where the slopes of the tangent lines of F are smaller.



19. $f''(x)=6x+12x^2\Rightarrow f'(x)=6\cdot\dfrac{x^2}{2}+12\cdot\dfrac{x^3}{3}+C=3x^2+4x^3+C\Rightarrow$
$f(x)=3\cdot\dfrac{x^3}{3}+4\cdot\dfrac{x^4}{4}+Cx+D=x^3+x^4+Cx+D$

20. $f''(x)=2+x^3+x^6\Rightarrow f'(x)=2x+\dfrac{1}{4}x^4+\dfrac{1}{7}x^7+C\Rightarrow\ f(x)=x^2+\dfrac{1}{20}x^5+\dfrac{1}{56}x^8+Cx+D$

21. $f''(x)=1+x^{4/5}\Rightarrow f'(x)=x+\dfrac{5}{9}x^{9/5}+C\Rightarrow\ f(x)=\dfrac{1}{2}x^2+\dfrac{5}{9}\cdot\dfrac{5}{14}x^{14/5}+Cx+D=\dfrac{1}{2}x^2+\dfrac{25}{126}x^{14/5}+Cx+D$

22. $f''(x)=\cos x\Rightarrow f'(x)=\sin x+C\Rightarrow\ f(x)=-\cos x+Cx+D$

23. $f'''(t)=60t^2\Rightarrow f''(t)=20t^3+C\Rightarrow f'(t)=5t^4+Ct+D\Rightarrow\ f(t)=t^5+\dfrac{1}{2}Ct^2+Dt+E$

24. $f'''(t)=t-\sqrt{t}\Rightarrow f''(t)=\dfrac{1}{2}t^2-\dfrac{2}{3}t^{3/2}+C\Rightarrow f'(t)=\dfrac{1}{6}t^3-\dfrac{4}{15}t^{5/2}+Ct+D\Rightarrow$
$f(t)=\dfrac{1}{24}t^4-\dfrac{8}{105}t^{7/2}+\dfrac{1}{2}Ct^2+Dt+E$

25. $f'(x)=1-6x\Rightarrow f(x)=x-3x^2+C$. $f(0)=C$ and $f(0)=8\Rightarrow C=8$, so $f(x)=x-3x^2+8$.

26. $f'(x)=8x^3+12x+3 \Rightarrow f(x)=2x^4+6x^2+3x+C$. $f(1)=11+C$ and $f(1)=6 \Rightarrow$
$11+C=6 \Rightarrow C=-5$, so $f(x)=2x^4+6x^2+3x-5$.

27. $f'(x)=\sqrt{x}(6+5x)=6x^{1/2}+5x^{3/2} \Rightarrow f(x)=4x^{3/2}+2x^{5/2}+C$.
$f(1)=6+C$ and $f(1)=10 \Rightarrow C=4$, so $f(x)=4x^{3/2}+2x^{5/2}+4$.

28. $f'(x)=2x-3/x^4=2x-3x^{-4} \Rightarrow f(x)=x^2+x^{-3}+C$ because we're given that $x>0$.
$f(1)=2+C$ and $f(1)=3 \Rightarrow C=1$, so $f(x)=x^2+1/x^3+1$.

29. $f'(t)=2\cos t+\sec^2 t \Rightarrow f(t)=2\sin t+\tan t+C$ because $-\pi/2<t<\pi/2$.
$f\left(\frac{\pi}{3}\right)=2\left(\sqrt{3}/2\right)+\sqrt{3}+C=2\sqrt{3}+C$ and $f\left(\frac{\pi}{3}\right)=4 \Rightarrow C=4-2\sqrt{3}$, so $f(t)=2\sin t+\tan t+4-2\sqrt{3}$.

30. $f'(x)=3x^{-2} \Rightarrow f(x)=\begin{cases} -3/x+C_1 & \text{if } x>0 \\ -3/x+C_2 & \text{if } x<0 \end{cases}$ $f(1)=-3+C_1=0 \Rightarrow C_1=3$,
$f(-1)=3+C_2=0 \Rightarrow C_2=-3$. So $f(x)=\begin{cases} -3/x+3 & \text{if } x>0 \\ -3/x-3 & \text{if } x<0 \end{cases}$

31. $f'(x)=2/x \Rightarrow f(x)=2\ln|x|+C=2\ln(-x)+C$ (since $x<0$). Now $f(-1)=2\ln 1+C=2(0)+C=7 \Rightarrow C=7$. Therefore, $f(x)=2\ln(-x)+7$, $x<0$.

32. $f'(x)=4/\sqrt{1-x^2} \Rightarrow f(x)=4\sin^{-1}x+C$. $f\left(\frac{1}{2}\right)=4\sin^{-1}\left(\frac{1}{2}\right)+C=4\cdot\frac{\pi}{6}+C$ and $f\left(\frac{1}{2}\right)=1 \Rightarrow$
$\frac{2\pi}{3}+C=1 \Rightarrow C=1-\frac{2\pi}{3}$, so $f(x)=4\sin^{-1}x+1-\frac{2\pi}{3}$.

33. $f''(x)=24x^2+2x+10 \Rightarrow f'(x)=8x^3+x^2+10x+C$. $f'(1)=8+1+10+C$ and $f'(1)=-3 \Rightarrow 19+C=-3 \Rightarrow$
$C=-22$, so $f'(x)=8x^3+x^2+10x-22$ and hence, $f(x)=2x^4+\frac{1}{3}x^3+5x^2-22x+D$. $f(1)=2+\frac{1}{3}+5-22+D$ and
$f(1)=5 \Rightarrow D=22-\frac{7}{3}=\frac{59}{3}$, so $f(x)=2x^4+\frac{1}{3}x^3+5x^2-22x+\frac{59}{3}$.

34. $f''(x)=4-6x-40x^3 \Rightarrow f'(x)=4x-3x^2-10x^4+C$. $f'(0)=C$ and $f'(0)=1 \Rightarrow C=1$, so
$f'(x)=4x-3x^2-10x^4+1$ and hence, $f(x)=2x^2-x^3-2x^5+x+D$. $f(0)=D$ and $f(0)=2 \Rightarrow D=2$, so
$f(x)=2x^2-x^3-2x^5+x+2$.

35.

$f''(\theta)=\sin\theta+\cos\theta \Rightarrow f'(\theta)=-\cos\theta+\sin\theta+C$. $f'(0)=-1+C$ and $f'(0)=4\Rightarrow C=5$, so $f'(\theta)=-\cos\theta+\sin\theta+5$ and hence, $f(\theta)=-\sin\theta-\cos\theta+5\theta+\mathrm{D}$. $f(0)=-1+D$ and $f(0)=3\Rightarrow D=4$, so $f(\theta)=-\sin\theta-\cos\theta+5\theta+4$.

36. $f''(t)=3/\sqrt{t}=3t^{-1/2}\Rightarrow f'(t)=6t^{1/2}+C$. $f'(4)=12+C$ and $f'(4)=7\Rightarrow C=-5$, so $f'(t)=6t^{1/2}-5$ and hence, $f(t)=4t^{3/2}-5t+D$. $f(4)=32-20+D$ and $f(4)=20\Rightarrow D=8$, so $f(t)=4t^{3/2}-5t+8$.

37. $f''(x)=2-12x\Rightarrow f'(x)=2x-6x^2+C\Rightarrow f(x)=x^2-2x^3+Cx+D$.
$f(0)=D$ and $f(0)=9\Rightarrow D=9$. $f(2)=4-16+2C+9=2C-3$ and $f(2)=15\Rightarrow 2C=18\Rightarrow C=9$, so $f(x)=x^2-2x^3+9x+9$.

38. $f''(x)=20x^3+12x^2+4\Rightarrow f'(x)=5x^4+4x^3+4x+C\Rightarrow f(x)=x^5+x^4+2x^2+Cx+D$. $f(0)=D$ and $f(0)=8\Rightarrow D=8$. $f(1)=1+1+2+C+8=C+12$ and $f(1)=5\Rightarrow C=-7$, so $f(x)=x^5+x^4+2x^2-7x+8$.

39. $f''(x)=2+\cos x\Rightarrow f'(x)=2x+\sin x+C\Rightarrow f(x)=x^2-\cos x+Cx+D$. $f(0)=-1+D$ and $f(0)=-1\Rightarrow D=0$. $f\left(\frac{\pi}{2}\right)=\pi^2/4+\left(\frac{\pi}{2}\right)C$ and $f\left(\frac{\pi}{2}\right)=0\Rightarrow \left(\frac{\pi}{2}\right)C=-\pi^2/4\Rightarrow C=-\frac{\pi}{2}$, so $f(x)=x^2-\cos x-\left(\frac{\pi}{2}\right)x$.

40. $f''(t)=2e^t+3\sin t\Rightarrow f'(t)=2e^t-3\cos t+C\Rightarrow f(t)=2e^t-3\sin t+Ct+D$. $f(0)=2+D$ and $f(0)=0\Rightarrow D=-2$. $f(\pi)=2e^\pi+\pi C-2$ and $f(\pi)=0\Rightarrow \pi C=2-2e^\pi\Rightarrow C=\dfrac{2-2e^\pi}{\pi}$, so $f(t)=2e^t-3\sin t+\dfrac{2-2e^\pi}{\pi}t-2$.

41. $f''(x)=x^{-2}$, $x>0\Rightarrow f'(x)=-1/x+C\Rightarrow f(x)=-\ln|x|+Cx+D=-\ln x+Cx+D$ (since $x>0$). $f(1)=0\Rightarrow C+D=0$ and $f(2)=0\Rightarrow -\ln 2+2C+D=0\Rightarrow$
$-\ln 2+2C-C=0$ [since $D=-C$] $\Rightarrow -\ln 2+C=0\Rightarrow C=\ln 2$ and $D=-\ln 2$.
So $f(x)=-\ln x+(\ln 2)x-\ln 2$.

42. $f'''(x)=\sin x\Rightarrow f''(x)=-\cos x+C\Rightarrow 1=f''(0)=-1+C\Rightarrow C=2$, so $f''(x)=-\cos x+2\Rightarrow f'(x)=-\sin x+2x+D\Rightarrow 1=f'(0)=D\Rightarrow f'(x)=-\sin x+2x+1\Rightarrow f(x)=\cos x+x^2+x+E$ $\Rightarrow 1=f(0)=1+E\Rightarrow E=0$, so $f(x)=\cos x+x^2+x$.

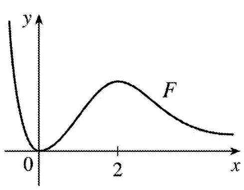
43. Given $f'(x)=2x+1$, we have $f(x)=x^2+x+C$. Since f passes through $(1,6)$, $f(1)=6\Rightarrow 1^2+1+C=6$ $\Rightarrow C=4$. Therefore, $f(x)=x^2+x+4$ and $f(2)=2^2+2+4=10$.

44. $f'(x)=x^3 \Rightarrow f(x)=\frac{1}{4}x^4+C$. $x+y=0 \Rightarrow y=-x \Rightarrow m=-1$. Now $m=f'(x) \Rightarrow -1=x^3 \Rightarrow x=-1 \Rightarrow y=1$ (from the equation of the tangent line), so $(-1,1)$ is a point on the graph of f . From f , $1=\frac{1}{4}(-1)^4+C \Rightarrow C=\frac{3}{4}$. Therefore, the function is $f(x)=\frac{1}{4}x^4+\frac{3}{4}$.

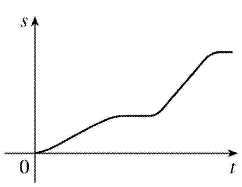
45. b is the antiderivative of f . For small x , f is negative, so the graph of its antiderivative must be decreasing. But both a and c are increasing for small x , so only b can be f 's antiderivative. Also, f is positive where b is increasing, which supports our conclusion.

46. We know right away that c cannot be f 's antiderivative, since the slope of c is not zero at the x–value where $f=0$. Now f is positive when a is increasing and negative when a is decreasing, so a is the antiderivative of f .

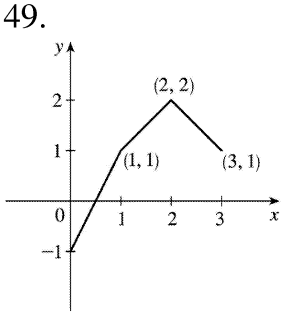
47. The graph of F will have a minimum at 0 and a maximum at 2 , since $f=F'$ goes from negative to positive at $x=0$, and from positive to negative at $x=2$.



48. The position function is the antiderivative of the velocity function, so its graph has to be horizontal where the velocity function is equal to 0 .



49.

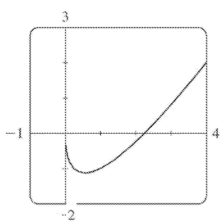


$f'(x)=\begin{cases} 2 & \text{if} 0\le x<1 \\ 1 & \text{if} 1<x<2 \\ -1 & \text{if} 2<x\le 3 \end{cases} \Rightarrow f(x)=\begin{cases} 2x+C & \text{if} 0\le x<1 \\ x+D & \text{if} 1<x<2 \\ -x+E & \text{if} 2<x\le 3 \end{cases}$ $f(0)=-1\Rightarrow 2(0)+C=-1\Rightarrow C=-1$.

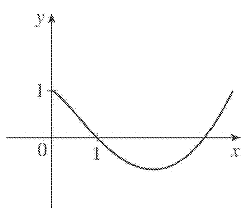
Starting at the point $(0,-1)$ and moving to the right on a line with slope 2 gets us to the point $(1,1)$. The slope for $1<x<2$ is 1 , so we get to the point $(2,2)$. Here we have used the fact that f is continuous. We can include the point $x=1$ on either the first or the second part of f . The line connecting $(1,1)$ to $(2,2)$ is $y=x$, so $D=0$. The slope for $2<x\le 3$ is -1 , so we get to $(3,1)$. $f(3)=1\Rightarrow -3+E=1\Rightarrow E=4$. Thus,

$$f(x)=\begin{cases} 2x-1 & \text{if } 0\le x\le 1 \\ x & \text{if } 1<x<2 \\ -x+4 & \text{if } 2\le x\le 3 \end{cases}$$

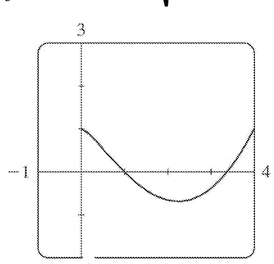
Note that $f'(x)$ does not exist at $x=1$ or at $x=2$.

50. **(a)** 

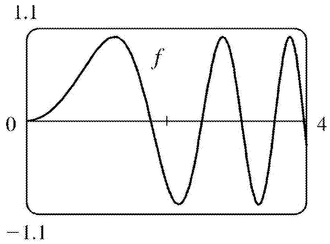
(b) Since $F(0)=1$, we can start our graph at $(0,1)$. f has a minimum at about $x=0.5$, so its derivative is zero there. f is decreasing on $(0,0.5)$, so its derivative is negative and hence, F is CD on $(0,0.5)$ and has an IP at $x\approx 0.5$. On $(0.5,2.2)$, f is negative and increasing (f' is positive), so F is decreasing and CU. On $(2.2,\infty)$, f is positive and increasing, so F is increasing and CU.

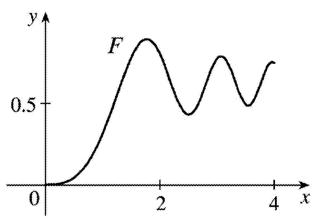


(c) $f(x)=2x-3\sqrt{x}\Rightarrow F(x)=x^2-3\cdot \frac{2}{3}x^{3/2}+C$. $F(0)=C$ and $F(0)=1\Rightarrow C=1$, so $F(x)=x^2-2x^{3/2}+1$.

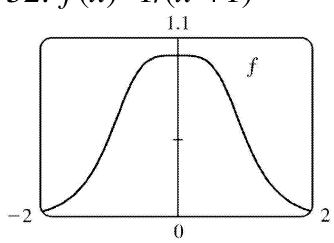
(d) 

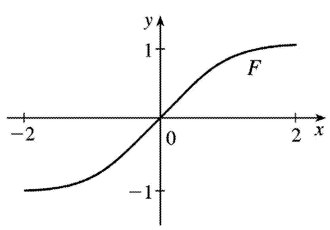
51. $f(x)=\sin(x^2)$, $0\leq x\leq 4$



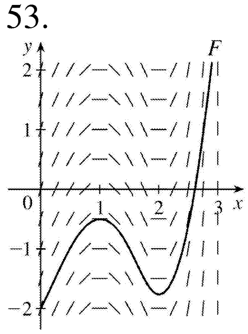


52. $f(x)=1/(x^4+1)$

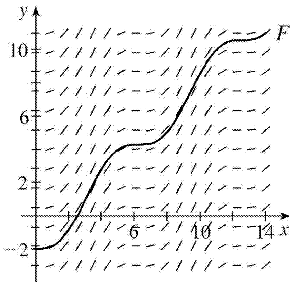




53.



54.

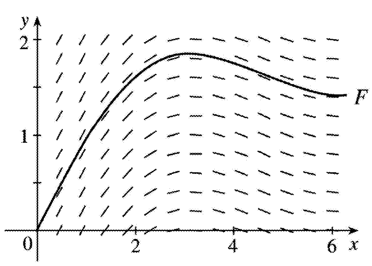


55.

x	$f(x)$
0	1
0.5	0.959
1.0	0.841
1.5	0.665
2.0	0.455
2.5	0.239
3.0	0.047

x	$f(x)$
3.5	−0.100
4.0	−0.189
4.5	−0.217
5.0	−0.192
5.5	−0.128
6.0	−0.047

We compute slopes [values of $f(x)=(\sin x) / x$ for $0 < x < 2\pi$] as in the table $\left[\lim_{x \to 0^+} f(x)=1 \right]$ and draw a direction field as in Example 6. Then we use the direction field to graph F starting at $(0,0)$

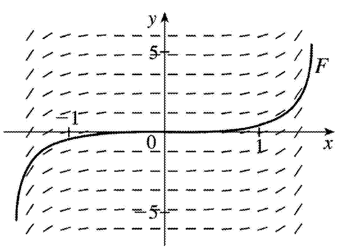


56.

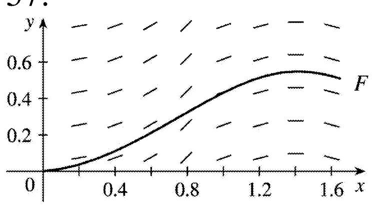
x	$f(x)$
0	0
± 0.2	0.041
± 0.4	0.169
± 0.6	0.410
± 0.8	0.824
± 1.0	1.557
± 1.2	3.087

± 1.4	8.117
± 1.5	21.152

We compute slopes [values of $f(x) = x \tan x$ for $-\pi/2 < x \ \ \pi/2$] as in the table and draw a direction field as in Example 6. Then we use the direction field to graph F starting at $(0,0)$ and extending in both directions. Note that if f is an even function, then the antiderivative F that passes through the origin is an odd function.]

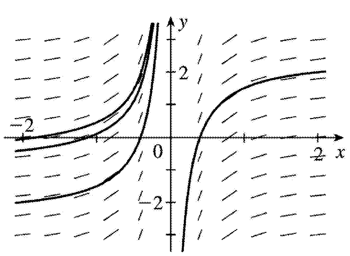


57.



Remember that the given table values of f are the slopes of F at any x . For example, at $x=1.4$, the slope of F is $f(1.4)=0$.

58. **(a)**



(b) The general antiderivative of $f(x)=x^{-2}$ is $F(x)=\begin{cases} -1/x+C_1 & \text{if } x<0 \\ -1/x+C_2 & \text{if } x>0 \end{cases}$ since $f(x)$ is not defined at $x=0$. The graph of the general antiderivatives of $f(x)$ looks like the graph in part (a), as expected.

59. $v(t)=s'(t)=\sin t-\cos t \Rightarrow s(t)=-\cos t-\sin t+C$. $s(0)=-1+C$ and $s(0)=0 \Rightarrow C=1$, so $s(t)=-\cos t-\sin t+1$.

60. $v(t)=s'(t)=1.5\sqrt{t} \Rightarrow s(t)=t^{3/2}+C$. $s(4)=8+C$ and $s(4)=10 \Rightarrow C=2$, so $s(t)=t^{3/2}+2$.

61. $a(t)=v'(t)=t-2 \Rightarrow v(t)=\frac{1}{2}t^2-2t+C$. $v(0)=C$ and $v(0)=3 \Rightarrow C=3$, so $v(t)=\frac{1}{2}t^2-2t+3$ and $s(t)=\frac{1}{6}t^3-t^2+3t+D$. $s(0)=D$ and $s(0)=1 \Rightarrow D=1$, and $s(t)=\frac{1}{6}t^3-t^2+3t+1$.

62. $a(t)=v'(t)=\cos t+\sin t \Rightarrow v(t)=\sin t-\cos t+C \Rightarrow 5=v(0)=-1+C \Rightarrow C=6$, so $v(t)=\sin t-\cos t+6 \Rightarrow$ $s(t)=-\cos t-\sin t+6t+D \Rightarrow 0=s(0)=-1+D \Rightarrow D=1$, so $s(t)=-\cos t-\sin t+6t+1$.

63. $a(t)=v'(t)=10\sin t+3\cos t \Rightarrow v(t)=-10\cos t+3\sin t+C \Rightarrow s(t)=-10\sin t-3\cos t+Ct+D$. $s(0)=-3+D=0$ and $s(2\pi)=-3+2\pi C+D=12 \Rightarrow D=3$ and $C=\frac{6}{\pi}$. Thus, $s(t)=-10\sin t-3\cos t+\frac{6}{\pi}t+3$.

64. $a(t)=v'(t)=10+3t-3t^2 \Rightarrow v(t)=10t+\frac{3}{2}t^2-t^3+C \Rightarrow s(t)=5t^2+\frac{1}{2}t^3-\frac{1}{4}t^4+Ct+D \Rightarrow 0=s(0)=D$ and $10=s(2)=20+4-4+2C \Rightarrow C=-5$, so $s(t)=-5t+5t^2+\frac{1}{2}t^3-\frac{1}{4}t^4$.

65. **(a)** We first observe that since the stone is dropped 450 m above the ground, $v(0)=0$ and $s(0)=450$.

$v'(t)=a(t)=-9.8 \Rightarrow v(t)=-9.8t+C$. Now $v(0)=0 \Rightarrow C=0$, so $v(t)=-9.8t \Rightarrow s(t)=-4.9t^2+D$. Last, $s(0)=450 \Rightarrow D=450 \Rightarrow s(t)=450-4.9t^2$.

(b) The stone reaches the ground when $s(t)=0$. $450-4.9t^2=0 \Rightarrow t^2=450/4.9 \Rightarrow t_1=\sqrt{450/4.9} \approx 9.58$ s.

(c) The velocity with which the stone strikes the ground is $v\left(t_1\right)=-9.8\sqrt{450/4.9} \approx -93.9$ m / s.

(d) This is just reworking parts (a) and (b) with $v(0)=-5$. Using $v(t)=-9.8t+C$, $v(0)=-5 \Rightarrow 0+C=-5 \Rightarrow$ $v(t)=-9.8t-5$. So $s(t)=-4.9t^2-5t+D$ and $s(0)=450 \Rightarrow D=450 \Rightarrow s(t)=-4.9t^2-5t+450$. Solving $s(t)=0$ by using the quadratic formula gives us $t=\left(5 \pm \sqrt{8845}\right)/(-9.8) \Rightarrow t_1 \approx 9.09$ s.

66. $v'(t)=a(t)=a \Rightarrow v(t)=at+C$ and $v_0=v(0)=C \Rightarrow v(t)=at+v_0 \Rightarrow$ $s(t)=\frac{1}{2}at^2+v_0t+D \Rightarrow s_0=s(0)=D \Rightarrow s(t)=\frac{1}{2}at^2+v_0t+s_0$

67. By Exercise with $a=-9.8$, $s(t)=-4.9t^2+v_0t+s_0$ and $v(t)=s'(t)=-9.8t+v_0$. So $[v(t)]^2=\left(-9.8t+v_0\right)^2=(9.8)^2t^2-19.6v_0t+v_0^2=v_0^2+96.04t^2-19.6v_0t=v_0^2-19.6\left(-4.9t^2+v_0t\right)$. But $-4.9t^2+v_0t$ is just $s(t)$ without the s_0 term; that is, $s(t)-s_0$. Thus, $[v(t)]^2=v_0^2-19.6\left[s(t)-s_0\right]$.

68. For the first ball, $s_1(t)=-16t^2+48t+432$ from Example 8. For the second ball, $a(t)=-32 \Rightarrow$ $v(t)=-32t+C$, but $v(1)=-32(1)+C=24 \Rightarrow C=56$, so $v(t)=-32t+56 \Rightarrow s(t)=-16t^2+56t+D$, but $s(1)=-16(1)^2+56(1)+D=432 \Rightarrow D=392$, and $s_2(t)=-16t^2+56t+392$. The balls pass each other when $s_1(t)=s_2(t) \Rightarrow -16t^2+48t+432=-16t^2+56t+392 \Leftrightarrow 8t=40 \Leftrightarrow t=5$ s.

Another solution: From Exercise , we have $s_1(t)=-16t^2+48t+432$ and $s_2(t)=-16t^2+24t+432$. We now want to solve $s_1(t)=s_2(t-1) \Rightarrow -16t^2+48t+432=-16(t-1)^2+24(t-1)+432 \Rightarrow 48t=32t-16+24t-24 \Rightarrow 40=8t$ $\Rightarrow t=5$ s.

69. Using Exercise with $a=-32$, $v_0=0$, and $s_0=h$ (the height of the cliff), we know that the height at time t is $s(t)=-16t^2+h$. $v(t)=s'(t)=-32t$ and $v(t)=-120 \Rightarrow -32t=-120 \Rightarrow t=3.75$, so $0=s(3.75)=-16(3.75)^2+h \Rightarrow h=16(3.75)^2=225$ ft.

70. **(a)** $EIy''=mg(L-x)+\frac{1}{2}\rho g(L-x)^2 \Rightarrow EIy'=-\frac{1}{2}mg(L-x)^2-\frac{1}{6}\rho g(L-x)^3+C \Rightarrow$ $EIy=\frac{1}{6}mg(L-x)^3+\frac{1}{24}\rho g(L-x)^4+Cx+D$. Since the left end of the board is fixed, we must have $y=y'=0$ when $x=0$. Thus, $0=-\frac{1}{2}mgL^2-\frac{1}{6}\rho gL^3+C$ and $0=\frac{1}{6}mgL^3+\frac{1}{24}\rho gL^4+D$. It follows that $EIy=\frac{1}{6}mg(L-x)^3+\frac{1}{24}\rho g(L-x)^4+\left(\frac{1}{2}mgL^2+\frac{1}{6}\rho gL^3\right)x-\left(\frac{1}{6}mgL^3+\frac{1}{24}\rho gL^4\right)$ and

$f(x)=y=\frac{1}{EI}\left[\frac{1}{6}mg(L-x)^3+\frac{1}{24}\rho g(L-x)^4+\left(\frac{1}{2}mgL^2+\frac{1}{6}\rho gL^3\right)x-\left(\frac{1}{6}mgL^3+\frac{1}{24}\rho gL^4\right)\right]$

(b) $f(L)<0$, so the end of the board is a *distance* approximately $-f(L)$ below the horizontal. From our result in (a), we calculate

$$\begin{aligned} -f(L) &= \frac{-1}{EI}\left[\frac{1}{2}mgL^3+\frac{1}{6}\rho gL^4-\frac{1}{6}mgL^3-\frac{1}{24}\rho gL^4\right] \\ &= \frac{-1}{EI}\left(\frac{1}{3}mgL^3+\frac{1}{8}\rho gL^4\right)=-\frac{gL^3}{EI}\left(\frac{m}{3}+\frac{\rho L}{8}\right) \end{aligned}$$

Note: This is positive because g is negative.

71. Marginal cost $=1.92-0.002x=C'(x) \Rightarrow C(x)=1.92x-0.001x^2+K$. But $C(1)=1.92-0.001+K=562 \Rightarrow$ $K=560.081$. Therefore, $C(x)=1.92x-0.001x^2+560.081 \Rightarrow C(100)=742.081$, so the cost of producing

100 items is \$742.08 .

72. Let the mass, measured from one end, be $m(x)$. Then $m(0)=0$ and $\rho=\dfrac{dm}{dx}=x^{-1/2}\Rightarrow m(x)=2x^{1/2}+C$ and $m(0)=C=0$, so $m(x)=2\sqrt{x}$. Thus, the mass of the 100–centimeter rod is $m(100)=2\sqrt{100}=20$ g.

73. Taking the upward direction to be positive we have that for $0\le t\le 10$ (using the subscript 1 to refer to $0\le t\le 10$), $a_1(t)=-(9-0.9t)=v_1'(t)\Rightarrow v_1(t)=-9t+0.45t^2+v_0$, but $v_1(0)=v_0=-10\Rightarrow$ $v_1(t)=-9t+0.45t^2-10=s_1'(t)\Rightarrow s_1(t)=-\dfrac{9}{2}t^2+0.15t^3-10t+s_0$. But $s_1(0)=500=s_0\Rightarrow$ $s_1(t)=-\dfrac{9}{2}t^2+0.15t^3-10t+500$. $s_1(10)=-450+150-100+500=100$, so it takes more than 10 seconds for the raindrop to fall. Now for $t>10$, $a(t)=0=v'(t)\Rightarrow v(t)=$ constant $=v_1(10)=-9(10)+0.45(10)^2-10=-55$ $\Rightarrow v(t)=-55$. At 55 ft / s, it will take $100/55\approx 1.8$ s to fall the last 100 ft. Hence, the total time is $10+\dfrac{100}{55}=\dfrac{130}{11}\approx 11.8$ s.

74. $v'(t)=a(t)=-22$. The initial velocity is 50 mi / h $=\dfrac{50\cdot 5280}{3600}=\dfrac{220}{3}$ ft / s, so $v(t)=-22t+\dfrac{220}{3}$. The car stops when $v(t)=0\Leftrightarrow t=\dfrac{220}{3\cdot 22}=\dfrac{10}{3}$. Since $s(t)=-11t^2+\dfrac{220}{3}t$, the distance covered is $s\left(\dfrac{10}{3}\right)=-11\left(\dfrac{10}{3}\right)^2+\dfrac{220}{3}\cdot\dfrac{10}{3}=\dfrac{1100}{9}=122.\overline{2}$ ft.

75. $a(t)=k$, the initial velocity is 30 mi / h $=30\cdot\dfrac{5280}{3600}=44$ ft / s, and the final velocity (after 5 seconds) is 50 mi / h $=50\cdot\dfrac{5280}{3600}=\dfrac{220}{3}$ ft / s . So $v(t)=kt+C$ and $v(0)=44\Rightarrow C=44$. Thus, $v(t)=kt+44$ $\Rightarrow v(5)=5k+44$. But $v(5)=\dfrac{220}{3}$, so $5k+44=\dfrac{220}{3}\Rightarrow 5k=\dfrac{88}{3}\Rightarrow k=\dfrac{88}{15}\approx 5.87$ ft / s^2 .

76. $a(t)=-16\Rightarrow v(t)=-16t+v_0$ where v_0 is the car’s speed (in ft / s) when the brakes were applied. The car stops when $-16t+v_0=0\Leftrightarrow t=\dfrac{1}{16}v_0$. Now $s(t)=\dfrac{1}{2}(-16)t^2+v_0t=-8t^2+v_0t$. The car travels 200 ft in the time that it takes to stop, so $s\left(\dfrac{1}{16}v_0\right)=200\Rightarrow 200=-8\left(\dfrac{1}{16}v_0\right)^2+v_0\left(\dfrac{1}{16}v_0\right)=\dfrac{1}{32}v_0^2\Rightarrow$ $v_0^2=32\cdot 200=6400\Rightarrow v_0=80$ ft / s ($54.\overline{54}$ mi / h).

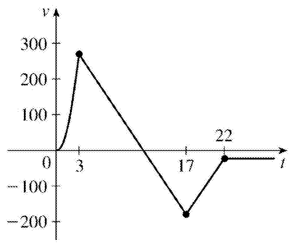
77. Let the acceleration be $a(t)=k\ \text{km}/\text{h}^2$. We have $v(0)=100\ \text{km}/\text{h}$ and we can take the initial position $s(0)$ to be 0 . We want the time t_f for which $v(t)=0$ to satisfy $s(t)<0.08$ km. In general, $v'(t)=a(t)=k$, so $v(t)=kt+C$, where $C=v(0)=100$. Now $s'(t)=v(t)=kt+100$, so $s(t)=\frac{1}{2}kt^2+100t+D$, where $D=s(0)=0$. Thus, $s(t)=\frac{1}{2}kt^2+100t$. Since $v(t_f)=0$, we have $kt_f+100=0$ or $t_f=-100/k$, so $s(t_f)=\frac{1}{2}k\left(-\frac{100}{k}\right)^2+100\left(-\frac{100}{k}\right)=10,000\left(\frac{1}{2k}-\frac{1}{k}\right)=-\frac{5{,}000}{k}$. The condition $s(t_f)$ must satisfy is $-\frac{5{,}000}{k}<0.08\Rightarrow-\frac{5{,}000}{0.08}>k$ [k is negative] $\Rightarrow k<-62,500\ \text{km}/\text{h}^2$, or equivalently, $k<-\frac{3125}{648}\approx-4.82\ \text{m}/\text{s}^2$.

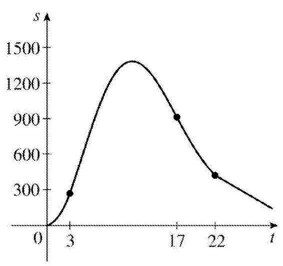
78. **(a)** For $0\le t\le 3$ we have $a(t)=60t\Rightarrow v(t)=30t^2+C\Rightarrow v(0)=0=C\Rightarrow v(t)=30t^2$, so $s(t)=10t^3+C\Rightarrow s(0)=0=C\Rightarrow s(t)=10t^3$. Note that $v(3)=270$ and $s(3)=270$.
For $3<t\le 17$: $a(t)=-g=-32\ \text{ft}/\text{s}\Rightarrow v(t)=-32(t-3)+C\Rightarrow v(3)=270=C\Rightarrow v(t)=-32(t-3)+270\Rightarrow s(t)=-16(t-3)^2+270(t-3)+C\Rightarrow s(3)=270=C\Rightarrow s(t)=-16(t-3)^2+270(t-3)+270$. Note that $v(17)=-178$ and $s(17)=914$.
For $17<t\le 22$: The velocity increases linearly from -178 ft / s to -18 ft / s during this period, so $\frac{\Delta v}{\Delta t}=\frac{-18-(-178)}{22-17}=\frac{160}{5}=32$. Thus, $v(t)=32(t-17)-178\Rightarrow s(t)=16(t-17)^2-178(t-17)+914$ and $s(22)=424$ ft.
For $t>22$: $v(t)=-18\Rightarrow s(t)=-18(t-22)+C$. But $s(22)=424=C\Rightarrow s(t)=-18(t-22)+424$.
Therefore, until the rocket lands, we have

$$v(t)=\begin{cases} 30t^2 & \text{if } 0\le t\le 3\\ -32(t-3)+270 & \text{if } 3<t\le 17\\ 32(t-17)-178 & \text{if } 17<t\le 22\\ -18 & \text{if } t>22\end{cases}$$

and

$$s(t)=\begin{cases} 10t^3 & \text{if } 0\le t\le 3\\ -16(t-3)^2+270(t-3)+270 & \text{if } 3<t\le 17\\ 16(t-17)^2-178(t-17)+914 & \text{if } 17<t\le 22\\ -18(t-22)+424 & \text{if } t>22\end{cases}$$





(b) To find the maximum height, set $v(t)$ on $3<t\leq 17$ equal to 0 . $-32(t-3)+270=0\Rightarrow t_1=11.4375$ s and the maximum height is $s\left(t_1\right)=-16\left(t_1-3\right)^2+270\left(t_1-3\right)+270=1409.0625$ ft.

(c) To find the time to land, set $s(t)=-18(t-22)+424=0$. Then $t-22=\dfrac{424}{18}=23.\overline{5}$, so $t\approx 45.6$ s.

79. **(a)** First note that $90\text{ mi / h}=90\times\dfrac{5280}{3600}\text{ ft / s}=132\text{ ft / s}$. Then $a(t)=4\text{ ft / s}^2\Rightarrow v(t)=4t+C$, but $v(0)=0\Rightarrow C=0$. Now $4t=132$ when $t=\dfrac{132}{4}=33$ s, so it takes 33 s to reach 132 ft / s. Therefore, taking $s(0)=0$, we have $s(t)=2t^2$, $0\leq t\leq 33$. So $s(33)=2178$ ft. 15 minutes $=15(60)=900$ s, so for $33<t\leq 933$ we have $v(t)=132$ ft / s $\Rightarrow s(933)=132(900)+2178=120,978$ ft $=22.9125$ mi.

(b) As in part (a), the train accelerates for 33 s and travels 2178 ft while doing so. Similarly, it decelerates for 33 s and travels 2178 ft at the end of its trip. During the remaining $900-66=834$ s it travels at 132 ft / s, so the distance traveled is $132\cdot 834=110,088$ ft. Thus, the total distance is $2178+110,088+2178=114,444$ ft $=21.675$ mi.

(c) 45 mi $=45(5280)=237,600$ ft. Subtract $2(2178)$ to take care of the speeding up and slowing down, and we have $233,244$ ft at 132 ft / s for a trip of $233,244/132=1767$ s at 90 mi / h. The total time is $1767+2(33)=1833\text{s}=30\text{ min }33\text{s}=30.55$ min.

(d) $37.5(60)=2250$ s. $2250-2(33)=2184$ s at maximum speed. $2184(132)+2(2178)=292,644$ total feet or $292,644/5280=55.425$ mi.

1. $f(x)=x^2-4x+1$, $[0,4]$. Since f is a polynomial, it is continuous and differentiable on R , so it is continuous on $[0,4]$ and differentiable on $(0,4)$. Also, $f(0)=1=f(4)$. $f'(c)=0 \Leftrightarrow 2c-4=0 \Leftrightarrow c=2$, which is in the open interval $(0,4)$, so $c=2$ satisfies the conclusion of Rolle's Theorem.

2. $f(x)=x^3-3x^2+2x+5$, $[0,2]$. f is continuous on $[0,2]$ and differentiable on $(0,2)$. Also, $f(0)=5=f(2)$. $f'(c)=0 \Leftrightarrow 3c^2-6c+2=0 \Leftrightarrow c=\dfrac{6\pm\sqrt{36-24}}{6}=1\pm\dfrac{1}{3}\sqrt{3}$, both in $(0,2)$.

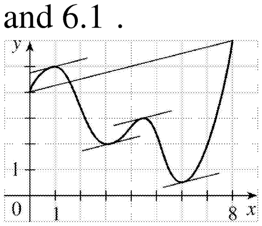
3. $f(x)=\sin 2\pi x$, $[-1,1]$. f , being the composite of the sine function and the polynomial $2\pi x$, is continuous and differentiable on R , so it is continuous on $[-1,1]$ and differentiable on $(-1,1)$. Also, $f(-1)=0=f(1)$. $f'(c)=0 \Leftrightarrow 2\pi\cos 2\pi c=0 \Leftrightarrow \cos 2\pi c=0 \Leftrightarrow 2\pi c=\pm\dfrac{\pi}{2}+2\pi n \Leftrightarrow c=\pm\dfrac{1}{4}+n$. If $n=0$ or ± 1 , then $c=\pm\dfrac{1}{4}$, $\pm\dfrac{3}{4}$ is in $(-1,1)$.

4. $f(x)=x\sqrt{x+6}$, $[-6,0]$. f is continuous on its domain, $[-6,\infty)$, and differentiable on $(-6,\infty)$, so it is continuous on $[-6,0]$ and differentiable on $(-6,0)$. Also, $f(-6)=0=f(0)$. $f'(c)=0 \Leftrightarrow \dfrac{3c+12}{2\sqrt{c+6}}=0 \Leftrightarrow c=-4$, which is in $(-6,0)$.

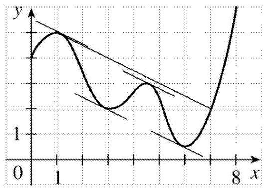
5. $f(x)=1-x^{2/3}$. $f(-1)=1-(-1)^{2/3}=1-1=0=f(1)$. $f'(x)=-\dfrac{2}{3}x^{-1/3}$, so $f'(c)=0$ has no solution. This does not contradict Rolle's Theorem, since $f'(0)$ does not exist, and so f is not differentiable on $(-1,1)$.

6. $f(x)=(x-1)^{-2}$. $f(0)=(0-1)^{-2}=1=(2-1)^{-2}=f(2)$. $f'(x)=-2(x-1)^{-3} \Rightarrow f'(x)$ is never 0 . This does not contradict Rolle's Theorem since $f'(1)$ does not exist.

7. $\dfrac{f(8)-f(0)}{8-0}=\dfrac{6-4}{8}=\dfrac{1}{4}$. The values of c which satisfy $f'(c)=\dfrac{1}{4}$ seem to be about $c=0.8$, 3.2 , 4.4 , and 6.1 .

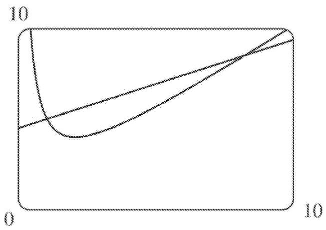


8. $\dfrac{f(7)-f(1)}{7-1}=\dfrac{2-5}{6}=-\dfrac{1}{2}$. The values of c which satisfy $f'(c)=-\dfrac{1}{2}$ seem to be about $c=1.1$, 2.8 , 4.6 , and 5.8 .



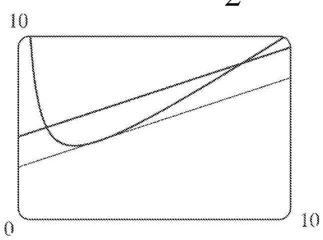
9.

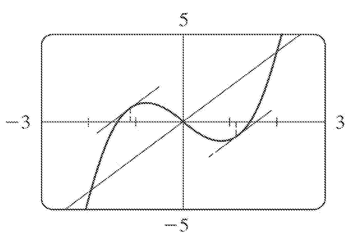
(a), (b) The equation of the secant line is $y-5=\frac{8.5-5}{8-1}(x-1)\Leftrightarrow y=\frac{1}{2}x+\frac{9}{2}$.



(c) $f(x)=x+4/x\Rightarrow f'(x)=1-4/x^2$.

So $f'(c)=\frac{1}{2}\Rightarrow c^2=8\Rightarrow c=2\sqrt{2}$, and $f(c)=2\sqrt{2}+\frac{4}{2\sqrt{2}}=3\sqrt{2}$. Thus, an equation of the tangent line is $y-3\sqrt{2}=\frac{1}{2}\left(x-2\sqrt{2}\right)\Leftrightarrow y=\frac{1}{2}x+2\sqrt{2}$.





10. **(a)**

It seems that the tangent lines are parallel to the secant at $x\approx\pm1.2$.

(b) The slope of the secant line is 2 , and its equation is $y=2x$. $f(x)=x^3-2x\Rightarrow f'(x)=3x^2-2$, so we solve $f'(c)=2\Rightarrow 3c^2=4\Rightarrow c=\pm\frac{2\sqrt{3}}{3}\approx1.155$. Our estimates were off by about 0.045 in each case.

11. $f(x)=3x^2+2x+5$, $[-1,1]$. f is continuous on $[-1,1]$ and differentiable on $(-1,1)$ since polynomials are continuous and differentiable on R . $f'(c)=\frac{f(b)-f(a)}{b-a}\Leftrightarrow$

$6c+2=\frac{f(1)-f(-1)}{1-(-1)}=\frac{10-6}{2}=2\Leftrightarrow 6c=0\Leftrightarrow c=0$, which is in $(-1,1)$.

12. $f(x)=x^3+x-1$, $[0,2]$. f is continuous on $[0,2]$ and differentiable on $(0,2)$. $f'(c)=\frac{f(2)-f(0)}{2-0}\Leftrightarrow$ $3c^2+1=\frac{9-(-1)}{2}\Leftrightarrow 3c^2=5-1\Leftrightarrow c^2=\frac{4}{3}\Leftrightarrow c=\pm\frac{2}{\sqrt{3}}$, but only $\frac{2}{\sqrt{3}}$ is in $(0,2)$.

13. $f(x)=e^{-2x}$, $[0,3]$. f is continuous and differentiable on R , so it is continuous on $[0,3]$ and differentiable on $(0,3)$. $f'(c)=\frac{f(b)-f(a)}{b-a}\Leftrightarrow -2e^{-2c}=\frac{e^{-6}-e^{0}}{3-0}\Leftrightarrow e^{-2c}=\frac{1-e^{-6}}{6}\Leftrightarrow -2c=\ln\left(\frac{1-e^{-6}}{6}\right)\Leftrightarrow$ $c=-\frac{1}{2}\ln\left(\frac{1-e^{-6}}{6}\right)\approx 0.897$, which is in $(0,3)$.

14. $f(x)=\frac{x}{x+2}$, $[1,4]$. f is continuous on $[1,4]$ and differentiable on $(1,4)$. $f'(c)=\frac{f(b)-f(a)}{b-a}\Leftrightarrow$ $\frac{2}{(c+2)^2}=\frac{\frac{2}{3}-\frac{1}{3}}{4-1}\Leftrightarrow (c+2)^2=18\Leftrightarrow c=-2\pm 3\sqrt{2}$. $-2+3\sqrt{2}\approx 2.24$ is in $(1,4)$.

15. $f(x)=|x-1|$. $f(3)-f(0)=|3-1|-|0-1|=1$. Since $f'(c)=-1$ if $c<1$ and $f'(c)=1$ if $c>1$, $f'(c)(3-0)=\pm 3$ and so is never equal to 1 . This does not contradict the Mean Value Theorem since $f'(1)$ does not exist.

16. $f(x)=\frac{x+1}{x-1}$. $f(2)-f(0)=3-(-1)=4$. $f'(x)=\frac{1(x-1)-1(x+1)}{(x-1)^2}=\frac{-2}{(x-1)^2}$. Since $f'(x)<0$ for all x (except $x=1$), $f'(c)(2-0)$ is always <0 and hence cannot equal 4 . This does not contradict the Mean Value Theorem since f is not continuous at $x=1$.

17. Let $f(x)=1+2x+x^3+4x^5$. Then $f(-1)=-6<0$ and $f(0)=1>0$. Since f is a polynomial, it is continuous, so the Intermediate Value Theorem says that there is a number c between -1 and 0 such that $f(c)=0$. Thus, the given equation has a real root. Suppose the equation has distinct real roots a and b with $a<b$. Then $f(a)=f(b)=0$. Since f is a polynomial, it is differentiable on (a,b) and continuous on $[a,b]$. By Rolle's Theorem, there is a number r in (a,b) such that $f'(r)=0$. But $f'(x)=2+3x^2+20x^4\geq 2$ for all x , so $f'(x)$ can never be 0 . This contradiction shows that the equation can't have two distinct real roots. Hence, it has exactly one real root.

18. Let $f(x)=2x-1-\sin x$. Then $f(0)=-1<0$ and $f(\pi/2)=\pi-2>0$. f is the sum of the polynomial $2x-1$ and the scalar multiple $(-1)\cdot\sin x$ of the trigonometric function $\sin x$, so f is continuous (and differentiable) for all x . By the Intermediate Value Theorem, there is a number c in $(0,\pi/2)$ such that $f(c)=0$. Thus, the given equation has at least one real root. If the equation has distinct real roots a and b with $a<b$, then $f(a)=f(b)=0$. Since f is continuous on $[a,b]$ and differentiable on (a,b) , Rolle's Theorem implies that there is a number r in (a,b) such that $f'(r)=0$. But $f'(r)=2-\cos r>0$ since $\cos r\leq 1$. This contradiction shows that the given equation can't have two distinct real roots, so it has exactly one real root.

19. Let $f(x)=x^3-15x+c$ for x in $[-2,2]$. If f has two real roots a and b in $[-2,2]$, with $a<b$, then $f(a)=f(b)=0$. Since the polynomial f is continuous on $[a,b]$ and differentiable on (a,b) , Rolle's Theorem implies that there is a number r in (a,b) such that $f'(r)=0$. Now $f'(r)=3r^2-15$. Since r is in (a,b) , which is contained in $[-2,2]$, we have $|r|<2$, so $r^2<4$. It follows that $3r^2-15<3\cdot 4-15=-3<0$. This contradicts $f'(r)=0$, so the given equation can't have two real roots in $[-2,2]$. Hence, it has at most one real root in $[-2,2]$.

20. $f(x)=x^4+4x+c$. Suppose that $f(x)=0$ has three distinct real roots a , b , d where $a<b<d$. Then $f(a)=f(b)=f(d)=0$. By Rolle's Theorem there are numbers c_1 and c_2 with $a<c_1<b$ and $b<c_2<d$ and $0=f'\left(c_1\right)=f'\left(c_2\right)$, so $f'(x)=0$ must have at least two real solutions. However $0=f'(x)=4x^3+4=4\left(x^3+1\right)=4(x+1)\left(x^2-x+1\right)$ has as its only real solution $x=-1$. Thus, $f(x)$ can have at most two real roots.

21. **(a)** Suppose that a cubic polynomial $P(x)$ has roots $a_1<a_2<a_3<a_4$, so $P\left(a_1\right)=P\left(a_2\right)=P\left(a_3\right)=P\left(a_4\right)$. By Rolle's Theorem there are numbers c_1 , c_2 , c_3 with $a_1<c_1<a_2$, $a_2<c_2<a_3$ and $a_3<c_3<a_4$ and $P'\left(c_1\right)=P'\left(c_2\right)=P'\left(c_3\right)=0$. Thus, the second-degree polynomial $P'(x)$ has three distinct real roots, which is impossible.
(b) We prove by induction that a polynomial of degree n has at most n real roots. This is certainly true for $n=1$. Suppose that the result is true for all polynomials of degree n and let $P(x)$ be a polynomial of degree $n+1$. Suppose that $P(x)$ has more than $n+1$ real roots, say $a_1<a_2<a_3<\cdots<a_{n+1}<a_{n+2}$. Then $P\left(a_1\right)=P\left(a_2\right)=\cdots=P\left(a_{n+2}\right)=0$. By Rolle's Theorem there are real numbers $c_1,\cdots,c_{n+1}$ with $a_1<c_1<a_2,\cdots,a_{n+1}<c_{n+1}<a_{n+2}$ and $P'\left(c_1\right)=\cdots=P'\left(c_{n+1}\right)=0$ Thus, the n th degree polynomial $P'(x)$ has at least $n+1$ roots. This contradiction shows that $P(x)$ has at most $n+1$ real roots.

22. **(a)** Suppose that $f(a)=f(b)=0$ where $a<b$. By Rolle's Theorem applied to f on $[a,b]$ there is a number c such that $a<c<b$ and $f'(c)=0$.
(b) Suppose that $f(a)=f(b)=f(c)=0$ where $a<b<c$. By Rolle's Theorem applied to $f(x)$ on $[a,b]$ and $[b,c]$ there are numbers $a<d<b$ and $b<e<c$ with $f'(d)=0$ and $f'(e)=0$. By Rolle's Theorem applied to $f'(x)$ on $[d,e]$ there is a number g with $d<g<e$ such that $f''(g)=0$.
(c) Suppose that f is n times differentiable on R and has $n+1$ distinct real roots. Then $f^{(n)}$ has at least one real root.

23. By the Mean Value Theorem, $f(4)-f(1)=f'(c)(4-1)$ for some $c\in(1,4)$. But for every $c\in(1,4)$ we have $f'(c)\geq 2$. Putting $f'(c)\geq 2$ into the above equation and substituting $f(1)=10$, we get $f(4)=f(1)+f'(c)(4-1)=10+3f'(c)\geq 10+3\cdot 2=16$. So the smallest possible value of $f(4)$ is 16 .

24. If $3\leq f'(x)\leq 5$ for all x , then by the Mean Value Theorem, $f(8)-f(2)=f'(c)\cdot(8-2)$ for some c in $[2,8]$.
(f is differentiable for all x , so, in particular, f is differentiable on $(2,8)$ and continuous on $[2,8]$. Thus, the hypotheses of the Mean Value Theorem are satisfied.) Since $f(8)-f(2)=6f'(c)$ and $3\leq f'(c)\leq 5$, it follows that $6\cdot 3\leq 6f'(c)\leq 6\cdot 5\Rightarrow 18\leq f(8)-f(2)\leq 30$.

25. Suppose that such a function f exists. By the Mean Value Theorem there is a number $0<c<2$ with $f'(c)=\dfrac{f(2)-f(0)}{2-0}=\dfrac{5}{2}$. But this is impossible since $f'(x)\leq 2<\dfrac{5}{2}$ for all x , so no such function can exist.

26. Let $h=f-g$. Then since f and g are continuous on $[a,b]$ and differentiable on (a,b) , so is h , and thus h satisfies the assumptions of the Mean Value Theorem. Therefore, there is a number c with $a<c<b$ such that $h(b)=h(b)-h(a)=h'(c)(b-a)$. Since $h'(c)<0$, $h'(c)(b-a)<0$, so $f(b)-g(b)=h(b)<0$ and hence $f(b)<g(b)$.

27. We use Exercise 26 with $f(x)=\sqrt{1+x}$, $g(x)=1+\frac{1}{2}x$, and $a=0$. Notice that $f(0)=1=g(0)$ and $f'(x)=\dfrac{1}{2\sqrt{1+x}}<\dfrac{1}{2}=g'(x)$ for $x>0$. So by Exercise 26, $f(b)<g(b)\Rightarrow\sqrt{1+b}<1+\frac{1}{2}b$ for $b>0$.
Another method: Apply the Mean Value Theorem directly to either $f(x)=1+\frac{1}{2}x-\sqrt{1+x}$ or $g(x)=\sqrt{1+x}$ on $[0,b]$.

28. f satisfies the conditions for the Mean Value Theorem, so we use this theorem on the interval $[-b,b]$: $\dfrac{f(b)-f(-b)}{b-(-b)}=f'(c)$ for some $c\in(-b,b)$. But since f is odd, $f(-b)=-f(b)$. Substituting this into the above equation, we get $\dfrac{f(b)+f(b)}{2b}=f'(c)\Rightarrow\dfrac{f(b)}{b}=f'(c)$.

29. Let $f(x)=\sin x$ and let $b<a$. Then $f(x)$ is continuous on $[b,a]$ and differentiable on (b,a) . By the Mean Value Theorem, there is a number $c\in(b,a)$ with
$\sin a-\sin b=f(a)-f(b)=f'(c)(a-b)=(\cos c)(a-b)$. Thus, $|\sin a-\sin b|\le|\cos c|\,|b-a|\le|a-b|$. If $a<b$, then $|\sin a-\sin b|=|\sin b-\sin a|\le|b-a|=|a-b|$. If $a=b$, both sides of the inequality are 0 .

30. Suppose that $f'(x)=c$. Let $g(x)=cx$, so $g'(x)=c$. Then, by Corollary 7, $f(x)=g(x)+d$, where d is a constant, so $f(x)=cx+d$.

31. For $x>0$, $f(x)=g(x)$, so $f'(x)=g'(x)$. For $x<0$, $f'(x)=(1/x)'=-1/x^2$ and $g'(x)=(1+1/x)'=-1/x^2$, so again $f'(x)=g'(x)$. However, the domain of $g(x)$ is not an interval so we cannot conclude that $f-g$ is constant (in fact it is not).

32. Let $f(x)=2\sin^{-1}x-\cos^{-1}\left(1-2x^2\right)$. Then
$f'(x)=\dfrac{2}{\sqrt{1-x^2}}-\dfrac{4x}{\sqrt{1-\left(1-2x^2\right)^2}}=\dfrac{2}{\sqrt{1-x^2}}-\dfrac{4x}{2x\sqrt{1-x^2}}=0$ (since $x\ge0$). Thus, $f'(x)=0$ for all $x\in(0,1)$. Thus, $f(x)=C$ on $(0,1)$. To find C , let $x=0.5$. Thus,
$2\sin^{-1}(0.5)-\cos^{-1}(0.5)=2\left(\dfrac{\pi}{6}\right)-\dfrac{\pi}{3}=0=C$. We conclude that $f(x)=0$ for x in $(0,1)$. By continuity of f , $f(x)=0$ on $[0,1]$. Therefore, we see that $f(x)=2\sin^{-1}x-\cos^{-1}\left(1-2x^2\right)=0\Rightarrow$
$2\sin^{-1}x=\cos^{-1}\left(1-2x^2\right)$.

33. Let $f(x)=\arcsin\left(\dfrac{x-1}{x+1}\right)-2\arctan\sqrt{x}+\dfrac{\pi}{2}$. Note that the domain of f is $[0,\infty)$. Thus,
$f'(x)=\dfrac{1}{\sqrt{1-\left(\dfrac{x-1}{x+1}\right)^2}}\,\dfrac{(x+1)-(x-1)}{(x+1)^2}-\dfrac{2}{1+x}\cdot\dfrac{1}{2\sqrt{x}}=\dfrac{1}{\sqrt{x}\,(x+1)}-\dfrac{1}{\sqrt{x}\,(x+1)}=0$. Then
$f(x)=C$ on $(0,\infty)$ by Theorem 5. By continuity of f , $f(x)=C$ on $[0,\infty)$. To find C , we let $x=0\Rightarrow$ $\arcsin(-1)-2\arctan(0)+\dfrac{\pi}{2}=C\Rightarrow-\dfrac{\pi}{2}-0+\dfrac{\pi}{2}=0=C$. Thus, $f(x)=0\Rightarrow\arcsin\left(\dfrac{x-1}{x+1}\right)=2\arctan\sqrt{x}-\dfrac{\pi}{2}$.

34. Let $v(t)$ be the velocity of the car t hours after 2:00 P.M. Then

$\frac{v(1/6)-v(0)}{1/6-0}=\frac{50-30}{1/6}=120$. By the Mean Value Theorem, there is a number c such that $0<c<\frac{1}{6}$ with $v'(c)=120$. Since $v'(t)$ is the acceleration at time t , the acceleration c hours after 2:00 P.M. is exactly 120 mi / h ^{2} .

35. Let $g(t)$ and $h(t)$ be the position functions of the two runners and let $f(t)=g(t)-h(t)$. By hypothesis, $f(0)=g(0)-h(0)=0$ and $f(b)=g(b)-h(b)=0$, where b is the finishing time. Then by the Mean Value Theorem, there is a time c , with $0<c<b$, such that $f'(c)=\frac{f(b)-f(0)}{b-0}$. But $f(b)=f(0)=0$, so $f'(c)=0$. Since $f'(c)=g'(c)-h'(c)=0$, we have $g'(c)=h'(c)$. So at time c , both runners have the same speed $g'(c)=h'(c)$.

36. Assume that f is differentiable (and hence continuous) on R and that $f'(x)\neq 1$ for all x . Suppose f has more than one fixed point. Then there are numbers a and b such that $a<b$, $f(a)=a$, and $f(b)=b$. Applying the Mean Value Theorem to the function f on $[a,b]$, we find that there is a number c in (a,b) such that $f'(c)=\frac{f(b)-f(a)}{b-a}$. But then $f'(c)=\frac{b-a}{b-a}=1$, contradicting our assumption that $f'(x)\neq 1$ for every real number x . This shows that our supposition was wrong, that is, that f cannot have more than one fixed point.

1. **(a)** f is increasing on $(0,6)$ and $(8,9)$.
(b) f is decreasing on $(6,8)$.
(c) f is concave upward on $(2,4)$ and $(7,9)$.
(d) f is concave downward on $(0,2)$ and $(4,7)$.
(e) The points of inflection are $(2,3)$, $(4,4.5)$ and $(7,4)$ (where the concavity changes).

2. **(a)** f is increasing on $(1,\approx 3.8)$ and $(5,\approx 6.5)$.
(b) f is decreasing on $(0,1)$, $(\approx 3.8,5)$, $(\approx 6.5,8)$, and $(8,9)$.
(c) f is concave upward on $(0,3)$ and $(8,9)$.
(d) f is concave downward on $(3,5)$ and $(5,8)$.
(e) The point of inflection is $(3,\approx 1.8)$ (where the concavity changes).

3. **(a)** Use the Increasing/Decreasing (I/D) Test.
(b) Use the Concavity Test.
(c) At any value of x where the concavity changes, we have an inflection point at $(x,f(x))$.

4. **(a)** See the First Derivative Test.
(b) See the Second Derivative Test and the note that precedes Example 7.

5. **(a)** Since $f'(x)>0$ on $(1,5)$, f is increasing on this interval. Since $f'(x)<0$ on $(0,1)$ and $(5,6)$, f is decreasing on these intervals.
(b) Since $f'(x)=0$ at $x=1$ and f' changes from negative to positive there, f changes from decreasing to increasing and has a local minimum at $x=1$. Since $f'(x)=0$ at $x=5$ and f' changes from positive to negative there, f changes from increasing to decreasing and has a local maximum at $x=5$.

6. **(a)** $f'(x)>0$ and f is increasing on $(0,1)$ and $(3,5)$. $f'(x)<0$ and f is decreasing on $(1,3)$ and $(5,6)$.
(b) Since $f'(x)=0$ at $x=1$ and $x=5$ and f' changes from positive to negative at both values, f changes from increasing to decreasing and has local maxima at $x=1$ and $x=5$. Since $f'(x)=0$ at $x=3$ and f' changes from negative to positive there, f changes from decreasing to increasing and has a local minimum at $x=3$.

7. There is an inflection point at $x=1$ because $f''(x)$ changes from negative to positive there, and so the graph of f changes from concave downward to concave upward. There is an inflection point at $x=7$ because $f''(x)$ changes from positive to negative there, and so the graph of f changes from concave upward to concave downward.

8. **(a)** f is increasing on the intervals where

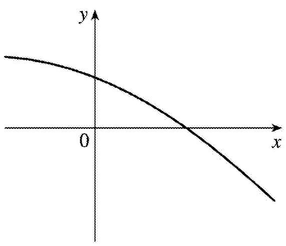
$f'(x)>0$, namely, $(2,4)$ and $(6,9)$.

(b) f has a local maximum where it changes from increasing to decreasing, that is, where f' changes from positive to negative (at $x=4$). Similarly, where f' changes from negative to positive, f has a local minimum (at $x=2$ and at $x=6$).

(c) When f' is increasing, its derivative f'' is positive and hence, f is concave upward. This happens on $(1,3)$, $(5,7)$, and $(8,9)$. Similarly, f is concave downward when f' is decreasing — that is, on $(0,1)$, $(3,5)$, and $(7,8)$.

(d) f has inflection points at $x=1$, 3 , 5 , 7 , and 8 , since the direction of concavity changes at each of these values.

9. The function must be always decreasing and concave downward.



10. **(a)** The rate of increase of the population is initially very small, then gets larger until it reaches a maximum at about $t=8$ hours, and decreases toward 0 as the population begins to level off.

(b) The rate of increase has its maximum value at $t=8$ hours.

(c) The population function is concave upward on $(0,8)$ and concave downward on $(8,18)$.

(d) At $t=8$, the population is about 350 , so the inflection point is about $(8,350)$.

11. **(a)** $f(x)=x^3-12x+1 \Rightarrow f'(x)=3x^2-12=3(x+2)(x-2)$.

We don't need to include "3" in the chart to determine the sign of $f'(x)$.

Interval	$x+2$	$x-2$	$f'(x)$	f
$x<-2$	$-$	$-$	$+$	increasing on $(-\infty,-2)$
$-2<x<2$	$+$	$-$	$-$	decreasing on $(-2,2)$
$x>2$	$+$	$+$	$+$	increasing on $(2,\infty)$

So f is increasing on $(-\infty,-2)$ and $(2,\infty)$ and f is decreasing on $(-2,2)$.

(b) f changes from increasing to decreasing at $x=-2$ and from decreasing to increasing at $x=2$. Thus, $f(-2)=17$ is a local maximum value and $f(2)=-15$ is a local minimum value.

(c) $f''(x)=6x$. $f''(x)>0 \Leftrightarrow x>0$ and $f''(x)<0 \Leftrightarrow x<0$. Thus, f is concave upward on $(0,\infty)$ and concave downward on $(-\infty,0)$. There is an inflection point where the concavity changes, at $(0,\mathrm{f}(0))=(0,1)$.

12. **(a)** $f(x)=5-3x^2+x^3 \Rightarrow f'(x)=-6x+3x^2=3x(x-2)$. Thus, $f'(x)>0 \Leftrightarrow x<0$ or $x>2$ and $f'(x)<0 \Leftrightarrow 0<x<2$. So f is increasing on $(-\infty,0)$ and $(2,\infty)$ and f is decreasing on $(0,2)$.
(b) f changes from increasing to decreasing at $x=0$ and from decreasing to increasing at $x=2$. Thus, $f(0)=5$ is a local maximum value and $f(2)=1$ is a local minimum value.
(c) $f''(x)=-6+6x=6(x-1)$. $f''(x)>0 \Leftrightarrow x>1$ and $f''(x)<0 \Leftrightarrow x<1$. Thus, f is concave upward on $(1,\infty)$ and concave downward on $(-\infty,1)$. There is an inflection point at $(1,3)$.

13. **(a)** $f(x)=x^4-2x^2+3 \Rightarrow f'(x)=4x^3-4x=4x\left(x^2-1\right)=4x\,(x+1)\,(x-1)$.

Interval	$x+1$	x	$x-1$	$f'(x)$	f
$x<-1$	$-$	$-$	$-$	$-$	decreasing on $(-\infty,-1)$
$-1<x<0$	$+$	$-$	$-$	$+$	increasing on $(-1,0)$
$0<x<1$	$+$	$+$	$-$	$-$	decreasing on $(0,1)$
$x>1$	$+$	$+$	$+$	$+$	increasing on $(1,\infty)$

So f is increasing on $(-1,0)$ and $(1,\infty)$ and f is decreasing on $(-\infty,-1)$ and $(0,1)$.
(b) f changes from increasing to decreasing at $x=0$ and from decreasing to increasing at $x=-1$ and $x=1$. Thus, $f(0)=3$ is a local maximum value and $f(\pm 1)=2$ are local minimum values.
(c) $f''(x)=12x^2-4=12\left(x^2-\dfrac{1}{3}\right)=12\left(x+1/\sqrt{3}\right)\left(x-1/\sqrt{3}\right)$. $f''(x)>0 \Leftrightarrow x<-1/\sqrt{3}$ or $x>1/\sqrt{3}$ and $f''(x)<0 \Leftrightarrow -1/\sqrt{3}<x<1/\sqrt{3}$. Thus, f is concave upward on $\left(-\infty,-\sqrt{3}/3\right)$ and $\left(\sqrt{3}/3,\infty\right)$ and concave downward on $\left(-\sqrt{3}/3,\sqrt{3}/3\right)$. There are inflection points at $\left(\pm\sqrt{3}/3,\dfrac{22}{9}\right)$.

14. **(a)** $f(x)=\dfrac{x^2}{x^2+3} \Rightarrow f'(x)=\dfrac{\left(x^2+3\right)(2x)-x^2(2x)}{\left(x^2+3\right)^2}=\dfrac{6x}{\left(x^2+3\right)^2}$. The denominator is positive so the sign of $f'(x)$ is determined by the sign of x . Thus, $f'(x)>0 \Leftrightarrow x>0$ and $f'(x)<0 \Leftrightarrow x<0$. So f is increasing on $(0,\infty)$ and f is decreasing on $(-\infty,0)$.
(b) f changes from decreasing to increasing at $x=0$. Thus, $f(0)=0$ is a local minimum value.
(c)

$$f''(x)=\frac{x^2+3^{\,2}(6)-6x\cdot 2\left(x^2+3\right)(2x)}{\left[\left(x^2+3\right)^2\right]^2}=\frac{6\left(x^2+3\right)\left[x^2+3-4x^2\right]}{\left(x^2+3\right)^4}$$

$$=\frac{6\left(3-3x^2\right)}{\left(x^2+3\right)^3}=\frac{-18(x+1)(x-1)}{\left(x^2+3\right)^3}\ .$$

$f''(x)>0 \Leftrightarrow -1<x<1$ and $f''(x)<0 \Leftrightarrow x<-1$ or $x>1$. Thus, f is concave upward on $(-1,1)$ and concave downward on $(-\infty,-1)$ and $(1,\infty)$. There are inflection points at $\left(\pm 1,\frac{1}{4}\right)$.

15. **(a)** $f(x)=x-2\sin x$ on $(0,3\pi) \Rightarrow f'(x)=1-2\cos x$. $f'(x)>0 \Leftrightarrow 1-2\cos x>0 \Leftrightarrow \cos x<\frac{1}{2} \Leftrightarrow \frac{\pi}{3}<x<\frac{5\pi}{3}$ or $\frac{7\pi}{3}<x<3\pi$. $f'(x)<0 \Leftrightarrow \cos x>\frac{1}{2} \Leftrightarrow 0<x<\frac{\pi}{3}$ or $\frac{5\pi}{3}<x<\frac{7\pi}{3}$. So f is increasing on $\left(\frac{\pi}{3},\frac{5\pi}{3}\right)$ and $\left(\frac{7\pi}{3},3\pi\right)$, and f is decreasing on $\left(0,\frac{\pi}{3}\right)$ and $\left(\frac{5\pi}{3},\frac{7\pi}{3}\right)$.

(b) f changes from increasing to decreasing at $x=\frac{5\pi}{3}$, and from decreasing to increasing at $x=\frac{\pi}{3}$ and at $x=\frac{7\pi}{3}$. Thus, $f\left(\frac{5\pi}{3}\right)=\frac{5\pi}{3}+\sqrt{3}\approx 6.97$ is a local maximum value and $f\left(\frac{\pi}{3}\right)=\frac{\pi}{3}-\sqrt{3}\approx -0.68$ and $f\left(\frac{7\pi}{3}\right)=\frac{7\pi}{3}-\sqrt{3}\approx 5.60$ are local minimum values.

(c) $f''(x)=2\sin x>0 \Leftrightarrow 0<x<\pi$ and $2\pi<x<3\pi$, $f''(x)<0 \Leftrightarrow \pi<x<2\pi$. Thus, f is concave upward on $(0,\pi)$ and $(2\pi,3\pi)$, and f is concave downward on $(\pi,2\pi)$. There are inflection points at (π,π) and $(2\pi,2\pi)$.

16. **(a)** $f(x)=\cos^2 x-2\sin x$, $0\le x\le 2\pi$. $f'(x)=-2\cos x\sin x-2\cos x=-2\cos x(1+\sin x)$. Note that $1+\sin x\ge 0$ [since $\sin x\ge -1$], with equality $\Leftrightarrow \sin x=-1 \Leftrightarrow x=3\pi/2$ [since $0\le x\le 2\pi$] $\Rightarrow \cos x=0$. Thus, $f'(x)>0 \Leftrightarrow \cos x<0 \Leftrightarrow \pi/2<x<3\pi/2$ and $f'(x)<0 \Leftrightarrow \cos x>0 \Leftrightarrow 0<x<\pi/2$ or $3\pi/2<x<2\pi$. Thus, f is increasing on $(\pi/2,3\pi/2)$ and f is decreasing on $(0,\pi/2)$ and $(3\pi/2,2\pi)$.

(b) f changes from decreasing to increasing at $x=\pi/2$ and from increasing to decreasing at $x=3\pi/2$. Thus, $f(\pi/2)=-2$ is a local minimum value and $f(3\pi/2)=2$ is a local maximum value.

(c)

$$\begin{aligned} f''(x) &=2\sin x(1+\sin x)-2\cos^2 x=2\sin x+2\sin^2 x-2\left(1-\sin^2 x\right)\\ &=4\sin^2 x+2\sin x-2=2(2\sin x-1)(\sin x+1)\end{aligned}$$

so $f''(x)>0 \Leftrightarrow \sin x>\frac{1}{2} \Leftrightarrow \frac{\pi}{6}<x<\frac{5\pi}{6}$, and $f''(x)<0 \Leftrightarrow \sin x<\frac{1}{2}$ and $\sin x\ne -1 \Leftrightarrow 0<x<\frac{\pi}{6}$ or $\frac{5\pi}{6}<x<\frac{3\pi}{2}$ or $\frac{3\pi}{2}<x<2\pi$. Thus, f is concave upward on $\left(\frac{\pi}{6},\frac{5\pi}{6}\right)$ and concave downward on

$\left(0,\frac{\pi}{6}\right)$, $\left(\frac{5\pi}{6},\frac{3\pi}{2}\right)$, and $\left(\frac{3\pi}{2},2\pi\right)$. There are inflection points at $\left(\frac{\pi}{6},-\frac{1}{4}\right)$ and $\left(\frac{5\pi}{6},-\frac{1}{4}\right)$.

17. **(a)** $y=f(x)=xe^{x}\Rightarrow f'(x)=xe^{x}+e^{x}=e^{x}(x+1)$. So $f'(x)>0\Leftrightarrow x+1>0\Leftrightarrow x>-1$. Thus, f is increasing on $(-1,\infty)$ and decreasing on $(-\infty,-1)$.

(b) f changes from decreasing to increasing at its only critical number, $x=-1$. Thus, $f(-1)=-e^{-1}$ is a local minimum value.

(c) $f'(x)=e^{x}(x+1)\Rightarrow f''(x)=e^{x}(1)+(x+1)e^{x}=e^{x}(x+2)$. So $f''(x)>0\Leftrightarrow x+2>0\Leftrightarrow x>-2$. Thus, f is concave upward on $(-2,\infty)$ and concave downward on $(-\infty,-2)$. Since the concavity changes direction at $x=-2$, the point $\left(-2,-2e^{-2}\right)$ is an inflection point.

18. **(a)** $y=f(x)=x^{2}e^{x}\Rightarrow f'(x)=x^{2}e^{x}+2xe^{x}=x(x+2)e^{x}$. So $f'(x)>0\Leftrightarrow x(x+2)>0\Leftrightarrow$ either $x<-2$ or $x>0$. Therefore f is increasing on $(-\infty,-2)$ and $(0,\infty)$, and decreasing on $(-2,0)$.

(b) f changes from increasing to decreasing at $x=-2$, so $f(-2)=4e^{-2}$ is a local maximum value. f changes from decreasing to increasing at $x=0$, so $f(0)=0$ is a local minimum value.

(c) $f'(x)=\left(x^{2}+2x\right)e^{x}\Rightarrow f''(x)=\left(x^{2}+2x\right)e^{x}+e^{x}(2x+2)=e^{x}\left(x^{2}+4x+2\right)$. $f''(x)=0\Leftrightarrow x^{2}+4x+2=0\Leftrightarrow x=-2\pm\sqrt{2}$. $f''(x)<0\Leftrightarrow -2-\sqrt{2}<x<-2+\sqrt{2}$, so f is concave downward on $\left(-2-\sqrt{2},-2+\sqrt{2}\right)$ and concave upward on $\left(-\infty,-2-\sqrt{2}\right)$ and $\left(-2+\sqrt{2},\infty\right)$. There are inflection points at $\left(-2-\sqrt{2},\mathrm{f}\left(-2-\sqrt{2}\right)\right)\approx(-3.41,0.38)$ and $\left(-2+\sqrt{2},\mathrm{f}\left(-2+\sqrt{2}\right)\right)\approx(-0.59,0.19)$.

19. **(a)** $y=f(x)=\dfrac{\ln x}{\sqrt{x}}$. (Note that f is only defined for $x>0$.)

$$f'(x)=\frac{\sqrt{x}(1/x)-\ln x\left(\frac{1}{2}x^{-1/2}\right)}{x}=\frac{\frac{1}{\sqrt{x}}-\frac{\ln x}{2\sqrt{x}}}{x}\cdot\frac{2\sqrt{x}}{2\sqrt{x}}=\frac{2-\ln x}{2x^{3/2}}>0\Leftrightarrow 2-\ln x>0\Leftrightarrow \ln x<2\Leftrightarrow x<e^{2}.$$

Therefore f is increasing on $\left(0,\mathrm{e}^{2}\right)$ and decreasing on $\left(e^{2},\infty\right)$.

(b) f changes from increasing to decreasing at $x=e^{2}$, so $f\left(e^{2}\right)=\dfrac{\ln e^{2}}{\sqrt{e^{2}}}=\dfrac{2}{e}$ is a local maximum value.

(c)

$$f''(x)=\frac{2x^{3/2}(-1/x)-(2-\ln x)\left(3x^{1/2}\right)}{\left(2x^{3/2}\right)^2}=\frac{-2x^{1/2}+3x^{1/2}(\ln x-2)}{4x^3}$$
$$=\frac{x^{1/2}(-2+3\ln x-6)}{4x^3}=\frac{3\ln x-8}{4x^{5/2}}$$

$f''(x)=0 \Leftrightarrow \ln x=\frac{8}{3} \Leftrightarrow x=e^{8/3}$. $f''(x)>0 \Leftrightarrow x>e^{8/3}$, so f is concave upward on $(e^{8/3},\infty)$ and concave downward on $(0,e^{8/3})$. There is an inflection point at $\left(e^{8/3},\frac{8}{3}e^{-4/3}\right)\approx(14.39,0.70)$.

20. **(a)** $y=f(x)=x\ln x$. (Note that f is only defined for $x>0$.)
$f'(x)=x(1/x)+\ln x=1+\ln x$. $f'(x)>0 \Leftrightarrow \ln x+1>0 \Leftrightarrow \ln x>-1 \Leftrightarrow x>e^{-1}$. Therefore f is increasing on $(1/e,\infty)$ and decreasing on $(0,1/e)$.
(b) f changes from decreasing to increasing at $x=1/e$, so $f(1/e)=-1/e$ is a local minimum value.
(c) $f''(x)=1/x>0$ for $x>0$. So f is concave upward on its entire domain, and has no inflection point.

21. $f(x)=x^5-5x+3 \Rightarrow f'(x)=5x^4-5=5\left(x^2+1\right)(x+1)(x-1)$.
First Derivative Test: $f'(x)<0 \Rightarrow -1<x<1$ and $f'(x)>0 \Rightarrow x>1$ or $x<-1$. Since f' changes from positive to negative at $x=-1$, $f(-1)=7$ is a local maximum value; and since f' changes from negative to positive at $x=1$, $f(1)=-1$ is a local minimum value.
Second Derivative Test: $f''(x)=20x^3$. $f'(x)=0 \Leftrightarrow x=\pm 1$. $f''(-1)=-20<0 \Rightarrow f(-1)=7$ is a local maximum value. $f''(1)=20>0 \Rightarrow f(1)=-1$ is a local minimum value.
Preference: For this function, the two tests are equally easy.

22. $f(x)=\dfrac{x}{x^2+4} \Rightarrow f'(x)=\dfrac{\left(x^2+4\right)\cdot 1-x(2x)}{x^2+4\;^2}=\dfrac{4-x^2}{\left(\dfrac{()}{x^2+4}\right)^2}=\dfrac{(2+x)(2-x)}{\left(x^2+4\right)^2}$.

First Derivative Test: $f'(x)>0 \Rightarrow -2<x<2$ and $f'(x)<0 \Rightarrow x>2$ or $x<-2$. Since f' changes from positive to negative at $x=2$, $f(2)=\frac{1}{4}$ is a local maximum value; and since f' changes from negative to positive at $x=-2$, $f(-2)=-\frac{1}{4}$ is a local minimum value. *Second Derivative Test:*

$$f''(x) = \frac{\left(x^2+4\right)^2(-2x)-\left(4-x^2\right)\cdot 2\left(x^2+4\right)(2x)}{\left[\left(x^2+4\right)^2\right]^2}$$
$$= \frac{-2x\left(x^2+4\right)\left[\left(x^2+4\right)+2\left(4-x^2\right)\right]}{\left(x^2+4\right)^4} = \frac{-2x\left(12-x^2\right)}{\left(x^2+4\right)^3}$$

$f'(x)=0 \Leftrightarrow x=\pm 2$. $f''(-2)=\frac{1}{16}>0 \Rightarrow f(-2)=-\frac{1}{4}$ is a local minimum value.
$f''(2)=-\frac{1}{16}<0 \Rightarrow f(2)=\frac{1}{4}$ is a local maximum value.
Preference: Since calculating the second derivative is fairly difficult, the First Derivative Test is easier to use for this function.

23. $f(x)=x+\sqrt{1-x} \Rightarrow f'(x)=1+\frac{1}{2}(1-x)^{-1/2}(-1)=1-\frac{1}{2\sqrt{1-x}}$. Note that f is defined for $1-x\geq 0$; that is, for $x\leq 1$. $f'(x)=0 \Rightarrow 2\sqrt{1-x}=1 \Rightarrow \sqrt{1-x}=\frac{1}{2} \Rightarrow 1-x=\frac{1}{4} \Rightarrow x=\frac{3}{4}$. f' does not exist at $x=1$, but we can't have a local maximum or minimum at an endpoint.
First Derivative Test: $f'(x)>0 \Rightarrow x<\frac{3}{4}$ and $f'(x)<0 \Rightarrow \frac{3}{4}<x<1$. Since f' changes from positive to negative at $x=\frac{3}{4}$, $f\left(\frac{3}{4}\right)=\frac{5}{4}$ is a local maximum value.
Second Derivative Test: $f''(x)=-\frac{1}{2}\left(-\frac{1}{2}\right)(1-x)^{-3/2}(-1)=-\frac{1}{4\left(\sqrt{1-x}\right)^3}$. $f''\left(\frac{3}{4}\right)=-2<0 \Rightarrow$
$f\left(\frac{3}{4}\right)=\frac{5}{4}$ is a local maximum value.
Preference: The First Derivative Test may be slightly easier to apply in this case.

24. **(a)** $f(x)=x^4(x-1)^3 \Rightarrow f'(x)=x^4\cdot 3(x-1)^2+(x-1)^3\cdot 4x^3=x^3(x-1)^2\left[3x+4(x-1)\right]=x^3(x-1)^2(7x-4)$
The critical numbers are 0 , 1 , and $\frac{4}{7}$.
(b)
$$f''(x) = 3x^2(x-1)^2(7x-4)+x^3\cdot 2(x-1)(7x-4)+x^3(x-1)^2\cdot 7$$
$$= x^2(x-1)\left[3(x-1)(7x-4)+2x(7x-4)+7x(x-1)\right]$$

Now $f''(0)=f''(1)=0$, so the Second Derivative Test gives no information for $x=0$ or $x=1$.

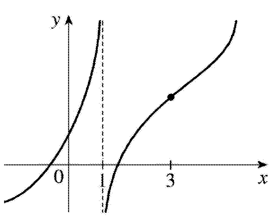
$f''\left(\frac{4}{7}\right)=\left(\frac{4}{7}\right)^2\left(\frac{4}{7}-1\right)\left[0+0+7\left(\frac{4}{7}\right)\left(\frac{4}{7}-1\right)\right]=\left(\frac{4}{7}\right)^2\left(-\frac{3}{7}\right)(4)\left(-\frac{3}{7}\right)>0$, so there is a local minimum at $x=\frac{4}{7}$.

(c) f' is positive on $(-\infty,0)$, negative on $\left(0,\frac{4}{7}\right)$, positive on $\left(\frac{4}{7},1\right)$, and positive on $(1,\infty)$. So f has a local maximum at $x=0$, a local minimum at $x=\frac{4}{7}$, and no local maximum or minimum at $x=1$.

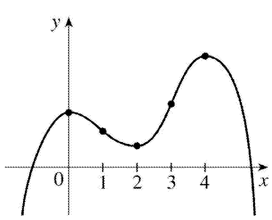
25. **(a)** By the Second Derivative Test, if $f'(2)=0$ and $f''(2)=-5<0$, f has a local maximum at $x=2$.

(b) If $f'(6)=0$, we know that f has a horizontal tangent at $x=6$. Knowing that $f''(6)=0$ does not provide any additional information since the Second Derivative Test fails. For example, the first and second derivatives of $y=(x-6)^4$, $y=-(x-6)^4$, and $y=(x-6)^3$ all equal zero for $x=6$, but the first has a local minimum at $x=6$, the second has a local maximum at $x=6$, and the third has an inflection point at $x=6$.

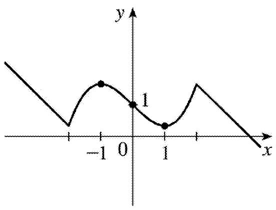
26. $f'(x)>0$ for all $x\neq 1$ with vertical asymptote $x=1$, so f is increasing on $(-\infty,1)$ and $(1,\infty)$. $f''(x)>0$ if $x<1$ or $x>3$, and $f''(x)<0$ if $1<x<3$, so f is concave upward on $(-\infty,1)$ and $(3,\infty)$, and concave downward on $(1,3)$. There is an inflection point when $x=3$.



27. $f'(0)=f'(2)=f'(4)=0 \Leftrightarrow$ horizontal tangents at $x=0$, 2 , 4 . $f'(x)>0$ if $x<0$ or $2<x<4 \Rightarrow f$ is increasing on $(-\infty,0)$ and $(2,4)$. $f'(x)<0$ if $0<x<2$ or $x>4 \Rightarrow f$ is decreasing on $(0,2)$ and $(4,\infty)$. $f''(x)>0$ if $1<x<3 \Rightarrow f$ is concave upward on $(1,3)$. $f''(x)<0$ if $x<1$ or $x>3 \Rightarrow f$ is concave downward on $(-\infty,1)$ and $(3,\infty)$. There are inflection points when $x=1$ and 3 .

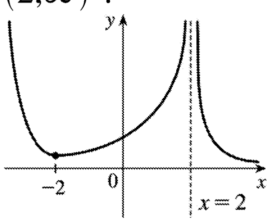


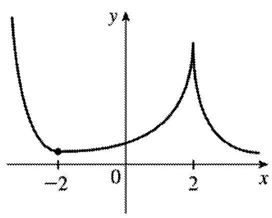
28.



$f'(1)=f'(-1)=0 \Rightarrow$ horizontal tangents at $x=\pm 1$. $f'(x)<0$ if $|x|<1 \Rightarrow f$ is decreasing on $(-1,1)$. $f'(x)>0$ if $1<|x|<2 \Rightarrow f$ is increasing on $(-2,-1)$ and $(1,2)$. $f'(x)=-1$ if $|x|>2 \Rightarrow$ the graph of f has constant slope -1 on $(-\infty,-2)$ and $(2,\infty)$. $f''(x)<0$ if $-2<x<0 \Rightarrow f$ is concave downward on $(-2,0)$. Inflection point $(0,1)$.

29. $f'(x)>0$ if $|x|<2 \Rightarrow f$ is increasing on $(-2,2)$. $f'(x)<0$ if $|x|>2 \Rightarrow f$ is decreasing on $(-\infty,-2)$ and $(2,\infty)$. $f'(-2)=0 \Rightarrow$ horizontal tangent at $x=-2$. $\lim_{x\to 2} \left|f'(x)\right|=\infty \Rightarrow$ there is a vertical asymptote or vertical tangent (cusp) at $x=2$. $f''(x)>0$ if $x\neq 2 \Rightarrow f$ is concave upward on $(-\infty,2)$ and $(2,\infty)$.





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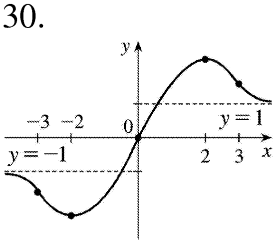
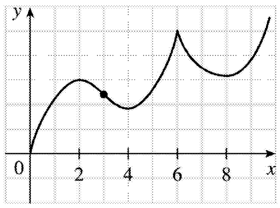


$f'(x)>0$ if $|x|<2 \Rightarrow f$ is increasing on $(-2,2)$. $f'(x)<0$ if $|x|>2 \Rightarrow f$ is decreasing on $(-\infty,-2)$ and $(2,\infty)$. $f'(2)=0$, so f has a horizontal tangent (and local maximum) at $x=2$. $\lim_{x\to\infty} f(x)=1 \Rightarrow y=1$ is a horizontal asymptote. $f(-x)=-f(x) \Rightarrow f$ is an odd function (its graph is symmetric about the origin). Finally, $f''(x)<0$ if $0<x<3$ and $f''(x)>0$ if $x>3$, so f is CD on $(0,3)$ and CU on $(3,\infty)$.

31. **(a)** f is increasing where f' is positive, that is, on $(0,2)$, $(4,6)$, and $(8,\infty)$; and decreasing where f' is negative, that is, on $(2,4)$ and $(6,8)$.

(b) f has local maxima where f' changes from positive to negative, at $x=2$ and at $x=6$, and local minima where f' changes from negative to positive, at $x=4$ and at $x=8$.

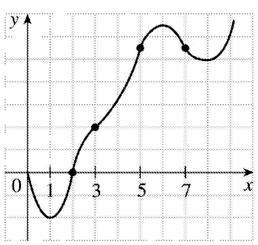
(c) f is concave upward (CU) where f' is increasing, that is, on $(3,6)$ and $(6,\infty)$, and concave downward (CD) where f' is decreasing, that is, on $(0,3)$.
(d) There is a point of in.ection where f changes from being CD to being CU, that is, at $x = 3$.

(e) 

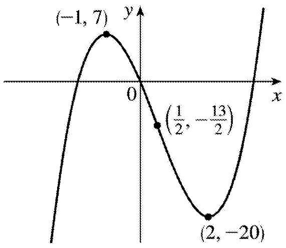
32. **(a)** f is increasing where f' is positive, on $(1,6)$ and $(8,\infty)$, and decreasing where f' is negative, on $(0,1)$ and $(6,8)$.

(b) f has a local maximum where f' changes from positive to negative, at $x=6$, and local minima where f' changes from negative to positive, at $x=1$ and at $x=8$.

(c) f is concave upward where f' is increasing, that is, on $(0\ \ 2)$, $(3,\ 5)$; and $(7,\ \infty)$ and concave downward where f' is decreasing, that is, on $(2\ \ 3)$, $(5,\ 7)$.
(d) There are points of inflection where f changes its direction of concavity, at $x=2$, $x=3$, $x=5$ and $x=7$.

(e) 

33. **(a)** $f(x)=2x^3-3x^2-12x \Rightarrow f'(x)=6x^2-6x-12=6\left(x^2-x-2\right)=6(x-2)(x+1)$. $f'(x)>0 \Leftrightarrow x<-1$ or $x>2$ and $f'(x)<0 \Leftrightarrow -1<x<2$. So f is increasing on $(-\infty,-1)$ and $(2,\infty)$, and f is decreasing on $(-1,2)$.
(b) Since f changes from increasing to decreasing at $x=-1$, $f(-1)=7$ is a local maximum value. Since f changes from decreasing to increasing at $x=2$, $f(2)=-20$ is a local minimum value.
(c) $f''(x)=6(2x-1) \Rightarrow f''(x)>0$ on $\left(\frac{1}{2},\infty\right)$ and $f''(x)<0$ on $\left(-\infty,\frac{1}{2}\right)$. So f is concave upward on $\left(\frac{1}{2},\infty\right)$ and concave downward on $\left(-\infty,\frac{1}{2}\right)$. There is a change in concavity at $x=\frac{1}{2}$, and we have an inflection point at $\left(\frac{1}{2},-\frac{13}{2}\right)$.
(d)

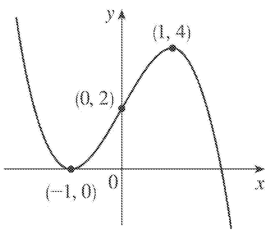


34. **(a)** $f(x)=2+3x-x^3 \Rightarrow f'(x)=3-3x^2=-3\left(x^2-1\right)=-3(x+1)(x-1)$. $f'(x)>0 \Leftrightarrow -1<x<1$ and $f'(x)<0 \Leftrightarrow x<-1$ or $x>1$. So f is increasing on $(-1,1)$ and f is decreasing on $(-\infty,-1)$ and $(1,\infty)$.

(b) $f(-1)=0$ is a local minimum value and $f(1)=4$ is a local maximum value.

(c) $f''(x)=-6x \Rightarrow f''(x)>0$ on $(-\infty,0)$ and $f''(x)<0$ on $(0,\infty)$. So f is concave upward on $(-\infty,0)$ and concave downward on $(0,\infty)$. There is an inflection point at $(0,2)$.

(d)



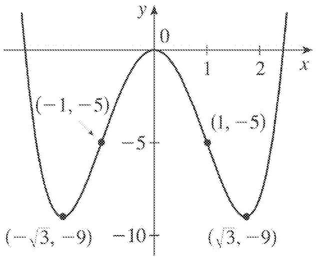
35. **(a)** $f(x)=x^4-6x^2 \Rightarrow f'(x)=4x^3-12x=4x\left(x^2-3\right)=0$ when $x=0$, $\pm\sqrt{3}$.

Interval	$4x$	x^2-3	$f'(x)$	f
$x<-\sqrt{3}$	−	+	−	decreasing on $(-\infty,-\sqrt{3})$
$-\sqrt{3}<x<0$	−	−	+	increasing on $(-\sqrt{3},0)$
$0<x<\sqrt{3}$	+	−	−	decreasing on $(0,\sqrt{3})$
$x>\sqrt{3}$	+	+	+	increasing on $(\sqrt{3},\infty)$

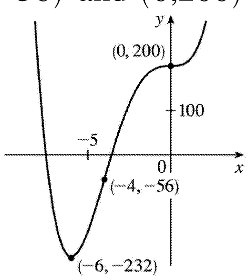
(b) Local minimum values $f(\pm\sqrt{3})=-9$, local maximum value $f(0)=0$

(c) $f''(x)=12x^2-12=12\left(x^2-1\right)>0 \Leftrightarrow x^2>1 \Leftrightarrow |x|>1 \Leftrightarrow x>1$ or $x<-1$, so f is CU on $(-\infty,-1)$, $(1,\infty)$ and CD on $(-1,1)$. Inflection points at $(\pm 1,-5)$

(d)



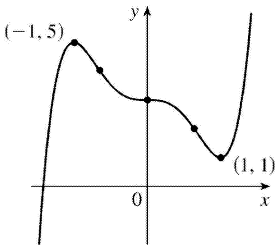
36. **(a)** $g(x)=200+8x^3+x^4 \Rightarrow g'(x)=24x^2+4x^3=4x^2(6+x)=0$ when $x=-6$ and when $x=0$. $g'(x)>0 \Leftrightarrow x>-6$ ($x\neq 0$) and $g'(x)<0 \Leftrightarrow x<-6$, so g is decreasing on $(-\infty,-6)$ and g is increasing on $(-6,\infty)$, with a horizontal tangent at $x=0$.
(b) $g(-6)=-232$ is a local minimum value. There is no local maximum value.
(c) $g''(x)=48x+12x^2=12x(4+x)=0$ when $x=-4$ and when $x=0$. $g''(x)>0 \Leftrightarrow x<-4$ or $x>0$ and $g''(x)<0 \Leftrightarrow -4<x<0$, so g is CU on $(-\infty,-4)$ and $(0,\infty)$, and g is CD on $(-4,0)$. Inflection points at $(-4,-56)$ and $(0,200)$

(d) 

37. **(a)** $h(x)=3x^5-5x^3+3 \Rightarrow h'(x)=15x^4-15x^2=15x^2\left(x^2-1\right)=0$ when $x=0$, ± 1 . Since $15x^2$ is nonnegative, $h'(x)>0 \Leftrightarrow x^2>1 \Leftrightarrow |x|>1 \Leftrightarrow x>1$ or $x<-1$, so h is increasing on $(-\infty,-1)$ and $(1,\infty)$ and decreasing on $(-1,1)$, with a horizontal tangent at $x=0$.
(b) Local maximum value $h(-1)=5$, local minimum value $h(1)=1$
(c)

$$\begin{aligned} h''(x) &= 60x^3-30x=30x\left(2x^2-1\right) \\ &= 60x\left(x+\frac{1}{\sqrt{2}}\right)\left(x-\frac{1}{\sqrt{2}}\right) \Rightarrow \end{aligned}$$

$h''(x)>0$ when $x>\frac{1}{\sqrt{2}}$ or $-\frac{1}{\sqrt{2}}<x<0$, so h is CU on $\left(-\frac{1}{\sqrt{2}},0\right)$ and $\left(\frac{1}{\sqrt{2}},\infty\right)$ and CD on $\left(-\infty,-\frac{1}{\sqrt{2}}\right)$ and $\left(0,\frac{1}{\sqrt{2}}\right)$. Inflection points at $(0,3)$ and $\left(\pm\frac{1}{\sqrt{2}},3\mp\frac{7}{8}\sqrt{2}\right)$.
(d)



38. **(a)** $h(x)=\left(x^2-1\right)^3 \Rightarrow h'(x)=6x\left(x^2-1\right)^2 \geq 0 \Leftrightarrow x>0$ ($x\neq 1$), so h is increasing on $(0,\infty)$ and decreasing on $(-\infty,0)$.

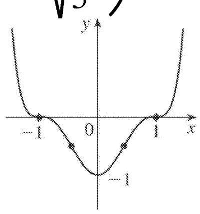
(b) $h(0)=-1$ is a local minimum value.

(c) $h''(x)=6\left(x^2-1\right)^2+24x^2\left(x^2-1\right)=6\left(x^2-1\right)\left(5x^2-1\right)$. The roots ± 1 and $\pm\dfrac{1}{\sqrt{5}}$ divide R into five intervals.

Interval	x^2	$5x^2-1$	$h''(x)$	Concavity
$x<-1$	+	+	+	upward
$-1<x<-\dfrac{1}{\sqrt{5}}$	–	+	–	downward
$-\dfrac{1}{\sqrt{5}}<x<\dfrac{1}{\sqrt{5}}$	–	–	+	upward
$\dfrac{1}{\sqrt{5}}<x<1$	–	+	–	downward
$x>1$	+	+	+	upward

From the table, we see that h is CU on $(-\infty,-1)$, $\left(-\dfrac{1}{\sqrt{5}},\dfrac{1}{\sqrt{5}}\right)$ and $(1,\infty)$, and CD on $\left(-1,-\dfrac{1}{\sqrt{5}}\right)$ and $\left(\dfrac{1}{\sqrt{5}},1\right)$. Inflection points at $(\pm 1,0)$ and $\left(\pm\dfrac{1}{\sqrt{5}},-\dfrac{64}{125}\right)$

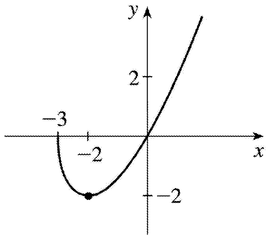
(d)

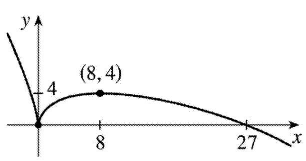


39. **(a)** $A(x)=x\sqrt{x+3} \Rightarrow A'(x)=x\cdot\dfrac{1}{2}(x+3)^{-1/2}+\sqrt{x+3}\cdot 1=\dfrac{x}{2\sqrt{x+3}}+\sqrt{x+3}=\dfrac{x+2(x+3)}{2\sqrt{x+3}}=\dfrac{3x+6}{2\sqrt{x+3}}$.

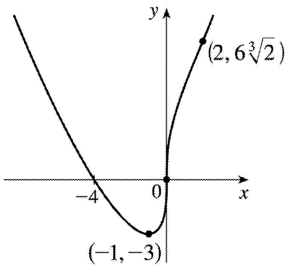
The domain of A is $[-3,\infty)$. $A'(x)>0$ for $x>-2$ and $A'(x)<0$ for $-3<x<-2$, so A is increasing on

$(-2,\infty)$ and decreasing on $(-3,-2)$.
(b) $A(-2)=-2$ is a local minimum value.
(c) $A''(x)=\dfrac{2\sqrt{x+3}\cdot 3-(3x+6)\cdot\dfrac{1}{\sqrt{x+3}}}{\left(2\sqrt{x+3}\right)^2}=\dfrac{6(x+3)-(3x+6)}{4(x+3)^{3/2}}=\dfrac{3x+12}{4(x+3)^{3/2}}=\dfrac{3(x+4)}{4(x+3)^{3/2}}$ $A''(x)>0$ for all $x>-3$, so A is concave upward on $(-3,\infty)$. There is no inflection point.

(d) 

40. **(a)** $B(x)=3x^{2/3}-x\Rightarrow B'(x)=2x^{-1/3}-1=\dfrac{2}{\sqrt[3]{x}}-1=\dfrac{2-\sqrt[3]{x}}{\sqrt[3]{x}}$. $B'(x)>0$ if $0<x<8$ and $B'(x)<0$ if $x<0$ or $x>8$, so B is decreasing on $(-\infty,0)$ and $(8,\infty)$, and B is increasing on $(0,8)$.
(b) $B(0)=0$ is a local minimum value.
$B(8)=4$ is a local maximum value.
(c) $B''(x)=-\dfrac{2}{3}x^{-4/3}=\dfrac{-2}{3x^{4/3}}$, so $B''(x)<0$ for all $x\neq 0$. B is concave downward on $(-\infty,0)$ and $(0,\infty)$. There is no inflection point.
(d) 

41. **(a)** $C(x)=x^{1/3}(x+4)=x^{4/3}+4x^{1/3}\Rightarrow C'(x)=\dfrac{4}{3}x^{1/3}+\dfrac{4}{3}x^{-2/3}=\dfrac{4}{3}x^{-2/3}(x+1)=\dfrac{4(x+1)}{3\sqrt[3]{x^2}}$. $C'(x)>0$ if $-1<x<0$ or $x>0$ and $C'(x)<0$ for $x<-1$, so C is increasing on $(-1,\infty)$ and C is decreasing on $(-\infty,-1)$.
(b) $C(-1)=-3$ is a local minimum value.
(c) $C''(x)=\dfrac{4}{9}x^{-2/3}-\dfrac{8}{9}x^{-5/3}=\dfrac{4}{9}x^{-5/3}(x-2)=\dfrac{4(x-2)}{9\sqrt[3]{x^5}}$. $C''(x)<0$ for $0<x<2$ and $C''(x)>0$ for $x<0$ and $x>2$, so C is concave downward on $(0,2)$ and concave upward on $(-\infty,0)$ and $(2,\infty)$. There are inflection points at $(0,0)$ and $\left(2,6\sqrt[3]{2}\right)\approx(2,7.56)$.
(d)



42. **(a)** $f(x)=\ln\left(x^4+27\right) \Rightarrow f'(x)=\dfrac{4x^3}{x^4+27}$. $f'(x)>0$ if $x>0$ and $f'(x)<0$ if $x<0$, so f is increasing on $(0,\infty)$ and f is decreasing on $(-\infty,0)$.

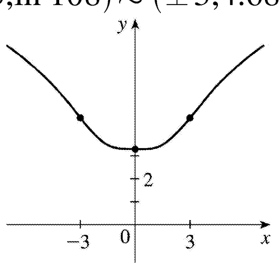
(b) $f(0)=\ln 27\approx 3.3$ is a local minimum value.

(c)

$$f''(x)=\frac{\left(x^4+27\right)\left(12x^2\right)-4x^3\left(4x^3\right)}{\left(x^4+27\right)^2}=\frac{4x^2\left[3\left(x^4+27\right)-4x^4\right]}{\left(x^4+27\right)^2}$$

$$=\frac{4x^2\left(81-x^4\right)}{\left(x^4+27\right)^2}=\frac{-4x^2\left(x^2+9\right)(x+3)(x-3)}{\left(x^4+27\right)^2}$$

$f''(x)>0$ if $-3<x<0$ and $0<x<3$, and $f''(x)<0$ if $x<-3$ or $x>3$. Thus, f is concave upward on $(-3,0)$ and $(0,3)$ and f is concave downward on $(-\infty,-3)$ and $(3,\infty)$. There are inflection points at $(\pm 3,\ln 108)\approx(\pm 3,4.68)$.



(d)

43. **(a)** $f(\theta)=2\cos\theta-\cos 2\theta$, $0\le\theta\le 2\pi$.

$f'(\theta)=-2\sin\theta+2\sin 2\theta=-2\sin\theta+2(2\sin\theta\cos\theta)=2\sin\theta(2\cos\theta-1)$.

Interval	$\sin\theta$	$2\cos\theta-1$	$f'(\theta)$	f
$0<\theta<\dfrac{\pi}{3}$	+	+	+	increasing on $(0,\dfrac{\pi}{3})$
$\dfrac{\pi}{3}<\theta<\pi$	+	–	–	decreasing on $(\dfrac{\pi}{3},\pi)$

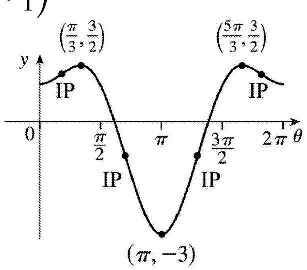
$\pi<\theta<\frac{5\pi}{3}$	$-$	$-$	$+$	increasing on $(\pi,\frac{5\pi}{3})$
$\frac{5\pi}{3}<\theta<2\pi$	$-$	$+$	$-$	decreasing on $(\frac{5\pi}{3},2\pi)$

(b) $f\left(\frac{\pi}{3}\right)=\frac{3}{2}$ and $f\left(\frac{5\pi}{3}\right)=\frac{3}{2}$ are local maximum values and $f(\pi)=-3$ is a local minimum value.

(c) $f'(\theta)=-2\sin\theta+2\sin 2\theta\Rightarrow$

$$f''(\theta)=-2\cos\theta+4\cos 2\theta=-2\cos\theta+4(2\cos^2\theta-1)$$
$$=2(4\cos^2\theta-\cos\theta-2)$$

$f''(\theta)=0\Leftrightarrow\cos\theta=\frac{1\pm\sqrt{33}}{8}\Leftrightarrow\theta=\cos^{-1}\left(\frac{1\pm\sqrt{33}}{8}\right)\Leftrightarrow\theta=\cos^{-1}\left(\frac{1+\sqrt{33}}{8}\right)\approx 0.5678$, $2\pi-\cos^{-1}\left(\frac{1+\sqrt{33}}{8}\right)\approx 5.7154$, $\cos^{-1}\left(\frac{1-\sqrt{33}}{8}\right)\approx 2.2057$, or $2\pi-\cos^{-1}\left(\frac{1-\sqrt{33}}{8}\right)\approx 4.0775$. Denote these four values of θ by θ_1, θ_4, θ_2, and θ_3, respectively. Then f is CU on $\left(0,\theta_1\right)$, CD on $\left(\theta_1,\theta_2\right)$,
CU on $\left(\theta_2,\theta_3\right)$, CD on $\left(\theta_3,\theta_4\right)$, and CU on $\left(\theta_4,2\pi\right)$. To find the *exact* y-coordinate for $\theta=\theta_1$, we have $f(\theta_1)=2\cos\theta_1-\cos 2\theta_1=2\cos\theta_1-\left(2\cos^2\theta_1-1\right)=2\left(\frac{1+\sqrt{33}}{8}\right)-2\left(\frac{1+\sqrt{33}}{8}\right)^2+1$
$=\frac{1}{4}+\frac{1}{4}\sqrt{33}-\frac{1}{32}-\frac{1}{16}\sqrt{33}-\frac{33}{32}+1=\frac{3}{16}+\frac{3}{16}\sqrt{33}=\frac{3}{16}\left(1+\sqrt{33}\right)=y_1\approx 1.26$. Similarly, $f(\theta_2)=\frac{3}{16}\left(1-\sqrt{33}\right)=y_2\approx -0.89$. So f has inflection points at $\left(\theta_1,y_1\right)$, $\left(\theta_2,y_2\right)$, $\left(\theta_3,y_2\right)$, and $\left(\theta_4,y_1\right)$.

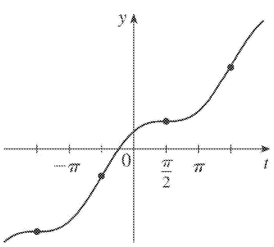


(d)

44. **(a)** $f(t)=t+\cos t$, $-2\pi\le t\le 2\pi\Rightarrow f'(t)=1-\sin t\ge 0$ for all t and $f'(t)=0$ when $\sin t=1\Leftrightarrow t=-\frac{3\pi}{2}$ or $\frac{\pi}{2}$, so f is increasing on $(-2\pi,2\pi)$.

(b) No maximum or minimum

(c) $f''(t) = -\cos t > 0 \quad \Leftrightarrow t \in \left(-\frac{3\pi}{2}, -\frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$, so f is CU on these intervals and CD 0n $\left(-2\pi, -\frac{3\pi}{2}\right)$, $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, and $\left(\frac{3\pi}{2}, 2\pi\right)$. Points of inflection at $\left(\pm\frac{3\pi}{2}, \pm\frac{3\pi}{2}\right)$ and $\left(\pm\frac{\pi}{2}, \pm\frac{\pi}{2}\right)$

(d) 

45. $f(x) = \frac{x^2}{x^2-1} = \frac{x^2}{(x+1)(x-1)}$ has domain $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$.

(a) $\lim_{x \to \pm\infty} f(x) = \lim_{x \to \pm\infty} \frac{x^2/x^2}{\left(x^2-1\right)/x^2} = \lim_{x \to \pm\infty} \frac{1}{1-1/x^2} = \frac{1}{1-0} = 1$, so $y=1$ is a HA.

$\lim_{x \to -1^-} \frac{x^2}{x^2-1} = \infty$ since $x^2 \to 1$ and $\left(x^2-1\right) \to 0^+$ as $x \to -1^-$, so $x=-1$ is a VA.

$\lim_{x \to 1^+} \frac{x^2}{x^2-1} = \infty$ since $x^2 \to 1$ and $\left(x^2-1\right) \to 0^+$ as $x \to 1^+$, so $x=1$ is a VA.

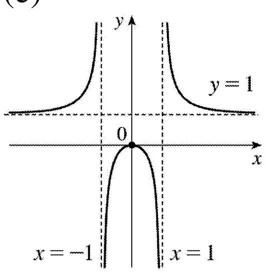
(b) $f(x) = \frac{x^2}{x^2-1} \Rightarrow f'(x) = \frac{\left(x^2-1\right)(2x) - x^2(2x)}{\left(x^2-1\right)^2} = \frac{2x\left[\left(x^2-1\right) - x^2\right]}{\left(x^2-1\right)^2} = \frac{-2x}{\left(x^2-1\right)^2}$. Since $\left(x^2-1\right)^2$ is positive for all x in the domain of f , the sign of the derivative is determined by the sign of $-2x$. Thus, $f'(x)>0$ if $x<0 \,(x \neq -1)$ and $f'(x)<0$ if $x>0 \,(x \neq 1)$. So f is increasing on $(-\infty, -1)$ and $(-1, 0)$, and f is decreasing on $(0, 1)$ and $(1, \infty)$.

(c) $f'(x) = 0 \Rightarrow x = 0$ and $f(0) = 0$ is a local maximum value.

(d)
$$f''(x)=\frac{\left(x^2-1\right)^2(-2)-(-2x)\cdot 2\left(x^2-1\right)(2x)}{\left(\left(x^2-1\right)^2\right)^2}$$
$$=\frac{2\left(x^2-1\right)\left(-\left(x^2-1\right)+4x^2\right)}{\left(x^2-1\right)^4}=\frac{2\left(3x^2+1\right)}{\left(x^2-1\right)^3}\ .$$

The sign of $f''(x)$ is determined by the denominator; that is, $f''(x)>0$ if $|x|>1$ and $f''(x)<0$ if $|x|<1$. Thus, f is CU on $(-\infty,-1)$ and $(1,\infty)$, and f is CD on $(-1,1)$. There are no inflection points.

(e)



46. $f(x)=\dfrac{x^2}{(x-2)^2}$ has domain $(-\infty,2)\cup(2,\infty)$.

(a) $\displaystyle\lim_{x\to\pm\infty}\frac{x^2}{x^2-4x+4}=\lim_{x\to\pm\infty}\frac{x^2/x^2}{\left(x^2-4x+4\right)/x^2}=\lim_{x\to\pm\infty}\frac{1}{1-4/x+4/x^2}=\frac{1}{1-0+0}=1$,

so $y=1$ is a HA. $\displaystyle\lim_{x\to 2^+}\frac{x^2}{(x-2)^2}=\infty$ since $x^2\to 4$ and $(x-2)^2\to 0^+$ as $x\to 2^+$, so $x=2$ is a VA.

(b) $f(x)=\dfrac{x^2}{(x-2)^2}\Rightarrow f'(x)=\dfrac{(x-2)^2(2x)-x^2\cdot 2(x-2)}{\left[(x-2)^2\right]^2}=\dfrac{2x(x-2)[(x-2)-x]}{(x-2)^4}=\dfrac{-4x}{(x-2)^3}$. $f'(x)>0$ if $0<x<2$ and $f'(x)<0$ if $x<0$ or $x>2$, so f is increasing on $(0,2)$ and f is decreasing on $(-\infty,0)$ and $(2,\infty)$.

(c) $f(0)=0$ is a local minimum value.

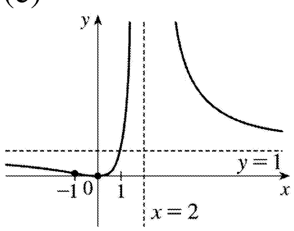
(d)

$$f''(x) = \frac{(x-2)^3(-4)-(-4x)\cdot 3(x-2)^2}{\left[(x-2)^3\right]^2}$$

$$= \frac{4(x-2)^2[-(x-2)+3x]}{(x-2)^6} = \frac{8(x+1)}{(x-2)^4}$$

$f''(x)>0$ if $x>-1$ ($x\neq 2$) and $f''(x)<0$ if $x<-1$. Thus, f is CU on $(-1,2)$ and $(2,\infty)$, and f is CD on $(-\infty,-1)$. There is an inflection point at $\left(-1,\frac{1}{9}\right)$.

(e)



47. **(a)** $\lim_{x\to-\infty}\left(\sqrt{x^2+1}-x\right)=\infty$ and

$$\lim_{x\to\infty}\left(\sqrt{x^2+1}-x\right)=\lim_{x\to\infty}\left(\sqrt{x^2+1}-x\right)\frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}+x}=\lim_{x\to\infty}\frac{1}{\sqrt{x^2+1}+x}=0$$, so $y=0$ is a HA.

(b) $f(x)=\sqrt{x^2+1}-x\Rightarrow f'(x)=\frac{x}{\sqrt{x^2+1}}-1$. Since $\frac{x}{\sqrt{x^2+1}}<1$ for all x , $f'(x)<0$, so f is decreasing on R .

(c) No minimum or maximum

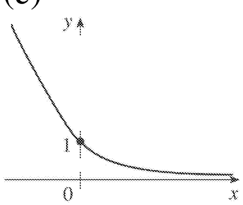
(d)

$$f''(x)=\frac{\left(x^2+1\right)^{1/2}(1)-x\cdot\frac{1}{2}\left(x^2+1\right)^{-1/2}(2x)}{\left(\sqrt{x^2+1}\right)^2}$$

$$=\frac{\left(x^2+1\right)^{1/2}-\frac{x^2}{\left(x^2+1\right)^{1/2}}}{x^2+1}=\frac{\left(x^2+1\right)-x^2}{\left(x^2+1\right)^{3/2}}$$

$$=\frac{1}{\left(x^2+1\right)^{3/2}}>0$$, so f is CU on R . No IP

(e)



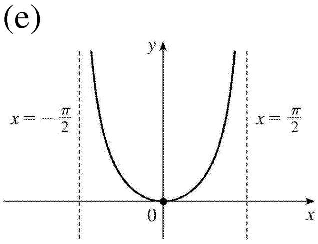
48. **(a)** $\lim_{x\to\pi/2^-} x\tan x=\infty$ and $\lim_{x\to-\pi/2^+} x\tan x=\infty$, so $x=\frac{\pi}{2}$ and $x=-\frac{\pi}{2}$ are VA.

(b) $f(x)=x\tan x$, $-\frac{\pi}{2}<x<\frac{\pi}{2}$. $f'(x)=x\sec^2 x+\tan x>0\Leftrightarrow 0<x<\frac{\pi}{2}$, so f increases on $\left(0,\frac{\pi}{2}\right)$ and decreases on $\left(-\frac{\pi}{2},0\right)$.

(c) $f(0)=0$ is a local minimum value.

(d) $f''(x)=2\sec^2 x+2x\tan x\sec^2 x>0$ for $-\frac{\pi}{2}<x<\frac{\pi}{2}$, so f is CU on $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$. No IP

(e)



49. $f(x)=\ln(1-\ln x)$ is defined when x>0 (so that ln\:\italic{x} is defined) and $1-\ln x>0$ [so that $\ln(1-\ln x)$ is defined]. The second condition is equivalent to $1>\ln x \Leftrightarrow x<e$, so f has domain $(0,e)$.

(a) As $x\to 0^+,\ln x\to-\infty$, so $1-\ln x\to\infty$ and $f(x)\to\infty$. As $x\to e^-,\ln x\to 1^-$, so $1-\ln x\to 0^+$ and

$f(x)\to-\infty$. Thus, $x=0$ and $x=e$ are vertical asymptotes. There is no horizontal asymptote.

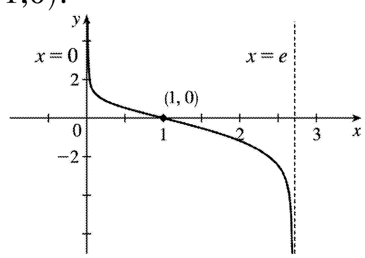
(b) $f'(x)=\dfrac{1}{1-\ln x}\left(-\dfrac{1}{x}\right)=-\dfrac{1}{x(1-\ln x)}<0$ on $(0,e)$. Thus, f is decreasing on its domain, $(0,e)$.

(c) $f'(x)\neq 0$ on $(0,e)$, so f has no local maximum or minimum value.

(d)

$$f''(x)=-\frac{-[x(1-\ln x)]'}{[x(1-\ln x)]^2}=\frac{x(-1/x)+(1-\ln x)}{x^2(1-\ln x)^2}$$
$$=-\frac{\ln x}{x^2(1-\ln x)^2}$$

so $f''(x)>0\Leftrightarrow \ln x<0\Leftrightarrow 0<x<1$. Thus, f is CU on $(0,1)$ and CD on $(1,e)$. There is an inflection point at $(1,0)$.

(e) 

50. $f(x)=\dfrac{e^x}{1+e^x}$ has domain R .

(a) $\lim\limits_{x\to\infty} f(x)=\lim\limits_{x\to\infty}\dfrac{e^x/e^x}{\left(1+e^x\right)/e^x}=\lim\limits_{x\to\infty}\dfrac{1}{e^{-x}+1}=\dfrac{1}{0+1}=1$, so $y=1$ is a HA.

$\lim\limits_{x\to-\infty} f(x)=\lim\limits_{x\to-\infty}\dfrac{e^x}{1+e^x}=\dfrac{0}{1+0}=0$, so $y=0$ is a HA. No VA.

(b) $f'(x)=\dfrac{\left(1+e^x\right)e^x-e^x\cdot e^x}{\left(1+e^x\right)^2}=\dfrac{e^x}{\left(1+e^x\right)^2}>0$ for all x . Thus, f is increasing on R .

(c) There is no local maximum or minimum.

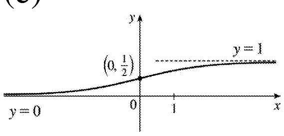
(d)

$$f''(x)=\frac{\left(1+e^{x}\right)^{2}e^{x}-e^{x}\cdot 2\left(1+e^{x}\right)e^{x}}{\left[\left(1+e^{x}\right)^{2}\right]^{2}}$$

$$=\frac{e^{x}\left(1+e^{x}\right)\left[\left(1+e^{x}\right)-2e^{x}\right]}{\left(1+e^{x}\right)^{4}}=\frac{e^{x}\left(1-e^{x}\right)}{\left(1+e^{x}\right)^{3}}$$

$f''(x)>0\Leftrightarrow 1-e^{x}>0\Leftrightarrow x<0$, so f is CU on $(-\infty,0)$ and CD on $(0,\infty)$. There is an inflection point at $\left(0,\frac{1}{2}\right)$.

(e)



51. **(a)** $\lim_{x\to\pm\infty} e^{-1/(x+1)}=1$ since $-1/(x+1)\to 0$, so $y=1$ is a HA. $\lim_{x\to -1^{+}} e^{-1/(x+1)}=0$ since $-1/(x+1)\to-\infty$, $\lim_{x\to -1^{-}} e^{-1/(x+1)}=\infty$ since $-1/(x+1)\to\infty$, so $x=-1$ is a VA.

(b) $f(x)=e^{-1/(x+1)}\Rightarrow f'(x)=e^{-1/(x+1)}\left[-(-1)\,\frac{1}{(x+1)^{2}}\right]$ [Reciprocal Rule] $=e^{-1/(x+1)}/(x+1)^{2}\Rightarrow f'(x)>0$ for all x except -1 , so f is increasing on $(-\infty,-1)$ and $(-1,\infty)$.

(c) No local maximum or minimum

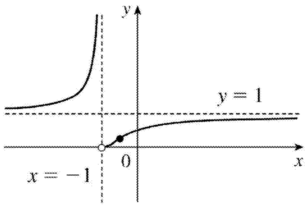
(d)

$$f''(x)=\frac{(x+1)^{2}e^{-1/(x+1)}\left[1/(x+1)^{2}\right]-e^{-1/(x+1)}\left[2(x+1)\right]}{\left[(x+1)^{2}\right]^{2}}$$

$$=\frac{e^{-1/(x+1)}\left[1-(2x+2)\right]}{(x+1)^{4}}=-\frac{e^{-1/(x+1)}(2x+1)}{(x+1)^{4}}\Rightarrow$$

$f''(x)>0\Leftrightarrow 2x+1<0\Leftrightarrow x<-\frac{1}{2}$, so f is CU on $(-\infty,-1)$ and $\left(-1,-\frac{1}{2}\right)$, and CD on $\left(-\frac{1}{2},\infty\right)$. f has an IP at $\left(-\frac{1}{2},\mathrm{e}^{-2}\right)$.

(e)



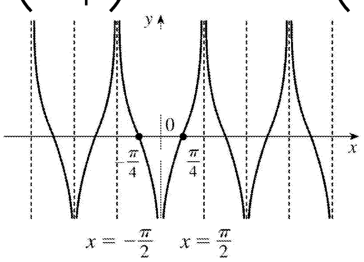
52. **(a)** f is periodic with period π, so we consider only $-\frac{\pi}{2}<x<\frac{\pi}{2}$. $\lim_{x\to 0} \ln\left(\tan^2 x\right)=-\infty$, $\lim_{x\to(\pi/2)^-} \ln\left(\tan^2 x\right)=\infty$, and $\lim_{x\to(-\pi/2)^+} \ln\left(\tan^2 x\right)=\infty$, so $x=0$, $x=\pm\frac{\pi}{2}$ are VA.

(b) $f(x)=\ln\left(\tan^2 x\right)\Rightarrow f'(x)=\dfrac{2\tan x\sec^2 x}{\tan^2 x}=2\dfrac{\sec^2 x}{\tan x}>0\Leftrightarrow\tan x>0\Leftrightarrow 0<x<\frac{\pi}{2}$, so f is increasing on $\left(0,\frac{\pi}{2}\right)$ and decreasing on $\left(-\frac{\pi}{2},0\right)$.

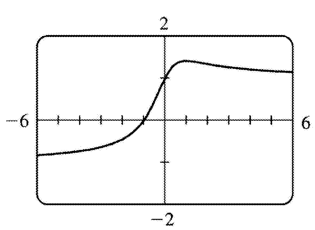
(c) No maximum or minimum

(d) $f'(x)=\dfrac{2}{\sin x\cos x}=\dfrac{4}{\sin 2x}\Rightarrow f''(x)=\dfrac{-8\cos 2x}{\sin^2 2x}<0$ $\Leftrightarrow\cos 2x>0\Leftrightarrow-\frac{\pi}{4}<x<\frac{\pi}{4}$, so f is CD on $\left(-\frac{\pi}{4},0\right)$ and $\left(0,\frac{\pi}{4}\right)$, and CU on $\left(-\frac{\pi}{2},-\frac{\pi}{4}\right)$ and $\left(\frac{\pi}{4},\frac{\pi}{2}\right)$. IP are $\left(\pm\frac{\pi}{4},0\right)$.

(e)



53. **(a)**

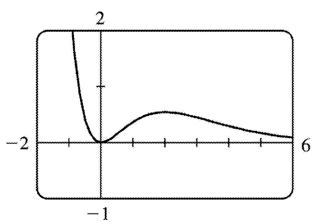


From the graph, we get an estimate of $f(1)\approx 1.41$ as a local maximum value, and no local minimum value.

$f(x)=\dfrac{x+1}{\sqrt{x^2+1}}\Rightarrow f'(x)=\dfrac{1-x}{\left(x^2+1\right)^{3/2}}$.

$f'(x)=0 \Leftrightarrow x=1$. $f(1)=\frac{2}{\sqrt{2}}=\sqrt{2}$ is the exact value.

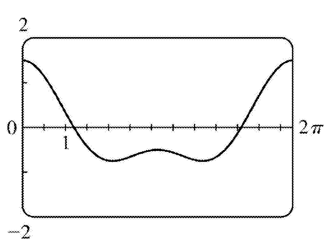
(b) From the graph in part (a), f increases most rapidly somewhere between $x=-\frac{1}{2}$ and $x=-\frac{1}{4}$. To find the exact value, we need to find the maximum value of f' , which we can do by finding the critical numbers of f' . $f''(x)=\frac{2x^2-3x-1}{\left(x^2+1\right)^{5/2}}=0 \Leftrightarrow x=\frac{3\pm\sqrt{17}}{4}$. $x=\frac{3+\sqrt{17}}{4}$ corresponds to the *minimum* value of f' . The maximum value of f' is at $\left(\frac{3-\sqrt{17}}{4},\sqrt{\frac{7}{6}-\frac{\sqrt{17}}{6}}\right)\approx(-0.28,0.69)$.



54. **(a)**
Tracing the graph gives us estimates of $f(0)=0$ for a local minimum value and $f(2)=0.54$ for a local maximum value.
$f(x)=x^2e^{-x} \Rightarrow f'(x)=xe^{-x}(2-x)$. $f'(x)=0 \Leftrightarrow x=0$ or 2 . $f(0)=0$ and $f(2)=4e^{-2}$ are the exact values.

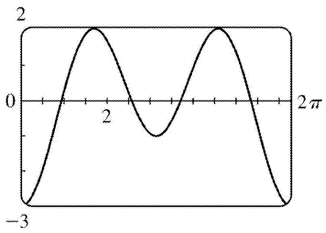
(b) From the graph in part (a), f increases most rapidly around $x=\frac{3}{4}$. To find the exact value, we need to find the maximum value of f' , which we can do by finding the critical numbers of f' . $f''(x)=e^{-x}\left(x^2-4x+2\right)=0 \Rightarrow x=2\pm\sqrt{2}$. $x=2+\sqrt{2}$ corresponds to the *minimum* value of f' . The maximum value of f' is at $\left(2-\sqrt{2},(2-\sqrt{2})^2e^{-2+\sqrt{2}}\right)\approx(0.59,0.19)$.

55. $f(x)=\cos x+\frac{1}{2}\cos 2x \Rightarrow f'(x)=-\sin x-\sin 2x \Rightarrow f''(x)=-\cos x-2\cos 2x$



(a)
From the graph of f , it seems that f is CD on $(0,1)$, CU on $(1,2.5)$, CD on $(2.5,3.7)$, CU on $(3.7,5.3)$, and CD on $(5.3,2\pi)$. The points of inflection appear to be at $(1,0.4)$, $(2.5,-0.6)$,

$(3.7,-0.6)$, and $(5.3,0.4)$.

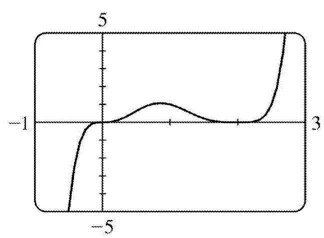


(b)

From the graph of f'' (and zooming in near the zeros), it seems that f is CD on $(0,0.94)$, CU on $(0.94,2.57)$, CD on $(2.57,3.71)$, CU on $(3.71,5.35)$, and CD on $(5.35,2\pi)$. Refined estimates of the inflection points are $(0.94,0.44)$, $(2.57,-0.63)$, $(3.71,-0.63)$, and $(5.35,0.44)$.

56. $f(x)=x^3(x-2)^4 \Rightarrow f'(x)=x^3 \cdot 4(x-2)^3+(x-2)^4 \cdot 3x^2=x^2(x-2)^3[4x+3(x-2)]=x^2(x-2)^3(7x-6) \Rightarrow$

$$\begin{aligned} f''(x) &=(2x)(x-2)^3(7x-6)+x^2 \cdot 3(x-2)^2(7x-6)+x^2(x-2)^3(7) \\ &=x(x-2)^2[2(x-2)(7x-6)+3x(7x-6)+7x(x-2)] \\ &=x(x-2)^2\left[42x^2-72x+24\right]=6x(x-2)^2\left(7x^2-12x+4\right) \end{aligned}$$



(a)

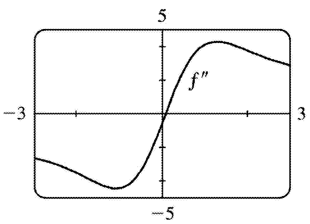
From the graph of f , it seems that f is CD on $(-\infty,0)$, CU on $(0,0.5)$, CD on $(0.5,1.3)$, and CU on $(1.3,\infty)$. The points of inflection appear to be at $(0,0)$, $(0.5,0.5)$, and $(1.3,0.6)$.

(b)

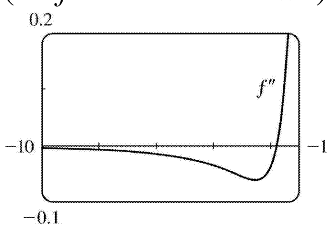
From the graph of f'' (and zooming in near the zeros), it seems that f is CD on $(-\infty,0)$, CU on $(0,0.45)$, CD on $(0.45,1.26)$, and CU on $(1.26,\infty)$. Refined estimates of the inflection points are $(0,0)$, $(0.45,0.53)$, and $(1.26,0.60)$.

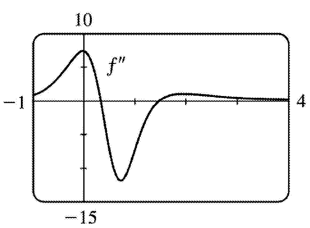
57. In Maple, we define f and then use the command plot (diff (diff (f, x), x), x= –3..3);. In Mathematica, we define f and then use Plot [Dt [Dt [f, x], x], {x, –3, 3}]. We see that $f''>0$ for

$x>0.1$ and $f''<0$ for $x<0.1$. So f is concave up on $(0.1,\infty)$ and concave down on $(-\infty,0.1)$.



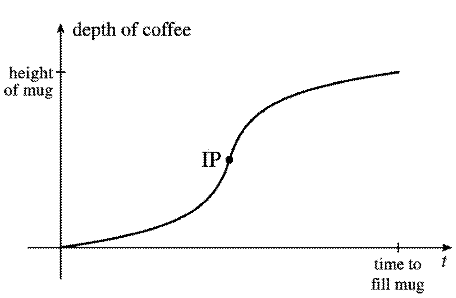
58. It appears that f'' is positive (and thus f is concave up) on $(-1.8,0.3)$ and $(1.5,\infty)$ and negative (so f is concave down) on $(-\infty,-1.8)$ and $(0.3,1.5)$.



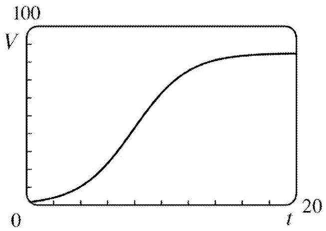


59. Most students learn more in the third hour of studying than in the eighth hour, so $K(3)-K(2)$ is larger than $K(8)-K(7)$. In other words, as you begin studying for a test, the rate of knowledge gain is large and then starts to taper off, so $K'(t)$ decreases and the graph of K is concave downward.

60. At first the depth increases slowly because the base of the mug is wide. But as the mug narrows, the coffee rises more quickly. Thus, the depth d increases at an increasing rate and its graph is concave upward. The rate of increase of d has a maximum where the mug is narrowest; that is, when the mug is half full. It is there that the inflection point (IP) occurs. Then the rate of increase of d starts to decrease as the mug widens and the graph becomes concave down.



61.



From the graph, we estimate that the most rapid increase in the percentage of households in the United States with at least one VCR occurs at about $t=8$. To maximize the first derivative, we need to determine the values for which the second derivative is 0. We'll use $V(t)=\dfrac{a}{1+be^{ct}}$, and substitute $a=85$, $b=53$, and $c=-0.5$ later.

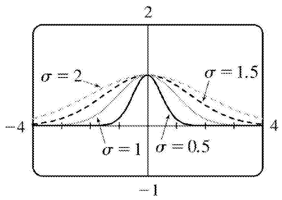
$V'(t)=-\dfrac{a\left(bce^{ct}\right)}{\left(1+be^{ct}\right)^2}$ [by the Reciprocal Rule] and

$$V''(t)=-abc\cdot\frac{\left(1+be^{ct}\right)^2\cdot ce^{ct}-e^{ct}\cdot 2\left(1+be^{ct}\right)\cdot bce^{ct}}{\left[\left(1+be^{ct}\right)^2\right]^2}$$

$$=\frac{-abc\cdot ce^{ct}\left(1+be^{ct}\right)\left[\left(1+be^{ct}\right)-2be^{ct}\right]}{\left(1+be^{ct}\right)^4}=\frac{-abc^2e^{ct}\left(1-be^{ct}\right)}{\left(1+be^{ct}\right)^3}$$

So $V''(t)=0\Leftrightarrow 1=be^{ct}\Leftrightarrow e^{ct}=1/b$. Now graph $y=e^{-0.5t}$ and $y=\dfrac{1}{53}$. These graphs intersect at $t\approx 7.94$ years, which corresponds to roughly midyear 1988.

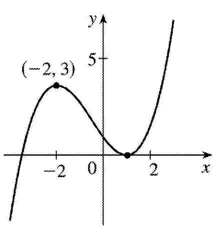
62. **(a)** As $|x|\to\infty$, $t=-x^2/\left(2\sigma^2\right)\to-\infty$, and $e^t\to 0$. The HA is $y=0$. Since t takes on its maximum value at $x=0$, so does e^t . Showing this result using derivatives, we have $f(x)=e^{-x^2/\left(2\sigma^2\right)}\Rightarrow$
$f'(x)=e^{-x^2/\left(2\sigma^2\right)}\left(-x/\sigma^2\right)$. $f'(x)=0\Leftrightarrow x=0$. Because f' changes from positive to negative at $x=0$, $f(0)=1$ is a local maximum. For inflection points, we find
$f''(x)=-\dfrac{1}{\sigma^2}\left[e^{-x^2/\left(2\sigma^2\right)}\cdot 1+xe^{-x^2/\left(2\sigma^2\right)}\left(-x/\sigma^2\right)\right]=\dfrac{-1}{\sigma^2}e^{-x^2/\left(2\sigma^2\right)}\left(1-x^2/\sigma^2\right)$.
$f''(x)=0\Leftrightarrow x^2=\sigma^2\Leftrightarrow x=\pm\sigma$. $f''(x)<0\Leftrightarrow x^2<\sigma^2\Leftrightarrow-\sigma<x<\sigma$. So f is CD on $(-\sigma,\sigma)$ and CU on $(-\infty,-\sigma)$ and (σ,∞) . IP at $\left(\pm\sigma,e^{-1/2}\right)$.
(b) Since we have IP at $x=\pm\sigma$, the inflection points move away from the y -axis as σ increases.



(c)

From the graph, we see that as σ increases, the graph tends to spread out and there is more area between the curve and the x –axis.

63. $f(x)=ax^3+bx^2+cx+d \Rightarrow f'(x)=3ax^2+2bx+c$. We are given that $f(1)=0$ and $f(-2)=3$, so $f(1)=a+b+c+d=0$ and $f(-2)=-8a+4b-2c+d=3$. Also $f'(1)=3a+2b+c=0$ and $f'(-2)=12a-4b+c=0$ by Fermat's Theorem. Solving these four equations, we get $a=\frac{2}{9}$, $b=\frac{1}{3}$, $c=-\frac{4}{3}$, $d=\frac{7}{9}$, so the function is $f(x)=\frac{1}{9}\left(2x^3+3x^2-12x+7\right)$.



64. $f(x)=axe^{bx^2} \Rightarrow f'(x)=a\left[xe^{bx^2}\cdot 2bx+e^{bx^2}\cdot 1\right]=ae^{bx^2}\left(2bx^2+1\right)$. For $f(2)=1$ to be a maximum value, we must have $f'(2)=0$. $f(2)=1 \Rightarrow 1=2ae^{4b}$ and $f'(2)=0 \Rightarrow 0=(8b+1)ae^{4b}$. So $8b+1=0 \Rightarrow b=-\frac{1}{8}$ and now $1=2ae^{-1/2} \Rightarrow a=\left.\sqrt{e}\right/2$.

65. Suppose that f is differentiable on an interval I and $f'(x)>0$ for all x in I except $x=c$. To show that f is increasing on I , let x_1 , x_2 be two numbers in I with $x_1<x_2$.

Case 2 $x_1<x_2<c$. Let J be the interval $\{x\in I \mid x<c\}$. By applying the Increasing/Decreasing Test to f on J , we see that f is increasing on J , so $f(x_1)<f(x_2)$.

Case 2 $c<x_1<x_2$. Apply the Increasing/Decreasing Test to f on $K=\{x\in I \mid x>c\}$.

Case 3 $x_1<x_2=c$. Apply the proof of the Increasing/Decreasing Test, using the Mean Value Theorem (MVT) on the interval $\left[x_1,x_2\right]$ and noting that the MVT does not require f to be differentiable at the endpoints of $\left[x_1,x_2\right]$.

Case 4 $c=x_1<x_2$. Same proof as in Case 3.

Case 5 $x_1<c<x_2$. By Cases 3 and 4, f is increasing on $\left[x_1,\text{c}\right]$ and on $\left[\text{c},\text{x}_2\right]$, so $f(x_1)<f(c)<f(x_2)$. In all cases, we have shown that $f(x_1)<f(x_2)$. Since x_1 , x_2 were any numbers in I with $x_1<x_2$, we have shown that f is increasing on I .

66. **(a)** We will make use of the converse of the Concavity Test (along with the stated assumptions); that is, if f is concave upward on I , then $f''>0$ on I . If f and g are CU on I , then $f''>0$ and $g''>0$ on I , so $(f+g)''=f''+g''>0$ on $I\Rightarrow f+g$ is CU on I .
(b) Since f is positive and CU on I , $f>0$ and $f''>0$ on I . So $g(x)=[f(x)]^2\Rightarrow g'=2ff'\Rightarrow$ $g''=2f'f'+2ff''=2\left(f'\right)^2+2ff''>0\Rightarrow g$ is CU on I .

67. **(a)** Since f and g are positive, increasing, and CU on I with f'' and g'' never equal to 0 , we have $f>0$, $f'\geq 0$, $f''>0$, $g>0$, $g'\geq 0$, $g''>0$ on I . Then $(fg)'=f'g+fg'\Rightarrow$ $(fg)''=f''g+2f'g'+fg''\geq f''g+fg''>0$ on $I\Rightarrow fg$ is CU on I .
(b) In part (a), if f and g are both decreasing instead of increasing, then $f'\leq 0$ and $g'\leq 0$ on I , so we still have $2f'g'\geq 0$ on I . Thus, $(fg)''=f''g+2f'g'+fg''\geq f''g+fg''>0$ on $I\Rightarrow fg$ is CU on I as in part (a).
(c) Suppose f is increasing and g is decreasing . Then $f'\geq 0$ and $g'\leq 0$ on I , so $2f'g'\leq 0$ on I and the argument in parts (a) and (b) fails.
Example 1.
$I=(0,\infty)$, $f(x)=x^3$, $g(x)=1/x$. Then $(fg)(x)=x^2$, so $(fg)'(x)=2x$ and $(fg)''(x)=2>0$ on I . Thus, fg is CU on I .

Example 2.

$I=(0,\infty)$, $f(x)=4x\sqrt{x}$, $g(x)=1/x$. Then $(fg)(x)=4\sqrt{x}$, so $(fg)'(x)=2/\sqrt{x}$ and $(fg)''(x)=-1/\sqrt{x^3}<0$ on I . Thus, fg is CD on I .

Example 3.

$I=(0,\infty)$, $f(x)=x^2$, $g(x)=1/x$. Thus, $(fg)(x)=x$, so fg is linear on I .

68. Since f and g are CU on $(-\infty,\infty)$, $f''>0$ and $g''>0$ on $(-\infty,\infty)$.
$h(x)=f(g(x))\Rightarrow h'(x)=f'(g(x))g'(x)\Rightarrow$

$h''(x)=f''(g(x))g'(x)g'(x)+f'(g(x))g''(x)=f''(g(x))\left[g'(x)\right]^2+f'(g(x))g''(x)>0$ if $f'>0$. So h is CU if f is increasing.

69. $f(x)=\tan x-x \Rightarrow f'(x)=\sec^2 x-1>0$ for $0<x<\frac{\pi}{2}$ since $\sec^2 x>1$ for $0<x<\frac{\pi}{2}$. So f is increasing on $\left(0,\frac{\pi}{2}\right)$. Thus, $f(x)>f(0)=0$ for $0<x<\frac{\pi}{2} \Rightarrow \tan x-x>0 \Rightarrow \tan x>x$ for $0<x<\frac{\pi}{2}$.

70. **(a)** Let $f(x)=e^x-1-x$. Now $f(0)=e^0-1=0$, and for $x\geq 0$, we have $f'(x)=e^x-1\geq 0$. Now, since $f(0)=0$ and f is increasing on $[0,\infty)$, $f(x)\geq 0$ for $x\geq 0 \Rightarrow e^x-1-x\geq 0 \Rightarrow e^x\geq 1+x$.

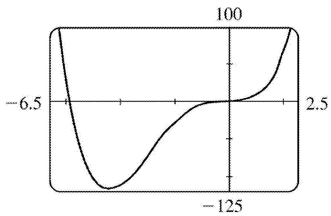
(b) Let $f(x)=e^x-1-x-\frac{1}{2}x^2$. Thus, $f'(x)=e^x-1-x$, which is positive for $x\geq 0$ by part (a). Thus, $f(x)$ is increasing on $(0,\infty)$, so on that interval, $0=f(0)\leq f(x)=e^x-1-x-\frac{1}{2}x^2 \Rightarrow e^x\geq 1+x+\frac{1}{2}x^2$.

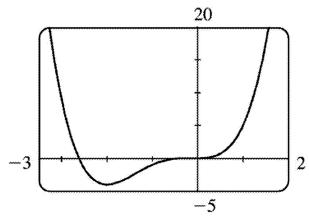
(c) By part (a), the result holds for $n=1$. Suppose that $e^x\geq 1+x+\frac{x^2}{2!}+\cdots+\frac{x^k}{k!}$ for $x\geq 0$. Let $f(x)=e^x-1-x-\frac{x^2}{2!}-\cdots-\frac{x^k}{k!}-\frac{x^{k+1}}{(k+1)!}$. Then $f'(x)=e^x-1-x-\cdots-\frac{x^k}{k!}\geq 0$ by assumption. Hence, $f(x)$ is increasing on $(0,\infty)$. So $0\leq x$ implies that $0=f(0)\leq f(x)=e^x-1-x-\cdots-\frac{x^k}{k!}-\frac{x^{k+1}}{(k+1)!}$, and hence $e^x\geq 1+x+\cdots+\frac{x^k}{k!}+\frac{x^{k+1}}{(k+1)!}$ for $x\geq 0$. Therefore, for $x\geq 0$, $e^x\geq 1+x+\frac{x^2}{2!}+\cdots+\frac{x^n}{n!}$ for every positive integer n , by mathematical induction.

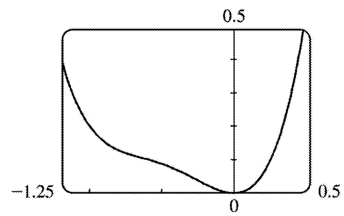
71. Let the cubic function be $f(x)=ax^3+bx^2+cx+d \Rightarrow f'(x)=3ax^2+2bx+c \Rightarrow f''(x)=6ax+2b$. So f is CU when $6ax+2b>0 \Leftrightarrow x>-b/(3a)$, CD when $x<-b/(3a)$, and so the only point of inflection occurs when $x=-b/(3a)$. If the graph has three x –intercepts x_1 , x_2 and x_3 , then the expression for $f(x)$ must factor as $f(x)=a\left(x-x_1\right)\left(x-x_2\right)\left(x-x_3\right)$. Multiplying these factors together gives us $f(x)=a\left[x^3-\left(x_1+x_2+x_3\right)x^2+\left(x_1x_2+x_1x_3+x_2x_3\right)x-x_1x_2x_3\right]$. \ Equating the coefficients of the x^2 –terms for the two forms of f gives us $b=-a\left(x_1+x_2+x_3\right)$. Hence, the x– coordinate of the point of inflection is $-\frac{b}{3a}=-\frac{-a\left(x_1+x_2+x_3\right)}{3a}=\frac{x_1+x_2+x_3}{3}$.

72. $P(x)=x^4+cx^3+x^2 \Rightarrow P'(x)=4x^3+3cx^2+2x \Rightarrow P''(x)=12x^2+6cx+2$. The graph of $P''(x)$ is a parabola. \ If $P''(x)$ has two roots, then it changes sign twice and so has two inflection points. This

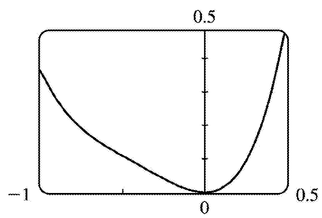
happens when the discriminant of $P''(x)$ is positive, that is, $(6c)^2-4\cdot 12\cdot 2>0 \Leftrightarrow 36c^2-96>0 \Leftrightarrow$ $|c|>\frac{2\sqrt{6}}{3}\approx 1.63$. If $36c^2-96=0 \Leftrightarrow c=\pm\frac{2\sqrt{6}}{3}$, $P''(x)$ is 0 at one point, but there is still no inflection point since $P''(x)$ never changes sign, and if $36c^2-96<0 \Leftrightarrow |c|<\frac{2\sqrt{6}}{3}$, then $P''(x)$ never changes sign, and so there is no inflection point.

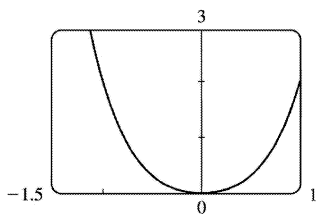


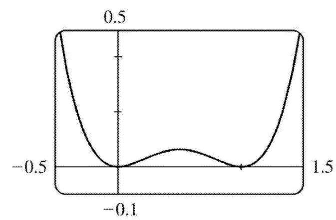




$c=6$ $c=3$ $c=1.8$







$c=\frac{2\sqrt{6}}{3}$ $c=0$ $c=-2$

For large positive c , the graph of f has two inflection points and a large dip to the left of the y– axis. As c decreases, the graph of f becomes flatter for $x<0$, and eventually the dip rises above the x– axis, and then disappears entirely, along with the inflection points. As c continues to decrease, the dip and the inflection points reappear, to the right of the origin.

73. By hypothesis $g=f'$ is differentiable on an open interval containing c . Since $(c,f(c))$ is a point of inflection, the concavity changes at $x=c$, so $f''(x)$ changes signs at $x=c$. Hence, by the First Derivative Test, f' has a local extremum at $x=c$. Thus, by Fermat’s Theorem $f''(c)=0$.

74. $f(x)=x^4 \Rightarrow f'(x)=4x^3 \Rightarrow f''(x)=12x^2 \Rightarrow f''(0)=0$. For $x<0$, $f''(x)>0$, so f is CU on $(-\infty,0)$; for $x>0$, $f''(x)>0$, so f is also CU on $(0,\infty)$. Since f does not change concavity at 0 , $(0,0)$ is not an inflection point.

75. Using the fact that $|x|=\sqrt{x^2}$, we have that $g(x)=x\sqrt{x^2} \Rightarrow g'(x)=\sqrt{x^2}+\sqrt{x^2}=2\sqrt{x^2}=2|x| \Rightarrow$ $g''(x)=2x\left(x^2\right)^{-1/2}=\frac{2x}{|x|}<0$ for $x<0$ and $g''(x)>0$ for $x>0$, so $(0,0)$ is an inflection point. But

$g''(0)$ does not exist.

76. There must exist some interval containing c on which f''' is positive, since $f'''(c)$ is positive and f''' is continuous. On this interval, f'' is increasing (since f''' is positive), so $f''=(f')'$ changes from negative to positive at c . So by the First Derivative Test, f' has a local minimum at $x=c$ and thus cannot change sign there, so f has no maximum or minimum at c . But since f'' changes from negative to positive at c , f has a point of inflection at c (it changes from concave down to concave up).

1. **(a)** $\lim_{x\to a} \frac{f(x)}{g(x)}$ is an indeterminate form of type $\frac{0}{0}$.

(b) $\lim_{x\to a} \frac{f(x)}{p(x)}$ =0 because the numerator approaches 0 while the denominator becomes large.

(c) $\lim_{x\to a} \frac{h(x)}{p(x)}$ =0 because the numerator approaches a finite number while the denominator becomes large.

(d) If $\lim_{x\to a} p(x)=\infty$ and $f(x)\to 0$ through positive values, then $\lim_{x\to a} \frac{p(x)}{f(x)}=\infty$. If $f(x)\to 0$ through negative values, then $\lim_{x\to a} \frac{p(x)}{f(x)}=-\infty$. If $f(x)\to 0$ through both positive and negative values, then the limit might not exist.

(e) $\lim_{x\to a} \frac{p(x)}{q(x)}$ is an indeterminate form of type $\frac{\infty}{\infty}$.

2. **(a)** $\lim_{x\to a} [f(x)p(x)]$ is an indeterminate form of type $0\cdot \infty$.

(b) When x is near a , $p(x)$ is large and $h(x)$ is near 1 , so $h(x)p(x)$ is large. Thus, $\lim_{x\to a} [h(x)p(x)]=\infty$.

(c) When x is near a , $p(x)$ and $q(x)$ are both large, so $p(x)q(x)$ is large. Thus, $\lim_{x\to a} [p(x)q(x)]=\infty$.

3. **(a)** When x is near a , $f(x)$ is near 0 and $p(x)$ is large, so $f(x)-p(x)$ is large negative. Thus, $\lim_{x\to a} [f(x)-p(x)]=-\infty$.

(b) $\lim_{x\to a} [p(x)-q(x)]$ is an indeterminate form of type $\infty-\infty$.

(c) When x is near a , $p(x)$ and $q(x)$ are both large, so $p(x)+q(x)$ is large. Thus, $\lim_{x\to a} [p(x)+q(x)]=\infty$.

4. **(a)** $\lim_{x\to a} [f(x)]^{g(x)}$ is an indeterminate form of type 0^0 .

(b) If $y=[f(x)]^{p(x)}$, then $\ln y=p(x)\ln f(x)$. When x is near a , $p(x)\to\infty$ and $\ln f(x)\to -\infty$, so $\ln y\to -\infty$. Therefore, $\lim_{x\to a} [f(x)]^{p(x)}=\lim_{x\to a} y=\lim_{x\to a} e^{\ln y}=0$, provided f^p is defined.

(c) $\lim_{x\to a} [h(x)]^{p(x)}$ is an indeterminate form of type 1^∞ .

(d) $\lim_{x\to a} [p(x)]^{f(x)}$ is an indeterminate form of type ∞^0 .

(e) If $y=[p(x)]^{q(x)}$, then $\ln y=q(x)\ln p(x)$. When x is near a , $q(x)\to\infty$ and $\ln p(x)\to\infty$, so

$\ln y \to \infty$. Therefore, $\lim_{x \to a} [p(x)]^{q(x)} = \lim_{x \to a} y = \lim_{x \to a} e^{\ln y} = \infty$.

(f) $\lim_{x \to a} \sqrt[q(x)]{p(x)} = \lim_{x \to a} [p(x)]^{1/q(x)}$ is an indeterminate form of type ∞^0 .

5. This limit has the form $\frac{0}{0}$. We can simply factor the numerator to evaluate this limit.

$$\lim_{x \to -1} \frac{x^2 - 1}{x+1} = \lim_{x \to -1} \frac{(x+1)(x-1)}{x+1} = \lim_{x \to -1} (x-1) = -2$$

6. $\lim_{x \to -2} \frac{x+2}{x^2+3x+2} = \lim_{x \to -2} \frac{x+2}{(x+1)(x+2)} = \lim_{x \to -2} \frac{1}{x+1} = -1$

7. This limit has the form $\frac{0}{0}$. $\lim_{x \to 1} \frac{x^9 - 1}{x^5 - 1} = \lim_{x \to 1} \frac{9x^8}{5x^4} = \frac{9}{5} \lim_{x \to 1} x^4 = \frac{9}{5}(1) = \frac{9}{5}$

8. $\lim_{x \to 1} \frac{x^a - 1}{x^b - 1} = \lim_{x \to 1} \frac{ax^{a-1}}{bx^{b-1}} = \frac{a}{b}$

9. This limit has the form $\frac{0}{0}$. $\lim_{x \to (\pi/2)^+} \frac{\cos x}{1 - \sin x} = \lim_{x \to (\pi/2)^+} \frac{-\sin x}{-\cos x} = \lim_{x \to (\pi/2)^+} \tan x = -\infty$.

10. $\lim_{x \to 0} \frac{x + \tan x}{\sin x} = \lim_{x \to 0} \frac{1 + \sec^2 x}{\cos x} = \frac{1+1^2}{1} = 2$

11. This limit has the form $\frac{0}{0}$. $\lim_{t \to 0} \frac{e^t - 1}{t^3} = \lim_{t \to 0} \frac{e^t}{3t^2} = \infty$ since $e^t \to 1$ and $3t^2 \to 0^+$ as $t \to 0$.

12. $\lim_{t \to 0} \frac{e^{3t} - 1}{t} = \lim_{t \to 0} \frac{3e^{3t}}{1} = 3$

13. This limit has the form $\frac{0}{0}$. $\lim_{x \to 0} \frac{\tan px}{\tan qx} = \lim_{x \to 0} \frac{p \sec^2 px}{q \sec^2 qx} = \frac{p(1)^2}{q(1)^2} = \frac{p}{q}$

14. $\lim_{\theta\to\pi/2} \dfrac{1-\sin\theta}{\csc\theta} = \dfrac{0}{1} = 0$. L'Hospital's Rule does not apply.

15. This limit has the form $\dfrac{\infty}{\infty}$. $\lim_{x\to\infty} \dfrac{\ln x}{x} = \lim_{x\to\infty} \dfrac{1/x}{1} = 0$

16. $\lim_{x\to\infty} \dfrac{e^x}{x} = \lim_{x\to\infty} \dfrac{e^x}{1} = \lim_{x\to\infty} e^x = \infty$

17. $\lim_{x\to 0^+} [(\ln x)/x] = -\infty$ since $\ln x \to -\infty$ as $x\to 0^+$ and dividing by small values of x just increases the magnitude of the quotient $(\ln x)/x$. L'Hospital's Rule does not apply.

18. $\lim_{x\to\infty} \dfrac{\ln\ln x}{x} = \lim_{x\to\infty} \dfrac{\frac{1}{\ln x}\cdot\frac{1}{x}}{1} = \lim_{x\to\infty} \dfrac{1}{x\ln x} = 0$

19. This limit has the form $\dfrac{0}{0}$. $\lim_{t\to 0} \dfrac{5^t-3^t}{t} = \lim_{t\to 0} \dfrac{5^t\ln 5 - 3^t\ln 3}{1} = \ln 5 - \ln 3 = \ln \dfrac{5}{3}$

20. $\lim_{x\to 1} \dfrac{\ln x}{\sin\pi x} = \lim_{x\to 1} \dfrac{1/x}{\pi\cos\pi x} = \dfrac{1}{\pi(-1)} = -\dfrac{1}{\pi}$

21. This limit has the form $\dfrac{0}{0}$. $\lim_{x\to 0} \dfrac{e^x-1-x}{x^2} = \lim_{x\to 0} \dfrac{e^x-1}{2x} = \lim_{x\to 0} \dfrac{e^x}{2} = \dfrac{1}{2}$

22. $\lim_{x\to 0} \dfrac{e^x-1-x-x^2/2}{x^3} = \lim_{x\to 0} \dfrac{e^x-1-x}{3x^2} = \lim_{x\to 0} \dfrac{e^x-1}{6x} = \lim_{x\to 0} \dfrac{e^x}{6} = \dfrac{1}{6}$

23. This limit has the form $\dfrac{\infty}{\infty}$. $\lim_{x\to\infty} \dfrac{e^x}{x^3} = \lim_{x\to\infty} \dfrac{e^x}{3x^2} = \lim_{x\to\infty} \dfrac{e^x}{6x} = \lim_{x\to\infty} \dfrac{e^x}{6} = \infty$

24. $\lim_{x\to 0} \dfrac{\sin x}{\sinh x} = \lim_{x\to 0} \dfrac{\cos x}{\cosh x} = \dfrac{1}{1} = 1$

25. This limit has the form $\frac{0}{0}$. $\lim_{x\to 0} \frac{\sin^{-1} x}{x} = \lim_{x\to 0} \frac{1/\sqrt{1-x^2}}{1} = \lim_{x\to 0} \frac{1}{\sqrt{1-x^2}} = \frac{1}{1} = 1$

26. $\lim_{x\to 0} \frac{\sin x - x}{x^3} = \lim_{x\to 0} \frac{\cos x - 1}{3x^2} = \lim_{x\to 0} \frac{-\sin x}{6x} = \lim_{x\to 0} \frac{-\cos x}{6} = -\frac{1}{6}$

27. This limit has the form $\frac{0}{0}$. $\lim_{x\to 0} \frac{1-\cos x}{x^2} = \lim_{x\to 0} \frac{\sin x}{2x} = \lim_{x\to 0} \frac{\cos x}{2} = \frac{1}{2}$

28. $\lim_{x\to\infty} \frac{(\ln x)^2}{x} = \lim_{x\to\infty} \frac{2(\ln x)(1/x)}{1} = 2\lim_{x\to\infty} \frac{\ln x}{x} = 2\lim_{x\to\infty} \frac{1/x}{1} = 2(0) = 0$

29. $\lim_{x\to 0} \frac{x+\sin x}{x+\cos x} = \frac{0+0}{0+1} = \frac{0}{1} = 0$. L'Hospital's Rule does not apply.

30. $\lim_{x\to 0} \frac{\cos mx - \cos nx}{x^2} = \lim_{x\to 0} \frac{-m\sin mx + n\sin nx}{2x} = \lim_{x\to 0} \frac{-m^2\cos mx + n^2\cos nx}{2} = \frac{1}{2}\left(n^2 - m^2\right)$

31. This limit has the form $\frac{\infty}{\infty}$. $\lim_{x\to\infty} \frac{x}{\ln\left(1+2e^x\right)} = \lim_{x\to\infty} \frac{1}{\frac{1}{1+2e^x}\cdot 2e^x} = \lim_{x\to\infty} \frac{1+2e^x}{2e^x} = \lim_{x\to\infty} \frac{2e^x}{2e^x} = 1$

32. $\lim_{x\to 0} \frac{x}{\tan^{-1}(4x)} = \lim_{x\to 0} \frac{1}{\frac{1}{1+(4x)^2}\cdot 4} = \lim_{x\to 0} \frac{1+16x^2}{4} = \frac{1}{4}$

33. This limit has the form $\frac{0}{0}$. $\lim_{x\to 1} \frac{1-x+\ln x}{1+\cos \pi x} = \lim_{x\to 1} \frac{-1+1/x}{-\pi\sin \pi x} = \lim_{x\to 1} \frac{-1/x^2}{-\pi^2\cos \pi x} = \frac{-1}{-\pi^2(-1)} = -\frac{1}{\pi^2}$

34. $\lim_{x\to\infty} \frac{\sqrt{x^2+2}}{\sqrt{2x^2+1}} = \lim_{x\to\infty} \sqrt{\frac{x^2+2}{2x^2+1}} = \sqrt{\lim_{x\to\infty} \frac{x^2+2}{2x^2+1}} = \sqrt{\lim_{x\to\infty} \frac{1+2/x^2}{2+1/x^2}} = \sqrt{\frac{1}{2}}$

35. This limit has the form $\frac{0}{0}$. $\lim_{x\to 1} \frac{x^a - ax + a - 1}{(x-1)^2} = \lim_{x\to 1} \frac{ax^{a-1} - a}{2(x-1)} = \lim_{x\to 1} \frac{a(a-1)x^{a-2}}{2} = \frac{a(a-1)}{2}$

36. $\lim_{x\to 0} \frac{1-e^{-2x}}{\sec x} = \frac{1-1}{1} = 0$. L'Hospital's Rule does not apply.

37. This limit has the form $0\cdot(-\infty)$. We need to write this product as a quotient, but keep in mind that we will have to differentiate both the numerator and the denominator. If we differentiate $\frac{1}{\ln x}$, we get a complicated expression that results in a more difficult limit. Instead we write the quotient as $\frac{\ln x}{x^{-1/2}}$.

$$\lim_{x\to 0^+} \sqrt{x}\,\ln x = \lim_{x\to 0^+} \frac{\ln x}{x^{-1/2}} = \lim_{x\to 0^+} \frac{1/x}{-\frac{1}{2}x^{-3/2}} \cdot \frac{-2x^{3/2}}{-2x^{3/2}} = \lim_{x\to 0^+} \left(-2\sqrt{x}\right) = 0$$

38. $\lim_{x\to -\infty} x^2 e^x = \lim_{x\to -\infty} \frac{x^2}{e^{-x}} = \lim_{x\to -\infty} \frac{2x}{-e^{-x}} = \lim_{x\to -\infty} \frac{2}{e^{-x}} = \lim_{x\to -\infty} 2e^x = 0$

39. This limit has the form $\infty\cdot 0$. We'll change it to the form $\frac{0}{0}$.

$$\lim_{x\to 0} \cot 2x \sin 6x = \lim_{x\to 0} \frac{\sin 6x}{\tan 2x} = \lim_{x\to 0} \frac{6\cos 6x}{2\sec^2 2x} = \frac{6(1)}{2(1)^2} = 3$$

40.

$$\lim_{x\to 0^+} \sin x \ln x = \lim_{x\to 0^+} \frac{\ln x}{\csc x} = \lim_{x\to 0^+} \frac{1/x}{-\csc x \cot x} = -\lim_{x\to 0^+} \left(\frac{\sin x}{x} \cdot \tan x \right)$$

$$= -\left(\lim_{x\to 0^+} \frac{\sin x}{x} \right) \left(\lim_{x\to 0^+} \tan x \right) = -1\cdot 0 = 0$$

41. This limit has the form $\infty\cdot 0$. $\lim_{x\to \infty} x^3 e^{-x^2} = \lim_{x\to \infty} \frac{x^3}{e^{x^2}} = \lim_{x\to \infty} \frac{3x^2}{2xe^{x^2}} = \lim_{x\to \infty} \frac{3x}{2e^{x^2}} = \lim_{x\to \infty} \frac{3}{4xe^{x^2}} = 0$

42.

$\lim_{x\to\pi/4}(1-\tan x)\sec x=(1-1)\sqrt{2}=0$. L'Hospital's Rule does not apply.

43. This limit has the form $0\cdot(-\infty)$.

$$\lim_{x\to 1^+}\ln x\tan(\pi x/2)=\lim_{x\to 1^+}\frac{\ln x}{\cot(\pi x/2)}=\lim_{x\to 1^+}\frac{1/x}{(-\pi/2)\csc^2(\pi x/2)}=\frac{1}{(-\pi/2)(1)^2}=-\frac{2}{\pi}$$

44. $\lim_{x\to\infty}x\tan(1/x)=\lim_{x\to\infty}\frac{\tan(1/x)}{1/x}=\lim_{x\to\infty}\frac{\sec^2(1/x)\left(-1/x^2\right)}{-1/x^2}=\lim_{x\to\infty}\sec^2(1/x)=1^2=1$

45.

$$\lim_{x\to 0}\left(\frac{1}{x}-\csc x\right)=\lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{\sin x}\right)=\lim_{x\to 0}\frac{\sin x-x}{x\sin x}$$

$$=\lim_{x\to 0}\frac{\cos x-1}{x\cos x+\sin x}=\lim_{x\to 0}\frac{-\sin x}{2\cos x-x\sin x}=\frac{0}{2}=0$$

46. $\lim_{x\to 0}(\csc x-\cot x)=\lim_{x\to 0}\left(\frac{1}{\sin x}-\frac{\cos x}{\sin x}\right)=\lim_{x\to 0}\frac{1-\cos x}{\sin x}=\lim_{x\to 0}\frac{\sin x}{\cos x}=0$

47. We will multiply and divide by the conjugate of the expression to change the form of the expression.

$$\lim_{x\to\infty}\left(\sqrt{x^2+x}-x\right)=\lim_{x\to\infty}\left(\frac{\sqrt{x^2+x}-x}{1}\cdot\frac{\sqrt{x^2+x}+x}{\sqrt{x^2+x}+x}\right)=\lim_{x\to\infty}\frac{\left(x^2+x\right)-x^2}{\sqrt{x^2+x}+x}$$

$$=\lim_{x\to\infty}\frac{x}{\sqrt{x^2+x}+x}=\lim_{x\to\infty}\frac{1}{\sqrt{1+1/x}+1}=\frac{1}{\sqrt{1}+1}=\frac{1}{2}.$$

As an alternate solution, write $\sqrt{x^2+x}-x$ as $\sqrt{x^2+x}-\sqrt{x^2}$, factor out $\sqrt{x^2}$, rewrite as $\left(\sqrt{1+1/x}-1\right)/(1/x)$, and apply l'Hospital's Rule.

48.

$$\lim_{x\to 1}\left(\frac{1}{\ln x}-\frac{1}{x-1}\right)=\lim_{x\to 1}\frac{x-1-\ln x}{(x-1)\ln x}=\lim_{x\to 1}\frac{1-1/x}{(x-1)(1/x)+\ln x}\cdot\frac{x}{x}$$

$$=\lim_{x\to 1}\frac{x-1}{x-1+x\ln x}=\lim_{x\to 1}\frac{1}{1+1+\ln x}=\frac{1}{2+0}=\frac{1}{2}$$

49. The limit has the form $\infty-\infty$ and we will change the form to a product by factoring out x.
$\lim_{x\to\infty}(x-\ln x)=\lim_{x\to\infty}x\left(1-\frac{\ln x}{x}\right)=\infty$ since $\lim_{x\to\infty}\frac{\ln x}{x}=\lim_{x\to\infty}\frac{1/x}{1}=0$.

50. As $x\to\infty$, $1/x\to 0$, and $e^{1/x}\to 1$. So the limit has the form $\infty-\infty$ and we will change the form to a product by factoring out x.
$\lim_{x\to\infty}\left(xe^{1/x}-x\right)=\lim_{x\to\infty}x\left(e^{1/x}-1\right)=\lim_{x\to\infty}\frac{e^{1/x}-1}{1/x}=\lim_{x\to\infty}\frac{e^{1/x}\left(-1/x^2\right)}{-1/x^2}=\lim_{x\to\infty}e^{1/x}=e^0=1$

51. $y=x^{x^2}\Rightarrow\ln y=x^2\ln x$, so $\lim_{x\to0^+}\ln y=\lim_{x\to0^+}x^2\ln x=\lim_{x\to0^+}\frac{\ln x}{1/x^2}=\lim_{x\to0^+}\frac{1/x}{-2/x^3}=\lim_{x\to0^+}\left(-\frac{1}{2}x^2\right)=0\Rightarrow$
$\lim_{x\to0^+}x^{x^2}=\lim_{x\to0^+}e^{\ln y}=e^0=1$.

52. $y=(\tan 2x)^x\Rightarrow\ln y=x\cdot\ln\tan 2x$, so

$$\lim_{x\to0^+}\ln y=\lim_{x\to0^+}x\cdot\ln\tan 2x=\lim_{x\to0^+}\frac{\ln\tan 2x}{1/x}$$

$$=\lim_{x\to0^+}\frac{(1/\tan 2x)(2\sec^2 2x)}{-1/x^2}=\lim_{x\to0^+}\frac{-2x^2\cos 2x}{\sin 2x\cos^2 2x}=\lim_{x\to0^+}\frac{2x}{\sin 2x}\cdot\lim_{x\to0^+}\frac{-x}{\cos 2x}=1\cdot 0=0\Rightarrow$$

$$\lim_{x\to0^+}(\tan 2x)^x=\lim_{x\to0^+}e^{\ln y}=e^0=1.$$

53. $y=(1-2x)^{1/x}\Rightarrow\ln y=\frac{1}{x}\ln(1-2x)$, so $\lim_{x\to0}\ln y=\lim_{x\to0}\frac{\ln(1-2x)}{x}=\lim_{x\to0}\frac{-2/(1-2x)}{1}=-2\Rightarrow$
$\lim_{x\to0}(1-2x)^{1/x}=\lim_{x\to0}e^{\ln y}=e^{-2}$.

54. $y=\left(1+\frac{a}{x}\right)^{bx}\Rightarrow\ln y=bx\ln\left(1+\frac{a}{x}\right)$, so

$$\lim_{x\to\infty}\ln y=\lim_{x\to\infty}\frac{b\ln(1+a/x)}{1/x}=\lim_{x\to\infty}\frac{b\left(\frac{1}{1+a/x}\right)\left(-\frac{a}{x^2}\right)}{-1/x^2}=\lim_{x\to\infty}\frac{ab}{1+a/x}=ab\Rightarrow$$

$\lim\limits_{x\to\infty}\left(1+\frac{a}{x}\right)^{bx}=\lim\limits_{x\to\infty}e^{\ln y}=e^{ab}$.

55. $y=\left(1+\frac{3}{x}+\frac{5}{x^2}\right)^x\Rightarrow \ln y=x\ln\left(1+\frac{3}{x}+\frac{5}{x^2}\right)\Rightarrow$

$$\lim_{x\to\infty}\ln y=\lim_{x\to\infty}\frac{\ln\left(1+\frac{3}{x}+\frac{5}{x^2}\right)}{1/x}=\lim_{x\to\infty}\frac{\left(-\frac{3}{x^2}-\frac{10}{x^3}\right)\Big/\left(1+\frac{3}{x}+\frac{5}{x^2}\right)}{-1/x^2}=\lim_{x\to\infty}\frac{3+\frac{10}{x}}{1+\frac{3}{x}+\frac{5}{x^2}}=3,$$

so $\lim\limits_{x\to\infty}\left(1+\frac{3}{x}+\frac{5}{x^2}\right)^x=\lim\limits_{x\to\infty}e^{\ln y}=e^3$.

56. $y=x^{(\ln 2)/(1+\ln x)}\Rightarrow \ln y=\frac{\ln 2}{1+\ln x}\ln x\Rightarrow$

$\lim\limits_{x\to\infty}\ln y=\lim\limits_{x\to\infty}\frac{(\ln 2)(\ln x)}{1+\ln x}=\lim\limits_{x\to\infty}\frac{(\ln 2)(1/x)}{1/x}=\lim\limits_{x\to\infty}\ln 2=\ln 2$,

so $\lim\limits_{x\to\infty}x^{(\ln 2)/(1+\ln x)}=\lim\limits_{x\to\infty}e^{\ln y}=e^{\ln 2}=2$.

57. $y=x^{1/x}\Rightarrow \ln y=(1/x)\ln x\Rightarrow \lim\limits_{x\to\infty}\ln y=\lim\limits_{x\to\infty}\frac{\ln x}{x}=\lim\limits_{x\to\infty}\frac{1/x}{1}=0\Rightarrow \lim\limits_{x\to\infty}x^{1/x}=\lim\limits_{x\to\infty}e^{\ln y}=e^0=1$

58. $y=\left(e^x+x\right)^{1/x}\Rightarrow \ln y=\frac{1}{x}\ln\left(e^x+x\right)$, so

$$\lim_{x\to\infty}\ln y=\lim_{x\to\infty}\frac{\ln\left(e^x+x\right)}{x}=\lim_{x\to\infty}\frac{e^x+1}{e^x+x}=\lim_{x\to\infty}\frac{e^x}{e^x+1}=\lim_{x\to\infty}\frac{e^x}{e^x}=1\Rightarrow\lim_{x\to\infty}\left(e^x+x\right)^{1/x}=\lim_{x\to\infty}e^{\ln y}=e^1=e .$$

59. $y=\left(\frac{x}{x+1}\right)^x\Rightarrow \ln y=x\ln\left(\frac{x}{x+1}\right)\Rightarrow$

$$\lim_{x\to\infty}\ln y=\lim_{x\to\infty}x\ln\left(\frac{x}{x+1}\right)=\lim_{x\to\infty}\frac{\ln x-\ln(x+1)}{1/x}=\lim_{x\to\infty}\frac{1/x-1/(x+1)}{-1/x^2}$$

$$=\lim_{x\to\infty}\left(-x+\frac{x^2}{x+1}\right)=\lim_{x\to\infty}\frac{-x}{x+1}=-1$$

so

$$\lim_{x\to\infty}\left(\frac{x}{x+1}\right)^x=\lim_{x\to\infty}e^{\ln y}=e^{-1}$$

Or: $\lim_{x\to\infty}\left(\frac{x}{x+1}\right)^x=\lim_{x\to\infty}\left[\left(\frac{x+1}{x}\right)^{-1}\right]^x=\left[\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x\right]^{-1}=e^{-1}$

60. $y=(\cos 3x)^{5/x}\Rightarrow \ln y=\frac{5}{x}\ln(\cos 3x)\Rightarrow \lim_{x\to 0}\ln y=5\lim_{x\to 0}\frac{\ln(\cos 3x)}{x}=5\lim_{x\to 0}\frac{-3\tan 3x}{1}=0$,

so $\lim_{x\to 0}(\cos 3x)^{5/x}=e^0=1$.

61. $y=(\cos x)^{1/x^2}\Rightarrow \ln y=\frac{1}{x^2}\ln\cos x\Rightarrow \lim_{x\to 0^+}\ln y=\lim_{x\to 0^+}\frac{\ln\cos x}{x^2}=\lim_{x\to 0^+}\frac{-\tan x}{2x}=\lim_{x\to 0^+}\frac{-\sec^2 x}{2}=-\frac{1}{2}\Rightarrow$

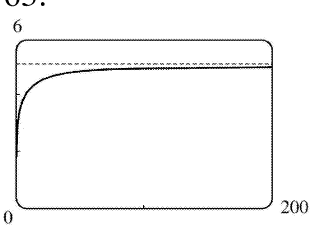
$\lim_{x\to 0^+}(\cos x)^{1/x^2}=\lim_{x\to 0^+}e^{\ln y}=e^{-1/2}=1/\sqrt{e}$

62. $y=\left(\frac{2x-3}{2x+5}\right)^{2x+1}\Rightarrow \ln y=(2x+1)\ln\left(\frac{2x-3}{2x+5}\right)\Rightarrow$

$$\lim_{x\to\infty}\ln y=\lim_{x\to\infty}\frac{\ln(2x-3)-\ln(2x+5)}{1/(2x+1)}=\lim_{x\to\infty}\frac{2/(2x-3)-2/(2x+5)}{-2/(2x+1)^2}=\lim_{x\to\infty}\frac{-8(2x+1)^2}{(2x-3)(2x+5)}$$

$$=\lim_{x\to\infty}\frac{-8(2+1/x)^2}{(2-3/x)(2+5/x)}=-8\Rightarrow \lim_{x\to\infty}\left(\frac{2x-3}{2x+5}\right)^{2x+1}=e^{-8}$$

63.



From the graph, it appears that $\lim_{x\to\infty}x[\ln(x+5)-\ln x]=5$.

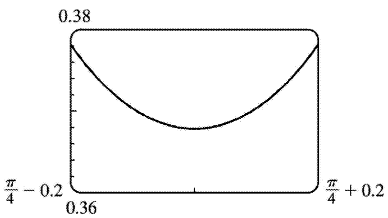
To prove this, we first note that

$\ln(x+5)-\ln x=\ln\frac{x+5}{x}=\ln\left(1+\frac{5}{x}\right)\to\ln 1=0$ as $x\to\infty$. Thus,

$$\lim_{x\to\infty} x[\ln(x+5)-\ln x] = \lim_{x\to\infty}\frac{\ln(x+5)-\ln x}{1/x} = \lim_{x\to\infty}\frac{\frac{1}{x+5}-\frac{1}{x}}{-1/x^2}$$

$$= \lim_{x\to\infty}\left[\frac{x-(x+5)}{x(x+5)}\cdot\frac{-x^2}{1}\right] = \lim_{x\to\infty}\frac{5x^2}{x^2+5x} = 5$$

64.



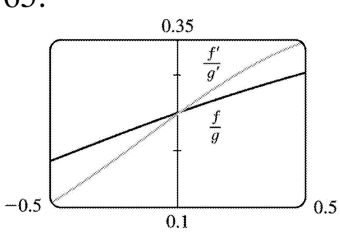
From the graph, it appears that $\lim_{x\to\pi/4}(\tan x)^{\tan 2x}\approx 0.368$.

The limit has the form 1^{∞} . Now $y=(\tan x)^{\tan 2x}\Rightarrow$
$\ln y=\tan 2x\ln(\tan x)$, so

$$\lim_{x\to\pi/4}\ln y = \lim_{x\to\pi/4}\frac{\ln(\tan x)}{\cot 2x} = \lim_{x\to\pi/4}\frac{\sec^2 x/\tan x}{-2\csc^2 2x} = \frac{2/1}{-2(1)} = -1 \Rightarrow$$

$$\lim_{x\to\pi/4}(\tan x)^{\tan 2x} = \lim_{x\to\pi/4} e^{\ln y} = e^{-1} = 1/e \approx 0.3679 .$$

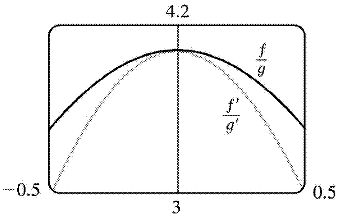
65.



From the graph, it appears that $\lim_{x\to 0}\frac{f(x)}{g(x)} = \lim_{x\to 0}\frac{f'(x)}{g'(x)} = 0.25$. We calculate

$$\lim_{x\to 0}\frac{f(x)}{g(x)} = \lim_{x\to 0}\frac{e^x-1}{x^3+4x} = \lim_{x\to 0}\frac{e^x}{3x^2+4} = \frac{1}{4} .$$

66.



From the graph, it appears that $\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{f'(x)}{g'(x)} = 4$. We calculate

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{2x\sin x}{\sec x - 1} = \lim_{x\to 0} \frac{2(x\cos x + \sin x)}{\sec x\tan x}$$

$$= \lim_{x\to 0} \frac{2(-x\sin x + \cos x + \cos x)}{\sec x\left(\sec^2 x\right) + \tan x(\sec x\tan x)} = \frac{4}{1} = 4$$

67. $\lim_{x\to\infty} \frac{e^x}{x^n} = \lim_{x\to\infty} \frac{e^x}{nx^{n-1}} = \lim_{x\to\infty} \frac{e^x}{n(n-1)x^{n-2}} = \cdots = \lim_{x\to\infty} \frac{e^x}{n!} = \infty$

68. $\lim_{x\to\infty} \frac{\ln x}{x^p} = \lim_{x\to\infty} \frac{1/x}{px^{p-1}} = \lim_{x\to\infty} \frac{1}{px^p} = 0$ since $p>0$.

69. First we will find $\lim_{n\to\infty} \left(1+\frac{i}{n}\right)^{nt}$, which is of the form 1^∞ . $y=\left(1+\frac{i}{n}\right)^{nt} \Rightarrow$ $\ln y = nt\ln\left(1+\frac{i}{n}\right)$, so

$$\lim_{n\to\infty} \ln y = \lim_{n\to\infty} nt\ln\left(1+\frac{i}{n}\right) = t\lim_{n\to\infty} \frac{\ln(1+i/n)}{1/n} = t\lim_{n\to\infty} \frac{\left(-i/n^2\right)}{(1+i/n)\left(-1/n^2\right)} = t\lim_{n\to\infty} \frac{i}{1+i/n} = ti \Rightarrow \lim_{n\to\infty} y = e^{it}$$

. Thus, as $n\to\infty$, $A=A_0\left(1+\frac{i}{n}\right)^{nt} \to A_0e^{it}$.

70. **(a)**

$$\lim_{t\to\infty} v = \lim_{t\to\infty} \frac{mg}{c}\left(1-e^{-ct/m}\right) = \frac{mg}{c}\lim_{t\to\infty}\left(1-e^{-ct/m}\right)$$

$$= \frac{mg}{c}(1-0) = \frac{mg}{c} ,$$

which is the speed the object approaches as time goes on, the so–called limiting velocity.

(b) $\lim_{m\to\infty} v$

$$=\lim_{m\to\infty} \frac{mg}{c}\left(1-e^{-ct/m}\right)=\frac{g}{c}\lim_{m\to\infty}\frac{1-e^{-ct/m}}{1/m}=\frac{g}{c}\lim_{m\to\infty}\frac{-e^{-ct/m}\left(ct/m^2\right)}{-1/m^2}$$

$$=\frac{g}{c}(ct)\lim_{m\to\infty} e^{-ct/m}=gt(1)$$

The speed of a very heavy falling object is approximately proportional to the elapsed time t , provided it can fall for time t in an environment where the given model continues to hold. .

71. We see that both numerator and denominator approach 0 , so we can use l'Hospital's Rule:

$$\lim_{x\to a}\frac{\sqrt{2a^3x-x^4}-a\sqrt[3]{aax}}{a-\sqrt[4]{ax^3}} = \lim_{x\to a}\frac{\frac{1}{2}\left(2a^3x-x^4\right)^{-1/2}\left(2a^3-4x^3\right)-a\left(\frac{1}{3}\right)(aax)^{-2/3}a^2}{-\frac{1}{4}\left(ax^3\right)^{-3/4}\left(3ax^2\right)}$$

$$=\frac{\frac{1}{2}\left(2a^3a-a^4\right)^{-1/2}\left(2a^3-4a^3\right)-\frac{1}{3}a^3\left(a^2a\right)^{-2/3}}{-\frac{1}{4}\left(aa^3\right)^{-3/4}\left(3aa^2\right)}$$

$$=\frac{\left(a^4\right)^{-1/2}\left(-a^3\right)-\frac{1}{3}a^3\left(a^3\right)^{-2/3}}{-\frac{3}{4}a^3\left(a^4\right)^{-3/4}}=\frac{-a-\frac{1}{3}a}{-\frac{3}{4}}=\frac{4}{3}\left(\frac{4}{3}a\right)=\frac{16}{9}a$$

72. Let the radius of the circle be r . We see that $A(\theta)$ is the area of the whole figure (a sector of the circle with

radius 1), minus the area of $\triangle OPR$. But the area of the sector of the circle is $\frac{1}{2}r^2\theta$ (see Reference

Page 1), and the area of the triangle is $\frac{1}{2}r|PQ|=\frac{1}{2}r(r\sin\theta)=\frac{1}{2}r^2\sin\theta$. So we have

$A(\theta)=\frac{1}{2}r^2\theta-\frac{1}{2}r^2\sin\theta=\frac{1}{2}r^2(\theta-\sin\theta)$. Now by elementary trigonometry,

$B(\theta)=\frac{1}{2}|QR|\,|PQ|=\frac{1}{2}(r-|OQ|)\,|PQ|=\frac{1}{2}(r-r\cos\theta)(r\sin\theta)=\frac{1}{2}r^2(1-\cos\theta)\sin\theta$.

So the limit we want is

$$\lim_{\theta\to 0^+}\frac{A(\theta)}{B(\theta)} = \lim_{\theta\to 0^+}\frac{\frac{1}{2}r^2(\theta-\sin\theta)}{\frac{1}{2}r^2(1-\cos\theta)\sin\theta}=\lim_{\theta\to 0^+}\frac{1-\cos\theta}{(1-\cos\theta)\cos\theta+\sin\theta(\sin\theta)}$$

$$=\lim_{\theta\to 0^+}\frac{1-\cos\theta}{\cos\theta-\cos^2\theta+\sin^2\theta}=\lim_{\theta\to 0^+}\frac{\sin\theta}{-\sin\theta-2\cos\theta(-\sin\theta)+2\sin\theta(\cos\theta)}$$

$$=\lim_{\theta\to 0^+}\frac{\sin\theta}{-\sin\theta+4\sin\theta\cos\theta}=\lim_{\theta\to 0^+}\frac{1}{-1+4\cos\theta}=\frac{1}{-1+4\cos 0}=\frac{1}{3}$$

73. Since $f(2)=0$, the given limit has the form $\frac{0}{0}$.

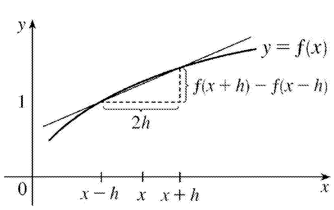
$$\lim_{x\to 0}\frac{f(2+3x)+f(2+5x)}{x}=\lim_{x\to 0}\frac{f'(2+3x)\cdot 3+f'(2+5x)\cdot 5}{1}=f'(2)\cdot 3+f'(2)\cdot 5=8f'(2)=8\cdot 7=56$$

74. $L=\lim_{x\to 0}\left(\frac{\sin 2x}{x^3}+a+\frac{b}{x^2}\right)=\lim_{x\to 0}\frac{\sin 2x+ax^3+bx}{x^3}=\lim_{x\to 0}\frac{2\cos 2x+3ax^2+b}{3x^2}$. As $x\to 0$, $3x^2\to 0$, and $\left(2\cos 2x+3ax^2+b\right)\to b+2$, so the last limit exists only if $b+2=0$, that is, $b=-2$. Thus, $\lim_{x\to 0}\frac{2\cos 2x+3ax^2-2}{3x^2}=\lim_{x\to 0}\frac{-4\sin 2x+6ax}{6x}=\lim_{x\to 0}\frac{-8\cos 2x+6a}{6}=\frac{6a-8}{6}$, which is equal to 0 if and only if $a=\frac{4}{3}$. Hence, $L=0$ if and only if $b=-2$ and $a=\frac{4}{3}$.

75. Since $\lim_{h\to 0}[f(x+h)-f(x-h)]=f(x)-f(x)=0$ (f is differentiable and hence continuous) and $\lim_{h\to 0}2h=0$, we use l'Hospital's Rule:

$$\lim_{h\to 0}\frac{f(x+h)-f(x-h)}{2h}=\lim_{h\to 0}\frac{f'(x+h)(1)-f'(x-h)(-1)}{2}=\frac{f'(x)+f'(x)}{2}=\frac{2f'(x)}{2}=f'(x)$$

$\frac{f(x+h)-f(x-h)}{2h}$ is the slope of the secant line between $(x-h,f(x-h))$ and $(x+h,f(x+h))$. As $h\to 0$, this line gets closer to the tangent line and its slope approaches $f'(x)$.



76. Since $\lim_{h\to 0}\left[f(x+h)-2f(x)+f(x-h)\right]=f(x)-2f(x)+f(x)=0$ (f is differentiable and hence continuous) and

$\lim_{h\to 0} h^2=0$, we can apply l'Hospital's Rule:

$$\lim_{h\to 0}\frac{f(x+h)-2f(x)+f(x-h)}{h^2}=\lim_{h\to 0}\frac{f'(x+h)-f'(x-h)}{2h}=f''(x)$$

At the last step, we have applied the result of Exercise to $f'(x)$.

77. **(a)** We show that $\lim_{x\to 0}\frac{f(x)}{x^n}=0$ for every integer $n\geq 0$. Let $y=\frac{1}{x^2}$. Then

$$\lim_{x\to 0}\frac{f(x)}{x^{2n}}=\lim_{x\to 0}\frac{e^{-1/x^2}}{\left(x^2\right)^n}=\lim_{y\to\infty}\frac{y^n}{e^y}=\lim_{y\to\infty}\frac{ny^{n-1}}{e^y}=\cdots=\lim_{y\to\infty}\frac{n!}{e^y}=0\Rightarrow$$

$\lim_{x\to 0}\frac{f(x)}{x^n}=\lim_{x\to 0}x^n\frac{f(x)}{x^{2n}}=\lim_{x\to 0}x^n\lim_{x\to 0}\frac{f(x)}{x^{2n}}=0$. Thus, $f'(0)=\lim_{x\to 0}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0}\frac{f(x)}{x}=0$.

(b) Using the Chain Rule and the Quotient Rule we see that $f^{(n)}(x)$ exists for $x\neq 0$. In fact, we prove by induction that for each $n\geq 0$, there is a polynomial p_n and a non–negative integer k_n with $f^{(n)}(x)=p_n(x)f(x)/x^{k_n}$ for $x\neq 0$. This is true for $n=0$; suppose it is true for the n th derivative. Then $f'(x)=f(x)(2/x^3)$, so

$$\begin{aligned}
f^{(n+1)}(x) &= \left[x^{k_n}\left[p_n'(x)f(x)+p_n(x)f'(x)\right]-k_nx^{k_n-1}p_n(x)f(x)\right]x^{-2k_n}\\
&= \left[x^{k_n}p_n'(x)+p_n(x)\left(2/x^3\right)-k_nx^{k_n-1}p_n(x)\right]f(x)x^{-2k_n}\\
&= \left[x^{k_n+3}p_n'(x)+2p_n(x)-k_nx^{k_n+2}p_n(x)\right]f(x)x^{-\left(2k_n+3\right)}
\end{aligned}$$

which has the desired form.

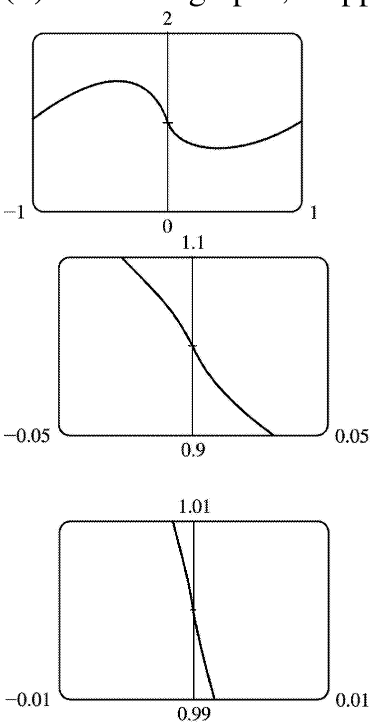
Now we show by induction that $f^{(n)}(0)=0$ for all n . By part (a), $f'(0)=0$. Suppose that $f^{(n)}(0)=0$. Then

$$f^{(n+1)}(0)=\lim_{x\to 0}\frac{f^{(n)}(x)-f^{(n)}(0)}{x-0}=\lim_{x\to 0}\frac{f^{(n)}(x)}{x}=\lim_{x\to 0}\frac{p_n(x)f(x)/x^{k_n}}{x}=\lim_{x\to 0}\frac{p_n(x)f(x)}{x^{k_n+1}}$$

$$= \lim_{x\to 0} p_n(x) \lim_{x\to 0} \frac{f(x)}{x^{k_n+1}} = p_n(0)\cdot 0 = 0$$

78. **(a)** For f to be continuous, we need $\lim_{x\to 0} f(x) = f(0) = 1$. We note that for $x\neq 0$, $\ln f(x) = \ln|x|^x = x\ln|x|$. So $\lim_{x\to 0} \ln f(x) = \lim_{x\to 0} x\ln|x| = \lim_{x\to 0} \frac{\ln|x|}{1/x} = \lim_{x\to 0} \frac{1/x}{-1/x^2} = 0$. Therefore, $\lim_{x\to 0} f(x) = \lim_{x\to 0} e^{\ln f(x)} = e^0 = 1$. So f is continuous at 0 .

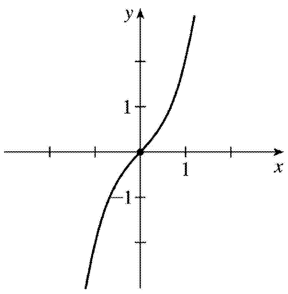
(b) From the graphs, it appears that f is differentiable at 0 .



(c) To find f' , we use logarithmic differentiation: $\ln f(x) = x\ln|x| \Rightarrow \frac{f'(x)}{f(x)} = x\left(\frac{1}{x}\right) + \ln|x| \Rightarrow$ $f'(x) = f(x)(1+\ln|x|) = |x|^x(1+\ln|x|)$, $x\neq 0$. Now $f'(x)\to -\infty$ as $x\to 0$, so the curve has a vertical tangent at $(0,1)$ and is therefore not differentiable there. The fact cannot be seen in the graphs in part (b) because $\ln|x|\to -\infty$ very slowly as $x\to 0$.

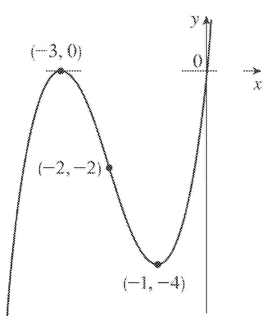
1. $y=f(x)=x^3+x=x\left(x^2+1\right)$ **A.** f is a polynomial, so $D=R$. **B.** x –intercept $=0$, y– intercept $=f(0)=0$ **C.** $f(-x)=-f(x)$, so f is odd; the curve is symmetric about the origin. **D.** f is a polynomial, so there is no asymptote. **E.** $f'(x)=3x^2+1>0$, so f is increasing on $(-\infty,\infty)$. **F.** There is no critical number and hence, no local maximum or minimum value. **G.** $f''(x)=6x>0$ on $(0,\infty)$ and $f''(x)<0$ on $(-\infty,0)$, so f is CU on $(0,\infty)$ and CD on $(-\infty,0)$. Since the concavity changes at $x=0$, there is an inflection point at $(0,0)$.

H.



2. $y=f(x)=x^3+6x^2+9x=x(x+3)^2$ **A.** $D=R$ **B.** x –intercepts are -3 and 0 , y –intercept $=0$ **C.** No symmetry **D.** No asymptote **E.** $f'(x)=3x^2+12x+9=3(x+1)(x+3)<0\Leftrightarrow -3<x<-1$, so f is decreasing on $(-3,-1)$ and increasing on $(-\infty,-3)$ and $(-1,\infty)$. **F.** Local maximum value $f(-3)=0$, local minimum value $f(-1)=-4$ **G.** $f''(x)=6x+12=6(x+2)>0\Leftrightarrow x>-2$, so f is CU on $(-2,\infty)$ and CD on $(-\infty,-2)$. IP at $(-2,-2)$

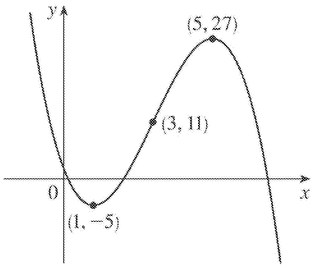
H.



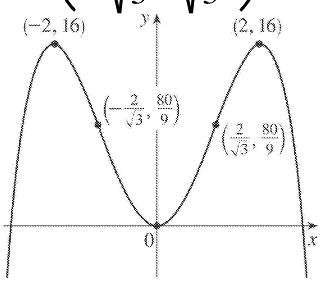
3. $y=f(x)=2-15x+9x^2-x^3=-(x-2)\left(x^2-7x+1\right)$ **A.** $D=R$ **B.** y –intercept: $f(0)=2$; x– intercepts: $f(x)=0$ $\Rightarrow x=2$ or (by the quadratic formula) $x=\dfrac{7\pm\sqrt{45}}{2}\approx 0.15$, 6.85

C. No symmetry **D.** No asymptote **E.** $f'(x)=-15+18x-3x^2=-3\left(x^2-6x+5\right)=-3(x-1)(x-5)>0\Leftrightarrow 1<x<5$ so f is increasing on $(1,5)$ and decreasing on $(-\infty,1)$ and $(5,\infty)$. **F.** Local maximum value $f(5)=27$, local minimum value $f(1)=-5$ **G.** $f''(x)=18-6x=-6(x-3)>0\Leftrightarrow x<3$, so f is CU on $(-\infty,3)$ and CD on $(3,\infty)$. IP at $(3,11)$

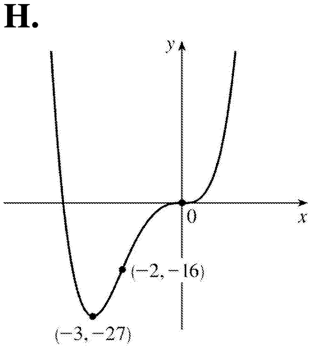
H.



4. $y=f(x)=8x^2-x^4=x^2\left(8-x^2\right)$ **A.** $D=R$ **B.** y –intercept: $f(0)=0$; x– intercepts: $f(x)=0\Rightarrow x=0$, $\pm 2\sqrt{2}$ ($\approx\pm 2.83$) **C.** $f(-x)=f(x)$, so f is even and symmetric about the y –axis. **D.** No asymptote **E.** $f'(x)=16x-4x^3=4x\left(4-x^2\right)=4x(2+x)(2-x)>0\Leftrightarrow x<-2$ or $0<x<2$, so f is increasing on $(-\infty,-2)$ and $(0,2)$ and decreasing on $(-2,0)$ and $(2,\infty)$. **F.** Local maximum value $f(\pm 2)=16$, local minimum value $f(0)=0$ **G.** $f''(x)=16-12x^2=4\left(4-3x^2\right)=0\Leftrightarrow x=\pm\dfrac{2}{\sqrt{3}}$. $f''(x)>0\Leftrightarrow -\dfrac{2}{\sqrt{3}}<x<\dfrac{2}{\sqrt{3}}$, so f is CU on $\left(-\dfrac{2}{\sqrt{3}},\dfrac{2}{\sqrt{3}}\right)$ and CD on $\left(-\infty,-\dfrac{2}{\sqrt{3}}\right)$ and $\left(\dfrac{2}{\sqrt{3}},\infty\right)$. IP at $\left(\pm\dfrac{2}{\sqrt{3}},\dfrac{80}{9}\right)$ **H.**



5. $y=f(x)=x^4+4x^3=x^3(x+4)$ **A.** $D=R$ **B.** y –intercept: $f(0)=0$; x –intercepts: $f(x)=0\Leftrightarrow x=-4,0$ **C.** No symmetry **D.** No asymptote **E.** $f'(x)=4x^3+12x^2=4x^2(x+3)>0\Leftrightarrow x>-3$, so f is increasing on $(-3,\infty)$ and decreasing on $(-\infty,-3)$. **F.** Local minimum value $f(-3)=-27$, no local maximum **G.**
$f''(x)=12x^2+24x=12x(x+2)<0\Leftrightarrow -2<x<0$,
so f is CD on $(-2,0)$ and CU on $(-\infty,-2)$ and $(0,\infty)$.
IP at $(0,0)$ and $(-2,-16)$
H.



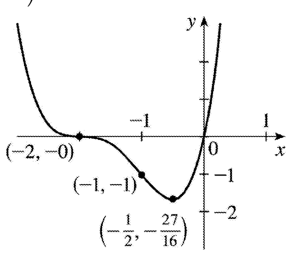
6. $y=f(x)=x(x+2)^3$ **A.** $D=R$ **B.** $y-$ intercept: $f(0)=0$; x –intercepts: $f(x)=0 \Leftrightarrow x=-2,0$ **C.** No symmetry **D.** No asymptote **E.** $f'(x)=3x(x+2)^2+(x+2)^3=(x+2)^2[3x+(x+2)]=(x+2)^2(4x+2)$. $f'(x)>0 \Leftrightarrow x>-\frac{1}{2}$, and $f'(x)<0 \Leftrightarrow x<-2$ or $-2<x<-\frac{1}{2}$, so f is increasing on $\left(-\frac{1}{2},\infty\right)$ and decreasing on $(-\infty,-2)$ **F.** Local minimum value $f\left(-\frac{1}{2}\right)=-\frac{27}{16}$, no local maximum

G.

$$\begin{aligned} f''(x) &= (x+2)^2(4)+(4x+2)(2)(x+2) \\ &= 2(x+2)[(x+2)(2)+4x+2] \\ &= 2(x+2)(6x+6)=12(x+1)(x+2) \end{aligned}$$

$f''(x)<0 \Leftrightarrow -2<x<-1$, so f is CD on $(-2,-1)$ and CU on $(-\infty,-2)$ and $(-1,\infty)$. IP at $(-2,0)$ and $(-1,-1)$

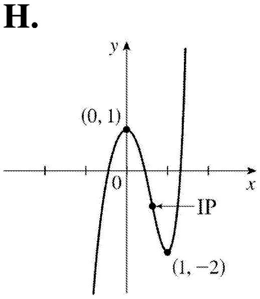
H.



7. $y=f(x)=2x^5-5x^2+1$ **A.** $D=R$ **B.** $y-$ intercept: $f(0)=1$ **C.** No symmetry **D.** No asymptote **E.** $f'(x)=10x^4-10x=10x(x^3-1)=10x(x-1)(x^2+x+1)$, so $f'(x)<0 \Leftrightarrow 0<x<1$ and $f'(x)>0 \Leftrightarrow x<0$ or $x>1$. Thus, f is increasing on $(-\infty,0)$ and $(1,\infty)$ and decreasing on $(0,1)$.

F. Local maximum value $f(0)=1$, local minimum value $f(1)=-2$ **G.** $f''(x)=40x^3-10=10(4x^3-1)$ so $f''(x)=0 \Leftrightarrow x=1/\sqrt[3]{4}$. $f''(x)>0 \Leftrightarrow x>1/\sqrt[3]{4}$ and $f''(x)<0 \Leftrightarrow x<1/\sqrt[3]{4}$, < so f is CD on $\left(-\infty,1/\sqrt[3]{4}\right)$ and CU on $\left(1/\sqrt[3]{4},\infty\right)$. IP at $\left(\frac{1}{\sqrt[3]{4}},1-\frac{9}{2\left(\sqrt[3]{4}\right)^2}\right)\approx(0.630,-0.786)$

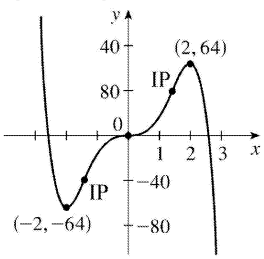
H.



8.

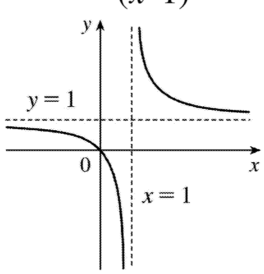
$y=f(x)=20x^3-3x^5$ **A.** $D=R$ **B.** $y-$ intercept: $f(0)=0$; x –intercepts: $f(x)=0 \Leftrightarrow -3x^3\left(x^2-\dfrac{20}{3}\right)=0 \Leftrightarrow x=0$ or $\pm\sqrt{20/3}\approx\pm 2.582$ **C.** $f(-x)=-f(x)$, so f is odd;
the curve is symmetric about the origin. **D.** No asymptote **E.**
$f'(x)=60x^2-15x^4=-15x^2(x^2-4)=-15x^2(x+2)(x-2)$, so $f'(x)>0 \Leftrightarrow -2<x<0$ or $0<x<2$ and $f'(x)<0 \Leftrightarrow$ $x<-2$ or $x>2$. Thus, f is increasing on $(-2,0)$ and $(0,2)$ and f is decreasing on $(-\infty,-2)$ and $(2,\infty)$.
F. Local minimum value $f(-2)=-64$, local maximum value $f(2)=64$ **G.**
$f''(x)=120x-60x^3=-60x(x^2-2)$. $f''(x)>0 \Leftrightarrow x<-\sqrt{2}$ or $0<x<\sqrt{2}$; $f''(x)<0 \Leftrightarrow -\sqrt{2}<x<0$ or $x>\sqrt{2}$.
Thus, f is CU on $(-\infty,-\sqrt{2})$
and $(0,\sqrt{2})$, and f is CD on $(-\sqrt{2},0)$ and $(\sqrt{2},\infty)$. IP at $(-\sqrt{2},-28\sqrt{2})\approx(-1.414,-39.598)$, $(0,0)$, and $(\sqrt{2},28\sqrt{2})$

H.



9. $y=f(x)=x/(x-1)$ **A.** $D=\{x \mid x\neq 1\}=(-\infty,1)\cup(1,\infty)$ **B.** x –intercept $=0$, y –intercept $=f(0)=0$ **C.** No symmetry **D.** $\lim\limits_{x\to\pm\infty}\dfrac{x}{x-1}=1$, so $y=1$ is a HA. $\lim\limits_{x\to 1^-}\dfrac{x}{x-1}=-\infty$, $\lim\limits_{x\to 1^+}\dfrac{x}{x-1}=\infty$, so $x=1$ is a VA. **E.**
$f'(x)=\dfrac{(x-1)-x}{(x-1)^2}=\dfrac{-1}{(x-1)^2}<0$ for $x\neq 1$, so f is decreasing on $(-\infty,1)$ and $(1,\infty)$. **F.** No extreme values
G. $f''(x)=\dfrac{2}{(x-1)^3}>0 \Leftrightarrow x>1$, so f is CU on $(1,\infty)$ and CD on $(-\infty,1)$. No IP

H.



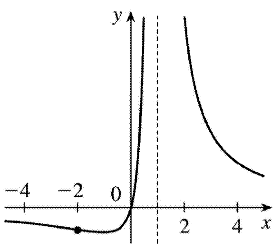
10. $y=x/(x-1)^2$ **A.** $D=\{x \mid x\neq 1\}=(-\infty,1)\cup(1,\infty)$ **B.** $x-$ intercept $=0$, y –intercept $=f(0)=0$ **C.** No symmetry **D.** $\lim\limits_{x\to\pm\infty}\dfrac{x}{(x-1)^2}=0$, so $y=0$ is a HA. $\lim\limits_{x\to 1}\dfrac{x}{(x-1)^2}=\infty$, so $x=1$ is a VA. **E.**

$f'(x)=\dfrac{(x-1)^2(1)-x(2)(x-1)}{(x-1)^4}=\dfrac{-x-1}{(x-1)^3}$. This is negative on $(-\infty,-1)$ and $(1,\infty)$ and positive on $(-1,1)$, so $f(x)$ is decreasing on $(-\infty,-1)$ and $(1,\infty)$ and increasing on $(-1,1)$.

F. Local minimum value $f(-1)=-\dfrac{1}{4}$, no local maximum. **G.**

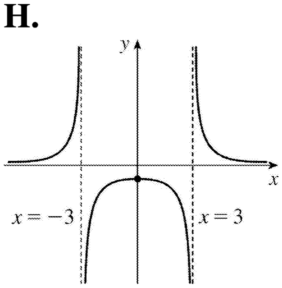
$f''(x)=\dfrac{(x-1)^3(-1)+(x+1)(3)(x-1)^2}{(x-1)^6}=\dfrac{2(x+2)}{(x-1)^4}$. This is negative on $(-\infty,-2)$, and positive on $(-2,1)$ and $(1,\infty)$. So f is CD on $(-\infty,-2)$ and CU on $(-2,1)$ and $(1,\infty)$. IP at $\left(-2,-\dfrac{2}{9}\right)$

H.



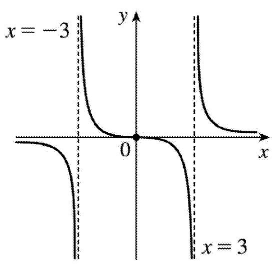
11. $y=f(x)=1/\left(x^2-9\right)$ **A.** $D=\{x\,|\,x\neq\pm3\}=(-\infty,-3)\cup(-3,3)\cup(3,\infty)$ **B.** y –intercept $=f(0)=-\dfrac{1}{9}$, no x –intercept **C.** $f(-x)=f(x)\Rightarrow f$ is even; the curve is symmetric about the y– axis. **D.** $\lim\limits_{x\to\pm\infty}\dfrac{1}{x^2-9}=0$, so $y=0$ is a HA. $\lim\limits_{x\to3^-}\dfrac{1}{x^2-9}=-\infty$, $\lim\limits_{x\to3^+}\dfrac{1}{x^2-9}=\infty$, $\lim\limits_{x\to-3^-}\dfrac{1}{x^2-9}=\infty$, $\lim\limits_{x\to-3^+}\dfrac{1}{x^2-9}=-\infty$, so $x=3$ and $x=-3$ are VA. **E.** $f'(x)=-\dfrac{2x}{\left(x^2-9\right)^2}>0\Leftrightarrow x<0$ ($x\neq-3$) so f is increasing on $(-\infty,-3)$ and $(-3,0)$ and decreasing on $(0,3)$ and $(3,\infty)$. **F.** Local maximum value $f(0)=-\dfrac{1}{9}$. **G.** $y''=\dfrac{-2\left(x^2-9\right)^2+(2x)2\left(x^2-9\right)(2x)}{\left(x^2-9\right)^4}=\dfrac{6\left(x^2+3\right)}{\left(x^2-9\right)^3}>0\Leftrightarrow x^2>9\Leftrightarrow x>3$ or $x<-3$, so f is CU on $(-\infty,-3)$ and $(3,\infty)$ and CD on $(-3,3)$. No IP

H.



12. $y=f(x)=x/\left(x^2-9\right)$ **A.** $D=\{x|x\neq\pm3\}=(-\infty,-3)\cup(-3,3)\cup(3,\infty)$ **B.** x –intercept $=0$, y– intercept $=f(0)=0$. **C.** $f(-x)=-f(x)$, so f is odd; the curve is symmetric about the origin. **D.** $\lim\limits_{x\to\pm\infty}\dfrac{x}{x^2-9}=0$, so $y=0$ is a HA. $\lim\limits_{x\to3^+}\dfrac{x}{x^2-9}=\infty$, $\lim\limits_{x\to3^-}\dfrac{x}{x^2-9}=-\infty$, $\lim\limits_{x\to-3^+}\dfrac{x}{x^2-9}=\infty$, $\lim\limits_{x\to-3^-}\dfrac{x}{x^2-9}=-\infty$, so $x=3$ and $x=-3$ are VA. **E.** $f'(x)=\dfrac{\left(x^2-9\right)-x(2x)}{\left(x^2-9\right)^2}=-\dfrac{x^2+9}{\left(x^2-9\right)^2}<0$ ($x\neq\pm3$) so f is decreasing on $(-\infty,-3)$, $(-3,3)$, and $(3,\infty)$. **F.** No extreme values **G.** $f''(x)=-\dfrac{2x\left(x^2-9\right)^2-\left(x^2+9\right)\cdot2\left(x^2-9\right)(2x)}{\left(x^2-9\right)^4}=\dfrac{2x\left(x^2+27\right)}{\left(x^2-9\right)^3}>0$ when $-3<x<0$ or $x>3$, so f is CU on $(-3,0)$ and $(3,\infty)$; CD on $(-\infty,-3)$ and $(0,3)$ > IP at $(0,0)$

H.



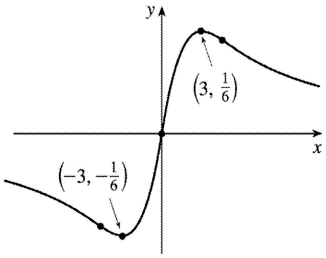
13. $y=f(x)=x/\left(x^2+9\right)$ **A.** $D=R$ **B.** y –intercept: $f(0)=0$; x –intercept: $f(x)=0\Leftrightarrow x=0$ **C.** $f(-x)=-f(x)$, so f is odd and the curve is symmetric about the origin. **D.** $\lim\limits_{x\to\pm\infty}\left[x/\left(x^2+9\right)\right]=0$, so $y=0$ is a HA; no VA **E.** $f'(x)=\dfrac{\left(x^2+9\right)(1)-x(2x)}{\left(x^2+9\right)^2}=\dfrac{9-x^2}{\left(x^2+9\right)^2}=\dfrac{(3+x)(3-x)}{\left(x^2+9\right)^2}>0\Leftrightarrow-3<x<3$, so f is increasing on $(-3,3)$ and decreasing on $(-\infty,-3)$ and $(3,\infty)$.

F. Local minimum value $f(-3)=-\dfrac{1}{6}$, local maximum value $f(3)=\dfrac{1}{6}$ **G.** $f''(x)$

$$=\frac{\left(x^2+9\right)^2(-2x)-\left(9-x^2\right)\cdot2\left(x^2+9\right)(2x)}{\left[\left(x^2+9\right)^2\right]^2}=\frac{(2x)\left(x^2+9\right)\left[-\left(x^2+9\right)-2\left(9-x^2\right)\right]}{\left(x^2+9\right)^4}=\frac{2x\left(x^2-27\right)}{\left(x^2+9\right)^3}=0\Leftrightarrow x=0,$$

$\pm\sqrt{27}=\pm3\sqrt{3}$ $f''(x)>0\Leftrightarrow-3\sqrt{3}<x<0$ or $x>3\sqrt{3}$, so f is CU on $\left(-3\sqrt{3},0\right)$ and $\left(3\sqrt{3},\infty\right)$, and CD on $\left(-\infty,-3\sqrt{3}\right)$ and $\left(0,3\sqrt{3}\right)$. There are three inflection points: $(0,0)$ and $\left(\pm3\sqrt{3},\pm\dfrac{1}{12}\sqrt{3}\right)$.

H.

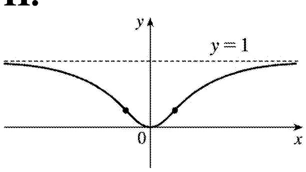


14. $y=f(x)=x^2/\left(x^2+9\right)$ **A.** $D=R$ **B.** y –intercept: $f(0)=0$; x –intercept: $f(x)=0 \Leftrightarrow x=0$ **C.** $f(-x)=f(x)$, so f is even and symmetric about the y –axis. **D.** $\lim\limits_{x\to\pm\infty}\left[x^2/\left(x^2+9\right)\right]=1$, so $y=1$ is a HA; no VA **E.** $f'(x)=\dfrac{\left(x^2+9\right)(2x)-x^2(2x)}{\left(x^2+9\right)^2}=\dfrac{18x}{\left(x^2+9\right)^2}>0 \Leftrightarrow x>0$, so f is increasing on $(0,\infty)$ and decreasing on $(-\infty,0)$. **F.** Local minimum value $f(0)=0$; no local maximum **G.** $f''(x)$

$$=\frac{\left(x^2+9\right)^2(18)-18x\cdot 2\left(x^2+9\right)\cdot 2x}{\left[\left(x^2+9\right)^2\right]^2}=\frac{18\left(x^2+9\right)\left[\left(x^2+9\right)-4x^2\right]}{\left(x^2+9\right)^4}=\frac{18\left(9-3x^2\right)}{\left(x^2+9\right)^3}$$

$=\dfrac{-54\left(x+\sqrt{3}\right)\left(x-\sqrt{3}\right)}{\left(x^2+9\right)^3}>0 \Leftrightarrow -\sqrt{3}<x<\sqrt{3}$ so f is CU on $\left(-\sqrt{3},\sqrt{3}\right)$ and CD on $\left(-\infty,-\sqrt{3}\right)$ and $\left(\sqrt{3},\infty\right)$. There are two inflection points: $\left(\pm\sqrt{3},\dfrac{1}{4}\right)$.

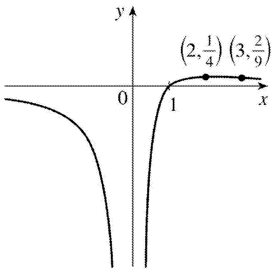
H.



15. $y=f(x)=\dfrac{x-1}{x^2}$ **A.** $D=\{x\mid x\neq 0\}=(-\infty,0)\cup(0,\infty)$ **B.** No y –intercept; x –intercept: $f(x)=0 \Leftrightarrow x=1$ **C.** No symmetry **D.** $\lim\limits_{x\to\pm\infty}\dfrac{x-1}{x^2}=0$, so $y=0$ is a HA. $\lim\limits_{x\to 0}\dfrac{x-1}{x^2}=-\infty$, so $x=0$ is a VA. **E.** $f'(x)=\dfrac{x^2\cdot 1-(x-1)\cdot 2x}{\left(x^2\right)^2}=\dfrac{-x^2+2x}{x^4}=\dfrac{-(x-2)}{x^3}$, so $f'(x)>0 \Leftrightarrow 0<x<2$ and $f'(x)<0 \Leftrightarrow x<0$ or $x>2$. Thus, f is increasing on $(0,2)$ and decreasing on $(-\infty,0)$ and $(2,\infty)$. **F.** No local minimum, local maximum value $f(2)=\dfrac{1}{4}$. **G.** $f''(x)=\dfrac{x^3\cdot(-1)-[-(x-2)]\cdot 3x^2}{\left(x^3\right)^2}=\dfrac{2x^3-6x^2}{x^6}=\dfrac{2(x-3)}{x^4}$. $f''(x)$ is negative on

$(-\infty,0)$ and $(0,3)$ and positive on $(3,\infty)$, so f is CD on $(-\infty,0)$ and $(0,3)$ and CU on $(3,\infty)$. IP at $\left(3,\frac{2}{9}\right)$

H.

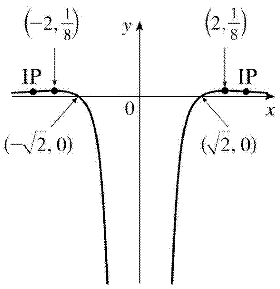


16. $y=f(x)=\frac{x^2-2}{x^4}$ **A.** $D=\{x \mid x\neq 0\}=(-\infty,0)\cup(0,\infty)$ **B.** No y –intercept; x –intercepts: $f(x)=0\Leftrightarrow x=\pm\sqrt{2}$ **C.** $f(-x)=f(x)$, so f is even; the curve is symmetric about the y –axis. **D.** $\lim\limits_{x\to\pm\infty}\frac{x^2-2}{x^4}=0$, so $y=0$ is a HA. $\lim\limits_{x\to 0}\frac{x^2-2}{x^4}=-\infty$, so $x=0$ is a VA. **E.**

$f'(x)=\frac{x^4\cdot 2x-\left(x^2-2\right)(4x^3)}{\left(x^4\right)^2}=\frac{-2x^5+8x^3}{x^8}=\frac{-2(x^2-4)}{x^5}=\frac{-2(x+2)(x-2)}{x^5}$. $f'(x)$ is negative on $(-2,0)$ and $(2,\infty)$ and positive on $(-\infty,-2)$ and $(0,2)$, so f is decreasing on $(-2,0)$ and $(2,\infty)$ and increasing on $(-\infty,-2)$ and $(0,2)$. **F.** Local maximum value $f(\pm 2)=\frac{1}{8}$, no local minimum. G.

$f''(x)=\frac{x^5\cdot(-4x)+2\left(x^2-4\right)\cdot 5x^4}{\left(x^5\right)^2}=\frac{2x^4\left[-2x^2+5\left(x^2-4\right)\right]}{x^{10}}=\frac{2\left(3x^2-20\right)}{x^6}$ $f''(x)$ is positive on $\left(-\infty,-\sqrt{\frac{20}{3}}\right)$ and $\left(\sqrt{\frac{20}{3}},\infty\right)$ and negative on $\left(-\sqrt{\frac{20}{3}},0\right)$ and $\left(0,\sqrt{\frac{20}{3}}\right)$, so f is CU on $\left(-\infty,-\sqrt{\frac{20}{3}}\right)$ and $\left(\sqrt{\frac{20}{3}},\infty\right)$ and CD on $\left(-\sqrt{\frac{20}{3}},0\right)$ and $\left(0,\sqrt{\frac{20}{3}}\right)$. IP at $\left(\pm\sqrt{\frac{20}{3}},\frac{21}{200}\right)\approx(\pm 2.5820,0.105)$

H.

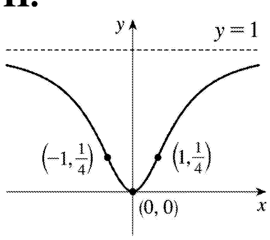


17. $y=f(x)=\dfrac{x^2}{x^2+3}=\dfrac{\left(x^2+3\right)-3}{x^2+3}=1-\dfrac{3}{x^2+3}$ **A.** $D=R$ **B.** $y-$ intercept: $f(0)=0$; x –intercepts: $f(x)=0\Leftrightarrow x=0$ **C.** $f(-x)=f(x)$, so f is even; the graph is symmetric about the y –axis. **D.** $\lim\limits_{x\to\pm\infty}\dfrac{x^2}{x^2+3}=1$, so $y=1$ is a HA. No VA. **E.** Using the Reciprocal Rule, $f'(x)=-3\cdot\dfrac{-2x}{\left(x^2+3\right)^2}=\dfrac{6x}{\left(x^2+3\right)^2}$. $f'(x)>0\Leftrightarrow x>0$ and $f'(x)<0\Leftrightarrow x<0$, so f is decreasing on $(-\infty,0)$ and increasing on $(0,\infty)$. **F.** Local minimum value $f(0)=0$, no local maximum. **G.** $f''(x)=\dfrac{\left(x^2+3\right)^2\cdot 6-6x\cdot 2\left(x^2+3\right)\cdot 2x}{\left[\left(x^2+3\right)^2\right]^2}$

$=\dfrac{6\left(x^2+3\right)\left[\left(x^2+3\right)-4x^2\right]}{\left(x^2+3\right)^4}=\dfrac{6\left(3-3x^2\right)}{\left(x^2+3\right)^3}=\dfrac{-18(x+1)(x-1)}{\left(x^2+3\right)^3}$

$f''(x)$ is negative on $(-\infty,-1)$ and $(1,\infty)$ and positive on $(-1,1)$, so f is CD on $(-\infty,-1)$ and $(1,\infty)$ and CU on $(-1,1)$. IP at $\left(\pm 1,\dfrac{1}{4}\right)$

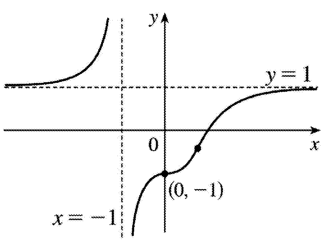
H.



18. $y=f(x)=\dfrac{x^3-1}{x^3+1}$ **A.** $D=\{x\mid x\neq-1\}=(-\infty,-1)\cup(-1,\infty)$ **B.** x –intercept $=1$, y –intercept $=f(0)=-1$ **C.** No symmetry **D.** $\lim\limits_{x\to\pm\infty}\dfrac{x^3-1}{x^3+1}=\lim\limits_{x\to\pm\infty}\dfrac{1-1/x^3}{1+1/x^3}=1$, so $y=1$ is a HA.

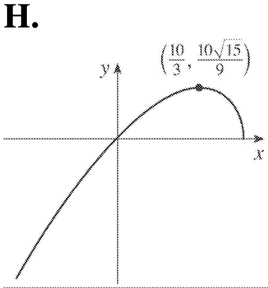
$\lim\limits_{x\to -1^-} \dfrac{x^3-1}{x^3+1}=\infty$ and $\lim\limits_{x\to -1^+} \dfrac{x^3-1}{x^3+1}=-\infty$, so $x=-1$ is a VA. **E.**
$f'(x)=\dfrac{\left(x^3+1\right)\left(3x^2\right)-\left(x^3-1\right)\left(3x^2\right)}{\left(x^3+1\right)^2}=\dfrac{6x^2}{\left(x^3+1\right)^2}>0$ ($x\neq -1$) so f is increasing on $(-\infty,-1)$ and $(-1,\infty)$. **F.** No extreme values **G.** $y''=\dfrac{12x\left(x^3+1\right)^2-6x^2\cdot 2\left(x^3+1\right)\cdot 3x^2}{\left(x^3+1\right)^4}=\dfrac{12x\left(1-2x^3\right)}{\left(x^3+1\right)^3}>0\Leftrightarrow x<-1$ or $0<x<\dfrac{1}{\sqrt[3]{2}}$, <so f is CU on $(-\infty,-1)$ and $\left(0,\dfrac{1}{\sqrt[3]{2}}\right)$ and CD on $(-1,0)$ and $\left(\dfrac{1}{\sqrt[3]{2}},\infty\right)$. IP at $(0,-1)$, $\left(\dfrac{1}{\sqrt[3]{2}},-\dfrac{1}{3}\right)$

H.



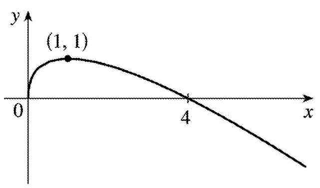
19. $y=f(x)=x\sqrt{5-x}$ **A.** The domain is $\{x\mid 5-x\geq 0\}=(-\infty,5]$ **B.** $y-$ intercept: $f(0)=0$; x –intercepts: $f(x)=0\Leftrightarrow x=0$, 5 **C.** No symmetry **D.** No asymptote **E.**
$f'(x)=x\cdot\dfrac{1}{2}(5-x)^{-1/2}(-1)+(5-x)^{1/2}\cdot 1=\dfrac{1}{2}(5-x)^{-1/2}[-x+2(5-x)]=\dfrac{10-3x}{2\sqrt{5-x}}>0\Leftrightarrow x<\dfrac{10}{3}$, so f is increasing on $\left(-\infty,\dfrac{10}{3}\right)$ and decreasing on $\left(\dfrac{10}{3},5\right)$. **F.** Local maximum value $f\left(\dfrac{10}{3}\right)=\dfrac{10}{9}\sqrt{15}\approx 4.3$; no local minimum **G.** $f''(x)$
$=\dfrac{2(5-x)^{1/2}(-3)-(10-3x)\cdot 2\left(\dfrac{1}{2}\right)(5-x)^{-1/2}(-1)}{\left(2\sqrt{5-x}\right)^2}=\dfrac{(5-x)^{-1/2}[-6(5-x)+(10-3x)]}{4(5-x)}=\dfrac{3x-20}{4(5-x)^{3/2}}$ $f''(x)<0$ for $x<5$, so f is CD on $(-\infty,5)$. No IP

H.



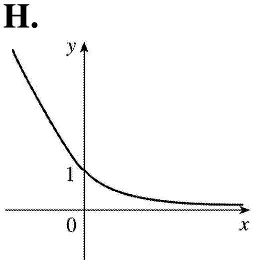
20. $y=f(x)=2\sqrt{x}-x$ **A.** $D=[0,\infty)$] **B.** $y-$ intercept: $f(0)=0$; x –intercepts: $f(x)=0\Rightarrow 2\sqrt{x}=x\Rightarrow 4x=x^2\Rightarrow$ $4x-x^2=0\Rightarrow x(4-x)=0\Rightarrow x=0$, 4 **C.** No symmetry **D.** No asymptote **E.** $f'(x)=\frac{1}{\sqrt{x}}-1=\frac{1}{\sqrt{x}}\left(1-\sqrt{x}\right)$. This is positive for $x<1$ and negative for $x>1$, so f is increasing on $(0,1)$ and decreasing on $(1,\infty)$. **F.** Local maximum value $f(1)=1$, no local minimum. **G.** $f''(x)=\left(x^{-1/2}-1\right)'=-\frac{1}{2}x^{-3/2}=\frac{-1}{2x^{3/2}}<0$ for $x>0$, so f is CD on $(0,\infty)$. No IP

H.



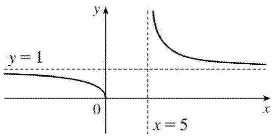
21. $y=f(x)=\sqrt{x^2+1}-x$ **A.** $D=R$ **B.** No x –intercept, y –intercept =1 **C.** No symmetry **D.** $\lim_{x\to-\infty}\left(\sqrt{x^2+1}-x\right)=\infty$ and $\lim_{x\to\infty}\left(\sqrt{x^2+1}-x\right)=\lim_{x\to\infty}\left(\sqrt{x^2+1}-x\right)\frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}+x}=\lim_{x\to\infty}\frac{1}{\sqrt{x^2+1}+x}=0$, so $y=0$ is a HA. **E.** $f'(x)=\frac{x}{\sqrt{x^2+1}}-1=\frac{x-\sqrt{x^2+1}}{\sqrt{x^2+1}}\Rightarrow f'(x)<0$, so f is decreasing on R. **F.** No extreme values **G.** $f''(x)=\frac{1}{\left(x^2+1\right)^{3/2}}>0$, so f is CU on R . No IP

H.



22. $y=f(x)=\sqrt{x/(x-5)}$ **A.** $D=\{x\,|\,x/(x-5)\geq 0\}=(-\infty,0]\cup(5,\infty)$. **B.** Intercepts are 0 . **C.** No symmetry **D.** $\lim_{x\to\pm\infty}\sqrt{\frac{x}{x-5}}=\lim_{x\to\pm\infty}\sqrt{\frac{1}{1-5/x}}=1$, so $y=1$ is a HA. $\lim_{x\to 5^+}\sqrt{\frac{x}{x-5}}=\infty$, so $x=5$ is a VA. **E.** $f'(x)=\frac{1}{2}\left(\frac{x}{x-5}\right)^{-1/2}\frac{(-5)}{(x-5)^2}=-\frac{5}{2}\left[x(x-5)^3\right]^{-1/2}<0$, so f is decreasing on $(-\infty,0)$ and $(5,\infty)$. **F.** No extreme values **G.** $f''(x)=\frac{5}{4}\left[x(x-5)^3\right]^{-3/2}(x-5)^2(4x-5)>0$ for $x>5$, and $f''(x)<0$ for $x<0$, so f

is CU on $(5,\infty)$ and CD on $(-\infty,0)$. No IP
H.

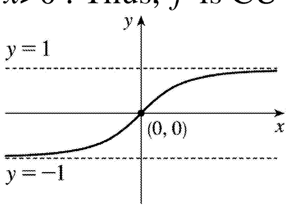


23. $y=f(x)=x/\sqrt{x^2+1}$ **A.** $D=R$ **B.** y –intercept: $f(0)=0$; x –intercepts: $f(x)=0 \Rightarrow x=0$ **C.** $f(-x)=-f(x)$, so f is odd; the graph is symmetric about the origin. **D.** $\lim_{x\to\infty} f(x)$

$$=\lim_{x\to\infty}\frac{x}{\sqrt{x^2+1}}=\lim_{x\to\infty}\frac{x/x}{\sqrt{x^2+1}/x}=\lim_{x\to\infty}\frac{x/x}{\sqrt{x^2+1}/\sqrt{x^2}}=\lim_{x\to\infty}\frac{1}{\sqrt{1+1/x^2}}\quad \lim_{x\to-\infty}f(x)$$

$$=\lim_{x\to-\infty}\frac{x}{\sqrt{x^2+1}}=\lim_{x\to-\infty}\frac{x/x}{\sqrt{x^2+1}/x}=\lim_{x\to-\infty}\frac{x/x}{\sqrt{x^2+1}\Big/\left(-\sqrt{x^2}\right)}=\lim_{x\to-\infty}\frac{1}{-\sqrt{1+1/x^2}}=\frac{1}{-\sqrt{1+0}}=-1$$ so

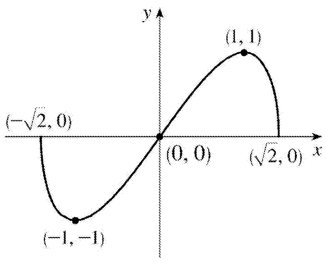
$y=\pm 1$ are HA. No VA. **E.** $f'(x)=\dfrac{\sqrt{x^2+1}-x\cdot\dfrac{2x}{2\sqrt{x^2+1}}}{[(x^2+1)^{1/2}]^2}=\dfrac{x^2+1-x^2}{\left(x^2+1\right)^{3/2}}=\dfrac{1}{\left(x^2+1\right)^{3/2}}>0$ for all x , so f is increasing on R . $f''(x)=-\dfrac{3}{2}\left(x^2+1\right)^{-5/2}\cdot 2x=\dfrac{-3x}{(x^2+1)^{5/2}}$, so $f''(x)>0$ for $x<0$ and $f''(x)<0$ for $x>0$. Thus, f is CU on $(-\infty,0)$ and CD on $(0,\infty)$. IP at $(0,0)$ **H.**



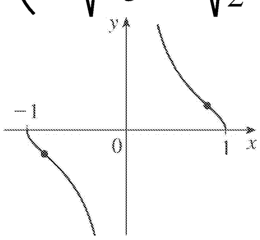
24. $y=f(x)=x\sqrt{2-x^2}$ **A.** $D=\left[-\sqrt{2},\sqrt{2}\right]$ **B.** y –intercept: $f(0)=0$; x –intercepts: $f(x)=0\Rightarrow$ $x=0$, $\pm\sqrt{2}$. **C.** $f(-x)=-f(x)$, so f is odd; the graph is symmetric about the origin. **D.** No asymptote **E.** $f'(x)=x\cdot\dfrac{-x}{\sqrt{2-x^2}}+\sqrt{2-x^2}=\dfrac{-x^2+2-x^2}{\sqrt{2-x^2}}=\dfrac{2(1+x)(1-x)}{\sqrt{2-x^2}}$. $f'(x)$ is negative for $-\sqrt{2}<x<-1$ and $1<x<\sqrt{2}$, and positive for $-1<x<1$, so f is decreasing on $\left(-\sqrt{2},-1\right)$ and $\left(1,\sqrt{2}\right)$ and increasing on $(-1,1)$. **F.** Local minimum value $f(-1)=-1$, local maximum value $f(1)=1$. **G** $f''(x)$

$$=\frac{\sqrt{2-x^2}(-4x)-(2-2x^2)\frac{-x}{\sqrt{2-x^2}}}{\left[(2-x^2)^{1/2}\right]^2}=\frac{(2-x^2)(-4x)+(2-2x^2)x}{(2-x^2)^{3/2}}=\frac{2x^3-6x}{(2-x^2)^{3/2}}=\frac{2x(x^2-3)}{(2-x^2)^{3/2}}$$. Since $x^2-3<0$ for x in $\left[-\sqrt{2},\sqrt{2}\right]$, $f''(x)>0$ for $-\sqrt{2}<x<0$ and $f''(x)<0$ for $0<x<\sqrt{2}$. Thus, f is CU on $\left(-\sqrt{2},0\right)$ and CD on $\left(0,\sqrt{2}\right)$. The only IP is $(0,0)$.

H.



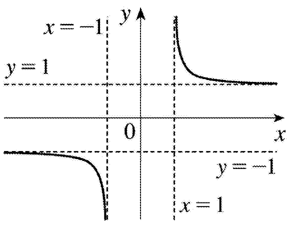
25. $y=f(x)=\sqrt{1-x^2}/x$ **A.** $D=\{x\mid |x|\leq 1, x\neq 0\}=[-1,0)\cup(0,1]$ **B.** $x-$ intercepts ± 1 , no y –intercept **C.** $f(-x)=-f(x)$, so the curve is symmetric about $(0,0)$. **D.** $\lim\limits_{x\to 0^+}\frac{\sqrt{1-x^2}}{x}=\infty$, $\lim\limits_{x\to 0^-}\frac{\sqrt{1-x^2}}{x}=-\infty$, so $x=0$ is a VA. **E.** $f'(x)=\frac{\left(-x^2/\sqrt{1-x^2}\right)-\sqrt{1-x^2}}{x^2}=-\frac{1}{x^2\sqrt{1-x^2}}<0$, so f is decreasing on $(-1,0)$ and $(0,1)$. **F.** No extreme values **G.** $f''(x)=\frac{2-3x^2}{x^3\left(1-x^2\right)^{3/2}}>0\Leftrightarrow -1<x<-\sqrt{\frac{2}{3}}$ or $0<x<\sqrt{\frac{2}{3}}$, so f is CU on $\left(-1,-\sqrt{\frac{2}{3}}\right)$ and $\left(0,\sqrt{\frac{2}{3}}\right)$ and CD on $\left(-\sqrt{\frac{2}{3}},0\right)$ and $\left(\sqrt{\frac{2}{3}},1\right)$. IP at $\left(\pm\sqrt{\frac{2}{3}},\pm\frac{1}{\sqrt{2}}\right)$ **H.**



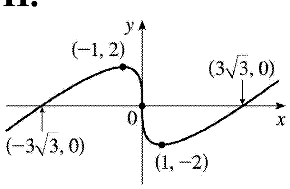
26. $y=f(x)=x/\sqrt{x^2-1}$ **A.** $D=(-\infty,-1)\cup(1,\infty)$ **B.** No intercepts **C.** $f(-x)=-f(x)$, so f is odd; the graph is symmetric about the origin. **D.**

$\lim_{x\to\infty} \frac{x}{\sqrt{x^2-1}}=1$ and $\lim_{x\to-\infty} \frac{x}{\sqrt{x^2-1}}=-1$, so $y=\pm 1$ are HA. $\lim_{x\to 1^+} f(x)=+\infty$ and $\lim_{x\to -1^-} f(x)=-\infty$, so $x=\pm 1$ are VA. **E.** $f'(x)=\frac{\sqrt{x^2-1}-x\cdot\frac{x}{\sqrt{x^2-1}}}{\left[\left(x^2-1\right)^{1/2}\right]^2}=\frac{x^2-1-x^2}{\left(x^2-1\right)^{3/2}}=\frac{-1}{\left(x^2-1\right)^{3/2}}<0$, so f is decreasing on $(-\infty,-1)$ and $(1,\infty)$. **F.** No extreme values **G.** $f''(x)=(-1)\left(-\frac{3}{2}\right)\left(x^2-1\right)^{-5/2}\cdot 2x=\frac{3x}{\left(x^2-1\right)^{5/2}}$

$f''(x)<0$ on $(-\infty,-1)$ and $f''(x)>0$ on $(1,\infty)$, so f is CD on $(-\infty,-1)$ and CU on $(1,\infty)$. No IP
H.

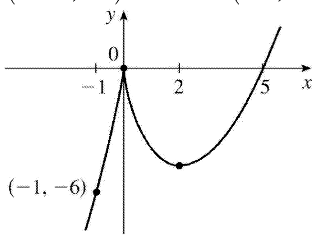


27. $y=f(x)=x-3x^{1/3}$ **A.** $D=R$ **B.** $y-$ intercept: $f(0)=0$; x –intercepts: $f(x)=0\Rightarrow x=3x^{1/3}\Rightarrow x^3=27x\Rightarrow x^3-27x=0\Rightarrow x(x^2-27)=0\Rightarrow x=0$, $\pm 3\sqrt{3}$ **C.** $f(-x)=-f(x)$, so f is odd; the graph is symmetric about the origin. **D.** No asymptote **E.** $f'(x)=1-x^{-2/3}=1-\frac{1}{x^{2/3}}=\frac{x^{2/3}-1}{x^{2/3}}$. $f'(x)>0$ when $|x|>1$ and $f'(x)<0$ when $0<|x|<1$, so f is increasing on $(-\infty,-1)$ and $(1,\infty)$, and decreasing on $(-1,0)$ and $(0,1)$. **F.** Local maximum value $f(-1)=2$, local minimum value $f(1)=-2$ **G.** $f''(x)=\frac{2}{3}x^{-5/3}<0$ when $x<0$ and $f''(x)>0$ when $x>0$, so f is CD on $(-\infty,0)$ and CU on $(0,\infty)$. IP at $(0,0)$
H.

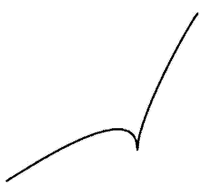


28. $y=f(x)=x^{5/3}-5x^{2/3}=x^{2/3}(x-5)$ **A.** $D=R$ **B.** x –intercepts 0 , 5 ; y –intercept 0 **C.** No symmetry **D.** $\lim_{x\to\pm\infty} x^{2/3}(x-5)=\pm\infty$, so there is no asymptote **E.** $f'(x)=\frac{5}{3}x^{2/3}-\frac{10}{3}x^{-1/3}=\frac{5}{3}x^{-1/3}(x-2)>0\Leftrightarrow x<0$ or $x>2$, so f is increasing on $(-\infty,0)$, $(2,\infty)$ and decreasing on $(0,2)$. **F.** Local maximum value $f(0)=0$, local minimum value $f(2)=-3\sqrt[3]{4}$ **G.**

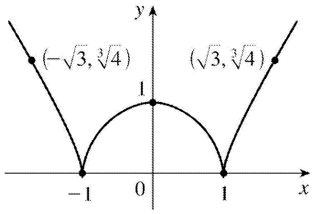
$f''(x)=\frac{10}{9}x^{-1/3}+\frac{10}{9}x^{-4/3}=\frac{10}{9}x^{-4/3}(x+1)>0\Leftrightarrow x>-1$, so f is CU on $(-1,0)$ and $(0,\infty)$, CD on $(-\infty,-1)$. IP at $(-1,-6)$ **H.**



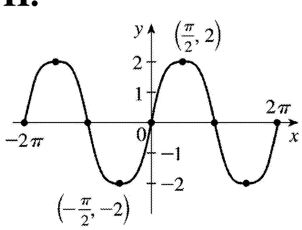
29. $y=f(x)=x+\sqrt{|x|}$ **A.** $D=R$ **B.** x –intercepts 0 , –1 ; y –intercept 0 **C.** No symmetry **D.** $\lim_{x\to\infty}\left(x+\sqrt{|x|}\right)=\infty$, $\lim_{x\to-\infty}\left(x+\sqrt{|x|}\right)=-\infty$. No asymptote **E.** For $x>0$, $f(x)=x+\sqrt{x}\Rightarrow$ $f'(x)=1+\frac{1}{2\sqrt{x}}>0$, so f increases on $(0,\infty)$. For $x<0$, $f(x)=x+\sqrt{-x}\Rightarrow f'(x)=1-\frac{1}{2\sqrt{-x}}>0\Leftrightarrow 2\sqrt{-x}>1$ $\Leftrightarrow -x>\frac{1}{4}\Leftrightarrow x<-\frac{1}{4}$, so f increases on $\left(-\infty,-\frac{1}{4}\right)$ and decreases on $\left(-\frac{1}{4},0\right)$. **F.** Local maximum value $f\left(-\frac{1}{4}\right)=\frac{1}{4}$, local minimum value $f(0)=0$ **G.** For $x>0$, $f''(x)=-\frac{1}{4}x^{-3/2}\Rightarrow f''(x)<0$, so f is CD on $(0,\infty)$. For $x<0$, $f''(x)=-\frac{1}{4}(-x)^{-3/2}\Rightarrow f''(x)<0$, so f is CD on $(-\infty,0)$. No IP
H.



30. $y=f(x)=\sqrt[3]{\left(x^2-1\right)^2}=\left(x^2-1\right)^{2/3}$ **A.** $D=R$ **B.** x –intercepts ±1 , y –intercept 1 **C.** $f(-x)=f(x)$, so the curve is symmetric about the y– axis. **D.** $\lim_{x\to\pm\infty}\left(x^2-1\right)^{2/3}=\infty$, no asymptote **E.** $f'(x)=\frac{4}{3}x\left(x^2-1\right)^{-1/3}\Rightarrow f'(x)>0\Leftrightarrow x>1$ or $-1<x<0$, $f'(x)<0\Leftrightarrow x<-1$ or $0<x<1$. So f is increasing on $(-1,0)$, $(1,\infty)$ and decreasing on $(-\infty,-1)$, $(0,1)$. **F.** Local minimum values $f(-1)=f(1)=0$, local maximum value $f(0)=1$ **G.** $f''(x)=\frac{4}{3}\left(x^2-1\right)^{-1/3}+\frac{4}{3}x\left(-\frac{1}{3}\right)\left(x^2-1\right)^{-4/3}(2x)$ $=\frac{4}{9}\left(x^2-3\right)\left(x^2-1\right)^{-4/3}>0\Leftrightarrow |x|>\sqrt{3}$ so f is CU on $(-\infty,-\sqrt{3})$, $(\sqrt{3},\infty)$ and CD on $(-\sqrt{3},-1)$, $(-1,1)$, $(1,\sqrt{3})$. IPs at $\left(\pm\sqrt{3},\sqrt[3]{4}\right)$ **H.**



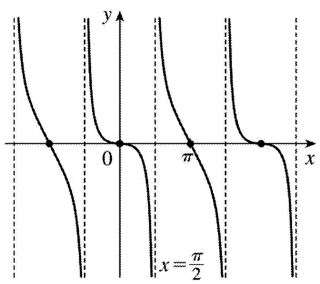
31. $y=f(x)=3\sin x-\sin^3 x$ **A.** $D=R$ **B.** $y-$ intercept: $f(0)=0$; x –intercepts: $f(x)=0 \Rightarrow \sin x(3-\sin^2 x)=0 \Rightarrow$ $\sin x=0$ [since $\sin^2 x \leq 1<3$] $\Rightarrow x=n\pi$, n an integer.
C. $f(-x)=-f(x)$, so f is odd; the graph (shown for $-2\pi \leq x \leq 2\pi$) is symmetric about the origin and periodic with period 2π . **D.** No asymptote **E.** $f'(x)=3\cos x-3\sin^2 x \cos x=3\cos x\left(1-\sin^2 x\right)=3\cos^3 x$.
$f'(x)>0 \Leftrightarrow \cos x>0 \Leftrightarrow x\in\left(2n\pi-\frac{\pi}{2}, 2n\pi+\frac{\pi}{2}\right)$ for each integer n , and $f'(x)<0 \Leftrightarrow$
$\cos x<0 \Leftrightarrow x\in\left(2n\pi+\frac{\pi}{2}, 2n\pi+\frac{3\pi}{2}\right)$ for each integer n . Thus, f is increasing on
$\left(2n\pi-\frac{\pi}{2}, 2n\pi+\frac{\pi}{2}\right)$ for each integer n , and f is decreasing on $\left(2n\pi+\frac{\pi}{2}, 2n\pi+\frac{3\pi}{2}\right)$ for each integer n . **F.** f has local maximum values $f(2n\pi+\frac{\pi}{2})=2$ and local minimum values
$f(2n\pi+\frac{3\pi}{2})=-2$.
G. $f''(x)=-9\sin x \cos^2 x=-9\sin x(1-\sin^2 x)=-9\sin x(1-\sin x)(1+\sin x)$. $f''(x)<0 \Leftrightarrow \sin x>0$ and $\sin x\neq\pm 1 \Leftrightarrow x\in\left(2n\pi, 2n\pi+\frac{\pi}{2}\right)\cup\left(2n\pi+\frac{\pi}{2}, 2n\pi+\pi\right)$ for some integer n . $f''(x)>0 \Leftrightarrow \sin x<0$ and $\sin x\neq\pm 1 \Leftrightarrow x\in\left((2n-1)\pi, (2n-1)\pi+\frac{\pi}{2}\right)\cup\left((2n-1)\pi+\frac{\pi}{2}, 2n\pi\right)$ for some integer n . Thus, f is CD on the intervals $\left(2n\pi, \left(2n+\frac{1}{2}\right)\pi\right)$ and $\left(\left(2n+\frac{1}{2}\right)\pi, (2n+1)\pi\right)$ for each integer n , and f is CU on the intervals $\left((2n-1)\pi, \left(2n-\frac{1}{2}\right)\pi\right)$ and $\left(\left(2n-\frac{1}{2}\right)\pi, 2n\pi\right)$ for each integer n . f has inflection points at $(n\pi, 0)$ for each integer n .
H.



32. $y=f(x)=\sin x-\tan x$ **A.** $D=\left\{ x | x\neq(2n+1)\frac{\pi}{2} \right\}$ **B.** $y=0 \Leftrightarrow \sin x=\tan x=\frac{\sin x}{\cos x} \Leftrightarrow \sin x=0$ or $\cos x=1$

$\Leftrightarrow x=n\pi$ (x– intercepts), y –intercept $=f(0)=0$ **C.** $f(-x)=-f(x)$, so the curve is symmetric about $(0,0)$. Also periodic with period 2π **D.** $\lim_{x\to(\pi/2)^-}(\sin x-\tan x)=-\infty$ and $\lim_{x\to(\pi/2)^+}(\sin x-\tan x)=\infty$, so $x=n\pi+\frac{\pi}{2}$ are VA. **E.** $f'(x)=\cos x-\sec^2 x\leq 0$, so f decreases on each interval in its domain, that is, on $\left((2n-1)\frac{\pi}{2},(2n+1)\frac{\pi}{2}\right)$. **F.** No extreme values **G.** $f''(x)=-\sin x-2\sec^2 x\tan x=\sin x\left(1+2\sec^3 x\right)$. Note that $1+2\sec^3 x\neq 0$ since $\sec^3 x\neq -\frac{1}{2}$. $f''(x)>0$ for $-\frac{\pi}{2}<x<0$ and $\frac{3\pi}{2}<x<2\pi$, so f is CU on $\left(\left(n-\frac{1}{2}\right)\pi,n\pi\right)$ and CD on $\left(n\pi,\left(n+\frac{1}{2}\right)\pi\right)$. f has IPs at $(n\pi,0)$. Note also that $f'(0)=0$, but $f'(\pi)=-2$.

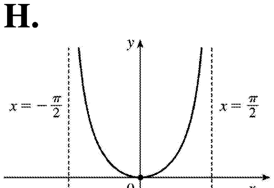
H.



33. $y=f(x)=x\tan x$, $-\frac{\pi}{2}<x<\frac{\pi}{2}$ **A.** $D=\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ **B.** Intercepts are 0 **C.** $f(-x)=f(x)$, so the curve is symmetric about the y– axis. **D.** $\lim_{x\to(\pi/2)^-}x\tan x=\infty$ and $\lim_{x\to-(\pi/2)^+}x\tan x=\infty$, so $x=\frac{\pi}{2}$ and $x=-\frac{\pi}{2}$ are VA. **E.** $f'(x)=\tan x+x\sec^2 x>0\Leftrightarrow 0<x<\frac{\pi}{2}$, so f increases on $\left(0,\frac{\pi}{2}\right)$ and decreases on $\left(-\frac{\pi}{2},0\right)$.

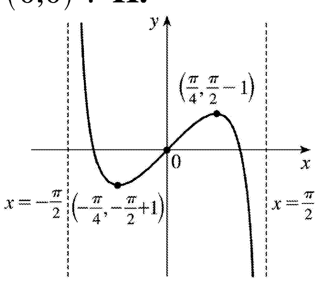
F. Absolute and local minimum value $f(0)=0$. **G.** $y''=2\sec^2 x+2x\tan x\sec^2 x>0$ for $-\frac{\pi}{2}<x<\frac{\pi}{2}$, so f is CU on $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$. No IP

H.

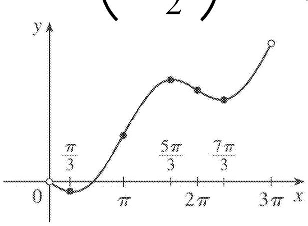


34. $y=f(x)=2x-\tan x$, $-\frac{\pi}{2}<x<\frac{\pi}{2}$ **A.** $D=\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ **B.** y– intercept: $f(0)=0$; x –intercepts: $f(0)=0$

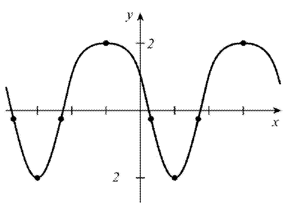
$\Leftrightarrow 2x=\tan x \Leftrightarrow x=0$ or $x\approx\pm 1.17$ **C.** $f(-x)=-f(x)$, so f is odd; the graph is symmetric about the origin. **D.** $\lim_{x\to(-\pi/2)^+}(2x-\tan x)=\infty$ and $\lim_{x\to(\pi/2)^-}(2x-\tan x)=-\infty$, so $x=\pm\frac{\pi}{2}$ are VA. No HA. **E.** $f'(x)=2-\sec^2 x<0 \Leftrightarrow |\sec x|>\sqrt{2}$ and $f'(x)>0 \Leftrightarrow |\sec x|<\sqrt{2}$, so f is decreasing on $\left(-\frac{\pi}{2},-\frac{\pi}{4}\right)$, increasing on $\left(-\frac{\pi}{4},\frac{\pi}{4}\right)$, and decreasing again on $\left(\frac{\pi}{4},\frac{\pi}{2}\right)$ **F.** Local maximum value $f\left(\frac{\pi}{4}\right)=\frac{\pi}{2}-1$, local minimum value $f\left(-\frac{\pi}{4}\right)=-\frac{\pi}{2}+1$ **G.** $f''(x)$ $=-2\sec x\cdot \sec x \tan x=-2\tan x \sec^2 x=-2\tan x(\tan^2 x+1)$ so $f''(x)>0 \Leftrightarrow \tan x<0 \Leftrightarrow -\frac{\pi}{2}<x<0$, and $f''(x)<0 \Leftrightarrow \tan x>0 \Leftrightarrow 0<x<\frac{\pi}{2}$. Thus, f is CU on $\left(-\frac{\pi}{2},0\right)$ and CD on $\left(0,\frac{\pi}{2}\right)$. f has an IP at $(0,0)$. **H.**



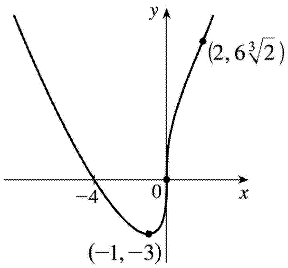
35. $y=f(x)=\frac{1}{2}x-\sin x$, $0<x<3\pi$ **A.** $D=(0,3\pi)$ **B.** No y –intercept. The x –intercept, approximately 1.9 , can be found using Newton's Method. **C.** No symmetry **D.** No asymptote **E.** $f'(x)=\frac{1}{2}-\cos\:,x>0\Leftrightarrow$ $\cos x<\frac{1}{2} \Leftrightarrow \frac{\pi}{3}<x<\frac{5\pi}{3}$ or $\frac{7\pi}{3}<x<3\pi$, so f is increasing on $\left(\frac{\pi}{3},\frac{5\pi}{3}\right)$ and $\left(\frac{7\pi}{3},3\pi\right)$ and decreasing on $\left(0,\frac{\pi}{3}\right)$ and $\left(\frac{5\pi}{3},\frac{7\pi}{3}\right)$. **F.** Local minimum value $f\left(\frac{\pi}{3}\right)=\frac{\pi}{6}-\frac{\sqrt{3}}{2}$, local maximum value $f\left(\frac{5\pi}{3}\right)=\frac{5\pi}{6}+\frac{\sqrt{3}}{2}$, local minimum value $f\left(\frac{7\pi}{3}\right)=\frac{7\pi}{6}-\frac{\sqrt{3}}{2}$ **G.** $f''(x)=\sin x>0 \Leftrightarrow 0<x<\pi$ or $2\pi<x<3\pi$, so f is CU on $(0,\pi)$ and $(2\pi,3\pi)$ and CD on $(\pi,2\pi)$. IPs at $\left(\pi,\frac{\pi}{2}\right)$ and $(2\pi,\pi)$. **H.**



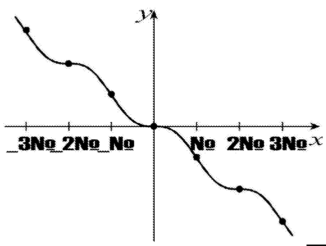
36. $y=f(x)=\cos^2 x-2\sin x$ **A.** $D=R$ **B.** $y-$ intercept: $f(0)=1$ **C.** No symmetry, but f has period 2π . **D.** No asymptote **E.** $y'=2\cos x(-\sin x)-2\cos x=-2\cos x(\sin x+1)$. $y'=0\Leftrightarrow\cos x=0$ or $\sin x=-1\Leftrightarrow$ $x=(2n+1)\frac{\pi}{2}$. $y'>0$ when $\cos x<0$ since $\sin x+1\geq 0$ for all x . So $y'>0$ and f is increasing on $\left((4n+1)\frac{\pi}{2},(4n+3)\frac{\pi}{2}\right)$; $y'<0$ and f is decreasing on $\left((4n-1)\frac{\pi}{2},(4n+1)\frac{\pi}{2}\right)$. **F.** Local maximum values $f\left((4n+3)\frac{\pi}{2}\right)=2$, local minimum values $f\left((4n+1)\frac{\pi}{2}\right)=-2$ **G.**
$y'=-2\cos x(\sin x+1)=-\sin 2x-2\cos x\Rightarrow y''=-2\cos 2x+2\sin x=-2\left(1-2\sin^2 x\right)+2\sin x$ $=4\sin^2 x+2\sin x-2=2(2\sin x-1)(\sin x+1)\,y''=0\Leftrightarrow\sin x=\frac{1}{2}$ or $-1\Rightarrow x=\frac{\pi}{6}+2n\pi$, $\frac{5\pi}{6}+2n\pi$, or $\frac{3\pi}{2}+2n\pi$. $y''>0$ and f is CU on $\left(\frac{\pi}{6}+2n\pi,\frac{5\pi}{6}+2n\pi\right)$; $y''\leq 0$ and f is CD on $\left(\frac{5\pi}{6}+2n\pi,\frac{\pi}{6}+2(n+1)\pi\right)$ .IPs at $\left(\frac{\pi}{6}+2n\pi,-\frac{1}{4}\right)$ and $\left(\frac{5\pi}{6}+2n\pi,-\frac{1}{4}\right)$
. **H.**



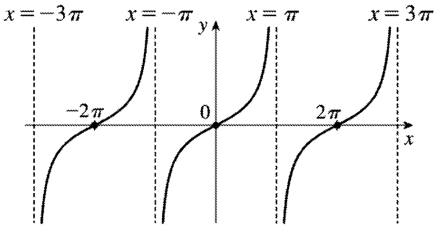
37. $y=f(x)=\sin 2x-2\sin x$ **A.** $D=R$ **B.** y –intercept $=f(0)=0$. $y=0\Leftrightarrow 2\sin x=\sin 2x=2\sin x\cos x\Leftrightarrow$ $\sin x=0$ or $\cos x=1\Leftrightarrow x=n\pi$ ($x-$ intercepts) **C.** $f(-x)=-f(x)$, so the curve is symmetric about $(0,0)$. *Note:* f is periodic with period 2π , so we determine E — G for $-\pi\leq x\leq\pi$. **D.** No asymptotes **E.** $f'(x)=2\cos 2x-2\cos x=2\left(2\cos^2 x-1-\cos x\right)=2(2\cos x+1)(\cos x-1)>0\Leftrightarrow\cos x<-\frac{1}{2}\Leftrightarrow-\pi<x<-\frac{2\pi}{3}$ or $\frac{2\pi}{3}<x<\pi$, so f is increasing on $\left(-\pi,-\frac{2\pi}{3}\right)$, $\left(\frac{2\pi}{3},\pi\right)$ and decreasing on $\left(-\frac{2\pi}{3},\frac{2\pi}{3}\right)$. **F.** Local maximum value $f\left(-\frac{2\pi}{3}\right)=\frac{3\sqrt{3}}{2}$, local minimum value $f\left(\frac{2\pi}{3}\right)=-\frac{3\sqrt{3}}{2}$ **G.**
$f''(x)=-4\sin 2x+2\sin x=2\sin x(1-4\cos x)=0$ when $x=0$, $\pm\pi$ or $\cos x=\frac{1}{4}$. If $\alpha=\cos^{-1}\frac{1}{4}$, then f is CU on $(-\alpha,0)$ and (α,π) and CD on $(-\pi,-\alpha)$ and $(0,\alpha)$. IPs at $(0,0)$, $(\pm\pi,0)$, $\left(\alpha,-\frac{3\sqrt{15}}{8}\right)$, $\left(-\alpha,\frac{3\sqrt{15}}{8}\right)$.
H.



38. $f(x)=\sin x-x$ **A.** $D=R$ **B.** $x-$ intercept $=0=y-$ intercept **C.** $f(-x)=\sin(-x)-(-x)=-(\sin x-x)=-f(x)$, so f is odd. **D.** No asymptote **E.** $f'(x)=\cos x-1\leq 0$ for all x, so f is decreasing on $(-\infty,\infty)$. **F.** No extreme values **G.** $f''(x)=-\sin x\Rightarrow f''(x)>0\Leftrightarrow \sin x<0\Leftrightarrow (2n-1)\pi<x<2n\pi$, so f is CU on $((2n-1)\pi,2n\pi)$ and CD on $(2n\pi,(2n+1)\pi)$, n an integer. Points of inflection occur when $x=n\pi$.
H.



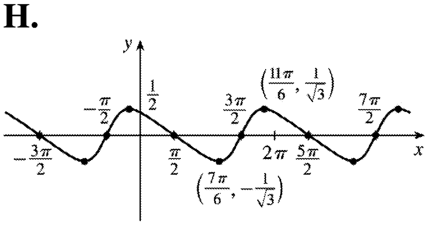
39. $y=f(x)=\dfrac{\sin x}{1+\cos x}\left[\overset{\substack{\text{when}\\ \cos x\neq 1}}{=}\dfrac{\sin x}{1+\cos x}\cdot\dfrac{1-\cos x}{1-\cos x}=\dfrac{\sin x(1-\cos x)}{\sin^2 x}=\dfrac{1-\cos x}{\sin x}=\csc x-\cot x\right]$ **A.** The domain of f is the set of all real numbers except odd integer multiples of π. **B.** y–intercept: $f(0)=0$; x–intercepts: $x=n\pi$, n an even integer. **C.** $f(-x)=-f(x)$, so f is an odd function; the graph is symmetric about the origin and has period 2π. **D.** When n is an odd integer, $\lim_{x\to(n\pi)^-} f(x)=\infty$ and $\lim_{x\to(n\pi)^+} f(x)=-\infty$, so $x=n\pi$ is a VA for each odd integer n. No HA. **E.**
$f'(x)=\dfrac{(1+\cos x)\cdot\cos x-\sin x(-\sin x)}{(1+\cos x)^2}=\dfrac{1+\cos x}{(1+\cos x)^2}=\dfrac{1}{1+\cos x}$. $f'(x)>0$ for all x except odd multiples of π, so f is increasing on $((2k-1)\pi,(2k+1)\pi)$ for each integer k. **F.** No extreme values **G.**
$f''(x)=\dfrac{\sin x}{(1+\cos x)^2}>0\Rightarrow \sin x>0\Rightarrow x\in(2k\pi,(2k+1)\pi)$ and $f''(x)<0$ on $((2k-1)\pi,2k\pi)$ for each integer k. f is CU on $(2k\pi,(2k+1)\pi)$ and CD on $((2k-1)\pi,2k\pi)$ for each integer k. f has IPs at $(2k\pi,0)$ for each integer k.
H



40. $y=f(x)=\cos x/(2+\sin x)$ **A.** $D=R$ *Note:* f is periodic with period 2π , so we determine B—G on $[0,2\pi]$. **B.** x –intercepts $\frac{\pi}{2},\frac{3\pi}{2}$, y –intercept $=f(0)=\frac{1}{2}$ **C.** No symmetry other than periodicity **D.** No asymptote **E.** $f'(x)=\frac{(2+\sin x)(-\sin x)-\cos x(\cos x)}{(2+\sin x)^2}=-\frac{2\sin x+1}{(2+\sin x)^2}$. $f'(x)>0\Leftrightarrow 2\sin x+1<0\Leftrightarrow$ $\sin x<-\frac{1}{2}\Leftrightarrow\frac{7\pi}{6}<x<\frac{11\pi}{6}$, so f is increasing on $\left(\frac{7\pi}{6},\frac{11\pi}{6}\right)$ and decreasing on $\left(0,\frac{7\pi}{6}\right)$, $\left(\frac{11\pi}{6},2\pi\right)$. **F.** Local minimum value $f\left(\frac{7\pi}{6}\right)=-\frac{1}{\sqrt{3}}$, local maximum value $f\left(\frac{11\pi}{6}\right)=\frac{1}{\sqrt{3}}$

G. $f''(x)=-\frac{(2+\sin x)^2(2\cos x)-(2\sin x+1)2(2+\sin x)\cos x}{(2+\sin x)^4}=-\frac{2\cos x(1-\sin x)}{(2+\sin x)^3}>0\Leftrightarrow\cos x<0\Leftrightarrow$ $\frac{\pi}{2}<x<\frac{3\pi}{2}$, so f is CU on $\left(\frac{\pi}{2},\frac{3\pi}{2}\right)$ and CD on $\left(0,\frac{\pi}{2}\right)$ and $\left(\frac{3\pi}{2},2\pi\right)$. IP at $\left(\frac{\pi}{2},0\right)$, $\left(\frac{3\pi}{2},0\right)$

H.



41. $y=1/\left(1+e^{-x}\right)$ **A.** $D=R$ **B.** No x –intercept; y –intercept $=f(0)=\frac{1}{2}$. **C.** No symmetry **D.** $\lim_{x\to\infty}1/\left(1+e^{-x}\right)=\frac{1}{1+0}=1$ and $\lim_{x\to-\infty}1/\left(1+e^{-x}\right)=0$ (since $\lim_{x\to-\infty}e^{-x}=\infty$), so f has horizontal asymptotes $y=0$ and $y=1$. **E.** $f'(x)=-\left(1+e^{-x}\right)^{-2}\left(-e^{-x}\right)=e^{-x}/\left(1+e^{-x}\right)^2$. This is positive for all x , so f is increasing on R . **F.** No extreme values

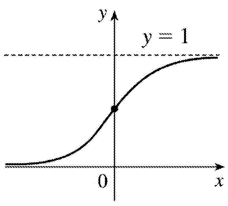
G.

$$f''(x)=\frac{\left(1+e^{-x}\right)^2\left(-e^{-x}\right)-e^{-x}(2)\left(1+e^{-x}\right)\left(-e^{-x}\right)}{\left(1+e^{-x}\right)^4}$$

$$=\frac{e^{-x}\left(e^{-x}-1\right)}{\left(1+e^{-x}\right)^3}$$

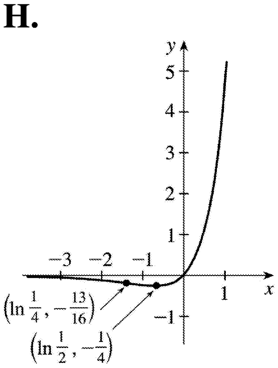
The second factor in the numerator is negative for $x>0$ and positive for $x<0$, and the other factors are always positive, so f is CU on $(-\infty,0)$ and CD on $(0,\infty)$. f has an inflection point at $\left(0,\frac{1}{2}\right)$.

H.



42. $y=f(x)=e^{2x}-e^{x}$ **A.** $D=R$ **B.** $y-$ intercept: $f(0)=0$; x –intercepts: $f(x)=0\Rightarrow e^{2x}=e^{x}\Rightarrow e^{x}=1\Rightarrow x=0$. **C.** No symmetry **D.** $\lim_{x\to-\infty} e^{2x}-e^{x}=0$, so $y=0$ is a HA. No VA. **E.** $f'(x)=2e^{2x}-e^{x}=e^{x}\left(2e^{x}-1\right)$, so $f'(x)>0$ $\Leftrightarrow e^{x}>\frac{1}{2}\Leftrightarrow x>\ln\frac{1}{2}=-\ln 2$ and $f'(x)<0\Leftrightarrow e^{x}<\frac{1}{2}\Leftrightarrow x<\ln\frac{1}{2}$, so f is decreasing on $\left(-\infty,\ln\frac{1}{2}\right)$ and increasing on $\left(\ln\frac{1}{2},\infty\right)$. **F.** Local minimum value $f\left(\ln\frac{1}{2}\right)=e^{2\ln(1/2)}-e^{\ln(1/2)}=\left(\frac{1}{2}\right)^2-\frac{1}{2}=-\frac{1}{4}$
G. $f''(x)=4e^{2x}-e^{x}=e^{x}\left(4e^{x}-1\right)$, so $f''(x)>0\Leftrightarrow$
$e^{x}>\frac{1}{4}\Leftrightarrow x>\ln\frac{1}{4}$ and $f''(x)<0\Leftrightarrow x<\ln\frac{1}{4}$.

H.



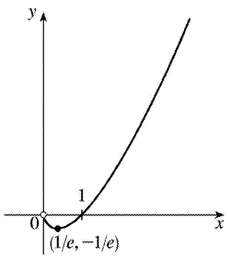
Thus, f is CD on $\left(-\infty,\ln\frac{1}{4}\right)$ and CU on $\left(\ln\frac{1}{4},\infty\right)$. f has an IP at $\left(\ln\frac{1}{4},\left(\frac{1}{4}\right)^2-\frac{1}{4}\right)=\left(\ln\frac{1}{4},-\frac{3}{16}\right)$.

43. $y=f(x)=x\ln x$ **A.** $D=(0,\infty)$ **B.** $x-$ intercept when $\ln x=0\Leftrightarrow x=1$, no $y-$ intercept

C. No symmetry **D.** $\lim_{x\to\infty} x\ln x=\infty$,

$\lim_{x\to 0^+} x\ln x=\lim_{x\to 0^+} \dfrac{\ln x}{1/x}=\lim_{x\to 0^+} \dfrac{1/x}{-1/x^2}=\lim_{x\to 0^+} (-x)=0$, no asymptote. **E.** $f'(x)=\ln x+1=0$ when $\ln x=-1 \Leftrightarrow x=e^{-1}$. $f'(x)>0 \Leftrightarrow \ln x>-1 \Leftrightarrow x>e^{-1}$, so f is increasing on $(1/e,\infty)$ and decreasing on $(0,1/e)$. **F.** $f(1/e)=-1/e$ is an absolute and local minimum value. **G.** $f''(x)=1/x>0$, so f is CU on $(0,\infty)$. No IP

H.



44. $y=f(x)=e^x/x$ **A.** $D=\{x|x\neq 0\}$ **B.** No intercept **C.** No symmetry **D.** $\lim_{x\to\infty} \dfrac{e^x}{x}=\lim_{x\to\infty} \dfrac{e^x}{1}=\infty$,

$\lim_{x\to -\infty} \dfrac{e^x}{x}=0$, so $y=0$ is a HA. \ $\lim_{x\to 0^+} \dfrac{e^x}{x}=\infty$, $\lim_{x\to 0^-} \dfrac{e^x}{x}=-\infty$, so $x=0$ is a VA.

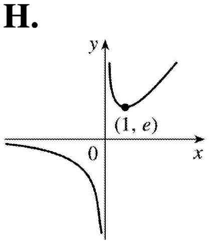
E. $f'(x)=\dfrac{xe^x-e^x}{x^2}>0 \Leftrightarrow (x-1)e^x>0 \Leftrightarrow x>1$,

so f is increasing on $(1,\infty)$, and decreasing on $(-\infty,0)$ and $(0,1)$. **F.** $f(1)=e$ is a local minimum value.

G. $f''(x)=\dfrac{x^2\left(xe^x\right)-2x\left(xe^x-e^x\right)}{x^4}=\dfrac{e^x\left(x^2-2x+2\right)}{x^3}>0$

$\Leftrightarrow x>0$ since $x^2-2x+2>0$ for all x . So f is CU on $(0,\infty)$ and CD on $(-\infty,0)$. No IP

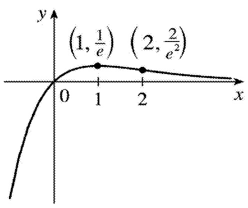
H.



45. $y=f(x)=xe^{-x}$ **A.** $D=R$ **B.** Intercepts are 0 **C.** No symmetry **D.** $\lim_{x\to\infty} xe^{-x}=\lim_{x\to\infty} \dfrac{x}{e^x}=\lim_{x\to\infty} \dfrac{1}{e^x}=0$, so $y=0$ is a HA.

$\lim_{x\to -\infty} xe^{-x}=-\infty$ **E.** $f'(x)=e^{-x}-xe^{-x}=e^{-x}(1-x)>0 \Leftrightarrow x<1$, so f is increasing on $(-\infty,1)$ and decreasing on $(1,\infty)$. **F.** Absolute and local maximum value $f(1)=1/\mathrm{e}$.

H.



G. $f''(x)=e^{-x}(x-2)>0 \Leftrightarrow x>2$, so f is CU on $(2,\infty)$ and CD on $(-\infty,2)$. IP at $\left(2,2/e^2\right)$

46. $y=f(x)=\ln\left(x^2-3x+2\right)=\ln[(x-1)(x-2)]$

A. $D=\left\{ x \text{in} R: x^2-3x+2>0\right\} =(-\infty,1)\cup(2,\infty)$.

B. y –intercept: $f(0)=\ln 2$; x –intercepts: $f(x)=0 \Leftrightarrow x^2-3x+2=e^0 \Leftrightarrow x^2-3x+1=0 \Leftrightarrow x=\dfrac{3\pm\sqrt{5}}{2} \Rightarrow x\approx 0.38$, 2.62 **C.** No symmetry **D.** $\lim_{x\to 1^-} f(x)=\lim_{x\to 2^+} f(x)=-\infty$, so $x=1$ and $x=2$ are VAs. No HA. **E.**

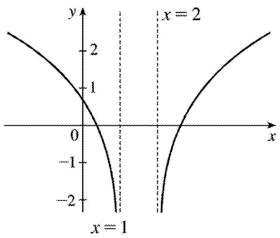
$f'(x)=\dfrac{2x-3}{x^2-3x+2}=\dfrac{2(x-3/2)}{(x-1)(x-2)}$, so $f'(x)<0$ for $x<1$ and $f'(x)>0$ for $x>2$. Thus, f is decreasing on $(-\infty,1)$ and increasing on $(2,\infty)$. **F.** No extreme values

G.

$$\begin{aligned} f''(x) &= \frac{\left(x^2-3x+2\right)\cdot 2-(2x-3)^2}{\left(x^2-3x+2\right)^2} \\ &= \frac{2x^2-6x+4-4x^2+12x-9}{\left(x^2-3x+2\right)^2} \\ &= \frac{-2x^2+6x-5}{\left(x^2-3x+2\right)^2} \end{aligned}$$

The numerator is negative for all x and the denominator is positive, so $f''(x)<0$ for all x in the domain of f . Thus, f is CD on $(-\infty,1)$ and $(2,\infty)$. No IP

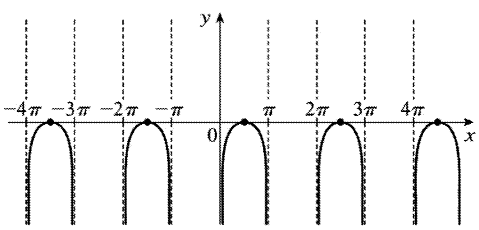
H.



47. $y=f(x)=\ln(\sin x)$
A.

$$D=\{x \text{ in } R \mid \sin x>0\} = \bigcup_{n=-\infty}^{\infty} (2n\pi,(2n+1)\pi)$$
$$=\cdots \cup(-4\pi,-3\pi)\cup(-2\pi,-\pi)\cup(0,\pi)\cup(2\pi,3\pi)\cup\cdots$$

B. No y –intercept; x –intercepts: $f(x)=0 \Leftrightarrow \ln(\sin x)=0 \Leftrightarrow \sin x=e^0=1 \Leftrightarrow x=2n\pi+\dfrac{\pi}{2}$ for each integer n .
C. f is periodic with period 2π . **D.** $\lim\limits_{x\to(2n\pi)^+} f(x)=-\infty$ and $\lim\limits_{x\to[(2n+1)\pi]^-} f(x)=-\infty$, so the lines $x=n\pi$ are VAs for all integers n . **E.** $f'(x)=\dfrac{\cos x}{\sin x}=\cot x$, so $f'(x)>0$ when $2n\pi<x<2n\pi+\dfrac{\pi}{2}$ for each integer n , and $f'(x)<0$ when $2n\pi+\dfrac{\pi}{2}<x<(2n+1)\pi$. Thus, f is increasing on $\left(2n\pi,2n\pi+\dfrac{\pi}{2}\right)$ and decreasing on $\left(2n\pi+\dfrac{\pi}{2},(2n+1)\pi\right)$ for each integer n . **F.** Local maximum values $f\left(2n\pi+\dfrac{\pi}{2}\right)=0$, no local minimum. **G.** $f''(x)=-\csc^2 x<0$, so f is CD on $(2n\pi,(2n+1)\pi)$ for each integer n. No IP
H.

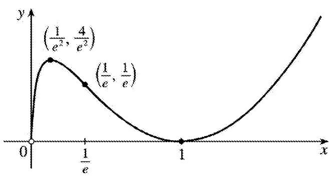


48. $y=f(x)=x(\ln x)^2$ **A.** $D=(0,\infty)$ **B.** x –intercept $=1$, no y –intercept **C.** No symmetry
D. $\lim\limits_{x\to\infty} x(\ln x)^2=\infty$, $\lim\limits_{x\to0^+} x(\ln x)^2=\lim\limits_{x\to0^+}\dfrac{(\ln x)^2}{1/x}=\lim\limits_{x\to0^+}\dfrac{2(\ln x)(1/x)}{-1/x^2}=\lim\limits_{x\to0^+}\dfrac{2\ln x}{-1/x}=\lim\limits_{x\to0^+}\dfrac{2/x}{1/x^2}=\lim\limits_{x\to0^+} 2x=0$, no asymptote **E.** $f'(x)=(\ln x)^2+2\ln x=(\ln x)(\ln x+2)=0$ when $\ln x=0 \Leftrightarrow x=1$ and when $\ln x=-2 \Leftrightarrow x=e^{-2}$.
$f'(x)>0$ when $0<x<e^{-2}$ and when $x>1$, so
f is increasing on $\left(0,e^{-2}\right)$ and $(1,\infty)$ and decreasing on $\left(e^{-2},1\right)$.
F. Local maximum value $f\left(e^{-2}\right)=4e^{-2}$,

local minimum value $f(1)=0$

G. $f''(x)=2(\ln x)(1/x)+2/x=(2/x)(\ln x+1)=0$ when $\ln x=-1 \Leftrightarrow x=e^{-1}$. $f''(x)>0 \Leftrightarrow x>1/e$, so f is CU on $(1/e,\infty)$, CD on $(0,1/e)$.
IP at $(1/e,1/e)$

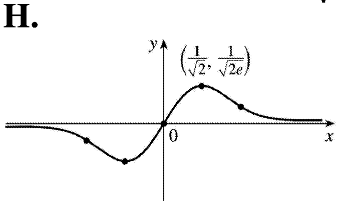
H.



49. $y=f(x)=xe^{-x^2}$ **A.** $D=R$ **B.** Intercepts are 0 **C.** $f(-x)=-f(x)$, so the curve is symmetric about the origin. **D.** $\lim_{x\to\pm\infty} xe^{-x^2}=\lim_{x\to\pm\infty}\frac{x}{e^{x^2}}=\lim_{x\to\pm\infty}\frac{1}{2xe^{x^2}}=0$, so $y=0$ is a HA. **E.**

$f'(x)=e^{-x^2}-2x^2e^{-x^2}=e^{-x^2}\left(1-2x^2\right)>0 \Leftrightarrow x^2<\frac{1}{2} \Leftrightarrow |x|<\frac{1}{\sqrt{2}}$, so f is increasing on $\left(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)$ and decreasing on $\left(-\infty,-\frac{1}{\sqrt{2}}\right)$ and $\left(\frac{1}{\sqrt{2}},\infty\right)$. **F.** Local maximum value $f\left(\frac{1}{\sqrt{2}}\right)=1/\sqrt{2e}$, local minimum value $f\left(-\frac{1}{\sqrt{2}}\right)=-1/\sqrt{2e}$ **G.** $f''(x)=-2xe^{-x^2}\left(1-2x^2\right)-4xe^{-x^2}=2xe^{-x^2}\left(2x^2-3\right)>0$ $\Leftrightarrow x>\sqrt{\frac{3}{2}}$ or $-\sqrt{\frac{3}{2}}<x<0$, so f is CU on $\left(\sqrt{\frac{3}{2}},\infty\right)$ and $\left(-\sqrt{\frac{3}{2}},0\right)$ and CD on $\left(-\infty,-\sqrt{\frac{3}{2}}\right)$ and $\left(0,\sqrt{\frac{3}{2}}\right)$.
IP are $(0,0)$ and $\left(\pm\sqrt{\frac{3}{2}},\pm\sqrt{\frac{3}{2}}\,e^{-3/2}\right)$.

H.



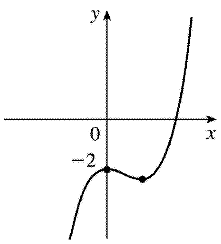
50. $y=f(x)=e^x-3e^{-x}-4x$ **A.** $D=R$ **B.** $y-$ intercept $=-2$; x –intercept ≈ 2.22 **C.** No symmetry **D.** $\lim_{x\to\infty}\left(e^x-3e^{-x}-4x\right)=\lim_{x\to\infty} x\left(\frac{e^x}{x}-3\frac{e^{-x}}{x}-4\right)=\infty$, since $\lim_{x\to\infty}\frac{e^x}{x}=\lim_{x\to\infty}\frac{e^x}{1}=\infty$.
Similarly,

$\lim\limits_{x\to-\infty}\left(e^{x}-3e^{-x}-4x\right)=-\infty$. No HA; no VA

E. $f'(x)=e^{x}+3e^{-x}-4=e^{-x}\left(e^{2x}-4e^{x}+3\right)=e^{-x}\left(e^{x}-3\right)\left(e^{x}-1\right)>0\Leftrightarrow e^{x}>3$ or $e^{x}<1\Leftrightarrow$ $x>\ln 3$ or $x<0$. So f is increasing on $(-\infty,0)$ and $(\ln 3,\infty)$ and decreasing on $(0,\ln 3)$. **F.** Local maximum value $f(0)=-2$,

local minimum value $f(\ln 3)=2-4\ln 3$ **G.** $f''(x)=e^{x}-3e^{-x}=e^{-x}\left(e^{2x}-3\right)>0\Leftrightarrow e^{2x}>3\Leftrightarrow x>\frac{1}{2}\ln 3$, so f is CU on $\left(\frac{1}{2}\ln 3,\infty\right)$ and CD on $\left(-\infty,\frac{1}{2}\ln 3\right)$. IP at $\left(\frac{1}{2}\ln 3,-2\ln 3\right)$.

H.



51. $y=f(x)=e^{3x}+e^{-2x}$ **A.** $D=R$ **B.** $y-$ intercept $=f(0)=2$; no x –intercept **C.** No symmetry **D.** No asymptotes

E. $f'(x)=3e^{3x}-2e^{-2x}$, so $f'(x)>0\Leftrightarrow 3e^{3x}>2e^{-2x}$ [multiply by e^{2x}] $\Leftrightarrow e^{5x}>\frac{2}{3}\Leftrightarrow 5x>\ln\frac{2}{3}\Leftrightarrow$

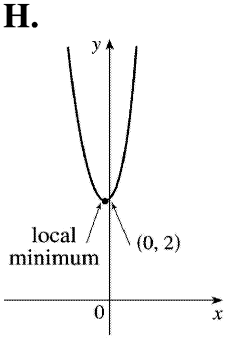
$x>\frac{1}{5}\ln\frac{2}{3}\approx-0.081$. Similarly, $f'(x)<0\Leftrightarrow x<\frac{1}{5}\ln\frac{2}{3}$.

f is decreasing on $\left(-\infty,\frac{1}{5}\ln\frac{2}{3}\right)$ and increasing on $\left(\frac{1}{5}\ln\frac{2}{3},\infty\right)$.

F. Local minimum value $f\left(\frac{1}{5}\ln\frac{2}{3}\right)=\left(\frac{2}{3}\right)^{3/5}+\left(\frac{2}{3}\right)^{-2/5}\approx 1.96$; no local maximum.

G. $f''(x)=9e^{3x}+4e^{-2x}$, so $f''(x)>0$ for all x , and f is CU on $(-\infty,\infty)$. No IP

H.



52. $y=f(x)=\tan^{-1}\left(\frac{x-1}{x+1}\right)$ **A.** $D=\{x\mid x\neq-1\}$

B. x –intercept $=1$, y –intercept

$=f(0)=\tan^{-1}(-1)=-\dfrac{\pi}{4}$ **C.** No symmetry **D.**

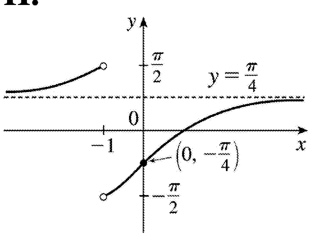
$\lim\limits_{x\to\pm\infty}\tan^{-1}\left(\dfrac{x-1}{x+1}\right)=\lim\limits_{x\to\pm\infty}\tan^{-1}\left(\dfrac{1-1/x}{1+1/x}\right)=\tan^{-1}1=\dfrac{\pi}{4}$, so $y=\dfrac{\pi}{4}$ is a HA. Also

$\lim\limits_{x\to-1^+}\tan^{-1}\left(\dfrac{x-1}{x+1}\right)=-\dfrac{\pi}{2}$ and $\lim\limits_{x\to-1^-}\tan^{-1}\left(\dfrac{x-1}{x+1}\right)=\dfrac{\pi}{2}$. **E.**

$$f'(x)=\frac{1}{1+\left[(x-1)/(x+1)\right]^2}\,\frac{(x+1)-(x-1)}{(x+1)^2}$$
$$=\frac{2}{(x+1)^2+(x-1)^2}=\frac{1}{x^2+1}>0$$

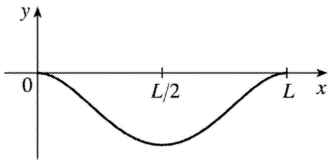
so f is increasing on $(-\infty,-1)$ and $(-1,\infty)$. **F.** No extreme values

H.

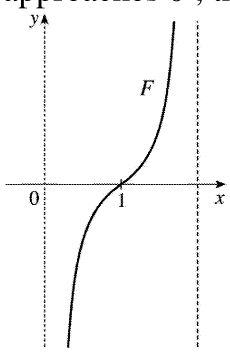


G. $f''(x)=-2x/\left(x^2+1\right)^2>0\Leftrightarrow x<0$, so f is CU on $(-\infty,-1)$ and $(-1,0)$, and CD on $(0,\infty)$.
IP at $\left(0,-\dfrac{\pi}{4}\right)$

53. $y=-\dfrac{W}{24EI}x^4+\dfrac{WL}{12EI}x^3-\dfrac{WL^2}{24EI}x^2=-\dfrac{W}{24EI}x^2\left(x^2-2Lx+L^2\right)=\dfrac{-W}{24EI}x^2(x-L)^2=cx^2(x-L)^2$ where $c=-\dfrac{W}{24EI}$ is a negative constant and $0\le x\le L$. We sketch $f(x)=cx^2(x-L)^2$ for $c=-1$. $f(0)=f(L)=0$
$f'(x)=cx^2\left[2(x-L)\right]+(x-L)^2(2cx)=2cx(x-L)\left[x+(x-L)\right]=2cx(x-L)(2x-L)$. So for $0<x<L$, $f'(x)>0\Leftrightarrow x(x-L)(2x-L)<0$ (since $c<0$) $\Leftrightarrow L/2<x<L$ and $f'(x)<0\Leftrightarrow 0<x<L/2$. So f is increasing on $(L/2,\mathrm{L})$ and decreasing on $(0,\mathrm{L}/2)$, and there is a local and absolute minimum at $(L/2,\mathrm{f}(L/2))=\left(L/2,\mathrm{cL}^4/16\right)$.
$f'(x)=2c\left[x(x-L)(2x-L)\right]\Rightarrow f''(x)=2c\left[1(x-L)(2x-L)+x(1)(2x-L)+x(x-L)(2)\right]=2c\left(6x^2-6Lx+L^2\right)=0\Leftrightarrow$
$x=\dfrac{6L\pm\sqrt{12L^2}}{12}=\dfrac{1}{2}L\pm\dfrac{\sqrt{3}}{6}L$, and these are the x –coordinates of the two inflection points.



54. $F(x)=-\dfrac{k}{x^2}+\dfrac{k}{(x-2)^2}$, where $k>0$ and $0<x<2$. For $0<x<2$, $x-2<0$, so $F'(x)=\dfrac{2k}{x^3}-\dfrac{2k}{(x-2)^3}>0$ and F is increasing. $\lim\limits_{x\to 0^+} F(x)=-\infty$ and $\lim\limits_{x\to 2^-} F(x)=\infty$, so $x=0$ and $x=2$ are vertical asymptotes. Notice that when the middle particle is at $x=1$, the net force acting on it is 0 . When $x>1$, the net force is positive, meaning that it acts to the right. And if the particle approaches $x=2$, the force on it rapidly becomes very large. When $x<1$, the net force is negative, so it acts to the left. If the particle approaches 0 , the force becomes very large to the left.



55.

$y=\dfrac{x^2+1}{x+1}$. Long division gives us:

$$
\begin{array}{r|l}
 & \quad\;\; x-1 \\ \hline
x^2+2x & x^2 \qquad +\; 1 \\
 & \underline{x^2+x\qquad} \\
 & \qquad -x+1 \\
 & \qquad \underline{-x+1} \\
 & \qquad\qquad 2
\end{array}
$$

Thus, $y=f(x)=\dfrac{x^2+1}{x+1}=x-1+\dfrac{2}{x+1}$ and $f(x)-(x-1)=\dfrac{2}{x+1}=\dfrac{\frac{2}{x}}{1+\frac{1}{x}}\to 0$ as $x\to\pm\infty$. So the line $y=x-1$ is a slant asymptote (SA).

56.

$y=\dfrac{2x^3+x^2+x+3}{x^2+2x}$. Long division gives us:

$$
\begin{array}{r|r}
 & 2x-3 \\ \hline
x^2+2x & 2x^3+x^2+x+3 \\
 & 2x^3+4x^2 \\ \cline{2-2}
 & -3x^2+x \\
 & -3x^2-6x \\ \cline{2-2}
 & 7x+3
\end{array}
$$

Thus, $y=f(x)=\dfrac{2x^3+x^2+x+3}{x^2+2x}=2x-3+\dfrac{7x+3}{x^2+2x}$ and $f(x)-(2x-3)=\dfrac{7x+3}{x^2+2x}=\dfrac{\dfrac{7}{x}+\dfrac{3}{x^2}}{1+\dfrac{2}{x}}\to 0$ as $x\to\pm\infty$. So the line $y=2x-3$ is a SA.

57.

$y=\dfrac{4x^3-2x^2+5}{2x^2+x-3}$. Long division gives us:

$$
\begin{array}{r|r}
 & 2x-2 \\ \hline
2x^2+x-3 & 4x^3-2x^2+3 \\
 & 4x^3+2x^2-6x \\ \cline{2-2}
 & -4x^2+6x+5 \\
 & -4x^2-2x+6 \\ \cline{2-2}
 & 8x-3
\end{array}
$$

Thus, $y=f(x)=\dfrac{4x^3-2x^2+5}{2x^2+x-3}=2x-2+\dfrac{8x-1}{2x^2+x-3}$ and

$f(x)-(2x-2)=\dfrac{8x-1}{2x^2+x-3}=\dfrac{\dfrac{8}{x}-\dfrac{1}{x^2}}{2+\dfrac{1}{x}-\dfrac{3}{x^2}}\to 0$ as $x\to\pm\infty$. So the line $y=2x-2$ is a SA.

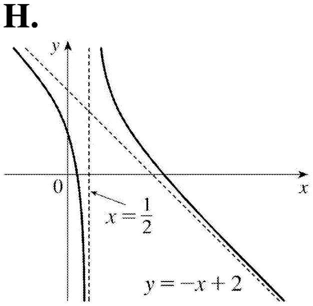
58.

$y=\dfrac{5x^4+x^2+x}{x^3-x^2+2}$. Long division gives us:

$$
\begin{array}{r|l}
 & \qquad\qquad\qquad\qquad 5x+2 \\
x^3-x^2+2 & 5x^4 \qquad\quad +x^2 \quad +x \\
 & 5x^4-5x^3 \qquad +10x \\ \hline
 & \quad -5x^3+x^2-9x \\
 & \quad\ \ 5x^3-5x^2 \qquad +10 \\ \hline
 & \qquad\qquad 6x^2-9x-10
\end{array}
$$

Thus, $y=f(x)=\dfrac{5x^4+x^2+x}{x^3-x^2+2}=5x+5+\dfrac{6x^2-9x-10}{x^3-x^2+2}$ and $f(x)-(5x+5)=\dfrac{6x^2-9x-10}{x^3-x^2+2}=\dfrac{\frac{6}{x}-\frac{9}{x^2}-\frac{10}{x^3}}{1-\frac{1}{x}+\frac{2}{x^3}}\to 0$ as $x\to\pm\infty$. So the line $y=5x+5$ is a SA.

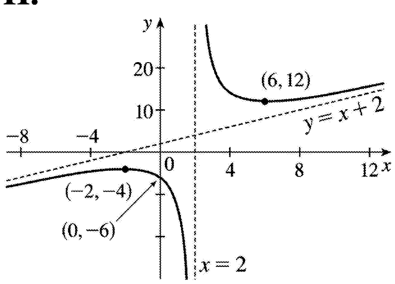
59. $y=f(x)=\dfrac{-2x^2+5x-1}{2x-1}=-x+2+\dfrac{1}{2x-1}$ **A.** $D=\left\{x\in R\mid x\neq\dfrac{1}{2}\right\}=\left(-\infty,\dfrac{1}{2}\right)\cup\left(\dfrac{1}{2},\infty\right)$
B. y –intercept: $f(0)=1$; x –intercepts: $f(x)=0\Rightarrow -2x^2+5x-1=0\Rightarrow x=\dfrac{-5\pm\sqrt{17}}{-4}\Rightarrow x\approx 0.22$, 2.28 . **C.** No symmetry **D.** $\lim\limits_{x\to(1/2)^-}f(x)=-\infty$ and $\lim\limits_{x\to(1/2)^+}f(x)=\infty$, so $x=\dfrac{1}{2}$ is a VA.
$\lim\limits_{x\to\pm\infty}\left[f(x)-(-x+2)\right]=\lim\limits_{x\to\pm\infty}\dfrac{1}{2x-1}=0$, so the line $y=-x+2$ is a SA. **E.** $f'(x)=-1-\dfrac{2}{(2x-1)^2}<0$ for $x\neq\dfrac{1}{2}$, so f is decreasing on $\left(-\infty,\dfrac{1}{2}\right)$ and $\left(\dfrac{1}{2},\infty\right)$. **F.** No extreme values **G.**
$f'(x)=-1-2(2x-1)^{-2}\Rightarrow f''(x)=-2(-2)(2x-1)^{-3}(2)=\dfrac{8}{(2x-1)^3}$, so $f''(x)>0$ when $x>\dfrac{1}{2}$ and $f''(x)<0$ when $x<\dfrac{1}{2}$. Thus, f is CU on $\left(\dfrac{1}{2},\infty\right)$ and CD on $\left(-\infty,\dfrac{1}{2}\right)$. No IP
H.



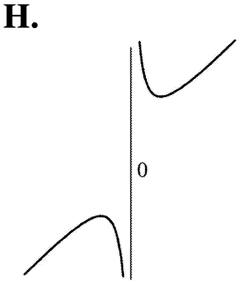
60. $y=f(x)=\dfrac{x^2+12}{x-2}=x+2+\dfrac{16}{x-2}$ **A.** $D=\{x\in R \mid x\neq 2\}=(-\infty,2)\cup(2,\infty)$ **B.** y-intercept: $f(0)=-6$; no x-intercepts. **C.** No symmetry **D.** $\lim\limits_{x\to 2^-} f(x)=-\infty$ and $\lim\limits_{x\to 2^+} f(x)=\infty$, so $x=2$ is a VA.

$\lim\limits_{x\to\pm\infty}\left[f(x)-(x+2)\right]=\lim\limits_{x\to\pm\infty}\dfrac{16}{x-2}=0$, so the line $y=x+2$ is a slant asymptote. **E.**

$f'(x)=1-\dfrac{16}{(x-2)^2}=\dfrac{x^2-4x-12}{(x-2)^2}=\dfrac{(x-6)(x+2)}{(x-2)^2}$, so $f'(x)>0$ when $x<-2$ or $x>6$ and $f'(x)<0$ when $-2<x<2$ or $2<x<6$. Thus, f is increasing on $(-\infty,-2)$ and $(6,\infty)$ and decreasing on $(-2,2)$ and $(2,6)$. **F.** Local maximum value $f(-2)=-4$, local minimum value $f(6)=12$ **G.**
$f''(x)=16(-2)(x-2)^{-3}=\dfrac{32}{(x-2)^3}$, so $f''(x)>0$ for $x>2$ and $f''(x)<0$ for $x<2$. f is CU on $(2,\infty)$ and CD on $(-\infty,2)$. No IP
H.



61. $y=f(x)=\left(x^2+4\right)/x=x+4/x$ **A.** $D=\{x \mid x\neq 0\}=(-\infty,0)\cup(0,\infty)$ **B.** No intercept **C.** $f(-x)=-f(x)\Rightarrow$ symmetry about the origin **D.** $\lim\limits_{x\to\infty}(x+4/x)=\infty$ but $f(x)-x=4/x\to 0$ as $x\to\pm\infty$, so $y=x$ is a slant asymptote. $\lim\limits_{x\to 0^+}(x+4/x)=\infty$ and $\lim\limits_{x\to 0^-}(x+4/x)=-\infty$, so $x=0$ is a VA. **E.** $f'(x)=1-4/x^2>0\Leftrightarrow x^2>4\Leftrightarrow x>2$ or $x<-2$, so f is increasing on $(-\infty,-2)$ and $(2,\infty)$ and decreasing on $(-2,0)$ and $(0,2)$. **F.** Local maximum value $f(-2)=-4$, local minimum value $f(2)=4$ **G.** $f''(x)=8/x^3>0\Leftrightarrow x>0$ so f is CU on $(0,\infty)$ and CD on $(-\infty,0)$. No IP
H.



62.

$y=f(x)=e^x-x$ **A.** $D=R$ **B.** No $x-$ intercept; y –intercept $=1$ **C.** No symmetry **D.** $\lim\limits_{x\to-\infty}\left(e^x-x\right)=\infty$,

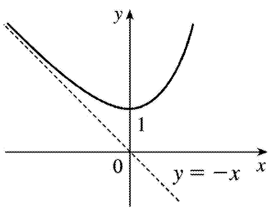
$\lim\limits_{x\to\infty}\left(e^x-x\right)=\lim\limits_{x\to\infty} x\left(\frac{e^x}{x}-1\right)=\infty$ since $\lim\limits_{x\to\infty}\frac{e^x}{x}=\lim\limits_{x\to\infty}\frac{e^x}{1}=\infty$.

$y=-x$ is a slant asymptote since $\left(e^x-x\right)-(-x)=e^x\to 0$ as $x\to-\infty$. **E.** $f'(x)=e^x-1>0 \Leftrightarrow e^x>1 \Leftrightarrow x>0$, so f is increasing on $(0,\infty)$ and decreasing on $(-\infty,0)$.

F. $f(0)=1$ is a local and absolute minimum value.

G. $f''(x)=e^x>0$ for all x , so f is CU on R . No IP

H.



63. $y=f(x)=\frac{2x^3+x^2+1}{x^2+1}=2x+1+\frac{-2x}{x^2+1}$ **A.** $D=R$ **B.** y –intercept: $f(0)=1$; x –intercept: $f(x)=0\Rightarrow$

$0=2x^3+x^2+1=(x+1)(2x^2-x+1)\Rightarrow x=-1$ **C.** No symmetry **D.** No VA

$\lim\limits_{x\to\pm\infty}\left[f(x)-(2x+1)\right]=\lim\limits_{x\to\pm\infty}\frac{-2x}{x^2+1}=\lim\limits_{x\to\pm\infty}\frac{-2/x}{1+1/x^2}=0$, so the line $y=2x+1$ is a slant asymptote. **E.**

${}'(x)=2+\frac{\left(x^2+1\right)(-2)-(-2x)(2x)}{\left(x^2+1\right)^2}=\frac{2\left(x^4+2x^2+1\right)-2x^2-2+4x^2}{\left(x^2+1\right)^2}=\frac{2x^4+6x^2}{(x^2+1)^2}=\frac{2x^2(x^2+3)}{(x^2+1)^2}$ so $f'(x)>0$ if $x\neq0$. Thus, f is increasing on $(-\infty,0)$ and $(0,\infty)$. Since f is continuous at 0 , f is increasing on R .

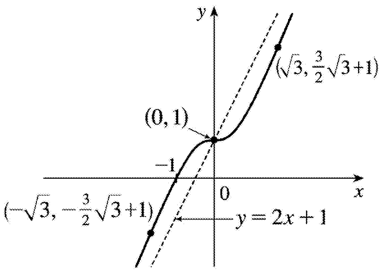
F. No extreme values $f''(x)=\frac{\left(x^2+1\right)^2\cdot\left(8x^3+12x\right)-\left(2x^4+6x^2\right)\cdot 2\left(x^2+1\right)(2x)}{\left[\left(x^2+1\right)^2\right]^2}$

$=\frac{4x\left(x^2+1\right)\left(x^2+1\right)\left(2x^2+3\right)-2x^4-6x^2]}{\left(x^2+1\right)^4}=\frac{4x\left(-x^2+3\right)}{\left(x^2+1\right)^3}$ so $f''(x)>0$ for $x<-\sqrt{3}$ and $0<x<\sqrt{3}$, and

$f''(x)<0$ for $-\sqrt{3}<x<0$ and $x>\sqrt{3}$. f is CU on $\left(-\infty,-\sqrt{3}\right)$ and $\left(0,\sqrt{3}\right)$, and CD on $\left(-\sqrt{3},0\right)$ and $\left(\sqrt{3},\infty\right)$. There are three IPs: $(0,1)$, $\left(-\sqrt{3},-\frac{3}{2}\sqrt{3}+1\right)\approx(-1.73,-1.60)$, and

$\left(\sqrt{3},\frac{3}{2}\sqrt{3}+1\right)\approx(1.73,3.60)$.

H.



64. $y=f(x)=\dfrac{(x+1)^3}{(x-1)^2}=\dfrac{x^3+3x^2+3x+1}{x^2-2x+1}=x+5+\dfrac{12x-4}{(x-1)^2}$ **A.** $D=\{x\in R \mid x\neq 1\}=(-\infty,1)\cup(1,\infty)$ **B.** y –intercept: $f(0)=1$; x –intercept: $f(x)=0\Rightarrow x=-1$ **C.** No symmetry **D.** $\lim_{x\to 1} f(x)=\infty$, so $x=1$ is a VA.

$\lim_{x\to\pm\infty}[f(x)-(x+5)]=\lim_{x\to\pm\infty}\dfrac{12x-4}{x^2-2x+1}=\lim_{x\to\pm\infty}\dfrac{\frac{12}{x}-\frac{4}{x^2}}{1-\frac{2}{x}^2+\frac{1}{x^2}}=0$, so the line $y=x+5$ is a SA. $f'(x)$

$=\dfrac{(x-1)^2\cdot 3(x+1)^2-(x+1)^3\cdot 2(x-1)}{\left[(x-1)^2\right]^2}=\dfrac{(x-1)(x+1)^2[3(x-1)-2(x+1)]}{(x-1)^4}=\dfrac{(x+1)^2(x-5)}{(x-1)^3}$ so $f'(x)>0$ when $x<-1$, $-1<x<1$, or $x>5$, and $f'(x)<0$ when $1<x<5$. f is increasing on $(-\infty,1)$ and $(5,\infty)$ and decreasing on $(1,5)$.

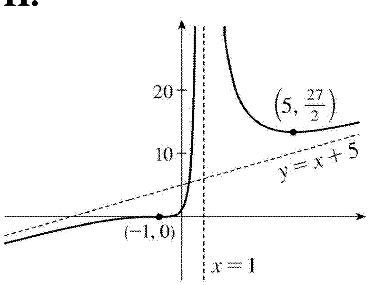
F. Local minimum value $f(5)=\dfrac{216}{16}=\dfrac{27}{2}$, no local maximum **G.** $f''(x)$

$=\dfrac{(x-1)^3\left[(x-1)^2+(x-5)\cdot 2(x+1)\right]-(x+1)^2(x-5)\cdot 3(x-1)^2}{\left[(x-1)^3\right]^2}$

$=\dfrac{(x-1)^2(x+1)\{(x-1)[(x+1)+2(x-5)]-3(x+1)(x-5)\}}{(x-1)^6}=\dfrac{(x+1)\left\{(x-1)[3x-9]-3\left(x^2-4x-5\right)\right\}}{(x-1)^4}=\dfrac{(x+1)(24)}{(x-1)^4}$

</ so $f''(x)>0$ if $-1<x<1$ or $x>1$, and $f''(x)<0$ if $x<-1$. Thus, f is CU on $(-1,1)$ and $(1,\infty)$ and CD on $(-\infty,-1)$. IP at $(-1,0)$

H.



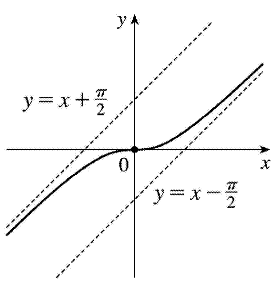
65. $y=f(x)=x-\tan^{-1}x$, $f^{\prime}(x)=1-\frac{1}{1+x^2}=\frac{1+x^2-1}{1+x^2}=\frac{x^2}{1+x^2}$,

$f^{\prime\prime}(x)=\frac{\left(1+x^2\right)(2x)-x^2(2x)}{\left(1+x^2\right)^2}=\frac{2x\left(1+x^2-x^2\right)}{\left(1+x^2\right)^2}=\frac{2x}{\left(1+x^2\right)^2}$.

$\lim\limits_{x\to\infty}\left[f(x)-\left(x-\frac{\pi}{2}\right)\right]=\lim\limits_{x\to\infty}\left(\frac{\pi}{2}-\tan^{-1}x\right)=\frac{\pi}{2}-\frac{\pi}{2}=0$, so $y=x-\frac{\pi}{2}$ is a SA. Also,

$$\lim_{x\to-\infty}\left[f(x)-\left(x+\frac{\pi}{2}\right)\right]=\lim_{x\to-\infty}\left(-\frac{\pi}{2}-\tan^{-1}x\right)$$
$$=-\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)=0$$

so $y=x+\frac{\pi}{2}$ is also a SA. $f^{\prime}(x)\geq 0$ for all x , with equality $\Leftrightarrow x=0$, so f is increasing on R . $f^{\prime\prime}(x)$ has the same sign as x , so f is CD on $(-\infty,0)$ and CU on $(0,\infty)$. $f(-x)=-f(x)$, so f is an odd function; its graph is symmetric about the origin. f has no local extreme values. Its only IP is at $(0,0)$.

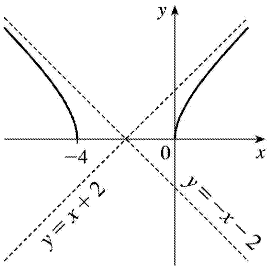


66. $y=f(x)=\sqrt{x^2+4x}=\sqrt{x(x+4)}$. $x(x+4)\geq 0\Leftrightarrow x\leq -4$ or $x\geq 0$, so $D=(-\infty,-4]\cup[0,\infty)$. y –intercept: $f(0)=0$; x –intercepts: $f(x)=0\Rightarrow x=-4$, 0 . $\sqrt{x^2+4x}\mp(x+2)$

$=\frac{\sqrt{x^2+4x}\mp(x+2)}{1}\cdot\frac{\sqrt{x^2+4x}\pm(x+2)}{\sqrt{x^2+4x}\pm(x+2)}=\frac{(x^2+4x)-(x^2+4x+4)}{\sqrt{x^2+4x}\pm(x+2)}=\frac{-4}{\sqrt{x^2+4x}\pm(x+2)}$ so

$\lim\limits_{x\to\pm\infty}[f(x)\mp(x+2)]=0$. Thus, the graph of f approaches the slant asymptote $y=x+2$ as $x\to\infty$ and it approaches the slant asymptote $y=-(x+2)$ as $x\to-\infty$. $f^{\prime}(x)=\frac{x+2}{\sqrt{x^2+4x}}$, so $f^{\prime}(x)<0$ for $x<-4$ and $f^{\prime}(x)>0$ for $x>0$; that is, f is decreasing on $(-\infty,-4)$ and increasing on $(0,\infty)$. There are no local extreme values. $f^{\prime}(x)=(x+2)(x^2+4x)^{-1/2}\Rightarrow$

$f''(x)=(x+2)\cdot\left(-\frac{1}{2}\right)(x^2+4x)^{-3/2}\cdot(2x+4)+(x^2+4x)^{-1/2}=(x^2+4x)^{-3/2}\left[-(x+2)^2+(x^2+4x)\right]=-4\left(x^2+4x\right)^{-3/2}<0$ on D so f is CD on $(-\infty,-4)$ and $(0,\infty)$. No IP



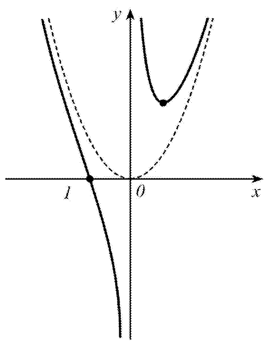
67. $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1\Rightarrow y=\pm\frac{b}{a}\sqrt{x^2-a^2}$. Now

$$\lim_{x\to\infty}\left[\frac{b}{a}\sqrt{x^2-a^2}-\frac{b}{a}x\right]=\frac{b}{a}\cdot\lim_{x\to\infty}\left(\sqrt{x^2-a^2}-x\right)\frac{\sqrt{x^2-a^2}+x}{\sqrt{x^2-a^2}+x}=\frac{b}{a}\cdot\lim_{x\to\infty}\frac{-a^2}{\sqrt{x^2-a^2}+x}=0,$$

which shows that $y=\frac{b}{a}x$ is a slant asymptote. Similarly,

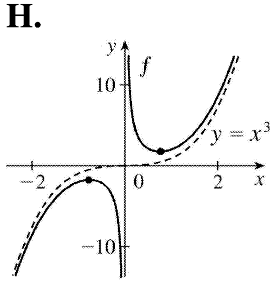
$\lim_{x\to\infty}\left[-\frac{b}{a}\sqrt{x^2-a^2}-\left(-\frac{b}{a}x\right)\right]=-\frac{b}{a}\cdot\lim_{x\to\infty}\frac{-a^2}{\sqrt{x^2-a^2}+x}=0$, so $y=-\frac{b}{a}x$ is a slant asymptote.

68. $f(x)-x^2=\frac{x^3+1}{x}-x^2=\frac{x^3+1-x^3}{x}=\frac{1}{x}$, and $\lim_{x\to\pm\infty}\frac{1}{x}=0$. Therefore, $\lim_{x\to\pm\infty}\left[f(x)-x^2\right]=0$, and so the graph of f is asymptotic to that of $y=x^2$. For purposes of differentiation, we will use $f(x)=x^2+1/x$.
A. $D=\{x|x\neq 0\}$ **B.** No $y-$ intercept; to find the x –intercept, we set $y=0\Leftrightarrow x=-1$.
C. No symmetry **D.** $\lim_{x\to0^+}\frac{x^3+1}{x}=\infty$ and $\lim_{x\to0^-}\frac{x^3+1}{x}=-\infty$, so $x=0$ is a vertical asymptote. Also, the graph is asymptotic to the parabola $y=x^2$, as shown above. **E.** $f'(x)=2x-1/x^2>0\Leftrightarrow x>\frac{1}{\sqrt[3]{2}}$, so f is increasing on $\left(\frac{1}{\sqrt[3]{2}},\infty\right)$ and decreasing on $(-\infty,0)$ and $\left(0,\frac{1}{\sqrt[3]{2}}\right)$. **F.** Local minimum value $f\left(\frac{1}{\sqrt[3]{2}}\right)=\frac{3\sqrt[3]{3}}{2}$, no local maximum **G.** $f''(x)=2+2/x^3>0\Leftrightarrow x<-1$ or $x>0$, so f is CU on $(-\infty,-1)$ and $(0,\infty)$, and CD on $(-1,0)$. IP at $(-1,0)$
H.

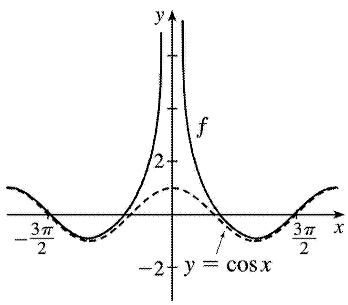


69. $\lim\limits_{x\to\pm\infty}\left[f(x)-x^3\right]=\lim\limits_{x\to\pm\infty}\dfrac{x^4+1}{x}-\dfrac{x^4}{x}=\lim\limits_{x\to\pm\infty}\dfrac{1}{x}=0$, so the graph of f is asymptotic to that of $y=x^3$.

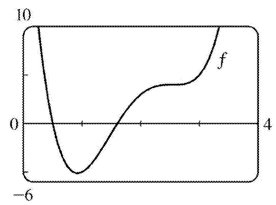
A. $D=\{x\,|\,x\neq 0\}$ **B.** No intercept **C.** f is symmetric about the origin. **D.** $\lim\limits_{x\to 0^-}\left(x^3+\dfrac{1}{x}\right)=-\infty$ and $\lim\limits_{x\to 0^+}\left(x^3+\dfrac{1}{x}\right)=\infty$, so $x=0$ is a vertical asymptote, and as shown above, the graph of f is asymptotic to that of $y=x^3$. **E.** $f'(x)=3x^2-1/x^2>0\Leftrightarrow x^4>\dfrac{1}{3}\Leftrightarrow |x|>\dfrac{1}{\sqrt[4]{3}}$, so f is increasing on $\left(-\infty,-\dfrac{1}{\sqrt[4]{3}}\right)$ and $\left(\dfrac{1}{\sqrt[4]{3}},\infty\right)$ and decreasing on $\left(-\dfrac{1}{\sqrt[4]{3}},0\right)$ and $\left(0,\dfrac{1}{\sqrt[4]{3}}\right)$. **F.** Local maximum value $f\left(-\dfrac{1}{\sqrt[4]{3}}\right)=-4\cdot 3^{-5/4}$, local minimum value $f\left(\dfrac{1}{\sqrt[4]{3}}\right)=4\cdot 3^{-5/4}$
G. $f''(x)=6x+2/x^3>0\Leftrightarrow x>0$, so f is CU on $(0,\infty)$ and CD on $(-\infty,0)$. No IP
H.

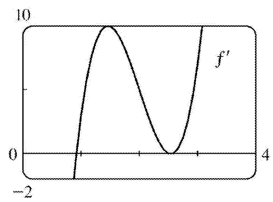


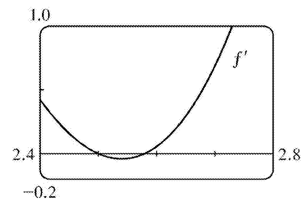
70. $\lim\limits_{x\to\pm\infty}\left[f(x)-\cos x\right]=\lim\limits_{x\to\pm\infty}1/x^2=0$, so the graph of f is asymptotic to that of $\cos x$. The intercepts can only be found approximately. $f(x)=f(-x)$, so f is even. $\lim\limits_{x\to 0}\left(\cos x+\dfrac{1}{x^2}\right)=\infty$, so $x=0$ is a vertical asymptote. We don't need to calculate the derivatives, since we know the asymptotic behavior of the curve.

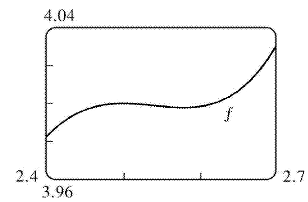

y
f
2
$-\frac{3\pi}{2}$
$\frac{3\pi}{2}$
x
−2
$y = \cos x$

1. $f(x)=4x^4-32x^3+89x^2-95x+29 \Rightarrow f'(x)=16x^3-96x^2+178x-95 \Rightarrow$
$f''(x)=48x^2-192x+178$. $f(x)=0 \Leftrightarrow x\approx 0.5$, 1.60 ; $f'(x)=0 \Leftrightarrow x\approx 0.92$, 2.5 , 2.58 and
$f''(x)=0 \Leftrightarrow x\approx 1.46$, 2.54 .

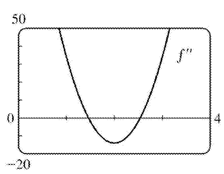




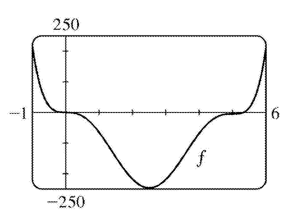


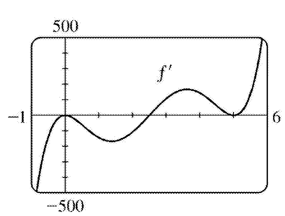


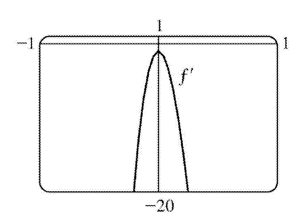
From the graphs of f' , we estimate that $f'<0$ and that f is decreasing on $(-\infty,0.92)$ and $(2.5,2.58)$, and that $f'>0$ and f is increasing on $(0.92,2.5)$ and $(2.58,\infty)$ with local minimum values $f(0.92)\approx -5.12$ and $f(2.58)\approx 3.998$ and local maximum value $f(2.5)=4$. The graphs of f' make it clear that f has a maximum and a minimum near $x=2.5$, shown more clearly in the fourth graph. From the graph of f'' , we estimate that $f''>0$ and that f is CU on $(-\infty,1.46)$ and $(2.54,\infty)$, and that $f''<0$ and f is CD on $(1.46,2.54)$. There are inflection points at about $(1.46,-1.40)$ and $(2.54,3.999)$.

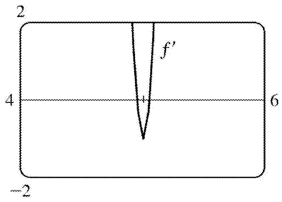


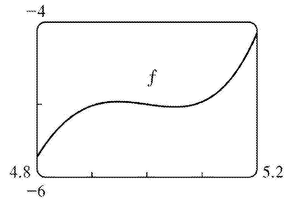
2. $f(x)=x^6-15x^5+75x^4-125x^3-x \Rightarrow f'(x)=6x^5-75x^4+300x^3-375x^2-1 \Rightarrow$
$f''(x)=30x^4-300x^3+900x^2-750x$. $f(x)=0 \Leftrightarrow x=0$ or $x\approx 5.33$; $f'(x)=0 \Leftrightarrow x\approx 2.50$, 4.95 , or 5.05;
$f''(x)=0 \Leftrightarrow x=0$, 5 or $x\approx 1.38$, 3.62 .

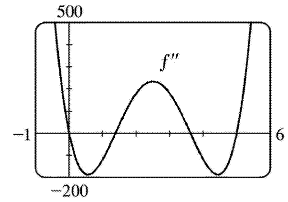






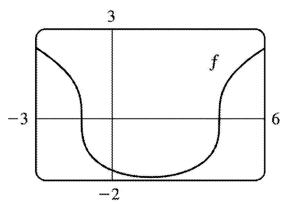


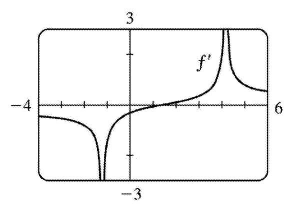


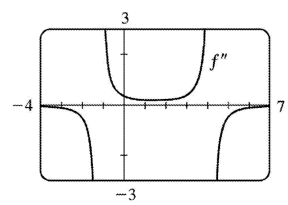


From the graphs of f', we estimate that f is decreasing on $(-\infty, 2.50)$, increasing on $(2.50, 4.95)$, decreasing on $(4.95, 5.05)$, and increasing on $(5.05, \infty)$, with local minimum values $f(2.50) \approx -246.6$ and $f(5.05) \approx -5.03$, and local maximum value $f(4.95) \approx -4.965$ (notice the second graph of f). From the graph of f'', we estimate that f is CU on $(-\infty, 0)$, CD on $(0, 1.38)$, CU on $(1.38, 3.62)$, CD on $(3.62, 5)$, and CU on $(5, \infty)$. There are inflection points at $(0,0)$ and $(5,-5)$, and at about $(1.38, -126.38)$ and $(3.62, -128.62)$.

3. $f(x) = \sqrt[3]{x^2 - 3x - 5} \Rightarrow f'(x) = \frac{1}{3} \frac{2x-3}{\left(x^2-3x-5\right)^{2/3}} \Rightarrow f''(x) = -\frac{2}{9} \frac{x^2-3x+24}{\left(x^2-3x-5\right)^{5/3}}$





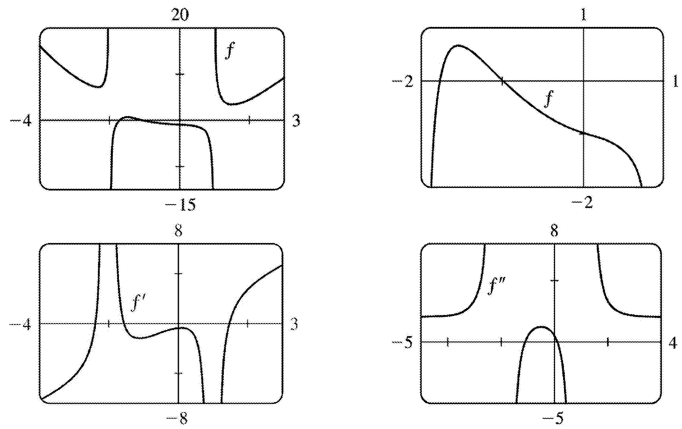


Note: With some CAS's, including Maple, it is necessary to define $f(x) = \frac{x^2-3x-5}{\left|x^2-3x-5\right|} \left|x^2-3x-5\right|^{1/3}$, since the CAS does not compute real cube roots of negative numbers. We estimate from the graph of f' that f is increasing on $(1.5, \infty)$, and decreasing on $(-\infty, 1.5)$. f has no maximum. Minimum value: $f(1.5) \approx -1.9$.

From the graph of f'', we estimate that f is CU on $(-1.2, 4.2)$ and CD on $(-\infty, -1.2)$ and $(4.2, \infty)$. IP at $(-1.2, 0)$ and $(4.2, 0)$.

4. $f(x) = \frac{x^4 + x^3 - 2x^2 + 2}{x^2 + x - 2} \Rightarrow f'(x) = 2\frac{x^5 + 2x^4 - 3x^3 - 4x^2 + 2x - 1}{\left(x^2+x-2\right)^2} \Rightarrow$

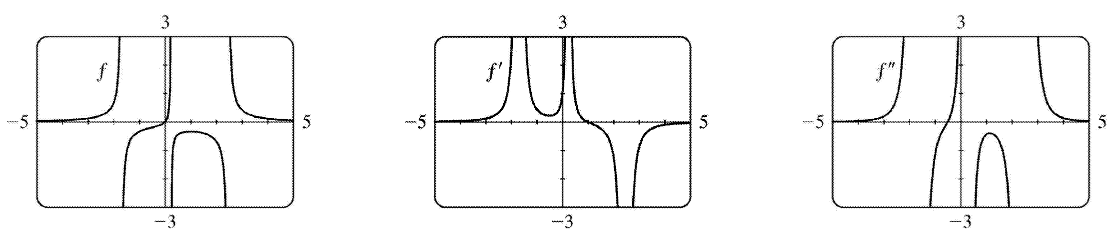
$$f''(x)=2\,\frac{x^6+3x^5-3x^4-11x^3+12x^2+18x-2}{\left(x^2+x-2\right)^3}$$



We estimate from the graph of f' that f is increasing on $(-2.4,-2)$, $(-2,-1.5)$ and $(1.5,\infty)$ and decreasing on $(-\infty,-2.4)$, $(-1.5,1)$ and $(1,1.5)$. Local maximum value: $f(-1.5)\approx 0.7$.

Local minimum values: $f(-2.4)\approx 7.2$, $f(1.5)\approx 3.4$. From the graph of f'' , we estimate that f is CU on $(-\infty,-2)$, $(-1.1,0.1)$ and $(1,\infty)$ and CD on $(-2,-1.1)$ and $(0.1,1)$.
f has IP at $(-1.1,0.2)$ and $(0.1,-1.1)$.

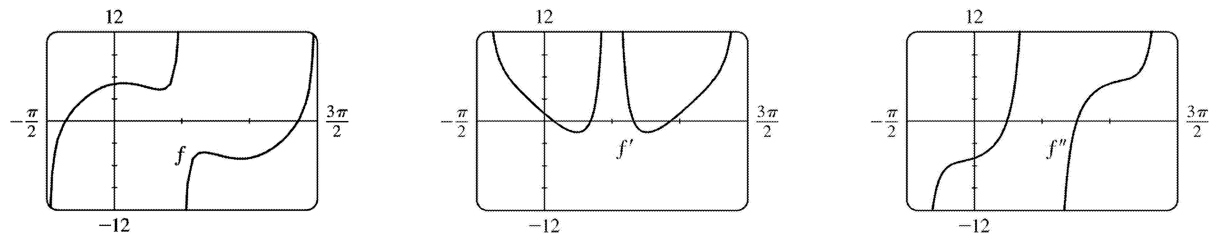
5. $f(x)=\dfrac{x}{x^3-x^2-4x+1}\Rightarrow f'(x)=\dfrac{-2x^3+x^2+1}{\left(x^3-x^2-4x+1\right)^2}\Rightarrow f''(x)=\dfrac{2\left(3x^5-3x^4+5x^3-6x^2+3x+4\right)}{\left(x^3-x^2-4x+1\right)^3}$



We estimate from the graph of f that $y=0$ is a horizontal asymptote, and that there are vertical asymptotes at $x=-1.7$, $x=0.24$, and $x=2.46$. From the graph of f' , we estimate that f is increasing on $(-\infty,-1.7)$, $(-1.7,0.24)$, and $(0.24,1)$, and that f is decreasing on $(1,2.46)$ and $(2.46,\infty)$. There is a local maximum value at $f(1)=-\dfrac{1}{3}$. From the graph of f'' , we estimate that f is CU on $(-\infty,-1.7)$, $(-0.506,0.24)$, and $(2.46,\infty)$, and that f is CD on $(-1.7,-0.506)$ and $(0.24,2.46)$.

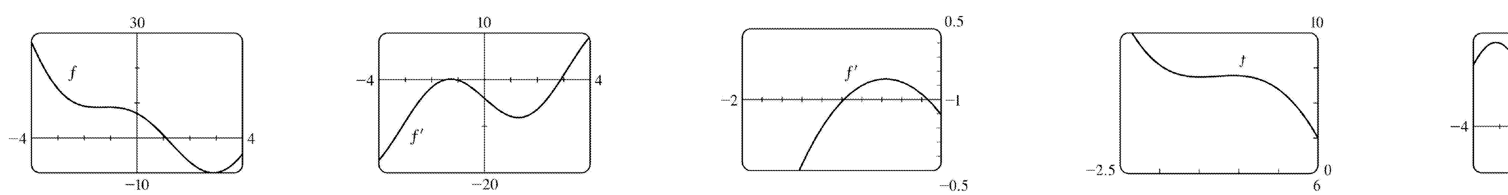
There is an inflection point at $(-0.506,-0.192)$.

6. $f(x)=\tan x+5\cos x \Rightarrow f'(x)=\sec^2 x-5\sin x \Rightarrow f''(x)=2\sec^2 x\tan x-5\cos x$. Since f is periodic with period 2π , and defined for all x except odd multiples of $\frac{\pi}{2}$, we graph f and its derivatives on $\left[-\frac{\pi}{2},\frac{3\pi}{2}\right]$.



We estimate from the graph of f' that f is increasing on $\left(-\frac{\pi}{2},0.21\right)$, $\left(1.07,\frac{\pi}{2}\right)$, $\left(\frac{\pi}{2},2.07\right)$, and $\left(2.93,\frac{3\pi}{2}\right)$, and decreasing on $(0.21,1.07)$ and $(2.07,2.93)$. Local minimum values: $f(1.07)\approx 4.23$, $f(2.93)\approx -5.10$. Local maximum values: $f(0.21)\approx 5.10$, $f(2.07)\approx -4.23$. From the graph of f'' , we estimate that f is CU on $\left(0.76,\frac{\pi}{2}\right)$ and $\left(2.38,\frac{3\pi}{2}\right)$, and CD on $\left(-\frac{\pi}{2},0.76\right)$ and $\left(\frac{\pi}{2},2.38\right)$. f has IP at $(0.76,4.57)$ and $(2.38,-4.57)$.

7. $f(x)=x^2-4x+7\cos x$, $-4\leq x\leq 4$. $f'(x)=2x-4-7\sin x \Rightarrow f''(x)=2-7\cos x$. $f(x)=0 \Leftrightarrow x\approx 1.10$; $f'(x)=0 \Leftrightarrow x\approx -1.49$, -1.07 , or 2.89 ; $f''(x)=0 \Leftrightarrow x=\pm\cos^{-1}\left(\frac{2}{7}\right)\approx\pm 1.28$



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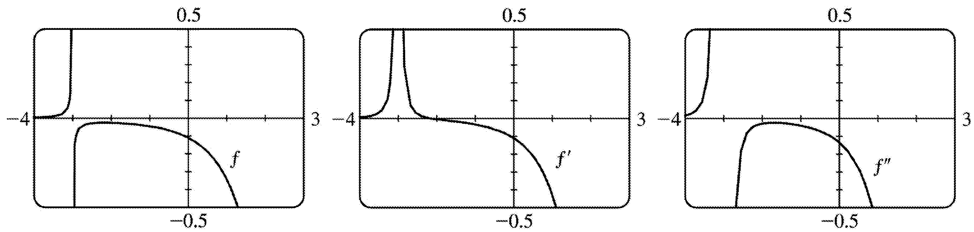
From the graphs of f' , we estimate that f is decreasing ($f'<0$) on $(-4,-1.49)$, increasing on $(-1.49,-1.07)$, decreasing on $(-1.07,2.89)$, and increasing on $(2.89,4)$, with local minimum values $f(-1.49)\approx 8.75$ and $f(2.89)\approx -9.99$ and local maximum value $f(-1.07)\approx 8.79$ (notice the second graph of f). From the graph of f'' , we estimate that f is CU ($f''>0$) on $(-4,-1.28)$, CD on $(-1.28,1.28)$, and CU on $(1.28,4)$. There are inflection points at about $(-1.28,8.77)$ and

$(1.28,-1.48)$.

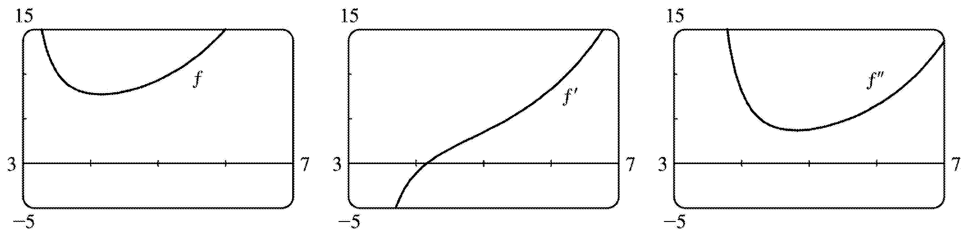
8. $f(x)=\dfrac{e^{x}}{x^{2}-9} \Rightarrow f'(x)=\dfrac{e^{x}(x^{2}-2x-9)}{\left(x^{2}-9\right)^{2}} \Rightarrow f''(x)=\dfrac{e^{x}(x^{4}-4x^{3}-12x^{2}+36x+99)}{\left(x^{2}-9\right)^{3}}$

There are vertical asymptotes at $x=\pm 3$. It is difficult to show all the pertinent features in one viewing rectangle, so we'll show f , f' , and f'' for $x<3$ and also for $x>3$.

For $x<3$

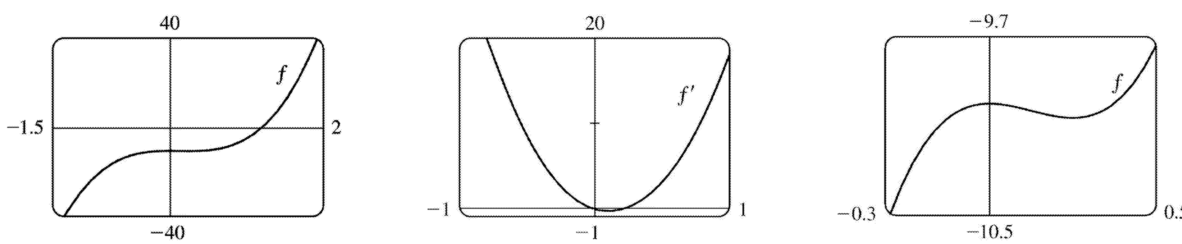


For $x>3$



We estimate from the graphs of f' and f that f is increasing on $(-\infty,-3)$, $(-3,-2.16)$, and $(4.16,\infty)$ and decreasing on $(-2.16,3)$ and $(3,4.16)$. There is a local maximum value of $f(-2.16)\approx-0.03$ and a local minimum value of $f(4.16)\approx 7.71$. From the graphs of f'' , we see that f is CU on $(-\infty,-3)$ and $(3,\infty)$ and CD on $(-3,3)$. There is no inflection point.

9. $f(x)=8x^{3}-3x^{2}-10 \Rightarrow f'(x)=24x^{2}-6x \Rightarrow f''(x)=48x-6$

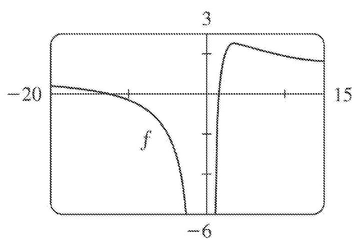


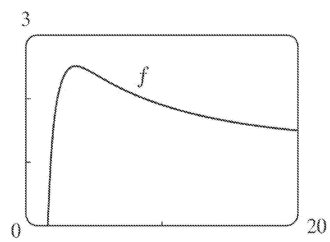
From the graphs, it appears that $f(x)=8x^{3}-3x^{2}-10$ increases on $(-\infty,0)$ and $(0.25,\infty)$ and decreases on $(0,0.25)$; that f has a local maximum value of $f(0)=-10.0$ and a local minimum value of $f(0.25)\approx-10.1$; that f is CU on $(0.1,\infty)$ and CD on $(-\infty,0.1)$; and that f has an IP at $(0.1,-10)$.

To find the exact values, note that $f'(x)=24x^{2}-6x=6x(4x-1)$, which is positive (f is increasing) for

$(-\infty,0)$ and $\left(\frac{1}{4},\infty\right)$, and negative (f is decreasing) on $\left(0,\frac{1}{4}\right)$. By the FDT, f has a local maximum at $x=0$: $f(0)=-10$; and f has a local minimum at $\frac{1}{4}$: $f\left(\frac{1}{4}\right)=\frac{1}{8}-\frac{3}{16}-10=-\frac{161}{16}$. $f''(x)=48x-6=6(8x-1)$, which is positive (f is CU) on $\left(\frac{1}{8},\infty\right)$ and negative (f is CD) on $\left(-\infty,\frac{1}{8}\right)$. f has an IP at $\left(\frac{1}{8},f\left(\frac{1}{8}\right)\right)=\left(\frac{1}{8},-\frac{321}{32}\right)$.

10.





From the graphs, it appears that f increases on $(0,3.6)$ and decreases on $(-\infty,0)$ and $(3.6,\infty)$; that f has a local maximum of $f(3.6)\approx 2.5$ and no local minima; that f is CU on $(5.5,\infty)$ and CD on $(-\infty,0)$ and $(0,5.5)$; and that f has an IP at $(5.5,2.3)$. $f(x)=\frac{x^2+11x-20}{x^2}=1+\frac{11}{x}-\frac{20}{x^2}\Rightarrow$ $f'(x)=-11x^{-2}+40x^{-3}=-x^{-3}(11x-40)$, which is positive (f is increasing) on $\left(0,\frac{40}{11}\right)$, and negative (f is decreasing) on $(-\infty,0)$ and on $\left(\frac{40}{11},\infty\right)$. By the FDT, f has a local maximum at $x=\frac{40}{11}$:

$$f\left(\frac{40}{11}\right)=\frac{\left(\frac{40}{11}\right)^2+11\left(\frac{40}{11}\right)-20}{\left(\frac{40}{11}\right)^2}=\frac{1600+11\cdot 11\cdot 40-20\cdot 121}{1600}=\frac{201}{80}$$

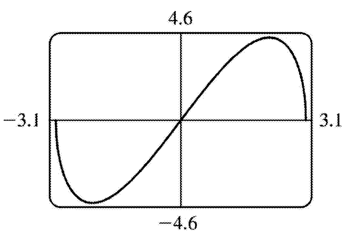
; and f has no local minimum. $f'(x)=-11x^{-2}+40x^{-3}\Rightarrow f''(x)=22x^{-3}-120x^{-4}=2x^{-4}(11x-60)$, which is positive (f is CU) on $\left(\frac{60}{11},\infty\right)$, and negative (f is CD) on $(-\infty,0)$ and $\left(0,\frac{60}{11}\right)$. f has an IP at $\left(\frac{60}{11},f\left(\frac{60}{11}\right)\right)=\left(\frac{60}{11},\frac{211}{90}\right)$.

11. From the graph, it appears that f increases on $(-2.1,2.1)$ and decreases on $(-3,-2.1)$ and $(2.1,3)$; that f has a local maximum of $f(2.1)\approx 4.5$ and a local minimum of $f(-2.1)\approx -4.5$; that f is CU on $(-3.0,0)$ and CD on $(0,3.0)$, and that f has an IP at $(0,0)$. $f(x)=x\sqrt{9-x^2}\Rightarrow$

$f'(x) = \frac{-x^2}{\sqrt{9-x^2}} + \sqrt{9-x^2} = \frac{9-2x^2}{\sqrt{9-x^2}}$, which is positive (f is increasing) on $\left(\frac{-3\sqrt{2}}{2}, \frac{3\sqrt{2}}{2} \right)$ and negative (f is decreasing) on $\left(-3, \frac{-3\sqrt{2}}{2} \right)$ and $\left(\frac{3\sqrt{2}}{2}, 3 \right)$. By the FDT, f has a local maximum value of $f\left(\frac{3\sqrt{2}}{2} \right) = \frac{3\sqrt{2}}{2} \sqrt{9 - \left(\frac{3\sqrt{2}}{2} \right)^2} = \frac{9}{2}$; and f has a local minimum value of $f\left(\frac{-3\sqrt{2}}{2} \right) = -\frac{9}{2}$ (since f is an odd function). $f'(x) = \frac{-x^2}{\sqrt{9-x^2}} + \sqrt{9-x^2} \Rightarrow f''(x)$

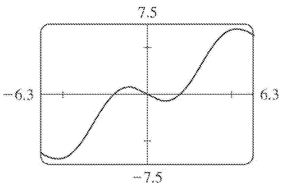
$= \frac{\sqrt{9-x^2}(-2x) + x^2\left(\frac{1}{2}\right)\left(9-x^2\right)^{-1/2}(-2x)}{9-x^2} - x\left(9-x^2\right)^{-1/2} = \frac{-2x - x^3\left(9-x^2\right)^{-1} - x}{\sqrt{9-x^2}}$

$= \frac{-3x}{\sqrt{9-x^2}} - \frac{x^3}{\left(9-x^2\right)^{3/2}} = \frac{x\left(2x^2-27\right)}{\left(9-x^2\right)^{3/2}}$ which is positive (f is CU) on $(-3,0)$ and negative (f is CD) on $(0,3)$. f has an IP at $(0,0)$.

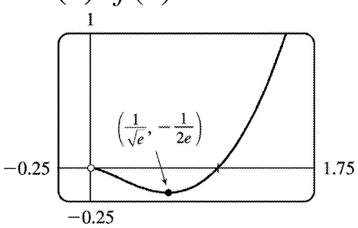


12. From the graph, it appears that f increases on $(-5.2,-1.0)$ and $(1.0,5.2)$ and decreases on $(-2\pi,-5.2)$, $(-1.0,1.0)$, and $(5.2,2\pi)$; that f has local maximum values of $f(-1.0)\approx 0.7$ and $f(5.2)\approx 7.0$ and local minimum values of $f(-5.2)\approx -7.0$ and $f(1.0)\approx -0.7$; that f is CU on $(-2\pi,-3.1)$ and $(0,3.1)$ and CD on $(-3.1,0)$ and $(3.1,2\pi)$, and that f has IP at $(0,0)$, $(-3.1,-3.1)$ and $(3.1,3.1)$. $f(x)=x-2\sin x \Rightarrow f'(x)=1-2\cos x$, which is positive (f is increasing) when $\cos x < \frac{1}{2}$, that is, on $\left(-\frac{5\pi}{3}, -\frac{\pi}{3} \right)$ and $\left(\frac{\pi}{3}, \frac{5\pi}{3} \right)$, and negative (f is decreasing) on $\left(-2\pi, -\frac{5\pi}{3} \right)$, $\left(-\frac{\pi}{3}, \frac{\pi}{3} \right)$, and $\left(\frac{5\pi}{3}, 2\pi \right)$. By the FDT, f has local maximum values of $f\left(-\frac{\pi}{3} \right) = \frac{\pi}{3} + \sqrt{3}$ and $f\left(\frac{5\pi}{3} \right) = \frac{5\pi}{3} + \sqrt{3}$, and local minimum values of $f\left(-\frac{5\pi}{3} \right) = -\frac{5\pi}{3} - \sqrt{3}$ and $f\left(\frac{\pi}{3} \right) = -\frac{\pi}{3} - \sqrt{3}$. $f'(x)=1-2\cos x \Rightarrow f''(x)=2\sin x$, which is positive (f is CU) on $(-2\pi,-\pi)$

and $(0,\pi)$ and negative (f is CD) on $(-\pi,0)$ and $(\pi,2\pi)$. f has IP at $(0,0)$, $(-\pi,-\pi)$ and (π,π) .



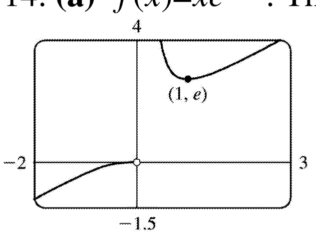
13. **(a)** $f(x)=x^2\ln x$. The domain of f is $(0,\infty)$.



(b) $\lim\limits_{x\to 0^+} x^2\ln x=\lim\limits_{x\to 0^+}\dfrac{\ln x}{1/x^2}=\lim\limits_{x\to 0^+}\dfrac{1/x}{-2/x^3}=\lim\limits_{x\to 0^+}\left(-\dfrac{x^2}{2}\right)=0$. There is a hole at $(0,0)$.

(c) It appears that there is an IP at about $(0.2,-0.06)$ and a local minimum at $(0.6,-0.18)$. $f(x)=x^2\ln x$ $\Rightarrow f'(x)=x^2(1/x)+(\ln x)(2x)=x(2\ln x+1)>0\Leftrightarrow \ln x>-\dfrac{1}{2}\Leftrightarrow x>e^{-1/2}$, so f is increasing on $\left(1/\sqrt{e},\infty\right)$, decreasing on $\left(0,1/\sqrt{e}\right)$. By the FDT, $f\left(1/\sqrt{e}\right)=-1/(2e)$ is a local minimum value. This point is approximately $(0.6065,-0.1839)$, which agrees with our estimate.
$f''(x)=x(2/x)+(2\ln x+1)=2\ln x+3>0\Leftrightarrow \ln x>-\dfrac{3}{2}\Leftrightarrow x>e^{-3/2}$, so f is CU on $\left(e^{-3/2},\infty\right)$ and CD on $\left(0,\mathrm{e}^{-3/2}\right)$. IP is $\left(e^{-3/2},-3/\left(2e^3\right)\right)\approx(0.2231,-0.0747)$.

14. **(a)** $f(x)=xe^{1/x}$. The domain of f is $(-\infty,0)\cup(0,\infty)$.



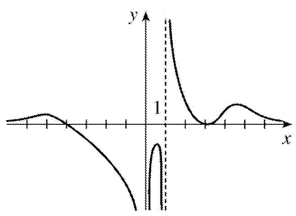
(b) $\lim\limits_{x\to 0^+} xe^{1/x}=\lim\limits_{x\to 0^+}\dfrac{e^{1/x}}{1/x}=\lim\limits_{x\to 0^+}\dfrac{e^{1/x}\left(-1/x^2\right)}{-1/x^2}=\lim\limits_{x\to 0^+} e^{1/x}=\infty$, so $x=0$ is a VA.
Also

$\lim_{x\to 0^-} xe^{1/x} = 0$ since $1/x \to -\infty \Rightarrow e^{1/x} \to 0$.

(c) It appears that there is a local minimum at $(1,2.7)$. There are no IP and f is CD on $(-\infty,0)$ and CU on $(0,\infty)$.

$f(x)=xe^{1/x} \Rightarrow f'(x)=xe^{1/x}\left(-\dfrac{1}{x^2}\right)+e^{1/x}=e^{1/x}\left(1-\dfrac{1}{x}\right)>0 \Leftrightarrow \dfrac{1}{x}<1 \Leftrightarrow x<0$ or $x>1$, so f is increasing on $(-\infty,0)$ and $(1,\infty)$, and decreasing on $(0,1)$. By the FDT, $f(1)=e$ is a local minimum value, which agrees with our estimate.

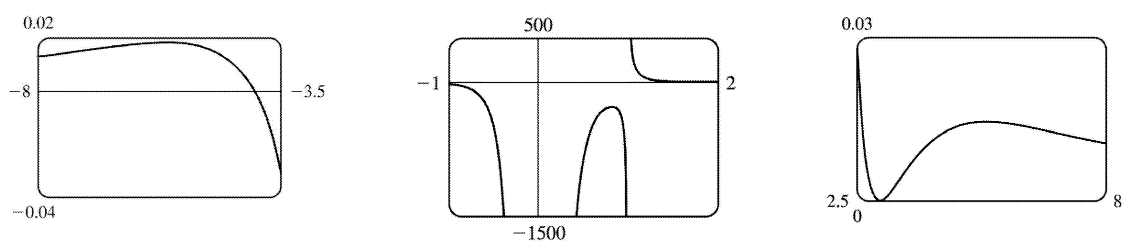
$f''(x)=e^{1/x}\left(1/x^2\right)+(1-1/x)e^{1/x}\left(-1/x^2\right)=\left(e^{1/x}/x^2\right)(1-1+1/x)=e^{1/x}/x^3>0 \Leftrightarrow x>0$, so f is CU on $(0,\infty)$ and CD on $(-\infty,0)$. No IP.

15. 

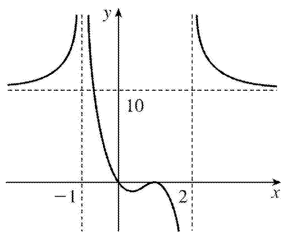
$f(x)=\dfrac{(x+4)(x-3)^2}{x^4(x-1)}$ has VA at $x=0$ and at $x=1$ since $\lim_{x\to 0} f(x)=-\infty$, $\lim_{x\to 1^-} f(x)=-\infty$ and $\lim_{x\to 1^+} f(x)=\infty$.

$$f(x)=\frac{\dfrac{x+4}{x}\cdot\dfrac{(x-3)^2}{x^2}}{\dfrac{x^4}{x^3}\cdot(x-1)} \qquad \text{[dividing numerator and denominator by } x^3\text{]}$$

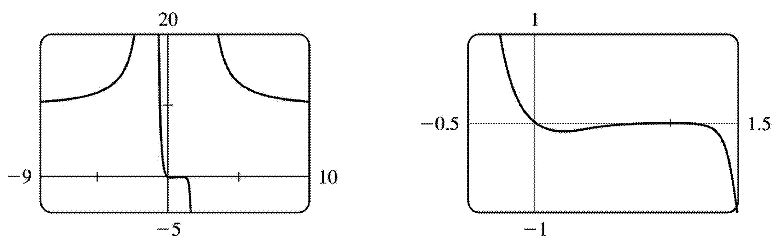
$=\dfrac{(1+4/x)(1-3/x)^2}{x(x-1)} \to 0$ as $x\to\pm\infty$, so f is asymptotic to the x –axis. Since f is undefined at $x=0$, it has no y – intercept. $f(x)=0 \Rightarrow (x+4)(x-3)^2=0 \Rightarrow x=-4$ or $x=3$, so f has x –intercepts -4 and 3 . Note, however, that the graph of f is only tangent to the $x-$ axis and does not cross it at $x=3$, since f is positive as $x\to 3^-$ and as $x\to 3^+$.



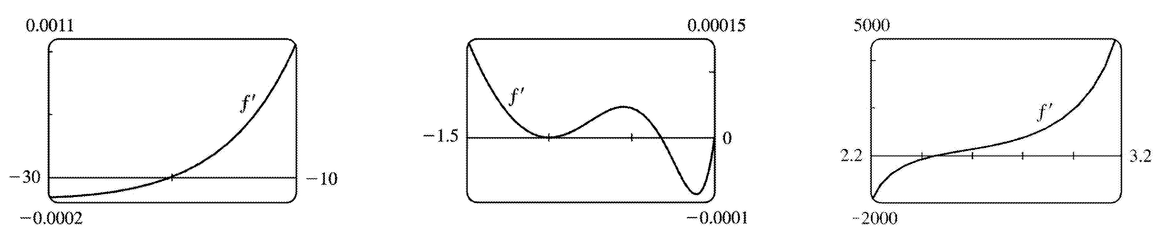
From these graphs, it appears that f has three maximum values and one minimum value. The maximum values are approximately $f(-5.6)=0.0182$, $f(0.82)=-281.5$ and $f(5.2)=0.0145$ and we know (since the graph is tangent to the $x-$ axis at $x=3$) that the minimum value is $f(3)=0$.

16. 

$f(x)=\dfrac{10x(x-1)^4}{(x-2)^3(x+1)^2}$ has VA at $x=-1$ and at $x=2$ since $\lim\limits_{x\to -1} f(x)=\infty$, $\lim\limits_{x\to 2^-} f(x)=-\infty$ and $\lim\limits_{x\to 2^+} f(x)=\infty$.
$f(x)=\dfrac{10(1-1/x)^4}{(1-2/x)^3(1+1/x)^2}\to 10$ as $x\to\pm\infty$, so f is asymptotic to the line $y=10$. $f(0)=0$, so f has a $y-$ intercept at 0 . $f(x)=0\Rightarrow 10x(x-1)^4=0\Rightarrow x=0$ or $x=1$. So f has x –intercepts 0 and 1 . Note, however, that f does not change sign at $x=1$, so the graph is tangent to the $x-$ axis and does not cross it. We know (since the graph is tangent to the $x-$ axis at $x=1$) that the maximum value is $f(1)=0$. From the graphs it appears that the minimum value is about $f(0.2)=-0.1$.



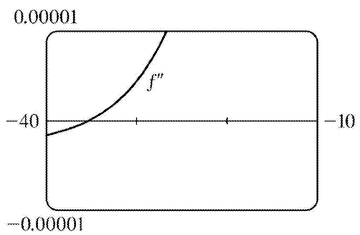
17. $f(x)=\dfrac{x^2(x+1)^3}{(x-2)^2(x-4)^4}\Rightarrow f'(x)=-\dfrac{x(x+1)^2\left(x^3+18x^2-44x-16\right)}{(x-2)^3(x-4)^5}$ (from CAS).

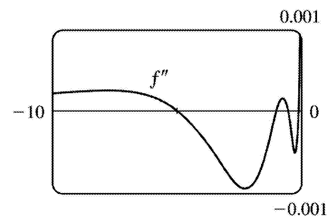


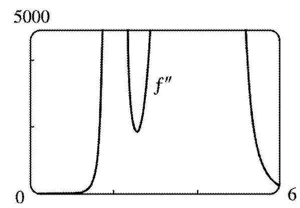
From the graphs of f' , it seems that the critical points which indicate extrema occur at $x\approx -20$, -0.3

, and 2.5 , as estimated in Example 3. (There is another critical point at $x=-1$, but the sign of f' does not change there.) We differentiate again, obtaining

$$f''(x)=2\,\frac{(x+1)\left(x^6+36x^5+6x^4-628x^3+684x^2+672x+64\right)}{(x-2)^4(x-4)^6}\ .$$

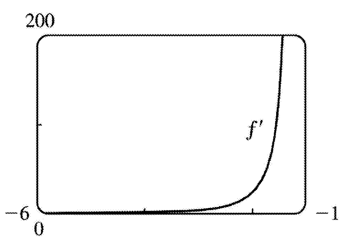


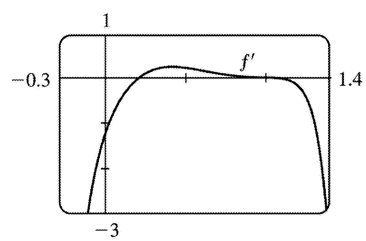


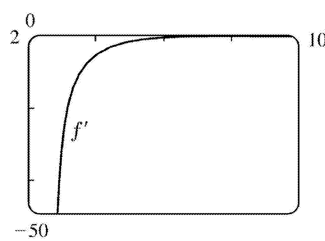


From the graphs of f'' , it appears that f is CU on $(-35.3,-5.0)$, $(-1,-0.5)$, $(-0.1,2)$, $(2,4)$ and $(4,\infty)$ and CD on $(-\infty,-35.3)$, $(-5.0,-1)$ and $(-0.5,-0.1)$. We check back on the graphs of f to find the $y-$ coordinates of the inflection points, and find that these points are approximately $(-35.3,-0.015)$, $(-5.0,-0.005)$, $(-1,0)$, $(-0.5,0.00001)$, and $(-0.1,0.0000066)$.

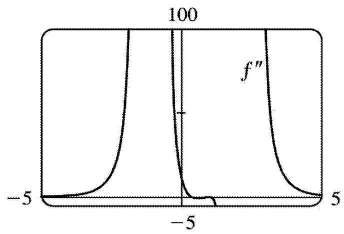
18. $f(x)=\dfrac{10x(x-1)^4}{(x-2)^3(x+1)^2}\Rightarrow f'(x)=-20\,\dfrac{(x-1)^3(5x-1)}{(x-2)^4(x+1)^3}$ (from CAS).

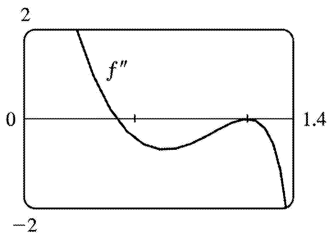






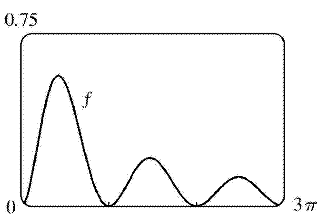
From the graphs of f' , we estimate that f is increasing on $(-\infty,-1)$ and $(0.2,1)$ and decreasing on $(-1,0.2)$, $(1,2)$ and $(2,\infty)$. Differentiating $f'(x)$, we get $f''(x)=60\,\dfrac{(x-1)^2\left(5x^3-8x^2+17x-6\right)}{(x-2)^5(x+1)^4}$.

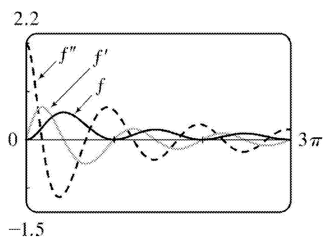




From the graphs of f'', it seems that f is CU on $(-\infty,-1.0)$, $(-1.0,0.4)$ and $(2.0,\infty)$, and CD on $(0.4,2)$. There is an inflection point at about $(0.4,-0.06)$.

19. $y=f(x)=\dfrac{\sin^2 x}{\sqrt{x^2+1}}$ with $0\le x\le 3\pi$. From a CAS, $y'=\dfrac{\sin x\left[2\left(x^2+1\right)\cos x-x\sin x\right]}{\left(x^2+1\right)^{3/2}}$ and
$y''=\dfrac{\left(4x^4+6x^2+5\right)\cos^2 x-4x\left(x^2+1\right)\sin x\cos x-2x^4-2x^2-3}{\left(x^2+1\right)^{5/2}}$.



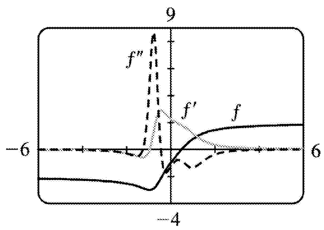


From the graph of f' and the formula for y', we determine that $y'=0$ when $x=\pi$, 2π, 3π, or $x\approx 1.3$, 4.6, or 7.8. So f is increasing on $(0,1.3)$, $(\pi,4.6)$, and $(2\pi,7.8)$. f is decreasing on $(1.3,\pi)$, $(4.6,2\pi)$, and $(7.8,3\pi)$. Local maximum values: $f(1.3)\approx 0.6$, $f(4.6)\approx 0.21$, and $f(7.8)\approx 0.13$. Local minimum values: $f(\pi)=f(2\pi)=0$. From the graph of f'', we see that $y''=0$ $\Leftrightarrow x\approx 0.6$, 2.1, 3.8, 5.4, 7.0, or 8.6. So f is CU on $(0,0.6)$, $(2.1,3.8)$, $(5.4,7.0)$, and $(8.6,3\pi)$. f is CD on $(0.6,2.1)$, $(3.8,5.4)$, and $(7.0,8.6)$. There are IP at $(0.6,0.25)$, $(2.1,0.31)$, $(3.8,0.10)$, $(5.4,0.11)$, $(7.0,0.061)$, and $(8.6,0.065)$.

20. $f(x)=\dfrac{2x-1}{\sqrt[4]{x^4+x+1}}\Rightarrow$

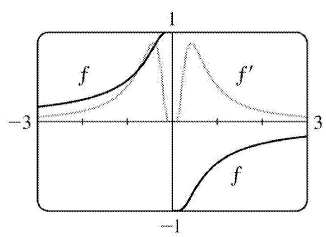
$f'(x)=\dfrac{4x^3+6x+9}{4\left(x^4+x+1\right)^{5/4}}\Rightarrow$

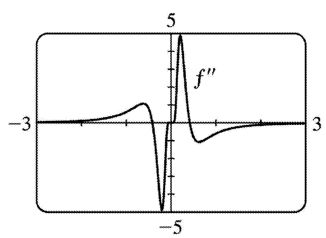
$$f''(x) = -\frac{32x^6+96x^4+152x^3-48x^2+6x+21}{16\left(x^4+x+1\right)^{9/4}}$$



From the graph of f', f appears to be decreasing on $(-\infty,-0.94)$ and increasing on $(-0.94,\infty)$. There is a local minimum value of $f(-0.94)\approx-3.01$. From the graph of f'', f appears to be CU on $(-1.25,-0.44)$ and CD on $(-\infty,-1.25)$ and $(-0.44,\infty)$. There are inflection points at $(-1.25,-2.87)$ and $(-0.44,-2.14)$.

21. $y=f(x)=\dfrac{1-e^{1/x}}{1+e^{1/x}}$. From a CAS, $y'=\dfrac{2e^{1/x}}{x^2\left(1+e^{1/x}\right)^2}$ and $y''=\dfrac{-2e^{1/x}\left(1-e^{1/x}+2x+2xe^{1/x}\right)}{x^4\left(1+e^{1/x}\right)^3}$.

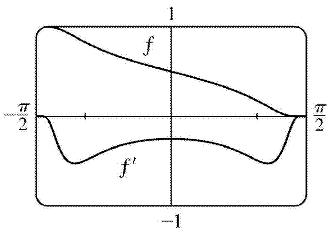


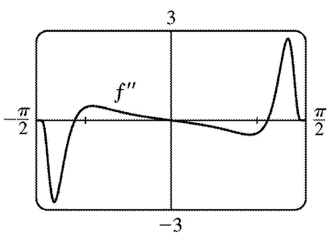


f is an odd function defined on $(-\infty,0)\cup(0,\infty)$. Its graph has no x- or y-intercepts. Since $\lim_{x\to\pm\infty} f(x)=0$, the x-axis is a HA. $f'(x)>0$ for $x\neq0$, so f is increasing on $(-\infty,0)$ and $(0,\infty)$. It has no local extreme values. $f''(x)=0$ for $x\approx\pm0.417$, so f is CU on $(-\infty,-0.417)$, CD on $(-0.417,0)$, CU on $(0,0.417)$, and CD on $(0.417,\infty)$. f has IPs at $(-0.417,0.834)$ and $(0.417,-0.834)$.

22. $y=f(x)=\dfrac{1}{1+e^{\tan x}}$. From a CAS,

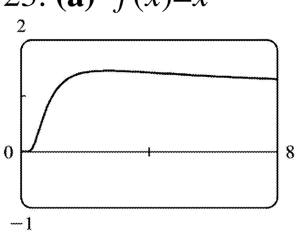
$y'=-\dfrac{e^{\tan x}}{\cos^2 x\left(1+e^{\tan x}\right)^2}$ and $y''=-\dfrac{e^{\tan x}\left[e^{\tan x}(2\sin x\cos x-1)+2\sin x\cos x+1\right]}{\cos^4 x\left(1+e^{\tan x}\right)^3}$. f is a periodic function with period π that has positive values throughout its domain, which consists of all real numbers except odd multiples of $\dfrac{\pi}{2}$ (that is, $\pm\dfrac{\pi}{2}$, $\pm\dfrac{3\pi}{2}$, $\pm\dfrac{5\pi}{2}$, and so on). f has y –intercept $\dfrac{1}{2}$, but no x –intercepts. We graph f , f' , and f'' on one period, $\left(-\dfrac{\pi}{2},\dfrac{\pi}{2}\right)$.





Since $f'(x)<0$ for all x in the domain of f , f is decreasing on the intervals between odd multiples of $\dfrac{\pi}{2}$. $f''(x)=0$ for $x=0+n\pi$ and for $x\approx\pm 1.124+n\pi$, so f is CD on $\left(-\dfrac{\pi}{2},-1.124\right)$, CU on $(-1.124,0)$, CD on $(0,1.124)$, and CU on $\left(1.124,\dfrac{\pi}{2}\right)$. Since f is periodic, this behavior repeats on every interval of length π . f has IPs at $(-1.124+n\pi,0.890)$, $\left(n\pi,\dfrac{1}{2}\right)$, and $(1.124+n\pi,0.110)$.

23. **(a)** $f(x)=x^{1/x}$

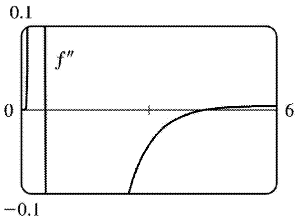


(b) Recall that $a^b=e^{b\ln a}$. $\lim\limits_{x\to 0^+} x^{1/x}=\lim\limits_{x\to 0^+} e^{(1/x)\ln x}$. As $x\to 0^+$, $\dfrac{\ln x}{x}\to-\infty$, so $x^{1/x}=e^{(1/x)\ln x}\to 0$. This indicates that there is a hole at $(0,0)$. As $x\to\infty$, we have the indeterminate form ∞^0 .

$\lim_{x\to\infty} x^{1/x} = \lim_{x\to\infty} e^{(1/x)\ln x}$, but $\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1/x}{1} = 0$, so $\lim_{x\to\infty} x^{1/x} = e^0 = 1$. This indicates that $y=1$ is a HA.

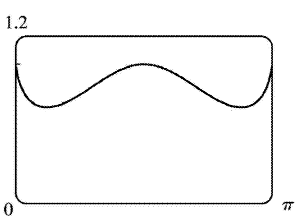
(c) Estimated maximum: $(2.72, 1.45)$. No estimated minimum. We use logarithmic differentiation to find any critical numbers. $y = x^{1/x} \Rightarrow \ln y = \frac{1}{x}\ln x \Rightarrow \frac{y'}{y} = \frac{1}{x}\cdot\frac{1}{x} + (\ln x)\left(-\frac{1}{x^2}\right) \Rightarrow$

$y' = x^{1/x}\left(\frac{1-\ln x}{x^2}\right) = 0 \Rightarrow \ln x = 1 \Rightarrow x = e$. For $0<x<e$, $y'>0$ and for $x>e$, $y'<0$, so $f(e) = e^{1/e}$ is a local maximum value. This point is approximately $(2.7183, 1.4447)$, which agrees with our estimate.

(d) 

From the graph, we see that $f''(x)=0$ at $x\approx 0.58$ and $x\approx 4.37$. Since f'' changes sign at these values, they are x-coordinates of inflection points.

24. **(a)** $f(x) = (\sin x)^{\sin x}$ is continuous where $\sin x > 0$, that is, on intervals of the form $(2n\pi, (2n+1)\pi)$, so we have graphed f on $(0, \pi)$.

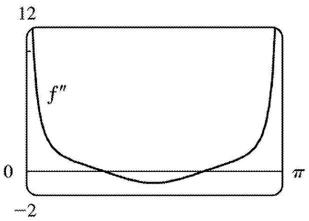


(b) $y = (\sin x)^{\sin x} \Rightarrow \ln y = \sin x \ln \sin x$, so

$\lim_{x\to 0^+} \ln y = \lim_{x\to 0^+} \sin x \ln \sin x = \lim_{x\to 0^+} \frac{\ln \sin x}{\csc x} = \lim_{x\to 0^+} \frac{\cot x}{-\csc x \cot x} = \lim_{x\to 0^+} (-\sin x) = 0 \Rightarrow \lim_{x\to 0^+} y = e^0 = 1$.

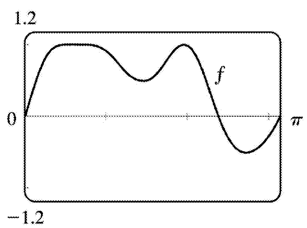
(c) It appears that we have a local maximum at $(1.57, 1)$ and local minima at $(0.38, 0.69)$ and $(2.76, 0.69)$. $y = (\sin x)^{\sin x} \Rightarrow \ln y = \sin x \ln \sin x \Rightarrow$

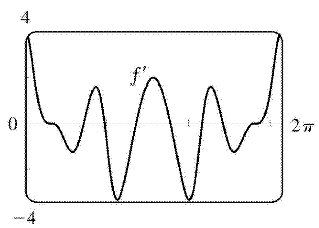
$\frac{y'}{y}=(\sin x)\left(\frac{\cos x}{\sin x}\right)+(\ln \sin x)\cos x=\cos x(1+\ln \sin x)\Rightarrow y'=(\sin x)^{\sin x}(\cos x)(1+\ln \sin x)$. $y'=0\Rightarrow$ $\cos x=0$ or $\ln \sin x=-1\Rightarrow x_2=\frac{\pi}{2}$ or $\sin x=e^{-1}$. On $(0,\pi)$, $\sin x=e^{-1}\Rightarrow x_1=\sin^{-1}\left(e^{-1}\right)$ and $x_3=\pi-\sin^{-1}\left(e^{-1}\right)$. Approximating these points gives us $\left(x_1,f\left(x_1\right)\right)\approx(0.3767,0.6922)$, $\left(x_2,f\left(x_2\right)\right)\approx(1.5708,1)$, and $\left(x_3,f\left(x_3\right)\right)\approx(2.7649,0.6922)$. The approximations confirm our estimates.

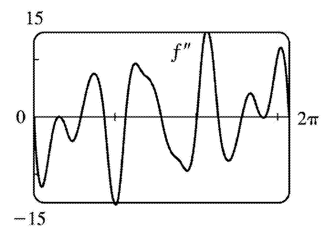
(d) 

From the graph, we see that $f''(x)=0$ at $x\approx 0.94$ and $x\approx 2.20$. Since f'' changes sign at these values, they are x –coordinates of inflection points.

25.

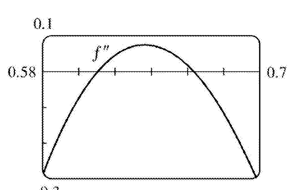


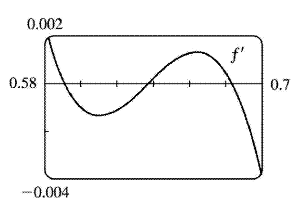


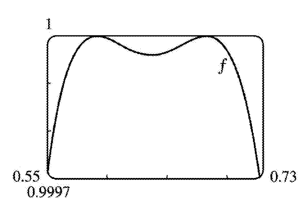


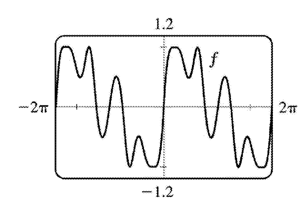
From the graph of $f(x)=\sin(x+\sin 3x)$ in the viewing rectangle $[0,\pi]$ by $[-1.2,1.2]$, it looks like f has two maxima and two minima. If we calculate and graph $f'(x)=[\cos(x+\sin 3x)](1+3\cos 3x)$ on $[0,2\pi]$,

we see that the graph of f' appears to be almost tangent to the x– axis at about $x=0.7$. The graph of $f''=-[\sin(x+\sin 3x)](1+3\cos 3x)^2+\cos(x+\sin 3x)(-9\sin 3x)$ is even more interesting near this x– value: it seems to just touch the x– axis.



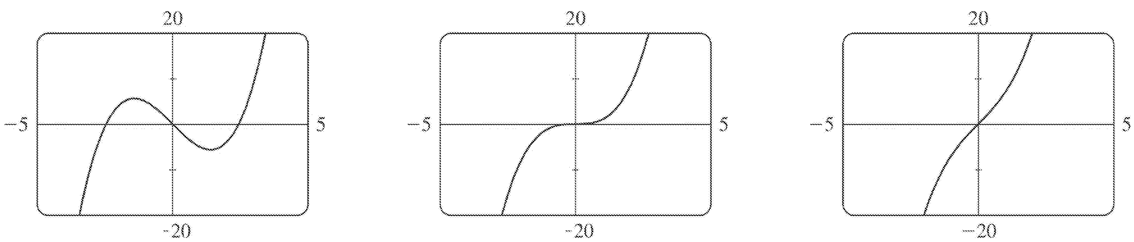






If we zoom in on this place on the graph of f'', we see that f'' actually does cross the axis twice near $x=0.65$, indicating a change in concavity for a very short interval. If we look at the graph of f' on the same interval, we see that it changes sign three times near $x=0.65$, indicating that what we had thought was a broad extremum at about $x=0.7$ actually consists of three extrema (two maxima and a minimum). These maximum values are roughly $f(0.59)=1$ and $f(0.68)=1$, and the minimum value is roughly $f(0.64)=0.99996$. There are also a maximum value of about $f(1.96)=1$ and minimum values of about $f(1.46)=0.49$ and $f(2.73)=-0.51$. The points of inflection on $(0,\pi)$ are about $(0.61,0.99998)$, $(0.66,0.99998)$, $(1.17,0.72)$, $(1.75,0.77)$, and $(2.28,0.34)$. On $(\pi,2\pi)$, they are about $(4.01,-0.34)$, $(4.54,-0.77)$, $(5.11,-0.72)$, $(5.62,-0.99998)$, and $(5.67,-0.99998)$. There are also IP at $(0,0)$ and $(\pi,0)$. Note that the function is odd and periodic with period 2π, and it is also rotationally symmetric about all points of the form $((2n+1)\pi,0)$, n an integer.

26. $f(x)=x^3+cx=x\left(x^2+c\right) \Rightarrow f'(x)=3x^2+c \Rightarrow f''(x)=6x$



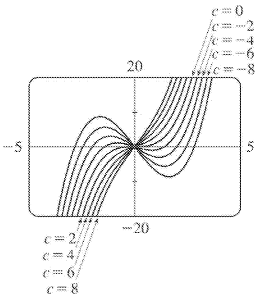
$c=-6$ $c=0$ $c=6$

x-intercepts: When $c\geq 0$, 0 is the only x-intercept. When $c<0$, the x-intercepts are 0 and $\pm\sqrt{-c}$. y-intercept $=f(0)=0$. f is odd, so the graph is symmetric with respect to the origin. $f''(x)<0$ for $x<0$ and $f''(x)>0$ for $x>0$, so f is CD on $(-\infty,0)$ and CU on $(0,\infty)$. The origin is the only inflection point.

If $c>0$, then $f'(x)>0$ for all x, so f is increasing and has no local maximum or minimum.

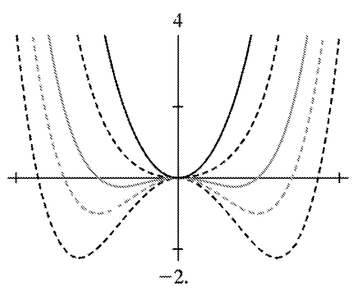
If $c=0$, then $f'(x)\geq 0$ with equality at $x=0$, so again f is increasing and has no local maximum or minimum.

If $c<0$, then $f'(x)=3\left[x^2-(-c/3)\right]=3\left(x+\sqrt{-c/3}\right)\left(x-\sqrt{-c/3}\right)$, so $f'(x)>0$ on $\left(-\infty,-\sqrt{-c/3}\right)$ and $\left(\sqrt{-c/3},\infty\right)$; $f'(x)<0$ on $\left(-\sqrt{-c/3},\sqrt{-c/3}\right)$. It follows that $f\left(-\sqrt{-c/3}\right)=-\frac{2}{3}c\sqrt{-c/3}$ is a local maximum value and $f\left(\sqrt{-c/3}\right)=\frac{2}{3}c\sqrt{-c/3}$ is a local minimum value. As c decreases (toward more negative values), the local maximum and minimum move further apart. There is no absolute maximum or minimum value. The only transitional value of c corresponding to a change in character of the graph is $c=0$.



27. $f(x)=x^4+cx^2=x^2\left(x^2+c\right)$. Note that f is an even function. For $c\geq 0$, the only x –intercept is the point $(0,0)$. We calculate $f'(x)=4x^3+2cx=4x\left(x^2+\frac{1}{2}c\right)\Rightarrow f''(x)=12x^2+2c$. If $c\geq 0$, $x=0$ is the only critical point and there is no inflection point. As we can see from the examples, there is no change in the basic shape of the graph for $c\geq 0$; it merely becomes steeper as c increases. For $c=0$, the graph is the simple curve

$y=x^4$. For $c<0$, there are x– intercepts at 0 and at $\pm\sqrt{-c}$. Also, there is a maximum at $(0,0)$, and there are minima at $\left(\pm\sqrt{-\frac{1}{2}c},-\frac{1}{4}c^2\right)$. As $c\to-\infty$, the x– coordinates of these minima get larger in absolute value, and the minimum points move downward. There are inflection points at $\left(\pm\sqrt{-\frac{1}{6}c},-\frac{5}{36}c^2\right)$, which also move away from the origin as $c\to-\infty$.



28. We need only consider the function $f(x)=x^2\sqrt{c^2-x^2}$ for $c\geq 0$, because if c is replaced by $-c$, the function is unchanged. For $c=0$, the graph consists of the single point $(0,0)$. The domain of f is $[-c,c]$, and the graph of f is symmetric about the y– axis.

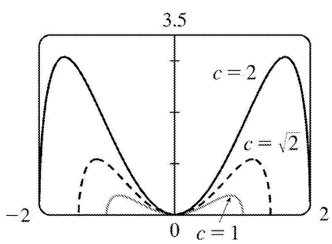
$$f'(x)=2x\sqrt{c^2-x^2}+x^2\frac{-2x}{2\sqrt{c^2-x^2}}=2x\sqrt{c^2-x^2}-\frac{x^3}{\sqrt{c^2-x^2}}=\frac{2x\left(c^2-x^2\right)-x^3}{\sqrt{c^2-x^2}}=-\frac{3x\left(x^2-\frac{2}{3}c^2\right)}{\sqrt{c^2-x^2}}$$. So we see that all members of the family of curves have horizontal tangents at $x=0$, since $f'(0)=0$ for all $c>0$.

Also, the tangents to all the curves become very steep as $x\to\pm c$, since

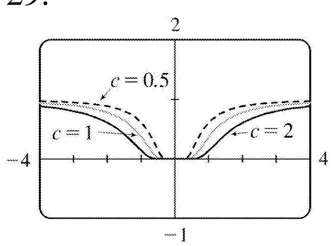
$\lim_{x\to -c^+} f'(x)=\infty$ and $\lim_{x\to c^-} f'(x)=-\infty$. We set $f'(x)=0 \Leftrightarrow x=0$ or $x^2-\frac{2}{3}c^2=0$, so the absolute maximum values are $f\left(\pm\sqrt{\frac{2}{3}}\,c\right)=\frac{2}{3\sqrt{3}}c^3$.

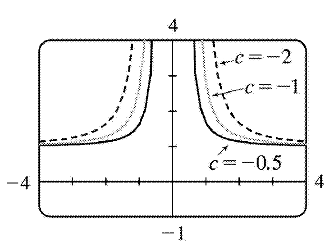
$$f''(x)=\frac{\left(-9x^2+2c^2\right)\sqrt{c^2-x^2}-\left(-3x^3+2c^2x\right)\left(-x/\sqrt{c^2-x^2}\right)}{c^2-x^2}=\frac{6x^4-9c^2x^2+2c^4}{\left(c^2-x^2\right)^{3/2}} .$$

Using the quadratic formula, we find that $f''(x)=0 \Leftrightarrow x^2=\frac{9c^2\pm c^2\sqrt{33}}{12}$. Since $-c<x<c$, we take $x^2=\frac{9-\sqrt{33}}{12}c^2$, so the inflection points are $\left(\pm\sqrt{\frac{9-\sqrt{33}}{12}}\,c,\ \frac{\left(9-\sqrt{33}\right)\left(\sqrt{33}-3\right)}{144}c^3\right)$.
From these calculations we can see that the maxima and the points of inflection get both horizontally and vertically further from the origin as c increases. Since all of the functions have two maxima and two inflection points, we see that the basic shape of the curve does not change as c changes.



29.



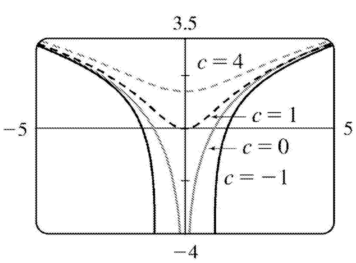


$c=0$ is a transitional value — we get the graph of $y=1$. For $c>0$, we see that there is a HA at $y=1$, and that the graph spreads out as c increases. At first glance there appears to be a minimum at $(0,0)$, but $f(0)$ is undefined, so there is no minimum or maximum. For $c<0$, we still have the HA at $y=1$, but the range is $(1,\infty)$ rather than $(0,1)$. We also have a VA at $x=0$. $f(x)=e^{-c/x^2} \Rightarrow$ $f'(x)=e^{-c/x^2}\left(-2c/x^3\right) \Rightarrow f''(x)=\dfrac{2c\left(2c-3x^2\right)}{x^6 e^{c/x^2}}$. $f'(x)\neq 0$ and $f'(x)$ exists for all $x\neq 0$ (and 0 is not in the domain of f), so there are no maxima or minima. $f''(x)=0 \Rightarrow x=\pm\sqrt{2c/3}$, so if $c>0$, the inflection points spread out as c increases,

and if $c<0$, there are no IP. For $c>0$, there are IP at $\left(\pm\sqrt{2c/3}, e^{-3/2}\right)$. Note that the y-coordinate of the IP is constant.

30. We see that if $c\leq 0$, $f(x)=\ln\left(x^2+c\right)$ is only defined for $x^2>-c \Rightarrow |x|>\sqrt{-c}$, and $\lim\limits_{x\to\sqrt{-c}^+} f(x)=\lim\limits_{x\to-\sqrt{-c}^-} f(x)=-\infty$, since $\ln y\to-\infty$ as $y\to 0$. Thus, for $c<0$, there are vertical asymptotes at $x=\pm\sqrt{c}$, and as c decreases (that is, $|c|$ increases), the asymptotes get further apart. For $c=0$, $\lim\limits_{x\to 0} f(x)=-\infty$, so there is a vertical asymptote at $x=0$. If $c>0$, there are no asymptotes. To find the extrema

and inflection points, we differentiate: $f(x)=\ln\left(x^2+c\right) \Rightarrow f'(x)=\dfrac{1}{x^2+c}(2x)$, so by the First Derivative Test there is a local and absolute minimum at $x=0$. Differentiating again, we get $f''(x)=\dfrac{1}{x^2+c}(2)+2x\left[-\left(x^2+c\right)^{-2}(2x)\right]=\dfrac{2\left(c-x^2\right)}{\left(x^2+c\right)^2}$. Now if $c\leq 0$, f'' is always negative, so f is concave down on both of the intervals on which it is defined. If $c>0$, then f'' changes sign when $c=x^2 \Leftrightarrow x=\pm\sqrt{c}$. So for $c>0$ there are inflection points at $x=\pm\sqrt{c}$, and as c increases, the inflection points get further apart.



31. Note that $c=0$ is a transitional value at which the graph consists of the x-axis. Also, we can see that if we substitute $-c$ for c, the function $f(x)=\dfrac{cx}{1+c^2x^2}$ will be reflected in the x-axis, so we investigate only positive

values of c (except $c=-1$, as a demonstration of this reflective property). Also, f is an odd function.

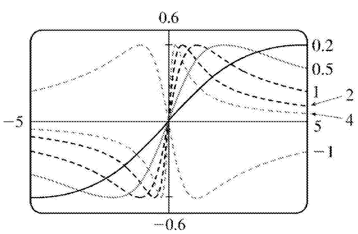
$\lim\limits_{x\to\pm\infty} f(x)=0$, so $y=0$ is a horizontal asymptote for all c . We calculate

$f'(x)=\dfrac{\left(1+c^2x^2\right)c-cx\left(2c^2x\right)}{\left(1+c^2x^2\right)^2}=-\dfrac{c\left(c^2x^2-1\right)}{\left(1+c^2x^2\right)^2}$. $f'(x)=0\Leftrightarrow c^2x^2-1=0\Leftrightarrow x=\pm 1/c$. So there is an absolute maximum value of $f(1/c)=\dfrac{1}{2}$ and an absolute minimum value of $f(-1/c)=-\dfrac{1}{2}$. These extrema have the same value regardless of c , but the maximum points move closer to the $y-$ axis as c increases.

$$f''(x)=\frac{\left(-2c^3x\right)\left(1+c^2x^2\right)^2-\left(-c^3x^2+c\right)}{\left(1+c^2x^2\right)^4}$$

$$=\frac{\left(-2c^3x\right)\left(1+c^2x^2\right)+\left(c^3x^2-c\right)\left(4c^2x\right)}{\left(1+c^2x^2\right)^3}=\frac{2c^3x\left(c^2x^2-3\right)}{\left(1+c^2x^2\right)^3}$$

$f''(x)=0\Leftrightarrow x=0$ or $\pm\sqrt{3}/c$, so there are inflection points at $(0,0)$ and at $\left(\pm\sqrt{3}/\mathrm{c},\pm\sqrt{3}/4\right)$.

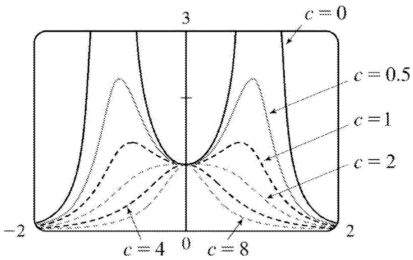


Again, the $y-$ coordinate of the inflection points does not depend on c , but as c increases, both inflection points approach the $y-$ axis.

32. Note that $f(x)=\dfrac{1}{\left(1-x^2\right)^2+cx^2}$ is an even function, and also that $\lim\limits_{x\to\pm\infty} f(x)=0$ for any value of c , so $y=0$ is a horizontal asymptote. We calculate the derivatives:

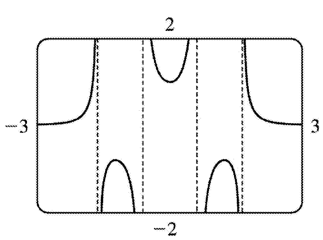
$f'(x)=\dfrac{-4\left(1-x^2\right)x+2cx}{\left[\left(1-x^2\right)^2+cx^2\right]^2}=\dfrac{4x\left[x^2+\left(\frac{1}{2}c-1\right)\right]}{\left[\left(1-x^2\right)^2+cx^2\right]^2}$, and

$f''(x)=2\,\dfrac{10x^6+(9c-18)x^4+\left(3c^2-12c+6\right)x^2+2-c}{\left[x^4+(c-2)x^2+1\right]^3}$.

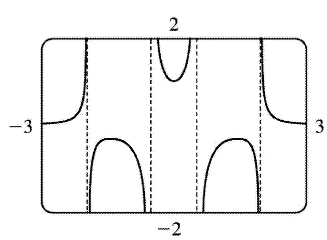


We first consider the case $c>0$. Then the denominator of $f^{\prime}$ is positive, that is, $\left(1-x^2\right)^2+cx^2>0$ for all x , so f has domain R and also $f>0$. If $\frac{1}{2}c-1\geq 0$; that is, $c\geq 2$, then the only critical point is $f(0)=1$, a maximum. Graphing a few examples for $c\geq 2$ shows that there are two IP which approach the $y-$ axis as $c\rightarrow\infty$. $c=2$ and $c=0$ are transitional values of c at which the shape of the curve changes. For $0<c<2$, there are three critical points: $f(0)=1$, a minimum value, and $f\left(\pm\sqrt{1-\frac{1}{2}c}\right)=\frac{1}{c(1-c/4)}$, both maximum values. As c decreases from 2 to 0 , the maximum values get larger and larger, and the $x-$ values at which they occur go from 0 to ± 1 . Graphs show that there are four inflection points for $0<c<2$, and that they get farther away from the origin, both vertically and horizontally, as $c\rightarrow 0^+$. For $c=0$, the function is simply asymptotic to the $x-$ axis and to the lines $x=\pm 1$, approaching $+\infty$ from both sides of each. The y –intercept is 1 , and $(0,1)$ is a local minimum. There are no inflection points. Now if $c<0$, we can write

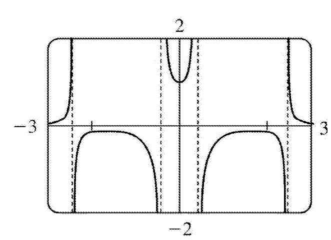
$f(x)=\frac{1}{\left(1-x^2\right)^2+cx^2}=\frac{1}{\left(1-x^2\right)^2-\left(\sqrt{-c}\,x\right)^2}=\frac{1}{\left(x^2-\sqrt{-c}\,x-1\right)\left(x^2+\sqrt{-c}\,x-1\right)}$. So f has vertical asymptotes where $x^2\pm\sqrt{-c}\,x-1=0\Leftrightarrow x=\left|\left(-\sqrt{-c}\pm\sqrt{4-c}\right)\right/2$ or $x=\left|\left(\sqrt{-c}\pm\sqrt{4-c}\right)\right/2$. As c decreases, the two exterior asymptotes move away from the origin, while the two interior ones move toward it. We graph a few examples to see the behavior of the graph near the asymptotes, and the nature of the critical points $x=0$ and $x=\pm\sqrt{1-\frac{1}{2}c}$:



$c=-1$



$c=-1$

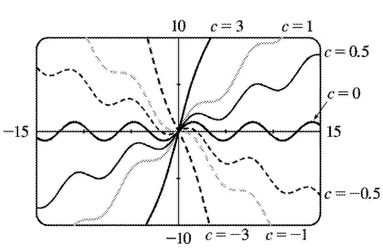


$c=-1$

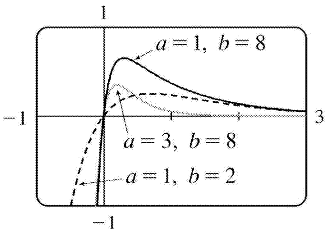
We see that there is one local minimum value, $f(0)=1$, and there are two local maximum values,

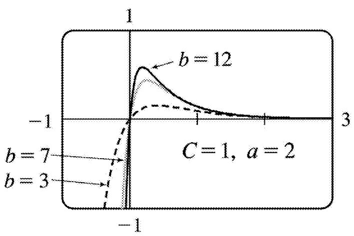
$f\left(\pm\sqrt{1-\frac{1}{2}c}\right)=\frac{1}{c(1-c/4)}$ as before. As c decreases, the $x-$ values at which these maxima occur get larger, and the maximum values themselves approach 0 , though they are always negative.

33. $f(x)=cx+\sin x \Rightarrow f'(x)=c+\cos x \Rightarrow f''(x)=-\sin x$
$f(-x)=-f(x)$, so f is an odd function and its graph is symmetric with respect to the origin.
$f(x)=0 \Leftrightarrow \sin x=-cx$, so 0 is always an x –intercept.
$f'(x)=0 \Leftrightarrow \cos x=-c$, so there is no critical number when $|c|>1$. If $|c|\leq 1$, then there are infinitely many critical numbers. If x_1 is the unique solution of $\cos x=-c$ in the interval $[0,\pi]$, then the critical numbers are $2n\pi \pm x_1$, where n ranges over the integers. (Special cases: When $c=1$, $x_1=0$; when $c=0$, $x=\frac{\pi}{2}$; and when $c=-1$, $x_1=\pi$.)
$f''(x)<0 \Leftrightarrow \sin x>0$, so f is CD on intervals of the form $(2n\pi,(2n+1)\pi)$. f is CU on intervals of the form $((2n-1)\pi,2n\pi)$. The inflection points of f are the points $(2n\pi,2n\pi c)$, where n is an integer.
If $c\geq 1$, then $f'(x)\geq 0$ for all x , so f is increasing and has no extremum. If $c\leq -1$, then $f'(x)\leq 0$ for all x , so f is decreasing and has no extremum. If $|c|<1$, then $f'(x)>0 \Leftrightarrow \cos x>-c \Leftrightarrow x$ is in an interval of the form $\left(2n\pi-x_1,2n\pi+x_1\right)$ for some integer n . These are the intervals on which f is increasing. Similarly, we
find that f is decreasing on the intervals of the form $\left(2n\pi+x_1,2(n+1)\pi-x_1\right)$. Thus, f has local maxima at the points $2n\pi+x_1$, where f has the values $c(2n\pi+x_1)+\sin x_1=c(2n\pi+x_1)+\sqrt{1-c^2}$, and f has local minima at the points $2n\pi-x_1$, where we have
$f(2n\pi-x_1)=c(2n\pi-x_1)-\sin x_1=c(2n\pi-x_1)-\sqrt{1-c^2}$ .The transitional values of c are -1 and 1 . The inflection points move vertically, but not horizontally, when c changes. When $|c|\geq 1$, there is no extremum. For $|c|<1$, the maxima are spaced 2π apart horizontally, as are the minima. The horizontal spacing between maxima and adjacent minima is regular (and equals π) when $c=0$, but the horizontal space between a local maximum and the nearest local minimum shrinks as $|c|$ approaches 1 .



34. For $f(t)=C\left(e^{-at}-e^{-bt}\right)$, C affects only vertical stretching, so we let $C=1$. From the first figure, we notice that the graphs all pass through the origin, approach the t –axis as t increases, and approach $-\infty$ as $t\to-\infty$. Next we let $a=2$ and produce the second figure.





Here, as b increases, the slope of the tangent at the origin increases and the local maximum value increases. $f(t)=e^{-2t}-e^{-bt}\Rightarrow f'(t)=be^{-bt}-2e^{-2t}$. $f'(0)=b-2$, which increases as b increases. $f'(t)=0\Rightarrow be^{-bt}=2e^{-2t}\Rightarrow\dfrac{b}{2}=e^{(b-2)t}\Rightarrow\ln\dfrac{b}{2}=(b-2)t\Rightarrow t=t_1=\dfrac{\ln b-\ln 2}{b-2}$, which decreases as b increases (the maximum is getting closer to the y –axis). $f(t_1)=\dfrac{(b-2)2^{2/(b-2)}}{b^{1+2/(b-2)}}$. We can show that this value increases as b increases by considering it to be a function of b and graphing its derivative with respect to b , which is always positive.

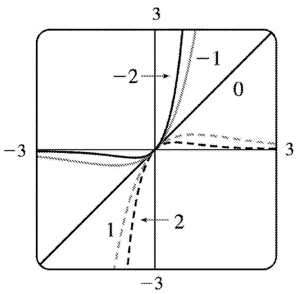
35. If $c<0$, then $\lim\limits_{x\to-\infty}f(x)=\lim\limits_{x\to-\infty}\dfrac{x}{e^{cx}}=\lim\limits_{x\to-\infty}\dfrac{1}{ce^{cx}}=0$, and $\lim\limits_{x\to\infty}f(x)=\infty$.

If $c>0$, then $\lim\limits_{x\to-\infty}f(x)=-\infty$, and $\lim\limits_{x\to\infty}f(x)=\lim\limits_{x\to\infty}\dfrac{1}{ce^{cx}}=0$.

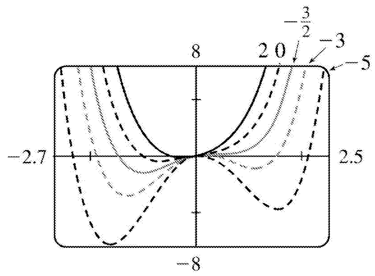
If $c=0$, then $f(x)=x$, so $\lim\limits_{x\to\pm\infty}f(x)=\pm\infty$ respectively.

So we see that $c=0$ is a transitional value. We now exclude the case $c=0$, since we know how the function behaves in that case. To find the maxima and minima of f , we differentiate: $f(x)=xe^{-cx}\Rightarrow$ $f'(x)=x\left(-ce^{-cx}\right)+e^{-cx}=(1-cx)\,e^{-cx}$. This is 0 when $1-cx=0\Leftrightarrow x=1/c$. If $c<0$ then this represents a minimum value of $f\,(1/c)=1/\,(ce)$, since $f'(x)$ changes from negative to positive at $x=1/c$;

and if $c>0$, it represents a maximum value. As $|c|$ increases, the maximum or minimum point gets closer to the origin. To find the inflection points, we differentiate again: $f'(x)=e^{-cx}(1-cx)\Rightarrow$ $f''(x)=e^{-cx}(-c)+(1-cx)\left(-ce^{-cx}\right)=(cx-2)ce^{-cx}$. This changes sign when $cx-2=0\Leftrightarrow x=2/c$. So as $|c|$ increases, the points of inflection get closer to the origin.

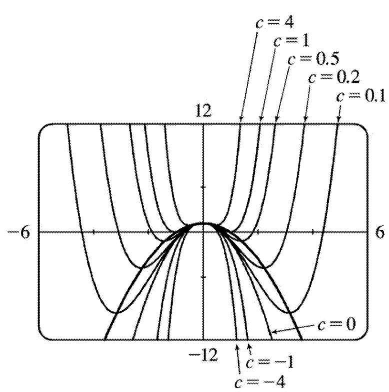


36. For $c=0$, there is no inflection point; the curve is CU everywhere. If c increases, the curve simply becomes steeper, and there are still no inflection points. If c starts at 0 and decreases, a slight upward bulge appears near $x=0$, so that there are two inflection points for any $c<0$. This can be seen algebraically by calculating the second derivative: $f(x)=x^4+cx^2+x\Rightarrow f'(x)=4x^3+2cx+1\Rightarrow$ $f''(x)=12x^2+2c$. Thus, $f''(x)>0$ when $c>0$. For $c<0$, there are inflection points when $x=\pm\sqrt{-\frac{1}{6}c}$. For $c=0$, the graph has one critical number, at the absolute minimum somewhere around $x=-0.6$. As c increases, the number of critical points does not change. If c instead decreases from 0 , we see that the graph eventually sprouts another local minimum, to the right of the origin, somewhere between $x=1$ and $x=2$. Consequently, there is also a maximum near $x=0$.
After a bit of experimentation, we find that at $c=-1.5$, there appear to be two critical numbers: the absolute minimum at about $x=-1$, and a horizontal tangent with no extremum at about $x=0.5$. For any c smaller than this there will be 3 critical points, as shown in the graphs with $c=-3$ and with $c=-5$. To prove this algebraically, we calculate $f'(x)=4x^3+2cx+1$. Now if we substitute our value of $c=-1.5$, the formula for $f'(x)$ becomes $4x^3-3x+1=(x+1)(2x-1)^2$. This has a double root at $x=\frac{1}{2}$, indicating that the function has two critical points: $x=-1$ and $x=\frac{1}{2}$, just as we had guessed from the graph.



37. **(a)** $f(x)=cx^4-2x^2+1$. For $c=0$, $f(x)=-2x^2+1$, a parabola whose vertex, $(0,1)$, is the absolute maximum. For $c>0$, $f(x)=cx^4-2x^2+1$ opens upward with two minimum points. As $c\to 0$, the minimum points spread apart and move downward; they are below the x –axis for $0<c<1$ and above for $c>1$. For $c<0$, the graph opens downward, and has an absolute maximum at $x=0$ and no local minimum.

(b) $f'(x)=4cx^3-4x=4cx\left(x^2-1/c\right)$ ($c\neq 0$). If $c\leq 0$, 0 is the only critical number. $f''(x)=12cx^2-4$, so $f''(0)=-4$ and there is a local maximum at $(0,f(0))=(0,1)$, which lies on $y=1-x^2$. If $c>0$, the critical numbers are 0 and $\pm 1/\sqrt{c}$. As before, there is a local maximum at $(0,f(0))=(0,1)$, which lies on $y=1-x^2$. $f''(\pm 1/\sqrt{c})=12-4=8>0$, so there is a local minimum at $x=\pm 1/\sqrt{c}$. Here $f(\pm 1/\sqrt{c})=c\left(1/c^2\right)-2/c+1=-1/c+1$. But $\left(\pm 1/\sqrt{c},-1/c+1\right)$ lies on $y=1-x^2$ since $1-(\pm 1/\sqrt{c})^2=1-1/c$.



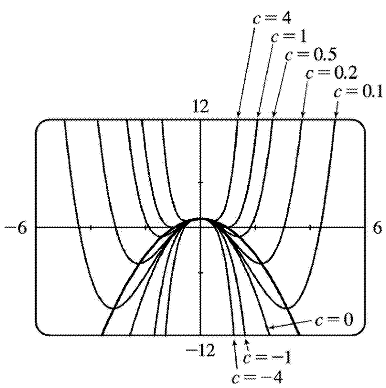
38. **(a)** $f(x)=2x^3+cx^2+2x\Rightarrow f'(x)=6x^2+2cx+2=2\left(3x^2+cx+1\right)$. $f'(x)=0\Leftrightarrow x=\dfrac{-c\pm\sqrt{c^2-12}}{6}$. So f has critical points $\Leftrightarrow c^2-12\geq 0\Leftrightarrow |c|\geq 2\sqrt{3}$. For $c=\pm 2\sqrt{3}$, $f'(x)\geq 0$ on $(-\infty,\infty)$, so f' does not change signs at $-c/6$, and there is no extremum. If $c^2-12>0$, then f' changes from positive to negative at $x=\dfrac{-c-\sqrt{c^2-12}}{6}$ and from negative to positive at $x=\dfrac{-c+\sqrt{c^2-12}}{6}$. So f has a local maximum at

$x=\dfrac{-c-\sqrt{c^2-12}}{6}$ and a local minimum at $x=\dfrac{-c+\sqrt{c^2-12}}{6}$.

(b) Let x_0 be a critical number for $f(x)$. Then $f'(x_0)=0\Rightarrow 3x_0^2+cx_0+1=0\Leftrightarrow c=\dfrac{-1-3x_0^2}{x_0}$. Now

$$f(x_0) = 2x_0^3+cx_0^2+2x_0=2x_0^3+x_0^2\left(\frac{-1-3x_0^2}{x_0}\right)+2x_0 = 2x_0^3-x_0-3x_0^3+2x_0=x_0-x_0^3$$

So the point is $\left(x_0,y_0\right)=\left(x_0,x_0-x_0^3\right)$; that is, the point lies on the curve $y=x-x^3$.



1. **(a)**

First Number	Second Number	Product
1	22	22
2	21	42
3	20	60
4	19	76
5	18	90
6	17	102
7	16	112
8	15	120
9	14	126
10	13	130
11	12	132

We needn't consider pairs where the first number is larger than the second, since we can just interchange the numbers in such cases. The answer appears to be 11 and 12 , but we have considered only integers in the table.

(b) Call the two numbers x and y . Then $x+y=23$, so $y=23-x$. Call the product P . Then $P=xy=x(23-x)=23x-x^2$, so we wish to maximize the function $P(x)=23x-x^2$. Since $P'(x)=23-2x$, we see that $P'(x)=0 \Leftrightarrow x=\frac{23}{2}=11.5$. Thus, the maximum value of P is $P(11.5)=(11.5)^2=132.25$ and it occurs when $x=y=11.5$.
Or: Note that $P''(x)=-2<0$ for all x , so P is everywhere concave downward and the local maximum at $x=11.5$ must be an absolute maximum.

2. The two numbers are $x+100$ and x . Minimize $f(x)=(x+100)x=x^2+100x$. $f'(x)=2x+100=0 \Rightarrow x=-50$. Since $f''(x)=2>0$, there is an absolute minimum at $x=-50$. The two numbers are 50 and -50 .

3. The two numbers are x and $\frac{100}{x}$, where $x>0$. Minimize $f(x)=x+\frac{100}{x}$. $f'(x)=1-\frac{100}{x^2}=\frac{x^2-100}{x^2}$.
The critical number is $x=10$. Since $f'(x)<0$ for $0<x<10$ and $f'(x)>0$ for $x>10$, there is an absolute minimum at $x=10$. The numbers are 10 and 10 .

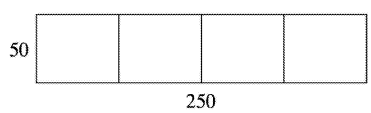
4. Let $x>0$ and let $f(x)=x+1/x$. We wish to minimize $f(x)$. Now
$f'(x)=1-\frac{1}{x^2}=\frac{1}{x^2}\left(x^2-1\right)=\frac{1}{x^2}(x+1)(x-1)$, so the only critical number in $(0,\infty)$ is 1 .

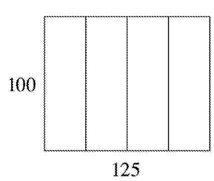
$f'(x)<0$ for $0<x<1$ and $f'(x)>0$ for $x>1$, so f has an absolute minimum at $x=1$, and $f(1)=2$.
Or: $f''(x)=2/x^3>0$ for all $x>0$, so f is concave upward everywhere and the critical point $(1,2)$ must correspond to a local minimum for f .

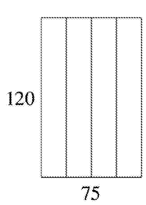
5. If the rectangle has dimensions x and y , then its perimeter is $2x+2y=100$ m, so $y=50-x$. Thus, the area is $A=xy=x(50-x)$. We wish to maximize the function $A(x)=x(50-x)=50x-x^2$, where $0<x<50$. Since $A'(x)=50-2x=-2(x-25)$, $A'(x)>0$ for $0<x<25$ and $A'(x)<0$ for $25<x<50$. Thus, A has an absolute maximum at $x=25$, and $A(25)=25^2=625$ m^2 . The dimensions of the rectangle that maximize its area are $x=y=25$ m. (The rectangle is a square.)

6. If the rectangle has dimensions x and y , then its area is $xy=1000$ m^2 , so $y=1000/x$. The perimeter $P=2x+2y=2x+2000/x$. We wish to minimize the function $P(x)=2x+2000/x$ for $x>0$.
$P'(x)=2-2000/x^2=\left(2/x^2\right)\left(x^2-1000\right)$, so the only critical number in the domain of P is $x=\sqrt{1000}$.
$P''(x)=4000/x^3>0$, so P is concave upward throughout its domain and $P\left(\sqrt{1000}\right)=4\sqrt{1000}$ is an absolute minimum value. The dimensions of the rectangle with minimal perimeter are
$x=y=\sqrt{1000}=10\sqrt{10}$ m.
(The rectangle is a square.)

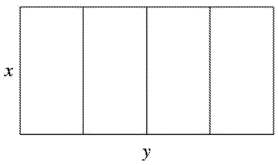
7. **(a)**







The areas of the three figures are 12 , 500 , 12 , 500 , and 9000 ft^2 . There appears to be a maximum area of at least 12 , 500 ft^2 .
(b) Let x denote the length of each of two sides and three dividers.
Let y denote the length of the other two sides.

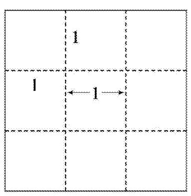


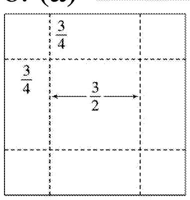
(c) Area A= length × width $= y \cdot x$

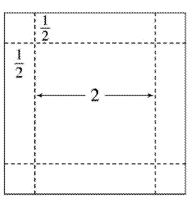
(d) Length of fencing $=750 \Rightarrow 5x+2y=750$

(e) $5x+2y=750 \Rightarrow y=375-\frac{5}{2}x \Rightarrow A(x)=\left(375-\frac{5}{2}x\right)x=375x-\frac{5}{2}x^2$

(f) $A'(x)=375-5x=0 \Rightarrow x=75$. Since $A''(x)=-5<0$ there is an absolute maximum when $x=75$. Then $y=\frac{375}{2}=187.5$. The largest area is $75\left(\frac{375}{2}\right)=14,062.5\ \text{ft}^2$. These values of x and y are between the values in the first and second figures in part (a). Our original estimate was low.

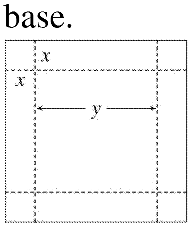
8. **(a)** 





The volumes of the resulting boxes are 1 , 1.6875 , and 2 ft^3 . There appears to be a maximum volume of at least 2 ft^3 .

(b) Let x denote the length of the side of the square being cut out. Let y denote the length of the base.



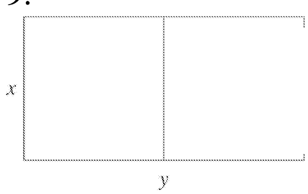
(c) Volume V= length × width × height $\Rightarrow V=y \cdot y \cdot x=xy^2$

(d) Length of cardboard $=3 \Rightarrow x+y+x=3 \Rightarrow y+2x=3$

(e) $y+2x=3 \Rightarrow y=3-2x \Rightarrow V(x)=x(3-2x)^2$

(f) $V(x)=x(3-2x)^2 \Rightarrow$
$V'(x)=x\cdot 2(3-2x)(-2)+(3-2x)^2\cdot 1=(3-2x)[-4x+(3-2x)]=(3-2x)(-6x+3)$,
so the critical numbers are $x=\frac{3}{2}$ and $x=\frac{1}{2}$. Now $0\le x\le \frac{3}{2}$ and $V(0)=V\left(\frac{3}{2}\right)=0$, so the maximum is $V\left(\frac{1}{2}\right)=\left(\frac{1}{2}\right)(2)^2=2\text{ ft}^3$, which is the value found from our third figure in part (a).

9.

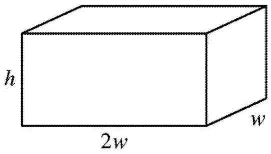


$xy=1.5\times 10^6$, so $y=1.5\times 10^6/x$. Minimize the amount of fencing, which is $3x+2y=3x+2\left(1.5\times 10^6/x\right)=3x+3\times 10^6/x=F(x)$. $F'(x)=3-3\times 10^6/x^2=3\left(x^2-10^6\right)/x^2$. The critical number is $x=10^3$ and $F'(x)<0$ for $0<x<10^3$ and $F'(x)>0$ if $x>10^3$, so the absolute minimum occurs when $x=10^3$ and $y=1.5\times 10^3$. The field should be 1000 feet by 1500 feet with the middle fence parallel to the short side of the field.

10. Let b be the length of the base of the box and h the height. The volume is $32,000=b^2h \Rightarrow h=32,000/b^2$. The surface area of the open box is $S=b^2+4hb=b^2+4(32,000/b^2)b=b^2+4\,(32,000)\,/b$. So $S'(b)=2b-4(32,000)/b^2=2\left(b^3-64,000\right)/b^2=0 \Leftrightarrow b=\sqrt[3]{64,000}=40$. This gives an absolute minimum since $S'(b)<0$ if $0<b<40$ and $S'(b)>0$ if $b>40$. The box should be $40\times 40\times 20$.

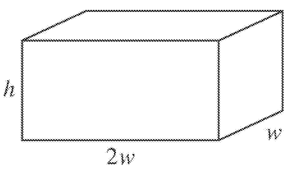
11. Let b be the length of the base of the box and h the height. The surface area is $1200=b^2+4hb \Rightarrow h=\left(1200-b^2\right)/(4b)$. The volume is $V=b^2h=b^2\left(1200-b^2\right)/4b=300b-b^3/4 \Rightarrow V'(b)=300-\frac{3}{4}b^2$. $V'(b)=0 \Rightarrow 300=\frac{3}{4}b^2 \Rightarrow b^2=400 \Rightarrow b=\sqrt{400}=20$. Since $V'(b)>0$ for $0<b<20$ and $V'(b)<0$ for $b>20$, there is an absolute maximum when $b=20$ by the First Derivative Test for Absolute Extreme Values (see page 280). If $b=20$, then $h=\left(1200-20^2\right)/(4\cdot 20)=10$, so the largest possible volume is $b^2h=(20)^2(10)=4000\text{ cm}^3$.

12.



$V=lwh \Rightarrow 10=(2w)(w)h=2w^2h$, so $h=5/w^2$. The cost is $10\left(2w^2\right)+6\left[2(2wh)+2(hw)\right]=20w^2+36wh$, so $C(w)=20w^2+36w\left(5/w^2\right)=20w^2+180/w$. $C'(w)=40w-180/w^2=40\left(w^3-\frac{9}{2}\right)/w^2 \Rightarrow w=\sqrt[3]{\frac{9}{2}}$ is the critical number. There is an absolute minimum for C when $w=\sqrt[3]{\frac{9}{2}}$ since $C'(w)<0$ for $0<w<\sqrt[3]{\frac{9}{2}}$ and $C'(w)>0$ for $w>\sqrt[3]{\frac{9}{2}}$. $C\left(\sqrt[3]{\frac{9}{2}}\right)=20\left(\sqrt[3]{\frac{9}{2}}\right)^2+\frac{180}{\sqrt[3]{9/2}}\approx\$163.54$.

13.



$10=(2w)(w)h=2w^2h$, so $h=5/w^2$. The cost is $C(w)=10\left(2w^2\right)+6\left[2(2wh)+2hw\right]+6\left(2w^2\right)=$ $32w^2+36wh=32w^2+180/w C'(w)=64w-180/w^2=4\left(16w^3-45\right)/w^2 \Rightarrow w=\sqrt[3]{\frac{45}{16}}$ is the critical number. $C'(w)<0$ for $0<w<\sqrt[3]{\frac{45}{16}}$ and $C'(w)>0$ for $w>\sqrt[3]{\frac{45}{16}}$. The minimum cost is $C\left(\sqrt[3]{\frac{45}{16}}\right)=32(2.8125)^{2/3}+180/\sqrt{2.8125}\approx\$191.28$.

14. **(a)** Let the rectangle have sides x and y and area A , so $A=xy$ or $y=A/x$. The problem is to minimize the perimeter $=2x+2y=2x+2A/x=P(x)$. Now $P'(x)=2-2A/x^2=2\left(x^2-A\right)/x^2$. So the critical number is $x=\sqrt{A}$. Since $P'(x)<0$ for $0<x<\sqrt{A}$ and $P'(x)>0$ for $x>\sqrt{A}$, there is an absolute minimum at $x=\sqrt{A}$. The sides of the rectangle are $\sqrt{A}$ and $A/\sqrt{A}=\sqrt{A}$, so the rectangle is a square.

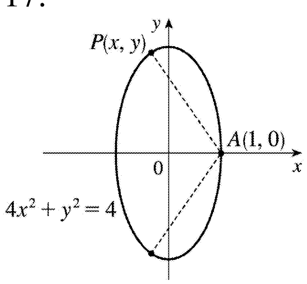
(b) Let p be the perimeter and x and y the lengths of the sides, so $p=2x+2y \Rightarrow 2y=p-2x \Rightarrow y=\frac{1}{2}p-x$. The area is $A(x)=x\left(\frac{1}{2}p-x\right)=\frac{1}{2}px-x^2$. Now $A'(x)=0 \Rightarrow \frac{1}{2}p-2x=0 \Rightarrow 2x=\frac{1}{2}p \Rightarrow x=\frac{1}{4}p$. Since $A''(x)=-2<0$, there is an absolute maximum for A when $x=\frac{1}{4}p$ by the Second Derivative Test. The sides of the rectangle are

$\frac{1}{4}p$ and $\frac{1}{2}p-\frac{1}{4}p=\frac{1}{4}p$, so the rectangle is a square.

15. The distance from a point (x,y) on the line $y=4x+7$ to the origin is $\sqrt{(x-0)^2+(y-0)^2}=\sqrt{x^2+y^2}$. However, it is easier to work with the *square* of the distance; that is, $D(x)=\left(\sqrt{x^2+y^2}\right)^2=x^2+y^2=x^2+(4x+7)^2$. Because the distance is positive, its minimum value will occur at the same point as the minimum value of D .
$D'(x)=2x+2(4x+7)(4)=34x+56$, so $D'(x)=0\Leftrightarrow x=-\frac{28}{17}$.
$D''(x)=34>0$, so D is concave upward for all x . Thus, D has an absolute minimum at $x=-\frac{28}{17}$. The point closest to the origin is $(x,y)=\left(-\frac{28}{17},4\left(-\frac{28}{17}\right)+7\right)=\left(-\frac{28}{17},\frac{7}{17}\right)$.

16. The square of the distance from a point (x,y) on the line $y=-6x+9$ to the point $(-3,1)$ is $D(x)=(x+3)^2+(y-1)^2=(x+3)^2+(-6x+8)^2=37x^2-90x+73$. $D'(x)=74x-90$, so $D'(x)=0\Leftrightarrow x=\frac{45}{37}$.
$D''(x)=74>0$, so D is concave upward for all x . Thus, D has an absolute minimum at $x=\frac{45}{37}$. The point on the line closest to $(-3,1)$ is $\left(\frac{45}{37},\frac{63}{37}\right)$.

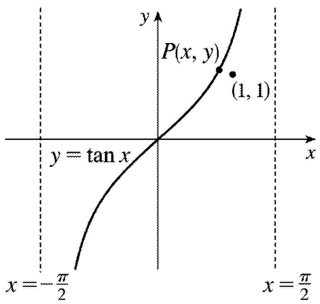
17.



From the figure, we see that there are two points that are farthest away from $A(1,0)$. The distance d from A to an arbitrary point $P(x,y)$ on the ellipse is $d=\sqrt{(x-1)^2+(y-0)^2}$ and the square of the distance is $S=d^2=x^2-2x+1+y^2=x^2-2x+1+\left(4-4x^2\right)=-3x^2-2x+5$. $S'=-6x-2$ and $S'=0\Rightarrow x=-\frac{1}{3}$. Now $S''=-6<0$, so we know that S has a maximum at $x=-\frac{1}{3}$. Since $-1\leq x\leq 1$, $S(-1)=4$, $S\left(-\frac{1}{3}\right)=\frac{16}{3}$, and $S(1)=0$, we see that the maximum distance is $\sqrt{\frac{16}{3}}$. The corresponding y –values are

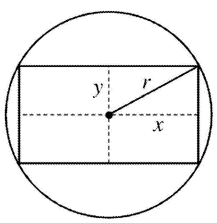
$y=\pm\sqrt{4-4\left(-\frac{1}{3}\right)^2}=\pm\sqrt{\frac{32}{9}}=\pm\frac{4}{3}\sqrt{2}\approx\pm1.89$. The points are $\left(-\frac{1}{3},\pm\frac{4}{3}\sqrt{2}\right)$.

18.



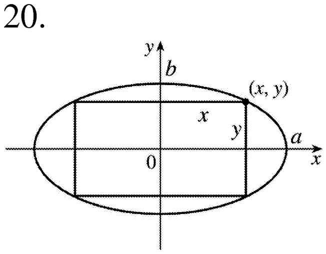
The distance d from $(1,1)$ to an arbitrary point P(x,y) on the curve $y=\tan x$ is $d=\sqrt{(x-1)^2+(y-1)^2}$ and the square of the distance is $S=d^2=(x-1)^2+(\tan x-1)^2$. $S'=2(x-1)+2(\tan x-1)\sec^2 x$. Graphing S' on $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ gives us a zero at $x\approx0.82$, and so $\tan x\approx1.08$. The point on $y=\tan x$ that is closest to $(1,1)$ is approximately $(0.82,1.08)$.

19.



The area of the rectangle is $(2x)(2y)=4xy$. Also $r^2=x^2+y^2$ so $y=\sqrt{r^2-x^2}$, so the area is $A(x)=4x\sqrt{r^2-x^2}$. Now $A'(x)=4\left(\sqrt{r^2-x^2}-\frac{x^2}{\sqrt{r^2-x^2}}\right)=4\frac{r^2-2x^2}{\sqrt{r^2-x^2}}$. The critical number is $x=\frac{1}{\sqrt{2}}r$. Clearly this gives a maximum. $y=\sqrt{r^2-\left(\frac{1}{\sqrt{2}}r\right)^2}=\sqrt{\frac{1}{2}r^2}=\frac{1}{\sqrt{2}}r=x$, which tells us that the rectangle is a square. The dimensions are $2x=\sqrt{2}r$ and $2y=\sqrt{2}r$.

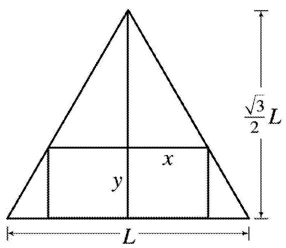
20.



The area of the rectangle is $(2x)(2y)=4xy$. Now

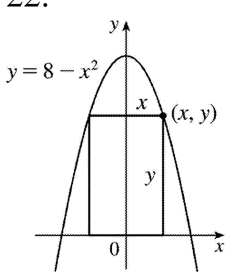
$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ gives $y=\frac{b}{a}\sqrt{a^2-x^2}$, so we maximize $A(x)=4\frac{b}{a}x\sqrt{a^2-x^2}$ $A'(x)=$ $\frac{4b}{a}\left[x\cdot\frac{1}{2}\left(a^2-x^2\right)^{-1/2}(-2x)+\left(a^2-x^2\right)^{1/2}\cdot 1\right]=\frac{4b}{a}\left(a^2-x^2\right)^{-1/2}\left[-x^2+a^2-x^2\right]=\frac{4b}{a\sqrt{a^2-x^2}}\left[a^2-2x^2\right]$ So the critical number is $x=\frac{1}{\sqrt{2}}a$, and this clearly gives a maximum. Then $y=\frac{1}{\sqrt{2}}b$, so the maximum area is $4\left(\frac{1}{\sqrt{2}}a\right)\left(\frac{1}{\sqrt{2}}b\right)=2ab$.

21.



The height h of the equilateral triangle with sides of length L is $\frac{\sqrt{3}}{2}L$, since $h^2+(L/2)^2=L^2\Rightarrow$ $h^2=L^2-\frac{1}{4}L^2=\frac{3}{4}L^2\Rightarrow h=\frac{\sqrt{3}}{2}L$. Using similar triangles, $\frac{\frac{\sqrt{3}}{2}L-y}{x}=\frac{\frac{\sqrt{3}}{2}L}{L/2}=\sqrt{3}\Rightarrow\sqrt{3}x=\frac{\sqrt{3}}{2}L-y\Rightarrow$ $y=\frac{\sqrt{3}}{2}L-\sqrt{3}x\Rightarrow y=\frac{\sqrt{3}}{2}(L-2x)$. The area of the inscribed rectangle is $A(x)=(2x)y=\sqrt{3}x(L-2x)=\sqrt{3}Lx-2\sqrt{3}x^2$, where $0\leq x\leq L/2$. Now $0=A'(x)=\sqrt{3}L-4\sqrt{3}x\Rightarrow$ $x=\sqrt{3}L/\left(4\sqrt{3}\right)=L/4$. Since $A(0)=A(L/2)=0$, the maximum occurs when $x=L/4$, and $y=\frac{\sqrt{3}}{2}L-\frac{\sqrt{3}}{4}L=\frac{\sqrt{3}}{4}L$, so the dimensions are $L/2$ and $\frac{\sqrt{3}}{4}L$.

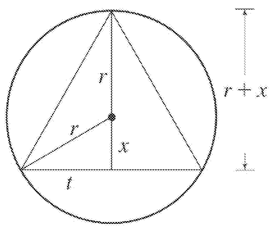
22.



The rectangle has area $A(x)=2xy=2x\left(8-x^2\right)=16x-2x^3$, where $0\leq x\leq 2\sqrt{2}$. Now $A'(x)=16-6x^2=0\Rightarrow$

$x=2\sqrt{\frac{2}{3}}$. Since $A(0)=A(2\sqrt{2})=0$, there is a maximum when $x=2\sqrt{\frac{2}{3}}$. Then $y=\frac{16}{3}$, so the rectangle has dimensions $4\sqrt{\frac{2}{3}}$ and $\frac{16}{3}$.

23.

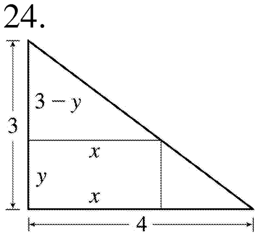


The area of the triangle is $A(x)=\frac{1}{2}(2t)(r+x)=t(r+x)=\sqrt{r^2-x^2}\,(r+x)$. 0=

$A'(x)=r\,\frac{-2x}{2\sqrt{r^2-x^2}}+\sqrt{r^2-x^2}+x\,\frac{-2x}{2\sqrt{r^2-x^2}}=-\frac{x^2+rx}{\sqrt{r^2-x^2}}+\sqrt{r^2-x^2}\Rightarrow\frac{x^2+rx}{\sqrt{r^2-x^2}}=\sqrt{r^2-x^2}\Rightarrow x^2+rx=r^2-x^2$

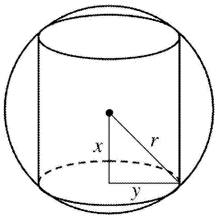
$\Rightarrow 0=2x^2+rx-r^2=(2x-r)(x+r)\Rightarrow x=\frac{1}{2}r$ or $x=-r$. Now $A(r)=0=A(-r)\Rightarrow$ the maximum occurs where $x=\frac{1}{2}r$, so the triangle has height $r+\frac{1}{2}r=\frac{3}{2}r$ and base $2\sqrt{r^2-\left(\frac{1}{2}r\right)^2}=2\sqrt{\frac{3}{4}r^2}=\sqrt{3}\,r$.

24.



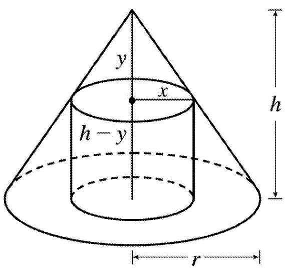
The rectangle has area xy . By similar triangles $\frac{3-y}{x}=\frac{3}{4}\Rightarrow -4y+12=3x$ or $y=-\frac{3}{4}x+3$. So the area is $A(x)=x\left(-\frac{3}{4}x+3\right)=-\frac{3}{4}x^2+3x$ where $0\leq x\leq 4$. Now $0=A'(x)=-\frac{3}{2}x+3\Rightarrow x=2$ and $y=\frac{3}{2}$. Since $A(0)=A(4)=0$, the maximum area is $A(2)=2\left(\frac{3}{2}\right)=3\text{ cm}^2$.

25.



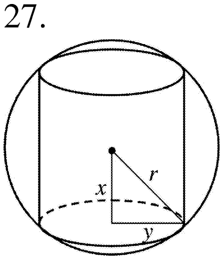
The cylinder has volume $V=\pi y^2(2x)$. Also $x^2+y^2=r^2 \Rightarrow y^2=r^2-x^2$, so $V(x)=\pi\left(r^2-x^2\right)(2x)=2\pi\left(r^2x-x^3\right)$, where $0\leq x\leq r$. $V'(x)=2\pi\left(r^2-3x^2\right)=0\Rightarrow x=r/\sqrt{3}$. Now $V(0)=V(r)=0$, so there is a maximum when $x=r/\sqrt{3}$ and $V\left(r/\sqrt{3}\right)=\pi\left(r^2-r^2/3\right)\left(2r/\sqrt{3}\right)=4\pi r^3/\left(3\sqrt{3}\right)$.

26.



By similar triangles, $y/x=h/r$, so $y=hx/r$. The volume of the cylinder is $\pi x^2(h-y)=\pi hx^2-(\pi h/r)x^3=V(x)$. Now $V'(x)=2\pi hx-(3\pi h/r)x^2=\pi hx(2-3x/r)$. So $V'(x)=0\Rightarrow x=0$ or $x=\frac{2}{3}r$. The maximum clearly occurs when $x=\frac{2}{3}r$ and then the volume is $\pi hx^2-(\pi h/r)x^3=\pi hx^2(1-x/r)=\pi\left(\frac{2}{3}r\right)^2h\left(1-\frac{2}{3}\right)=\frac{4}{27}\pi r^2h$.

27.



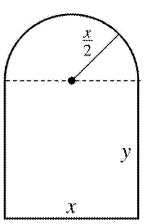
The cylinder has surface area 2(area of the base)+(lateral surface area)$=2\pi$(radius)$^2+2\pi$(radius)(height)$=2\pi y^2+2\pi y(2x)$. Now $x^2+y^2=r^2\Rightarrow y^2=r^2-x^2\Rightarrow y=\sqrt{r^2-x^2}$, so the surface area is $S(x)=2\pi\left(r^2-x^2\right)+4\pi x\sqrt{r^2-x^2}$, $0\leq x\leq r=2\pi r^2-2\pi x^2+4\pi\left(x\sqrt{r^2-x^2}\right)$ Thus, $S'(x)$ $=0-4\pi x+4\pi\left[x\cdot\frac{1}{2}\left(r^2-x^2\right)^{-1/2}(-2x)+\left(r^2-x^2\right)^{1/2}\cdot 1\right]$

$=4\pi\left[-x-\frac{x^2}{\sqrt{r^2-x^2}}+\sqrt{r^2-x^2}\right]=4\pi\cdot\frac{-x\sqrt{r^2-x^2}-x^2+r^2-x^2}{\sqrt{r^2-x^2}}$

$S'(x)=0\Rightarrow x\sqrt{r^2-x^2}=r^2-2x^2\ (\ *\ )\Rightarrow\left(x\sqrt{r^2-x^2}\right)^2=\left(r^2-2x^2\right)^2\Rightarrow x^2\left(r^2-x^2\right)=r^4-4r^2x^2+4x^4\Rightarrow$
$r^2x^2-x^4=r^4-4r^2x^2+4x^4\Rightarrow 5x^4-5r^2x^2+r^4=0$. This is a quadratic equation in x^2 . By the quadratic formula, $x^2=\frac{5\pm\sqrt{5}}{10}r^2$, but we reject the root with the + sign since it doesn't satisfy (*). So $x=\sqrt{\frac{5-\sqrt{5}}{10}}\,r$. Since $S(0)=S(r)=0$, the maximum surface area occurs at the critical number and $x^2=\frac{5-\sqrt{5}}{10}r^2\Rightarrow y^2=r^2-\frac{5-\sqrt{5}}{10}r^2=\frac{5+\sqrt{5}}{10}r^2\Rightarrow$ the surface area is

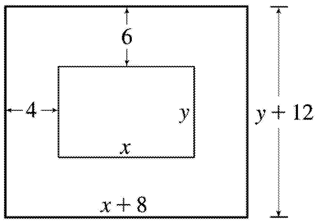
$2\pi\left(\frac{5+\sqrt{5}}{10}\right)r^2+4\pi\sqrt{\frac{5-\sqrt{5}}{10}}\sqrt{\frac{5+\sqrt{5}}{10}}r^2=\pi r^2\left[2\cdot\frac{5+\sqrt{5}}{10}+4\frac{\sqrt{(5-\sqrt{5})(5+\sqrt{5})}}{10}\right]$
$=\pi r^2\left[\frac{5+\sqrt{5}}{5}+\frac{2\sqrt{20}}{5}\right]=\pi r^2\left[\frac{5+\sqrt{5}+2\cdot 2\sqrt{5}}{5}\right]=\pi r^2\left[\frac{5+5\sqrt{5}}{5}\right]=\pi r^2\left(1+\sqrt{5}\right)$.

28.



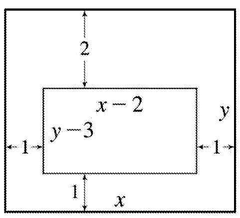
Perimeter=30$\Rightarrow 2y+x+\pi\left(\frac{x}{2}\right)=30\Rightarrow y=\frac{1}{2}\left(30-x-\frac{\pi x}{2}\right)=15-\frac{x}{2}-\frac{\pi x}{4}$. The area is the area of the rectangle plus the area of the semicircle, or $xy+\frac{1}{2}\pi\left(\frac{x}{2}\right)^2$, so
$A(x)=x\left(15-\frac{x}{2}-\frac{\pi x}{4}\right)+\frac{1}{8}\pi x^2=15x-\frac{1}{2}x^2-\frac{\pi}{8}x^2$. $A'(x)=15-\left(1+\frac{\pi}{4}\right)x=0\Rightarrow x=\frac{15}{1+\pi/4}=\frac{60}{4+\pi}$.
$A''(x)=-\left(1+\frac{\pi}{4}\right)<0$, so this gives a maximum. The dimensions are $x=\frac{60}{4+\pi}$ ft and
$y=15-\frac{30}{4+\pi}-\frac{15\pi}{4+\pi}=\frac{60+15\pi-30-15\pi}{4+\pi}=\frac{30}{4+\pi}$ ft, so the height of the rectangle is half the base.

29.



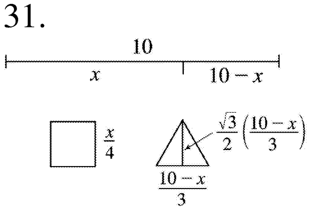
$xy=384 \Rightarrow y=384/x$. Total area is $A(x)=(8+x)(12+384/x)=12(40+x+256/x)$, so $A'(x)=12\left(1-256/x^2\right)=0 \Rightarrow x=16$. There is an absolute minimum when $x=16$ since $A'(x)<0$ for $0<x<16$ and $A'(x)>0$ for $x>16$. When $x=16$, $y=384/16=24$, so the dimensions are 24 cm and 36 cm.

30.



$xy=180$, so $y=180/x$. The printed area is $(x-2)(y-3)=(x-2)(180/x-3)=186-3x-360/x=A(x)$. $A'(x)=-3+360/x^2=0$ when $x^2=120 \Rightarrow x=2\sqrt{30}$. This gives an absolute maximum since $A'(x)>0$ for $0<x<2\sqrt{30}$ and $A'(x)<0$ for $x>2\sqrt{30}$. When $x=2\sqrt{30}$, $y=180/(2\sqrt{30})$, so the dimensions are $2\sqrt{30}$ in. and $90/\sqrt{30}$ in.

31.



Let x be the length of the wire used for the square. The total area is

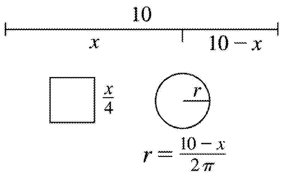
$$A(x)=\left(\frac{x}{4}\right)^2+\frac{1}{2}\left(\frac{10-x}{3}\right)\frac{\sqrt{3}}{2}\left(\frac{10-x}{3}\right)=\frac{1}{16}x^2+\frac{\sqrt{3}}{36}(10-x)^2,\ 0\leq x\leq 10$$

$A'(x)=\frac{1}{8}x-\frac{\sqrt{3}}{18}(10-x)=0 \Leftrightarrow \frac{9}{72}x+\frac{4\sqrt{3}}{72}x-\frac{40\sqrt{3}}{72}=0 \Leftrightarrow x=\frac{40\sqrt{3}}{9+4\sqrt{3}}$. Now $A(0)=\left(\frac{\sqrt{3}}{36}\right)100\approx 4.81$, $A(10)=\frac{100}{16}=6.25$ and $A\left(\frac{40\sqrt{3}}{9+4\sqrt{3}}\right)\approx 2.72$, so

(a) The maximum area occurs when $x=10$ m, and all the wire is used for the square.

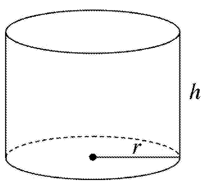
(b) The minimum area occurs when $x=\frac{40\sqrt{3}}{9+4\sqrt{3}}\approx 4.35$ m.

32.



Total area is $A(x)=\left(\frac{x}{4}\right)^2+\pi\left(\frac{10-x}{2\pi}\right)^2=\frac{x^2}{16}+\frac{(10-x)^2}{4\pi}$, $0\leq x\leq 10$.
$A'(x)=\frac{x}{8}-\frac{10-x}{2\pi}=\left(\frac{1}{2\pi}+\frac{1}{8}\right)x-\frac{5}{\pi}=0\Rightarrow x=40/(4+\pi)$. $A(0)=25/\pi\approx 7.96$, $A(10)=6.25$, and $A(40/(4+\pi))\approx 3.5$, so the maximum occurs when $x=0$ m and the minimum occurs when $x=40/(4+\pi)$ m.

33.

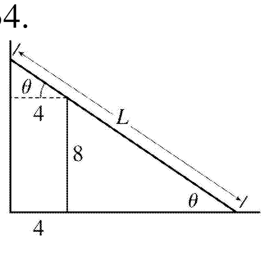


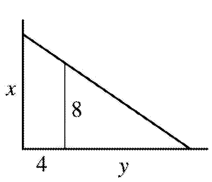
The volume is $V=\pi r^2 h$ and the surface area is $S(r)=\pi r^2+2\pi rh=\pi r^2+2\pi r\left(\frac{V}{\pi r^2}\right)=\pi r^2+\frac{2V}{r}$.

$S'(r)=2\pi r-\frac{2V}{r^2}=0\Rightarrow 2\pi r^3=2V\Rightarrow r=\sqrt[3]{\frac{V}{\pi}}$ cm.

This gives an absolute minimum since $S'(r)<0$ for $0<r<\sqrt[3]{\frac{V}{\pi}}$ and $S'(r)>0$ for $r>\sqrt[3]{\frac{V}{\pi}}$. When $r=\sqrt[3]{\frac{V}{\pi}}$, $h=\frac{V}{\pi r^2}=\frac{V}{\pi(V/\pi)^{2/3}}=\sqrt[3]{\frac{V}{\pi}}$ cm.

34.



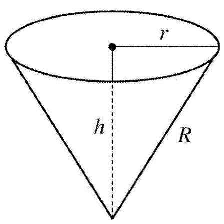


$L=8\theta+4\sec\theta$, $0<\theta<\frac{\pi}{2}$, $\frac{dL}{d\theta}=-8\theta\ \cot\theta+4\sec\theta\ \tan\theta=0$ when $\sec\theta\ \tan\theta=2\theta\ \cot\theta\Leftrightarrow\tan^{3}\theta=2\Leftrightarrow$ $\tan\theta=\sqrt[3]{2}\Leftrightarrow\theta=\tan^{-1}\sqrt[3]{2}$.

$dL/d\theta<0$ when $0<\theta<\tan^{-1}\sqrt[3]{2}$, $dL/d\theta>0$ when $\tan^{-1}\sqrt[3]{2}<\theta<\frac{\pi}{2}$, so L has an absolute minimum when $\theta=\tan^{-1}\sqrt[3]{2}$, and the shortest ladder has length $L=8\frac{\sqrt{1+2^{2/3}}}{2^{1/3}}+4\sqrt{1+2^{2/3}}\approx16.65$ ft.

Another method: Minimize $L^2=x^2+(4+y)^2$, where $\frac{x}{4+y}=\frac{8}{y}$.

35.

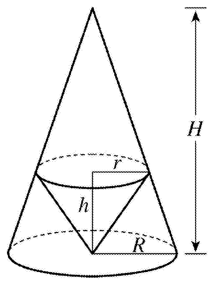


$h^2+r^2=R^2\Rightarrow V=\frac{\pi}{3}r^2h=\frac{\pi}{3}\left(R^2-h^2\right)h=\frac{\pi}{3}\left(R^2h-h^3\right)$. $V'(h)=\frac{\pi}{3}\left(R^2-3h^2\right)=0$ when $h=\frac{1}{\sqrt{3}}R$. This gives an absolute maximum, since $V'(h)>0$ for $0<h<\frac{1}{\sqrt{3}}R$ and $V'(h)<0$ for $h>\frac{1}{\sqrt{3}}R$. The maximum volume is $V\left(\frac{1}{\sqrt{3}}R\right)=\frac{\pi}{3}\left(\frac{1}{\sqrt{3}}R^3-\frac{1}{3\sqrt{3}}R^3\right)=\frac{2}{9\sqrt{3}}\pi R^3$.

36. The volume and surface area of a cone with radius r and height h are given by $V=\frac{1}{3}\pi r^2h$ and $S=\pi r\sqrt{r^2+h^2}$. We'll minimize $A=S^2$ subject to $V=27$. $V=27\Rightarrow\frac{1}{3}\pi r^2h=27\Rightarrow r^2=\frac{81}{\pi h}$ **(1)** .

$A=\pi^2r^2(r^2+h^2)=\pi^2\left(\frac{81}{\pi h}\right)\left(\frac{81}{\pi h}+h^2\right)=\frac{81^2}{h^2}+81\pi h$, so $A'=0\Rightarrow\frac{-2\cdot81^2}{h^3}+81\pi=0\Rightarrow81\pi=\frac{2\cdot81^2}{h^3}\Rightarrow$ $h^3=\frac{162}{\pi}\Rightarrow h=\sqrt[3]{\frac{162}{\pi}}=3\sqrt[3]{\frac{6}{\pi}}\approx3.722$. From **(1)** , $r^2=\frac{81}{\pi h}=\frac{81}{\pi\cdot3\sqrt[3]{6/\pi}}=\frac{27}{\sqrt[3]{6\pi^2}}\Rightarrow$ $r=\frac{3\sqrt{3}}{\sqrt[6]{6\pi^2}}\approx2.632$. $A''=6\cdot81^2/h^4>0$, so A and hence S has an absolute minimum at these values of r and h .

37.



By similar triangles, $\frac{H}{R}=\frac{H-h}{r}$ **(1)** . The volume of the inner cone is $V=\frac{1}{3}\pi r^2 h$, so we'll solve **(1)** for h . $\frac{Hr}{R}=H-h \Rightarrow h=H-\frac{Hr}{R}=\frac{HR-Hr}{R}=\frac{H}{R}(R-r)$ **(2)** .

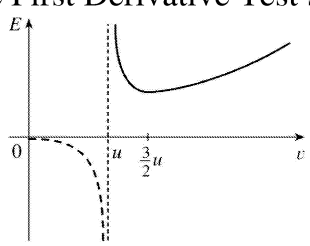
Thus, $V(r)=\frac{\pi}{3}r^2\cdot\frac{H}{R}(R-r)=\frac{\pi H}{3R}(Rr^2-r^3) \Rightarrow V'(r)=\frac{\pi H}{3R}(2Rr-3r^2)=\frac{\pi H}{3R}r(2R-3r)$.

$V'(r)=0 \Rightarrow r=0$ or $2R=3r \Rightarrow r=\frac{2}{3}R$ and from **(2)** , $h=\frac{H}{R}\left(R-\frac{2}{3}R\right)=\frac{H}{R}\left(\frac{1}{3}R\right)=\frac{1}{3}H$.

$V'(r)$ changes from positive to negative at $r=\frac{2}{3}R$, so the inner cone has a maximum volume of $V=\frac{1}{3}\pi r^2 h=\frac{1}{3}\pi\left(\frac{2}{3}R\right)^2\left(\frac{1}{3}H\right)=\frac{4}{27}\cdot\frac{1}{3}\pi R^2 H$, which is approximately 15% of the volume of the larger cone.

38. **(a)** $E(v)=\frac{aLv^3}{v-u} \Rightarrow E'(v)=aL\frac{(v-u)3v^2-v^3}{(v-u)^2}=0$ when $2v^3=3uv^2 \Rightarrow 2v=3u \Rightarrow v=\frac{3}{2}u$.

The First Derivative Test shows that this value of v gives the minimum value of E .

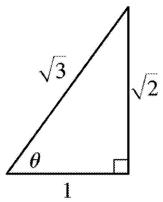
(b) 

39. $S=6sh-\frac{3}{2}s^2\cot\theta+3s^2\frac{\sqrt{3}}{2}\ \theta$

(a) $\frac{dS}{d\theta}=\frac{3}{2}s^2\ {}^2\theta-3s^2\frac{\sqrt{3}}{2}\ \theta\cot\theta$ or $\frac{3}{2}s^2\ \theta\left(\ \theta-\sqrt{3}\cot\theta\right)$.

(b) $\frac{dS}{d\theta}=0$ when $\theta-\sqrt{3}\cot\theta=0 \Rightarrow \frac{1}{\sin\theta}-\sqrt{3}\frac{\cos\theta}{\sin\theta}=0 \Rightarrow \cos\theta=\frac{1}{\sqrt{3}}$. The First Derivative Test shows that the minimum surface area occurs when $\theta=\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)\approx 55^\circ$.

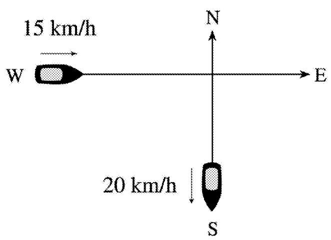
(c)



If $\cos\theta = \dfrac{1}{\sqrt{3}}$, then $\cot\theta = \dfrac{1}{\sqrt{2}}$ and $\theta = \dfrac{\sqrt{3}}{\sqrt{2}}$, so the surface area is $S <=$

$$6sh-\frac{3}{2}s^2\frac{1}{\sqrt{2}}+3s^2\frac{\sqrt{3}}{2}\frac{\sqrt{3}}{\sqrt{2}}=6sh-\frac{3}{2\sqrt{2}}s^2+\frac{9}{2\sqrt{2}}s^2=6sh+\frac{6}{2\sqrt{2}}s^2=6s\left(h+\frac{1}{2\sqrt{2}}s\right)$$

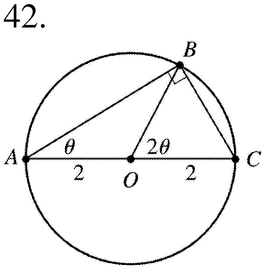
40.



Let t be the time, in hours, after 2:00 P.M. The position of the boat heading south at time t is $(0,-20t)$. The position of the boat heading east at time t is $(-15+15t,0)$. If $D(t)$ is the distance between the boats at time t , we minimize $f(t)=[D(t)]^2=20^2t^2+15^2(t-1)^2$. $f'(t)=800t+450(t-1)=1250t-450=0$ when $t=\dfrac{450}{1250}=0.36$ h. 0.36 h $\times\ \dfrac{60\text{min}}{\text{h}}=21.6$ min $=21$ min 36 s. Since $f''(t)>0$, this gives a minimum, so the boats are closest together at 2:21:36 P.M.

41. Here $T(x)=\dfrac{\sqrt{x^2+25}}{6}+\dfrac{5-x}{8}$, $0\le x\le 5 \Rightarrow T'(x)=\dfrac{x}{6\sqrt{x^2+25}}-\dfrac{1}{8}=0 \Leftrightarrow 8x=6\sqrt{x^2+25} \Leftrightarrow$ $16x^2=9\left(x^2+25\right) \Leftrightarrow x=\dfrac{15}{\sqrt{7}}$. But $\dfrac{15}{\sqrt{7}}>5$, so T has no critical number. Since $T(0)\approx 1.46$ and $T(5)\approx 1.18$, he should row directly to B .

42.

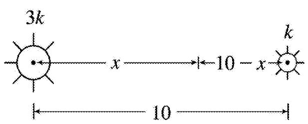


In isosceles triangle AOB , $\angle O=180^\circ-\theta-\theta$, so $\angle BOC=2\theta$. The distance rowed is $4\cos\theta$ while the distance walked is the length of arc $BC=2(2\theta)=4\theta$. The time taken is given by

$T(\theta)=\dfrac{4\cos\theta}{2}+\dfrac{4\theta}{4}=2\cos\theta+\theta$, $0\le\theta\le\dfrac{\pi}{2}$. $T'(\theta)=-2\sin\theta+1=0\Leftrightarrow\sin\theta=\dfrac{1}{2}\Rightarrow\theta=\dfrac{\pi}{6}$.

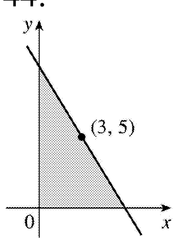
Check the value of T at $\theta=\dfrac{\pi}{6}$ and at the endpoints of the domain of T ; that is, $\theta=0$ and $\theta=\dfrac{\pi}{2}$. $T(0)=2$, $T\left(\dfrac{\pi}{6}\right)=\sqrt{3}+\dfrac{\pi}{6}\approx 2.26$, and $T\left(\dfrac{\pi}{2}\right)=\dfrac{\pi}{2}\approx 1.57$. Therefore, the minimum value of T is $\dfrac{\pi}{2}$ when $\theta=\dfrac{\pi}{2}$; that is, the woman should walk all the way. Note that $T''(\theta)=-2\cos\theta<0$ for $0\le\theta<\dfrac{\pi}{2}$, so $\theta=\dfrac{\pi}{6}$ gives a maximum time.

43.



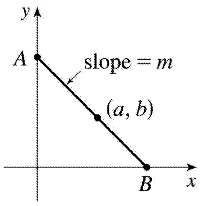
The total illumination is $I(x)=\dfrac{3k}{x^2}+\dfrac{k}{(10-x)^2}$, $0<x<10$. Then $I'(x)=\dfrac{-6k}{x^3}+\dfrac{2k}{(10-x)^3}=0\Rightarrow$ $6k(10-x)^3=2kx^3\Rightarrow 3(10-x)^3=x^3\Rightarrow\sqrt[3]{3}(10-x)=x\Rightarrow 10\sqrt[3]{3}-\sqrt[3]{3}x=x\Rightarrow 10\sqrt[3]{3}=x+\sqrt[3]{3}x\Rightarrow$ $10\sqrt[3]{3}=\left(1+\sqrt[3]{3}\right)x\Rightarrow x=\dfrac{10\sqrt[3]{3}}{1+\sqrt[3]{3}}\approx 5.9$ ft. This gives a minimum since $I''(x)>0$ for $0<x<10$.

44.



The line with slope m (where $m<0$) through $(3,5)$ has equation $y-5=m(x-3)$ or $y=mx+(5-3m)$. The $y-$ intercept is $5-3m$ and the $x-$ intercept is $-5/m+3$. So the triangle has area $A(m)=\dfrac{1}{2}(5-3m)(-5/m+3)=15-25/(2m)-\dfrac{9}{2}m$. Now $A'(m)=\dfrac{25}{2m^2}-\dfrac{9}{2}=0\Leftrightarrow m^2=\dfrac{25}{9}\Rightarrow m=-\dfrac{5}{3}$ (since $m<0$). $A''(m)=-\dfrac{25}{m^3}>0$, so there is an absolute minimum when $m=-\dfrac{5}{3}$. Thus, an equation of the line is $y-5=-\dfrac{5}{3}(x-3)$ or $y=-\dfrac{5}{3}x+10$.

45.



Every line segment in the first quadrant passing through (a,b) with endpoints on the x – and y –axes satisfies an equation of the form $y-b=m(x-a)$, where $m<0$. By setting $x=0$ and then $y=0$, we find its endpoints, $A(0,b-am)$ and $B\left(a-\frac{b}{m},0\right)$. The distance d from A to B is given by

$$d=\sqrt{\left[\left(a-\frac{b}{m}\right)-0\right]^2+[0-(b-am)]^2}\ .$$

It follows that the square of the length of the line segment, as a function of m , is given by

$$S(m)=\left(a-\frac{b}{m}\right)^2+(am-b)^2=a^2-\frac{2ab}{m}+\frac{b^2}{m^2}+a^2m^2-2abm+b^2\ .$$ Thus,

$$S'(m)=\frac{2ab}{m^2}-\frac{2b^2}{m^3}+2a^2m-2ab=\frac{2}{m^3}\left(abm-b^2+a^2m^4-abm^3\right)$$
$$=\frac{2}{m^3}\left[b(am-b)+am^3(am-b)\right]=\frac{2}{m^3}(am-b)\left(b+am^3\right)$$

Thus, $S'(m)=0\Leftrightarrow m=b/a$ or $m=-\sqrt[3]{\frac{b}{a}}$. Since $b/a>0$ and $m<0$, m must equal $-\sqrt[3]{\frac{b}{a}}$. Since $\frac{2}{m^3}<0$, we see that $S'(m)<0$ for $m<-\sqrt[3]{\frac{b}{a}}$ and $S'(m)>0$ for $m>-\sqrt[3]{\frac{b}{a}}$. Thus, S has its absolute minimum value when $m=-\sqrt[3]{\frac{b}{a}}$. That value is

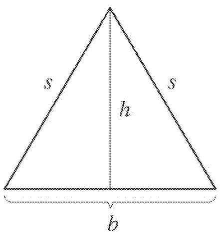
$$S\left(-\sqrt[3]{\frac{b}{a}}\right)=\left(a+b\sqrt[3]{\frac{a}{b}}\right)^2+\left(-a\sqrt[3]{\frac{b}{a}}-b\right)^2=\left(a+\sqrt[3]{ab^2}\right)^2+\left(\sqrt[3]{a^2b}+b\right)^2$$
$$=a^2+2a^{4/3}b^{2/3}+a^{2/3}b^{4/3}+a^{4/3}b^{2/3}+2a^{2/3}b^{4/3}+b^2=a^2+3a^{4/3}b^{2/3}+3a^{2/3}b^{4/3}+b^2$$

The last expression is of the form $x^3+3x^2y+3xy^2+y^3\left[=(x+y)^3\right]$ with $x=a^{2/3}$ and $y=b^{2/3}$, so we can write it as $(a^{2/3}+b^{2/3})^3$ and the shortest such line segment has length $\sqrt{S}=(a^{2/3}+b^{2/3})^{3/2}$.

46. $y=1+40x^3-3x^5\Rightarrow y'=120x^2-15x^4$, so the tangent line to the curve at $x=a$ has slope $m(a)=120a^2-15a^4$. Now $m'(a)=240a-60a^3=-60a(a^2-4)=-60a(a+2)(a-2)$, so $m'(a)>0$ for $a<-2$,

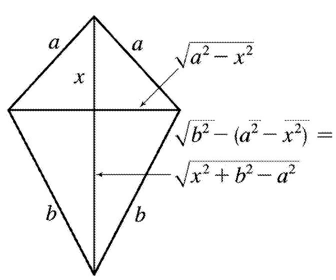
and $0<a<2$, and $m'(a)<0$ for $-2<a<0$ and $a>2$. Thus, m is increasing on $(-\infty,-2)$, decreasing on $(-2,0)$, increasing on $(0,2)$, and decreasing on $(2,\infty)$. Clearly, $m(a)\to-\infty$ as $a\to\pm\infty$, so the maximum value of $m(a)$ must be one of the two local maxima, $m(-2)$ or $m(2)$. But both $m(-2)$ and $m(2)$ equal $120\cdot 2^2-15\cdot 2^4=480-240=240$. So 240 is the largest slope, and it occurs at the points $(-2,-223)$ and $(2,225)$. *Note:* $a=0$ corresponds to a local *minimum* of m .

47.



Here $s^2=h^2+b^2/4$, so $h^2=s^2-b^2/4$. The area is $A=\frac{1}{2}b\sqrt{s^2-b^2/4}$. Let the perimeter be p , so $2s+b=p$ or $s=(p-b)/2\Rightarrow A(b)=\frac{1}{2}b\sqrt{(p-b)^2/4-b^2/4}=b\sqrt{p^2-2pb}/4$. Now

$A'(b)=\frac{\sqrt{p^2-2pb}}{4}-\frac{bp/4}{\sqrt{p^2-2pb}}=\frac{-3pb+p^2}{4\sqrt{p^2-2pb}}$. Therefore, $A'(b)=0\Rightarrow-3pb+p^2=0\Rightarrow b=p/3$. Since $A'(b)>0$ for $b<p/3$ and $A'(b)<0$ for $b>p/3$, there is an absolute maximum when $b=p/3$. But then $2s+p/3=p$, so $s=p/3\Rightarrow s=b\Rightarrow$ the triangle is equilateral.



48.

See the figure. The area is given by

$A(x)=\frac{1}{2}\left(2\sqrt{a^2-x^2}\right)x+\frac{1}{2}\left(2\sqrt{a^2-x^2}\right)\left(\sqrt{x^2+b^2-a^2}\right)=\sqrt{a^2-x^2}\left(x+\sqrt{x^2+b^2-a^2}\right)$ for $0\le x\le a$. Now

$A'(x)=\sqrt{a^2-x^2}\left(1+\frac{x}{\sqrt{x^2+b^2-a^2}}\right)+\left(x+\sqrt{x^2+b^2-a^2}\right)\frac{-x}{\sqrt{a^2-x^2}}=0\Leftrightarrow$

$\frac{x}{\sqrt{a^2-x^2}}\left(x+\sqrt{x^2+b^2-a^2}\right)=\sqrt{a^2-x^2}\left(\frac{x+\sqrt{x^2+b^2-a^2}}{\sqrt{x^2+b^2-a^2}}\right)$.

Except for the trivial case where $x=0$, $a=b$ and $A(x)=0$, we have

$x+\sqrt{x^2+b^2-a^2}>0$. Hence, cancelling this factor gives $\dfrac{x}{\sqrt{a^2-x^2}}=\dfrac{\sqrt{a^2-x^2}}{\sqrt{x^2+b^2-a^2}}\Rightarrow x\sqrt{x^2+b^2-a^2}=a^2-x^2$

$\Rightarrow x^2\left(x^2+b^2-a^2\right)=a^4-2a^2x^2+x^4\Rightarrow x^2\left(b^2-a^2\right)=a^4-2a^2x^2\Rightarrow x^2\left(b^2+a^2\right)=a^4\Rightarrow x=\dfrac{a^2}{\sqrt{a^2+b^2}}$.

Now we must check the value of A at this point as well as at the endpoints of the domain to see which gives the maximum value. $A(0)=a\sqrt{b^2-a^2}$, $A(a)=0$ and

$$A\left(\frac{a^2}{\sqrt{a^2+b^2}}\right)=\sqrt{a^2-\left(\frac{a^2}{\sqrt{a^2+b^2}}\right)^2}\left[\frac{a^2}{\sqrt{a^2+b^2}}+\sqrt{\left(\frac{a^2}{\sqrt{a^2+b^2}}\right)^2+b^2-a^2}\right]=$$

$$\frac{ab}{\sqrt{a^2+b^2}}\left[\frac{a^2}{\sqrt{a^2+b^2}}+\frac{b^2}{\sqrt{a^2+b^2}}\right]=\frac{ab\left(a^2+b^2\right)}{a^2+b^2}=ab=0$$

Since $b\geq\sqrt{b^2-a^2}$, $A\left(a^2/\sqrt{a^2+b^2}\right)\geq A(0)$. So there is an absolute maximum when $x=\dfrac{a^2}{\sqrt{a^2+b^2}}$.

In this case the horizontal piece should be $\dfrac{2ab}{\sqrt{a^2+b^2}}$ and the vertical piece should be

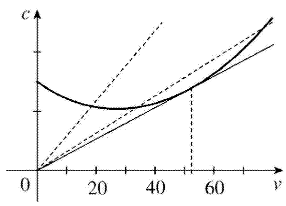
$\dfrac{a^2+b^2}{\sqrt{a^2+b^2}}=\sqrt{a^2+b^2}$.

49. Note that $|AD|=|AP|+|PD|\Rightarrow 5=x+|PD|\Rightarrow|PD|=5-x$. Using the Pythagorean Theorem for ΔPDB and ΔPDC gives us

$$\begin{aligned}L(x)&=|AP|+|BP|+|CP|=x+\sqrt{(5-x)^2+2^2}+\sqrt{(5-x)^2+3^2}\\&=x+\sqrt{x^2-10x+29}+\sqrt{x^2-10x+34}\end{aligned}$$

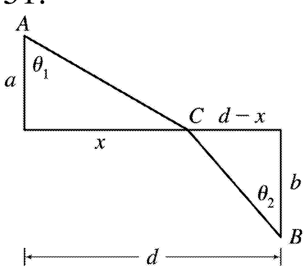
$\Rightarrow L'(x)=1+\dfrac{x-5}{\sqrt{x^2-10x+29}}+\dfrac{x-5}{\sqrt{x^2-10x+34}}$ From the graphs of L and L' , it seems that the minimum value of L is about $L(3.59)=9.35$ m.

50.



We note that since c is the consumption in gallons per hour, and v is the velocity in miles per hour, then $\frac{c}{v} = \frac{\text{gallons/hour}}{\text{miles/hour}} = \frac{\text{gallons}}{\text{mile}}$ gives us the consumption in gallons per mile, that is, the quantity G. To find the minimum, we calculate $\frac{dG}{dv} = \frac{d}{dv}\left(\frac{c}{v}\right) = \frac{v\frac{dc}{dv} - c\frac{dv}{dv}}{v^2} = \frac{v\frac{dc}{dv} - c}{v^2}$. This is 0 when $v\frac{dc}{dv} - c = 0 \Leftrightarrow \frac{dc}{dv} = \frac{c}{v}$. This implies that the tangent line of $c(v)$ passes through the origin, and this occurs when $v \approx 53$ mi / h. Note that the slope of the secant line through the origin and a point $(v, c(v))$ on the graph is equal to $G(v)$, and it is intuitively clear that G is minimized in the case where the secant is in fact a tangent.

51.



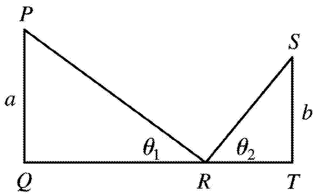
The total time is

$T(x)$ </ $= (\text{time from } A \text{ to } C) + (\text{time from } C \text{ to } B) = \frac{\sqrt{a^2+x^2}}{v_1} + \frac{\sqrt{b^2+(d-x)^2}}{v_2}, \; 0<x<d$

$$T'(x) = \frac{x}{v_1\sqrt{a^2+x^2}} - \frac{d-x}{v_2\sqrt{b^2+(d-x)^2}} = \frac{\sin\theta_1}{v_1} - \frac{\sin\theta_2}{v_2}$$

The minimum occurs when $T'(x) = 0 \Rightarrow \frac{\sin\theta_1}{v_1} = \frac{\sin\theta_2}{v_2}$.

52.



If $d=|QT|$, we minimize $f\left(\theta_1\right)=|PR|+|RS|=a\ \theta_1+b\ \theta_2$. Differentiating with respect to θ_1 , and setting $\frac{df}{d\theta_1}$ equal to 0 , we get $\frac{df}{d\theta_1}=0=-a\ \theta_1\cot\theta_1-b\ \theta_2\cot\theta_2\frac{d\theta_2}{d\theta_1}$.

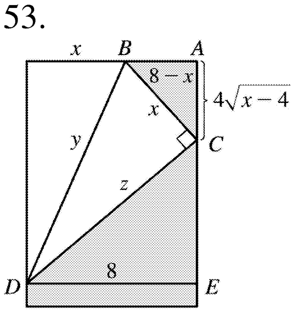
So we need to find an expression for $\frac{d\theta_2}{d\theta_1}$. We can do this by observing that $|QT|=$ constant $=a\cot\theta_1+b\cot\theta_2$.

Differentiating this equation implicitly with respect to θ_1 , we get $-a\ ^2\theta_1-b\ ^2\theta_2\frac{d\theta_2}{d\theta_1}=0\Rightarrow$ $\frac{d\theta_2}{d\theta_1}=-\frac{a\ ^2\theta_1}{b\ ^2\theta_2}$. We substitute this into the expression for $\frac{df}{d\theta_1}$ to get

$$-a\ \theta_1\cot\theta_1-b\ \theta_2\cot\theta_2\left(-\frac{a\ ^2\theta_1}{b\ ^2\theta_2}\right)=0\Leftrightarrow-a\csc\theta_1\cot\theta_1+a\frac{\csc^2\theta_1\cot\theta_2}{\csc\theta_2}=0\Leftrightarrow$$

$\cot\theta_1\csc\theta_2=\csc\theta_1\cot\theta_2\Leftrightarrow\frac{\cot\theta_1}{\csc\theta_1}=\frac{\cot\theta_2}{\csc\theta_2}\Leftrightarrow\cos\theta_1=\cos\theta_2$. Since θ_1 and θ_2 are both acute, we have $\theta_1=\theta_2$.

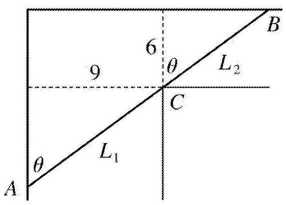
53.



$y^2=x^2+z^2$, but triangles *CDE* and *BCA* are similar, so $z/8=x\,/\left(4\sqrt{x-4}\right)\ \Rightarrow z=2x\,/\sqrt{x-4}$. Thus, we minimize $f(x)=y^2=x^2+4x^2/(x-4)=x^3/(x-4)$, $4<x\leq 8$.

$f'(x)=\dfrac{(x-4)\left(3x^2\right)-x^3}{(x-4)^2}=\dfrac{x^2\left[3(x-4)-x\right]}{(x-4)^2}=\dfrac{2x^2(x-6)}{(x-4)^2}=0$ when $x=6$. $f'(x)<0$ when $x<6$, $f'(x)>0$ when $x>6$, so the minimum occurs when $x=6$ in.

54.



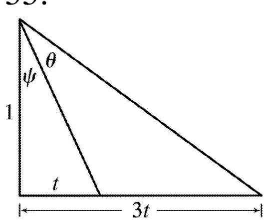
Paradoxically, we solve this maximum problem by solving a minimum problem. Let L be the length of the line ACB going from wall to wall touching the inner corner C . As $\theta\rightarrow 0$ or $\theta\rightarrow\dfrac{\pi}{2}$, we have $L\rightarrow\infty$ and there will be an angle that makes L a minimum. A pipe of this length will just fit around the corner.

From the diagram, $L=L_1+L_2=9\csc\theta+6\sec\theta\Rightarrow dL/d\theta=-9\csc\theta\ \cot\theta+6\sec\theta\ \tan\theta=0$ when $6\sec\theta\ \tan\theta=9\csc\theta\ \cot\theta\Leftrightarrow\tan^3\theta=\dfrac{9}{6}=1.5\Leftrightarrow\tan\theta=\sqrt{[3]1.5}$. Then $\sec^2\theta=1+\left(\dfrac{3}{2}\right)^{2/3}$ and $\csc^2\theta=1+\left(\dfrac{3}{2}\right)^{-2/3}$, so the longest pipe has length

$L=9\left[1+\left(\dfrac{3}{2}\right)^{-2/3}\right]^{1/2}+6\left[1+\left(\dfrac{3}{2}\right)^{2/3}\right]^{1/2}\approx 21.07$ ft.

Or, use $\theta=\tan^{-1}\left(\sqrt[3]{1.5}\right)\approx 0.852\Rightarrow L=9\ \theta+6\sec\theta\approx 21.07$ ft.

55.



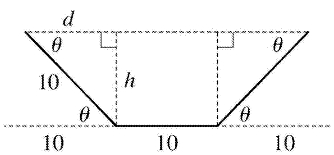
It suffices to maximize $\tan\theta$. Now $\dfrac{3t}{1}=\tan\left(\psi+\theta\right)=\dfrac{\tan\psi+\tan\theta}{1-\tan\psi\tan\theta}=\dfrac{t+\tan\theta}{1-t\tan\theta}$. So $3t\left(1-t\tan\theta\right)=t+\tan\theta\Rightarrow 2t=\left(1+3t^2\right)\tan\theta\Rightarrow\tan\theta=\dfrac{2t}{1+3t^2}$. Let $f(t)=\tan\theta=\dfrac{2t}{1+3t^2}\Rightarrow$

$f'(t)=\dfrac{2\left(1+3t^2\right)-2t(6t)}{\left(1+3t^2\right)^2}=\dfrac{2\left(1-3t^2\right)}{\left(1+3t^2\right)^2}=0\Leftrightarrow 1-3t^2=0\Leftrightarrow t=\dfrac{1}{\sqrt{3}}$ since $t\geq 0$.

Now

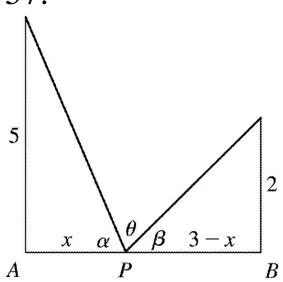
$f'(t)>0$ for $0\le t<\dfrac{1}{\sqrt{3}}$ and $f'(t)<0$ for $t>\dfrac{1}{\sqrt{3}}$, so f has an absolute maximum when $t=\dfrac{1}{\sqrt{3}}$ and $\tan\theta=\dfrac{2\left(1/\sqrt{3}\right)}{1+3\left(1/\sqrt{3}\right)^2}=\dfrac{1}{\sqrt{3}}\Rightarrow\theta=\dfrac{\pi}{6}$. Substituting for t and θ in $3t=\tan\left(\psi+\theta\right)$ gives us $\sqrt{3}=\tan\left(\psi+\dfrac{\pi}{6}\right)\Rightarrow\psi=\dfrac{\pi}{6}$.

56.



We maximize the cross–sectional area $A(\theta)=$
$10h+2\left(\dfrac{1}{2}dh\right)=10h+dh=10(10\sin\theta)+(10\cos\theta)(10\sin\theta) >=100(\sin\theta+\sin\theta\cos\theta),\ 0\le\theta\le\dfrac{\pi}{2}$
$A'(\theta)=100\left(\cos\theta+\cos^2\theta-\sin^2\theta\right)=100\left(\cos\theta+2\cos^2\theta-1\right)=100(2\cos\theta-1)(\cos\theta+1)=0$ when $\cos\theta=\dfrac{1}{2}\Leftrightarrow\theta=\dfrac{\pi}{3}$. ($\cos\theta\ne-1$ since $0\le\theta\le\dfrac{\pi}{2}$.)
Now $A(0)=0$, $A\left(\dfrac{\pi}{2}\right)=100$ and $A\left(\dfrac{\pi}{3}\right)=75\sqrt{3}\approx129.9$, so the maximum occurs when $\theta=\dfrac{\pi}{3}$.

57.



From the figure, $\tan\alpha=\dfrac{5}{x}$ and $\tan\beta=\dfrac{2}{3-x}$. Since $\alpha+\beta+\theta=180^\circ=\pi$,

$\theta=\pi-\tan^{-1}\left(\dfrac{5}{x}\right)-\tan^{-1}\left(\dfrac{2}{3-x}\right)\Rightarrow$

$$\frac{d\theta}{dx}=-\frac{1}{1+\left(\frac{5}{x}\right)^2}\left(-\frac{5}{x^2}\right)-\frac{1}{1+\left(\frac{2}{3-x}\right)^2}\left[\frac{2}{(3-x)^2}\right]$$

$$=\frac{x^2}{x^2+25}\cdot\frac{5}{x^2}-\frac{(3-x)^2}{(3-x)^2+4}\cdot\frac{2}{(3-x)^2}\ .$$

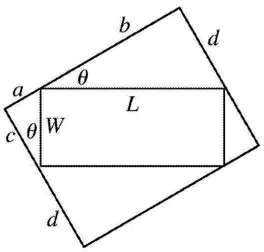
Now $\frac{d\theta}{dx}=0 \Rightarrow \frac{5}{x^2+25}=\frac{2}{x^2-6x+13} \Rightarrow 2x^2+50=5x^2-30x+65 \Rightarrow$
$3x^2-30x+15=0 \Rightarrow x^2-10x+5=0 \Rightarrow x=5\pm 2\sqrt{5}$. We reject the root with the + sign, since it is larger than 3 . $d\theta/dx>0$ for $x<5-2\sqrt{5}$ and $d\theta/dx<0$ for $x>5-2\sqrt{5}$, so θ is maximized when $|AP|=x=5-2\sqrt{5}\approx 0.53$.

58. Let x be the distance from the observer to the wall. Then, from the given figure, <
$\theta=\tan^{-1}\left(\frac{h+d}{x}\right)-\tan^{-1}\left(\frac{d}{x}\right)$, $x>0 \Rightarrow$

$$\begin{aligned}\frac{d\theta}{dx} &= \frac{1}{1+[(h+d)/x]^2}\left[-\frac{h+d}{x^2}\right]-\frac{1}{1+(d/x)^2}\left[-\frac{d}{x^2}\right]=-\frac{h+d}{x^2+(h+d)^2}+\frac{d}{x^2+d^2}\\ &= \frac{d\left[x^2+(h+d)^2\right]-(h+d)\left(x^2+d^2\right)}{\left[x^2+(h+d)^2\right]\left(x^2+d^2\right)}=\frac{h^2d+hd^2-hx^2}{\left[x^2+(h+d)^2\right]\left(x^2+d^2\right)}=0 \Leftrightarrow\end{aligned}$$

$hx^2=h^2d+hd^2 \Leftrightarrow x^2=hd+d^2 \Leftrightarrow x=\sqrt{d(h+d)}$. Since $d\theta/dx>0$ for all $x<\sqrt{d(h+d)}$ and $d\theta/dx<0$ for all $x>\sqrt{d(h+d)}$, the absolute maximum occurs when $x=\sqrt{d(h+d)}$.

59.



In the small triangle with sides a and c and hypotenuse W , $\sin\theta=\frac{a}{W}$ and $\cos\theta=\frac{c}{W}$. In the triangle with sides b and d and hypotenuse L , $\sin\theta=\frac{d}{L}$ and $\cos\theta=\frac{b}{L}$. Thus, $a=W\sin\theta$, $c=W\cos\theta$, $d=L\sin\theta$, and $b=L\cos\theta$, so the area of the circumscribed rectangle is

$$\begin{aligned}A(\theta) &= (a+b)(c+d)=(W\sin\theta+L\cos\theta)(W\cos\theta+L\sin\theta)=1-12pt\\ &= W^2\sin\theta\cos\theta+WL\sin^2\theta+LW\cos^2\theta+L^2\sin\theta\cos\theta=1-12pt\\ &= LW\sin^2\theta+LW\cos^2\theta+\left(L^2+W^2\right)\sin\theta\cos\theta=1-12pt\\ &= LW\left(\sin^2\theta+\cos^2\theta\right)+\left(L^2+W^2\right)\cdot\frac{1}{2}\cdot 2\sin\theta\cos\theta=1-12pt\\ &= LW+\frac{1}{2}\left(L^2+W^2\right)\sin 2\theta, 0\le\theta\le\frac{\pi}{2}\end{aligned}$$

This expression shows, without calculus, that the maximum value of $A(\theta)$ occurs when $\sin 2\theta=1 \Leftrightarrow$ $2\theta=\frac{\pi}{2}$
$\Rightarrow \theta=\frac{\pi}{4}$. So the maximum area is $A\left(\frac{\pi}{4}\right)=LW+\frac{1}{2}\left(L^2+W^2\right)=\frac{1}{2}\left(L^2+2LW+W^2\right)=\frac{1}{2}(L+W)^2$.

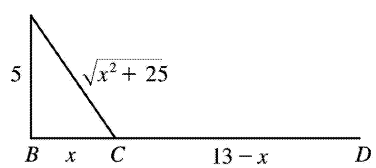
60. **(a)** Let D be the point such that $a=|AD|$. From the figure, $\sin\theta=\frac{b}{|BC|} \Rightarrow |BC|=b\ \theta$ and $\cos\theta=\frac{|BD|}{|BC|}=\frac{a-|AB|}{|BC|} \Rightarrow |BC|=(a-|AB|)\sec\theta$. Eliminating $|BC|$ gives $(a-|AB|)\sec\theta=b\ \theta \Rightarrow$ $b\cot\theta=a-|AB| \Rightarrow |AB|=a-b\cot\theta$. The total resistance is
$$R(\theta)=C\frac{|AB|}{r_1^4}+C\frac{|BC|}{r_2^4}=C\left(\frac{a-b\cot\theta}{r_1^4}+\frac{b\csc\theta}{r_2^4}\right).$$

(b) $$R'(\theta)=C\left(\frac{b\csc^2\theta}{r_1^4}-\frac{b\csc\theta\cot\theta}{r_2^4}\right)=bC\csc\theta\left(\frac{\csc\theta}{r_1^4}-\frac{\cot\theta}{r_2^4}\right).$$

$$R'(\theta)=0 \Leftrightarrow \frac{\csc\theta}{r_1^4}=\frac{\cot\theta}{r_2^4} \Leftrightarrow \frac{r_2^4}{r_1^4}=\frac{\cot\theta}{\csc\theta}=\cos\theta .$$

$R'(\theta)>0 \Leftrightarrow \frac{\csc\theta}{r_1^4}>\frac{\cot\theta}{r_2^4} \Rightarrow \cos\theta<\frac{r_2^4}{r_1^4}$ and $R'(\theta)<0$ when $\cos\theta>\frac{r_2^4}{r_1^4}$, so there is an absolute minimum when $\cos\theta=r_2^4/r_1^4$.

(c) When $r_2=\frac{2}{3}r_1$, we have $\cos\theta=\left(\frac{2}{3}\right)^4$, so $\theta=\cos^{-1}\left(\frac{2}{3}\right)^4\approx 79^\circ$.



61. **(a)**

If $k=$ energy / km over land, then energy / km over water $=1.4k$. So the total energy is $E=1.4k\sqrt{25+x^2}+k(13-x)$, $0\le x\le 13$, and so $\frac{dE}{dx}=\frac{1.4kx}{\left(25+x^2\right)^{1/2}}-k$.

Set $\frac{dE}{dx}=0$: $1.4kx=k\left(25+x^2\right)^{1/2} \Rightarrow 1.96x^2=x^2+25 \Rightarrow 0.96x^2=25 \Rightarrow x=\frac{5}{\sqrt{0.96}}\approx 5.1$. Testing against the value of E at the endpoints: $E(0)=1.4k(5)+13k=20k$, $E(5.1)\approx 17.9k$, $E(13)\approx 19.5k$. Thus, to minimize energy, the bird should fly to a point about 5.1 km from B .

(b) If W / L is large, the bird would fly to a point C that is closer to B than to D to minimize the energy used flying over water. If W / L is small, the bird would fly to a point C that is closer to D than to B to minimize the distance of the flight. $E=W\sqrt{25+x^2}+L(13-x)\Rightarrow \dfrac{dE}{dx}=\dfrac{Wx}{\sqrt{25+x^2}}-L=0$ when

$\dfrac{W}{L}=\dfrac{\sqrt{25+x^2}}{x}$. By the same sort of argument as in part (a), this ratio will give the minimal expenditure of energy if the bird heads for the point x km from B .

(c) For flight direct to D , $x=13$, so from part (b), $W / L=\dfrac{\sqrt{25+13^2}}{13}\approx 1.07$. There is no value of W / L for which the bird should fly directly to B . But note that $\lim\limits_{x\to 0^+} (W / L)=\infty$, so if the point at which E is a minimum is close to B , then W / L is large.

(d) Assuming that the birds instinctively choose the path that minimizes the energy expenditure, we can use the equation for $dE/dx=0$ from part (a) with $1.4k=c$, $x=4$, and $k=1$: $(c)(4)=1\cdot\left(25+4^2\right)^{1/2}\Rightarrow$ $c=\sqrt{41}/4\approx 1.6$.

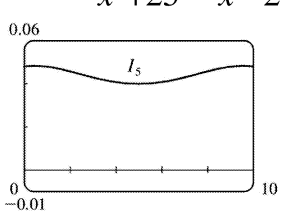
62. **(a)** $I(x)\propto \dfrac{\text{strength of source}}{(\text{distance from source})^2}$. Adding the intensities from the left and right lightbulbs,

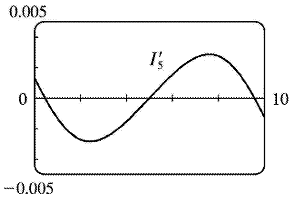
$$I(x)=\frac{k}{x^2+d^2}+\frac{k}{(10-x)^2+d^2}=\frac{k}{x^2+d^2}+\frac{k}{x^2-20x+100+d^2}\ .$$

(b) The magnitude of the constant k won't affect the location of the point of maximum intensity, so for convenience we take $k=1$. $I'(x)=-\dfrac{2x}{\left(x^2+d^2\right)^2}-\dfrac{2(x-10)}{\left(x^2-20x+100+d^2\right)^2}$.

Substituting $d=5$ into the equations for $I(x)$ and $I'(x)$, we get

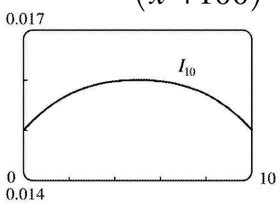
$$I_5(x)=\frac{1}{x^2+25}+\frac{1}{x^2-20x+125}\quad\text{and}\quad I_5'(x)=-\frac{2x}{\left(x^2+25\right)^2}-\frac{2(x-10)}{\left(x^2-20x+125\right)^2}$$

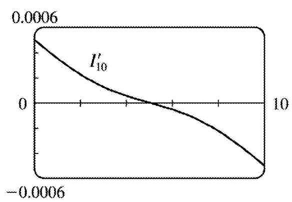




From the graphs, it appears that $I_5(x)$ has a minimum at $x=5$ m.

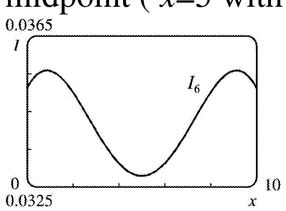
(c) Substituting $d=10$ into the equations for $I(x)$ and $I'(x)$ gives $I_{10}(x)=\dfrac{1}{x^2+100}+\dfrac{1}{x^2-20x+200}$ and $I_{10}'(x)=-\dfrac{2x}{\left(x^2+100\right)^2}-\dfrac{2(x-10)}{\left(x^2-20x+200\right)^2}$.

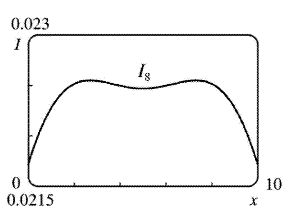


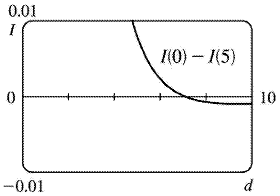


From the graphs, it seems that for $d=10$, the intensity is minimized at the endpoints, that is, $x=0$ and $x=10$. The midpoint is now the most brightly lit point!

(d) From the first figures in parts (b) and (c), we see that the minimal illumination changes from the midpoint ($x=5$ with $d=5$) to the endpoints ($x=0$ and $x=10$ with $d=10$).





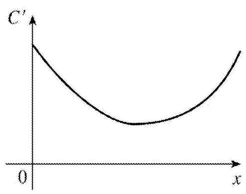


So we try $d=6$ (see the first figure) and we see that the minimum value still occurs at $x=5$. Next, we let $d=8$ (see the second figure) and we see that the minimum value occurs at the endpoints. It appears that for some value of d between 6 and 8 , we must have minima at both the midpoint and the endpoints, that is, $I(5)$ must equal $I(0)$. To find this value of d , we solve $I(0)=I(5)$ (with $k=1$):
$\dfrac{1}{d^2}+\dfrac{1}{100+d^2}=\dfrac{1}{25+d^2}+\dfrac{1}{25+d^2}=\dfrac{2}{25+d^2} \Rightarrow \left(25+d^2\right)\left(100+d^2\right)+d^2\left(25+d^2\right)=2d^2\left(100+d^2\right) \Rightarrow$
$2500+125d^2+d^4+25d^2+d^4=200d^2+2d^4 \Rightarrow 2500=50d^2 \Rightarrow d^2=50 \Rightarrow d=5\sqrt{2}\approx 7.071$ (for $0\le d\le 10$). The third figure, a graph of $I(0)–I(5)$ with d independent, confirms that $I(0)–I(5)=0$, that is, $I(0)=I(5)$, when $d=5\sqrt{2}$. Thus, the point of minimal illumination changes abruptly from the midpoint to the endpoints when $d=5\sqrt{2}$.

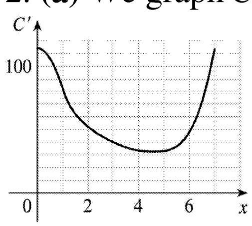
1. **(a)** $C(0)$ represents the fixed costs of production, such as rent, utilities, machinery etc., which are incurred even when nothing is produced.

(b) The inflection point is the point at which $C''(x)$ changes from negative to positive; that is, the marginal cost $C'(x)$ changes from decreasing to increasing. Thus, the marginal cost is minimized at the inflection point.

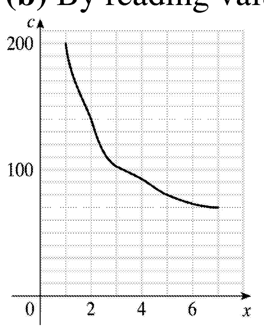
(c) The marginal cost function is $C'(x)$. We graph it as in Example 1 in Section.



2. **(a)** We graph C' as in Example 1 in Section.



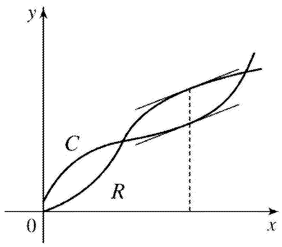
(b) By reading values of $C(x)$ from its graph, we can plot $c(x)=C(x)/x$.



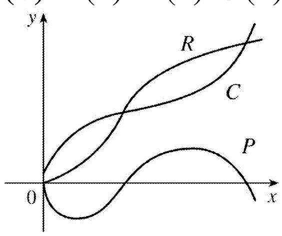
(c) Since the graph in part (b) is decreasing, we estimate that the minimum value of $c(x)$ occurs at $x=7$. The average cost and the marginal cost are equal at that value. See the box preceding Example 1.

3. $c(x)=21.4-0.002x$ and $c(x)=C(x)/x \Rightarrow C(x)=21.4x-0.002x^2$. $C'(x)=21.4-0.004x$ and $C'(1000)=17.4$. This means that the cost of producing the 1001 st unit is about \$17.40 .

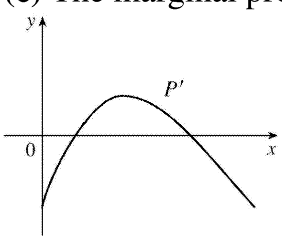
4. **(a)** Profit is maximized when the marginal revenue is equal to the marginal cost; that is, when R and C have equal slopes. See the box preceding Example 2.



(b) $P(x)=R(x)-C(x)$ is sketched.



(c) The marginal profit function is defined as $P'(x)$.



5. **(a)** The cost function is $C(x)=40,000+300x+x^2$, so the cost at a production level of 1000 is $C(1000)=\$1,340,000$. The average cost function is $c(x)=\frac{C(x)}{x}=\frac{40,000}{x}+300+x$ and $c(1000)=\$1340/$ unit. The marginal cost function is $C'(x)=300+2x$ and $C'(1000)=\$2300/$ unit.
(b) See the box preceding Example 1. We must have $C'(x)=c(x)\Leftrightarrow 300+2x=\frac{40,000}{x}+300+x\Leftrightarrow x=\frac{40,000}{x}\Rightarrow x^2=40,000\Rightarrow x=\sqrt{40,000}=200$. This gives a minimum value of the average cost function $c(x)$ since $c''(x)=\frac{80,000}{x^3}>0$.
(c) The minimum average cost is $c(200)=\$700/$ unit.

6. **(a)** $C(x)=25,000+120x+0.1x^2$, $C(1000)=\$245,000$. $c(x)=\frac{C(x)}{x}=\frac{25,000}{x}+120+0.1x$, $c(1000)=\$245/$ unit. $C'(x)=120+0.2x$, $C'(1000)=\$320/$ unit.
(b) We must have $C'(x)=c(x)\Leftrightarrow 120+0.2x=\frac{25,000}{x}+120+0.1x\Leftrightarrow 0.1x=\frac{25,000}{x}\Rightarrow 0.1x^2=25,000\Rightarrow$

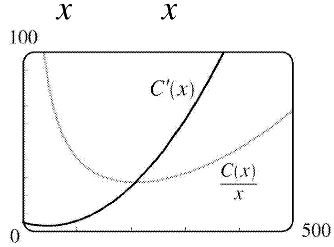
$x=\sqrt{250{,}000}=500$. This gives a minimum since $c''(x)=\dfrac{50{,}000}{x^3}>0$.

(c) The minimum average cost is $c(500)=\$220.00$/ unit.

7. **(a)** $C(x)=16{,}000+200x+4x^{3/2}$, $C(1000)=16{,}000+200{,}000+40{,}000\sqrt{10}\approx 216{,}000+126{,}491$, so $C(1000)\approx \$342{,}491$. $c(x)=C(x)/x=\dfrac{16{,}000}{x}+200+4x^{1/2}$, $c(1000)\approx \$342.49$/unit .
$C'(x)=200+6x^{1/2}$, $C'(1000)=200+60\sqrt{10}\approx \$389.74$/unit .
(b) We must have $C'(x)=c(x)\Leftrightarrow 200+6x^{1/2}=\dfrac{16{,}000}{x}+200+4x^{1/2}\Leftrightarrow 2x^{3/2}=16{,}000\Leftrightarrow x=(8{,}000)^{2/3}=400$ units. To check that this is a minimum, we calculate $c'(x)=\dfrac{-16{,}000}{x^2}+\dfrac{2}{\sqrt{x}}=\dfrac{2}{x^2}(x^{3/2}-8000)$. This is negative for $x<(8000)^{2/3}=400$, zero at $x=400$, and positive for $x>400$, so c is decreasing on $(0,400)$ and increasing on $(400,\infty)$. Thus, c has an absolute minimum at $x=400$.
(c) The minimum average cost is $c(400)=40+200+80=\$320$/ unit.

8. **(a)** $C(x)=10{,}000+340x-0.3x^2+0.0001x^3$, $C(1000)=\$150{,}000$.
$c(x)=C(x)/x=\dfrac{10{,}000}{x}+340-0.3x+0.0001x^2$, $c(1000)=\$150$/unit . $C'(x)=340-0.6x+0.0003x^2$,
$C'(1000)=\$40$/unit .
(b) We must have $C'(x)=c(x)\Leftrightarrow 340-0.6x+0.0003x^2=\dfrac{10{,}000}{x}+340-0.3x+0.0001x^2\Leftrightarrow$
$0.0002x^2=\dfrac{10{,}000}{x}+0.3x\Leftrightarrow 0.0002x^3-0.3x^2-10{,}000=0\Leftrightarrow x^3-1500x^2-50{,}000{,}000=0\Rightarrow$
$x\approx 1521.60\approx 1522$ units. This gives a minimum since $c''(x)=\dfrac{20{,}000}{x^3}+0.0002>0$.
(c) The minimum average cost is about $c(1521.60)\approx \$121.62$/ unit.

9. **(a)** $C(x)=3700+5x-0.04x^2+0.0003x^3\Rightarrow C'(x)=5-0.08x+0.0009x^2$ (marginal cost).
$c(x)=\dfrac{C(x)}{x}=\dfrac{3700}{x}+5-0.04x+0.0003x^2$ (average cost).

(b) 

The graphs intersect at $(208.51,27.45)$, so the production level that minimizes average cost is about

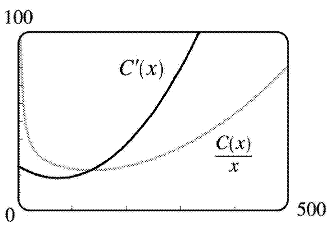
209 units.

(c) $c'(x)=-\dfrac{3700}{x^2}-0.04+0.0006x=0 \Rightarrow 3700+0.04x^2-0.0006x^3=0 \Rightarrow x_1 \approx 208.51$. $c\left(x_1\right) \approx \$27.45$/ unit.

(d) The marginal cost is given by $C'(x)$, so to find its minimum value we'll find the derivative of C' ; that is, C'' . $C''(x)=-0.08+0.0018x=0 \Rightarrow x_1=\dfrac{800}{18}=44.4\overline{4}$. $C'\left(x_1\right)=\$3.22$/ unit.

$C'''(x)=0.0018>0$ for all x , so this is the minimum marginal cost. C''' is the second derivative of C' . cost is given by $C'(x)$.

10. **(a)** $C(x)=339+25x-0.09x^2+0.0004x^3 \Rightarrow C'(x)=25-0.18x+0.0012x^2$ (marginal cost).
$c(x)=\dfrac{C(x)}{x}=\dfrac{339}{x}+25-0.09x+0.0004x^2$ (average cost).



(b)
The graphs intersect at $(135.56,22.65)$, so the production level that minimizes average cost is about 136 units.

(c) $c'(x)=-\dfrac{339}{x^2}-0.09+0.0008x=0 \Rightarrow x_1 \approx 135.56$. $c\left(x_1\right) \approx \$22.65$/ unit.

(d) $C''(x)=-0.18+0.0024x=0 \Rightarrow x=\dfrac{1800}{24}=75$. $C'(75)=\$18.25$/ unit.

$C'''(x)=0.0024>0$ for all x , so this is the minimum marginal cost.

11. $C(x)=680+4x+0.01x^2$, $p(x)=12 \Rightarrow R(x)=xp(x)=12x$. If the profit is maximum, then $R'(x)=C'(x)$ $\Rightarrow 12=4+0.02x \Rightarrow 0.02x=8 \Rightarrow x=400$. The profit is maximized if $P''(x)<0$, but since $P''(x)=R''(x)-C''(x)$, we can just check the condition $R''(x)<C''(x)$. Now $R''(x)=0<0.02=C''(x)$, so $x=400$ gives a maximum.

12. $C(x)=680+4x+0.01x^2$, $p(x)=12-x/500$. Then $R(x)=xp(x)=12x-x^2/500$. If the profit is maximum, then $R'(x)=C'(x) \Leftrightarrow 12-x/250=4+0.02x \Leftrightarrow 8=0.024x \Leftrightarrow x=8/0.024=\dfrac{1000}{3}$. The profit is maximized if $P''(x)<0$, but since $P''(x)=R''(x)-C''(x)$, we can just check the condition $R''(x)<C''(x)$.

Now $R''(x)=-\frac{1}{250}<0.02=C''(x)$, so $x=\frac{1000}{3}$ gives a maximum.

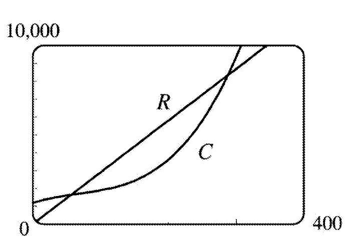
13. $C(x)=1450+36x-x^2+0.001x^3$, $p(x)=60-0.01x$. Then $R(x)=xp(x)=60x-0.01x^2$. If the profit is maximum, then $R'(x)=C'(x)\Leftrightarrow 60-0.02x=36-2x+0.003x^2\Rightarrow 0.003x^2-1.98x-24=0$. By the quadratic formula, $x=\frac{1.98\pm\sqrt{(-1.98)^2+4(0.003)(24)}}{2(0.003)}=\frac{1.98\pm\sqrt{4.2084}}{0.006}$. Since $x>0$, $x\approx(1.98+2.05)/0.006\approx 672$. Now $R''(x)=-0.02$ and $C''(x)=-2+0.006x\Rightarrow C''(672)=2.032\Rightarrow$ $R''(672)<C''(672)\Rightarrow$ there is a maximum at $x=672$.

14. $C(x)=16,000+500x-1.6x^2+0.004x^3$, $p(x)=1700-7x$. Then $R(x)=xp(x)=1700x-7x^2$. If the profit is maximum, then $R'(x)=C'(x)\Leftrightarrow 1700-14x=500-3.2x+0.012x^2\Leftrightarrow 0.012x^2+10.8x-1200=0\Leftrightarrow$ $x^2+900x-100,000=0\Leftrightarrow(x+1000)(x-100)=0\Leftrightarrow x=100$ (since $x>0$). The profit is maximized if $P''(x)<0$, but since $P''(x)=R''(x)-C''(x)$, we can just check the condition $R''(x)<C''(x)$. Now $R''(x)=-14<-3.2+0.024x=C''(x)$ for $x>0$, so there is a maximum at $x=100$.

15. $C(x)=0.001x^3-0.3x^2+6x+900$. The marginal cost is $C'(x)=0.003x^2-0.6x+6$. $C'(x)$ is increasing when $C''(x)>0\Leftrightarrow 0.006x-0.6>0\Leftrightarrow x>0.6/0.006=100$. So $C'(x)$ starts to increase when $x=100$.

16. $C(x)=0.0002x^3-0.25x^2+4x+1500$. The marginal cost is $C'(x)=0.0006x^2-0.50x+4$. $C'(x)$ is increasing when $C''(x)>0\Leftrightarrow 0.0012x-0.5>0\Leftrightarrow x>0.5/0.0012\approx 417$. So $C'(x)$ starts to increase when $x=417$.

17. **(a)** $C(x)=1200+12x-0.1x^2+0.0005x^3$. $R(x)=xp(x)=29x-0.00021x^2$. Since the profit is maximized when $R'(x)=C'(x)$, we examine the curves R and C in the figure, looking for $x-$ values at which the slopes of the tangent lines are equal. It appears that $x=200$ is a good estimate.



(b) $R'(x)=C'(x)\Rightarrow 29-0.00042x=12-0.2x+0.0015x^2\Rightarrow 0.0015x^2-0.19958x-17=0\Rightarrow x\approx 192.06$ (for $x>0$). As in Exercise 11, $R''(x)<C''(x)\Rightarrow -0.00042<-0.2+0.003x\Leftrightarrow 0.003x>0.19958\Leftrightarrow x>66.5$. Our value of 192 is in this range, so we have a maximum profit when we produce 192 yards of fabric.

18. **(a)** Cost=setupcost+manufacturingcost$\Rightarrow C(x)=500+m(x)=500+20x-5x^{3/4}+0.01x^2$. We can solve $x(p)=320-7.7p$ for p in terms of x to find the demand (or price) function. $x=320-7.7p\Rightarrow 7.7p=320-x$ $\Rightarrow p(x)=\dfrac{320-x}{7.7}$. $R(x)=xp(x)=\dfrac{320x-x^2}{7.7}$.

(b) $C'(x)=R'(x)\Rightarrow 20-\dfrac{15}{4}x^{-1/4}+0.02x=\dfrac{320-2x}{7.7}\Rightarrow x\approx 81.53$ planes, and $p(x)=\$30.97$ million. The maximum profit associated with these values is about \$463.59 million.

19. **(a)** We are given that the demand function p is linear and $p(27,000)=10$, $p(33,000)=8$, so the slope is $\dfrac{10-8}{27{,}000-33{,}000}=-\dfrac{1}{3000}$ and an equation of the line is $y-10=\left(-\dfrac{1}{3000}\right)(x-27{,}000)\Rightarrow$ $y=p(x)=-\dfrac{1}{3000}x+19=19-(x/3000)$.

(b) The revenue is $R(x)=xp(x)=19x-(x^2/3000)\Rightarrow R'(x)=19-(x/1500)=0$ when $x=28,500$. Since $R''(x)=-1/1500<0$, the maximum revenue occurs when $x=28,500\Rightarrow$ the price is $p(28,500)=\$9.50$.

20. **(a)** Let $p(x)$ be the demand function. Then $p(x)$ is linear and $y=p(x)$ passes through $(20,10)$ and $(18,11)$, so the slope is $-\dfrac{1}{2}$ and an equation of the line is $y-10=-\dfrac{1}{2}(x-20)\Leftrightarrow y=-\dfrac{1}{2}x+20$. Thus, the demand is $p(x)=-\dfrac{1}{2}x+20$ and the revenue is $R(x)=xp(x)=-\dfrac{1}{2}x^2+20x$.

(b) The cost is $C(x)=6x$, so the profit is $P(x)=R(x)-C(x)=-\dfrac{1}{2}x^2+14x$. Then $0=P'(x)=-x+14\Rightarrow x=14$. Since $P''(x)=-1<0$, the selling price for maximum profit is $p(14)=-\dfrac{1}{2}(14)+20=\$13$.

21. **(a)** As in Example 3, we see that the demand function p is linear. We are given that $p(1000)=450$ and deduce that $p(1100)=440$, since a \$10 reduction in price increases sales by 100 per week. The slope for p is $\dfrac{440-450}{1100-1000}=-\dfrac{1}{10}$, so an equation is $p-450=-\dfrac{1}{10}(x-1000)$ or $p(x)=-\dfrac{1}{10}x+550$.

(b) $R(x)=xp(x)=-\dfrac{1}{10}x^2+550x$. $R'(x)=-\dfrac{1}{5}x+550=0$ when $x=5(550)=2750$. $p(2750)=275$, so the rebate should be $450-275=\$175$.

(c) $C(x)=68,000+150x\Rightarrow$

$P(x)=R(x)-C(x)=-\frac{1}{10}x^2+550x-68,000-150x=-\frac{1}{10}x^2+400x-68,000$, $P'(x)=-\frac{1}{5}x+400=0$ when $x=2000$. $p(2000)=350$. Therefore, the rebate to maximize profits should be $450-350=\$100$.

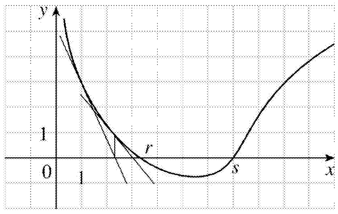
22. Let x denote the number of \$10 increases in rent. Then the price is $p(x)=800+10x$, and the number of units occupied is $100-x$. Now the revenue is

$$\begin{aligned} R(x) &= (\text{rental price per unit})\times(\text{number of unit srented}) \\ &= (800+10x)(100-x)=-10x^2+200x+80{,}000 \text{ for } 0\le x\le 100 \Rightarrow \end{aligned}$$

$R'(x)=-20x+200=0 \Leftrightarrow x=10$. This is a maximum since $R''(x)=-20<0$ for all x . Now we must check the value of $R(x)=(800+10x)(100-x)$ at $x=10$ and at the endpoints of the domain to see which value of x gives the maximum value of R . $R(0)=80,000$, $R(10)=(900)(90)=81,000$, and $R(100)=(1800)(0)=0$. Thus, the maximum revenue of \$81 , 000/ week occurs when 90 units are occupied at a rent of \$900/ week.

23. If the reorder quantity is x , then the manager places $\frac{800}{x}$ orders per year. Storage costs for the year are $\frac{1}{2}x\cdot 4=2x$ dollars. Handling costs are \$100 per delivery, for a total of $\frac{800}{x}\cdot 100=\frac{80{,}000}{x}$ dollars. The total costs for the year are $C(x)=2x+\frac{80{,}000}{x}$. To minimize $C(x)$, we calculate $C'(x)=2-\frac{80{,}000}{x^2}=\frac{2}{x^2}(x^2-40,000)$. This is negative when $x<200$, zero when $x=200$, and positive when $x>200$, so C is decreasing on $(0,200)$ and increasing on $(200,\infty)$, reaching its absolute minimum when $x=200$. Thus, the optimal reorder quantity is 200 cases. The manager will place 4 orders per year for a total cost of $C(200)=\$800$.

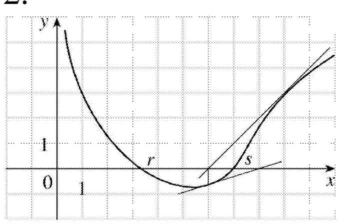
24. She will have A/n dollars after each withdrawal and 0 dollars just before the next withdrawal, so her average cash balance at any given time is $\frac{1}{2}(A/n+0)=A/(2n)$. The transaction costs for n withdrawals are nT . The lost interest cost on the average cash balance is $[A/(2n)]\cdot R$. Thus, the total cost for n transactions is $C(n)=nT+\frac{AR}{2n}$. Now $C'(n)=T-\frac{AR}{2n^2}$ and $C'(n)=0\Rightarrow \frac{AR}{2n^2}=T\Rightarrow n^2=\frac{AR}{2T}\Rightarrow$ $n=\sqrt{\frac{AR}{2T}}$, the value of n that minimizes total costs since $C''(n)=-\frac{AR}{n^3}<0$. Thus, the optimal average cash balance is $\frac{A}{2n}=\frac{A\sqrt{2T}}{2\sqrt{AR}}=\frac{\sqrt{AT}}{\sqrt{2R}}=\sqrt{\frac{AT}{2R}}$.

1. **(a)** 

The tangent line at $x=1$ intersects the x –axis at $x\approx 2.3$, so $x_2\approx 2.3$. The tangent line at $x=2.3$ intersects the x –axis at $x\approx 3$, so $x_3\approx 3.0$.

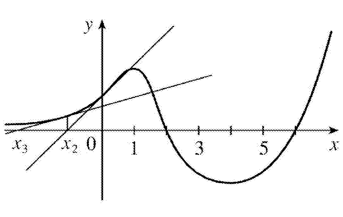
(b) $x_1=5$ would *not* be a better first approximation than $x_1=1$ since the tangent line is nearly horizontal. In fact, the second approximation for $x_1=5$ appears to be to the left of $x=1$.

2.



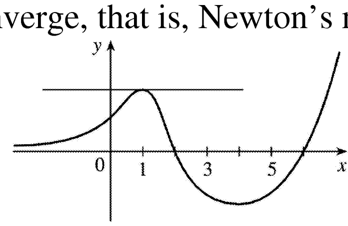
The tangent line at $x=9$ intersects the x –axis at $x\approx 6.0$, so $x_2\approx 6.0$. The tangent line at $x=6.0$ intersects the x –axis at $x\approx 8.0$, so $x_3\approx 8.0$.

3. Since $x_1=3$ and $y=5x-4$ is tangent to $y=f(x)$ at $x=3$, we simply need to find where the tangent line intersects the x– axis. $y=0\Rightarrow 5x_2-4=0\Rightarrow x_2=\frac{4}{5}$.



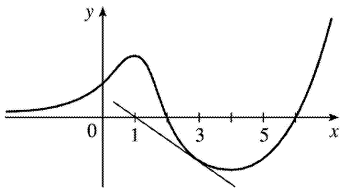
4. **(a)**

If $x_1=0$, then x_2 is negative, and x_3 is even more negative. The sequence of approximations does not converge, that is, Newton's method fails.



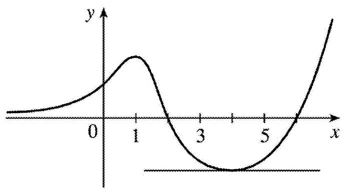
(b)

If $x_1=1$, the tangent line is horizontal and Newton's method fails.



(c)

If $x_1=3$, then $x_2=1$ and we have the same situation as in part (b). Newton's method fails again.



(d)

If $x_1=4$, the tangent line is horizontal and Newton's method fails.

5. $f(x)=x^3+2x-4 \Rightarrow f'(x)=3x^2+2$, so $x_{n+1}=x_n-\dfrac{x_n^3+2x_n-4}{3x_n^2+2}$. Now $x_1=1 \Rightarrow x_2=1-\dfrac{1+2-4}{3\cdot 1^2+2}=1-\dfrac{-1}{5}=1.2 \Rightarrow$

$x_3=1.2-\dfrac{(1.2)^3+2(1.2)-4}{3(1.2)^2+2} \approx 1.1797$.

6. $f(x)=x^3-x^2-1 \Rightarrow f'(x)=3x^2-2x$, so $x_{n+1}=x_n-\dfrac{f\left(x_n\right)}{f'\left(x_n\right)}=x_n-\dfrac{x_n^3-x_n^2-1}{3x_n^2-2x_n}$.

Now $x_1=1 \Rightarrow x_2=1-\dfrac{1-1-1}{3-2}=2 \Rightarrow x_3=2-\dfrac{2^3-2^2-1}{3\cdot 2^2-2\cdot 2}=1.625$.

7. $f(x)=x^4-20 \Rightarrow f'(x)=4x^3$, so $x_{n+1}=x_n-\dfrac{f\left(x_n\right)}{f'\left(x_n\right)}=x_n-\dfrac{x_n^4-20}{4x_n^3}$.

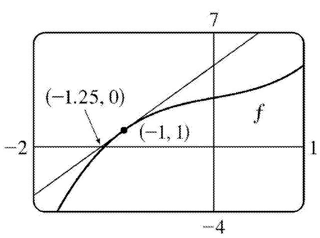
Now $x_1=2 \Rightarrow x_2=2-\dfrac{2^4-20}{4(2)^3}=2.125 \Rightarrow x_3=2.125-\dfrac{(2.125)^4-20}{4(2.125)^3} \approx 2.1148$.

8. $f(x)=x^5+2 \Rightarrow f'(x)=5x^4$, so

$x_{n+1}=x_n-\dfrac{x_n^5+2}{5x_n^4}$. Now $x_1=-1\Rightarrow x_2=-1-\dfrac{(-1)^5+2}{5\cdot(-1)^4}=-1-\dfrac{1}{5}=-1.2\Rightarrow x_3=-1.2-\dfrac{(-1.2)^5+2}{5(-1.2)^4}\approx-1.1529$.

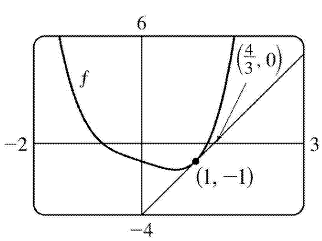
9. $f(x)=x^3+x+3\Rightarrow f'(x)=3x^2+1$, so $x_{n+1}=x_n-\dfrac{x_n^3+x_n+3}{3x_n^2+1}$. Now $x_1=-1\Rightarrow$

$x_2=-1-\dfrac{(-1)^3+(-1)+3}{3(-1)^2+1}=-1-\dfrac{-1-1+3}{3+1}=-1-\dfrac{1}{4}=-1.25$. Newton's method follows the tangent line at $(-1,1)$ down to its intersection with the x –axis at $(-1.25,0)$, giving the second approximation $x_2=-1.25$.



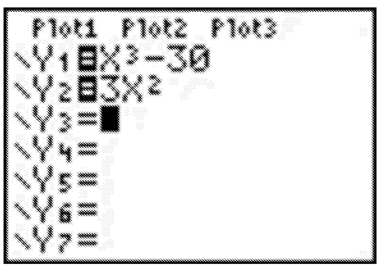
10. $f(x)=x^4-x-1\Rightarrow f'(x)=4x^3-1$, so $x_{n+1}=x_n-\dfrac{x_n^4-x_n-1}{4x_n^3-1}$. Now $x_1=1\Rightarrow x_2=1-\dfrac{1^4-1-1}{4\cdot1^3-1}=1-\dfrac{-1}{3}=\dfrac{4}{3}$.

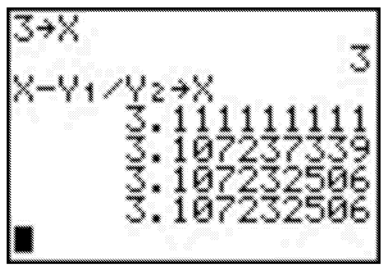
Newton's method follows the tangent line at $(1,-1)$ up to its intersection with the x –axis at $\left(\dfrac{4}{3},0\right)$, giving the second approximation $x_2=\dfrac{4}{3}$.



11. To approximate $x=\sqrt[3]{30}$ (so that $x^3=30$), we can take $f(x)=x^3-30$. So $f'(x)=3x^2$, and thus,

$x_{n+1}=x_n-\dfrac{x_n^3-30}{3x_n^2}$. Since $\sqrt[3]{27}=3$ and 27 is close to 30 , we'll use $x_1=3$. We need to find approximations until they agree to eight decimal places. $x_1=3\Rightarrow x_2\approx3.11111111$, $x_3\approx3.10723734$, $x_4\approx3.10723251\approx x_5$. So $\sqrt[3]{30}\approx3.10723251$, to eight decimal places. Here is a quick and easy method for finding the iterations for Newton's method on a programmable calculator. (The screens shown are from the TI–83 Plus, but the method is similar on other calculators.) Assign $f(x)=x^3-30$ to Y_1 , and $f'(x)=3x^2$ to Y_2 . Now store $x_1=3$ in X and then enter $X-Y_1/Y_2\rightarrow X$ to get $x_2=3.\overline{1}$. By successively pressing the ENTER key, you get the approximations x_3 , x_4 , … .





In Derive, load the utility file NEWTON$(x$^3–30$,x,3)$ and then APPROXIMATE to get . You can request a specific iteration by adding a fourth argument. For example, NEWTON$(x$^3–30$,x,3,\ 2)$ gives [3,3.11111111,3.10723733] .
In Maple, make the assignments f:=$x\rightarrow x$^3–30; , g:=$x\rightarrow x-f(x)/D(f)(x)$; , and x:=3.; . Repeatedly execute the command x:=$g(x)$; to generate successive approximations.

In Mathematica, make the assignments $f[x]$:=x^3–30 , $g[x]$:=$x-f[x]/f'[x]$, and x=3. Repeatedly execute the command x=$g[x]$ to generate successive approximations.

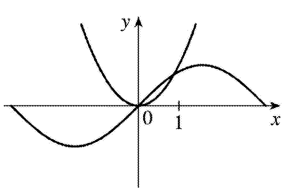
12. $f(x)=x^7-1000\Rightarrow f'(x)=7x^6$, so $x_{n+1}=x_n-\dfrac{x_n^7-1000}{7x_n^6}$. We need to find approximations until they agree to eight decimal places. $x_1=3\Rightarrow x_2\approx2.76739173$, $x_3\approx2.69008741$, $x_4\approx2.68275645$, $x_5\approx2.68269580\approx x_6$. So $\sqrt[7]{1000}\approx2.68269580$, to eight decimal places.

13. $f(x)=2x^3-6x^2+3x+1\Rightarrow f'(x)=6x^2-12x+3\Rightarrow$

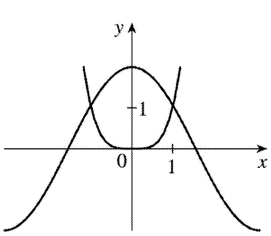
$x_{n+1}=x_n-\dfrac{2x_n^3-6x_n^2+3x_n+1}{6x_n^2-12x_n+3}$. We need to find approximations until they agree to six decimal places. $x_1=2.5\Rightarrow x_2\approx 2.285714$, $x_3\approx 2.228824$, $x_4\approx 2.224765$, $x_5\approx 2.224745\approx x_6$. So the root is 2.224745 , to six decimal places.

14. $f(x)=x^4+x-4\Rightarrow f'(x)=4x^3+1\Rightarrow x_{n+1}=x_n-\dfrac{x_n^4+x_n-4}{4x_n^3+1}$. $x_1=1.5\Rightarrow x_2\approx 1.323276$, $x_3\approx 1.285346$, $x_4\approx 1.283784$, $x_5\approx 1.283782\approx x_6$. So the root is 1.283782 , to six decimal places.

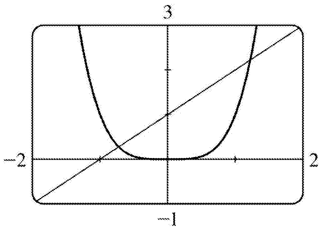
15. $\sin x=x^2$, so $f(x)=\sin x-x^2\Rightarrow f'(x)=\cos x-2x\Rightarrow x_{n+1}=x_n-\dfrac{\sin x_n-x_n^2}{\cos x_n-2x_n}$. From the figure, the positive root of $\sin x=x^2$ is near 1 . $x_1=1\Rightarrow x_2\approx 0.891396$, $x_3\approx 0.876985$, $x_4\approx 0.876726\approx x_5$. So the positive root is 0.876726 , to six decimal places.



16. $2\cos x=x^4$, so $f(x)=2\cos x-x^4\Rightarrow f'(x)=-2\sin x-4x^3\Rightarrow x_{n+1}=x_n-\dfrac{2\cos x_n-x_n^4}{-2\sin x_n-4x_n^3}$. From the figure, the positive root of $2\cos x=x^4$ is near 1.$x_1=1\Rightarrow x_2\approx 1.014184$, $x_3\approx 1.013958\approx x_4$. So the positive root is 1.013958 , to six decimal places.



17.

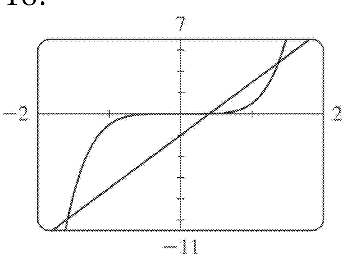


From the graph, we see that there appear to be points of intersection near $x=-0.7$ and $x=1.2$. Solving $x^4=1+x$ is the same as solving $f(x)=x^4-x-1=0$. $f(x)=x^4-x-1 \Rightarrow f'(x)=4x^3-1$, so $x_{n+1}=x_n-\dfrac{x_n^4-x_n-1}{4x_n^3-1}$.

$x_1=-0.7$	$x_1=1.2$
$x_2\approx-0.725253$	$x_2\approx1.221380$
$x_3\approx-0.724493$	$x_3\approx1.220745$
$x_4\approx-0.724492\approx x_5$	$x_4\approx1.220744\approx x_5$

To six decimal places, the roots of the equation are -0.724492 and 1.220744 .

18.



From the graph, we see that reasonable first approximations are $x=0.5$ and $x=\pm 1.5$. $f(x)=x^5-5x+2 \Rightarrow f'(x)=5x^4-5$, so $x_{n+1}=x_n-\dfrac{x_n^5-5x_n+2}{5x_n^4-5}$.

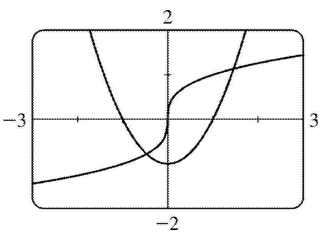
$x_1 = -1.5$
$x_2 \approx -1.593846$
$x_3 \approx -1.582241$
$x_4 \approx -1.582036 \approx x_5$

$x_1 = 0.5$
$x_2 = 0.4$
$x_3 \approx 0.402102 \approx x_4$

$x_1 = 1.5$
$x_2 \approx 1.396923$
$x_3 \approx 1.373078$
$x_4 \approx 1.371885$
$x_5 \approx 1.371882 \approx x_6$

To six decimal places, the roots are -1.582036 , 0.402102 , and 1.371882 .

19.



From the graph, we see that there appear to be points of intersection near $x=-0.5$ and $x=1.5$. Solving $\sqrt[3]{x}=x^2-1$ is the same as solving $f(x)=\sqrt[3]{x}-x^2+1=0$. $f(x)=\sqrt[3]{x}-x^2+1 \Rightarrow f'(x)=\frac{1}{3}x^{-2/3}-2x$, so

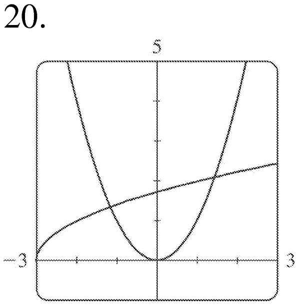
$$x_{n+1}=x_n-\frac{\sqrt[3]{x_n}-x_n^2+1}{\frac{1}{3}x_n^{-2/3}-2x_n} .$$

$x_1 = -0.5$
$x_2 \approx -0.471421$
$x_3 \approx -0.471074 \approx x_4$

$x_1 = 1.5$
$x_2 \approx 1.461653$
$x_3 \approx 1.461070 \approx x_4$

To six decimal places, the roots are -0.471074 and 1.461070 .

20.



From the graph, we see that there appear to be points of intersection near $x=-1.2$ and $x=1.5$. Solving

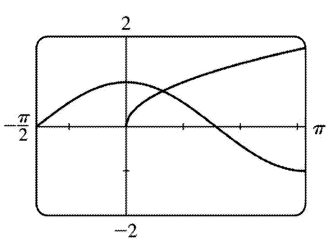
$\sqrt{x+3}=x^2$ is the same as solving $f(x)=x^2-\sqrt{x+3}=0$. $f(x)=x^2-\sqrt{x+3} \Rightarrow f'(x)=2x-\dfrac{1}{2\sqrt{x+3}}$, so

$$x_{n+1}=x_n-\frac{x_n^2-\sqrt{x_n+3}}{2x_n-1\Big/\left(2\sqrt{x_n+3}\right)} .$$

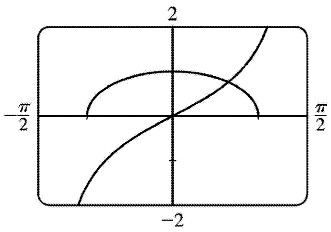
$x_1=-1.2$	$x_1=1.5$
$x_2\approx-1.164526$	$x_2\approx 1.453449$
$x_3\approx-1.164035\approx x_4$	$x_3\approx 1.452627\approx x_4$

To six decimal places, the roots of the equation are -1.164035 and 1.452627 .

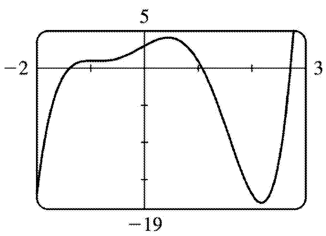
21. From the graph, there appears to be a point of intersection near $x=0.6$. Solving $\cos x=\sqrt{x}$ is the same as solving $f(x)=\cos x-\sqrt{x}=0$. $f(x)=\cos x-\sqrt{x} \Rightarrow f'(x)=-\sin x-1/\left(2\sqrt{x}\right)$, so $x_{n+1}=x_n-\dfrac{\cos x_n-\sqrt{x_n}}{-\sin x_n-1/\left(2\sqrt{x}\right)}$. Now $x_1=0.6\Rightarrow$ $x_2\approx 0.641928$, $x_3\approx 0.641714\approx x_4$. To six decimal places, the root of the equation is 0.641714 .



22. From the graph, there appears to be a point of intersection near $x=0.7$. Solving $\tan x=\sqrt{1-x^2}$ is the same as solving $f(x)=\tan x-\sqrt{1-x^2}=0$. $f(x)=\tan x-\sqrt{1-x^2} \Rightarrow f'(x)=\sec^2 x+x/\sqrt{1-x^2}$, so $x_{n+1}=x_n-\dfrac{\tan x_n-\sqrt{1-x_n^2}}{\sec^2 x_n+x_n/\sqrt{1-x_n^2}}$. $x_1=0.7\Rightarrow x_2\approx 0.652356$, $x_3\approx 0.649895$, $x_4\approx 0.649889\approx x_5$. To six decimal places, the root of the equation is 0.649889 .



23.

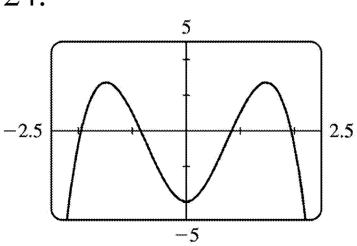


$f(x)=x^5-x^4-5x^3-x^2+4x+3 \Rightarrow f'(x)=5x^4-4x^3-15x^2-2x+4 \Rightarrow x_{n+1}=x_n-\dfrac{x_n^5-x_n^4-5x_n^3-x_n^2+4x_n+3}{5x_n^4-4x_n^3-15x_n^2-2x_n+4}$. From the graph of f , there appear to be roots near -1.4 , 1.1 , and 2.7 .

$x_1=-1.4$	$x_1=1.1$	$x_1=2.7$
$x_2\approx-1.39210970$	$x_2\approx1.07780402$	$x_2\approx2.72046250$
$x_3\approx-1.39194698$	$x_3\approx1.07739442$	$x_3\approx2.71987870$
$x_4\approx-1.39194691\approx x_5$	$x_4\approx1.07739428\approx x_5$	$x_4\approx2.71987822\approx x_5$

To eight decimal places, the roots of the equation are -1.39194691 , 1.07739428 , and 2.71987822 .

24.



Solving $x^2\left(4-x^2\right)=\dfrac{4}{x^2+1}$ is the same as solving $f(x)=4x^2-x^4-\dfrac{4}{x^2+1}=0$. $f'(x)=8x-4x^3+\dfrac{8x}{\left(x^2+1\right)^2} \Rightarrow$

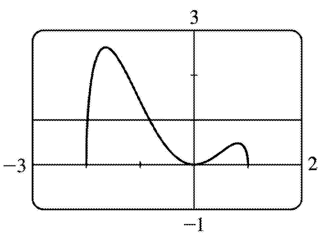
$x_{n+1}=x_n-\dfrac{4x_n^2-x_n^4-4/\left(x_n^2+1\right)}{8x_n-4x_n^3+8x_n/\left(x_n^2+1\right)^2}$. From the graph of $f(x)$, there appear to be roots near $x=\pm1.9$ and

$x=\pm 0.8$. Since f is even, we only need to find the positive roots.

$$\begin{array}{ll} x_1=0.8 & x_1=1.9 \\ x_2\approx 0.84287645 & x_2\approx 1.94689103 \\ x_3\approx 0.84310820 & x_3\approx 1.94383891 \\ x_4\approx 0.84310821\approx x_5 & x_4\approx 1.94382538\approx x_5 \end{array}$$

To eight decimal places, the roots of the equation are ± 0.84310821 and ± 1.94382538 .

25.



From the graph, $y=x^2\sqrt{2-x-x^2}$ and $y=1$ intersect twice, at $x\approx -2$ and at $x\approx -1$. $f(x)=x^2\sqrt{2-x-x^2}-1$ $\Rightarrow$

$$\begin{aligned} f'(x) &= x^2\cdot\frac{1}{2}\left(2-x-x^2\right)^{-1/2}(-1-2x)+\left(2-x-x^2\right)^{1/2}\cdot 2x \\ &= \frac{1}{2}x\left(2-x-x^2\right)^{-1/2}\left[x(-1-2x)+4\left(2-x-x^2\right)\right] \\ &= \frac{x\left(8-5x-6x^2\right)}{2\sqrt{(2+x)(1-x)}}\ , \end{aligned}$$

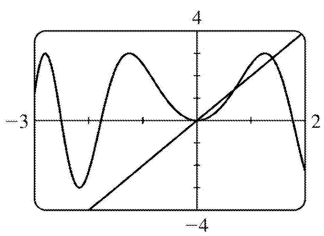
so $x_{n+1}=x_n-\dfrac{x^2\sqrt{2-x_n-x_n^2}-1}{\dfrac{x_n\left(8-5x_n-6x_n^2\right)}{2\sqrt{2(2+x_n)(1-x_n)}}}$. Trying $x_1=-2$ won't work because $f'(-2)$ is undefined, so we'll try $x_1=-1.95$.

$x_1 = -1.95$
$x_2 \approx -1.98580357$
$x_3 \approx -1.97899778$
$x_4 \approx -1.97807848$
$x_5 \approx -1.97806682$
$x_6 \approx -1.97806681 \approx x_7$

$x_1 = -0.8$
$x_2 \approx -0.82674444$
$x_3 \approx -0.82646236$
$x_4 \approx -0.82646233 \approx x_5$

To eight decimal places, the roots of the equation are -1.97806681 and -0.82646233 .

26.



From the equations $y=3\sin(x^2)$ and $y=2x$ and the graph, we deduce that one root of the equation $3\sin(x^2)=2x$ is $x=0$. We also see that the graphs intersect at approximately $x=0.7$ and $x=1.4$.

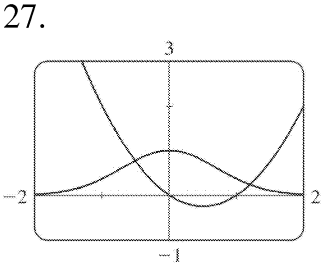
$f(x)=3\sin(x^2)-2x \Rightarrow f'(x)=3\cos(x^2)\cdot 2x-2$, so $x_{n+1}=x_n-\dfrac{3\sin(x_n^2)-2x_n}{6x_n\cos(x_n^2)-2}$.

$x_1=0.7$
$x_2 \approx 0.69303689$
$x_3 \approx 0.69299996 \approx x_4$

$x_1=1.4$
$x_2 \approx 1.39530295$
$x_3 \approx 1.39525078$
$x_4 \approx 1.39525077 \approx x_5$

To eight decimal places, the roots of the equation are 0.69299996 and 1.39525077 .

27.

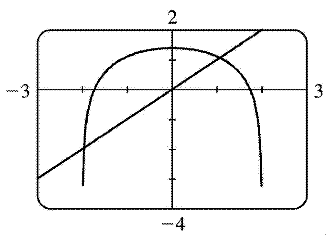


From the graph, we see that $y=e^{-x^2}$ and $y=x^2-x$ intersect twice. Good first approximations are $x=-0.5$ and $x=1.1$. $f(x)=e^{-x^2}-x^2+x \Rightarrow f'(x)=-2xe^{-x^2}-2x+1$, so $x_{n+1}=x_n-\dfrac{e^{-x_n^2}-x_n^2+x_n}{-2x_n e^{-x_n^2}-2x_n+1}$.

$x_1=-0.5$	$x_1=1.1$
$x_2\approx-0.51036446$	$x_2\approx1.20139754$
$x_3\approx-0.51031156\approx x_4$	$x_3\approx1.19844118$
	$x_4\approx1.19843871\approx x_5$

To eight decimal places, the roots of the equation are -0.51031156 and 1.19843871 .

28.



From the graph, $y=\ln\left(4-x^2\right)$ and $y=x$ intersect twice, at $x\approx-2$ and at $x\approx1$. $f(x)=\ln\left(4-x^2\right)-x \Rightarrow f'(x)=\dfrac{-2x}{4-x^2}-1$, so $x_{n+1}=x_n-\dfrac{\ln\left(4-x_n^2\right)-x_n}{\left[-2x_n/\left(4-x_n^2\right)\right]-1}$. Trying $x_1=-2$ won't work because it's not in the domain of $y=\ln\left(4-x^2\right)$. Trying $x_1=-1.9$ also fails after one iteration because the approximation x_2 is less than -2 . We try $x_1=-1.99$.

$x_1=-1.99$	$x_1=1.1$
$x_2\approx-1.97753026$	$x_2\approx1.05864851$
$x_3\approx-1.96741777$	$x_3\approx1.05800655$
$x_4\approx-1.96475281$	$x_4\approx1.05800640\approx x_5$
$x_5\approx-1.96463580$	
$x_6\approx-1.96463560\approx x_7$	

To eight decimal places, the roots of the equation are -1.96463560 and 1.05800640 .

29. **(a)** $f(x)=x^2-a \Rightarrow f'(x)=2x$, so Newton's method gives

$$x_{n+1}=x_n-\frac{x_n^2-a}{2x_n}=x_n-\frac{1}{2}x_n+\frac{a}{2x_n}=\frac{1}{2}x_n+\frac{a}{2x_n}=\frac{1}{2}\left(x_n+\frac{a}{x_n}\right).$$

(b) Using (a) with $a=1000$ and $x_1=\sqrt{900}=30$, we get $x_2\approx 31.666667$, $x_3\approx 31.622807$, and $x_4\approx 31.622777\approx x_5$. So $\sqrt{1000}\approx 31.622777$.

30. **(a)** $f(x)=\frac{1}{x}-a \Rightarrow f'(x)=-\frac{1}{x^2}$, so $x_{n+1}=x_n-\frac{1/x_n-a}{-1/x_n^2}=x_n+x_n-ax_n^2=2x_n-ax_n^2$.

(b) Using (a) with $a=1.6894$ and $x_1=\frac{1}{2}=0.5$, we get $x_2=0.5754$, $x_3\approx 0.588485$, and $x_4\approx 0.588789\approx x_5$. So $1/1.6984\approx 0.588789$.

31. $f(x)=x^3-3x+6 \Rightarrow f'(x)=3x^2-3$. If $x_1=1$, then $f'\left(x_1\right)=0$ and the tangent line used for approximating x_2 is horizontal. Attempting to find x_2 results in trying to divide by zero.

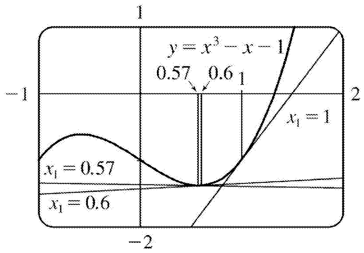
32. $x^3-x=1 \Leftrightarrow x^3-x-1=0$. $f(x)=x^3-x-1 \Rightarrow f'(x)=3x^2-1$, so $x_{n+1}=x_n-\frac{x_n^3-x_n-1}{3x_n^2-1}$.

(a) $x_1=1$, $x_2=1.5$, $x_3\approx 1.347826$, $x_4\approx 1.325200$, $x_5\approx 1.324718\approx x_6$

(b) $x_1=0.6$, $x_2=17.9$, $x_3\approx 11.946802$, $x_4\approx 7.985520$, $x_5\approx 5.356909$, $x_6\approx 3.624996$, $x_7\approx 2.505589$, $x_8\approx 1.820129$, $x_9\approx 1.461044$, $x_{10}\approx 1.339323$, $x_{11}\approx 1.324913$, $x_{12}\approx 1.324718\approx x_{13}$

(c) $x_1=0.57$, $x_2\approx -54.165455$, $x_3\approx -36.114293$, $x_4\approx -24.082094$, $x_5\approx -16.063387$, $x_6\approx -10.721483$, $x_7\approx -7.165534$, $x_8\approx -4.801704$, $x_9\approx -3.233425$, $x_{10}\approx -2.193674$, $x_{11}\approx -1.496867$, $x_{12}\approx -0.997546$, $x_{13}\approx -0.496305$, $x_{14}\approx -2.894162$, $x_{15}\approx -1.967962$, $x_{16}\approx -1.341355$, $x_{17}\approx -0.870187$, $x_{18}\approx -0.249949$, $x_{19}\approx -1.192219$, $x_{20}\approx -0.731952$, $x_{21}\approx 0.355213$, $x_{22}\approx -1.753322$, $x_{23}\approx -1.189420$, $x_{24}\approx -0.729123$, $x_{25}\approx 0.377844$, $x_{26}\approx -1.937872$, $x_{27}\approx -1.320350$, $x_{28}\approx -0.851919$, $x_{29}\approx -0.200959$, $x_{30}\approx -1.119386$, $x_{31}\approx -0.654291$,

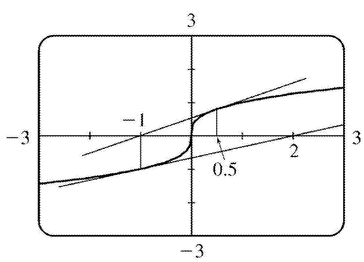
$x_{32} \approx 1.547010$, $x_{33} \approx 1.360051$, $x_{34} \approx 1.325828$, $x_{35} \approx 1.324719$, $x_{36} \approx 1.324718 \approx x_{37}$.



(d)

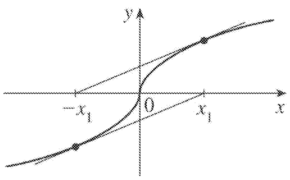
From the figure, we see that the tangent line corresponding to $x_1 = 1$ results in a sequence of approximations that converges quite quickly ($x_5 \approx x_6$). The tangent line corresponding to $x_1 = 0.6$ is close to being horizontal, so x_2 is quite far from the root. But the sequence still converges — just a little more slowly ($x_{12} \approx x_{13}$). Lastly, the tangent line corresponding to $x_1 = 0.57$ is very nearly horizontal, x_2 is farther away from the root, and the sequence takes more iterations to converge ($x_{36} \approx x_{37}$) .

33. For $f(x) = x^{1/3}$, $f'(x) = \frac{1}{3} x^{-2/3}$ and $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^{1/3}}{\frac{1}{3} x_n^{-2/3}} = x_n - 3x_n = -2x_n$. Therefore, each successive approximation becomes twice as large as the previous one in absolute value, so the sequence of approximations fails to converge to the root, which is 0 . In the figure, we have $x_1 = 0.5$, $x_2 = -2(0.5) = -1$, and $x_3 = -2(-1) = 2$.



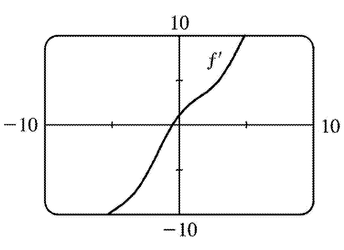
34. According to Newton's Method, for $x_n > 0$, $x_{n+1} = x_n - \frac{\sqrt{x_n}}{1/\left(2\sqrt{x_n}\right)} = x_n - 2x_n = -x_n$ and for $x_n < 0$,

$x_{n+1} = x_n - \frac{-\sqrt{-x_n}}{1/\left(2\sqrt{-x_n}\right)} = x_n - \left[-2\left(-x_n\right)\right] = -x_n$. So we can see that after choosing any value x_1 the subsequent values will alternate between $-x_1$ and x_1 and never approach the root.

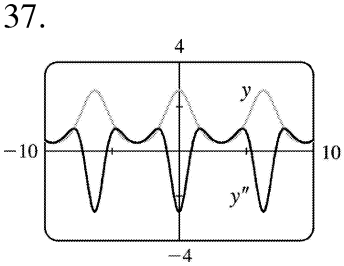


35. **(a)** $f(x)=3x^4-28x^3+6x^2+24x \Rightarrow f'(x)=12x^3-84x^2+12x+24 \Rightarrow f''(x)=36x^2-168x+12$. Now to solve $f'(x)=0$, try $x_1=\frac{1}{2} \Rightarrow x_2=x_1-\frac{f'(x_1)}{f''(x_1)}=\frac{2}{3} \Rightarrow x_3\approx 0.6455 \Rightarrow x_4\approx 0.6452 \Rightarrow x_5\approx 0.6452$. Now try $x_1=6 \Rightarrow x_2=7.12 \Rightarrow$
$x_3\approx 6.8353 \Rightarrow x_4\approx 6.8102 \Rightarrow x_5\approx 6.8100$. Finally try $x_1=-0.5 \Rightarrow x_2\approx -0.4571 \Rightarrow x_3\approx -0.4552 \Rightarrow$
$x_4\approx -0.4552$. Therefore, $x=-0.455$, 6.810 and 0.645 are all critical numbers correct to three decimal places.
(b) $f(-1)=13$, $f(7)=-1939$, $f(6.810)\approx -1949.07$, $f(-0.455)\approx -6.912$, $f(0.645)\approx 10.982$. Therefore, $f(6.810)\approx -1949.07$ is the absolute minimum correct to two decimal places.

36. $f(x)=x^2+\sin x \Rightarrow f'(x)=2x+\cos x$. $f'(x)$ exists for all x , so to find the minimum of f , we can examine the zeros of f' . From the graph of f' , we see that a good choice for x_1 is $x_1=-0.5$. Use $g(x)=2x+\cos x$ and $g'(x)=2-\sin x$ to obtain $x_2\approx -0.450627$, $x_3\approx -0.450184\approx x_4$. Since $f''(x)=2-\sin x>0$ for all x , $f(-0.450184)\approx -0.232466$ is the absolute minimum.

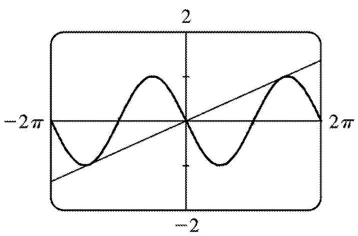


37.



From the figure, we see that $y=f(x)=e^{\cos x}$ is periodic with period 2π . To find the x –coordinates of the IP, we only need to approximate the zeros of y'' on $[0,\pi]$. $f'(x)=-e^{\cos x}\sin x \Rightarrow$ $f''(x)=e^{\cos x}\left(\sin^2 x-\cos x\right)$. Since $e^{\cos x}\neq 0$, we will use Newton's method with $g(x)=\sin^2 x-\cos x$, $g'(x)=2\sin x\cos x+\sin x$, and $x_1=1$. $x_2\approx 0.904173$, $x_3\approx 0.904557\approx x_4$. Thus, $(0.904557,1.855277)$ is the IP.

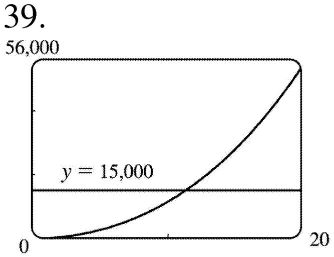
38.



$f(x)=-\sin x\Rightarrow f'(x)=-\cos x$. At $x=a$, the slope of the tangent line is $f'(a)=-\cos a$. The line through the origin and $(a,f(a))$ is $y=\dfrac{-\sin a-0}{a-0}x$. If this line is to be tangent to f at $x=a$, then its slope must equal $f'(a)$. Thus, $\dfrac{-\sin a}{a}=-\cos a\Rightarrow \tan a=a$. To solve this equation using Newton's method, let $g(x)=\tan x-x$,

$g'(x)=\sec^2 x-1$, and $x_{n+1}=x_n-\dfrac{\tan x_n-x_n}{\sec^2 x_n-1}$ with $x_1=4.5$ (estimated from the figure). $x_2\approx 4.493614$, $x_3\approx 4.493410$, $x_4\approx 4.493409\approx x_5$. Thus, the slope of the line that has the largest slope is $f'\left(x_5\right)\approx 0.217234$.

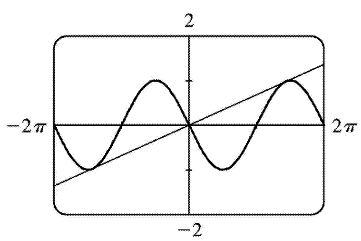
39.



The volume of the silo, in terms of its radius, is $V(r)=\pi r^2(30)+\frac{1}{2}\left(\frac{4}{3}\pi r^3\right)=30\pi r^2+\frac{2}{3}\pi r^3$. From a graph of V , we see that $V(r)=15,000$ at $r\approx 11$ ft. Now we use Newton's method to solve the equation $V(r)-15,000=0$. $\dfrac{dV}{dr}=60\pi r+2\pi r^2$, so

$r_{n+1}=r_n-\dfrac{30\pi r_n^2+\frac{2}{3}\pi r_n^3-15{,}000}{60\pi r_n+2\pi r_n^2}$. Taking $r_1=11$, we get $r_2\approx 11.2853$, $r_3\approx 11.2807\approx r_4$. So in order for the silo to hold 15 , 000 ft ^{3} of grain, its radius must be about 11.2807 ft.

40. Let the radius of the circle be r . Using $s=r\theta$, we have $5=r\theta$ and so $r=5/\theta$. From the Law of Cosines we get $4^2=r^2+r^2-2\cdot r\cdot r\cdot\cos\theta\Leftrightarrow 16=2r^2(1-\cos\theta)=2\left(5/\theta\right)^2(1-\cos\theta)$. Multiplying by θ^2 gives $16\theta^2=50(1-\cos\theta)$, so we take $f(\theta)=16\theta^2+50\cos\theta-50$ and $f'(\theta)=32\theta-50\sin\theta$. The formula for Newton's method is $\theta_{n+1}=\theta_n-\dfrac{16\theta_n^2+50\cos\theta_n-50}{32\theta_n-50\sin\theta_n}$. From the graph of f , we can use $\theta_1=2.2$, giving us $\theta_2\approx 2.2662$, $\theta_3\approx 2.2622\approx\theta_4$. So correct to four decimal places, the angle is 2.2622 radians $\approx 130^\circ$.



41. In this case, $A=18{,}000$, $R=375$, and $n=5(12)=60$. So the formula $A=\dfrac{R}{i}\left[1-(1+i)^{-n}\right]$ becomes $18{,}000=\dfrac{375}{x}\left[1-(1+x)^{-60}\right]\Leftrightarrow 48x=1-(1+x)^{-60}$ [multiply each term by $(1+x)^{60}$] $\Leftrightarrow$ $48x(1+x)^{60}-(1+x)^{60}+1=0$. Let the LHS be called $f(x)$, so that

$$\begin{aligned} f'(x) &= 48x(60)(1+x)^{59}+48(1+x)^{60}-60(1+x)^{59} \\ &= 12(1+x)^{59}\left[4x(60)+4(1+x)-5\right]=12(1+x)^{59}(244x-1) \end{aligned}$$

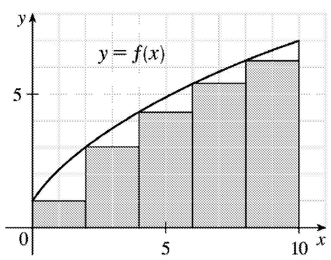
$x_{n+1}=x_n-\dfrac{48x_n\left(1+x_n\right)^{60}-\left(1+x_n\right)^{60}+1}{12\left(1+x_n\right)^{59}\left(244x_n-1\right)}$. An interest rate of 1% per month seems like a reasonable estimate for $x=i$. So let $x_1=1\%=0.01$, and we get $x_2\approx 0.0082202$, $x_3\approx 0.0076802$, $x_4\approx 0.0076291$, $x_5\approx 0.0076286\approx x_6$. Thus, the dealer is charging a monthly interest rate of 0.76286% (or 9.55% per year, compounded monthly).

42. **(a)** $p(x)=x^5-(2+r)x^4+(1+2r)x^3-(1-r)x^2+2(1-r)x+r-1 \Rightarrow$
$p'(x)=5x^4-4(2+r)x^3+3(1+2r)x^2-2(1-r)x+2(1-r)$. So we use
$x_{n+1}=x_n-\dfrac{x_n^5-(2+r)x_n^4+(1+2r)x_n^3-(1-r)x_n^2+2(1-r)x_n+r-1}{5x_n^4-4(2+r)x_n^3+3(1+2r)x_n^2-2(1-r)x_n+2(1-r)}$. We substitute in the value $r\approx 3.04042\times 10^{-6}$ in order to evaluate the approximations numerically. The libration point L_1 is slightly less than 1 AU from the Sun, so we take $x_1=0.95$ as our first approximation, and get $x_2\approx 0.96682$, $x_3\approx 0.97770$, $x_4\approx 0.98451$, $x_5\approx 0.98830$, $x_6\approx 0.98976$, $x_7\approx 0.98998$, $x_8\approx 0.98999\approx x_9$.
So, to five decimal places, L_1 is located 0.98999 AU from the Sun (or 0.01001 AU from Earth).

(b) In this case we use Newton's method with the function
$p(x)-2rx^2=x^5-(2+r)x^4+(1+2r)x^3-(1+r)x^2+2(1-r)x+r-1 \Rightarrow$
$\left[p(x)-2rx^2\right]'=5x^4-4(2+r)x^3+3(1+2r)x^2-2(1+r)x+2(1-r)$. So
$x_{n+1}=x_n-\dfrac{x_n^5-(2+r)x_n^4+(1+2r)x_n^3-(1+r)x_n^2+2(1-r)x_n+r-1}{5x_n^4-4(2+r)x_n^3+3(1+2r)x_n^2-2(1+r)x_n+2(1-r)}$. Again, we substitute $r\approx 3.04042\times 10^{-6}$. L_2 is slightly more than 1 AU from the Sun and, judging from the result of part (a), probably less than 0.02 AU from Earth. So we take $x_1=1.02$ and get $x_2\approx 1.01422$, $x_3\approx 1.01118$, $x_4\approx 1.01018$, $x_5\approx 1.01008\approx x_6$. So, to five decimal places, L_2 is located 1.01008 AU from the Sun (or 0.01008 AU from Earth).

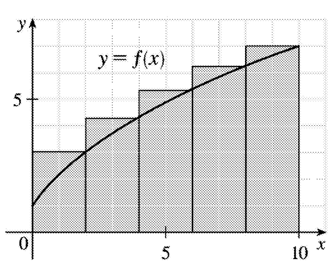
1. **(a)** Since f is *increasing* , we can obtain a *lower* estimate by using *left* endpoints. We are instructed to use five rectangles, so $n=5$.

$$\begin{aligned} L_5 &= \Sigma_{i=1}^{5} f\left(x_{i-1}\right) \Delta x \\ &= f\left(x_0\right) \cdot 2 + f\left(x_1\right) \cdot 2 + f\left(x_2\right) \cdot 2 + f\left(x_3\right) \cdot 2 + f\left(x_4\right) \cdot 2 \\ &= 2[f(0)+f(2)+f(4)+f(6)+f(8)] \\ &\approx 2(1+3+4.3+5.4+6.3) = 2(20) = 40 \end{aligned}$$



Since f is *increasing* , we can obtain an *upper* estimate by using *right* endpoints.

$$\begin{aligned} R_5 &= \Sigma_{i=1}^{5} f\left(x_i\right) \Delta x \\ &= 2\left[f\left(x_1\right) + f\left(x_2\right) + f\left(x_3\right) + f\left(x_4\right) + f\left(x_5\right)\right] \\ &= 2[f(2)+f(4)+f(6)+f(8)+f(10)] \\ &\approx 2(3+4.3+5.4+6.3+7) = 2(26) = 52 \end{aligned}$$

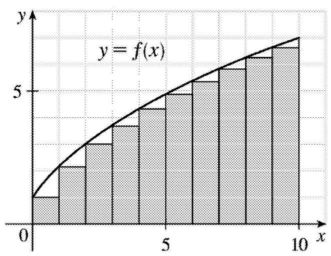


Comparing R_5 to L_5 , we see that we have added the area of the rightmost upper rectangle, $f(10) \cdot 2$, to the sum and subtracted the area of the leftmost lower rectangle, $f(0) \cdot 2$, from the sum. aaaaa

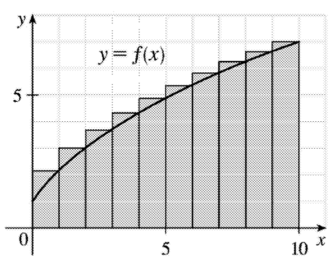
(b)

$$L_{10} = \Sigma_{i=1}^{10} f\left(x_{i-1}\right) \Delta x$$

$$=1\left[f\left(x_0\right)+f\left(x_1\right)+\cdots+f\left(x_9\right)\right]$$
$$=f(0)+f(1)+\cdots+f(9)$$
$$\approx 1+2.1+3+3.7+4.3+4.9+5.4+5.8+6.3+6.7$$
$$=43.2$$



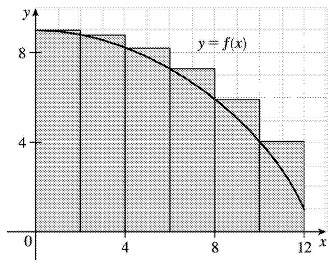
$$R_{10}=\sum_{i=1}^{10} f\left(x_i\right)\Delta x=f(1)+f(2)+\cdots+f(10)$$
$$=L_{10}+1\cdot f(10)-1\cdot f(0)$$ [add rightmost upper rectangle, subtract leftmost lower rectangle]
$$=43.2+7-1=49.2$$



2. **(a)**

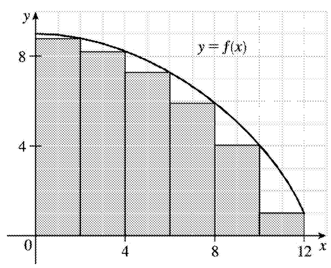
(i)

$$L_6=\sum_{i=1}^{6} f\left(x_{i-1}\right)\Delta x$$
$$=2\left[f\left(x_0\right)+f\left(x_1\right)+f\left(x_2\right)+f\left(x_3\right)+f\left(x_4\right)+f\left(x_5\right)\right]$$
$$=2[f(0)+f(2)+f(4)+f(6)+f(8)+f(10)]$$
$$\approx 2(9+8.8+8.2+7.3+5.9+4.1)$$
$$=2(43.3)=86.6$$



(ii) $R_6 = L_6 + 2 \cdot f(12) - 2 \cdot f(0)$

$\approx 86.6 + 2(1) - 2(9) = 70.6$

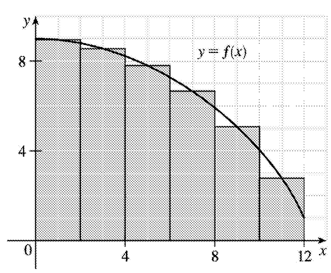


(iii) $M_6 = \sum_{i=1}^{6} f\left(x_i^*\right) \Delta x$

$= 2[f(1) + f(3) + f(5) + f(7) + f(9) + f(11)]$

$\approx 2(8.9 + 8.5 + 7.8 + 6.6 + 5.1 + 2.8)$

$= 2(39.7) = 79.4$



(b) Since f is *decreasing* , we obtain an *overestimate* by using *left* endpoints; that is, L_6 .

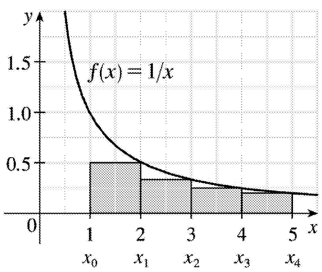
(c) Since f is *decreasing* , we obtain an *underestimate* by using *right* endpoints; that is, R_6 .

(d) M_6 gives the best estimate, since the area of each rectangle appears to be closer to the true area than the overestimates and underestimates in L_6 and R_6 .

3. **(a)**

$$R_4 = \Sigma_{i=1}^{4} f\left(x_i\right) \Delta x$$
$$= f\left(x_1\right) \cdot 1 + f\left(x_2\right) \cdot 1 + f\left(x_3\right) \cdot 1 + f\left(x_4\right) \cdot 1$$
$$= f(2) + f(3) + f(4) + f(5)$$
$$= \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} = \frac{77}{60} = 1.28\overline{3}$$

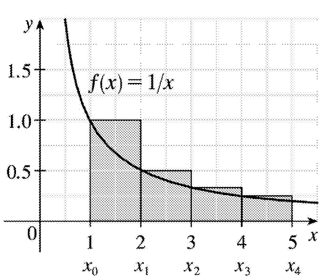
Since f is *decreasing* on $[1,5]$, an *underestimate* is obtained by using the *right* endpoint approximation, R_4 .



(b)

$$L_4 = \Sigma_{i=1}^{4} f\left(x_{i-1}\right) \Delta x$$
$$= f(1) + f(2) + f(3) + f(4)$$
$$= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{25}{12} = 2.08\overline{3}$$

L_4 is an overestimate. Alternatively, we could just add the area of the leftmost upper rectangle and subtract the area of the rightmost lower rectangle; that is, $L_4 = R_4 + f(1) \cdot 1 - f(5) \cdot 1$.

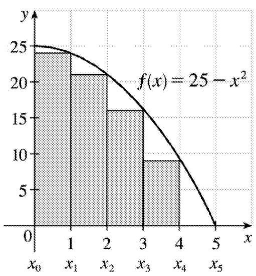


4. **(a)**

$$R_5 = \Sigma_{i=1}^{5} f\left(x_i\right) \Delta x$$

$=f\left(x_1\right)\cdot 1+f\left(x_2\right)\cdot 1+f\left(x_3\right)\cdot 1+f\left(x_4\right)\cdot 1+f\left(x_5\right)\cdot 1$
$=f(1)+f(2)+f(3)+f(4)+f(5)$
$=24+21+16+9+0=70$

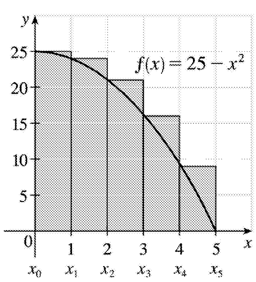
Since f is decreasing on $[0,5]$, R_5 is an underestimate.



(b)

$L_5=\sum_{i=1}^{5} f\left(x_{i-1}\right)\Delta x$
$=f(0)+f(1)+f(2)+f(3)+f(4)$
$=25+24+21+16+9=95$

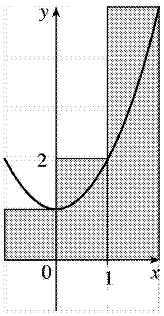
L_5 is an overestimate.

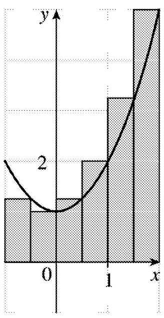


5. **(a)** $f(x)=1+x^2$ and $\Delta x=\dfrac{2-(-1)}{3}=1\Rightarrow R_3=1\cdot f(0)+1\cdot f(1)+1\cdot f(2)=1\cdot 1+1\cdot 2+1\cdot 5=8$.

$\Delta x=\dfrac{2-(-1)}{6}=0.5\Rightarrow$

$R_6=0.5[f(-0.5)+f(0)+f(0.5)+f(1)+f(1.5)+f(2)]$
$=0.5(1.25+1+1.25+2+3.25+5)$
$=0.5(13.75)=6.875$

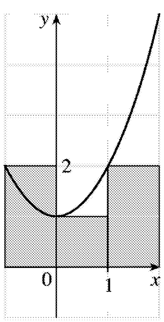


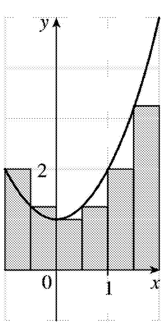


(b)

$L_3 = 1\cdot f(-1)+1\cdot f(0)+1\cdot f(1)=1\cdot 2+1\cdot 1+1\cdot 2=5$

$$\begin{aligned}L_6 &= 0.5[f(-1)+f(-0.5)+f(0)+f(0.5)+f(1)+f(1.5)]\\ &=0.5(2+1.25+1+1.25+2+3.25)\\ &=0.5(10.75)=5.375\end{aligned}$$

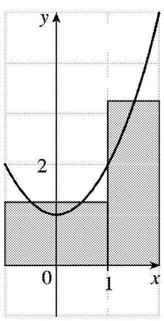


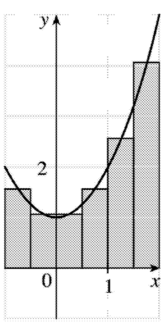


(c)

$$M_3 = 1\cdot f(-0.5) + 1\cdot f(0.5) + 1\cdot f(1.5)$$
$$= 1\cdot 1.25 + 1\cdot 1.25 + 1\cdot 3.25 = 5.75$$

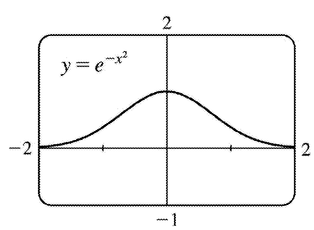
$$M_6 = 0.5[f(-0.75) + f(-0.25) + f(0.25)]$$
$$+ f(0.75) + f(1.25) + f(1.75)]$$
$$= 0.5(1.5625 + 1.0625 + 1.0625 + 1.5625 + 2.5625 + 4.0625)$$
$$= 0.5(11.875) = 5.9375$$





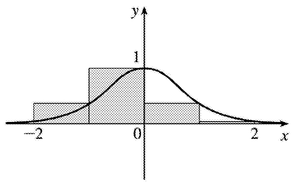
(d) M_6 appears to be the best estimate.

6. **(a)**

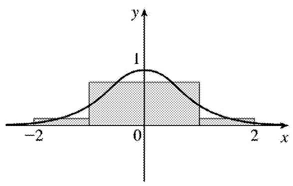


(b) $f(x) = e^{-x^2}$ and $\Delta x = \dfrac{2-(-2)}{4} = 1 \Rightarrow$

(i) $R_4 = 1\cdot f(-1) + 1\cdot f(0) + 1\cdot f(1) + 1\cdot f(2) = e^{-1} + 1 + e^{-1} + e^{-4} \approx 1.754$

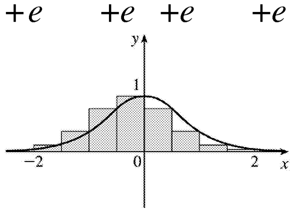


(ii) $M_4 = 1 \cdot f(-1.5) + 1 \cdot f(-0.5) + 1 \cdot f(0.5) + 1 \cdot f(1.5) = e^{-2.25} + e^{-0.25} + e^{-0.25} + e^{-2.25} \approx 1.768$

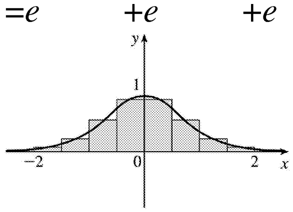


(c)

(i) $R_8 = 0.5\,[\,f(-1.5) + f(-1) + f(-0.5) + f(0)\ \big| + f(0.5) + f(1) + f(1.5) + f(2)\,] = e^{-2.25} + e^{-1} + e^{-0.25} + 1$
$+ e^{-0.25} + e^{-1} + e^{-2.25} + e^{-4} \approx 1.761$

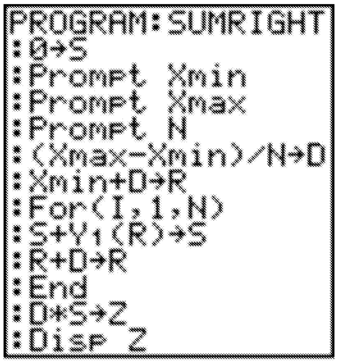


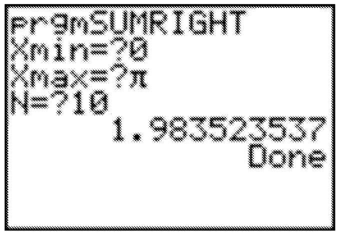
(ii) Due to the symmetry of the figure, we see that $M_8 = (0.5)(2)[f(0.25) + f(0.75) + f(1.25) + f(1.75)]$
$= e^{-0.0625} + e^{-0.5625} + e^{-1.5625} + e^{3.0625} \approx 1.766$



7. Here is one possible algorithm (ordered sequence of operations) for calculating the sums:
(a) Let SUM =0 , X_MIN =0 , X_MAX $=\pi$, N =10 (or 30 or 50 , depending on which sum we are calculating), DELTA_X =(XMAX−XMIN) / N, and RIGHT_ENDPOINT = X_MIN + DELTA_X.
(b) Repeat steps 2a, 2b in sequence until RIGHT_ENDPOINT > X_MAX.
(c) Add sin (RIGHTENDPOINT) to SUM.
(d) Add DELTA_X to RIGHT_ENDPOINT.
At the end of this procedure, (DELTAX)$\cdot$(SUM) is equal to the answer we are looking for. We find that $R_{10} = \frac{\pi}{10} \sum_{i=1}^{10} \sin\left(\frac{i\pi}{10}\right) \approx 1.9835$, $R_{30} = \frac{\pi}{30} \sum_{i=1}^{30} \sin\left(\frac{i\pi}{30}\right) \approx 1.9982$, and $R_{50} = \frac{\pi}{50} \sum_{i=1}^{50} \sin\left(\frac{i\pi}{50}\right) \approx 1.9993$. It appears that the exact area is 2 .
Shown below is program SUMRIGHT and its output from a TI−83 Plus calculator. To generalize the

program, we have input (rather than assigned) values for Xmin, Xmax, and N. Also, the function, $\sin x$, is assigned to Y_1, enabling us to evaluate any right sum merely by changing Y_1 and running the program.

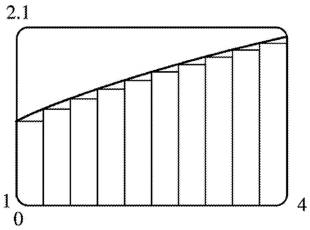




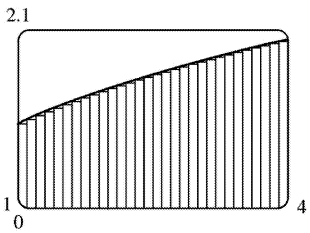
8. We can use the algorithm from Exercise 7 with X_MIN =1 , X_MAX =2 , and $1\big/(\text{RIGHTENDPOINT})^2$ instead of $\sin\,(\text{RIGHTENDPOINT})$ in step 2a. We find that $R_{10}=\dfrac{1}{10}\sum_{i=1}^{10}\dfrac{1}{(1+i/10)^2}\approx 0.4640$, $R_{30}=\dfrac{1}{30}\sum_{i=1}^{30}\dfrac{1}{(1+i/30)^2}\approx 0.4877$, and $R_{50}=\dfrac{1}{50}\sum_{i=1}^{50}\dfrac{1}{(1+i/50)^2}\approx 0.4926$. It appears that the exact area is $\dfrac{1}{2}$.

9. In Maple, we have to perform a number of steps before getting a numerical answer. After loading the student package we use the command sum: =leftsum(x^(1 / 2) , x=1..4,10 [or 30, or 50]) ; which gives us the expression in summation notation. To get a numerical approximation to the sum, we use evalf(left_sum);. Mathematica does not have a special command for these sums, so we must type them in manually. For example, the first left sum is given by
(3 / 10)*Sum[Sqrt[1 + 3 (i − 1) / 10] , {i, 1, 10}] ,], and we use the N command on the resulting output to get a numerical approximation.
In Derive, we use the LEFT_RIEMANN command to get the left sums, but must define the right sums ourselves. (We can define a new function using LEFT_RIEMANN with k ranging from 1 to n instead of from 0 to $n-1$.)

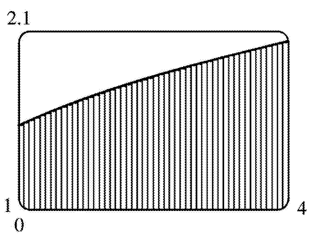
(a) With $f(x)=\sqrt{x}$, $1\le x\le 4$, the left sums are of the form $L_n=\dfrac{3}{n}\sum_{i=1}^{n}\sqrt{1+\dfrac{3(i-1)}{n}}$. Specifically, $L_{10}\approx 4.5148$, $L_{30}\approx 4.6165$, and $L_{50}\approx 4.6366$. The right sums are of the form $R_n=\dfrac{3}{n}\sum_{i=1}^{n}\sqrt{1+\dfrac{3i}{n}}$. Specifically, $R_{10}\approx 4.8148$, $R_{30}\approx 4.7165$, and $R_{50}\approx 4.6966$.
(b) In Maple, we use the leftbox and rightbox commands (with the same arguments as leftsum and rightsum above) to generate the graphs.



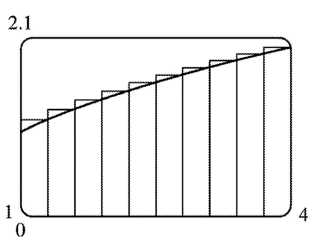
left endpoints, n=10



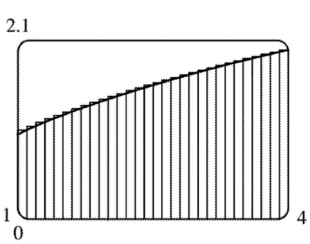
left endpoints, n=30



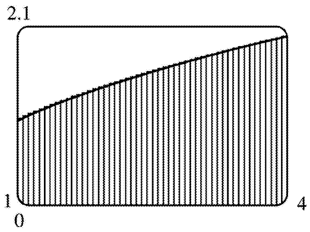
left endpoints, n=50



right endpoints, n=10



right endpoints, n=30

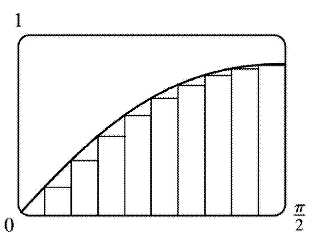


right endpoints, n=50

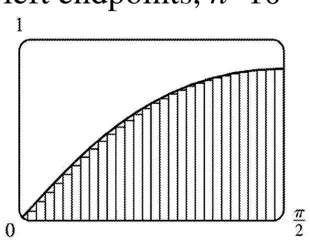
(c) We know that since $\sqrt{x}$ is an increasing function on $(1,4)$, all of the left sums are smaller than the actual area, and all of the right sums are larger than the actual area. Since the left sum with n=50 is about 4.637>4.6 and the right sum with n=50 is about 4.697<4.7 , we conclude that $4.6<L_{50}<$ exact area $<R_{50}<4.7$, so the exact area is between 4.6 and 4.7 .

10.

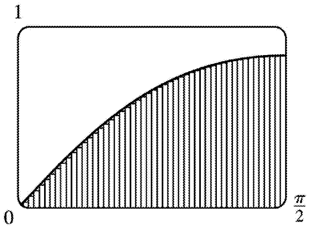
(a) With $f(x)=\sin(\sin x)$, $0\le x\le \frac{\pi}{2}$, the left sums are of the form $L_n=\frac{\pi}{2n}\sum_{i=1}^{n}\sin\left(\sin\frac{\pi(i-1)}{2n}\right)$. In particular, $L_{10}\approx0.8251$, $L_{30}\approx0.8710$, and $L_{50}\approx0.8799$. The right sums are of the form $R_n=\frac{\pi}{2n}\sum_{i=1}^{n}\sin\left(\sin\frac{\pi i}{2n}\right)$. In particular, $R_{10}\approx0.9573$, $R_{30}\approx0.9150$, and $R_{50}\approx0.9064$.
(b) In Maple, we use the leftbox and rightbox commands (with the same arguments as leftsum and rightsum above) to generate the graphs.



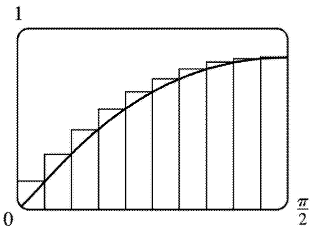
left endpoints, n=10



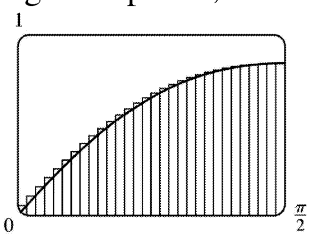
left endpoints, n=30



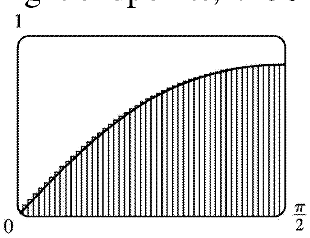
left endpoints, n=50



right endpoints, n=10



right endpoints, n=30



righnt endpoints, n=10

(c) We know that since sin (sin x) is an increasing function on $\left(0, \frac{\pi}{2}\right)$, all of the left sums are smaller than the actual area, and all of the right sums are larger than the actual area. Since the left sum with n=50 is about $0.8799>0.87$ and the right sum with n=50 is about $0.9064<0.91$, we conclude that $0.87<L_{50}<$ exact area $<R_{50}<0.91$, so the exact area is between 0.87 and 0.91 .

11. Since v is an increasing function, L_6 will give us a lower estimate and R_6 will give us an upper estimate.

$$L_6 = (0\text{ft/s})\,(0.5\text{s}) + (6.2)\,(0.5) + (10.8)\,(0.5) + (14.9)\,(0.5) + (18.1)\,(0.5) + (19.4)\,(0.5)$$
$$= 0.5\,(69.4) = 34.7 \text{ ft}$$

$$R_6 = 0.5\,(6.2+10.8+14.9+18.1+19.4+20.2) = 0.5\,(89.6) = 44.8 \text{ ft}$$

12.
(a)

$$d \approx L_5 = (30 \text{ ft / s })(12 \text{ s }) + 28 \cdot 12 + 25 \cdot 12 + 22 \cdot 12 + 24 \cdot 12$$
$$= (30+28+25+22+24) \cdot 12 = 129 \cdot 12 = 1548 \text{ ft}$$

(b) $d \approx R_5 = (28+25+22+24+27) \cdot 12 = 126 \cdot 12 = 1512 \text{ ft}$

(c) The estimates are neither lower nor upper estimates since v is neither an increasing nor a decreasing function of t .

13. Lower estimate for oil leakage: $R_5 = (7.6+6.8+6.2+5.7+5.3)(2) = (31.6)(2) = 63.2$ L.
Upper estimate for oil leakage: $L_5 = (8.7+7.6+6.8+6.2+5.7)(2) = (35)(2) = 70$ L.

14. We can find an upper estimate by using the final velocity for each time interval. Thus, the distance d traveled after 62 seconds can be approximated by
$d = \sum_{i=1}^{6} v\left(t_i\right) \Delta t_i = (185\text{ft/s})\ (10\text{s}) + 319 \cdot 5 + 447 \cdot 5 + 742 \cdot 12 + 1325 \cdot 27 + 1445 \cdot 3 = 54\ ,694$ ft

15. For a decreasing function, using left endpoints gives us an overestimate and using right endpoints results in an underestimate. We will use M_6 to get an estimate. $\Delta t = 1$, so

$$M_6 = 1[v(0.5)+v(1.5)+v(2.5)+v(3.5)+v(4.5)+v(5.5)]$$
$$\approx 55+40+28+18+10+4 = 155\text{ft}$$

For a very rough check on the above calculation, we can draw a line from $(0{,}70)$ to $(6{,}0)$ and calculate the area of the triangle: $\frac{1}{2}(70)(6) = 210$. This is clearly an overestimate, so our midpoint estimate of 155 is reasonable.

16. For an increasing function, using left endpoints gives us an underestimate and using right endpoints results in an overestimate. We will use M_6 to get an estimate. $\Delta t = \frac{30-0}{6} = 5 \text{ s} = \frac{5}{3600}$ h $= \frac{1}{720}$ h.

$$M_6 = \frac{1}{720}[v(2.5)+v(7.5)+v(12.5)+v(17.5)+v(22.5)+v(27.5)]$$
$$= \frac{1}{720}(31.25+66+88+103.5+113.75+119.25) = \frac{1}{720}(521.75) \approx 0.725\text{km}$$

For a very rough check on the above calculation, we can draw a line from $(0{,}0)$ to $(30{,}120)$ and calculate the area
of the triangle:

$\frac{1}{2}(30)(120)=1800$. Divide by 3600 to get 0.5 , which is clearly an underestimate, making our midpoint estimate of 0.725 seem reasonable. Of course, answers will vary due to different readings of the graph.

17. $f(x)=\sqrt[4]{x}$, $1\le x\le 16$. $\Delta x=(16-1)/n=15/n$ and $x_i=1+i\,\Delta x=1+15i/n$.

$A=\lim_{n\to\infty} R_n=\lim_{n\to\infty}\sum_{i=1}^{n} f\left(x_i\right)\Delta x=\lim_{n\to\infty}\sum_{i=1}^{n}\sqrt[4]{1+\frac{15i}{n}}\cdot\frac{15}{n}$.

18. $f(x)=\frac{\ln x}{x}$, $3\le x\le 10$. $\Delta x=(10-3)/n=7/n$ and $x_i=3+i\,\Delta x=3+7i/n$.

$A=\lim_{n\to\infty} R_n=\lim_{n\to\infty}\sum_{i=1}^{n} f\left(x_i\right)\Delta x=\lim_{n\to\infty}\sum_{i=1}^{n}\frac{\ln(3+7i/n)}{3+7i/n}\cdot\frac{7}{n}$.

19. $f(x)=x\cos x$, $0\le x\le\frac{\pi}{2}$. $\Delta x=(\frac{\pi}{2}-0)/n=\frac{\pi}{2}/n$ and $x_i=0+i\,\Delta x=\frac{\pi}{2}\,i/n$.

$A=\lim_{n\to\infty} R_n=\lim_{n\to\infty}\sum_{i=1}^{n} f\left(x_i\right)\Delta x=\lim_{n\to\infty}\sum_{i=1}^{n}\frac{i\pi}{2n}\cos\left(\frac{i\pi}{2n}\right)\cdot\frac{\pi}{2n}$.

20. $\lim_{n\to\infty}\sum_{i=1}^{n}\frac{2}{n}\left(5+\frac{2i}{n}\right)^{10}$ can be interpreted as the area of the region lying under the graph of $y=(5+x)^{10}$ on the interval $[0,2]$, since for $y=(5+x)^{10}$ on $[0,2]$ with $\Delta x=\frac{2-0}{n}=\frac{2}{n}$, $x_i=0+i\,\Delta x=\frac{2i}{n}$, and $x_i^*=x_i$, the expression for the area is $A=\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x=\lim_{n\to\infty}\sum_{i=1}^{n}\left(5+\frac{2i}{n}\right)^{10}\frac{2}{n}$. Note that the answer is not unique. We could use $y=x^{10}$ on $[5,7]$ or, in general, $y=((5-n)+x)^{10}$ on $[n,n+2]$.

21. $\lim_{n\to\infty}\sum_{i=1}^{n}\frac{\pi}{4n}\tan\frac{i\pi}{4n}$ can be interpreted as the area of the region lying under the graph of $y=\tan x$ on the interval $\left[0,\frac{\pi}{4}\right]$, since for $y=\tan x$ on $\left[0,\frac{\pi}{4}\right]$ with $\Delta x=\frac{\pi/4-0}{n}=\frac{\pi}{4n}$, $x_i=0+i\,\Delta x=\frac{i\pi}{4n}$, and $x_i^*=x_i$, the expression for the area is $A=\lim_{n\to\infty}\sum_{i=1}^{n} f\left(x_i^*\right)\Delta x=\lim_{n\to\infty}\sum_{i=1}^{n}\tan\left(\frac{i\pi}{4n}\right)\frac{\pi}{4n}$. Note that this answer is not unique, since the expression for the area is the same for the function $y=\tan(x-k\pi)$ on the interval $\left[k\pi,k\pi+\frac{\pi}{4}\right]$, where k is any integer.

22. **(a)**

$\Delta x=\dfrac{1-0}{n}=\dfrac{1}{n}$ and $x_i=0+i\,\Delta x=\dfrac{i}{n}$. $A=\lim\limits_{n\to\infty} R_n=\lim\limits_{n\to\infty}\sum_{i=1}^{n} f\left(x_i\right)\Delta x=\lim\limits_{n\to\infty}\sum_{i=1}^{n}\left(\dfrac{i}{n}\right)^3\cdot\dfrac{1}{n}$.

(b) $\lim\limits_{n\to\infty}\sum_{i=1}^{n}\dfrac{i^3}{n^3}\cdot\dfrac{1}{n}=\lim\limits_{n\to\infty}\dfrac{1}{n^4}\sum_{i=1}^{n} i^3=\lim\limits_{n\to\infty}\dfrac{1}{n^4}\left[\dfrac{n(n+1)}{2}\right]^2=\lim\limits_{n\to\infty}\dfrac{(n+1)^2}{4n^2}=\dfrac{1}{4}\lim\limits_{n\to\infty}\left(1+\dfrac{1}{n}\right)^2=\dfrac{1}{4}$

23. **(a)** $y=f(x)=x^5$. $\Delta x=\dfrac{2-0}{n}=\dfrac{2}{n}$ and $x_i=0+i\,\Delta x=\dfrac{2i}{n}$.

$A=\lim\limits_{n\to\infty} R_n=\lim\limits_{n\to\infty}\sum_{i=1}^{n} f\left(x_i\right)\Delta x=\lim\limits_{n\to\infty}\sum_{i=1}^{n}\left(\dfrac{2i}{n}\right)^5\cdot\dfrac{2}{n}=\lim\limits_{n\to\infty}\sum_{i=1}^{n}\dfrac{32i^5}{n^5}\cdot\dfrac{2}{n}=\lim\limits_{n\to\infty}\dfrac{64}{n^6}\sum_{i=1}^{n} i^5$.

(b) $\sum_{i=1}^{n} i^5=\dfrac{n^2(n+1)^2\left(2n^2+2n-1\right)}{12}$

(c) $\lim\limits_{n\to\infty}\dfrac{64}{n^6}\cdot\dfrac{n^2(n+1)^2\left(2n^2+2n-1\right)}{12}=\dfrac{64}{12}\lim\limits_{n\to\infty}\dfrac{\left(n^2+2n+1\right)\left(2n^2+2n-1\right)}{n^2\cdot n^2}$

$=\dfrac{16}{3}\lim\limits_{n\to\infty}\left(1+\dfrac{2}{n}+\dfrac{1}{n^2}\right)\left(2+\dfrac{2}{n}-\dfrac{1}{n^2}\right)=\dfrac{16}{3}\cdot 1\cdot 2=\dfrac{32}{3}$

24. From Example 3(a), we have $A=\lim\limits_{n\to\infty}\dfrac{2}{n}\sum_{i=1}^{n} e^{-2i/n}$. Using a CAS, $\sum_{i=1}^{n} e^{-2i/n}=\dfrac{e^{-2}\left(e^2-1\right)}{e^{2/n}-1}$ and

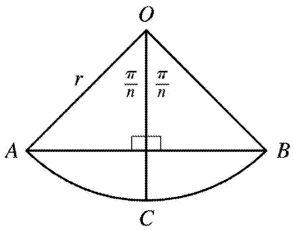
$\lim\limits_{n\to\infty}\dfrac{2}{n}\cdot\dfrac{e^{-2}\left(e^2-1\right)}{e^{2/n}-1}=e^{-2}\left(e^2-1\right)\approx 0.8647$, whereas the estimate from Example 3(b) using M_{10} was 0.8632 .

25. $y=f(x)=\cos x$. $\Delta x=\dfrac{b-0}{n}=\dfrac{b}{n}$ and $x_i=0+i\,\Delta x=\dfrac{bi}{n}$.

$$A=\lim_{n\to\infty} R_n=\lim_{n\to\infty}\sum_{i=1}^{n} f\left(x_i\right)\Delta x=\lim_{n\to\infty}\sum_{i=1}^{n}\cos\left(\frac{bi}{n}\right)\cdot\frac{b}{n}=\lim_{n\to\infty}\left[\frac{b\sin\left(b\left(\frac{1}{2n}+1\right)\right)}{2n\sin\left(\frac{b}{2n}\right)}-\frac{b}{2n}\right]=\sin b$$

If $b=\dfrac{\pi}{2}$, then $A=\sin\dfrac{\pi}{2}=1$.

26. **(a)**



The diagram shows one of the n congruent triangles, $\triangle AOB$, with central angle $2\pi/n$. O is the center of the circle and AB is one of the sides of the polygon. Radius OC is drawn so as to bisect $\angle AOB$. It follows that OC intersects AB at right angles and bisects AB. Thus, $\triangle AOB$ is divided into two right triangles with legs of length $\frac{1}{2}(AB)=r\sin(\pi/n)$ and $r\cos(\pi/n)$.

$\triangle AOB$ has area $2\cdot\frac{1}{2}[r\sin(\pi/n)][r\cos(\pi/n)]=r^2\sin(\pi/n)\cos(\pi/n)=\frac{1}{2}r^2\sin(2\pi/n)$, so $A_n=n\cdot$ area $(\triangle AOB)=\frac{1}{2}nr^2\sin(2\pi/n)$.

(b) To use Equation 3.4.2, $\lim\limits_{\theta\to 0}\frac{\sin\theta}{\theta}=1$, we need to have the same expression in the denominator as we have in the argument of the sine function — in this case, $2\pi/n$.

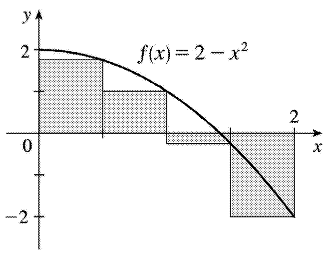
$\lim\limits_{n\to\infty}A_n=\lim\limits_{n\to\infty}\frac{1}{2}nr^2\sin(2\pi/n)=\lim\limits_{n\to\infty}\frac{1}{2}nr^2\frac{\sin(2\pi/n)}{2\pi/n}\cdot\frac{2\pi}{n}=\lim\limits_{n\to\infty}\frac{\sin(2\pi/n)}{2\pi/n}\pi r^2$. Let $\theta=\frac{2\pi}{n}$.

Then as $n\to\infty$, $\theta\to 0$, so $\lim\limits_{n\to\infty}\frac{\sin(2\pi/n)}{2\pi/n}\pi r^2=\lim\limits_{\theta\to 0}\frac{\sin\theta}{\theta}\pi r^2=(1)\pi r^2=\pi r^2$.

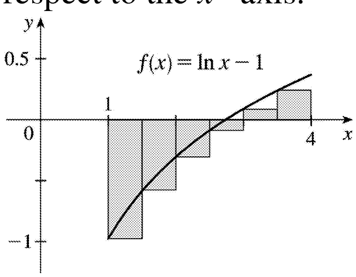
1.

$$R_4 = \sum_{i=1}^{4} f\left(x_i\right) \Delta x$$
$$= 0.5[f(0.5)+f(1)+f(1.5)+f(2)]$$
$$= 0.5[1.75+1+(-0.25)+(-2)]$$
$$= 0.5(0.5) = 0.25$$

The Riemann sum represents the sum of the areas of the two rectangles above the x -axis minus the sum of the areas of the two rectangles below the x -axis; that is, the *net area* of the rectangles with respect to the x -axis.



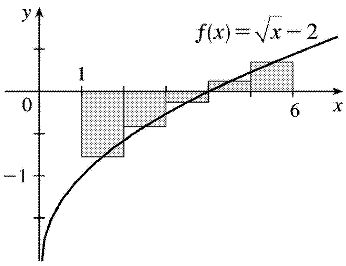
2. $L_6 = \sum_{i=1}^{6} f\left(x_{i-1}\right) \Delta x = 0.5[f(1)+f(1.5)+f(2)+f(2.5)+f(3)+f(3.5)]$
$\approx 0.5(-1-0.5945349-0.3068528-0.0837093+0.0986123+0.2527630) = 0.5(-1.6337217) \approx -0.816861$
The Riemann sum represents the sum of the areas of the two rectangles above the x -axis minus the sum of the areas of the four rectangles below the x -axis; that is, the *net area* of the rectangles with respect to the x -axis.



3.

$$M_5 = \sum_{i=1}^{5} f\left(\bar{x}_i\right) \Delta x$$
$$= 1[f(1.5)+f(2.5)+f(3.5)$$
$$+f(4.5)+f(5.5)]$$
$$\approx -0.856759$$

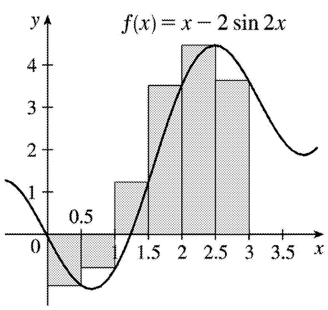
The Riemann sum represents the sum of the areas of the two rectangles above the x -axis minus the sum of the areas of the three rectangles below the x -axis.



4. **(a)**

$$R_6 = \sum_{i=1}^{6} f\left(x_i\right) \Delta x$$
$$= 0.5[f(0.5) + f(1) + f(1.5) + f(2)$$
$$+ f(2.5) + f(3)]$$
$$\approx 5.353254$$

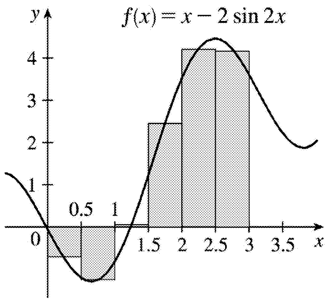
The Riemann sum represents the sum of the areas of the four rectangles above the x –axis minus the sum of the areas of the two rectangles below the x –axis.



(b)

$$M_6 = \sum_{i=1}^{6} f\left(\bar{x}_i\right) \Delta x$$
$$= 0.5[f(0.25) + f(0.75) + f(1.25) + f(1.75)$$
$$+ f(2.25) + f(2.75)]$$
$$\approx 4.458461$$

The Riemann sum represents the sum of the areas of the four rectangles above the x –axis minus the sum of the areas of the two rectangles below the x –axis.



5. $\Delta x=(b-a)/n=(8-0)/4=8/4=2$.

(a) Using the right endpoints to approximate $\int_0^8 f(x)dx$, we have

$\sum_{i=1}^4 f\left(x_i\right)\Delta x=2[f(2)+f(4)+f(6)+f(8)]\approx 2[1+2+(-2)+1]=4$.

(b) Using the left endpoints to approximate $\int_0^8 f(x)dx$, we have

$\sum_{i=1}^4 f\left(x_{i-1}\right)\Delta x=2[f(0)+f(2)+f(4)+f(6)]\approx 2[2+1+2+(-2)]=6$.

(c) Using the midpoint of each subinterval to approximate $\int_0^8 f(x)dx$, we have

$\sum_{i=1}^4 f\left(\bar{\mathrm{x}}_i\right)\Delta x=2[f(1)+f(3)+f(5)+f(7)]\approx 2[3+2+1+(-1)]=10$.

6. **(a)** Using the right endpoints to approximate $\int_{-3}^3 g(x)dx$, we have

$$\begin{aligned}\sum_{i=1}^6 g\left(x_i\right)\Delta x &= 1[g(-2)+g(-1)+g(0)+g(1)+g(2)+g(3)]\\ &\approx 1-0.5-1.5-1.5-0.5+2.5=-0.5\end{aligned}$$

(b) Using the left endpoints to approximate $\int_{-3}^3 g(x)dx$, we have

$$\begin{aligned}\sum_{i=1}^6 g\left(x_{i-1}\right)\Delta x &= 1[g(-3)+g(-2)+g(-1)+g(0)+g(1)+g(2)]\\ &\approx 2+1-0.5-1.5-1.5-0.5=-1\end{aligned}$$

(c) Using the midpoint of each subinterval to approximate $\int_{-3}^3 g(x)dx$, we have

$$\begin{aligned}\sum_{i=1}^6 g\left(\bar{\mathrm{x}}_i\right)\Delta x &= 1[g(-2.5)+g(-1.5)+g(-0.5)+g(0.5)+g(1.5)+g(2.5)]\\ &\approx 1.5+0-1-1.75-1+0.5=-1.75\end{aligned}$$

7. Since f is increasing, $L_5\leq\int_0^{25} f(x)dx\leq R_5$.

Lower estimate $= L_5 = \sum_{i=1}^{5} f\left(x_{i-1}\right)\Delta x = 5[f(0)+f(5)+f(10)+f(15)+f(20)]$

$= 5(-42-37-25-6+15) = 5(-95) = -475$

Upper estimate $= R_5 = \sum_{i=1}^{5} f\left(x_i\right)\Delta x = 5[f(5)+f(10)+f(15)+f(20)+f(25)]$

$= 5(-37-25-6+15+36) = 5(-17) = -85$

8. **(a)** Using the right endpoints to approximate $\int_0^6 f(x)\,dx$, we have

$\sum_{i=1}^{3} f\left(x_i\right)\Delta x = 2[f(2)+f(4)+f(6)] = 2(8.3+2.3-10.5) = 0.2$

(b) Using the left endpoints to approximate $\int_0^6 f(x)\,dx$, we have

$\sum_{i=1}^{3} f\left(x_{i-1}\right)\Delta x = 2[f(0)+f(2)+f(4)] = 2(9.3+8.3+2.3) = 39.8$

(c) Using the midpoint of each interval to approximate $\int_0^6 f(x)\,dx$, we have

$\sum_{i=1}^{3} f\left(\bar{x}_i\right)\Delta x = 2\,[f(1)+f(3)+f(5)] = 2(9.0+6.5-7.6) = 15.8$.

The estimate using the right endpoints must be less than $\int_0^6 f(x)\,dx$, since if we take x_i^* to be the right endpoint x_i of each interval, then $f\left(x_i\right)\le f(x)$ for all x on $\left[x_{i-1}, x_i\right]$, which implies that $f\left(x_i\right)\Delta x \le \int_{x_{i-1}}^{x_i} f(x)\,dx$, and so the sum $\sum_{i=1}^{3}\left[f\left(x_i\right)\Delta x\right] \le \sum_{i=1}^{3}\left[\int_{x_{i-1}}^{x_i} f(x)\,dx\right] = \int_0^6 f(x)\,dx$.
Similarly, if we take x_i^* to be the left endpoint x_{i-1} of each interval, then $f\left(x_{i-1}\right)\ge f(x)$ for all x on $\left[x_{i-1}, x_i\right]$, and so $\sum_{i=1}^{3}\left[f\left(x_{i-1}\right)\Delta x\right] \ge \int_0^6 f(x)\,dx$. We cannot say anything about the midpoint estimate.

9. $\Delta x = (10-2)/4 = 2$, so the endpoints are 2 , 4 , 6 , 8 , and 10 , and the midpoints are 3 , 5 , 7 , and 9 . The Midpoint Rule gives

$\int_2^{10}\sqrt{x^3+1}\;dx \approx \sum_{i=1}^{4} f\left(\bar{x}_i\right)\Delta x = 2\left(\sqrt{3^3+1}+\sqrt{5^3+1}+\sqrt{7^3+1}+\sqrt{9^3+1}\right) \approx 124.1644$.

10. $\Delta x = (\pi-0)/6 = \frac{\pi}{6}$, so the endpoints are 0 , $\frac{\pi}{6}$, $\frac{2\pi}{6}$, $\frac{3\pi}{6}$, $\frac{4\pi}{6}$, $\frac{5\pi}{6}$, and $\frac{6\pi}{6}$, and the midpoints are $\frac{\pi}{12}$, $\frac{3\pi}{12}$, $\frac{5\pi}{12}$, $\frac{7\pi}{12}$, $\frac{9\pi}{12}$, and $\frac{11\pi}{12}$. The Midpoint Rule gives

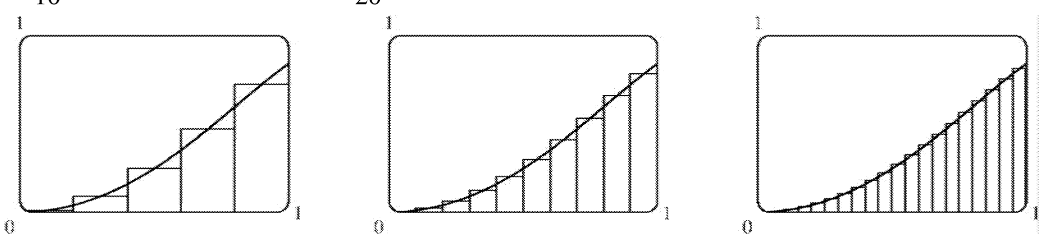
$\int_0^{\pi}\sec\,(x/3)\,dx \approx \sum_{i=1}^{6} f\left(\bar{x}_i\right)\Delta x = \frac{\pi}{6}\left(\sec\frac{\pi}{36}+\sec\frac{3\pi}{36}+\sec\frac{5\pi}{36}+\sec\frac{7\pi}{36}+\sec\frac{9\pi}{36}+\sec\frac{11\pi}{36}\right) \approx 3.9379$

.

11. $\Delta x=(1-0)/5=0.2$, so the endpoints are 0, 0.2, 0.4, 0.6, 0.8, and 1, and the midpoints are 0.1, 0.3, 0.5, 0.7, and 0.9. The Midpoint Rule gives
$\int_0^1 \sin(x^2)dx \approx \sum_{i=1}^{5} f\left(\bar{x}_i\right)\Delta x = 0.2\left[\sin(0.1)^2+\sin(0.3)^2+\sin(0.5)^2+\sin(0.7)^2+\sin(0.9)^2\right] \approx 0.3084$.

12. $\Delta x=(5-1)/4=1$, so the endpoints are 1, 2, 3, 4, and 5, and the midpoints are 1.5, 2.5, 3.5, and 4.5. The Midpoint Rule gives
$\int_1^5 x^2 e^{-x} dx \approx \sum_{n=1}^{4} f\left(\bar{x}_i\right)\Delta x = 1\left[(1.5)^2 e^{-1.5}+(2.5)^2 e^{-2.5}+(3.5)^2 e^{-3.5}+(4.5)^2 e^{-4.5}\right] \approx 1.6099$.

13. In Maple, we use the command with(student) to load the sum and box commands, then m:=middlesum(sin(x^),x=0..1,5); which gives us the sum in summation notation, then M:=evalf(m); which gives $M_5 \approx 0.30843908$, confirming the result of Exercise 11. The command middlebox(sin(x^),x=0..1,5) generates the graph. Repeating for $n=10$ and $n=20$ gives $M_{10} \approx 0.30981629$ and $M_{20} \approx 0.31015563$.



14. See the solution to Exercise 5.1.7 for a possible algorithm to calculate the sums. With $\Delta x=(1-0)/100=0.01$ and subinterval endpoints 1, 1.01, 1.02, … , 1.99, 2, we calculate that the left Riemann sum is $L_{100}=\sum_{i=1}^{100} \sin(x_{i-1})\Delta x \approx 0.30607$, and the right Riemann sum is
$R_{100}=\sum_{i=1}^{100} \sin\left(x_i^2\right)\Delta x \approx 0.31448$.
Since $f(x)=\sin\left(x^2\right)$ is an increasing function, we must have $L_{100} \le \int_0^1 \sin\left(x^2\right) dx \le R_{100}$, so $0.306 < L_{100} \le \int_0^1 \sin\left(x^2\right) dx \le R_{100} < 0.315$. Therefore, the approximate value $0.3084 \approx 0.31$ in Exercise 11 must be accurate to two decimal places.

15. We'll create the table of values to approximate $\int_0^{\pi} \sin x dx$ by using the program in the solution to Exercise 5.1.7 with $Y_1=\sin x$, Xmin=0 , $X =\pi$, and $n=5$, 10 , 50 , and 100 .

The values of R_n appear to be approaching 2 .

n	R_n

5	1.933766
10	1.983524
50	1.999342
100	1.999836

16. $\int_0^2 e^{-x^2}\,dx$ with $n=5$, 10 , 50 , and 100 .

n	L_n	R_n
5	1.077467	0.684794
10	0.980007	0.783670
50	0.901705	0.862438
100	0.891896	0.872262

The value of the integral lies between 0.872 and 0.892 . Note that $f(x)=e^{-x^2}$ is decreasing on $(0,2)$. We cannot make a similar statement for $\int_{-1}^2 e^{-x^2}\,dx$ since f is increasing on $(-1,0)$.

17. On $[0,\pi]$, $\lim_{n\to\infty}\sum_{i=1}^n x_i \sin x_i\,\Delta x=\int_0^\pi x\sin x\,dx$.

18. On $[1,5]$, $\lim_{n\to\infty}\sum_{i=1}^n \frac{e^{x_i}}{1+x_i}\,\Delta x=\int_1^5 \frac{e^x}{1+x}\,dx$.

19. On $[1,8]$, $\lim_{n\to\infty}\sum_{i=1}^n \sqrt{2x_i^*+\left(x_i^*\right)^2}\,\Delta x=\int_1^8 \sqrt{2x+x^2}\,dx$.

20. On [0,2] , $\lim_{n\to\infty}\sum_{i=1}^n \Delta x=\int_0^2 (4-3x^2+6x^5)dx$.

21. Note that $\Delta x=\frac{5-(-1)}{n}=\frac{6}{n}$ and $x_i=-1+i\,\Delta x=-1+\frac{6i}{n}$.

$$\int_{-1}^5 (1+3x)dx = \lim_{n\to\infty}\sum_{i=1}^n f\left(x_i\right)\Delta x=\lim_{n\to\infty}\sum_{i=1}^n\left[1+3\left(-1+\frac{6i}{n}\right)\right]\frac{6}{n}$$

$$= \lim_{n\to\infty}\frac{6}{n}\sum_{i=1}^n\left[-2+\frac{18i}{n}\right]=\lim_{n\to\infty}\frac{6}{n}\left[\sum_{i=1}^n(-2)+\sum_{i=1}^n\frac{18i}{n}\right]$$

$$= \lim_{n\to\infty} \frac{6}{n}\left[-2n+\frac{18}{n}\sum_{i=1}^{n} i\right] = \lim_{n\to\infty} \frac{6}{n}\left[-2n+\frac{18}{n}\cdot\frac{n(n+1)}{2}\right]$$

$$= \lim_{n\to\infty}\left[-12+\frac{108}{n^2}\cdot\frac{n(n+1)}{2}\right] = \lim_{n\to\infty}\left[-12+54\,\frac{n+1}{n}\right]$$

$$= \lim_{n\to\infty}\left[-12+54\left(1+\frac{1}{n}\right)\right] = -12+54\cdot 1 = 42$$

22.

$$\int_1^4 (x^2+2x-5)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x \;[\;\Delta x = 3/n \text{ and } x_i = 1+3i/n\;]$$

$$= \lim_{n\to\infty}\sum_{i=1}^{n}\left[\left(1+\frac{3i}{n}\right)^2+2\left(1+\frac{3i}{n}\right)-5\right]\left(\frac{3}{n}\right)$$

$$= \lim_{n\to\infty}\frac{3}{n}\left[\sum_{i=1}^{n}\left(1+\frac{6i}{n}+\frac{9i^2}{n^2}+2+\frac{6i}{n}-5\right)\right]$$

$$= \lim_{n\to\infty}\frac{3}{n}\left[\sum_{i=1}^{n}\left(\frac{9}{n^2}\cdot i^2+\frac{12}{n}\cdot i-2\right)\right]$$

$$= \lim_{n\to\infty}\frac{3}{n}\left[\frac{9}{n^2}\sum_{i=1}^{n} i^2+\frac{12}{n}\sum_{i=1}^{n} i-\sum_{i=1}^{n} 2\right]$$

$$= \lim_{n\to\infty}\left(\frac{27}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}+\frac{36}{n^2}\cdot\frac{n(n+1)}{2}-\frac{6}{n}\cdot n\right)$$

$$= \lim_{n\to\infty}\left(\frac{9}{2}\cdot\frac{n+1}{n}\cdot\frac{2n+1}{n}+18\cdot\frac{n+1}{n}-6\right)$$

$$= \lim_{n\to\infty}\left[\frac{9}{2}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)+18\left(1+\frac{1}{n}\right)-6\right] = \frac{9}{2}\cdot 1\cdot 2+18\cdot 1-6 = 21$$

23. Note that $\Delta x = \frac{2-0}{n} = \frac{2}{n}$ and $x_i = 0+i\Delta x = \frac{2i}{n}$.

$$\int_0^2 \left(2-x^2\right)dx = \lim_{n\to\infty}\sum_{i=1}^{n} f\left(x_i\right)\Delta x = \lim_{n\to\infty}\sum_{i=1}^{n}\left(2-\frac{4i^2}{n^2}\right)\left(\frac{2}{n}\right)$$

$$= \lim_{n\to\infty}\frac{2}{n}\left[\sum_{i=1}^{n} 2-\frac{4}{n^2}\sum_{i=1}^{n} i^2\right] = \lim_{n\to\infty}\frac{2}{n}\left(2n-\frac{4}{n^2}\sum_{i=1}^{n} i^2\right)$$

$$=\lim_{n\to\infty}\left[4-\frac{8}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}\right]=\lim_{n\to\infty}\left(4-\frac{4}{3}\cdot\frac{n+1}{n}\cdot\frac{2n+1}{n}\right)$$

$$=\lim_{n\to\infty}\left[4-\frac{4}{3}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)\right]=4-\frac{4}{3}\cdot 1\cdot 2=\frac{4}{3}$$

24.

$$\int_0^5\left(1+2x^3\right)dx=\lim_{n\to\infty}\Sigma_{i=1}^n f\left(x_i\right)\Delta x$$

$$=\lim_{n\to\infty}\Sigma_{i=1}^n\left(1+2\cdot\frac{125i^3}{n^3}\right)\left(\frac{5}{n}\right)=\lim_{n\to\infty}\frac{5}{n}\left[\Sigma_{i=1}^n 1+\frac{250}{n^3}\Sigma_{i=1}^n i^3\right]$$

$$=\lim_{n\to\infty}\frac{5}{n}\left(1\cdot n+\frac{250}{n^3}\Sigma_{i=1}^n i^3\right)=\lim_{n\to\infty}\left[5+\frac{1250}{n^4}\cdot\frac{n^2(n+1)^2}{4}\right]$$

$$=\lim_{n\to\infty}\left[5+312.5\cdot\frac{(n+1)^2}{n^2}\right]=\lim_{n\to\infty}\left[5+312.5\left(1+\frac{1}{n}\right)^2\right]$$

$$=5+312.5=317.5$$

25. Note that $\Delta x=\frac{2-1}{n}=\frac{1}{n}$ and $x_i=1+i\Delta x=1+i\,(1/n)=1+i/n$.

$$\int_1^2 x^3\,dx=\lim_{n\to\infty}\Sigma_{i=1}^n f\left(x_i\right)\Delta x=\lim_{n\to\infty}\Sigma_{i=1}^n\left(1+\frac{i}{n}\right)^3\left(\frac{1}{n}\right)=\lim_{n\to\infty}\frac{1}{n}\Sigma_{i=1}^n\left(\frac{n+i}{n}\right)^3$$

$$=\lim_{n\to\infty}\frac{1}{n^4}\Sigma_{i=1}^n\left(n^3+3n^2i+3ni^2+i^3\right)=\lim_{n\to\infty}\frac{1}{n^4}\left[\Sigma_{i=1}^n n^3+\Sigma_{i=1}^n 3n^2i+\Sigma_{i=1}^n 3ni^2+\Sigma_{i=1}^n i^3\right]$$

$$=\lim_{n\to\infty}\frac{1}{n^4}\left[n\cdot n^3+3n^2\Sigma_{i=1}^n i+3n\Sigma_{i=1}^n i^2+\Sigma_{i=1}^n i^3\right]$$

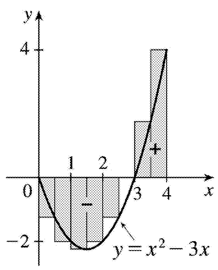
$$=\lim_{n\to\infty}\left[1+\frac{3}{n^2}\cdot\frac{n(n+1)}{2}+\frac{3}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}+\frac{1}{n^4}\cdot\frac{n^2(n+1)^2}{4}\right]$$

$$=\lim_{n\to\infty}\left[1+\frac{3}{2}\cdot\frac{n+1}{n}+\frac{1}{2}\cdot\frac{n+1}{n}\cdot\frac{2n+1}{n}+\frac{1}{4}\cdot\frac{(n+1)^2}{n^2}\right]$$

$$=\lim_{n\to\infty}\left[1+\frac{3}{2}\left(1+\frac{1}{n}\right)+\frac{1}{2}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)+\frac{1}{4}\left(1+\frac{1}{n}\right)^2\right]=1+\frac{3}{2}+\frac{1}{2}\cdot 2+\frac{1}{4}=3.75$$

26. **(a)** $\Delta x=(4-0)/8=0.5$ and $x_i^*=x_i=0.5i$.

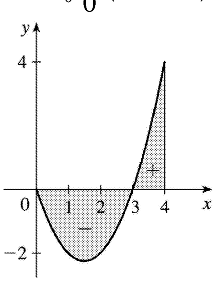
$$\int_0^4 \left(x^2-3x\right)dx\approx \sum_{i=1}^{8} f\left(x_i^*\right)\Delta x$$
$$=0.5\left\{\left[0.5^2-3(0.5)\right]+\left[1.0^2-3(1.0)\right]+\cdots +\left[3.5^2-3(3.5)\right]+\left[4.0^2-3(4.0)\right]\right\}$$
$$=\frac{1}{2}\left(-\frac{5}{4}-2-\frac{9}{4}-2-\frac{5}{4}+0+\frac{7}{4}+4\right)=-1.5$$

(b) 

(c) $\int_0^4 \left(x^2-3x\right)dx=\lim\limits_{n\to\infty}\sum_{i=1}^{n}\left[\left(\frac{4i}{n}\right)^2-3\left(\frac{4i}{n}\right)\right]\left(\frac{4}{n}\right)$

$$=\lim_{n\to\infty}\frac{4}{n}\left[\frac{16}{n^2}\sum_{i=1}^{n} i^2-\frac{12}{n}\sum_{i=1}^{n} i\right]$$
$$=\lim_{n\to\infty}\left[\frac{64}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}-\frac{48}{n^2}\cdot\frac{n(n+1)}{2}\right]$$
$$=\lim_{n\to\infty}\left[\frac{32}{3}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)-24\left(1+\frac{1}{n}\right)\right]$$
$$=\frac{32}{3}\cdot 2-24=-\frac{8}{3}$$

(d) $\int_0^4 \left(x^2-3x\right)dx=A_1-A_2$, where A_1 is the area marked + and A_2 is the area marked − .



27.

$$\int_a^b x\,dx = \lim_{n\to\infty} \frac{b-a}{n} \sum_{i=1}^n \left[a+\frac{b-a}{n} i \right] = \lim_{n\to\infty} \left[\frac{a(b-a)}{n} \sum_{i=1}^n 1 + \frac{(b-a)^2}{n^2} \sum_{i=1}^n i \right]$$

$$= \lim_{n\to\infty} \left[\frac{a(b-a)}{n} n + \frac{(b-a)^2}{n^2} \cdot \frac{n(n+1)}{2} \right] = a(b-a) + \lim_{n\to\infty} \frac{(b-a)^2}{2} \left(1+\frac{1}{n} \right)$$

$$= a(b-a) + \frac{1}{2}(b-a)^2 = (b-a)\left(a+\frac{1}{2}b-\frac{1}{2}a \right) = (b-a)\frac{1}{2}(b+a) = \frac{1}{2}\left(b^2-a^2\right)$$

28.

$$\int_a^b x^2\,dx = \lim_{n\to\infty} \frac{b-a}{n} \sum_{i=1}^n \left[a+\frac{b-a}{n} i \right]^2 = \lim_{n\to\infty} \frac{b-a}{n} \sum_{i=1}^n \left[a^2+2a\frac{b-a}{n} i + \frac{(b-a)^2}{n^2} i^2 \right]$$

$$= \lim_{n\to\infty} \left[\frac{(b-a)^3}{n^3} \sum_{i=1}^n i^2 + \frac{2a(b-a)^2}{n^2} \sum_{i=1}^n i + \frac{a^2(b-a)}{n} \sum_{i=1}^n 1 \right]$$

$$= \lim_{n\to\infty} \left[\frac{(b-a)^3}{n^3} \frac{n(n+1)(2n+1)}{6} + \frac{2a(b-a)^2}{n^2} \frac{n(n+1)}{2} + \frac{a^2(b-a)}{n} n \right]$$

$$= \lim_{n\to\infty} \left[\frac{(b-a)^3}{6} \cdot 1 \cdot \left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) + a(b-a)^2 \cdot 1 \cdot \left(1+\frac{1}{n}\right) + a^2(b-a) \right]$$

$$= \frac{(b-a)^3}{3} + a(b-a)^2 + a^2(b-a) = \frac{b^3-3ab^2+3a^2b-a^3}{3} + ab^2-2a^2b+a^3+a^2b-a^3$$

$$= \frac{b^3}{3} - \frac{a^3}{3} - ab^2+a^2b+ab^2-a^2b = \frac{b^3-a^3}{3}$$

29. $f(x)=\frac{x}{1+x^5}$, $a=2$, $b=6$, and $\Delta x=\frac{6-2}{n}=\frac{4}{n}$. Using Equation 3, we get $x_i^*=x_i=2+i\,\Delta x=2+\frac{4i}{n}$, so

$$\int_2^6 \frac{x}{1+x^5}\,dx = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \sum_{i=1}^n \frac{2+\frac{4i}{n}}{1+\left(2+\frac{4i}{n}\right)^5} \cdot \frac{4}{n} .$$

30. $\Delta x=\frac{10-1}{n}=\frac{9}{n}$ and $x_i=1+i\,\Delta x=1+\frac{9i}{n}$, so

$\int_1^{10}(x-4\ln x)dx=\lim\limits_{n\to\infty} R_n=\lim\limits_{n\to\infty} \sum_{i=1}^{n}\left[\left(1+\frac{9i}{n}\right)-4\ln\left(1+\frac{9i}{n}\right)\right]\cdot\frac{9}{n}$.

31. $\Delta x=(\pi-0)/n=\pi/n$ and $x_i^*=x_i=\pi i/n$.

$\int_0^{\pi}\sin 5x dx=\lim\limits_{n\to\infty}\sum_{i=1}^{n}\left(\sin 5x_i\right)\left(\frac{\pi}{n}\right)=\lim\limits_{n\to\infty}\sum_{i=1}^{n}\left(\sin\frac{5\pi i}{n}\right)\frac{\pi}{n}=\pi\lim\limits_{n\to\infty}\frac{1}{n}\cot\left(\frac{5\pi}{2n}\right)=\pi\left(\frac{2}{5\pi}\right)=\frac{2}{5}$

32. $\Delta x=(10-2)/n=8/n$ and $x_i^*=x_i=2+8i/n$.

$$\int_2^{10}x^6 dx=\lim_{n\to\infty}\sum_{i=1}^{n}\left(2+\frac{8i}{n}\right)^6\left(\frac{8}{n}\right)=8\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\left(2+\frac{8i}{n}\right)^6$$

$$=8\lim_{n\to\infty}\frac{1}{n}\cdot\frac{64\left(58{,}593n^6+164{,}052n^5+131{,}208n^4-27{,}776n^2+2048\right)}{21n^5}$$

$$=8\left(\frac{1{,}249{,}984}{7}\right)=\frac{9{,}999{,}872}{7}\approx 1{,}428{,}553.1$$

33. **(a)** Think of $\int_0^2 f(x)dx$ as the area of a trapezoid with bases 1 and 3 and height 2 . The area of a trapezoid is $A=\frac{1}{2}(b+B)h$, so $\int_0^2 f(x)dx=\frac{1}{2}(1+3)2=4$.

(b)

$$\int_0^5 f(x)dx=\underbrace{\int_0^2 f(x)dx}_{\text{trapezoid}}+\underbrace{\int_2^3 f(x)dx}_{\text{rectangle}}+\underbrace{\int_3^5 f(x)dx}_{\text{triangle}}$$

$$=\frac{1}{2}(1+3)2+3\cdot 1+\frac{1}{2}\cdot 2\cdot 3=4+3+3=10$$

(c) $\int_5^7 f(x)dx$ is the negative of the area of the triangle with base 2 and height 3 .

$\int_5^7 f(x)dx=-\frac{1}{2}\cdot 2\cdot 3=-3$.

(d) $\int_7^9 f(x)dx$ is the negative of the area of a trapezoid with bases 3 and 2 and height 2 , so it equals $-\frac{1}{2}(B+b)h=-\frac{1}{2}(3+2)2=-5$. Thus, $\int_0^9 f(x)dx=\int_0^5 f(x)dx+\int_5^7 f(x)dx+\int_7^9 f(x)dx=10+(-3)+(-5)=2$.

34. **(a)**

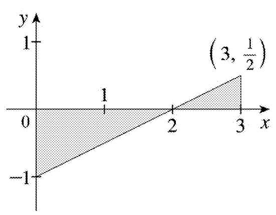
$\int_0^2 g(x)dx = \frac{1}{2} \cdot 4 \cdot 2 = 4$ (area of a triangle)

(b) $\int_2^6 g(x)dx = -\frac{1}{2}\pi(2)^2 = -2\pi$ (negative of the area of a semicircle)

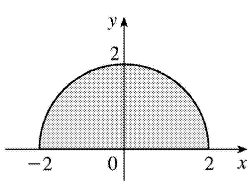
(c) $\int_6^7 g(x)dx = \frac{1}{2} \cdot 1 \cdot 1 = \frac{1}{2}$ (area of a triangle)

$\int_0^7 g(x)dx = \int_0^2 g(x)dx + \int_2^6 g(x)dx + \int_6^7 g(x)dx = 4 - 2\pi + \frac{1}{2} = 4.5 - 2\pi$

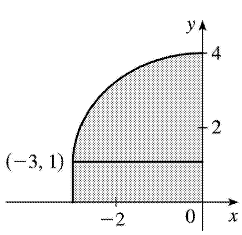
35. $\int_0^3 \left(\frac{1}{2}x - 1 \right) dx$ can be interpreted as the area of the triangle above the x-axis minus the area of the triangle below the x-axis; that is, $\frac{1}{2}(1)\left(\frac{1}{2}\right) - \frac{1}{2}(2)(1) = \frac{1}{4} - 1 = -\frac{3}{4}$.



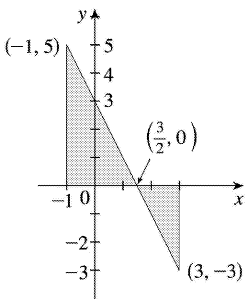
36. $\int_{-2}^2 \sqrt{4-x^2}\, dx$ can be interpreted as the area under the graph of $f(x) = \sqrt{4-x^2}$ between $x = -2$ and $x = 2$. This is equal to half the area of the circle with radius 2 , so $\int_{-2}^2 \sqrt{4-x^2}\, dx = \frac{1}{2}\pi \cdot 2^2 = 2\pi$.



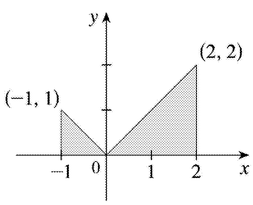
37. $\int_{-3}^0 \left(1 + \sqrt{9-x^2}\right) dx$ can be interpreted as the area under the graph of $f(x) = 1 + \sqrt{9-x^2}$ between $x = -3$ and $x = 0$. This is equal to one-quarter the area of the circle with radius 3 , plus the area of the rectangle, so $\int_{-3}^0 \left(1 + \sqrt{9-x^2}\right) dx = \frac{1}{4}\pi \cdot 3^2 + 1 \cdot 3 = 3 + \frac{9}{4}\pi$.



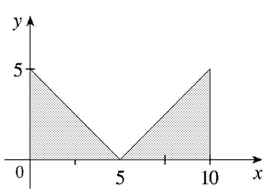
38. $\int_{-1}^{3}(3-2x)\,dx$ can be interpreted as the area of the triangle above the x –axis minus the area of the triangle below the x –axis; that is, $\frac{1}{2}\left(\frac{5}{2}\right)(5)-\frac{1}{2}\left(\frac{3}{2}\right)(3)=\frac{25}{4}-\frac{9}{4}=4$.



39. $\int_{-1}^{2}|x|\,dx$ can be interpreted as the sum of the areas of the two shaded triangles; that is, $\frac{1}{2}(1)(1)+\frac{1}{2}(2)(2)=\frac{1}{2}+\frac{4}{2}=\frac{5}{2}$.



40. $\int_{0}^{10}|x-5|\,dx$ can be interpreted as the sum of the areas of the two shaded triangles; that is, $2\left(\frac{1}{2}\right)(5)(5)=25$.



41.

$$\begin{aligned}\int_{9}^{4}\sqrt{t}\,dt &= -\int_{4}^{9}\sqrt{t}\,dt\\ &= -\int_{4}^{9}\sqrt{x}\,dx\\ &= -\frac{38}{3}\end{aligned}$$

42. $\int_1^1 x^2 \cos x\,dx=0$ since the limits of integration are equal.

43. $\int_0^1 \left(5-6x^2\right)dx=\int_0^1 5\,dx-6\int_0^1 x^2\,dx=5(1-0)-6\left(\frac{1}{3}\right)=5-2=3$

44. $\int_1^3 \left(2e^x-1\right)dx=2\int_1^3 e^x\,dx-\int_1^3 1\,dx=2\left(e^3-e\right)-1(3-1)=2e^3-2e-2$

45. $\int_1^3 e^{x+2}\,dx=\int_1^3 e^x\cdot e^2\,dx=e^2\int_1^3 e^x\,dx=e^2\left(e^3-e\right)=e^5-e^3$

46.

$$\int_0^{\pi/2}(2\cos x-5x)\,dx=\int_0^{\pi/2}2\cos x\,dx-\int_0^{\pi/2}5x\,dx=2\int_0^{\pi/2}\cos x\,dx-5\int_0^{\pi/2}x\,dx$$
$$=2(1)-5\,\frac{(\pi/2)^2-0^2}{2}=2-\frac{5\pi^2}{8}$$

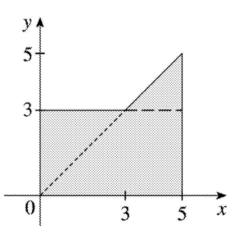
47.

$$\int_{-2}^{2}f(x)\,dx+\int_{2}^{5}f(x)\,dx-\int_{-2}^{-1}f(x)\,dx=\int_{-2}^{5}f(x)\,dx+\int_{-1}^{-2}f(x)\,dx$$
$$=\int_{-1}^{5}f(x)\,dx$$

48. $\int_1^4 f(x)\,dx=\int_1^5 f(x)\,dx-\int_4^5 f(x)\,dx=12-3.6=8.4$

49. $\int_0^9 [2f(x)+3g(x)]\,dx=2\int_0^9 f(x)\,dx+3\int_0^9 g(x)\,dx=2(37)+3(16)=122$

50. If $f(x)=\begin{cases} 3 & \text{for } x<3 \\ x & \text{for } x\geq 3\end{cases}$, then $\int_0^5 f(x)\,dx$ can be interpreted as the area of the shaded region, which consists of a 5–by–3 rectangle surmounted by an isosceles right triangle whose legs have length 2. Thus, $\int_0^5 f(x)\,dx=5(3)+\frac{1}{2}(2)(2)=17$.



51. $0 \le \sin x < 1$ on $\left[0, \frac{\pi}{4}\right]$, so $\sin^3 x \le \sin^2 x$ on $\left[0, \frac{\pi}{4}\right]$. Hence, $\int_0^{\pi/4} \sin^3 x\,dx \le \int_0^{\pi/4} \sin^2 x\,dx$ (Property 7).

52. $5-x \ge 3 \ge x+1$ on $[1,2]$, so $\sqrt{5-x} \ge \sqrt{x+1}$ and $\int_1^2 \sqrt{5-x}\,dx \ge \int_1^2 \sqrt{x+1}\,dx$.

53. If $-1 \le x \le 1$, then $0 \le x^2 \le 1$ and $1 \le 1+x^2 \le 2$, so $1 \le \sqrt{1+x^2} \le \sqrt{2}$ and $1[1-(-1)] \le \int_{-1}^1 \sqrt{1+x^2}\,dx \le \sqrt{2}[1-(-1)]$; that is, $2 \le \int_{-1}^1 \sqrt{1+x^2}\,dx \le 2\sqrt{2}$.

54. $\frac{1}{2} \le \sin x \le 1$ for $\frac{\pi}{6} \le x \le \frac{\pi}{2}$, so $\frac{1}{2}\left(\frac{\pi}{2} - \frac{\pi}{6}\right) \le \int_{\pi/6}^{\pi/2} \sin x\,dx \le 1\left(\frac{\pi}{2} - \frac{\pi}{6}\right)$; that is, $\frac{\pi}{6} \le \int_{\pi/6}^{\pi/2} \sin x\,dx \le \frac{\pi}{3}$.

55. If $1 \le x \le 2$, then $\frac{1}{2} \le \frac{1}{x} \le 1$, so $\frac{1}{2}(2-1) \le \int_1^2 \frac{1}{x}\,dx \le 1(2-1)$ or $\frac{1}{2} \le \int_1^2 \frac{1}{x}\,dx \le 1$.

56. If $0 \le x \le 2$, then $0 \le x^3 \le 8$, so $1 \le x^3+1 \le 9$ and $1 \le \sqrt{x^3+1} \le 3$. Thus, $1(2-0) \le \int_0^2 \sqrt{x^3+1}\,dx \le 3(2-0)$ that is, $2 \le \int_0^2 \sqrt{x^3+1}\,dx \le 6$.

57. If $\frac{\pi}{4} \le x \le \frac{\pi}{3}$, then $1 \le \tan x \le \sqrt{3}$, so $1\left(\frac{\pi}{3} - \frac{\pi}{4}\right) \le \int_{\pi/4}^{\pi/3} \tan x\,dx \le \sqrt{3}\left(\frac{\pi}{3} - \frac{\pi}{4}\right)$ or $\frac{\pi}{12} \le \int_{\pi/4}^{\pi/3} \tan x\,dx \le \frac{\pi}{12}\sqrt{3}$.

58. Let $f(x)=x^3-3x+3$ for $0 \le x \le 2$. Then $f'(x)=3x^2-3=3(x+1)(x-1)$, so f is decreasing on $(0,1)$ and increasing on $(1,2)$. f has the absolute minimum value $f(1)=1$. Since $f(0)=3$ and $f(2)=5$, the absolute maximum value of f is $f(2)=5$. Thus, $1 \le x^3-3x+3 \le 5$ for x in $[0,2]$. It follows from Property 8 that $1\cdot(2-0) \le \int_0^2 \left(x^3-3x+3\right)dx \le 5\cdot(2-0)$; that is, $2 \le \int_0^2 \left(x^3-3x+3\right)dx \le 10$.

59. The only critical number of $f(x)=xe^{-x}$ on $[0,2]$ is $x=1$. Since $f(0)=0$, $f(1)=e^{-1} \approx 0.368$, and $f(2)=2e^{-2} \approx 0.271$, we know that the absolute minimum value of f on $[0,2]$ is 0, and the absolute maximum is e^{-1}. By Property 8, $0 \le xe^{-x} \le e^{-1}$ for $0 \le x \le 2 \Rightarrow 0(2-0) \le \int_0^2 xe^{-x}\,dx \le e^{-1}(2-0) \Rightarrow$

$0 \le \int_0^2 xe^{-x}\,dx \le 2/e$.

60. If $\frac{1}{4}\pi \le x \le \frac{3}{4}\pi$, then $\frac{\sqrt{2}}{2} \le \sin x \le 1$ and $\frac{1}{2} \le \sin^2 x \le 1$, so
$\frac{1}{2}\left(\frac{3}{4}\pi - \frac{1}{4}\pi\right) \le \int_{\pi/4}^{3\pi/4} \sin^2 x\,dx \le 1\left(\frac{3}{4}\pi - \frac{1}{4}\pi\right)$; that is, $\frac{1}{4}\pi \le \int_{\pi/4}^{3\pi/4} \sin^2 x\,dx \le \frac{1}{2}\pi$.

61. $\sqrt{x^4+1} \ge \sqrt{x^4} = x^2$, so $\int_1^3 \sqrt{x^4+1}\,dx \ge \int_1^3 x^2\,dx = \frac{1}{3}\left(3^3 - 1^3\right) = \frac{26}{3}$.

62. $0 \le \sin x \le 1$ for $0 \le x \le \frac{\pi}{2}$, so $x\sin x \le x \Rightarrow \int_0^{\pi/2} x\sin x\,dx \le \int_0^{\pi/2} x\,dx = \frac{1}{2}\left[\left(\frac{\pi}{2}\right)^2 - 0^2\right] = \frac{\pi^2}{8}$.

63. Using a regular partition and right endpoints as in the proof of Property 2, we calculate
$\int_a^b cf(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^n cf\left(x_i\right)\Delta x_i = \lim_{n\to\infty} c\sum_{i=1}^n f\left(x_i\right)\Delta x_i = c\lim_{n\to\infty} \sum_{i=1}^n f\left(x_i\right)\Delta x_i = c\int_a^b f(x)\,dx$.

64. As in the proof of Property 2, we write $\int_a^b f(x)\,dx = {}_{n\to\infty}\sum_{i=1}^n f\left(x_i\right)\Delta x$. Now $f\left(x_i\right) \ge 0$ and $\Delta x \ge 0$, so $f\left(x_i\right)\Delta x \ge 0$ and therefore $\sum_{i=1}^n f\left(x_i\right)\Delta x \ge 0$. But the limit of nonnegative quantities is nonnegative by Theorem 2.3.2, so $\int_a^b f(x)\,dx \ge 0$.

65. Since $-|f(x)| \le f(x) \le |f(x)|$, it follows from Property 7 that

$$-\int_a^b |f(x)|\,dx \le \int_a^b f(x)\,dx \le \int_a^b |f(x)|\,dx \Rightarrow \left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx$$

Note that the definite integral is a real number, and so the following property applies: $-a \le b \le a \Rightarrow |b| \le a$ for all real numbers b and nonnegative numbers a .

66. $\left|\int_0^{2\pi} f(x)\sin 2x\,dx\right| \le \int_0^{2\pi} |f(x)\sin 2x|\,dx = \int_0^{2\pi} |f(x)|\,|\sin 2x|\,dx \le \int_0^{2\pi} |f(x)|\,dx$ by Property 7 , since $|\sin 2x| \le 1 \Rightarrow |f(x)|\,|\sin 2x| \le |f(x)|$.

67. $\lim_{n\to\infty} \sum_{i=1}^n \frac{i^4}{n^5} = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^n \left(\frac{i}{n}\right)^4 = \int_0^1 x^4\,dx$

68.

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \frac{1}{1+(i/n)^2} = \int_0^1 \frac{dx}{1+x^2}$$

69. Choose $x_i = 1+\frac{i}{n}$ and $x_i^* = \sqrt{x_{i-1}x_i} = \sqrt{\left(1+\frac{i-1}{n}\right)\left(1+\frac{i}{n}\right)}$. Then

$$\begin{aligned}
\int_1^2 x^{-2}dx &= \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \frac{1}{\left(1+\frac{i-1}{n}\right)\left(1+\frac{i}{n}\right)} \\
&= \lim_{n\to\infty} n\sum_{i=1}^{n} \frac{1}{(n+i-1)(n+i)} \\
&= \lim_{n\to\infty} n\sum_{i=1}^{n} \left(\frac{1}{n+i-1} - \frac{1}{n+i}\right) \quad \text{[by the hint]} \\
&= \lim_{n\to\infty} n\left(\sum_{i=0}^{n-1} \frac{1}{n+i} - \sum_{i=1}^{n} \frac{1}{n+i}\right) \\
&= \lim_{n\to\infty} n\left(\left[\frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{2n-1}\right] - \left[\frac{1}{n+1} + \cdots + \frac{1}{2n-1} + \frac{1}{2n}\right]\right) \\
&= \lim_{n\to\infty} n\left(\frac{1}{n} - \frac{1}{2n}\right) = \lim_{n\to\infty}\left(1-\frac{1}{2}\right) = \frac{1}{2}
\end{aligned}$$

1. The precise version of this statement is given by the Fundamental Theorem of Calculus. See the statement of this theorem and the paragraph that follows it at the end of Section 5.3.

2. **(a)**

$g(x) = \int_0^x f(t)\,dt$, so $g(0) = \int_0^0 f(t)\,dt = 0$.

$g(1) = \int_0^1 f(t)\,dt = \frac{1}{2} \cdot 1 \cdot 1$ [area of triangle] $= \frac{1}{2}$.

$g(2) = \int_0^2 f(t)\,dt = \int_0^1 f(t)\,dt + \int_1^2 f(t)\,dt$ [below the x –axis]

$= \frac{1}{2} - \frac{1}{2} \cdot 1 \cdot 1 = 0.$

$g(3) = g(2) + \int_2^3 f(t)\,dt = 0 - \frac{1}{2} \cdot 1 \cdot 1 = -\frac{1}{2}$.

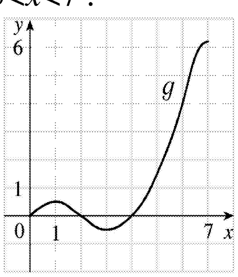
$g(4) = g(3) + \int_3^4 f(t)\,dt = -\frac{1}{2} + \frac{1}{2} \cdot 1 \cdot 1 = 0.$

$g(5) = g(4) + \int_4^5 f(t)\,dt = 0 + 1.5 = 1.5.$

$g(6) = g(5) + \int_5^6 f(t)\,dt = 1.5 + 2.5 = 4.$

(b) $g(7) = g(6) + \int_6^7 f(t)\,dt \approx 4 + 2.2$ [estimate from the graph] $= 6.2$.

(c) The answers from part (a) and part (b) indicate that g has a minimum at $x=3$ and a maximum at $x=7$. This makes sense from the graph of f since we are subtracting area on $1<x<3$ and adding area on $3<x<7$.

(d) 

3. **(a)**

$g(x) = \int_0^x f(t)\,dt$.

$g(0) = \int_0^0 f(t)\,dt = 0$

$g(1)=\int_0^1 f(t)\,dt=1\cdot 2=2$,

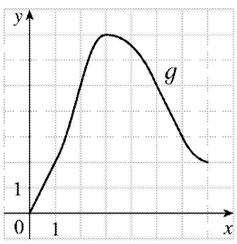
$g(2)=\int_0^2 f(t)\,dt=\int_0^1 f(t)\,dt+\int_1^2 f(t)\,dt=g(1)+\int_1^2 f(t)\,dt$

$=2+1\cdot 2+\frac{1}{2}\cdot 1\cdot 2=5$,

$g(3)=\int_0^3 f(t)\,dt=g(2)+\int_2^3 f(t)\,dt=5+\frac{1}{2}\cdot 1\cdot 4=7$,

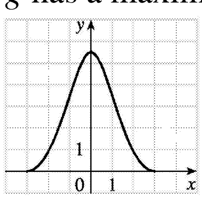
$g(6)=g(3)+\int_3^6 f(t)\,dt$

$=7+\left[-\left(\frac{1}{2}\cdot 2\cdot 2+1\cdot 2\right)\right]=7-4=3$

(b) 

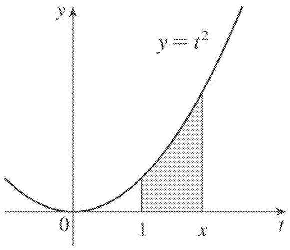
(c) g is increasing on $(0,3)$ because as x increases from 0 to 3 , we keep adding more area.
(d) g has a maximum value when we start subtracting area; that is, at $x=3$.

4. **(a)** $g(-3)=\int_{-3}^{-3} f(t)\,dt=0$, $g(3)=\int_{-3}^{3} f(t)\,dt=\int_{-3}^{0} f(t)\,dt+\int_0^3 f(t)\,dt=0$ by symmetry, since the area above the $x-$ axis is the same as the area below the axis.
(b) From the graph, it appears that to the nearest $\frac{1}{2}$, $g(-2)=\int_{-3}^{-2} f(t)\,dt\approx 1$, $g(-1)=\int_{-3}^{-1} f(t)\,dt\approx 3\frac{1}{2}$, and $g(0)=\int_{-3}^{0} f(t)\,dt\approx 5\frac{1}{2}$.
(c) g is increasing on $(-3,0)$ because as x increases from -3 to 0 , we keep adding more area.
(d) g has a maximum value when we start subtracting area; that is, at $x=0$.

(e) 

(f) The graph of $g'(x)$ is the same as that of $f(x)$, as indicated by FTC1.

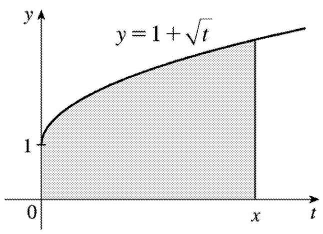
5.



(a) By FTC1 with $f(t)=t^2$ and $a=1$, $g(x)=\int_1^x t^2\,dt \Rightarrow g'(x)=f(x)=x^2$.

(b) Using FTC2, $g(x)=\int_1^x t^2\,dt=\left[\frac{1}{3}t^3\right]_1^x=\frac{1}{3}x^3-\frac{1}{3} \Rightarrow g'(x)=x^2$.

6.



(a) By FTC1 with $f(t)=1+\sqrt{t}$ and $a=0$, $g(x)=\int_0^x \left(1+\sqrt{t}\right)dt \Rightarrow g'(x)=f(x)=1+\sqrt{x}$.

(b) Using FTC2, $g(x)=\int_0^x \left(1+\sqrt{t}\right)dt=\left[t+\frac{2}{3}t^{3/2}\right]_0^x=x+\frac{2}{3}x^{3/2} \Rightarrow g'(x)=1+x^{1/2}=1+\sqrt{x}$.

7. $f(t)=\sqrt{1+2t}$ and $g(x)=\int_0^x \sqrt{1+2t}\,dt$, so by FTC1, $g'(x)=f(x)=\sqrt{1+2x}$.

8. $f(t)=\ln t$ and $g(x)=\int_1^x \ln t\,dt$, so by FTC1, $g'(x)=f(x)=\ln x$.

9. $f(t)=t^2\sin t$ and $g(y)=\int_2^y t^2\sin t\,dt$, so by FTC1, $g'(y)=f(y)=y^2\sin y$.

10. $f(x)=\dfrac{1}{x+x^2}$ and $g(u)=\int_3^u \dfrac{1}{x+x^2}\,dx$, so $g'(u)=f(u)=\dfrac{1}{u+u^2}$.

11. $F(x)=\int_x^2 \cos\left(t^2\right)dt=-\int_2^x \cos\left(t^2\right)dt \Rightarrow F'(x)=-\cos\left(x^2\right)$

12. $f(\theta)=\tan\theta$ and $F(x)=\int_x^{10}\tan\theta\,d\theta=-\int_{10}^x \tan\theta\,d\theta$, so by FTC1, $F'(x)=-f(x)=-\tan x$.

13. Let $u=\frac{1}{x}$. Then $\frac{du}{dx}=-\frac{1}{x^2}$. Also, $\frac{dh}{dx}=\frac{dh}{du}\frac{du}{dx}$, so

$$h'(x)=\frac{d}{dx}\int_2^{1/x}\arctan t\,dt=\frac{d}{du}\int_2^{u}\arctan t\,dt\cdot\frac{du}{dx}=\arctan u\frac{du}{dx}=-\frac{\arctan(1/x)}{x^2}.$$

14. Let $u=x^2$. Then $\frac{du}{dx}=2x$. Also, $\frac{dh}{dx}=\frac{dh}{du}\frac{du}{dx}$, so

$$h'(x)=\frac{d}{dx}\int_0^{x^2}\sqrt{1+r^3}\,dr=\frac{d}{du}\int_0^{u}\sqrt{1+r^3}\,dr\cdot\frac{du}{dx}=\sqrt{1+u^3}(2x)=2x\sqrt{1+(x^2)^3}=2x\sqrt{1+x^6}.$$

15. Let $u=\sqrt{x}$. Then $\frac{du}{dx}=\frac{1}{2\sqrt{x}}$. Also, $\frac{dy}{dx}=\frac{dy}{du}\frac{du}{dx}$, so

$$y'=\frac{d}{dx}\int_3^{\sqrt{x}}\frac{\cos t}{t}\,dt=\frac{d}{du}\int_3^{u}\frac{\cos t}{t}\,dt\cdot\frac{du}{dx}=\frac{\cos u}{u}\cdot\frac{1}{2\sqrt{x}}=\frac{\cos\sqrt{x}}{\sqrt{x}}\cdot\frac{1}{2\sqrt{x}}=\frac{\cos\sqrt{x}}{2x}.$$

16. Let $u=\cos x$. Then $\frac{du}{dx}=-\sin x$. Also, $\frac{dy}{dx}=\frac{dy}{du}\frac{du}{dx}$, so

$$\begin{aligned} y' &= \frac{d}{dx}\int_1^{\cos x}(t+\sin t)\,dt=\frac{d}{du}\int_1^{u}(t+\sin t)\,dt\cdot\frac{du}{dx} \\ &= (u+\sin u)\cdot(-\sin x)=-\sin x[\cos x+\sin(\cos x)] \end{aligned}$$

17. Let $w=1-3x$. Then $\frac{dw}{dx}=-3$. Also, $\frac{dy}{dx}=\frac{dy}{dw}\frac{dw}{dx}$, so

$$\begin{aligned} y' &= \frac{d}{dx}\int_{1-3x}^{1}\frac{u^3}{1+u^2}\,du=\frac{d}{dw}\int_{w}^{1}\frac{u^3}{1+u^2}\,du\cdot\frac{dw}{dx} \\ &= -\frac{d}{dw}\int_1^{w}\frac{u^3}{1+u^2}\,du\cdot\frac{dw}{dx}=-\frac{w^3}{1+w^2}(-3)=\frac{3(1-3x)^3}{1+(1-3x)^2} \end{aligned}$$

18. Let $u=e^x$. Then $\frac{du}{dx}=e^x$. Also, $\frac{dy}{dx}=\frac{dy}{du}\frac{du}{dx}$, so

$$y'=\frac{d}{dx}\int_{e^x}^{0}\sin^3 t\,dt=\frac{d}{du}\int_u^{0}\sin^3 t\,dt\cdot\frac{du}{dx}=-\frac{d}{du}\int_0^{u}\sin^3 t\,dt\cdot\frac{du}{dx}=-\sin^3 u\cdot e^x=-e^x\sin^3\left(e^x\right).$$

19. $\int_{-1}^{3}x^5\,dx=\left[\frac{x^6}{6}\right]_{-1}^{3}=\frac{3^6}{6}-\frac{(-1)^6}{6}=\frac{729-1}{6}=\frac{364}{3}$

20. $\int_{-2}^{5} 6\,dx = [6x]_{-2}^{5} = 6[5-(-2)] = 6(7) = 42$

21. $\int_{2}^{8} (4x+3)\,dx = \left[\frac{4}{2}x^2 + 3x\right]_{2}^{8} = \left(2\cdot 8^2 + 3\cdot 8\right) - \left(2\cdot 2^2 + 3\cdot 2\right) = 152 - 14 = 138$

22. $\int_{0}^{4} \left(1+3y-y^2\right) dy = \left[y + \frac{3}{2}y^2 - \frac{1}{3}y^3\right]_{0}^{4} = \left(4 + \frac{3}{2}\cdot 16 - \frac{1}{3}\cdot 64\right) - (0) = \frac{20}{3}$

23. $\int_{0}^{1} x^{4/5}\,dx = \left[\frac{5}{9}x^{9/5}\right]_{0}^{1} = \frac{5}{9} - 0 = \frac{5}{9}$

24. $\int_{1}^{8} \sqrt[3]{x}\,dx = \int_{1}^{8} x^{1/3}\,dx = \left[\frac{3}{4}x^{4/3}\right]_{1}^{8} = \frac{3}{4}(8^{4/3} - 1^{4/3}) = \frac{3}{4}(2^4 - 1) = \frac{3}{4}(16-1) = \frac{3}{4}(15) = \frac{45}{4}$

25. $\int_{1}^{2} \frac{3}{t^4}\,dt = 3\int_{1}^{2} t^{-4}\,dt = 3\left[\frac{t^{-3}}{-3}\right]_{1}^{2} = \frac{3}{-3}\left[\frac{1}{t^3}\right]_{1}^{2} = -1\left(\frac{1}{8} - 1\right) = \frac{7}{8}$

26. $\int_{-2}^{3} x^{-5}\,dx$ does not exist because the function $f(x) = x^{-5}$ has an infinite discontinuity at $x=0$; that is, f is discontinuous on the interval $[-2,3]$.

27. $\int_{-5}^{5} \frac{2}{x^3}\,dx$ does not exist because the function $f(x) = \frac{2}{x^3}$ has an infinite discontinuity at $x=0$; that is, f is discontinuous on the interval $[-5,5]$.

28. $\int_{\pi}^{2\pi} \cos\theta\,d\theta = [\sin\theta]_{\pi}^{2\pi} = \sin 2\pi - \sin\pi = 0 - 0 = 0$

29. $\int_{0}^{2} x(2+x^5)\,dx = \int_{0}^{2} (2x+x^6)\,dx = \left[x^2 + \frac{1}{7}x^7\right]_{0}^{2} = \left(4 + \frac{128}{7}\right) - (0+0) = \frac{156}{7}$

30. $\int_{1}^{4} \frac{1}{\sqrt{x}}\,dx = \int_{1}^{4} x^{-1/2}\,dx = \left[\frac{x^{1/2}}{1/2}\right]_{1}^{4} = \left[2x^{1/2}\right]_{1}^{4} = 2\sqrt{4} - 2\sqrt{1} = 4 - 2 = 2$

31. $\int_{0}^{\pi/4} \sec^2 t\,dt = [\tan t]_{0}^{\pi/4} = \tan\frac{\pi}{4} - \tan 0 = 1 - 0 = 1$

32.

$\int_0^1 \left(3+x\sqrt{x}\right)dx=\int_0^1\left(3+x^{3/2}\right)dx=\left[3x+\frac{2}{5}x^{5/2}\right]_0^1=\left[\left(3+\frac{2}{5}\right)-0\right]=\frac{17}{5}$

33. $\int_\pi^{2\pi}\csc^2\theta\,d\theta$ does not exist because the function $f(\theta)=\csc^2\theta$ has infinite discontinuities at $\theta=\pi$ and $\theta=2\pi$; that is, f is discontinuous on the interval $[\pi,2\pi]$.

34. $\int_0^{\pi/6}\csc\theta\cot\theta\,d\theta$ does not exist because the function $f(\theta)=\csc\theta\cot\theta$ has an infinite discontinuity at $\theta=0$;
that is, f is discontinuous on the interval $\left[0,\frac{\pi}{6}\right]$.

35. $\int_1^9\frac{1}{2x}\,dx=\frac{1}{2}\int_1^9\frac{1}{x}\,dx=\frac{1}{2}\left[\ln|x|\right]_1^9=\frac{1}{2}(\ln 9-\ln 1)=\frac{1}{2}\ln 9-0=\ln 9^{1/2}=\ln 3$

36. $\int_0^1 10^x\,dx=\left[\frac{10^x}{\ln 10}\right]_0^1=\frac{10}{\ln 10}-\frac{1}{\ln 10}=\frac{9}{\ln 10}$

37.
$$\int_{1/2}^{\sqrt{3}/2}\frac{6}{\sqrt{1-t^2}}\,dt=6\int_{1/2}^{\sqrt{3}/2}\frac{1}{\sqrt{1-t^2}}\,dt=6\left[\sin^{-1}t\right]_{1/2}^{\sqrt{3}/2}=6\left[\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)-\sin^{-1}\left(\frac{1}{2}\right)\right]$$
$$=6\left(\frac{\pi}{3}-\frac{\pi}{6}\right)=6\left(\frac{\pi}{6}\right)=\pi$$

38. $\int_0^1\frac{4}{t^2+1}\,dt=4\int_0^1\frac{1}{1+t^2}\,dt=4\left[\tan^{-1}t\right]_0^1=4\left(\tan^{-1}1-\tan^{-1}0\right)=4\left(\frac{\pi}{4}-0\right)=\pi$

39. $\int_{-1}^1 e^{u+1}\,du=\left[e^{u+1}\right]_{-1}^1=e^2-e^0=e^2-1$ [or start with $e^{u+1}=e^ue^1$]

40.
$$\int_1^2\frac{4+u^2}{u^3}\,du=\int_1^2\left(4u^{-3}+u^{-1}\right)du=\left[\frac{4}{-2}u^{-2}+\ln|u|\right]_1^2=\left[\frac{-2}{u^2}+\ln u\right]_1^2$$
$$=\left(-\frac{1}{2}+\ln 2\right)-(-2+\ln 1)=\frac{3}{2}+\ln 2$$

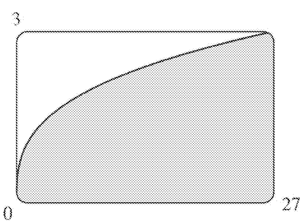
41. $\int_0^2 f(x)dx = \int_0^1 x^4 dx + \int_1^2 x^5 dx = \left[\frac{1}{5}x^5 \right]_0^1 + \left[\frac{1}{6}x^6 \right]_1^2 = \left(\frac{1}{5} - 0 \right) + \left(\frac{64}{6} - \frac{1}{6} \right) = 10.7$

42.

$$\int_{-\pi}^{\pi} f(x)dx = \int_{-\pi}^{0} x dx + \int_0^{\pi} \sin x dx = \left[\frac{1}{2}x^2 \right]_{-\pi}^{0} - [\cos x]_0^{\pi} = \left(0 - \frac{\pi^2}{2} \right) - (\cos \pi - \cos 0)$$

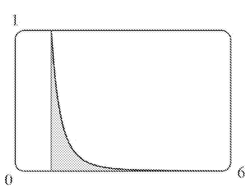
$$= -\frac{\pi^2}{2} - (-1-1) = 2 - \frac{\pi^2}{2}$$

43. From the graph, it appears that the area is about 60 . The actual area is $\int_0^{27} x^{1/3} dx = \left[\frac{3}{4}x^{4/3} \right]_0^{27} = \frac{3}{4} \cdot 81 - 0 = \frac{243}{4} = 60.75$. This is $\frac{3}{4}$ of the area of the viewing rectangle.

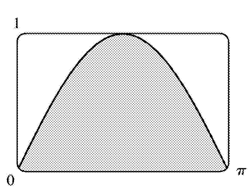


44. From the graph, it appears that the area is about $\frac{1}{3}$. The actual area is

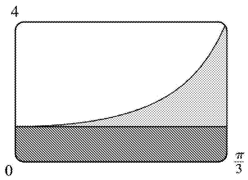
$$\int_1^6 x^{-4} dx = \left[\frac{x^{-3}}{-3} \right]_1^6 = \left[\frac{-1}{3x^3} \right]_1^6 = -\frac{1}{3 \cdot 216} + \frac{1}{3} = \frac{215}{648} \approx 0.3318 .$$



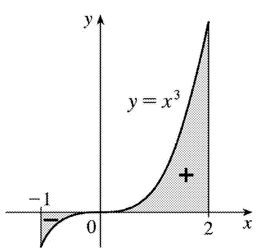
45. It appears that the area under the graph is about $\frac{2}{3}$ of the area of the viewing rectangle, or about $\frac{2}{3}\pi \approx 2.1$. The actual area is $\int_0^{\pi} \sin x dx = [-\cos x]_0^{\pi} = (-\cos \pi) - (-\cos 0) = -(-1) + 1 = 2$.



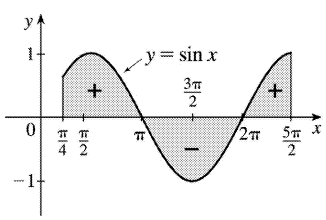
46. Splitting up the region as shown, we estimate that the area under the graph is $\frac{\pi}{3}+\frac{1}{4}\left(3\cdot\frac{\pi}{3}\right)\approx 1.8$. The actual area is $\int_0^{\pi/3}\sec^2 x\,dx=[\tan x]_0^{\pi/3}=\sqrt{3}-0=\sqrt{3}\approx 1.73$.



47. $\int_{-1}^{2}x^3\,dx=\left[\frac{1}{4}x^4\right]_{-1}^{2}=4-\frac{1}{4}=\frac{15}{4}=3.75$



48. $\int_{\pi/4}^{5\pi/2}\sin x\,dx=[-\cos x]_{\pi/4}^{5\pi/2}=0+\frac{\sqrt{2}}{2}=\frac{\sqrt{2}}{2}$



49. $g(x)=\int_{2x}^{3x}\frac{u^2-1}{u^2+1}\,du=\int_{2x}^{0}\frac{u^2-1}{u^2+1}\,du+\int_0^{3x}\frac{u^2-1}{u^2+1}\,du=-\int_0^{2x}\frac{u^2-1}{u^2+1}\,du+\int_0^{3x}\frac{u^2-1}{u^2+1}\,du\Rightarrow$

$g'(x)=-\frac{(2x)^2-1}{(2x)^2+1}\cdot\frac{d}{dx}(2x)+\frac{(3x)^2-1}{(3x)^2+1}\cdot\frac{d}{dx}(3x)=-2\cdot\frac{4x^2-1}{4x^2+1}+3\cdot\frac{9x^2-1}{9x^2+1}$

50. $g(x)=\int_{\tan x}^{x^2}\frac{1}{\sqrt{2+t^4}}\,dt=\int_{\tan x}^{1}\frac{dt}{\sqrt{2+t^4}}+\int_1^{x^2}\frac{dt}{\sqrt{2+t^4}}=-\int_1^{\tan x}\frac{dt}{\sqrt{2+t^4}}+\int_1^{x^2}\frac{dt}{\sqrt{2+t^4}}\Rightarrow$

$g'(x)=\frac{-1}{\sqrt{2+\tan^4 x}}\frac{d}{dx}(\tan x)+\frac{1}{\sqrt{2+x^8}}\frac{d}{dx}\left(x^2\right)=-\frac{\sec^2 x}{\sqrt{2+\tan^4 x}}+\frac{2x}{\sqrt{2+x^8}}$

51. $y=\int_{\sqrt{x}}^{x^3}\sqrt{t}\sin t\,dt=\int_{\sqrt{x}}^{1}\sqrt{t}\sin t\,dt+\int_{1}^{x^3}\sqrt{t}\sin t\,dt=-\int_{1}^{\sqrt{x}}\sqrt{t}\sin t\,dt+\int_{1}^{x^3}\sqrt{t}\sin t\,dt\Rightarrow$

$y'=-\sqrt[4]{x}\left(\sin\sqrt{x}\right)\cdot\frac{d}{dx}\left(\sqrt{x}\right)+x^{3/2}\sin\left(x^3\right)\cdot\frac{d}{dx}\left(x^3\right)=-\frac{\sqrt[4]{x}\sin\sqrt{x}}{2\sqrt{x}}+x^{3/2}\sin\left(x^3\right)\left(3x^2\right)$

$=3x^{7/2}\sin\left(x^3\right)-\frac{\sin\sqrt{x}}{2\sqrt[4]{x}}$

52. $y=\int_{\cos x}^{5x}\cos\left(u^2\right)du=\int_0^{5x}\cos\left(u^2\right)du-\int_0^{\cos x}\cos\left(u^2\right)du\Rightarrow$

$y'=\cos\left(25x^2\right)\cdot\frac{d}{dx}(5x)-\cos\left(\cos^2x\right)\cdot\frac{d}{dx}(\cos x)=\cos\left(25x^2\right)\cdot 5-\cos\left(\cos^2x\right)\cdot(-\sin x)$

$=5\cos\left(25x^2\right)+\sin x\cos\left(\cos^2x\right)$

53. $F(x)=\int_1^x f(t)\,dt\Rightarrow F'(x)=f(x)=\int_1^{x^2}\frac{\sqrt{1+u^4}}{u}\,du\left[\text{since } f(t)=\int_1^{t^2}\frac{\sqrt{1+u^4}}{u}\,du\right]\Rightarrow$

$F''(x)=f'(x)=\frac{\sqrt{1+(x^2)^4}}{x^2}\cdot\frac{d}{dx}\left(x^2\right)=\frac{\sqrt{1+x^8}}{x^2}\cdot 2x=\frac{2\sqrt{1+x^8}}{x}$. So $F''(2)=\sqrt{1+2^8}=\sqrt{257}$.

54. For the curve to be concave upward, we must have $y''>0$. $y=\int_0^x\frac{1}{1+t+t^2}\,dt\Rightarrow y'=\frac{1}{1+x+x^2}\Rightarrow$

$y''=\frac{-(1+2x)}{\left(1+x+x^2\right)^2}$. For this expression to be positive, we must have $(1+2x)<0$, since $\left(1+x+x^2\right)^2>0$ for all x. $(1+2x)<0\Leftrightarrow x<-\frac{1}{2}$. Thus, the curve is concave upward on $\left(-\infty,-\frac{1}{2}\right)$.

55. By FTC2, $\int_1^4 f'(x)\,dx=f(4)-f(1)$, so $17=f(4)-12\Rightarrow f(4)=17+12=29$.

56. **(a)** $\operatorname{erf}(x)=\frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt\Rightarrow\int_0^x e^{-t^2}dt=\frac{\sqrt{\pi}}{2}\operatorname{erf}(x)$. By Property 5 of definite integrals in Section 5.2, $\int_0^b e^{-t^2}dt=\int_0^a e^{-t^2}dt+\int_a^b e^{-t^2}dt$, so

$\int_a^b e^{-t^2}dt=\int_0^b e^{-t^2}dt-\int_0^a e^{-t^2}dt=\frac{\sqrt{\pi}}{2}\operatorname{erf}(b)-\frac{\sqrt{\pi}}{2}\operatorname{erf}(a)=\frac{1}{2}\sqrt{\pi}\,[\operatorname{erf}(b)-\operatorname{erf}(a)]$.

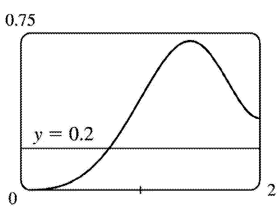
(b) $y=e^{x^2}\operatorname{erf}(x)\Rightarrow$

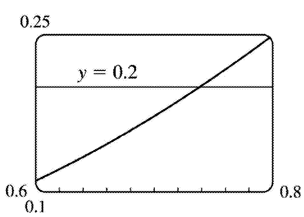
$y' = 2xe^{x^2}\operatorname{erf}(x) + e^{x^2}\operatorname{erf}'(x) = 2xy + e^{x^2} \cdot \frac{2}{\sqrt{\pi}} e^{-x^2}$ [by FTC1] $= 2xy + 2/\sqrt{\pi}$.

57. **(a)** The Fresnel function $S(x) = \int_0^x \sin\left(\frac{\pi}{2} t^2\right) dt$ has local maximum values where $0 = S'(x) = \sin\left(\frac{\pi}{2} x^2\right)$ and S' changes from positive to negative. For $x>0$, this happens when $\frac{\pi}{2} x^2 = (2n-1)\pi \Leftrightarrow x^2 = 2(2n-1) \Leftrightarrow x = \sqrt{4n-2}$, n any positive integer. For $x<0$, S' changes from positive to negative where $\frac{\pi}{2} x^2 = 2n\pi \Leftrightarrow x^2 = 4n \Leftrightarrow x = -2\sqrt{n}$. S' does not change sign at $x=0$.

(b) S is concave upward on those intervals where $S''(x)>0$. Differentiating our expression for $S'(x)$, we get $S''(x) = \cos\left(\frac{\pi}{2} x^2\right)\left(2\frac{\pi}{2} x\right) = \pi x \cos\left(\frac{\pi}{2} x^2\right)$. For $x>0$, $S''(x)>0$ where $\cos\left(\frac{\pi}{2} x^2\right) > 0 \Leftrightarrow 0 < \frac{\pi}{2} x^2 < \frac{\pi}{2}$ or $\left(2n - \frac{1}{2}\right)\pi < \frac{\pi}{2} x^2 < \left(2n + \frac{1}{2}\right)\pi$, n any integer $\Leftrightarrow 0<x<1$ or $\sqrt{4n-1} < x < \sqrt{4n+1}$, n any positive integer. For $x<0$, $S''(x)>0$ where $\cos\left(\frac{\pi}{2} x^2\right) < 0 \Leftrightarrow \left(2n - \frac{3}{2}\right)\pi < \frac{\pi}{2} x^2 < \left(2n - \frac{1}{2}\right)\pi$, n any integer $\Leftrightarrow 4n-3 < x^2 < 4n-1 \Leftrightarrow \sqrt{4n-3} < |x| < \sqrt{4n-1} \Rightarrow \sqrt{4n-3} < -x < \sqrt{4n-1} \Rightarrow -\sqrt{4n-3} > x > -\sqrt{4n-1}$, so the intervals of upward concavity for $x<0$ are $\left(-\sqrt{4n-1}, -\sqrt{4n-3}\right)$, n any positive integer. To summarize: S is concave upward on the intervals $(0,1)$, $\left(-\sqrt{3},-1\right)$, $\left(\sqrt{3},\sqrt{5}\right)$, $\left(-\sqrt{7},-\sqrt{5}\right)$, $\left(\sqrt{7},3\right)$, … .

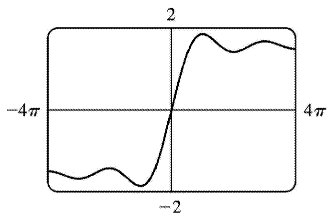
(c) In Maple, we use plot({int(sin(Pi*t^2/2),t=0..x),0.2},x=0..2); . Note that Maple recognizes the Fresnel function, calling it FresnelS(x) . In Mathematica, we use Plot[{Integrate[Sin[Pi*t^2/2],{t,0,x}],0.2},{x,0,2}] . In Derive, we load the utility file FRESNEL and plot FRESNEL_SIN(x) . From the graphs, we see that $\int_0^x \sin\left(\frac{\pi}{2} t^2\right) dt = 0.2$ at $x \approx 0.74$.





58. **(a)** In Maple, we should start by setting si:=int(sin(t)/t,t=0..x); . In Mathematica, the command is

si=Integrate[Sin[t]/t,{t,0,x}] . Note that both systems recognize this function; Maple calls it Si(x) and Mathematica calls it SinIntegral[x] . In Maple, the command to generate the graph is plot(si,x=–4*Pi..4*Pi); . In Mathematica, it is plot[si,{x,–4*Pi.,4*Pi}] . In Derive, we load the utility file EXP_INT and plot SI(x) .



(b) SI(x) has local maximum values where $\mathrm{Si}'(x)$ changes from positive to negative, passing through 0 . From the Fundamental Theorem we know that $\mathrm{Si}'(x)=\frac{d}{dx}\int_0^x \frac{\sin t}{t}\,dt=\frac{\sin x}{x}$, so we must have $\sin x=0$ for a maximum, and for $x>0$ we must have $x=(2n-1)\pi$, n any positive integer, for Si' to be changing from positive to negative at x . For $x<0$, we must have $x=2n\pi$, n any positive integer, for a maximum, since the denominator of $\mathrm{Si}'(x)$ is negative for $x<0$. Thus, the local maxima occur at $x=\pi,-2\pi,3\pi,-4\pi,5\pi,-6\pi,\ldots$.

(c) To find the first inflection point, we solve $\mathrm{Si}''(x)=\frac{\cos x}{x}-\frac{\sin x}{x^2}=0$. We can see from the graph that the first inflection point lies somewhere between $x=3$ and $x=5$. Using a root finder gives the value $x\approx 4.4934$. To find the $y-$ coordinate of the inflection point, we evaluate $\mathrm{Si}(4.4934)\approx 1.6556$. So the coordinates of the first inflection point to the right of the origin are about $(4.4934,1.6556)$. Alternatively, we could graph $S''(x)$ and estimate the first positive $x-$ value at which it changes sign.

(d) It seems from the graph that the function has horizontal asymptotes at $y\approx 1.5$, with $\lim_{x\to\pm\infty}\mathrm{Si}(x)\approx\pm 1.5$ respectively. Using the limit command, we get $\lim_{x\to\infty}\mathrm{Si}(x)=\frac{\pi}{2}$. Since Si(x) is an odd function, $\lim_{x\to-\infty}\mathrm{Si}(x)=-\frac{\pi}{2}$. So Si(x) has the horizontal asymptotes $y=\pm\frac{\pi}{2}$.

(e) We use the fsolve command in Maple (or FindRoot in Mathematica) to find that the solution is $x\approx 1.1$. Or, as in Exercise (c), we graph $y=\mathrm{Si}(x)$ and $y=1$ on the same screen to see where they intersect.

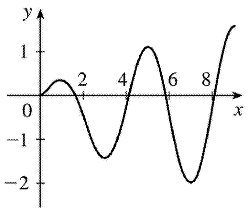
59. **(a)** By FTC1, $g'(x)=f(x)$. So $g'(x)=f(x)=0$ at $x=1,3,5,7$, and 9 . g has local maxima at $x=1$ and 5 (since $f=g'$ changes from positive to negative there) and local minima at $x=3$ and 7 . There is no local maximum or minimum at $x=9$, since f is not defined for $x>9$.

(b) We can see from the graph that $\left|\int_0^1 f\,dt\right|<\left|\int_1^3 f\,dt\right|<\left|\int_3^5 f\,dt\right|<\left|\int_5^7 f\,dt\right|<\left|\int_7^9 f\,dt\right|$. So

$g(1)=\left|\int_0^1 f\,dt\right|$, $g(5)=\int_0^5 f\,dt=g(1)-\left|\int_1^3 f\,dt\right|+\left|\int_3^5 f\,dt\right|$, and $g(9)=\int_0^9 f\,dt=g(5)-\left|\int_5^7 f\,dt\right|+\left|\int_7^9 f\,dt\right|$. Thus, $g(1)<g(5)<g(9)$, and so the absolute maximum of $g(x)$ occurs at $x=9$.

(c) g is concave downward on those intervals where $g''<0$. But $g'(x)=f(x)$, so $g''(x)=f'(x)$, which is negative on (approximately) $\left(\frac{1}{2},2\right)$, $(4,6)$ and $(8,9)$. So g is concave downward on these intervals.

(d)

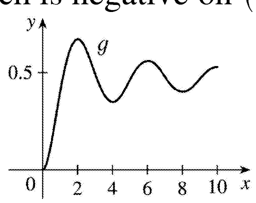


60. **(a)** By FTC1, $g'(x)=f(x)$. So $g'(x)=f(x)=0$ at $x=2$, 4 , 6 , 8 , and 10 . g has local maxima at $x=2$ and 6 (since $f=g'$ changes from positive to negative there) and local minima at $x=4$ and 8 . There is no local maximum or minimum at $x=10$, since f is not defined for $x>10$.

(b) We can see from the graph that $\left|\int_0^2 f\,dt\right|>\left|\int_2^4 f\,dt\right|>\left|\int_4^6 f\,dt\right|>\left|\int_6^8 f\,dt\right|>\left|\int_8^{10} f\,dt\right|$. So $g(2)=\left|\int_0^2 f\,dt\right|$, $g(6)=\int_0^6 f\,dt=g(2)-\left|\int_2^4 f\,dt\right|+\left|\int_4^6 f\,dt\right|$, and $g(10)=\int_0^{10} f\,dt=g(6)-\left|\int_6^8 f\,dt\right|+\left|\int_8^{10} f\,dt\right|$. Thus, $g(2)>g(6)>g(10)$, and so the absolute maximum of $g(x)$ occurs at $x=2$.

(c) g is concave downward on those intervals where $g''<0$. But $g'(x)=f(x)$, so $g''(x)=f'(x)$, which is negative on $(1,3)$, $(5,7)$ and $(9,10)$. So g is concave downward on these intervals.

(d)



61. $\lim\limits_{n\to\infty}\sum_{i=1}^{n}\frac{i^3}{n^4}=\lim\limits_{n\to\infty}\frac{1-0}{n}\sum_{i=1}^{n}\left(\frac{i}{n}\right)^3=\int_0^1 x^3\,dx=\left[\frac{x^4}{4}\right]_0^1=\frac{1}{4}$

62. $\lim\limits_{n\to\infty}\frac{1}{n}\left(\sqrt{\frac{1}{n}}+\sqrt{\frac{2}{n}}+\cdots+\sqrt{\frac{n}{n}}\right)=\lim\limits_{n\to\infty}\frac{1-0}{n}\sum_{i=1}^{n}\sqrt{\frac{i}{n}}=\int_0^1\sqrt{x}\,dx=\left[\frac{2x^{3/2}}{3}\right]_0^1=\frac{2}{3}-0=\frac{2}{3}$

63. Suppose $h<0$. Since f is continuous on $[x+\mathrm{h},\mathrm{x}]$, the Extreme Value Theorem says that there are

numbers u and v in $[x+h,x]$ such that $f(u)=m$ and $f(v)=M$, where m and M are the absolute minimum and maximum values of f on $[x+h,x]$. By Property 8 of integrals, $m(-h)\le \int_{x+h}^{x} f(t)\,dt\le M(-h)$; that is, $f(u)(-h)\le -\int_{x}^{x+h} f(t)\,dt\le f(v)(-h)$. Since $-h>0$, we can divide this inequality by $-h$: $f(u)\le \frac{1}{h}\int_{x}^{x+h} f(t)\,dt\le f(v)$. By Equation 2, $\frac{g(x+h)-g(x)}{h}=\frac{1}{h}\int_{x}^{x+h} f(t)\,dt$ for $h\ne 0$, and hence $f(u)\le \frac{g(x+h)-g(x)}{h}\le f(v)$, which is Equation 3 in the case where $h<0$.

64.

$$\begin{aligned}\frac{d}{dx}\int_{g(x)}^{h(x)} f(t)\,dt &= \frac{d}{dx}\left[\int_{g(x)}^{a} f(t)\,dt+\int_{a}^{h(x)} f(t)\,dt\right] \text{ (where } a \text{ is in the domain of } f\text{)}\\ &= \frac{d}{dx}\left[-\int_{a}^{g(x)} f(t)\,dt\right]+\frac{d}{dx}\left[\int_{a}^{h(x)} f(t)\,dt\right]=-f\left(g(x)\right)g'(x)+f\left(h(x)\right)h'(x)\\ &=f\left(h(x)\right)h'(x)-f\left(g(x)\right)g'(x)\end{aligned}$$

65. **(a)** Let $f(x)=\sqrt{x}\Rightarrow f'(x)=1/(2\sqrt{x})>0$ for $x>0\Rightarrow f$ is increasing on $(0,\infty)$. If $x\ge 0$, then $x^3\ge 0$, so $1+x^3\ge 1$ and since f is increasing, this means that $f\left(1+x^3\right)\ge f(1)\Rightarrow \sqrt{1+x^3}\ge 1$ for $x\ge 0$. Next let $g(t)=t^2-t\Rightarrow g'(t)=2t-1\Rightarrow g'(t)>0$ when $t\ge 1$. Thus, g is increasing on $(1,\infty)$. And since $g(1)=0$, $g(t)\ge 0$ when $t\ge 1$. Now let $t=\sqrt{1+x^3}$, where $x\ge 0$. $\sqrt{1+x^3}\ge 1$ (from above) $\Rightarrow t\ge 1\Rightarrow g(t)\ge 0\Rightarrow$ $\left(1+x^3\right)-\sqrt{1+x^3}\ge 0$ for $x\ge 0$. Therefore, $1\le \sqrt{1+x^3}\le 1+x^3$ for $x\ge 0$.

(b) From part (a) and Property 7: $\int_0^1 1\,dx\le \int_0^1 \sqrt{1+x^3}\,dx\le \int_0^1 \left(1+x^3\right)dx\Leftrightarrow$ $[x]_0^1\le \int_0^1 \sqrt{1+x^3}\,dx\le \left[x+\frac{1}{4}x^4\right]_0^1\Leftrightarrow 1\le \int_0^1 \sqrt{1+x^3}\,dx\le 1+\frac{1}{4}=1.25$.

66. **(a)** If $x<0$, then $g(x)=\int_0^x f(t)\,dt=\int_0^x 0\,dt=0$.

If $0\le x\le 1$, then $g(x)=\int_0^x f(t)\,dt=\int_0^x t\,dt=\left[\frac{1}{2}t^2\right]_0^x=\frac{1}{2}x^2$.

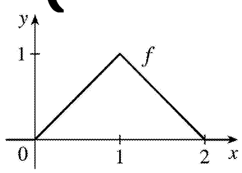
If $1<x\le 2$, then

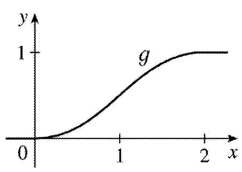
$$\begin{aligned}g(x) &=\int_0^x f(t)\,dt=\int_0^1 f(t)\,dt+\int_1^x f(t)\,dt\\ &=g(1)+\int_1^x (2-t)\,dt=\frac{1}{2}(1)^2+\left[2t-\frac{1}{2}t^2\right]_1^x\end{aligned}$$

$$=\frac{1}{2}+\left(2x-\frac{1}{2}x^2\right)-\left(2-\frac{1}{2}\right)=2x-\frac{1}{2}x^2-1 .$$

If $x>2$, then $g(x)=\int_0^x f(t)\,dt=g(2)+\int_2^x 0\,dt=1+0=1$. So

$$g(x)=\begin{cases} 0 & \text{if } x<0 \\ \frac{1}{2}x^2 & \text{if } 0\le x\le 1 \\ 2x-\frac{1}{2}x^2-1 & \text{if } 1<x\le 2 \\ 1 & \text{if } x>2 \end{cases}$$

(b) 



(c) f is not differentiable at its corners at $x=0$, 1 , and 2 . f is differentiable on $(-\infty,0)$, $(0,1)$, $(1,2)$ and $(2,\infty)$. g is differentiable on $(-\infty,\infty)$.

67. Using FTC1, we differentiate both sides of $6+\int_a^x \frac{f(t)}{t^2}\,dt=2\sqrt{x}$ to get $\frac{f(x)}{x^2}=2\,\frac{1}{2\sqrt{x}}\Rightarrow f(x)=x^{3/2}$.

To find a , we substitute $x=a$ in the original equation to obtain $6+\int_a^a \frac{f(t)}{t^2}\,dt=2\sqrt{a}\Rightarrow 6+0=2\sqrt{a}\Rightarrow$

$3=\sqrt{a}\Rightarrow a=9$.

68. $B=3A\Rightarrow \int_0^b e^x\,dx=3\int_0^a e^x\,dx\Rightarrow \left[e^x\right]_0^b=3\left[e^x\right]_0^a\Rightarrow e^b-1=3\left(e^a-1\right)\Rightarrow$
$e^b=3e^a-2\Rightarrow b=\ln\left(3e^a-2\right)$

69. **(a)** Let $F(t)=\int_0^t f(s)\,ds$. Then, by FTC1, $F'(t)=f(t)=$ rate of depreciation, so $F(t)$ represents the loss in value over the interval $[0,t]$.

(b) $C(t)=\frac{1}{t}\left[A+\int_0^t f(s)\,ds\right]=\frac{A+F(t)}{t}$ represents the average expenditure per unit of t during the interval $[0,t]$, assuming that there has been only one overhaul during that time period. The company wants to minimize average expenditure.

(c) $C(t)=\frac{1}{t}\left[A+\int_0^t f(s)\,ds\right]$. Using FTC1, we have $C'(t)=-\frac{1}{t^2}\left[A+\int_0^t f(s)\,ds\right]+\frac{1}{t}f(t)$.

$C'(t)=0\Rightarrow t\,f(t)=A+\int_0^t f(s)\,ds\Rightarrow f(t)=\frac{1}{t}\left[A+\int_0^t f(s)\,ds\right]=C(t)$.

70. **(a)** $C(t)=\frac{1}{t}\int_0^t[f(s)+g(s)]\,ds$. Using FTC1 and the Product Rule, we have

$C'(t)=\frac{1}{t}[f(t)+g(t)]-\frac{1}{t^2}\int_0^t[f(s)+g(s)]\,ds$. Set $C'(t)=0$: $\frac{1}{t}[f(t)+g(t)]-\frac{1}{t^2}\int_0^t[f(s)+g(s)]\,ds=0\Rightarrow$

$[f(t)+g(t)]-\frac{1}{t}\int_0^t[f(s)+g(s)]\,ds=0\Rightarrow[f(t)+g(t)]-C(t)=0\Rightarrow C(t)=f(t)+g(t)$.

(b) For $0\le t\le 30$, we have $D(t)=\int_0^t\left(\frac{V}{15}-\frac{V}{450}s\right)ds=\left[\frac{V}{15}s-\frac{V}{900}s^2\right]_0^t=\frac{V}{15}t-\frac{V}{900}t^2$.

So $D(t)=V\Rightarrow\frac{V}{15}t-\frac{V}{900}t^2=V\Rightarrow 60t-t^2=900\Rightarrow t^2-60t+900=0\Rightarrow$

$(t-30)^2=0\Rightarrow t=30$. So the length of time T is 30 months.

(c)

$$\begin{aligned}C(t)&=\frac{1}{t}\int_0^t\left(\frac{V}{15}-\frac{V}{450}s+\frac{V}{12{,}900}s^2\right)ds=\frac{1}{t}\left[\frac{V}{15}s-\frac{V}{900}s^2+\frac{V}{38{,}700}s^3\right]_0^t\\&=\frac{1}{t}\left(\frac{V}{15}t-\frac{V}{900}t^2+\frac{V}{38{,}700}t^3\right)=\frac{V}{15}-\frac{V}{900}t+\frac{V}{38{,}700}t^2\Rightarrow\end{aligned}$$

$C'(t)=-\frac{V}{900}+\frac{V}{19{,}350}t=0$ when $\frac{1}{19{,}350}t=\frac{1}{900}\Rightarrow t=21.5$.

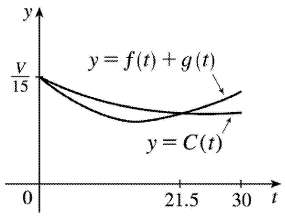
$C(21.5)=\frac{V}{15}-\frac{V}{900}(21.5)+\frac{V}{38{,}700}(21.5)^2\approx 0.05472V$, $C(0)=\frac{V}{15}\approx 0.06667V$, and

$C(30)=\frac{V}{15}-\frac{V}{900}(30)+\frac{V}{38{,}700}(30)^2\approx 0.05659V$, so the absolute minimum is $C(21.5)\approx 0.05472V$.

(d) As in part (c), we have $C(t)=\frac{V}{15}-\frac{V}{900}t+\frac{V}{38{,}700}t^2$, so $C(t)=f(t)+g(t)\Leftrightarrow$

$\frac{V}{15}-\frac{V}{900}t+\frac{V}{38{,}700}t^2=\frac{V}{15}-\frac{V}{450}t+\frac{V}{12{,}900}t^2\Leftrightarrow t^2\left(\frac{1}{12{,}900}-\frac{1}{38{,}700}\right)=t\left(\frac{1}{450}-\frac{1}{900}\right)\Leftrightarrow$

$t=\frac{1/900}{2/38{,}700}=\frac{43}{2}=21.5$. This is the value of t that we obtained as the critical number of C in part (c), so we have verified the result of (a) in this case.



1. $\frac{d}{dx}\left[\sqrt{x^2+1}+C\right]=\frac{d}{dx}\left[\left(x^2+1\right)^{1/2}+C\right]=\frac{1}{2}\left(x^2+1\right)^{-1/2}\cdot 2x=\frac{x}{\sqrt{x^2+1}}$

2. $\frac{d}{dx}\left[x\sin x+\cos x+C\right]=x\cos x+(\sin x)\cdot 1-\sin x=x\cos x$

3. $\frac{d}{dx}\left[\frac{x}{a^2\sqrt{a^2-x^2}}+C\right]=\frac{1}{a^2}\frac{\sqrt{a^2-x^2}-x(-x)\Big/\sqrt{a^2-x^2}}{a^2-x^2}=\frac{1}{a^2}\frac{\left(a^2-x^2\right)+x^2}{\left(a^2-x^2\right)^{3/2}}=\frac{1}{\sqrt{\left(a^2-x^2\right)^3}}$

4.

$$\frac{d}{dx}\left[-\frac{\sqrt{x^2+a^2}}{a^2x}+C\right]=-\frac{1}{a^2}\frac{d}{dx}\left[\frac{\sqrt{x^2+a^2}}{x}\right]=-\frac{x\left(x/\sqrt{x^2+a^2}\right)-\sqrt{x^2+a^2}\cdot 1}{a^2x^2}$$

$$=-\frac{x^2-\left(x^2+a^2\right)}{a^2x^2\sqrt{x^2+a^2}}=\frac{1}{x^2\sqrt{x^2+a^2}}$$

5. $\int x^{-3/4}dx=\frac{x^{-3/4+1}}{-3/4+1}+C=\frac{x^{1/4}}{1/4}+C=4x^{1/4}+C$

6. $\int\sqrt[3]{x}\,dx=\int x^{1/3}dx=\frac{x^{4/3}}{4/3}+C=\frac{3}{4}x^{4/3}+C$

7. $\int\left(x^3+6x+1\right)dx=\frac{x^4}{4}+6\frac{x^2}{2}+x+C=\frac{1}{4}x^4+3x^2+x+C$

8. $\int x\left(1+2x^4\right)dx=\int\left(x+2x^5\right)dx=\frac{x^2}{2}+2\frac{x^6}{6}+C=\frac{1}{2}x^2+\frac{1}{3}x^6+C$

9. $\int(1-t)\left(2+t^2\right)dt=\int\left(2-2t+t^2-t^3\right)dt=2t-2\frac{t^2}{2}+\frac{t^3}{3}-\frac{t^4}{4}+C=2t-t^2+\frac{1}{3}t^3-\frac{1}{4}t^4+C$

10. $\int\left(x^2+1+\frac{1}{x^2+1}\right)dx=\frac{x^3}{3}+x+\tan^{-1}x+C$

11. $\int (2-\sqrt{x})^2 dx = \int (4-4\sqrt{x}+x)\,dx = 4x-4\,\frac{x^{3/2}}{3/2}+\frac{x^2}{2}+C = 4x-\frac{8}{3}x^{3/2}+\frac{1}{2}x^2+C$

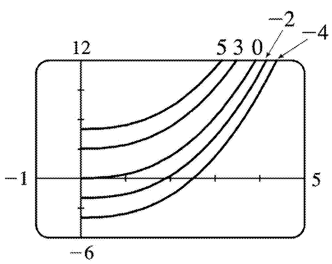
12. $\int \left(3e^u + \sec^2 u\right) du = 3e^u + \tan u + C$

13. $\int \frac{\sin x}{1-\sin^2 x}\,dx = \int \frac{\sin x}{\cos^2 x}\,dx = \int \frac{1}{\cos x}\cdot\frac{\sin x}{\cos x}\,dx = \int \sec x \tan x\,dx = \sec x + C$

14. $\int \frac{\sin 2x}{\sin x}\,dx = \int \frac{2\sin x\cos x}{\sin x}\,dx = \int 2\cos x\,dx = 2\sin x + C$

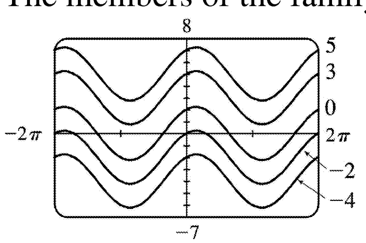
15. $\int x\sqrt{x}\,dx = \int x^{3/2}\,dx = \frac{2}{5}x^{5/2}+C$.

The members of the family in the figure correspond to $C=5$, 3 , 0 , -2 , and -4 .



16. $\int (\cos x - 2\sin x)\,dx = \sin x + 2\cos x + C$.

The members of the family in the figure correspond to $C=5$, 3 , 0 , -2 , and -4 .



17. $\int_0^2 (6x^2-4x+5)\,dx = \left[6\cdot\frac{1}{3}x^3-4\cdot\frac{1}{2}x^2+5x\right]_0^2 = \left[2x^3-2x^2+5x\right]_0^2 = (16-8+10)-0 = 18$

18.

$$\int_1^3 (1+2x-4x^3)\,dx = \left[x+2\cdot\frac{1}{2}x^2-4\cdot\frac{1}{4}x^4\right]_1^3 = \left[x+x^2-x^4\right]_1^3$$

$$=(3+9-81)-(1+1-1) = -69-1 = -70$$

19. $\int_{-1}^0 \left(2x-e^x\right)dx = \left[x^2-e^x\right]_{-1}^0 = (0-1)-\left(1-e^{-1}\right) = -2+1/e$

20. $\int_{-2}^{0}(u^5-u^3+u^2)\,du=\left[\frac{1}{6}u^6-\frac{1}{4}u^4+\frac{1}{3}u^3\right]_{-2}^{0}=0-\left(\frac{32}{3}-4-\frac{8}{3}\right)=-4$

21.

$$\int_{-2}^{2}(3u+1)^2\,du=\int_{-2}^{2}\left(9u^2+6u+1\right)du=\left[9\cdot\frac{1}{3}u^3+6\cdot\frac{1}{2}u^2+u\right]_{-2}^{2}=\left[3u^3+3u^2+u\right]_{-2}^{2}$$
$$=(24+12+2)-(-24+12-2)=38-(-14)=52$$

22.

$$\int_{0}^{4}(2v+5)(3v-1)\,dv=\int_{0}^{4}(6v^2+13v-5)\,dv=\left[6\cdot\frac{1}{3}v^3+13\cdot\frac{1}{2}v^2-5v\right]_{0}^{4}=\left[2v^3+\frac{13}{2}v^2-5v\right]_{0}^{4}$$
$$=(128+104-20)-0=212$$

23. $\int_{1}^{4}\sqrt{t}\,(1+t)\,dt=\int_{1}^{4}(t^{1/2}+t^{3/2})\,dt=\left[\frac{2}{3}t^{3/2}+\frac{2}{5}t^{5/2}\right]_{1}^{4}=\left(\frac{16}{3}+\frac{64}{5}\right)-\left(\frac{2}{3}+\frac{2}{5}\right)=\frac{14}{3}+\frac{62}{5}=\frac{256}{15}$

24. $\int_{0}^{9}\sqrt{2t}\,dt=\int_{0}^{9}\sqrt{2}\,t^{1/2}\,dt=\left[\sqrt{2}\cdot\frac{2}{3}t^{3/2}\right]_{0}^{9}=\sqrt{2}\cdot\frac{2}{3}\cdot 27-0=18\sqrt{2}$

25. $\int_{-2}^{-1}\left(4y^3+\frac{2}{y^3}\right)dy=\left[4\cdot\frac{1}{4}y^4+2\cdot\frac{1}{-2}y^{-2}\right]_{-2}^{-1}=\left[y^4-\frac{1}{y^2}\right]_{-2}^{-1}=(1-1)-\left(16-\frac{1}{4}\right)=-\frac{63}{4}$

26. $\int_{1}^{2}\frac{y+5y^7}{y^3}\,dy=\int_{1}^{2}(y^{-2}+5y^4)\,dy=\left[-y^{-1}+5\cdot\frac{1}{5}y^5\right]_{1}^{2}=\left[-\frac{1}{y}+y^5\right]_{1}^{2}=\left(-\frac{1}{2}+32\right)-(-1+1)=\frac{63}{2}$

27. $\int_{0}^{1}x\left(\sqrt[3]{x}+\sqrt[4]{x}\right)dx=\int_{0}^{1}(x^{4/3}+x^{5/4})\,dx=\left[\frac{3}{7}x^{7/3}+\frac{4}{9}x^{9/4}\right]_{0}^{1}=\left(\frac{3}{7}+\frac{4}{9}\right)-0=\frac{55}{63}$

28. $\int_{0}^{5}\left(2e^x+4\cos x\right)dx=\left[2e^x+4\sin x\right]_{0}^{5}=\left(2e^5+4\sin 5\right)-\left(2e^0+4\sin 0\right)=2e^5+4\sin 5-2\approx 290.99$

29. $\int_{1}^{4}\sqrt{5/x}\,dx=\sqrt{5}\int_{1}^{4}x^{-1/2}\,dx=\sqrt{5}\left[2\sqrt{x}\right]_{1}^{4}=\sqrt{5}\,(2\cdot 2-2\cdot 1)=2\sqrt{5}$

30.

$$\int_{1}^{9}\frac{3x-2}{\sqrt{x}}\,dx=\int_{1}^{9}(3x^{1/2}-2x^{-1/2})\,dx=\left[3\cdot\frac{2}{3}x^{3/2}-2\cdot 2x^{1/2}\right]_{1}^{9}=\left[2x^{3/2}-4x^{1/2}\right]_{1}^{9}$$

$=(54-12)-(2-4)=44$

31. $\int_0^{\pi}(4\sin\theta-3\cos\theta)\,d\theta=\left[-4\cos\theta-3\sin\theta\right]_0^{\pi}=(4-0)-(-4-0)=8$

32. $\int_{\pi/4}^{\pi/3}\sec\theta\,\tan\theta\,d\theta=\left[\sec\theta\right]_{\pi/4}^{\pi/3}=\sec\frac{\pi}{3}-\sec\frac{\pi}{4}=2-\sqrt{2}$

33.

$$\int_0^{\pi/4}\frac{1+\cos^2\theta}{\cos^2\theta}\,d\theta=\int_0^{\pi/4}\left(\frac{1}{\cos^2\theta}+\frac{\cos^2\theta}{\cos^2\theta}\right)d\theta=\int_0^{\pi/4}\left(\sec^2\theta+1\right)d\theta$$

$$=\left[\tan\theta+\theta\right]_0^{\pi/4}=\left(\tan\frac{\pi}{4}+\frac{\pi}{4}\right)-(0+0)=1+\frac{\pi}{4}$$

34.

$$\int_0^{\pi/3}\frac{\sin\theta+\sin\theta\,\tan^2\theta}{\sec^2\theta}\,d\theta=\int_0^{\pi/3}\frac{\sin\theta\,(1+\tan^2\theta)}{\sec^2\theta}\,d\theta=\int_0^{\pi/3}\frac{\sin\theta\,\sec^2\theta}{\sec^2\theta}\,d\theta=\int_0^{\pi/3}\sin\theta\,d\theta$$

$$=\left[-\cos\theta\right]_0^{\pi/3}=-\frac{1}{2}-(-1)=\frac{1}{2}$$

35.

$$\int_1^{64}\frac{1+\sqrt[3]{x}}{\sqrt{x}}\,dx=\int_1^{64}\left(\frac{1}{x^{1/2}}+\frac{x^{1/3}}{x^{1/2}}\right)dx=\int_1^{64}(x^{-1/2}+x^{(1/3)-(1/2)})dx=\int_1^{64}(x^{-1/2}+x^{-1/6})dx$$

$$=\left[2x^{1/2}+\frac{6}{5}x^{5/6}\right]_1^{64}=\left(16+\frac{192}{5}\right)-\left(2+\frac{6}{5}\right)=14+\frac{186}{5}=\frac{256}{5}$$

36. $\int_0^1(1+x^2)^3\,dx=\int_0^1(1+3x^2+3x^4+x^6)\,dx=\left[x+x^3+\frac{3}{5}x^5+\frac{1}{7}x^7\right]_0^1=\left(1+1+\frac{3}{5}+\frac{1}{7}\right)-0=\frac{96}{35}$

37.

$$\int_1^e\frac{x^2+x+1}{x}\,dx=\int_1^e\left(x+1+\frac{1}{x}\right)dx=\left[\frac{1}{2}x^2+x+\ln|x|\right]_1^e=\left(\frac{1}{2}e^2+e+\ln e\right)-\left(\frac{1}{2}+1+\ln 1\right)=\frac{1}{2}e^2+e-\frac{1}{2}$$

38.

$$\int_4^9\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)^2dx=\int_4^9\left(x+2+\frac{1}{x}\right)dx=\left[\frac{1}{2}x^2+2x+\ln|x|\right]_4^9=\frac{81}{2}+18+\ln 9-(8+8+\ln 4)=\frac{85}{2}+\ln\frac{9}{4}$$

39.

$$\int_{-1}^{2}(x-2|x|)\,dx=\int_{-1}^{0}[x-2(-x)]\,dx+\int_{0}^{2}[x-2(x)]\,dx=\int_{-1}^{0}3x\,dx+\int_{0}^{2}(-x)\,dx=3\left[\frac{1}{2}x^{2}\right]_{-1}^{0}-\left[\frac{1}{2}x^{2}\right]_{0}^{2}$$

$$=3\left(0-\frac{1}{2}\right)-(2-0)=-\frac{7}{2}=-3.5$$

40.

$$\int_{0}^{3\pi/2}|\sin x|\,dx=\int_{0}^{\pi}\sin x\,dx+\int_{\pi}^{3\pi/2}(-\sin x)\,dx=[-\cos x]_{0}^{\pi}+[\cos x]_{\pi}^{3\pi/2}$$

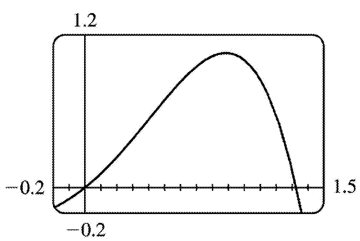
$$=[1-(-1)]+[0-(-1)]=2+1=3$$

41. The graph shows that $y=x+x^{2}-x^{4}$ has $x-$ intercepts at $x=0$ and at $x=a\approx 1.32$. So the area of the region that lies under the curve and above the $x-$ axis is

$$\int_{0}^{a}\left(x+x^{2}-x^{4}\right)dx=\left[\frac{1}{2}x^{2}+\frac{1}{3}x^{3}-\frac{1}{5}x^{5}\right]_{0}^{a}$$

$$=\left(\frac{1}{2}a^{2}+\frac{1}{3}a^{3}-\frac{1}{5}a^{5}\right)-0$$

$$\approx 0.84$$

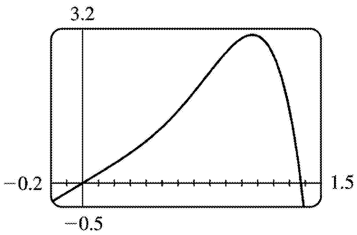


42. The graph shows that $y=2x+3x^{4}-2x^{6}$ has $x-$ intercepts at $x=0$ and at $x=a\approx 1.37$. So the area of the region that lies under the curve and above the $x-$ axis is

$$\int_{0}^{a}\left(2x+3x^{4}-2x^{6}\right)dx=\left[x^{2}+\frac{3}{5}x^{5}-\frac{2}{7}x^{7}\right]_{0}^{a}$$

$$=\left(a^{2}+\frac{3}{5}a^{5}-\frac{2}{7}a^{7}\right)-0$$

$$\approx 2.18$$



43. $A=\int_0^2 \left(2y-y^2\right) dy=\left[y^2-\frac{1}{3}y^3\right]_0^2=\left(4-\frac{8}{3}\right)-0=\frac{4}{3}$

44. $y=\sqrt[4]{x} \Rightarrow x=y^4$, so $A=\int_0^1 y^4 dy=\left[\frac{1}{5}y^5\right]_0^1=\frac{1}{5}$.

45. If $w'(t)$ is the rate of change of weight in pounds per year, then $w(t)$ represents the weight in pounds of the child at age t . We know from the Net Change Theorem that $\int_5^{10} w'(t)dt=w(10)-w(5)$, so the integral represents the increase in the child's weight (in pounds) between the ages of 5 and 10 .

46. $\int_a^b I(t)dt=\int_a^b Q'(t)dt=Q(b)-Q(a)$ by the Net Change Theorem, so it represents the change in the charge Q from time $t=a$ to $t=b$.

47. Since $r(t)$ is the rate at which oil leaks, we can write $r(t)=-V'(t)$, where $V(t)$ is the volume of oil at time t . Thus, by the Net Change Theorem,
$\int_0^{120} r(t)dt=-\int_0^{120} V'(t)dt=-[V(120)-V(0)]=V(0)-V(120)$, which is the number of gallons of oil that leaked from the tank in the first two hours (120 minutes).

48. By the Net Change Theorem, $\int_0^{15} n'(t)dt=n(15)-n(0)=n(15)-100$ represents the increase in the bee population in 15 weeks. So $100+\int_0^{15} n'(t)dt=n(15)$ represents the total bee population after 15 weeks.

49. By the Net Change Theorem, $\int_{1000}^{5000} R'(x)dx=R(5000)-R(1000)$, so it represents the increase in revenue when production is increased from 1000 units to 5000 units.

50. The slope of the trail is the rate of change of the elevation E , so $f(x)=E'(x)$. By the Net Change Theorem, $\int_3^5 f(x)dx=\int_3^5 E'(x)dx=E(5)-E(3)$ is the change in the elevation E between $x=3$ miles and $x=5$ miles from the start of the trail.

51. In general, the unit of measurement for $\int_a^b f(x)\,dx$ is the product of the unit for $f(x)$ and the unit for x . Since
$f(x)$ is measured in newtons and x is measured in meters, the units for $\int_0^{100} f(x)\,dx$ are newton–meters. (A newton–meter is abbreviated N–m and is called a joule.)

52. The units for $a(x)$ are pounds per foot and the units for x are feet, so the units for da/dx are pounds per foot per foot, denoted (lb / ft) / ft. The unit of measurement for $\int_2^8 a(x)\,dx$ is the product of pounds per foot and feet; that is, pounds.

53. **(a)** displacement $=\int_0^3 (3t-5)\,dt=\left[\frac{3}{2}t^2-5t\right]_0^3=\frac{27}{2}-15=-\frac{3}{2}$ m

(b)

distance traveled $=\int_0^3 |3t-5|\,dt=\int_0^{5/3}(5-3t)\,dt+\int_{5/3}^3 (3t-5)\,dt$

$=\left[5t-\frac{3}{2}t^2\right]_0^{5/3}+\left[\frac{3}{2}t^2-5t\right]_{5/3}^3=\frac{25}{3}-\frac{3}{2}\cdot\frac{25}{9}+\frac{27}{2}-15-\left(\frac{3}{2}\cdot\frac{25}{9}-\frac{25}{3}\right)=\frac{41}{6}$ m

54. **(a)** displacement $=\int_1^6 \left(t^2-2t-8\right)dt=\left[\frac{1}{3}t^3-t^2-8t\right]_1^6=(72-36-48)-\left(\frac{1}{3}-1-8\right)=-\frac{10}{3}$ m

(b)

distance traveled $=\int_1^6 |t^2-2t-8|\,dt=\int_1^6 |(t-4)(t+2)|\,dt$

$=\int_1^4 \left(-t^2+2t+8\right)dt+\int_4^6 \left(t^2-2t-8\right)dt=\left[-\frac{1}{3}t^3+t^2+8t\right]_1^4+\left[\frac{1}{3}t^3-t^2-8t\right]_4^6$

$=\left(-\frac{64}{3}+16+32\right)-\left(-\frac{1}{3}+1+8\right)+(72-36-48)-\left(\frac{64}{3}-16-32\right)=\frac{98}{3}$ m

55. **(a)** $v'(t)=a(t)=t+4\Rightarrow v(t)=\frac{1}{2}t^2+4t+C\Rightarrow v(0)=C=5\Rightarrow v(t)=\frac{1}{2}t^2+4t+5$ m / s

(b)

distance traveled $=\int_0^{10} |v(t)|\,dt=\int_0^{10}\left|\frac{1}{2}t^2+4t+5\right|dt=\int_0^{10}\left(\frac{1}{2}t^2+4t+5\right)dt$

$=\left[\frac{1}{6}t^3+2t^2+5t\right]_0^{10}=\frac{500}{3}+200+50=416\frac{2}{3}$ m

56. **(a)** $v'(t)=a(t)=2t+3\Rightarrow v(t)=t^2+3t+C\Rightarrow v(0)=C=-4\Rightarrow v(t)=t^2+3t-4$

(b)

distance traveled $=\int_0^3 \left| t^2+3t-4 \right| dt=\int_0^3 |(t+4)(t-1)|\,dt$

$=\int_0^1 \left(-t^2-3t+4\right)dt+\int_1^3 \left(t^2+3t-4\right)dt$

$=\left[-\frac{1}{3}t^3-\frac{3}{2}t^2+4t\right]_0^1+\left[\frac{1}{3}t^3+\frac{3}{2}t^2-4t\right]_1^3$

$=\left(-\frac{1}{3}-\frac{3}{2}+4\right)+\left(9+\frac{27}{2}-12\right)-\left(\frac{1}{3}+\frac{3}{2}-4\right)=\frac{89}{6}$ m

57. Since $m'(x)=\rho(x)$, $m=\int_0^4 \rho(x)\,dx=\int_0^4 \left(9+2\sqrt{x}\right)dx=\left[9x+\frac{4}{3}x^{3/2}\right]_0^4=36+\frac{32}{3}-0=\frac{140}{3}=46\frac{2}{3}$ kg.

58. By the Net Change Theorem, the amount of water that flows from the tank is $\int_0^{10} r(t)dt=\int_0^{10}(200-4t)\,dt=\left[200t-2t^2\right]_0^{10}=(2000-200)-0=1800$ liters.

59. Let s be the position of the car. We know from Equation 2 that $s(100)-s(0)=\int_0^{100} v(t)\,dt$. We use the Midpoint Rule for $0\leq t\leq 100$ with $n=5$. Note that the length of each of the five time intervals is 20 seconds $=\frac{20}{3600}$ hour$=\frac{1}{180}$ hour. So the distance traveled is

$$\int_0^{100} v(t)\,dt \approx \frac{1}{180}[v(10)+v(30)+v(50)+v(70)+v(90)]$$

$$= \frac{1}{180}(38+58+51+53+47)$$

$$= \frac{247}{180}\approx 1.4\text{miles}$$

60. **(a)** By the Net Change Theorem, the total amount spewed into the atmosphere is $Q(6)-Q(0)=\int_0^6 r(t)\,dt=Q(6)$ since $Q(0)=0$. The rate $r(t)$ is positive, so Q is an increasing function. Thus, an upper estimate for $Q(6)$ is R_6 and a lower estimate for $Q(6)$ is L_6 . $\Delta t=\frac{b-a}{n}=\frac{6-0}{6}=1$.

$R_6=\sum_{i=1}^{6} r\left(t_i\right)\Delta t=10+24+36+46+54+60=230$ tonnes.

$L_6=\sum_{i=1}^{6} r\left(t_{i-1}\right)\Delta t=R_6+r(0)-r(6)=230+2-60=172$ tonnes.

(b) $\Delta t=\frac{b-a}{n}=\frac{6-0}{3}=2$. $Q(6)\approx M_3=2[r(1)+r(3)+r(5)]=2(10+36+54)=2(100)=200$ tonnes.

61. From the Net Change Theorem, the increase in cost if the production level is raised from 2000 yards to 4000 yards is $C(4000)-C(2000)=\int_{2000}^{4000} C'(x)\,dx$.

$$\int_{2000}^{4000} C'(x)\,dx = \int_{2000}^{4000}\left(3-0.01x+0.000006x^2\right)dx$$
$$= \left[3x-0.005x^2+0.000002x^3\right]_{2000}^{4000}=60{,}000-2{,}000=\$58{,}000$$

62. By the Net Change Theorem, the amount of water after four days is

$$25{,}000+\int_0^4 r(t)\,dt \approx 25{,}000+M_4$$
$$=25{,}000+\frac{4-0}{4}\left[r(0.5)+r(1.5)+r(2.5)+r(3.5)\right]$$
$$\approx 25{,}000+\left[1500+1770+740+(-690)\right]=28{,}320 \text{ liters}$$

63. **(a)** We can find the area between the Lorenz curve and the line $y=x$ by subtracting the area under $y=L(x)$ from the area under $y=x$. Thus,

$$\text{coefficient of inequality} = \frac{\text{area between Lorenz curve and line } y=x}{\text{area under line } y=x} = \frac{\int_0^1 [x-L(x)]\,dx}{\int_0^1 x\,dx}$$
$$= \frac{\int_0^1 [x-L(x)]\,dx}{\left[x^2/2\right]_0^1} = \frac{\int_0^1 [x-L(x)]\,dx}{1/2} = 2\int_0^1 [x-L(x)]\,dx$$

(b) $L(x)=\frac{5}{12}x^2+\frac{7}{12}x \Rightarrow L(50\%)=L\left(\frac{1}{2}\right)=\frac{5}{48}+\frac{7}{24}=\frac{19}{48}=0.3958\overline{3}$, so the bottom 50% of the households receive at most about 40% of the income. Using the result in part (a),

$$\text{coefficientofinequality} = 2\int_0^1 [x-L(x)]\,dx=2\int_0^1\left(x-\frac{5}{12}x^2-\frac{7}{12}x\right)dx$$
$$= 2\int_0^1\left(\frac{5}{12}x-\frac{5}{12}x^2\right)dx=2\int_0^1\frac{5}{12}\left(x-x^2\right)dx$$
$$= \frac{5}{6}\left[\frac{1}{2}x^2-\frac{1}{3}x^3\right]_0^1=\frac{5}{6}\left(\frac{1}{2}-\frac{1}{3}\right)=\frac{5}{6}\left(\frac{1}{6}\right)=\frac{5}{36}$$

64. **(a)** From Exercise 4.1. (a), $v(t)=0.00146t^3-0.11553t^2+24.98169t-21.26872$.

(b) $h(125)-h(0)=\int_0^{125} v(t)\,dt=\left[0.000365t^4-0.03851t^3+12.490845t^2-21.26872t\right]_0^{125}\approx 206{,}407$ ft

1. Let $u=3x$. Then $du=3dx$, so $dx=\frac{1}{3}\,du$. Thus,
$\int\cos 3x\,dx=\int\cos u\left(\frac{1}{3}\,du\right)=\frac{1}{3}\int\cos u\,du=\frac{1}{3}\sin u+C=\frac{1}{3}\sin 3x+C$. Don't forget that it is often very easy to check an indefinite integration by differentiating your answer. In this case,
$\frac{d}{dx}\left(\frac{1}{3}\sin 3x+C\right)=\frac{1}{3}(\cos 3x)\cdot 3=\cos 3x$, the desired result.

2. Let $u=4+x^2$. Then $du=2x\,dx$ and $x\,dx=\frac{1}{2}\,du$, so
$\int x\left(4+x^2\right)^{10}dx=\int u^{10}\left(\frac{1}{2}\,du\right)=\frac{1}{2}\cdot\frac{1}{11}\,u^{11}+C=\frac{1}{22}\left(4+x^2\right)^{11}+C$.

3. Let $u=x^3+1$. Then $du=3x^2dx$ and $x^2dx=\frac{1}{3}\,du$, so
$\int x^2\sqrt{x^3+1}\,dx=\int\sqrt{u}\left(\frac{1}{3}\,du\right)=\frac{1}{3}\,\frac{u^{3/2}}{3/2}+C=\frac{1}{3}\cdot\frac{2}{3}\,u^{3/2}+C=\frac{2}{9}(x^3+1)^{3/2}+C$.

4. Let $u=\sqrt{x}$. Then $du=\frac{1}{2\sqrt{x}}\,dx$ and $\frac{1}{\sqrt{x}}\,dx=2du$, so
$\int\frac{\sin\sqrt{x}}{\sqrt{x}}\,dx=\int\sin u(2du)=2(-\cos u)+C=-2\cos\sqrt{x}+C$.

5. Let $u=1+2x$. Then $du=2dx$ and $dx=\frac{1}{2}\,du$, so
$\int\frac{4}{(1+2x)^3}\,dx=4\int u^{-3}\left(\frac{1}{2}\,du\right)=2\,\frac{u^{-2}}{-2}+C=-\frac{1}{u^2}+C=-\frac{1}{(1+2x)^2}+C$.

6. Let $u=\sin\theta$. Then $du=\cos\theta\,d\theta$, so $\int e^{\sin\theta}\cos\theta\,d\theta=\int e^u du=e^u+C=e^{\sin\theta}+C$.

7. Let $u=x^2+3$. Then $du=2x\,dx$, so $\int 2x(x^2+3)^4dx=\int u^4du=\frac{1}{5}\,u^5+C=\frac{1}{5}(x^2+3)^5+C$.

8. Let $u=x^3+5$. Then $du=3x^2dx$ and $x^2dx=\frac{1}{3}\,du$, so
$\int x^2(x^3+5)^9dx=\int u^9\left(\frac{1}{3}\,du\right)=\frac{1}{3}\cdot\frac{1}{10}\,u^{10}+C=\frac{1}{30}(x^3+5)^{10}+C$.

9. Let $u=3x-2$. Then $du=3dx$ and

$dx=\frac{1}{3}\,du$, so $\int(3x-2)^{20}dx=\int u^{20}\left(\frac{1}{3}\,du\right)=\frac{1}{3}\cdot\frac{1}{21}\,u^{21}+C=\frac{1}{63}\,(3x-2)^{21}+C$.

10. Let $u=2-x$. Then $du=-dx$ and $dx=-du$, so $\int(2-x)^{6}dx=\int u^{6}(-du)=-\frac{1}{7}\,u^{7}+C=-\frac{1}{7}\,(2-x)^{7}+C$.

11. Let $u=1+x+2x^{2}$. Then $du=(1+4x)\,dx$, so

$$\int\frac{1+4x}{\sqrt{1+x+2x^{2}}}\,dx=\int\frac{du}{\sqrt{u}}=\int u^{-1/2}du=\frac{u^{1/2}}{1/2}+C=2\sqrt{1+x+2x^{2}}+C\ .$$

12. Let $u=x^{2}+1$. Then $du=2x\,dx$ and $x\,dx=\frac{1}{2}\,du$, so

$$\int\frac{x}{(x^{2}+1)^{2}}\,dx=\int u^{-2}\left(\frac{1}{2}\,du\right)=\frac{1}{2}\cdot\frac{-1}{u}+C=\frac{-1}{2u}+C=\frac{-1}{2(x^{2}+1)}+C\ .$$

13. Let $u=5-3x$. Then $du=-3\,dx$ and $dx=-\frac{1}{3}\,du$, so

$$\int\frac{dx}{5-3x}=\int\frac{1}{u}\left(-\frac{1}{3}\,du\right)=-\frac{1}{3}\ln|u|+C=-\frac{1}{3}\ln|5-3x|+C\ .$$

14. Let $u=x^{2}+1$. Then $du=2x\,dx$ and $x\,dx=\frac{1}{2}\,du$, so

$$\int\frac{x}{x^{2}+1}\,dx=\int\frac{\frac{1}{2}\,du}{u}=\frac{1}{2}\ln|u|+C=\frac{1}{2}\ln\left|x^{2}+1\right|+C=\frac{1}{2}\ln\left(x^{2}+1\right)+C\ \left[\text{ since } x^{2}+1>0\ \right]$$

or $\ln(x^{2}+1)^{1/2}+C=\ln\sqrt{x^{2}+1}+C$.

15. Let $u=2y+1$. Then $du=2\,dy$ and $dy=\frac{1}{2}\,du$, so

$$\int\frac{3}{(2y+1)^{5}}\,dy=\int 3u^{-5}\left(\frac{1}{2}\,du\right)=\frac{3}{2}\cdot\frac{1}{-4}\,u^{-4}+C=\frac{-3}{8(2y+1)^{4}}+C\ .$$

16. Let $u=5t+4$. Then $du=5\,dt$ and $dt=\frac{1}{5}\,du$, so

$$\int\frac{1}{(5t+4)^{2.7}}\,dt=\int u^{-2.7}\left(\frac{1}{5}\,du\right)=\frac{1}{5}\cdot\frac{1}{-1.7}\,u^{-1.7}+C=\frac{-1}{8.5}\,u^{-1.7}+C=\frac{-2}{17(5t+4)^{1.7}}+C\ .$$

17. Let $u=4-t$. Then $du=-dt$ and $dt=-du$, so $\int\sqrt{4-t}\,dt=\int u^{1/2}(-du)=-\frac{2}{3}u^{3/2}+C=-\frac{2}{3}(4-t)^{3/2}+C$.

18. Let $u=2y^4-1$. Then $du=8y^3\,dy$ and $y^3\,dy=\frac{1}{8}\,du$, so
$\int y^3\sqrt{2y^4-1}\,dy=\int u^{1/2}\left(\frac{1}{8}\,du\right)=\frac{1}{8}\cdot\frac{2}{3}u^{3/2}+C=\frac{1}{12}(2y^4-1)^{3/2}+C$.

19. Let $u=\pi t$. Then $du=\pi\,dt$ and $dt=\frac{1}{\pi}\,du$, so
$\int\sin\pi t\,dt=\int\sin u\left(\frac{1}{\pi}\,du\right)=\frac{1}{\pi}(-\cos u)+C=-\frac{1}{\pi}\cos\pi t+C$.

20. Let $u=2\theta$. Then $du=2\,d\theta$ and $d\theta=\frac{1}{2}\,du$, so $\int\sec 2\theta\tan 2\theta\,d\theta=\int\sec u$
$\tan u\left(\frac{1}{2}\,du\right)=\frac{1}{2}\sec u+C=\frac{1}{2}\sec 2\theta+C$.

21. Let $u=\ln x$. Then $du=\frac{dx}{x}$, so $\int\frac{(\ln x)^2}{x}\,dx=\int u^2du=\frac{1}{3}u^3+C=\frac{1}{3}(\ln x)^3+C$.

22. Let $u=\tan^{-1}x$. Then $du=\frac{dx}{1+x^2}$, so $\int\frac{\tan^{-1}x}{1+x^2}\,dx=\int u\,du=\frac{u^2}{2}+C=\frac{\left(\tan^{-1}x\right)^2}{2}+C$.

23. Let $u=\sqrt{t}$. Then $du=\frac{dt}{2\sqrt{t}}$ and $\frac{1}{\sqrt{t}}\,dt=2\,du$, so $\int\frac{\cos\sqrt{t}}{\sqrt{t}}\,dt=\int\cos u(2\,du)=2\sin u+C=2\sin\sqrt{t}+C$
.

24. Let $u=1+x^{3/2}$. Then $du=\frac{3}{2}x^{1/2}dx$ and $\sqrt{x}\,dx=\frac{2}{3}\,du$, so
$\int\sqrt{x}\sin(1+x^{3/2})dx=\int\sin u\left(\frac{2}{3}\,du\right)=\frac{2}{3}\cdot(-\cos u)+C=-\frac{2}{3}\cos(1+x^{3/2})+C$.

25. Let $u=\sin\theta$. Then $du=\cos\theta\,d\theta$, so $\int\cos\theta\sin^6\theta\,d\theta=\int u^6du=\frac{1}{7}u^7+C=\frac{1}{7}\sin^7\theta+C$.

26. Let $u=1+\tan\theta$. Then $du=\sec^2\theta\,d\theta$, so $\int(1+\tan\theta)^5\sec^2\theta\,d\theta=\int u^5du=\frac{1}{6}u^6+C=\frac{1}{6}(1+\tan\theta)^6+C$.

27. Let $u=1+e^x$. Then $du=e^x dx$, so $\int e^x\sqrt{1+e^x}\,dx=\int \sqrt{u}\,du=\frac{2}{3}u^{3/2}+C=\frac{2}{3}(1+e^x)^{3/2}+C$.

Or: Let $u=\sqrt{1+e^x}$. Then $u^2=1+e^x$ and $2u\,du=e^x dx$, so
$\int e^x\sqrt{1+e^x}\,dx=\int u\cdot 2u\,du=\frac{2}{3}u^3+C=\frac{2}{3}(1+e^x)^{3/2}+C$.

28. Let $u=\cos t$. Then $du=-\sin t\,dt$ and $\sin t\,dt=-du$, so $\int e^{\cos t}\sin t\,dt=\int e^u(-du)=-e^u+C=-e^{\cos t}+C$.

29. Let $u=1+z^3$. Then $du=3z^2 dz$ and $z^2 dz=\frac{1}{3}du$, so
$\int \frac{z^2}{\sqrt[3]{1+z^3}}\,dz=\int u^{-1/3}\left(\frac{1}{3}du\right)=\frac{1}{3}\cdot\frac{3}{2}u^{2/3}+C=\frac{1}{2}(1+z^3)^{2/3}+C$.

30. Let $u=ax^2+2bx+c$. Then $du=2(ax+b)\,dx$ and $(ax+b)\,dx=\frac{1}{2}du$, so
$\int \frac{(ax+b)\,dx}{\sqrt{ax^2+2bx+c}}=\int \frac{\frac{1}{2}du}{\sqrt{u}}=\frac{1}{2}\int u^{-1/2}du=u^{1/2}+C=\sqrt{ax^2+2bx+c}+C$.

31. Let $u=\ln x$. Then $du=\frac{dx}{x}$, so $\int \frac{dx}{x\ln x}=\int \frac{du}{u}=\ln|u|+C=\ln|\ln x|+C$.

32. Let $u=e^x+1$. Then $du=e^x dx$, so $\int \frac{e^x}{e^x+1}\,dx=\int \frac{du}{u}=\ln|u|+C=\ln\left(e^x+1\right)+C$.

33. Let $u=\cot x$. Then $du=-\csc^2 x\,dx$ and $\csc^2 x\,dx=-du$, so
$\int \sqrt{\cot x}\csc^2 x\,dx=\int \sqrt{u}\,(-du)=-\frac{u^{3/2}}{3/2}+C=-\frac{2}{3}(\cot x)^{3/2}+C$.

34. Let $u=\frac{\pi}{x}$. Then $du=-\frac{\pi}{x^2}dx$ and $\frac{1}{x^2}dx=-\frac{1}{\pi}du$, so
$\int \frac{\cos(\pi/x)}{x^2}\,dx=\int \cos u\left(-\frac{1}{\pi}du\right)=-\frac{1}{\pi}\sin u+C=-\frac{1}{\pi}\sin\frac{\pi}{x}+C$.

35.

$\int \cot x dx=\int \frac{\cos x}{\sin x} dx$. Let $u=\sin x$. Then $du=\cos x dx$, so $\int \cot x dx=\int \frac{1}{u} du=\ln |u|+C=\ln |\sin x|+C$.

36. Let $u=\cos x$. Then $du=-\sin x dx$ and $\sin x dx=-du$, so
$\int \frac{\sin x}{1+\cos^2 x} dx=\int \frac{-du}{1+u^2}=-\tan^{-1} u+C=-\tan^{-1} (\cos x)+C$.

37. Let $u=\sec x$. Then $du=\sec x \tan x dx$, so
$\int \sec^3 x \tan x dx=\int \sec^2 x(\sec x \tan x)dx=\int u^2 du=\frac{1}{3} u^3+C=\frac{1}{3} \sec^3 x+C$.

38. Let $u=x^3+1$. Then $x^3=u-1$ and $du=3x^2 dx$, so

$$\begin{aligned}\int \sqrt[3]{x^3+1}\, x^5 dx &= \int \sqrt[3]{x^3+1} \cdot x^3 \cdot x^2 dx=\int u^{1/3}(u-1)\left(\frac{1}{3} du \right)=\frac{1}{3} \int (u^{4/3}-u^{1/3})du \\ &= \frac{1}{3} \left(\frac{3}{7} u^{7/3}-\frac{3}{4} u^{4/3}\right)+C=\frac{1}{7} (x^3+1)^{7/3}-\frac{1}{4} (x^3+1)^{4/3}+C\end{aligned}$$

39. Let $u=b+cx^{a+1}$. Then $du=(a+1)cx^a dx$, so
$\int x^a\sqrt{b+cx^{a+1}} dx=\int u^{1/2} \frac{1}{(a+1)c} du=\frac{1}{(a+1)c} \left(\frac{2}{3} u^{3/2}\right)+C=\frac{2}{3c(a+1)} \left(b+cx^{a+1}\right)^{3/2}+C$.

40. Let $u=\cos$ t. Then $du=-\sin t dt$ and $\sin t dt=-du$, so
$\int \sin t \sec^2(\cos t)dt=\int \sec^2 u\cdot (-du)=-\tan u+C=-\tan (\cos t)+C$.

41. Let $u=1+x^2$. Then $du=2x dx$, so

$$\int \frac{1+x}{1+x^2} dx=\int \frac{1}{1+x^2} dx+\int \frac{x}{1+x^2} dx=\tan^{-1} x+\int \frac{\frac{1}{2} du}{u}=\tan^{-1} x+\frac{1}{2} \ln |u|+C$$
$$=\tan^{-1} x+\frac{1}{2} \ln \left|1+x^2\right|+C=\tan^{-1} x+\frac{1}{2} \ln \left(1+x^2\right)+C \text{ [since } 1+x^2>0 \text{].}$$

42. Let $u=x^2$. Then $du=2x dx$, so $\int \frac{x}{1+x^4} dx=\int \frac{\frac{1}{2} du}{1+u^2}=\frac{1}{2} \tan^{-1} u+C=\frac{1}{2} \tan^{-1} \left(x^2\right)+C$.

43. Let $u=x+2$. Then $du=dx$, so

$$\int \frac{x}{\sqrt[4]{x+2}}\,dx = \int \frac{u-2}{\sqrt[4]{u}}\,du = \int \left(u^{3/4}-2u^{-1/4}\right)du = \frac{4}{7}u^{7/4}-2\cdot\frac{4}{3}u^{3/4}+C$$
$$= \frac{4}{7}(x+2)^{7/4}-\frac{8}{3}(x+2)^{3/4}+C$$

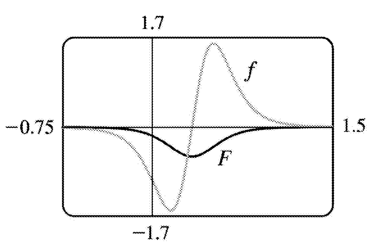
44. Let $u=1-x$. Then $x=1-u$ and $dx=-du$, so

$$\int \frac{x^2}{\sqrt{1-x}}\,dx = \int \frac{(1-u)^2}{\sqrt{u}}\,(-du) = -\int \frac{1-2u+u^2}{\sqrt{u}}\,du = -\int \left(u^{-1/2}-2u^{1/2}+u^{3/2}\right)du$$
$$= -\left(2u^{1/2}-2\cdot\frac{2}{3}u^{3/2}+\frac{2}{5}u^{5/2}\right)+C = -2\sqrt{1-x}+\frac{4}{3}(1-x)^{3/2}-\frac{2}{5}(1-x)^{5/2}+C$$

45. $f(x)=\dfrac{3x-1}{\left(3x^2-2x+1\right)^4}$.

$u=3x^2-2x+1 \Rightarrow du=(6x-2)dx=2(3x-1)dx$, so

$$\int \frac{3x-1}{\left(3x^2-2x+1\right)^4}\,dx = \int \frac{1}{u^4}\left(\frac{1}{2}\,du\right) = \frac{1}{2}\int u^{-4}\,du$$
$$= -\frac{1}{6}u^{-3}+C = -\frac{1}{6\left(3x^2-2x+1\right)^3}+C$$



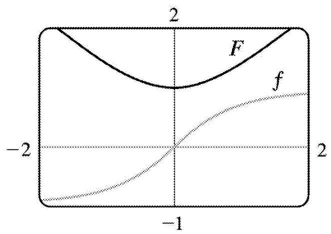
Notice that at $x=\dfrac{1}{3}$, f changes from negative to positive, and F has a local minimum.

46. $f(x)=\dfrac{x}{\sqrt{x^2+1}}$. $u=x^2+1 \Rightarrow du=2x\,dx$, so

$$\int \frac{x}{\sqrt{x^2+1}}\,dx = \int \frac{1}{\sqrt{u}}\left(\frac{1}{2}\,du\right) = \frac{1}{2}\int u^{-1/2}\,du$$

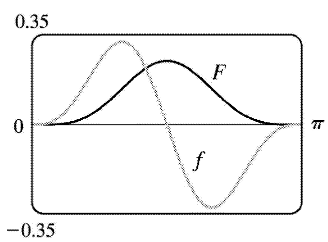
$$= u^{1/2}+C=\sqrt{x^2+1}+C.$$

Note that at $x=0$, f changes from negative to positive and F has a local minimum.



47. $f(x)=\sin^3 x\cos x$. $u=\sin x \Rightarrow du=\cos x\,dx$, so $\left[\int \sin^3 x\cos x\,dx=\int u^3\,du=\frac{1}{4}u^4+C=\frac{1}{4}\sin^4 x+C\right]$

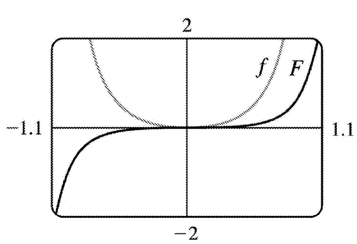
Note that at $x=\frac{\pi}{2}$, f changes from positive to negative and F has a local maximum. Also, both f and F are periodic with period π , so at $x=0$ and at $x=\pi$, f changes from negative to positive and F has local minima.



48. $f(\theta)=\tan^2\theta\sec^2\theta$. $u=\tan\theta \Rightarrow du=\sec^2\theta\,d\theta$, so

$$\int \tan^2\theta\sec^2\theta\,d\theta=\int u^2\,du=\frac{1}{3}u^3+C=\frac{1}{3}\tan^3\theta+C$$

Note that f is positive and F is increasing. At $x=0$, $f=0$ and F has a horizontal tangent.



49. Let $u=x-1$, so $du=dx$. When $x=0$, $u=-1$; when $x=2$, $u=1$. Thus, $\int_0^2 (x-1)^{25}\,dx=\int_{-1}^1 u^{25}\,du=0$ by Theorem 7(b), since $f(u)=u^{25}$ is an odd function.

50. Let $u=4+3x$, so $du=3dx$. When $x=0$, $u=4$; when $x=7$, $u=25$. Thus,

$$\int_0^7\sqrt{4+3x}\,dx=\int_4^{25}\sqrt{u}\left(\frac{1}{3}\,du\right)=\frac{1}{3}\left[\frac{u^{3/2}}{3/2}\right]_4^{25}=\frac{2}{9}\left(25^{3/2}-4^{3/2}\right)=\frac{2}{9}(125-8)=\frac{234}{9}=26$$

51. Let $u=1+2x^3$, so $du=6x^2dx$. When $x=0$, $u=1$; when $x=1$, $u=3$. Thus,

$$\int_0^1 x^2\left(1+2x^3\right)^5dx=\int_1^3 u^5\left(\frac{1}{6}\,du\right)=\frac{1}{6}\left[\frac{1}{6}u^6\right]_1^3=\frac{1}{36}\left(3^6-1^6\right)=\frac{1}{36}(729-1)=\frac{728}{36}=\frac{182}{9}$$

52. Let $u=x^2$, so $du=2xdx$. When $x=0$, $u=0$; when $x=\sqrt{\pi}$, $u=\pi$. Thus,
$\int_0^{\sqrt{\pi}} x\cos\left(x^2\right)dx=\int_0^{\pi}\cos u\left(\frac{1}{2}\,du\right)=\frac{1}{2}\left[\sin u\right]_0^{\pi}=\frac{1}{2}(\sin\pi-\sin 0)=\frac{1}{2}(0-0)=0$.

53. Let $u=t/4$, so $du=\frac{1}{4}\,dt$. When $t=0$, $u=0$; when $t=\pi$, $u=\pi/4$. Thus,
$\int_0^{\pi}\sec^2(t/4)\,dt=\int_0^{\pi/4}\sec^2u\,(4du)=4\left[\tan u\right]_0^{\pi/4}=4\left(\tan\frac{\pi}{4}-\tan 0\right)=4(1-0)=4$.

54. Let $u=\pi t$, so $du=\pi\,dt$. When $t=\frac{1}{6}$, $u=\frac{\pi}{6}$; when $t=\frac{1}{2}$, $u=\frac{\pi}{2}$. Thus,
$\int_{1/6}^{1/2}\csc\pi t\cot\pi t\,dt=\int_{\pi/6}^{\pi/2}\csc u\cot u\left(\frac{1}{\pi}\,du\right)=\frac{1}{\pi}\left[-\csc u\right]_{\pi/6}^{\pi/2}=-\frac{1}{\pi}(1-2)=\frac{1}{\pi}$.

55. $\int_{-\pi/6}^{\pi/6}\tan^3\theta\,d\theta=0$ by Theorem (b), since $f(\theta)=\tan^3\theta$ is an odd function.

56. $\int_0^2\frac{dx}{(2x-3)^2}$ does not exist since $f(x)=\frac{1}{(2x-3)^2}$ has an infinite discontinuity at $x=\frac{3}{2}$.

57. Let $u=1/x$, so $du=-1/x^2dx$. When $x=1$, $u=1$; when $x=2$, $u=\frac{1}{2}$. Thus,
$\int_1^2\frac{e^{1/x}}{x^2}\,dx=\int_1^{1/2}e^u(-du)=-\left[e^u\right]_1^{1/2}=-\left(e^{1/2}-e\right)=e-\sqrt{e}$.

58. Let $u=-x^2$, so $du=-2xdx$. When $x=0$, $u=0$; when $x=1$, $u=-1$. Thus,
$\int_0^1 xe^{-x^2}dx=\int_0^{-1}e^u\left(-\frac{1}{2}\,du\right)=-\frac{1}{2}\left[e^u\right]_0^{-1}=-\frac{1}{2}\left(e^{-1}-e^0\right)=\frac{1}{2}(1-1/e)$.

59. Let $u=\cos\theta$, so $du=-\sin\theta\, d\theta$. When $\theta=0$, $u=1$; when $\theta=\frac{\pi}{3}$, $u=\frac{1}{2}$. Thus,

$$\int_0^{\pi/3}\frac{\sin\theta}{\cos^2\theta}\,d\theta=\int_1^{1/2}\frac{-du}{u^2}=\int_{1/2}^{1}u^{-2}du=\left[-\frac{1}{u}\right]_{1/2}^{1}=-1-(-2)=1 .$$

60. $\int_{-\pi/2}^{\pi/2}\frac{x^2\sin x}{1+x^6}\,dx=0$ by Theorem (b), since $f(x)=\frac{x^2\sin x}{1+x^6}$ is an odd function.

61. Let $u=1+2x$, so $du=2dx$. When $x=0$, $u=1$; when $x=13$, $u=27$. Thus,

$$\int_0^{13}\frac{dx}{\sqrt[3]{(1+2x)^2}}=\int_1^{27}u^{-2/3}\left(\frac{1}{2}\,du\right)=\left[\frac{1}{2}\cdot 3u^{1/3}\right]_1^{27}=\frac{3}{2}\,(3-1)=3 .$$

62. Let $u=\sin x$, so $du=\cos x\,dx$. When $x=0$, $u=0$; when $x=\frac{\pi}{2}$, $u=1$. Thus,

$$\int_0^{\pi/2}\cos x\sin(\sin x)dx=\int_0^1\sin u\,du=[-\cos u]_0^1=-(\cos 1-1)=1-\cos 1 .$$

63. Let $u=x-1$, so $u+1=x$ and $du=dx$. When $x=1$, $u=0$; when $x=2$, $u=1$. Thus,

$$\int_1^2 x\sqrt{x-1}\,dx=\int_0^1(u+1)\sqrt{u}\,du=\int_0^1(u^{3/2}+u^{1/2})du=\left[\frac{2}{5}u^{5/2}+\frac{2}{3}u^{3/2}\right]_0^1=\frac{2}{5}+\frac{2}{3}=\frac{16}{15} .$$

64. Let $u=1+2x$, so $x=\frac{1}{2}(u-1)$ and $du=2dx$. When $x=0$, $u=1$; when $x=4$, $u=9$. Thus,

$$\begin{aligned}\int_0^4\frac{x\,dx}{\sqrt{1+2x}} &= \int_1^9\frac{\frac{1}{2}(u-1)}{\sqrt{u}}\,\frac{du}{2}=\frac{1}{4}\int_1^9(u^{1/2}-u^{-1/2})du=\frac{1}{4}\left[\frac{2}{3}u^{3/2}-2u^{1/2}\right]_1^9\\ &= \frac{1}{4}\cdot\frac{2}{3}\left[u^{3/2}-3u^{1/2}\right]_1^9=\frac{1}{6}\,[(27-9)-(1-3)]=\frac{20}{6}=\frac{10}{3}\end{aligned}$$

65. Let $u=\ln x$, so $du=\frac{dx}{x}$. When $x=e$, $u=1$; when $x=e^4$; $u=4$. Thus,

$$\int_e^{e^4}\frac{dx}{x\sqrt{\ln x}}=\int_1^4u^{-1/2}du=2\left[u^{1/2}\right]_1^4=2(2-1)=2 .$$

66. Let $u=\sin^{-1}x$, so $du=\frac{dx}{\sqrt{1-x^2}}$. When $x=0$, $u=0$; when $x=\frac{1}{2}$, $u=\frac{\pi}{6}$. Thus,

$\int_0^{1/2} \frac{\sin^{-1} x}{\sqrt{1-x^2}}\, dx=\int_0^{\pi/6} u\, du=\left[\frac{u^2}{2} \right]_0^{\pi/6} = \frac{\pi^2}{72}$.

67. $\int_0^4 \frac{dx}{(x-2)^3}$ does not exist since $f(x)= \frac{1}{(x-2)^3}$ has an infinite discontinuity at $x=2$.

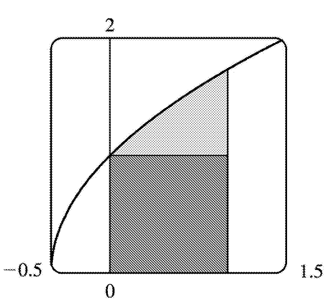
68. Assume $a>0$. Let $u=a^2-x^2$, so $du=-2x\,dx$. When $x=0$, $u=a^2$; when $x=a$, $u=0$. Thus,
$\int_0^a x\sqrt{a^2-x^2}\, dx=\int_{a^2}^0 u^{1/2}\left(-\frac{1}{2}\, du\right)=\frac{1}{2}\int_0^{a^2} u^{1/2} du=\frac{1}{2}\cdot\left[\frac{2}{3}\, u^{3/2}\right]_0^{a^2}=\frac{1}{3}\, a^3$.

69. Let $u=x^2+a^2$, so $du=2x\,dx$ and $x\,dx=\frac{1}{2}\, du$. When $x=0$, $u=a^2$; when $x=a$, $u=2a^2$. Thus,

$$\int_0^a x\sqrt{x^2+a^2}\, dx =\int_{a^2}^{2a^2} u^{1/2}\left(\frac{1}{2}\, du\right)=\frac{1}{2}\left[\frac{2}{3}\, u^{3/2}\right]_{a^2}^{2a^2}=\left[\frac{1}{3}\, u^{3/2}\right]_{a^2}^{2a^2}$$
$$=\frac{1}{3}\left[(2a^2)^{3/2}-(a^2)^{3/2}\right]=\frac{1}{3}\left(2\sqrt{2}-1\right) a^3$$

70. $\int_{-a}^{a} x\sqrt{x^2+a^2}\, dx=0$ by Theorem 7(b), since $f(x)=x\sqrt{x^2+a^2}$ is an odd function.

71. From the graph, it appears that the area under the curve is about
$1+\left(\text{a little more than } \frac{1}{2}\cdot 1\cdot 0.7\right)$, or about 1.4 . The exact area is given by $A=\int_0^1\sqrt{2x+1}\, dx$. Let $u=2x+1$, so $du=2\,dx$. The limits change to $2\cdot 0+1=1$ and $2\cdot 1+1=3$, and
$A=\int_1^3\sqrt{u}\left(\frac{1}{2}\, du\right)=\frac{1}{2}\left[\frac{2}{3}\, u^{3/2}\right]_1^3=\frac{1}{3}\left(3\sqrt{3}-1\right)=\sqrt{3}-\frac{1}{3}\approx 1.399$.

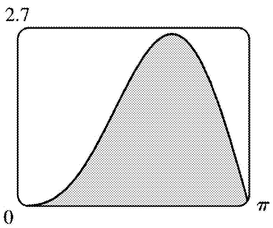


72. From the graph, it appears that the area under the curve is almost $\frac{1}{2}\cdot\pi\cdot 2.6$, or about 4 . The

exact area is given by

$$A=\int_0^{\pi}(2\sin x-\sin 2x)dx=-2\big[\cos x\big]_0^{\pi}-\int_0^{\pi}\sin 2x\,dx$$
$$=-2(-1-1)-0=4$$

Note: $\int_0^{\pi}\sin 2x\,dx=0$ since it is clear from the graph of $y=\sin 2x$ that $\int_{\pi/2}^{\pi}\sin 2x\,dx=-\int_0^{\pi/2}\sin 2x\,dx$.



73. First write the integral as a sum of two integrals: $I=\int_{-2}^{2}(x+3)\sqrt{4-x^2}\,dx=I_1+I_2=\int_{-2}^{2}x\sqrt{4-x^2}\,dx+\int_{-2}^{2}3\sqrt{4-x^2}\,dx$. $I_1=0$ by Theorem 7(b), since $f(x)=x\sqrt{4-x^2}$ is an odd function and we are integrating from $x=-2$ to $x=2$. We interpret I_2 as three times the area of a semicircle with radius 2 , so $I=0+3\cdot\frac{1}{2}\left(\pi\cdot 2^2\right)=6\pi$.

74. Let $u=x^2$. Then $du=2x\,dx$ and the limits are unchanged ($0^2=0$ and $1^2=1$), so $I=\int_0^1 x\sqrt{1-x^4}\,dx=\frac{1}{2}\int_0^1\sqrt{1-u^2}\,du$. But this integral can be interpreted as the area of a quarter–circle with radius 1 . So $I=\frac{1}{2}\cdot\frac{1}{4}\left(\pi\cdot 1^2\right)=\frac{1}{8}\pi$.

75. **First Figure**

Let $u=\sqrt{x}$, so $x=u^2$ and $dx=2u\,du$. When $x=0$, $u=0$; when $x=1$, $u=1$. Thus, $A_1=\int_0^1 e^{\sqrt{x}}\,dx=\int_0^1 e^u(2u\,du)=2\int_0^1 ue^u\,du$.

Second Figure

$A_2=\int_0^1 2xe^x\,dx=2\int_0^1 ue^u\,du$.

Third Figure

Let $u=\sin x$, so $du=\cos x\,dx$. When $x=0$, $u=0$; when

$x=\frac{\pi}{2}$, $u=1$. Thus, $A_3=\int_0^{\pi/2} e^{\sin x}\sin 2x\,dx=\int_0^{\pi/2} e^{\sin x}(2\sin x\cos x)\,dx=\int_0^1 e^u(2u\,du)=2\int_0^1 ue^u\,du$.

Since $A_1=A_2=A_3$, all three areas are equal.

76. Let $r(t)=ae^{bt}$ with $a=450.268$ and $b=1.12567$, and $n(t)=$ population after t hours. Since $r(t)=n'(t)$, $\int_0^3 r(t)\,dt=n(3)-n(0)$ is the total change in the population after three hours. Since we start with 400 bacteria, the population will be

$n(3)=400+\int_0^3 r(t)\,dt=400+\int_0^3 ae^{bt}\,dt=400+\frac{a}{b}\left[e^{bt}\right]_0^3=400+\frac{a}{b}\left(e^{3b}-1\right)\approx 400+11{,}313=11{,}713$ bacteria

77. The volume of inhaled air in the lungs at time t is

$$V(t)=\int_0^t f(u)\,du=\int_0^t \frac{1}{2}\sin\left(\frac{2\pi}{5}u\right)du=\int_0^{2\pi t/5}\frac{1}{2}\sin v\left(\frac{5}{2\pi}\,dv\right) v=\frac{2\pi}{5}u, dv=\frac{2\pi}{5}\,du]$$

$$=\frac{5}{4\pi}\left[-\cos v\right]_0^{2\pi t/5}=\frac{5}{4\pi}\left[-\cos\left(\frac{2\pi}{5}t\right)+1\right]=\frac{5}{4\pi}\left[1-\cos\left(\frac{2\pi}{5}t\right)\right]\text{liters}$$

78.

$$\text{Number of calculators }=x(4)-x(2)=\int_2^4 5000\left[1-100(t+10)^{-2}\right]dt$$

$$=5000\left[t+100(t+10)^{-1}\right]_2^4=5000\left[\left(4+\frac{100}{14}\right)-\left(2+\frac{100}{12}\right)\right]\approx 4048$$

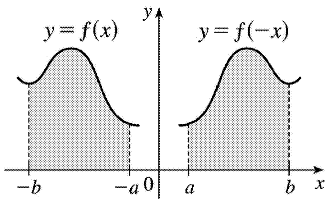
79. Let $u=2x$. Then $du=2dx$, so $\int_0^2 f(2x)\,dx=\int_0^4 f(u)\left(\frac{1}{2}\,du\right)=\frac{1}{2}\int_0^4 f(u)\,du=\frac{1}{2}(10)=5$.

80. Let $u=x^2$. Then $du=2x\,dx$, so $\int_0^3 xf\left(x^2\right)dx=\int_0^9 f(u)\left(\frac{1}{2}\,du\right)=\frac{1}{2}\int_0^9 f(u)\,du=\frac{1}{2}(4)=2$.

81. Let $u=-x$. Then $du=-dx$, so

$$\int_a^b f(-x)\,dx=\int_{-a}^{-b} f(u)(-du)=\int_{-b}^{-a} f(u)\,du=\int_{-b}^{-a} f(x)\,dx$$

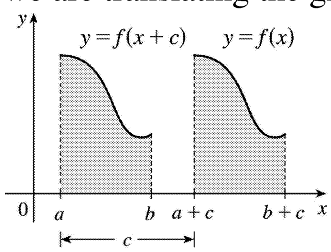
From the diagram, we see that the equality follows from the fact that we are reflecting the graph of f , and the limits of integration, about the $y-$ axis.



82. Let $u=x+c$. Then $du=dx$, so

$$\int_a^b f(x+c)\,dx=\int_{a+c}^{b+c} f(u)\,du=\int_{a+c}^{b+c} f(x)\,dx$$

From the diagram, we see that the equality follows from the fact that we are translating the graph of f , and the limits of integration, by a distance c .



83. Let $u=1-x$. Then $x=1-u$ and $dx=-du$, so

$$\int_0^1 x^a(1-x)^b\,dx=\int_1^0 (1-u)^a u^b(-du)=\int_0^1 u^b(1-u)^a\,du=\int_0^1 x^b(1-x)^a\,dx .$$

84. Let $u=\pi-x$. Then $du=-dx$. When $x=\pi$, $u=0$ and when $x=0$, $u=\pi$. So

$$\int_0^\pi xf(\sin x)\,dx = -\int_\pi^0 (\pi-u)f(\sin(\pi-u))\,du=\int_0^\pi(\pi-u)f(\sin u)\,du$$

$$= \pi\int_0^\pi f(\sin u)\,du-\int_0^\pi uf(\sin u)\,du=\pi\int_0^\pi f(\sin x)\,dx-\int_0^\pi xf(\sin x)\,dx$$

$$\Rightarrow 2\int_0^\pi xf(\sin x)\,dx=\pi\int_0^\pi f(\sin x)\,dx\Rightarrow \int_0^\pi xf(\sin x)\,dx=\frac{\pi}{2}\int_0^\pi f(\sin x)\,dx .$$

85. $\dfrac{x\sin x}{1+\cos^2 x}=x\cdot\dfrac{\sin x}{2-\sin^2 x}=xf(\sin x)$, where $f(t)=\dfrac{t}{2-t^2}$. By Exercise 84,

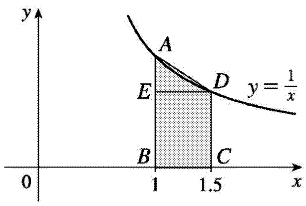
$$\int_0^\pi \frac{x\sin x}{1+\cos^2 x}\,dx=\int_0^\pi xf(\sin x)\,dx=\frac{\pi}{2}\int_0^\pi f(\sin x)\,dx=\frac{\pi}{2}\int_0^\pi \frac{\sin x}{1+\cos^2 x}\,dx$$

Let $u=\cos x$. Then $du=-\sin x\,dx$. When $x=\pi$, $u=-1$ and when $x=0$, $u=1$. So

$$\frac{\pi}{2}\int_0^\pi \frac{\sin x}{1+\cos^2 x}\,dx=-\frac{\pi}{2}\int_1^{-1}\frac{du}{1+u^2}=\frac{\pi}{2}\int_{-1}^{1}\frac{du}{1+u^2}=\frac{\pi}{2}\Big[\tan^{-1}u\Big]_{-1}^{1}=$$

$$\frac{\pi}{2}\left[\tan^{-1}1 - \tan^{-1}(-1)\right] = \frac{\pi}{2}\left[\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right] = \frac{\pi^2}{4}$$

1. **(a)**



We interpret ln 1.5 as the area under the curve $y=1/x$ from $x=1$ to $x=1.5$. The area of the rectangle $BCDE$ is $\frac{1}{2}\cdot\frac{2}{3}=\frac{1}{3}$. The area of the trapezoid $ABCD$ is $\frac{1}{2}\cdot\frac{1}{2}\left(1+\frac{2}{3}\right)=\frac{5}{12}$. Thus, by comparing areas, we observe that $\frac{1}{3}<\ln 1.5<\frac{5}{12}$.

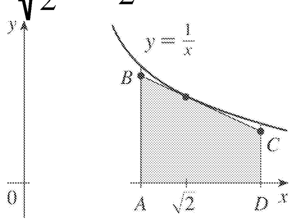
(b) With $f(t)=1/t$, $n=10$, and $\Delta x=0.05$, we have

$$\ln 1.5=\int_1^{1.5}(1/t)\,dt\approx(0.05)[f(1.025)+f(1.075)+\cdots+f(1.475)]$$

$$=(0.05)\left[\frac{1}{1.025}+\frac{1}{1.075}+\cdots+\frac{1}{1.475}\right]\approx 0.4054$$

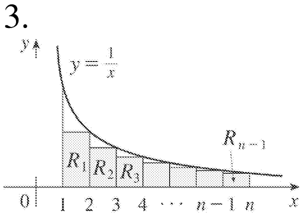
2. **(a)** $y=\frac{1}{t}$, $y'=-\frac{1}{t^2}$. The slope of AD is $\frac{1/2-1}{2-1}=-\frac{1}{2}$. Let c be the $t-$ coordinate of the point on $y=\frac{1}{t}$ with slope $-\frac{1}{2}$. Then $-\frac{1}{c^2}=-\frac{1}{2}\Rightarrow c^2=2\Rightarrow c=\sqrt{2}$ since $c>0$. Therefore, the tangent line is given by $y-\frac{1}{\sqrt{2}}=-\frac{1}{2}\left(t-\sqrt{2}\right)\Rightarrow y=-\frac{1}{2}t+\sqrt{2}$.

(b)

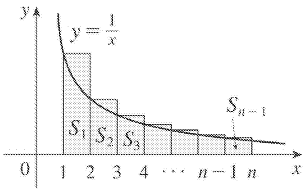


Since the graph of $y=1/t$ is concave upward, the graph lies above the tangent line, that is, above the line segment BC . Now $|AB|=-\frac{1}{2}+\sqrt{2}$ and $|CD|=-1+\sqrt{2}$. So the area of the trapezoid $ABCD$ is $\frac{1}{2}\left[\left(-\frac{1}{2}+\sqrt{2}\right)+\left(-1+\sqrt{2}\right)1\right]=-\frac{3}{4}+\sqrt{2}\approx 0.6642$. So ln 2> area of trapezoid $ABCD>0.66$.

3.



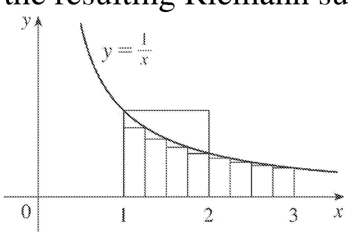
The area of R_i is $\dfrac{1}{i+1}$ and so $\dfrac{1}{2}+\dfrac{1}{3}+\cdots+\dfrac{1}{n}<\int_1^n \dfrac{1}{t}\,dt=\ln n$.



The area of S_i is $\dfrac{1}{i}$ and so $1+\dfrac{1}{2}+\cdots+\dfrac{1}{n-1}>\int_1^n \dfrac{1}{t}\,dt=\ln n$.

Thus, $\dfrac{1}{2}+\dfrac{1}{3}+\cdots+\dfrac{1}{n}<\ln n<1+\dfrac{1}{2}+\cdots+\dfrac{1}{n-1}$.

4. **(a)** From the diagram, we see that the area under the graph of $y=1/x$ between $x=1$ and $x=2$ is less than the area of the square, which is 1 . So $\ln 2=\int_1^2 (1/x)\,dx<1$. To show the other side of the inequality, we must find an area larger than 1 which lies under the graph of $y=1/x$ between $x=1$ and $x=3$. One way to do this is to partition the interval $[1,3]$ into 8 intervals of equal length and calculate the resulting Riemann sum, using the right endpoints:



$$\frac{1}{4}\left(\frac{1}{5/4}+\frac{1}{3/2}+\frac{1}{7/4}+\frac{1}{2}+\frac{1}{9/4}+\frac{1}{5/2}+\frac{1}{11/4}+\frac{1}{3}\right)=\frac{28{,}271}{27{,}720}>1$$

and therefore $1<\int_1^3 (1/x)\,dx=\ln 3$.

A slightly easier method uses the fact that since $y=1/x$ is concave upward, it lies above all its tangent lines. Drawing two such tangent lines at the points $\left(\dfrac{3}{2},\dfrac{2}{3}\right)$ and $\left(\dfrac{5}{2},\dfrac{2}{5}\right)$, we see that the area under the curve from $x=1$ to $x=3$ is more than the sum of the areas of the two trapezoids, that is, $\dfrac{2}{3}+\dfrac{2}{5}=\dfrac{16}{15}$. Thus, $1<\dfrac{16}{15}<\int_1^3 (1/x)\,dx=\ln 3$.

(b) By part (a), $\ln 2<1<\ln 3$. But e is defined such that $\ln e=1$, and because the natural logarithm function is increasing, we have $\ln 2<\ln e<\ln 3 \Leftrightarrow 2<e<3$.

5. If $f(x)=\ln\left(x^r\right)$, then $f'(x)=\left(1/x^r\right)\left(rx^{r-1}\right)=r/x$. But if $g(x)=r\ln x$, then $g'(x)=r/x$. So f and g must differ by a constant: $\ln\left(x^r\right)=r\ln x+C$. Put $x=1$: $\ln\left(1^r\right)=r\ln 1+C\Rightarrow C=0$, so $\ln\left(x^r\right)=r\ln x$.

6. Using the second law of logarithms and Equation 10, we have $\ln\left(e^x/e^y\right)=\ln e^x-\ln e^y=x-y=\ln\left(e^{x-y}\right)$. Since ln is a one–to–one function, it follows that $e^x/e^y=e^{x-y}$.

7. Using the third law of logarithms and Equation 10, we have $\ln e^{rx}=rx=r\ln e^x=\ln\left(e^x\right)^r$. Since ln is a one–to–one function, it follows that $e^{rx}=\left(e^x\right)^r$.

8. Using Definition 13 and the second law of exponents for e^x, we have
$$a^{x-y}=e^{(x-y)\ln a}=e^{x\ln a-y\ln a}=\frac{e^{x\ln a}}{e^{y\ln a}}=\frac{a^x}{a^y}.$$

9. Using Definition 13, the first law of logarithms, and the first law of exponents for e^x, we have $(ab)^x=e^{x\ln(ab)}=e^{x(\ln a+\ln b)}=e^{x\ln a+x\ln b}=e^{x\ln a}e^{x\ln b}=a^xb^x$.

10. Let $\log_a x=r$ and $\log_a y=s$. Then $a^r=x$ and $a^s=y$.

(a) $xy=a^ra^s=a^{r+s}\Rightarrow\log_a(xy)=r+s=\log_a x+\log_a y$

(b) $\dfrac{x}{y}=\dfrac{a^r}{a^s}=a^{r-s}\Rightarrow\log_a\dfrac{x}{y}=r-s=\log_a x-\log_a y$

(c) $x^y=\left(a^r\right)^y=a^{ry}\Rightarrow\log_a\left(x^y\right)=ry=y\log_a x$

1.

$$A=\int_{x=0}^{x=4}\left(y_T-y_B\right)dx=\int_0^4\left[\left(5x-x^2\right)-x\right]dx=\int_0^4\left(4x-x^2\right)dx$$

$$=\left[2x^2-\frac{1}{3}x^3\right]_0^4=\left(32-\frac{64}{3}\right)-(0)=\frac{32}{3}$$

2.

$$A=\int_0^2\left(\sqrt{x+2}-\frac{1}{x+1}\right)dx=\left[\frac{2}{3}(x+2)^{3/2}-\ln(x+1)\right]_0^2$$

$$=\left[\frac{2}{3}(4)^{3/2}-\ln 3\right]-\left[\frac{2}{3}(2)^{3/2}-\ln 1\right]=\frac{16}{3}-\ln 3-\frac{4}{3}\sqrt{2}$$

3.

$$A=\int_{y=-1}^{y=1}\left(x_R-x_L\right)dy=\int_{-1}^{1}\left[e^y-\left(y^2-2\right)\right]dy$$

$$=\int_{-1}^{1}\left(e^y-y^2+2\right)dy=\left[e^y-\frac{1}{3}y^3+2y\right]_{-1}^{1}=\left(e^1-\frac{1}{3}+2\right)-\left(e^{-1}+\frac{1}{3}-2\right)=e-\frac{1}{e}+\frac{10}{3}$$

4.

$$A=\int_0^3\left[\left(2y-y^2\right)-\left(y^2-4y\right)\right]dy=\int_0^3\left(-2y^2+6y\right)dy$$

$$=\left[-\frac{2}{3}y^3+3y^2\right]_0^3=(-18+27)-0=9$$

5.

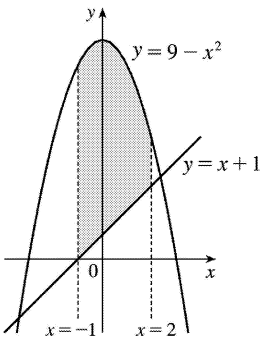
$$A=\int_{-1}^{2}\left[\left(9-x^2\right)-(x+1)\right]dx$$

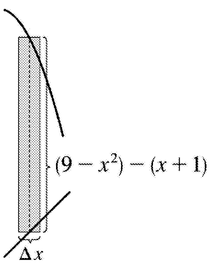
$$=\int_{-1}^{2}\left(8-x-x^2\right)dx$$

$$=\left[8x-\frac{x^2}{2}-\frac{x^3}{3}\right]_{-1}^{2}$$

$$=\left(16-2-\frac{8}{3}\right)-\left(-8-\frac{1}{2}+\frac{1}{3}\right)$$

$$=22-3+\frac{1}{2}=\frac{39}{2}$$





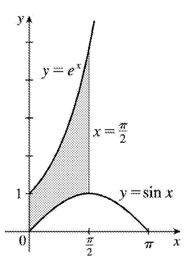
6.

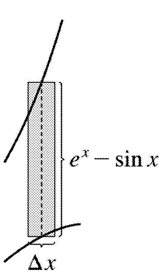
$$A=\int_0^{\pi/2}\left(e^x-\sin x\right)dx$$

$$=\left[e^x+\cos x\right]_0^{\pi/2}$$

$$=\left(e^{\pi/2}+0\right)-(1+1)$$

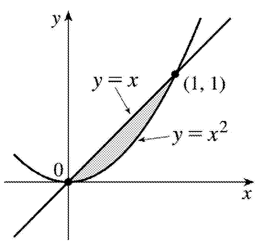
$$=e^{\pi/2}-2$$

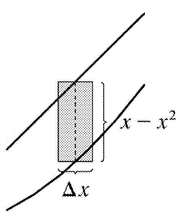




7. The curves intersect when $x=x^2 \Rightarrow x^2-x=0 \Leftrightarrow x(x-1)=0 \Leftrightarrow x=0\ ,\ 1$.

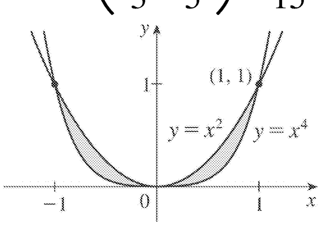
$$A = \int_0^1 \left(x-x^2\right) dx$$
$$= \left[\frac{1}{2}x^2 - \frac{1}{3}x^3 \right]_0^1$$
$$= \frac{1}{2} - \frac{1}{3}$$
$$= \frac{1}{6}$$

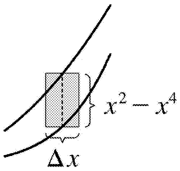




8.

$$A = \int_{-1}^1 \left(x^2-x^4\right) dx$$
$$=2\int_0^1 \left(x^2-x^4\right) dx$$
$$=2\left[\frac{1}{3}x^3 - \frac{1}{5}x^5 \right]_0^1$$
$$=2\left(\frac{1}{3} - \frac{1}{5} \right) = \frac{4}{15}$$



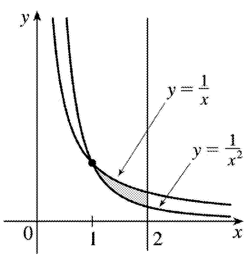


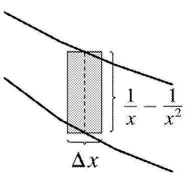
9.

$$A = \int_1^2 \left(\frac{1}{x} - \frac{1}{x^2} \right) dx = \left[\ln x + \frac{1}{x} \right]_1^2$$

$$= \left(\ln 2 + \frac{1}{2} \right) - (\ln 1 + 1)$$

$$= \ln 2 - \frac{1}{2} \approx 0.19$$

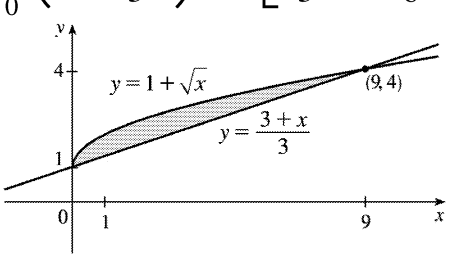


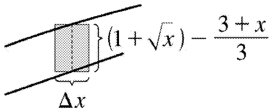


10. $1+\sqrt{x} = \frac{3+x}{3} = 1 + \frac{x}{3} \Rightarrow \sqrt{x} = \frac{x}{3} \Rightarrow x = \frac{x^2}{9} \Rightarrow 9x - x^2 = 0 \Rightarrow x(9-x) = 0 \Rightarrow x = 0$ or 9 , so

$$A = \int_0^9 \left[(1+\sqrt{x}) - \left(\frac{3+x}{3} \right) \right] dx = \int_0^9 \left[(1+\sqrt{x}) - \left(1 + \frac{x}{3} \right) \right] dx =$$

$$\int_0^9 \left(\sqrt{x} - \frac{1}{3}x \right) dx = \left[\frac{2}{3}x^{3/2} - \frac{1}{6}x^2 \right]_0^9 = 18 - \frac{27}{2} = \frac{9}{2}$$





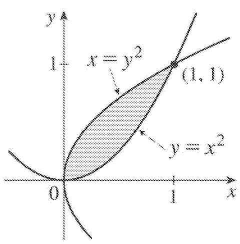
11.

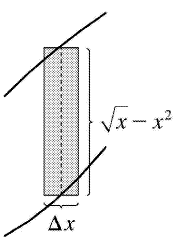
$$A=\int_0^1\left(\sqrt{x}-x^2\right)dx$$

$$=\left[\frac{2}{3}x^{3/2}-\frac{1}{3}x^3\right]_0^1$$

$$=\frac{2}{3}-\frac{1}{3}$$

$$=\frac{1}{3}$$



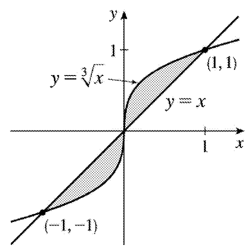


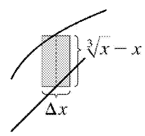
12. $x=\sqrt[3]{x}\Rightarrow x^3=x\Rightarrow x^3-x=0\Rightarrow x(x^2-1)=0\Rightarrow x(x+1)(x-1)=0\Rightarrow$
$x=-1$, 0 , or 1 , so

$$A=\int_{-1}^{1}\left|\sqrt[3]{x}-x\right|dx=\int_{-1}^{0}\left(x-\sqrt[3]{x}\right)dx+\int_0^1\left(\sqrt[3]{x}-x\right)dx=2\int_0^1\left(x^{1/3}-x\right)dx$$

[by symmetry]

$$=2\left[\frac{3}{4}x^{4/3}-\frac{1}{2}x^2\right]_0^1=2\left(\frac{3}{4}-\frac{1}{2}\right)=\frac{1}{2}$$

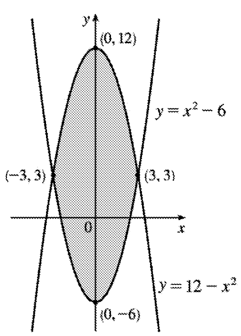


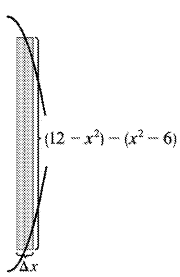


13. $12-x^2=x^2-6 \Leftrightarrow 2x^2=18 \Leftrightarrow x^2=9 \Leftrightarrow x^2=\pm 3$, so

$$A=\int_{-3}^{3}\left[\left(12-x^2\right)-\left(x^2-6\right)\right]dx=2\int_{0}^{3}\left(18-2x^2\right)dx \text{ [by symmetry]}$$

$$=2\left[18x-\frac{2}{3}x^3\right]_0^3=2[(54-18)-0]=2(36)=72$$





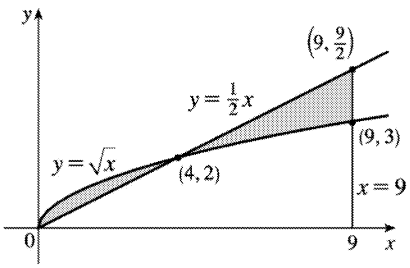
14. $x^3-x=3x \Rightarrow x^3-4x=0 \Rightarrow x(x^2-4)=0 \Rightarrow x(x+2)(x-2)=0 \Rightarrow x=0$, -2 , or 2 .
By symmetry,

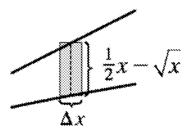
$$A=\int_{-2}^{2}\left|3x-\left(x^3-x\right)\right|dx=2\int_{0}^{2}\left[3x-\left(x^3-x\right)\right]dx=2\int_{0}^{2}\left(4x-x^3\right)dx=2\left[2x^2-\frac{1}{4}x^4\right]_0^2$$

$$=2(8-4)=8 \text{]}$$

15. $\frac{1}{2}x=\sqrt{x} \Rightarrow \frac{1}{4}x^2=x \Rightarrow x^2-4x=0 \Rightarrow x(x-4)=0 \Rightarrow x=0$ or 4 , so

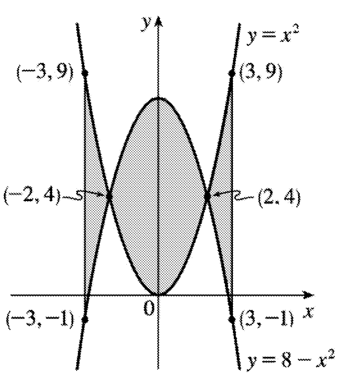
$$A=\int_0^4\left(\sqrt{x}-\frac{1}{2}x\right)dx+\int_4^9\left(\frac{1}{2}x-\sqrt{x}\right)dx=\left[\frac{2}{3}x^{3/2}-\frac{1}{4}x^2\right]_0^4+\left[\frac{1}{4}x^2-\frac{2}{3}x^{3/2}\right]_4^9$$
$$=\left[\left(\frac{16}{3}-4\right)-0\right]+\left[\left(\frac{81}{4}-18\right)-\left(4-\frac{16}{3}\right)\right]=\frac{81}{4}+\frac{32}{3}-26=\frac{59}{12}$$

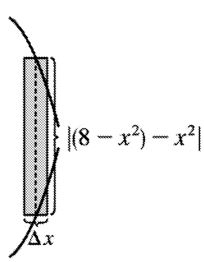




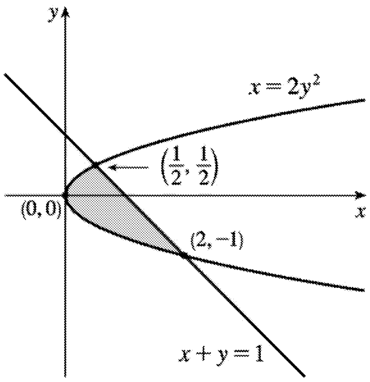
For $4<x<9$

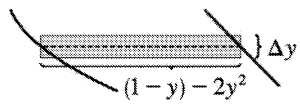
16.
$$A=\int_{-3}^3\left|\left(8-x^2\right)-x^2\right|dx=2\int_0^3\left|8-2x^2\right|dx=2\int_0^2\left(8-2x^2\right)dx+2\int_2^3\left(2x^2-8\right)dx$$
$$=2\left[8x-\frac{2}{3}x^3\right]_0^2+2\left[\frac{2}{3}x^3-8x\right]_2^3=2\left[\left(16-\frac{16}{3}\right)-0\right]+2\left[(18-24)-\left(\frac{16}{3}-16\right)\right]$$
$$=32-\frac{32}{3}+20-\frac{32}{3}=52-\frac{64}{3}=\frac{92}{3}$$





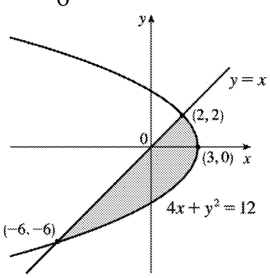
17. $2y^2=1-y\Leftrightarrow 2y^2+y-1=0\Leftrightarrow(2y-1)(y+1)=0\Leftrightarrow y=\frac{1}{2}$ or -1 , so $x=\frac{1}{2}$ or 2 and
$$A=\int_{-1}^{1/2}\left[(1-y)-2y^2\right]dy=\int_{-1}^{1/2}\left(1-y-2y^2\right)dy=\left[y-\frac{1}{2}y^2-\frac{2}{3}y^3\right]_{-1}^{1/2}$$
$$=\left(\frac{1}{2}-\frac{1}{8}-\frac{1}{12}\right)-\left(-1-\frac{1}{2}+\frac{2}{3}\right)=\frac{7}{24}-\left(-\frac{5}{6}\right)=\frac{7}{24}+\frac{20}{24}=\frac{27}{24}=\frac{9}{8}$$

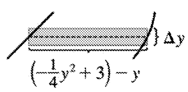




18. $4x+x^2=12 \Leftrightarrow (x+6)(x-2)=0 \Leftrightarrow x=-6$ or $x=2$, so $y=-6$ or $y=2$ and

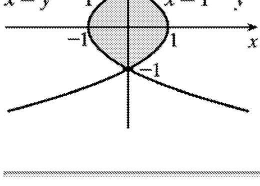
$$A=\int_{-6}^{2}\left[\left(-\frac{1}{4}y^2+3\right)-y\right]dy=\left[-\frac{1}{12}y^3-\frac{1}{2}y^2+3y\right]_{-6}^{2}=\left(-\frac{2}{3}-2+6\right)-(18-18-18)=22-\frac{2}{3}=\frac{64}{3}\ .$$

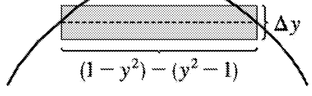




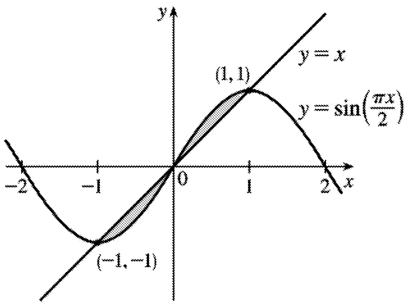
19. The curves intersect when $1-y^2=y^2-1 \Leftrightarrow 2=2y^2 \Leftrightarrow y^2=1 \Leftrightarrow y=\pm 1$.

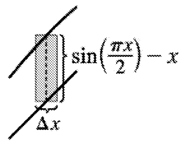
$$\begin{aligned}A &= \int_{-1}^{1}\left[\left(1-y^2\right)-\left(y^2-1\right)\right]dy \\ &= \int_{-1}^{1} 2\left(1-y^2\right)dy \\ &= 2\cdot 2\int_{0}^{1}\left(1-y^2\right)dy \\ &= 4\left[y-\frac{1}{3}y^3\right]_{0}^{1}=4\left(1-\frac{1}{3}\right)=\frac{8}{3}\end{aligned}$$





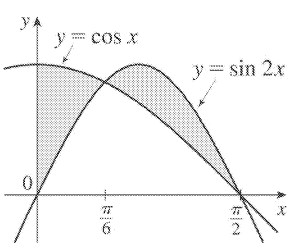
20. $A=2\int_0^1 \left[\sin\left(\frac{\pi x}{2}\right)-x\right]dx=2\left[-\frac{2}{\pi}\cos\left(\frac{\pi x}{2}\right)-\frac{x^2}{2}\right]_0^1=2\left[\left(0-\frac{1}{2}\right)-\left(-\frac{2}{\pi}-0\right)\right]=\frac{4}{\pi}-1$





21. Notice that $\cos x=\sin 2x=2\sin x\cos x \Leftrightarrow$
$2\sin x\cos x-\cos x=0 \Leftrightarrow \cos x(2\sin x-1)=0 \Leftrightarrow$
$2\sin x=1$ or $\cos x=0 \Leftrightarrow x=\frac{\pi}{6}$ or $\frac{\pi}{2}$.

$$A = \int_0^{\pi/6} (\cos x-\sin 2x)\,dx+\int_{\pi/6}^{\pi/2} (\sin 2x-\cos x)\,dx$$
$$= \left[\sin x+\frac{1}{2}\cos 2x\right]_0^{\pi/6}+\left[-\frac{1}{2}\cos 2x-\sin x\right]_{\pi/6}^{\pi/2}$$
$$= \frac{1}{2}+\frac{1}{2}\cdot\frac{1}{2}-\left(0+\frac{1}{2}\cdot 1\right)+\left(\frac{1}{2}-1\right)-\left(-\frac{1}{2}\cdot\frac{1}{2}-\frac{1}{2}\right)=\frac{1}{2}$$

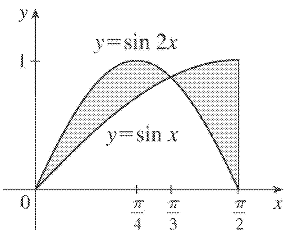


22. $\sin x=\sin 2x=2\sin x\cos x$ when $\sin x=0$ and when $\cos x=\frac{1}{2}$;

that is, when $x=0$ or $\frac{\pi}{3}$.

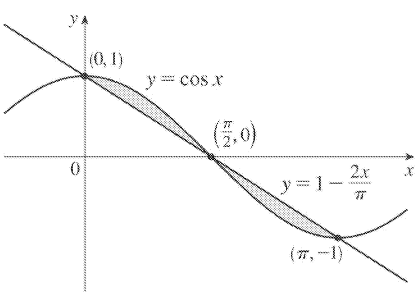
$$A = \int_0^{\pi/3} (\sin 2x-\sin x)\,dx+\int_{\pi/3}^{\pi/2} (\sin x-\sin 2x)\,dx$$
$$= \left[-\frac{1}{2}\cos 2x+\cos x\right]_0^{\pi/3}+\left[\frac{1}{2}\cos 2x-\cos x\right]_{\pi/3}^{\pi/2}$$

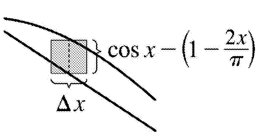
$$= \left[-\frac{1}{2}\left(-\frac{1}{2}\right)+\frac{1}{2} \right]-\left(-\frac{1}{2}+1\right)$$
$$+\left(-\frac{1}{2}-0\right)-\left[\frac{1}{2}\left(-\frac{1}{2}\right)-\frac{1}{2} \right]=\frac{1}{2}$$



23. From the graph, we see that the curves intersect at $x=0$, $x=\dfrac{\pi}{2}$, and $x=\pi$. By symmetry,

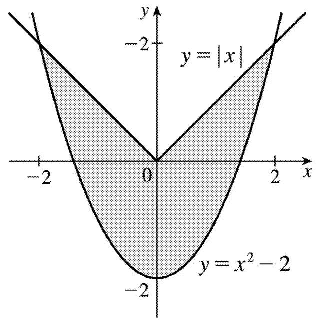
$$A=\int_0^{\pi}\left|\cos x-\left(1-\frac{2x}{\pi}\right)\right|dx=2\int_0^{\pi/2}\left[\cos x-\left(1-\frac{2x}{\pi}\right)\right]dx=2\int_0^{\pi/2}\left(\cos x-1+\frac{2x}{\pi}\right)dx$$
$$=2\left[\sin x-x+\frac{1}{\pi}x^2\right]_0^{\pi/2}=2\left[\left(1-\frac{\pi}{2}+\frac{1}{\pi}\cdot\frac{\pi^2}{4}\right)-0\right]=2\left(1-\frac{\pi}{2}+\frac{\pi}{4}\right)=2-\frac{\pi}{2}$$

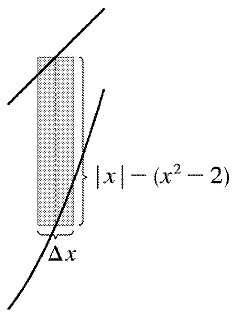




24. For $x>0$, $x=x^2-2\Rightarrow 0=x^2-x-2\Rightarrow 0=(x-2)(x+1)\Rightarrow x=2$. By symmetry,

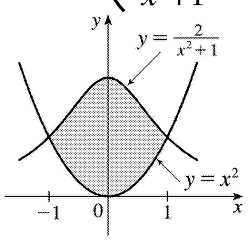
$$\int_{-2}^{2}\left[|x|-\left(x^2-2\right)\right]dx = 2\int_0^2\left[x-\left(x^2-2\right)\right]dx=2\int_0^2\left(x-x^2+2\right)dx=2\left[\frac{1}{2}x^2-\frac{1}{3}x^3+2x\right]_0^2=$$
$$2\left(2-\frac{8}{3}+4\right)=\frac{20}{3}$$

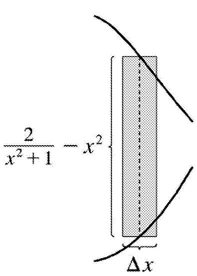




25. The curves intersect when $x^2 = \frac{2}{x^2+1} \Leftrightarrow x^4 + x^2 = 2 \Leftrightarrow x^4 + x^2 - 2 = 0 \Leftrightarrow \left(x^2+2\right)\left(x^2-1\right) = 0 \Leftrightarrow x^2 = 1 \Leftrightarrow x = \pm 1$.

$$A = \int_{-1}^{1} \left(\frac{2}{x^2+1} - x^2 \right) dx = 2\int_{0}^{1} \left(\frac{2}{x^2+1} - x^2 \right) dx = 2\left[2\tan^{-1} x - \frac{1}{3} x^3 \right]_0^1 = 2\left(2 \cdot \frac{\pi}{4} - \frac{1}{3} \right) = \pi - \frac{2}{3} \approx 2.47$$





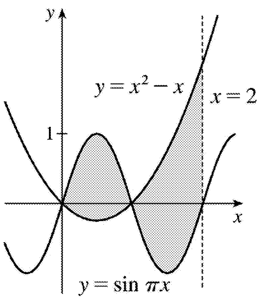
26.

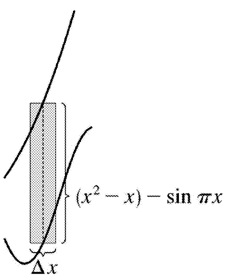
$$A = \int_0^1 \left[\sin \pi x - \left(x^2 - x\right) \right] dx + \int_1^2 \left[\left(x^2 - x\right) - \sin \pi x \right] dx$$

$$= \left[-\frac{1}{\pi}\cos \pi x - \frac{1}{3}x^3 + \frac{1}{2}x^2 \right]_0^1 + \left[\frac{1}{3}x^3 - \frac{1}{2}x^2 + \frac{1}{\pi}\cos \pi x \right]_1^2$$

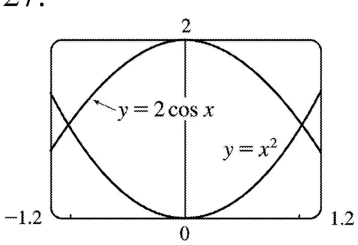
$$= \left(\frac{1}{\pi} - \frac{1}{3} + \frac{1}{2} \right) - \left(-\frac{1}{\pi} \right) + \left(\frac{8}{3} - 2 + \frac{1}{\pi} \right) - \left(\frac{1}{3} - \frac{1}{2} - \frac{1}{\pi} \right)$$

$$= \frac{4}{\pi} + 1$$





27.



From the graph, we see that the curves intersect at $x = \pm a \approx \pm 1.02$, with $2\cos x > x^2$ on $(-a, a)$. So the area of the region bounded by the curves is

$$A = \int_{-a}^{a} \left(2\cos x - x^2 \right) dx = 2\int_{0}^{a} \left(2\cos x - x^2 \right) dx$$

$$= 2 \left[2\sin x - \frac{1}{3}x^3 \right]_0^a \approx 2.70$$

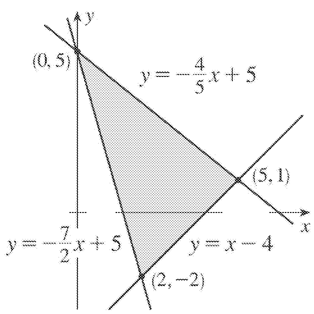
28.

$$A = \int_0^2 \left[\left(-\frac{4}{5}x+5 \right) - \left(-\frac{7}{2}x+5 \right) \right] dx + \int_2^5 \left[\left(-\frac{4}{5}x+5 \right) - (x-4) \right] dx$$

$$= \int_0^2 \frac{27}{10} x\,dx + \int_2^5 \left(-\frac{9}{5}x+9 \right) dx$$

$$= \left[\frac{27}{20}x^2 \right]_0^2 + \left[-\frac{9}{10}x^2+9x \right]_2^5$$

$$= \left(\frac{27}{5} - 0 \right) + \left(-\frac{45}{2} + 45 \right) - \left(-\frac{18}{5} + 18 \right) = \frac{27}{2}$$



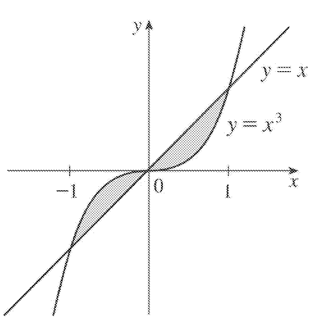
29.

$$A = \int_{-1}^{1} \left| x^3 - x \right| dx$$

$$= 2\int_0^1 \left(x - x^3 \right) dx \text{ [by symmetry]}$$

$$= 2\left[\frac{1}{2}x^2 - \frac{1}{4}x^4 \right]_0^1$$

$$= 2\left(\frac{1}{2} - \frac{1}{4} \right) = \frac{1}{2}$$



30. The curves intersect when $\sqrt{x+2}=x \Rightarrow x+2=x^2 \Rightarrow x^2-x-2=0 \Rightarrow$ $(x-2)(x+1)=0 \Rightarrow x=-1$ or 2 .

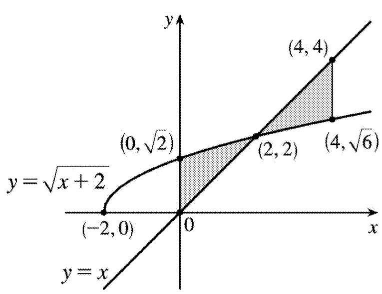
$$A=\int_0^4 \left|\sqrt{x+2}-x\right| dx$$

$$=\int_0^2 \left(\sqrt{x+2}-x\right) dx+\int_2^4 \left(x-\sqrt{x+2}\right) dx$$

$$=\left[\frac{2}{3}(x+2)^{3/2}-\frac{1}{2}x^2\right]_0^2+\left[\frac{1}{2}x^2-\frac{2}{3}(x+2)^{3/2}\right]_2^4$$

$$=\left(\frac{16}{3}-2\right)-\left(\frac{2}{3}\left(2\sqrt{2}\right)-0\right)+\left(8-\frac{2}{3}\left(6\sqrt{6}\right)\right)-\left(2-\frac{16}{3}\right)$$

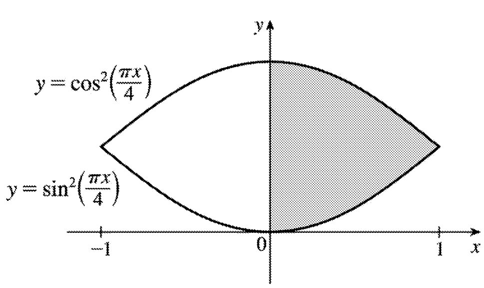
$$=4+\frac{32}{3}-\frac{4}{3}\sqrt{2}-4\sqrt{6}=\frac{44}{3}-4\sqrt{6}-\frac{4}{3}\sqrt{2}$$



31. Let $f(x)=\cos^2\left(\frac{\pi x}{4}\right)-\sin^2\left(\frac{\pi x}{4}\right)$ and $\Delta x=\frac{1-0}{4}$.

The shaded area is given by

$$A=\int_0^1 f(x)dx \approx M_4=\frac{1}{4}\left[f\left(\frac{1}{8}\right)+f\left(\frac{3}{8}\right)+f\left(\frac{5}{8}\right)+f\left(\frac{7}{8}\right)\right]\approx 0.6407$$



32. The curves intersect when $\sqrt[3]{16-x^3}=x \Rightarrow$

$16-x^3=x^3 \Rightarrow 2x^3=16 \Rightarrow x^3=8 \Rightarrow x=2.$

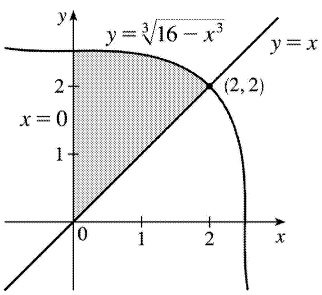
Let $f(x)=\sqrt[3]{16-x^3}-x$ and $\Delta x=\frac{2-0}{4}$.

The shaded area is given by

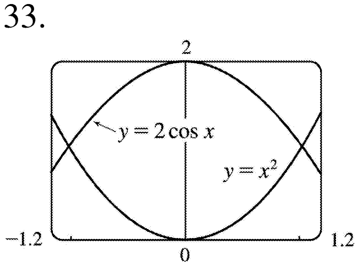
$$A=\int_0^2 f(x)\,dx \approx M_4$$

$$=\frac{2}{4}\left[f\left(\frac{1}{4}\right)+f\left(\frac{3}{4}\right)+f\left(\frac{5}{4}\right)+f\left(\frac{7}{4}\right)\right]$$

$$\approx 2.8144$$



33.

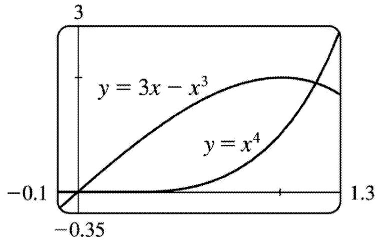


From the graph, we see that the curves intersect at $x=\pm a\approx\pm 1.02$, with $2\cos x>x^2$ on $(-a,a)$. So the area of the region bounded by the curves is

$$A = \int_{-a}^{a}\left(2\cos x-x^2\right)dx=2\int_0^a\left(2\cos x-x^2\right)dx$$

$$= 2\left[2\sin x-\frac{1}{3}x^3\right]_0^a\approx 2.70$$

34.

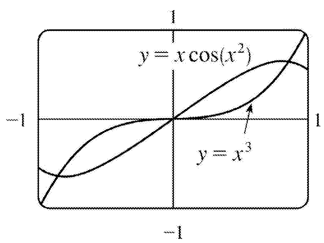


From the graph, we see that the curves intersect at $x=0$ and at $x=a\approx 1.17$, with $3x-x^3>x^4$ on $(0,a)$. So the area of the region bounded by the curves is

$$A = \int_0^a \left[\left(3x-x^3\right)-x^4\right]dx=\left[\frac{3}{2}x^2-\frac{1}{4}x^4-\frac{1}{5}x^5\right]_0^a$$

$$\approx 1.15$$

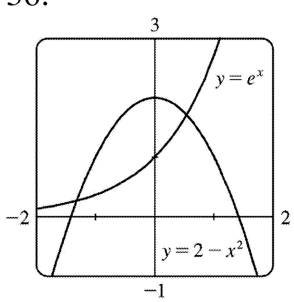
35.



From the graph, we see that the curves intersect at $x=\pm a\approx \pm 0.86$.
So the area of the region bounded by the curves is

$$A=2\int_0^a\left[x\cos\left(x^2\right)-x^3\right]dx=2\left[\frac{1}{2}\sin\left(x^2\right)-\frac{1}{4}x^4\right]_0^a\approx 0.40$$

36.



From the graph, we see that the curves intersect at $x=a\approx -1.32$ and $x=b\approx 0.54$, with $2-x^2>e^x$ on (a,b) . So the area of the region bounded by the curves is

$$A=\int_a^b\left[\left(2-x^2\right)-e^x\right]\mathrm{dx}=\left[2x-\frac{1}{3}x^3-e^x\right]_a^b\approx 1.45$$

37. As the figure illustrates, the curves $y=x$ and

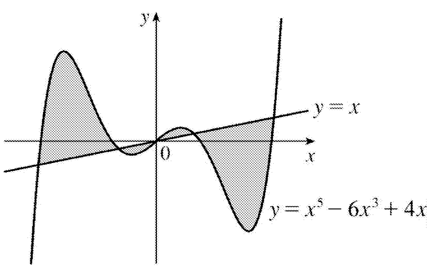
$y=x^5-6x^3+4x$ enclose a four–part region symmetric about the origin (since x^5-6x^3+4x and x are odd functions of x). The curves intersect at values of x where $x^5-6x^3+4x=x$; that is, where $x(x^4-6x^2+3)=0$. That happens at $x=0$ and where $x^2=\frac{6\pm\sqrt{36-12}}{2}=3\pm\sqrt{6}$; that is, at $x=-\sqrt{3+\sqrt{6}}$, $-\sqrt{3-\sqrt{6}}$, 0 , $\sqrt{3-\sqrt{6}}$, and $\sqrt{3+\sqrt{6}}$.

The exact area is

$$2\int_0^{\sqrt{3+\sqrt{6}}}\left|\left(x^5-6x^3+4x\right)-x\right|dx=2\int_0^{\sqrt{3+\sqrt{6}}}\left|x^5-6x^3+3x\right|dx$$

$$=2\int_0^{\sqrt{3-\sqrt{6}}}\left(x^5-6x^3+3x\right)dx+2\int_{\sqrt{3-\sqrt{6}}}^{\sqrt{3+\sqrt{6}}}\left(-x^5+6x^3-3x\right)dx$$

$$=12\sqrt{6}-9$$



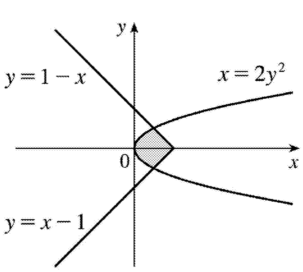
38. The inequality $x\geq 2y^2$ describes the region that lies on, or to the right of, the parabola $x=2y^2$. The inequality $x\leq 1-|y|$ describes the region

that lies on, or to the left of, the curve $x=1-|y|=\begin{cases}1-y \text{ if } y\geq 0\\ 1+y \text{ if } y<0\end{cases}$.

So the given region is the shaded region that lies between the curves.

The graphs of $x=1-y$ and $x=2y^2$ intersect when $1-y=2y^2\Leftrightarrow$

$2y^2+y-1=0\Leftrightarrow(2y-1)(y+1)=0\Rightarrow y=\frac{1}{2}$ (for $y\geq 0$).



By symmetry,

$$A=2\int_0^{1/2}\left[(1-y)-2y^2\right]dy=2\left[-\frac{2}{3}y^3-\frac{1}{2}y^2+y\right]_0^{1/2}=2\left[\left(-\frac{1}{12}-\frac{1}{8}+\frac{1}{2}\right)-0\right]=2\left(\frac{7}{24}\right)=\frac{7}{12}\ .$$

39. 1 second $=\frac{1}{3600}$ hour, so 10 s $=\frac{1}{360}$ h. With the given data, we can take $n=5$ to use the Midpoint Rule. $\Delta t=\frac{1/360-0}{5}=\frac{1}{1800}$, so

$$\begin{aligned}\text{distance}_{\text{Kelly}}-\text{distance}_{\text{Chris}} &= \int_0^{1/360} v_K\,dt-\int_0^{1/360} v_C\,dt=\int_0^{1/360}\left(v_K-v_C\right)dt\\ &\approx M_5=\frac{1}{1800}\Big[\left(v_K-v_C\right)(1)+\left(v_K-v_C\right)(3)+\left(v_K-v_C\right)(5)\\ &\quad+\left(v_K-v_C\right)(7)+\left(v_K-v_C\right)(9)\Big]\\ &= \frac{1}{1800}\left[(22-20)+(52-46)+(71-62)+(86-75)+(98-86)\right]\\ &= \frac{1}{1800}(2+6+9+11+12)=\frac{1}{1800}(40)=\frac{1}{45}\text{ mile,or}117\frac{1}{3}\text{ feet}\end{aligned}$$

40. If $x=$ distance from left end of pool and $w=w(x)=$ width at x , then the Midpoint Rule with $n=4$ and $\Delta x=\frac{b-a}{n}=\frac{8\cdot 2-0}{4}=4$ gives Area $=\int_0^{16} w\,dx\approx 4(6.2+6.8+5.0+4.8)=4(22.8)=91.2\ \mathrm{m}^2$.

41. We know that the area under curve A between $t=0$ and $t=x$ is $\int_0^x v_A(t)\,dt=s_A(x)$, where $v_A(t)$ is the velocity of car A and s_A is its displacement. Similarly, the area under curve B between $t=0$ and $t=x$ is $\int_0^x v_B(t)\,dt=s_B(x)$.

(a) After one minute, the area under curve A is greater than the area under curve B . So car A is ahead after one minute.

(b) The area of the shaded region has numerical value $s_A(1)-s_B(1)$, which is the distance by which A is ahead of B after 1 minute.

(c) After two minutes, car B is traveling faster than car A and has gained some ground, but the area under curve A from $t=0$ to $t=2$ is still greater than the corresponding area for curve B , so car A is still ahead.

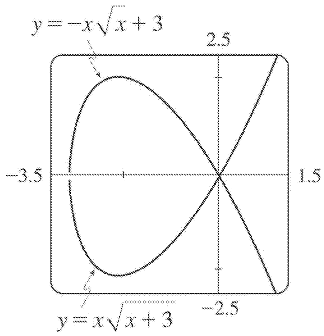
(d) From the graph, it appears that the area between curves A and B for $0\le t\le 1$ (when car A is going faster), which corresponds to the distance by which car A is ahead, seems to be about 3 squares. Therefore, the cars will be side by side at the time x where the area between the curves for $1 \leq t \leq x$ (when car B is going faster) is the same as the area for $0 \leq t \leq 1$. From the graph, it appears that this time is $x \approx 2.2$. So the cars are side by side when $t \approx 2.2$ minutes.

42. The area under $R'(x)$ from $x=50$ to $x=100$ represents the change in revenue, and the area under $C'(x)$ from $x=50$ to $x=100$ represents the change in cost. The shaded region represents the difference between these two values; that is, the increase in profit as the production level increases from 50 units to 100 units. We use the
Midpoint Rule with $n=5$ and $\Delta x=10$:

$$\begin{aligned} M_5 &= \Delta x\{+[R'(65)-C'(65)]+[R'(75)-C'(75)] \\ &\quad +[R'(85)-C'(85)]+[R'(95)-C'(95)]\} \\ &\approx 10(2.40-0.85+2.20-0.90+2.00-1.00+1.80-1.10+1.70-1.20) \\ &= 10(5.05)=50.5 \text{ thousand dollars} \end{aligned}$$

Using M_1 would give us $50(2-1)=50$ thousand dollars.

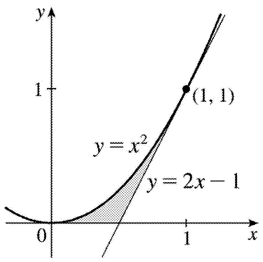
43.



To graph this function, we must first express it as a combination of explicit functions of y ; namely, $y=\pm x\sqrt{x+3}$. We can see from the graph that the loop extends from $x=-3$ to $x=0$, and that by symmetry, the area we seek is just twice the area under the top half of the curve on this interval, the equation of the top half being $y=-x\sqrt{x+3}$. So the area is $A=2\int_{-3}^{0}\left(-x\sqrt{x+3}\right)dx$. We substitute $u=x+3$, so $du=dx$ and the limits change to 0 and 3 , and we get

$$\begin{aligned} A &= -2\int_0^3 \left[(u-3)\sqrt{u}\right]du=-2\int_0^3\left(u^{3/2}-3u^{1/2}\right)du \\ &= -2\left[\frac{2}{5}u^{5/2}-2u^{3/2}\right]_0^3=-2\left[\frac{2}{5}\left(3^2\sqrt{3}\right)-2\left(3\sqrt{3}\right)\right]=\frac{24}{5}\sqrt{3} \end{aligned}$$

44.

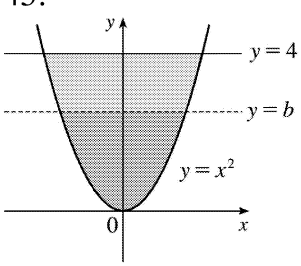


We start by finding the equation of the tangent line to $y=x^2$ at the point $(1,1)$: $y'=2x$, so the slope of the tangent is $2(1)=2$, and its equation is $y-1=2(x-1)$, or $y=2x-1$. We would need two integrals to integrate with respect to x , but only one to integrate with respect to y .

$$A=\int_0^1\left[\frac{1}{2}(y+1)-\sqrt{y}\right]dy=\left[\frac{1}{4}y^2+\frac{1}{2}y-\frac{2}{3}y^{3/2}\right]_0^1$$

$$=\frac{1}{4}+\frac{1}{2}-\frac{2}{3}=\frac{1}{12}$$

45.



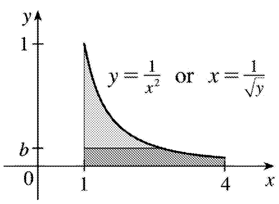
By the symmetry of the problem, we consider only the first quadrant, where $y=x^2\Rightarrow x=\sqrt{y}$. We are looking for a number b such that $\int_0^b\sqrt{y}\,dy=\int_b^4\sqrt{y}\,dy\Rightarrow\frac{2}{3}\left[y^{3/2}\right]_0^b=\frac{2}{3}\left[y^{3/2}\right]_b^4\Rightarrow b^{3/2}=4^{3/2}-b^{3/2}\Rightarrow 2b^{3/2}=8$ $\Rightarrow b^{3/2}=4\Rightarrow b=4^{2/3}\approx 2.52$.

46. **(a)** We want to choose a so that $\int_1^a\frac{1}{x^2}\,dx=\int_a^4\frac{1}{x^2}\,dx\Rightarrow\left[\frac{-1}{x}\right]_1^a=\left[\frac{-1}{x}\right]_a^4\Rightarrow-\frac{1}{a}+1=-\frac{1}{4}+\frac{1}{a}\Rightarrow$ $\frac{5}{4}=\frac{2}{a}\Rightarrow a=\frac{8}{5}$.

(b) The area under the curve $y=1/x^2$ from $x=1$ to $x=4$ is $\frac{3}{4}$. Now the line $y=b$ must intersect the curve $x=1/\sqrt{y}$ and not the line $x=4$, since the area under the line $y=1/4^2$ from $x=1$ to $x=4$ is only $\frac{3}{16}$, which is less than half of

$\frac{3}{4}$. We want to choose b so that the upper area in the diagram is half of the total area under the curve $y=\frac{1}{x^2}$ from $x=1$ to $x=4$. This implies that

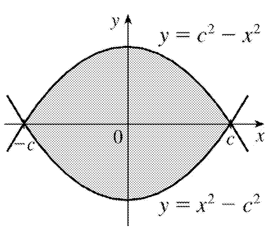
$\int_b^1 \left(1/\sqrt{y}-1\right)dy=\frac{1}{2}\cdot\frac{3}{4} \Rightarrow \left[2\sqrt{y}-y\right]_b^1=\frac{3}{8} \Rightarrow 1-2\sqrt{b}+b=\frac{3}{8} \Rightarrow b-2\sqrt{b}+\frac{5}{8}=0$. Letting $c=\sqrt{b}$, we get $c^2-2c+\frac{5}{8}=0 \Rightarrow 8c^2-16c+5=0$. Thus, $c=\frac{16\pm\sqrt{256-160}}{16}=1\pm\frac{\sqrt{6}}{4}$. But $c=\sqrt{b}<1 \Rightarrow c=1-\frac{\sqrt{6}}{4} \Rightarrow$ $b=c^2=1+\frac{3}{8}-\frac{\sqrt{6}}{2}=\frac{1}{8}\left(11-4\sqrt{6}\right)\approx 0.1503$.



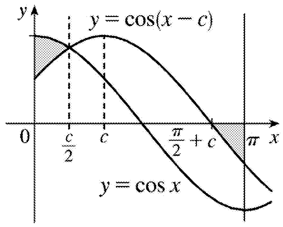
47. We first assume that $c>0$, since c can be replaced by $-c$ in both equations without changing the graphs, and if $c=0$ the curves do not enclose a region. We see from the graph that the enclosed area A lies between $x=-c$ and $x=c$, and by symmetry, it is equal to four times the area in the first quadrant. The enclosed area is

$$A = 4\int_0^c \left(c^2-x^2\right)dx=4\left[c^2x-\frac{1}{3}x^3\right]_0^c$$
$$= 4\left(c^3-\frac{1}{3}c^3\right)=4\left(\frac{2}{3}c^3\right)=\frac{8}{3}c^3$$

So $A=576 \Leftrightarrow \frac{8}{3}c^3=576 \Leftrightarrow c^3=216 \Leftrightarrow c=\sqrt[3]{216}=6$. Note that $c=-6$ is another solution, since the graphs are the same.



48.

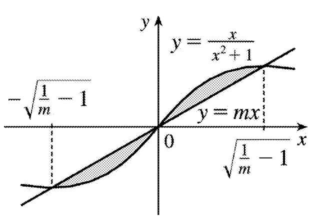


It appears from the diagram that the curves $y=\cos x$ and $y=\cos(x-c)$ intersect halfway between 0 and c , namely, when $x=c/2$. We can verify that this is indeed true by noting that $\cos(c/2-c)=\cos(-c/2)=\cos(c/2)$.

The point where $\cos(x-c)$ crosses the $x-$ axis is $x=\frac{\pi}{2}+c$. So we require that $\int_0^{c/2}[\cos x-\cos(x-c)]dx=-\int_{\pi/2+c}^{\pi}\cos(x-c)dx$ (the negative sign on the RHS is needed since the second area is beneath the $x-$ axis) $\Leftrightarrow$

$[\sin x-\sin(x-c)]_0^{c/2}=-[\sin(x-c)]_{\pi/2+c}^{\pi}\Rightarrow$
$[\sin(c/2)-\sin(-c/2)]-[-\sin(-c)]=-\sin(\pi-c)+\sin\left[\left(\frac{\pi}{2}+c\right)-c\right]\Leftrightarrow 2\sin(c/2)-\sin c=-\sin c+1$ . . So $2\sin(c/2)=1\Leftrightarrow\sin(c/2)=\frac{1}{2}\Leftrightarrow c/2=\frac{\pi}{6}\Leftrightarrow c=\frac{\pi}{3}$.

49. The curve and the line will determine a region when they intersect at two or more points. So we solve the equation $x/\left(x^2+1\right)=mx\Rightarrow x=x\left(mx^2+m\right)\Rightarrow$
$x\left(mx^2+m\right)-x=0\Rightarrow x\left(mx^2+m-1\right)=0\Rightarrow$
$x=0$ or $mx^2+m-1=0\Rightarrow x=0$ or $x^2=\frac{1-m}{m}\Rightarrow$

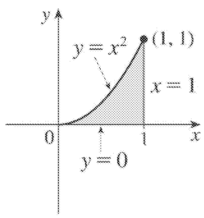


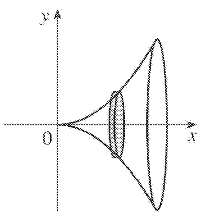
$x=0$ or $x=\pm\sqrt{\frac{1}{m}-1}$. Note that if $m=1$, this has only the solution $x=0$, and no region is determined. But if $1/m-1>0\Leftrightarrow 1/m>1\Leftrightarrow 0<m<1$, then there are two solutions. [Another way of seeing this is to observe that the slope of the tangent to $y=x/(x^2+1)$ at the origin is $y'=1$ and therefore we must have $0<m<1$.] Note that we cannot just integrate between the positive and negative roots, since the curve and the line cross at the origin. Since mx and $x/(x^2+1)$ are both odd functions, the total area is twice the area between the curves on the interval $\left[0,\sqrt{1/m-1}\,\right]$. So the total area enclosed is

$$2\int_0^{\sqrt{1/m-1}}\left[\frac{x}{x^2+1}-mx\right]dx=2\left[\frac{1}{2}\ln\left(x^2+1\right)-\frac{1}{2}mx^2\right]_0^{\sqrt{1/m-1}}=\left[\ln(1/m-1+1)-m(1/m-1)\right]-(\ln 1-0)=$$
$$\ln(1/m)-1+m=m-\ln m-1$$

1. A cross–section is circular with radius x^2 , so its area is $A(x)=\pi\left(x^2\right)^2$.

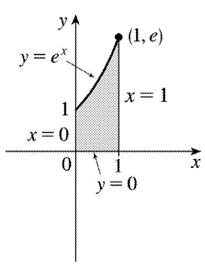
$$V=\int_0^1 A(x)\,dx=\int_0^1 \pi\left(x^2\right)^2 dx=\pi\int_0^1 x^4 dx=\pi\left[\frac{1}{5}x^5\right]_0^1=\frac{\pi}{5}$$

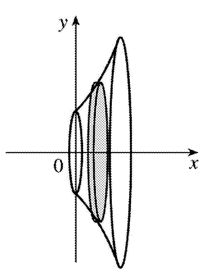




2. A cross–section is a disk with radius e^x , so its area is $A(x)=\pi\left(e^x\right)^2$.

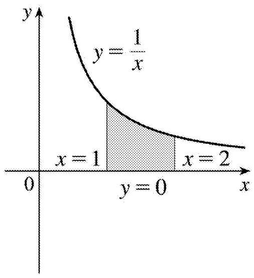
$$V=\int_0^1 A(x)\,dx=\int_0^1 \pi\left(e^x\right)^2 dx=\pi\int_0^1 e^{2x} dx=\frac{1}{2}\pi\left[e^{2x}\right]_0^1=\frac{\pi}{2}\left(e^2-1\right)$$

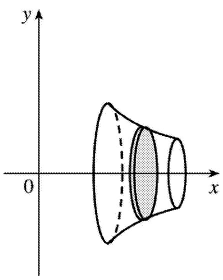




3. A cross–section is a disk with radius $1/x$, so its area is $A(x)=\pi(1/x)^2$.

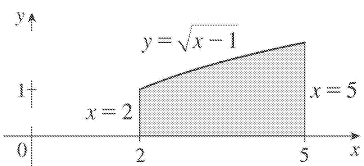
$$V=\int_1^2 A(x)\,dx=\int_1^2 \pi\left(\frac{1}{x}\right)^2 dx=\pi\int_1^2 \frac{1}{x^2}\,dx=\pi\left[-\frac{1}{x}\right]_1^2=\pi\left[-\frac{1}{2}-(-1)\right]=\frac{\pi}{2}$$

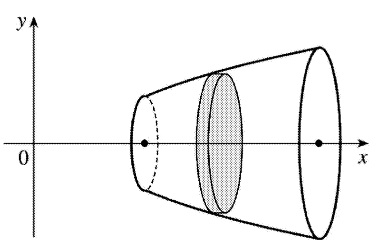




4. A cross–section is circular with radius $\sqrt{x-1}$, so its area is $A(x)=\pi\left(\sqrt{x-1}\right)^2=\pi(x-1)$.

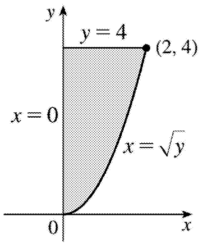
$$V=\int_2^5 A(x)\,dx=\int_2^5 \pi(x-1)\,dx=\pi\left[\frac{1}{2}x^2-x\right]_2^5=\pi\left(\frac{25}{2}-5-\frac{4}{2}+2\right)=\frac{15}{2}\pi$$

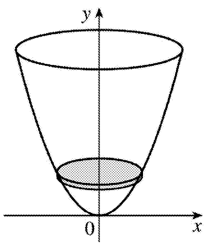




5. A cross–section is a disk with radius $\sqrt{y}$, so its area is $A(y)=\pi\left(\sqrt{y}\right)^2$.

$$V=\int_0^4 A(y)\,dy=\int_0^4 \pi\left(\sqrt{y}\right)^2 dy=\pi\int_0^4 y\,dy=\pi\left[\frac{1}{2}y^2\right]_0^4=8\pi$$

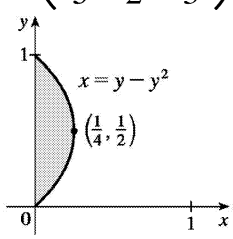


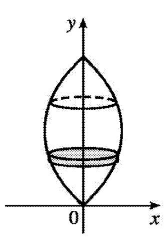


6. A cross–section is a disk with radius $y-y^2$, so its area is $A(y)=\pi\left(y-y^2\right)^2$.

$$V=\int_0^1 A(y)\,dy=\int_0^1 \pi\left(y-y^2\right)^2 dy=\pi\int_0^1\left(y^4-2y^3+y^2\right)dy=\pi\left[\frac{1}{5}y^5-\frac{1}{2}y^4+\frac{1}{3}y^3\right]_0^1$$

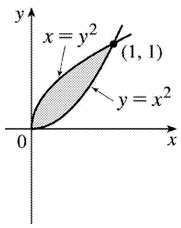
$$\pi\left(\frac{1}{5}-\frac{1}{2}+\frac{1}{3}\right)=\frac{\pi}{30}$$

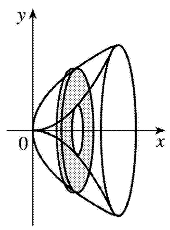




7. A cross–section is a washer (annulus) with inner radius x^2 and outer radius $\sqrt{x}$, so its area is $A(x)=\pi\left(\sqrt{x}\right)^2-\pi\left(x^2\right)^2=\pi\left(x-x^4\right)$.

$$V=\int_0^1 A(x)\,dx=\pi\int_0^1 \left(x-x^4\right)dx=\pi\left[\frac{1}{2}x^2-\frac{1}{5}x^5\right]_0^1=\pi\left(\frac{1}{2}-\frac{1}{5}\right)=\frac{3\pi}{10}$$

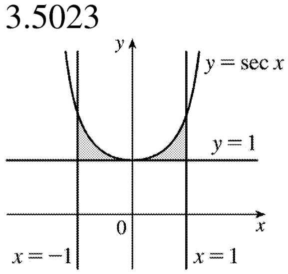


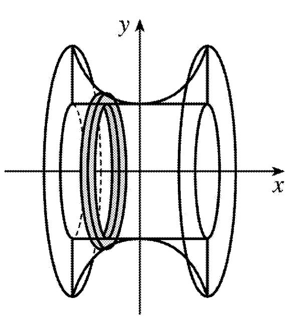


8. A cross–section is a washer with inner radius 1 and outer radius $\sec x$, so its area is
$A(x)=\pi(\sec x)^2-\pi(1)^2=\pi\left(\sec^2 x-1\right)$.

$$V=\int_{-1}^{1} A(x)\,dx=\int_{-1}^{1}\pi\left(\sec^2 x-1\right)dx=2\pi\int_0^1\left(\sec^2 x-1\right)dx=2\pi\left[\tan x-x\right]_0^1=2\pi(\tan 1-1)\approx$$

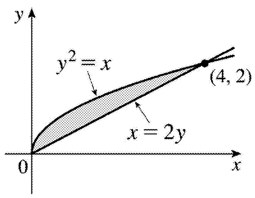
3.5023

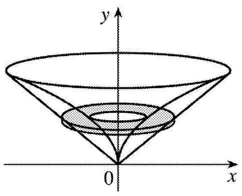




9. A cross–section is a washer with inner radius y^2 and outer radius $2y$, so its area is
$A(y)=\pi(2y)^2-\pi\left(y^2\right)^2=\pi\left(4y^2-y^4\right)$.

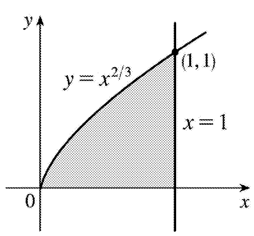
$$V=\int_0^2 A(y)\,dy=\pi\int_0^2\left(4y^2-y^4\right)dy=\pi\left[\frac{4}{3}y^3-\frac{1}{5}y^5\right]_0^2=\pi\left(\frac{32}{3}-\frac{32}{5}\right)=\frac{64\pi}{15}$$

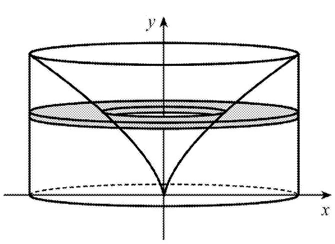




10. $y=x^{2/3}\Leftrightarrow x=y^{3/2}$, so a cross–section is a washer with inner radius $y^{3/2}$ and outer radius 1 , and its area is $A(y)=\pi(1)^2-\pi\left(y^{3/2}\right)^2=\pi\left(1-y^3\right)$.

$$V=\int_0^1 A(y)\,dy=\pi\int_0^1\left(1-y^3\right)dy=\pi\left[y-\frac{1}{4}y^4\right]_0^1=\frac{3}{4}\pi$$

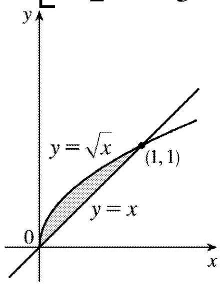


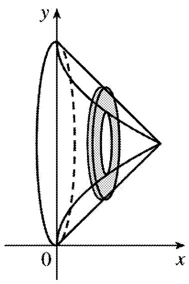


11. A cross–section is a washer with inner radius $1-\sqrt{x}$ and outer radius $1-x$, so its area is $A(x)=\pi(1-x)^2-\pi\left(1-\sqrt{x}\right)^2=\pi\left[\left(1-2x+x^2\right)-\left(1-2\sqrt{x}+x\right)\right]=\pi\left(-3x+x^2+2\sqrt{x}\right)$.

$$V=\int_0^1 A(x)\,dx=\pi\int_0^1\left(-3x+x^2+2\sqrt{x}\right)dx=$$

$$\pi\left[-\frac{3}{2}x^2+\frac{1}{3}x^3+\frac{4}{3}x^{3/2}\right]_0^1=\pi\left(-\frac{3}{2}+\frac{5}{3}\right)=\frac{\pi}{6}$$

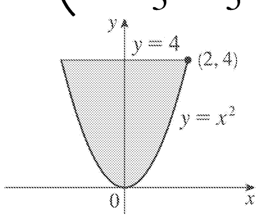


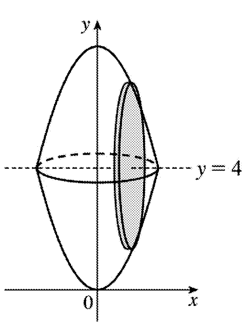


12. A cross–section is circular with radius $4-x^2$, so its area is $A(x)=\pi\left(4-x^2\right)^2=\pi\left(16-8x^2+x^4\right)$.

$$V=\int_{-2}^{2}A(x)dx=2\int_0^2A(x)dx=2\pi\int_0^2\left(16-8x^2+x^4\right)dx=2\pi\left[16x-\frac{8}{3}x^3+\frac{1}{5}x^5\right]_0^2=$$

$$2\pi\left(32-\frac{64}{3}+\frac{32}{5}\right)=64\pi\left(1-\frac{2}{3}+\frac{1}{5}\right)=64\pi\cdot\frac{8}{15}=\frac{512\pi}{15}$$



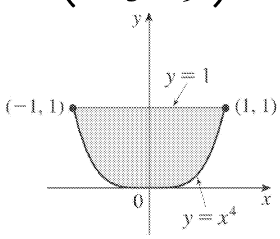


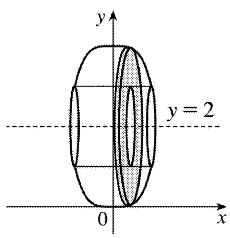
13. A cross–section is an annulus with inner radius 2–1 and outer radius $2-x^4$, so its area is

$$A(x)=\pi\left(2-x^4\right)^2-\pi\,(2-1)^2=\pi\left(3-4x^4+x^8\right).$$

$$V=\int_{-1}^{1}A(x)\,dx=2\int_0^1 A(x)\,dx=2\pi\int_0^1\left(3-4x^4+x^8\right)dx=2\pi\left[3x-\frac{4}{5}x^5+\frac{1}{9}x^9\right]_0^1=$$

$$2\pi\left(3-\frac{4}{5}+\frac{1}{9}\right)=\frac{208}{45}\pi$$



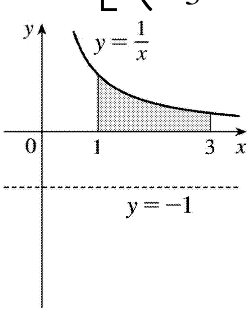


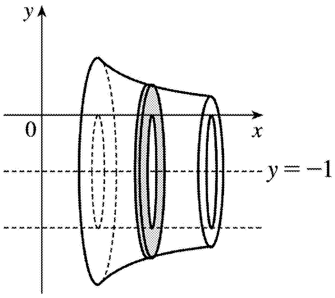
14.

$$V=\int_1^3\pi\left\{\left[\frac{1}{x}-(-1)\right]^2-[0-(-1)]^2\right\}dx=\pi\int_1^3\left[\left(\frac{1}{x}+1\right)^2-1^2\right]dx$$

$$=\pi\int_1^3\left(\frac{1}{x^2}+\frac{2}{x}\right)dx=\pi\left[-\frac{1}{x}+2\ln x\right]_1^3$$

$$=\pi\left[\left(-\frac{1}{3}+2\ln 3\right)-(-1+0)\right]=\pi\left(2\ln 3+\frac{2}{3}\right)=2\pi\left(\ln 3+\frac{1}{3}\right)$$

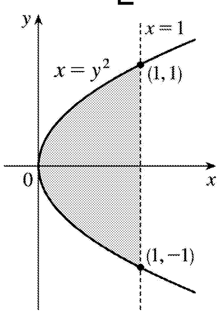


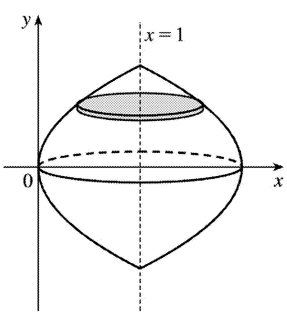


15.

$$V=\int_{-1}^{1}\pi\left(1-y^2\right)^2dy=2\int_{0}^{1}\pi\left(1-y^2\right)^2dy=2\pi\int_{0}^{1}\left(1-2y^2+y^4\right)dy$$

$$=2\pi\left[y-\frac{2}{3}y^3+\frac{1}{5}y^5\right]_0^1=2\pi\cdot\frac{8}{15}=\frac{16}{15}\pi$$

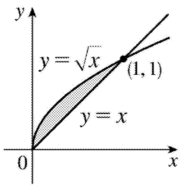


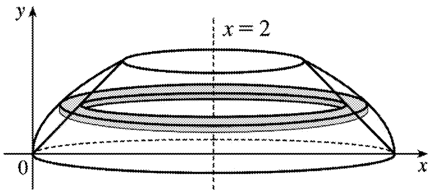


16. $y=\sqrt{x}\Rightarrow x=y^2$, so the outer radius is $2-y^2$.

$$V=\int_{0}^{1}\pi\left[\left(2-y^2\right)^2-(2-y)^2\right]dy=\pi\int_{0}^{1}\left[\left(4-4y^2+y^4\right)-\left(4-4y+y^2\right)\right]dy$$

$$=\pi\int_{0}^{1}\left(y^4-5y^2+4y\right)dy=\pi\left[\frac{1}{5}y^5-\frac{5}{3}y^3+2y^2\right]_0^1=\pi\left(\frac{1}{5}-\frac{5}{3}+2\right)=\frac{8}{15}\pi$$



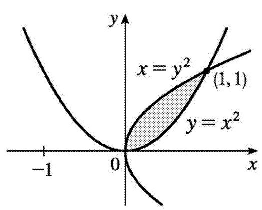


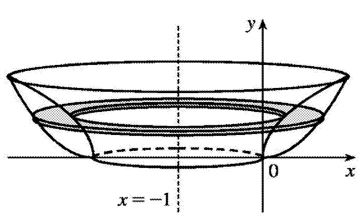
17. $y=x^2 \Rightarrow x=\sqrt{y}$ for $x \geq 0$. The outer radius is the distance from $x=-1$ to $x=\sqrt{y}$ and the inner radius is the distance from $x=-1$ to $x=y^2$.

$$V=\int_0^1 \pi \left\{ \left[\sqrt{y}-(-1)\right]^2-\left[y^2-(-1)\right]^2 \right\} dy=\pi \int_0^1 \left[\left(\sqrt{y}+1\right)^2-\left(y^2+1\right)^2 \right] dy$$

$$=\pi \int_0^1 \left(y+2\sqrt{y}+1-y^4-2y^2-1\right) dy=\pi \int_0^1 \left(y+2\sqrt{y}-y^4-2y^2\right) dy$$

$$=\pi \left[\frac{1}{2}y^2+\frac{4}{3}y^{3/2}-\frac{1}{5}y^5-\frac{2}{3}y^3 \right]_0^1=\pi \left(\frac{1}{2}+\frac{4}{3}-\frac{1}{5}-\frac{2}{3} \right)=\frac{29}{30}\pi$$



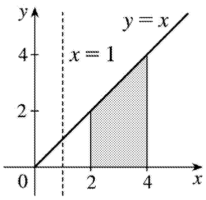


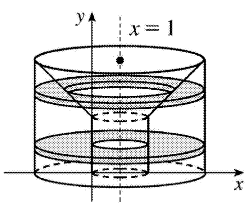
18. For $0 \leq y < 2$, a cross–section is an annulus with inner radius $2-1$ and outer radius $4-1$, the area of which is $A_1(y)=\pi\,(4-1)^2-\pi\,(2-1)^2$. For $2 \leq y \leq 4$, a cross–section is an annulus with inner radius $y-1$ and outer radius $4-1$, the area of which is $A_2(y)=\pi\,(4-1)^2-\pi\,(y-1)^2$.

$$V = \int_0^4 A(y)\,dy = \pi\int_0^2 \left[(4-1)^2-(2-1)^2\right]dy + \pi\int_2^4 \left[(4-1)^2-(y-1)^2\right]dy$$

$$= \pi\,[8y]_0^2 + \pi\int_2^4 \left(8+2y-y^2\right)dy = 16\pi + \pi\left[8y+y^2-\frac{1}{3}y^3\right]_2^4$$

$$= 16\pi + \pi\left[\left(32+16-\frac{64}{3}\right)-\left(16+4-\frac{8}{3}\right)\right] = \frac{76}{3}\pi$$





19. R_1 about OA (the line $y=0$): $V = \int_0^1 A(x)\,dx = \int_0^1 \pi (x^3)^2\,dx = \pi\int_0^1 x^6\,dx = \pi\left[\frac{1}{7}x^7\right]_0^1 = \frac{\pi}{7}$

20. R_1 about OC (the line $x=0$):

$$V = \int_0^1 A(y)\,dy = \int_0^1 \left[\pi(1)^2 - \pi\left(\sqrt[3]{y}\right)^2\right]dy = \pi\int_0^1 (1-y^{2/3})\,dy = \pi\left[y-\frac{3}{5}y^{5/3}\right]_0^1 = \pi\left(1-\frac{3}{5}\right) = \frac{2\pi}{5}$$

21. R_1 about AB (the line $x=1$):

$$V = \int_0^1 A(y)\,dy = \int_0^1 \pi\left(1-\sqrt[3]{y}\right)^2 dy = \pi\int_0^1 (1-2y^{1/3}+y^{2/3})\,dy$$

$$= \pi\left[y-\frac{3}{2}y^{4/3}+\frac{3}{5}y^{5/3}\right]_0^1 = \pi\left(1-\frac{3}{2}+\frac{3}{5}\right) = \frac{\pi}{10}$$

22. R_1 about BC (the line $y=1$):

$$V = \int_0^1 A(x)\,dx = \int_0^1 \left[\pi(1)^2 - \pi(1-x^3)^2\right]dx = \pi\int_0^1 \left[1-(1-2x^3+x^6)\right]dx$$

$$=\pi\int_0^1 (2x^3-x^6)\,dx=\pi\left[\frac{1}{2}x^4-\frac{1}{7}x^7\right]_0^1=\pi\left(\frac{1}{2}-\frac{1}{7}\right)=\frac{5\pi}{14}$$

23. R_2 about OA (the line $y=0$):

$$V=\int_0^1 A(x)\,dx=\int_0^1\left[\pi(1)^2-\pi\left(\sqrt{x}\right)^2\right]dx=\pi\int_0^1(1-x)\,dx=\pi\left[x-\frac{1}{2}x^2\right]_0^1=\pi\left(1-\frac{1}{2}\right)=\frac{\pi}{2}$$

24. R_2 about OC (the line $x=0$): $V=\int_0^1 A(y)\,dy=\int_0^1\pi(y^2)^2\,dy=\pi\int_0^1 y^4\,dy=\pi\left[\frac{1}{5}y^5\right]_0^1=\frac{\pi}{5}$

25. R_2 about AB (the line $x=1$):

$$V=\int_0^1 A(y)\,dy=\int_0^1\left[\pi(1)^2-\pi(1-y^2)^2\right]dy=\pi\int_0^1\left[1-(1-2y^2+y^4)\right]dy$$

$$=\pi\int_0^1(2y^2-y^4)\,dy=\pi\left[\frac{2}{3}y^3-\frac{1}{5}y^5\right]_0^1=\pi\left(\frac{2}{3}-\frac{1}{5}\right)=\frac{7\pi}{15}$$

26. R_2 about BC (the line $y=1$):

$$V=\int_0^1 A(x)\,dx=\int_0^1\pi\left(1-\sqrt{x}\right)^2dx=\pi\int_0^1(1-2x^{1/2}+x)\,dx$$

$$=\pi\left[x-\frac{4}{3}x^{3/2}+\frac{1}{2}x^2\right]_0^1=\pi\left(1-\frac{4}{3}+\frac{1}{2}\right)=\frac{\pi}{6}$$

27. R_3 about OA (the line $y=0$):

$$V=\int_0^1 A(x)\,dx=\int_0^1\left[\pi\left(\sqrt{x}\right)^2-\pi(x^3)^2\right]dx=\pi\int_0^1(x-x^6)\,dx=\pi\left[\frac{1}{2}x^2-\frac{1}{7}x^7\right]_0^1=\pi\left(\frac{1}{2}-\frac{1}{7}\right)=\frac{5\pi}{14}\ .$$

Note: Let $= {}_1+$ mathpi5_2 $+R_3$. If we rotate about any of the segments OA , OC , AB , or BC , we obtain a right circular cylinder of height 1 and radius 1 . Its volume is $\pi r^2 h=\pi(1)^2\cdot 1=\pi$. As a check for Exercises 19, 23, and 27, we can add the answers, and that sum must equal π . Thus,

$$\frac{\pi}{7}+\frac{\pi}{2}+\frac{5\pi}{14}=\left(\frac{2+7+5}{14}\right)\pi=\pi\ .$$

28.

R_3 about OC (the line $x=0$):

$$V=\int_0^1 A(y)\,dy=\int_0^1 \left[\pi\left(\sqrt[3]{y}\right)^2-\pi\left(y^2\right)^2\right]dy=\pi\int_0^1 \left(y^{2/3}-y^4\right)dy$$

$$=\pi\left[\frac{3}{5}y^{5/3}-\frac{1}{5}y^5\right]_0^1=\pi\left(\frac{3}{5}-\frac{1}{5}\right)=\frac{2\pi}{5}$$

Note: See the note in Exercise 27. For Exercises 20, 24, and 28, we have $\frac{2\pi}{5}+\frac{\pi}{5}+\frac{2\pi}{5}=\pi$.

29. R_3 about AB (the line $x=1$):

$$V=\int_0^1 A(y)\,dy=\int_0^1 \left[\pi\left(1-y^2\right)^2-\pi\left(1-\sqrt[3]{y}\right)^2\right]dy=\pi\int_0^1 \left[\left(1-2y^2+y^4\right)-\left(1-2y^{1/3}+y^{2/3}\right)\right]dy$$

$$=\pi\int_0^1 \left(-2y^2+y^4+2y^{1/3}-y^{2/3}\right)dy=\pi\left[-\frac{2}{3}y^3+\frac{1}{5}y^5+\frac{3}{2}y^{4/3}-\frac{3}{5}y^{5/3}\right]_0^1$$

$$=\pi\left(-\frac{2}{3}+\frac{1}{5}+\frac{3}{2}-\frac{3}{5}\right)=\frac{13\pi}{30}$$

Note: See the note in Exercise 27. For Exercises 21, 25, and 29, we have
$\frac{\pi}{10}+\frac{7\pi}{15}+\frac{13\pi}{30}=\left(\frac{3+14+13}{30}\right)\pi=\pi$.

30. R_3 about BC (the line $y=1$):

$$V=\int_0^1 A(x)\,dx=\int_0^1 \left[\pi\left(1-x^3\right)^2-\pi\left(1-\sqrt{x}\right)^2\right]dx$$

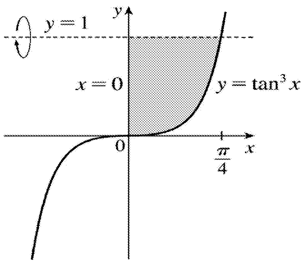
$$=\pi\int_0^1 \left[\left(1-2x^3+x^6\right)-\left(1-2x^{1/2}+x\right)\right]dx=\pi\int_0^1 \left(-2x^3+x^6+2x^{1/2}-x\right)dx$$

$$=\pi\left[-\frac{1}{2}x^4+\frac{1}{7}x^7+\frac{4}{3}x^{3/2}-\frac{1}{2}x^2\right]_0^1=\pi\left(-\frac{1}{2}+\frac{1}{7}+\frac{4}{3}-\frac{1}{2}\right)=\frac{10\pi}{21}$$

Note: See the note in Exercise 27. For Exercises 22, 26, and 30, we have
$\frac{5\pi}{14}+\frac{\pi}{6}+\frac{10\pi}{21}=\left(\frac{15+7+20}{42}\right)\pi=\pi$.

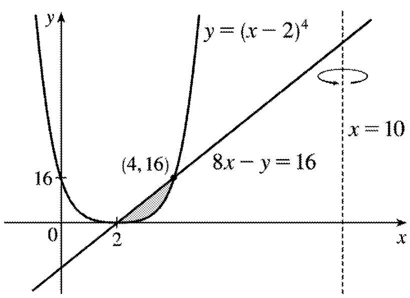
31.

$$V=\pi\int_0^{\pi/4}(1-\tan^3 x)^2\,dx$$



32. $y=(x-2)^4$ and $8x-y=16$ intersect when $(x-2)^4=8x-16=8(x-2)\Leftrightarrow(x-2)^4-8(x-2)=0\Leftrightarrow$ $(x-2)\left[(x-2)^3-8\right]=0\Leftrightarrow x-2=0$ or $x-2=2\Leftrightarrow x=2$ or 4 . $y=(x-2)^4\Rightarrow x-2=\pm\sqrt[4]{y}\Rightarrow x=2+\sqrt[4]{y}$ [since $x\geq 2$]. $8x-y=16\Rightarrow 8x=y+16\Rightarrow x=\frac{1}{8}y+2$.

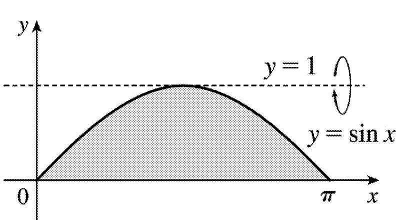
$$V=\pi\int_0^{16}\left\{\left[10-\left(\frac{1}{8}y+2\right)\right]^2-\left[10-\left(2+\sqrt[4]{y}\right)\right]^2\right\}dy$$



33.

$$V=\pi\int_0^{\pi}\left[(1-0)^2-(1-\sin x)^2\right]dx$$

$$=\pi\int_0^{\pi}\left[1^2-(1-\sin x)^2\right]dx$$



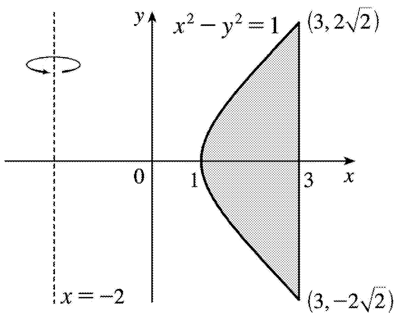
34.

$$V=\pi\int_0^{\pi}\left[(\sin x+2)^2-2^2\right]dx$$

35.

$$V=\pi\int_{-\sqrt{8}}^{\sqrt{8}}\left\{[3-(-2)]^2-\left[\sqrt{y^2+1}-(-2)\right]^2\right\}dy$$

$$=\pi\int_{-2\sqrt{2}}^{2\sqrt{2}}\left[5^2-\left(\sqrt{1+y^2}+2\right)^2\right]dy$$

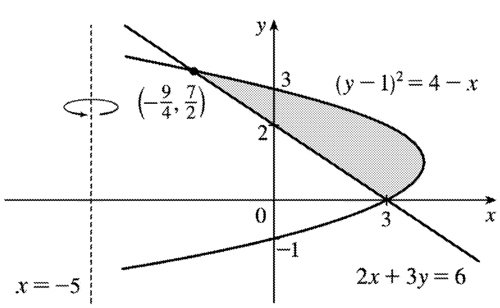


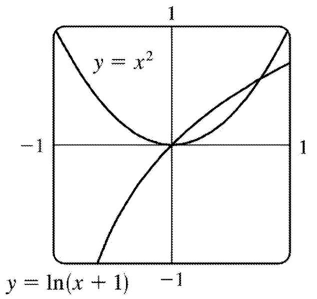
36. Solve the equations for x : $(y-1)^2=4-x\Leftrightarrow x=4-(y-1)^2$ and $2x+3y=6\Leftrightarrow x=3-\frac{3}{2}y$.

The points of intersection of the two curves are $(3,0)$ and $\left(-\frac{9}{4},\frac{7}{2}\right)$. Therefore,

$$V=\pi\int_0^{7/2}\left\{\left[4-(y-1)^2-(-5)\right]^2-\left[3-\frac{3}{2}y-(-5)\right]^2\right\}dy$$

$$=\pi\int_0^{7/2}\left\{\left[9-(y-1)^2\right]^2-\left(8-\frac{3}{2}y\right)^2\right\}dy$$

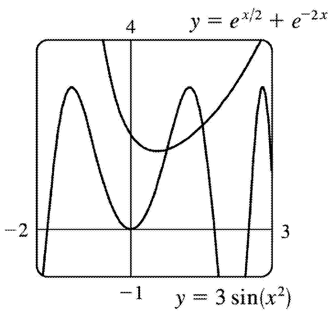




37. $y=x^2$ and $y=\ln(x+1)$ intersect at $x=0$ and at $x=a\approx 0.747$.

$$V=\pi\int_0^a \left\{ [\ln(x+1)]^2-\left(x^2\right)^2 \right\} dx\approx 0.132$$

38.



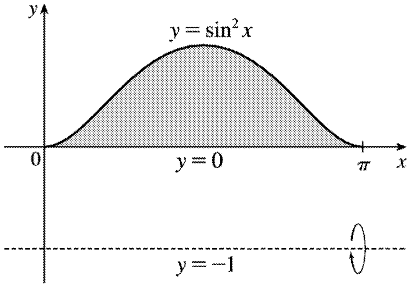
$y=3\sin(x^2)$ and $y=e^{x/2}+e^{-2x}$ intersect at $x=a\approx 0.772$ and at $x=b\approx 1.524$.

$$V=\pi\int_a^b \left\{ \left[3\sin(x^2)\right]^2-\left(e^{x/2}+e^{-2x}\right)^2 \right\} dx \approx 7.519$$

39.

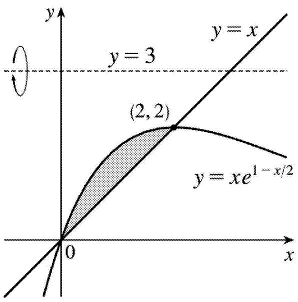
$$V=\pi\int_0^{\pi} \left\{ \left[\sin^2 x-(-1)\right]^2-[0-(-1)]^2 \right\} dx$$

$$=\frac{11}{8}\pi^2$$



40. V

$$=\pi\int_0^2\left[(3-x)^2-(3-xe^{1-x/2})^2\right]dx \overset{\text{CAS}}{=} \pi\left(-2e^2+24e-\frac{142}{3}\right)$$



41. $\pi\int_0^{\pi/2}\cos^2 x\,dx$ describes the volume of the solid obtained by rotating the region $R=\left\{(x,y)\,|\,0\le x\le\frac{\pi}{2}, 0\le y\le\cos x\right\}$ of the xy –plane about the x –axis.

42. $\pi\int_2^5 y\,dy=\pi\int_2^5(\sqrt{y})^2dy$ describes the volume of the solid obtained by rotating the region $R=\{(x,y)\,|\,2\le y\le 5, 0\le x\le\sqrt{y}\}$ of the xy –plane about the y –axis.

43. $\pi\int_0^1\left(y^4-y^8\right)dy=\pi\int_0^1\left[\left(y^2\right)^2-\left(y^4\right)^2\right]dy$ describes the volume of the solid obtained by rotating the region $R=\left\{(x,y)\,|\,0\le y\le 1, y^4\le x\le y^2\right\}$ of the xy –plane about the y –axis.

44. $\pi\int_0^{\pi/2}\left[(1+\cos x)^2-1^2\right]dx$ describes the volume of the solid obtained by rotating the region $=\left\{(x,y)\,|\,0\le x\le\frac{\pi}{2}, 1\le y\le 1+\cos x\right\}$ of the xy –plane about the x –axis.

Or: The solid could be obtained by rotating the region $'=\left\{(x,y)\,|\,0\le x\le\frac{\pi}{2}, 0\le y\square x\right\}$ about the line $y=-1$.

45. There are 10 subintervals over the 15–cm length, so we'll use $n=10/2=5$ for the Midpoint Rule.

$$V=\int_0^{15}A(x)\,dx\approx M_5=\frac{15-0}{5}[A(1.5)+A(4.5)+A(7.5)+A(10.5)+A(13.5)]$$

$$=3(18+79+106+128+39)=3\cdot 370=1110\text{ cm}^3$$

46.

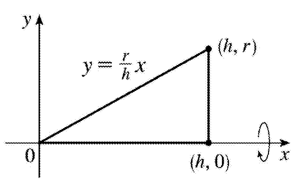
$$V=\int_0^{10} A(x)\,dx\approx M_5=\frac{10-0}{5}\,[A(1)+A(3)+A(5)+A(7)+A(9)]$$

$$=2(0.65+0.61+0.59+0.55+0.50)=2(2.90)=5.80\text{m}^3$$

47. We'll form a right circular cone with height h and base radius r by revolving the line $y=\frac{r}{h}x$ about the x –axis.

$$V=\pi\int_0^h \left(\frac{r}{h}x\right)^2 dx=\pi\int_0^h \frac{r^2}{h^2}x^2\,dx=\pi\frac{r^2}{h^2}\left[\frac{1}{3}x^3\right]_0^h$$

$$=\pi\frac{r^2}{h^2}\left(\frac{1}{3}h^3\right)=\frac{1}{3}\pi r^2 h$$



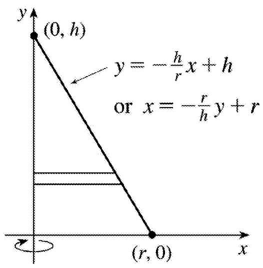
Another solution: Revolve $x=-\frac{r}{h}y+r$ about the y –axis.

$$V=\pi\int_0^h \left(-\frac{r}{h}y+r\right)^2 dy=\pi\int_0^h \left[\frac{r^2}{h^2}y^2-\frac{2r^2}{h}y+r^2\right]dy$$

$$=\pi\left[\frac{r^2}{3h^2}y^3-\frac{r^2}{h}y^2+r^2y\right]_0^h=\pi\left(\frac{1}{3}r^2h-r^2h+r^2h\right)=\frac{1}{3}\pi r^2 h$$

* Or use substitution with $u=r-\frac{r}{h}y$ and $du=-\frac{r}{h}dy$ to get

$$\pi\int_r^0 u^2\left(-\frac{h}{r}\right)du=-\pi\frac{h}{r}\left[\frac{1}{3}u^3\right]_r^0=-\pi\frac{h}{r}\left(-\frac{1}{3}r^3\right)=\frac{1}{3}\pi r^2 h\ .$$



48.

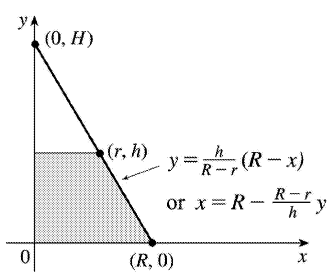
$$V = \pi \int_0^h \left(R - \frac{R-r}{h} y \right)^2 dy$$

$$= \pi \int_0^h \left[R^2 - \frac{2R(R-r)}{h} y + \left(\frac{R-r}{h} \right)^2 y^2 \right] dy$$

$$= \pi \left[R^2 y - \frac{R(R-r)}{h} y^2 + \frac{1}{3} \left(\frac{R-r}{h} \right)^2 y^3 \right]_0^h$$

$$= \pi \left[R^2 h - R(R-r)h + \frac{1}{3} (R-r)^2 h \right] \square$$

$$= \frac{1}{3} \pi h \left[3Rr + (R^2 - 2Rr + r^2) \right] = \frac{1}{3} \pi h (R^2 + Rr + r^2)$$



Another solution: $\frac{H}{R} = \frac{H-h}{r}$ by similar triangles. Therefore,

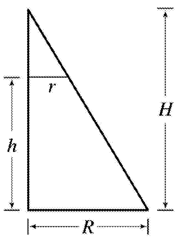
$Hr = HR - hR \Rightarrow hR = H(R-r) \Rightarrow H = \frac{hR}{R-r}$. Now

$$V = \frac{1}{3} \pi R^2 H - \frac{1}{3} \pi r^2 (H-h) \text{ [by Exercise 47]}$$

$$= \frac{1}{3} \pi R^2 \frac{hR}{R-r} - \frac{1}{3} \pi r^2 \frac{rh}{R-r} \quad \left[H - h = \frac{rH}{R} = \frac{rhR}{R(R-r)} \right]$$

$$= \frac{1}{3} \pi h \frac{R^3 - r^3}{R-r} = \frac{1}{3} \pi h \left(R^2 + Rr + r^2 \right)$$

$$=\frac{1}{3}\left[\pi R^2+\pi r^2+\sqrt{\left(\pi R^2\right)\left(\pi r^2\right)}\right]h=\frac{1}{3}\left(A_1+A_2+\sqrt{A_1A_2}\right)h$$



where A_1 and A_2 are the areas of the bases of the frustum. (See Exercise 50 for a related result.)

49. $x^2+y^2=r^2 \Leftrightarrow x^2=r^2-y^2$

$$V=\pi\int_{r-h}^{r}\left(r^2-y^2\right)dy=\pi\left[r^2y-\frac{y^3}{3}\right]_{r-h}^{r}$$

$$=\pi\left\{\left[r^3-\frac{r^3}{3}\right]-\left[r^2(r-h)-\frac{(r-h)^3}{3}\right]\right\}$$

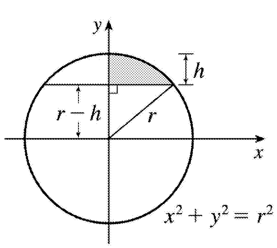
$$=\pi\left\{\frac{2}{3}r^3-\frac{1}{3}(r-h)\left[3r^2-(r-h)^2\right]\right\}$$

$$=\frac{1}{3}\pi\left\{2r^3-(r-h)\left[3r^2-\left(r^2-2rh+h^2\right)\right]\right\}$$

$$=\frac{1}{3}\pi\left\{2r^3-(r-h)\left[2r^2+2rh-h^2\right]\right\}$$

$$=\frac{1}{3}\pi\left(2r^3-2r^3-2r^2h+rh^2+2r^2h+2rh^2-h^3\right)$$

$$=\frac{1}{3}\pi\left(3rh^2-h^3\right)=\frac{1}{3}\pi h^2(3r-h)\text{ , or, equivalently, }\pi h^2\left(r-\frac{h}{3}\right)$$



50. An equation of the line is

$$x=\frac{\Delta x}{\Delta y}\ y+(x\ \text{–intercept})=\frac{a/2-b/2}{h-0}\ y+\frac{b}{2}=\frac{a-b}{2h}\ y+\frac{b}{2}\ .$$

$$V=\int_0^h A(y)\,dy=\int_0^h (2x)^2\,dy$$

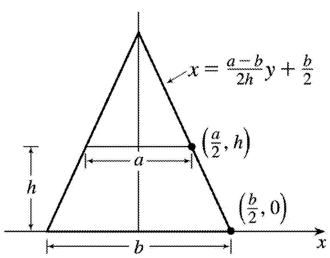
$$=\int_0^h\left[2\left(\frac{a-b}{2h}\ y+\frac{b}{2}\right)\right]^2 dy=\int_0^h\left[\frac{a-b}{h}\ y+b\right]^2 dy$$

$$=\int_0^h\left[\frac{(a-b)^2}{h^2}\ y^2+\frac{2b(a-b)}{h}\ y+b^2\right]dy$$

$$=\left[\frac{(a-b)^2}{3h^2}\ y^3+\frac{b(a-b)}{h}\ y^2+b^2 y\right]_0^h$$

$$=\frac{1}{3}(a-b)^2h+b(a-b)h+b^2h=\frac{1}{3}\left(a^2-2ab+b^2+3ab\right)h$$

$$=\frac{1}{3}\left(a^2+ab+b^2\right)h$$



Note that this can be written as $\frac{1}{3}\left(A_1+A_2+\sqrt{A_1A_2}\right)h$, as in Exercise 48.

If $a=b$, we get a rectangular solid with volume b^2h . If $a=0$, we get a square pyramid with volume $\frac{1}{3}\ b^2h$.

51. For a cross–section at height y , we see from similar triangles that $\frac{\alpha/2}{b/2}=\frac{h-y}{h}$, so $\alpha=b\left(1-\frac{y}{h}\right)$.

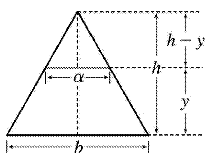
Similarly, for cross–sections having $2b$ as their base and β replacing α , $\beta=2b\left(1-\frac{y}{h}\right)$. So

$$V=\int_0^h A(y)\,dy=\int_0^h \left[b\left(1-\frac{y}{h}\right)\right]\left[2b\left(1-\frac{y}{h}\right)\right]dy$$

$$=\int_0^h 2b^2\left(1-\frac{y}{h}\right)^2 dy=2b^2\int_0^h\left(1-\frac{2y}{h}+\frac{y^2}{h^2}\right)dy$$

$$=2b^2\left[y-\frac{y^2}{h}+\frac{y^3}{3h^2}\right]_0^h=2b^2\left[h-h+\frac{1}{3}h\right]$$

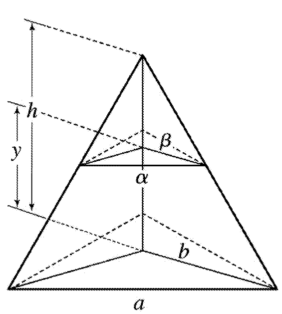
$=\frac{2}{3}b^2h$ [$=\frac{1}{3}Bh$ where B is the area of the base, as with any pyramid.]



52. Consider the triangle consisting of two vertices of the base and the center of the base. This triangle is similar to the corresponding triangle at a height y, so $a/b=\alpha/\beta \Rightarrow \alpha=a\beta/b$. Also by similar triangles, $b/h=\beta/(h-y)$
$\Rightarrow \beta=b(h-y)/h$. These two equations imply that $\alpha=a(1-y/h)$, and since the cross–section is an equilateral triangle, it has area $A(y)=\frac{1}{2}\cdot\alpha\cdot\frac{\sqrt{3}}{2}\alpha=\frac{a^2(1-y/h)^2}{4}\sqrt{3}$, so

$$V=\int_0^h A(y)\,dy=\frac{a^2\sqrt{3}}{4}\int_0^h\left(1-\frac{y}{h}\right)^2 dy$$

$$=\frac{a^2\sqrt{3}}{4}\left[-\frac{h}{3}\left(1-\frac{y}{h}\right)^3\right]_0^h=-\frac{\sqrt{3}}{12}a^2h(-1)=\frac{\sqrt{3}}{12}a^2h$$



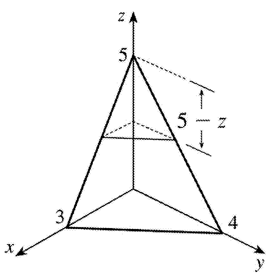
53. A cross–section at height z is a triangle similar to the base, so we'll multiply the legs of the base triangle, 3 and 4, by a proportionality factor of $(5-z)/5$. Thus, the triangle at height z has area

$A(z)=\frac{1}{2}\cdot 3\left(\frac{5-z}{5}\right)\cdot 4\left(\frac{5-z}{5}\right)=6\left(1-\frac{z}{5}\right)^2$, so

$$V = \int_0^5 A(z)\,dz=6\int_0^5 (1-z/5)^2 dz$$

$$= 6\int_1^0 u^2(-5\,du)\left[u=1-z/5,du=-\frac{1}{5}\,dz\right]$$

$$= -30\left[\frac{1}{3}u^3\right]_1^0=-30\left(-\frac{1}{3}\right)=10\text{cm}^3$$



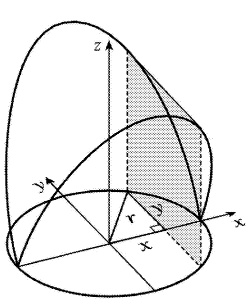
54. A cross–section is shaded in the diagram.

$A(x)=(2y)^2=\left(2\sqrt{r^2-x^2}\right)^2$, so

$$V = \int_{-r}^{r} A(x)\,dx=2\int_0^r 4\left(r^2-x^2\right)dx$$

$$= 8\left[r^2x-\frac{1}{3}x^3\right]_0^r$$

$$= 8\left(\frac{2}{3}r^3\right)=\frac{16}{3}r^3$$

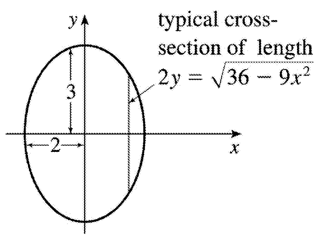


55. If l is a leg of the isosceles right triangle and $2y$ is the hypotenuse,
then $l^2+l^2=(2y)^2 \Rightarrow 2l^2=4y^2 \Rightarrow l^2=2y^2$.

$$V=\int_{-2}^{2} A(x)\,dx=2\int_{0}^{2} A(x)\,dx=2\int_{0}^{2} \frac{1}{2}\,(l)(l)\,dx=2\int_{0}^{2} y^2\,dx$$

$$=2\int_{0}^{2} \frac{1}{4}\left(36-9x^2\right)dx=\frac{9}{2}\int_{0}^{2}\left(4-x^2\right)dx$$

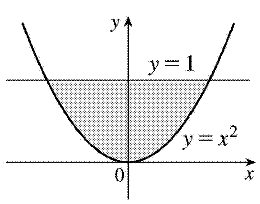
$$=\frac{9}{2}\left[4x-\frac{1}{3}x^3\right]_0^2=\frac{9}{2}\left(8-\frac{8}{3}\right)=24$$



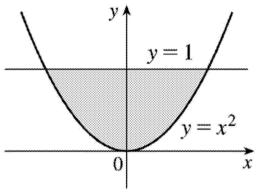
56. The cross–section of the base corresponding to the coordinate y has length $2x=2\sqrt{y}$. The corresponding equilateral triangle with side s has area

$$A(y)=s^2\left(\frac{\sqrt{3}}{4}\right)=(2x)^2\left(\frac{\sqrt{3}}{4}\right)=\left(2\sqrt{y}\right)^2\left(\frac{\sqrt{3}}{4}\right)=y\sqrt{3}$$. Therefore,

$$V=\int_{0}^{1} A(y)\,dy=\int_{0}^{1} y\sqrt{3}\,dy=\sqrt{3}\left[\frac{1}{2}y^2\right]_0^1=\frac{\sqrt{3}}{2}$$.



57. The cross–section of the base corresponding to the coordinate y has length $2x=2\sqrt{y}$. The square has area $A(y)=\left(2\sqrt{y}\right)^2=4y$, so $V=\int_{0}^{1} A(y)\,dy=\int_{0}^{1} 4y\,dy=\left[2y^2\right]_0^1=2$.

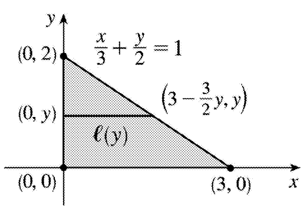


58. A typical cross–section perpendicular to the y –axis in the base has length $\ell(y)=3-\frac{3}{2}$ y. This length is the diameter of a cross–sectional semicircle in S , so

$$V=\int_0^2 A(y)\,dy=\int_0^2 \frac{\pi}{2}\left[\frac{\ell(y)}{2}\right]^2 dy=\frac{\pi}{8}\int_0^2\left(3-\frac{3}{2}y\right)^2 dy$$

$$=\frac{\pi}{8}\int_3^0 u^2\left(-\frac{2}{3}\right)du\left[u=3-\frac{3}{2}y,du=-\frac{3}{2}dy\right]$$

$$=-\frac{\pi}{12}\left[\frac{1}{3}u^3\right]_3^0=-\frac{\pi}{12}(-9)=\frac{3\pi}{4}$$



59. A typical cross–section perpendicular to the y –axis in the base has length $\ell(y)=3-\frac{3}{2}$ y. This length is the leg of an isosceles right triangle, so

$$A(y)=\frac{1}{2}[\ell(y)]^2\left[\frac{1}{2}bh \text{ with base = height}\right]$$

$$=\frac{1}{2}\left[3\left(1-\frac{1}{2}y\right)\right]^2=\frac{9}{2}\left(1-\frac{1}{2}y\right)^2$$

Thus,

$$V=\int_0^2 A(y)\,dy=\frac{9}{2}\int_1^0 u^2(-2du)\quad\left[u=1-\frac{1}{2}y,du=-\frac{1}{2}dy\right]$$

$$=-9\left[\frac{1}{3}u^3\right]_1^0=-9\left(-\frac{1}{3}\right)=3$$

60. **(a)** $V=\int_{-r}^{r} A(x)\,dx=2\int_{0}^{r} A(x)\,dx=2\int_{0}^{r} \frac{1}{2}\,h\left(2\sqrt{r^2-x^2}\right)dx=2h\int_{0}^{r}\sqrt{r^2-x^2}\,dx$

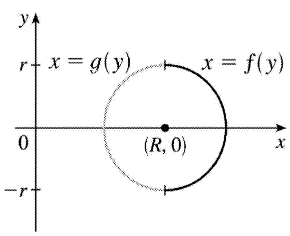
(b) Observe that the integral represents one quarter of the area of a circle of radius r , so
$V=2h\cdot\frac{1}{4}\pi r^2=\frac{1}{2}\pi h r^2$.

61. **(a)** The torus is obtained by rotating the circle $(x-R)^2+y^2=r^2$
about the $y-$ axis. Solving for x , we see that the right half of the
circle is given by $x=R+\sqrt{r^2-y^2}=f(y)$ and the left half by $x=R-\sqrt{r^2-y^2}=g(y)$. So

$$V=\pi\int_{-r}^{r}\left\{[f(y)]^2-[g(y)]^2\right\}dy$$

$$=2\pi\int_{0}^{r}\left[\left(R^2+2R\sqrt{r^2-y^2}+r^2-y^2\right)-\left(R^2-2R\sqrt{r^2-y^2}+r^2-y^2\right)\right]dy$$

$$=2\pi\int_{0}^{r}4R\sqrt{r^2-y^2}\,dy=8\pi R\int_{0}^{r}\sqrt{r^2-y^2}\,dy$$



(b) Observe that the integral represents a quarter of the area of a circle with radius r , so
$8\pi R\int_{0}^{r}\sqrt{r^2-y^2}\,dy=8\pi R\cdot\frac{1}{4}\pi r^2=2\pi^2 r^2 R$.

62. The cross–sections perpendicular to the $y-$ axis in Figure 17 are rectangles. The rectangle corresponding to the coordinate y has a base of length $2\sqrt{16-y^2}$ in the $xy-$ plane and a height of $\frac{1}{\sqrt{3}}\,y$, since $\angle BAC=30^\circ$ and $|BC|=\frac{1}{\sqrt{3}}\,|AB|$. Thus, $A(y)=\frac{2}{\sqrt{3}}\,y\sqrt{16-y^2}$ and

$$V=\int_{0}^{4}A(y)\,dy=\frac{2}{\sqrt{3}}\int_{0}^{4}\sqrt{16-y^2}\,y\,dy$$

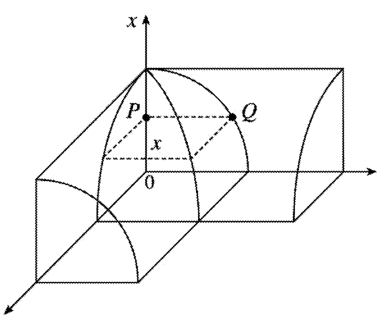
$$= \frac{2}{\sqrt{3}} \int_{16}^{0} u^{1/2}\left(-\frac{1}{2}\right) du \quad [\text{Put } u=16-y^2, \text{so } du=-2y\,dy]$$

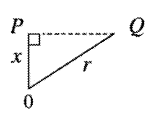
$$= \frac{1}{\sqrt{3}} \int_{0}^{16} u^{1/2} du = \frac{1}{\sqrt{3}} \; \frac{2}{3}\left[u^{3/2}\right]_0^{16} = \frac{2}{3\sqrt{3}}\;(64) = \frac{128}{3\sqrt{3}}$$

63. **(a)** Volume $\left(S_1\right)=\int_0^h A(z)\,dz=$ Volume $\left(S_2\right)$ since the cross–sectional area $A(z)$ at height z is the same for both solids.
(b) By Cavalieri's Principle, the volume of the cylinder in the figure is the same as that of a right circular cylinder with radius r and height h , that is, $\pi r^2 h$.
64. Each cross–section of the solid S in a plane perpendicular to the x– axis is a square (since the edges of the cut lie on the cylinders, which are perpendicular). One–quarter of this square and one –eighth of S are shown. The area of this quarter –square is $|PQ|^2=r^2-x^2$. Therefore, $A(x)=4\left(r^2-x^2\right)$ and the volume of S is

$$V = \int_{-r}^{r} A(x)\,dx=4\int_{-r}^{r}\left(r^2-x^2\right)dx$$

$$= 8\int_0^r\left(r^2-x^2\right)dx=8\left[r^2x-\frac{1}{3}x^3\right]_0^r=\frac{16}{3}r^3$$



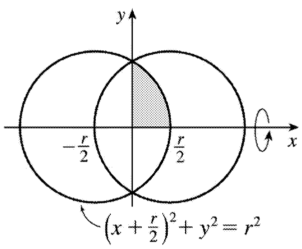


65. The volume is obtained by rotating the area common to two circles of radius r , as shown. The volume of the right half is

$$V_{\text{right}} = \pi\int_0^{r/2} y^2\,dx=\pi\int_0^{r/2}\left[r^2-\left(\frac{1}{2}r+x\right)^2\right]dx$$

$$= \pi \left[r^2 x - \frac{1}{3}\left(\frac{1}{2}r + x\right)^3 \right]_0^{r/2} = \pi \left[\left(\frac{1}{2}r^3 - \frac{1}{3}r^3\right) - \left(0 - \frac{1}{24}r^3\right) \right] = \frac{5}{24}\pi r^3$$

So by symmetry, the total volume is twice this, or $\frac{5}{12}\pi r^3$.



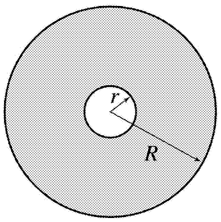
Another solution: We observe that the volume is the twice the volume of a cap of a sphere, so we can use the formula from Exercise 49 with $h = \frac{1}{2} r$:

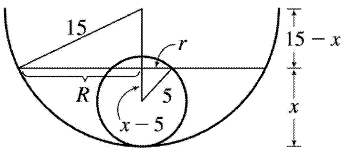
$$V = 2 \cdot \frac{1}{3}\pi h^2 (3r - h) = \frac{2}{3}\pi \left(\frac{1}{2}r\right)^2 \left(3r - \frac{1}{2}r\right) = \frac{5}{12}\pi r^3 .$$

66. We consider two cases: one in which the ball is not completely submerged and the other in which it is.

Case 1: $0 \le h \le 10$ The ball will not be completely submerged, and so a cross–section of the water parallel to the surface will be the shaded area shown in the first diagram. We can find the area of the cross–section at height x above the bottom of the bowl by using the Pythagorean Theorem: $R^2 = 15^2 - (15-x)^2$ and $r^2 = 5^2 - (x-5)^2$, so $A(x) = \pi\left(R^2 - r^2\right) = 20\pi x$. The volume of water when it has depth h is then $V(h) = \int_0^h A(x)\,dx = \int_0^h 20\pi x\,dx = \left[10\pi x^2\right]_0^h = 10\pi h^2 \text{ cm}^3$, $0 \le h \le 10$.

Case 2: $10 < h \le 15$ In this case we can find the volume by simply subtracting the volume displaced by the ball from the total volume inside the bowl underneath the surface of the water. The total volume underneath the
surface is just the volume of a cap of the bowl, so we use the formula from Exercise 49:
$V_{\text{cap}}(h) = \frac{1}{3}\pi h^2 (45 - h)$. The volume of
the small sphere is $V_{\text{ball}} = \frac{4}{3}\pi(5)^3 = \frac{500}{3}\pi$,
so the total volume is $V_{\text{cap}} - V_{\text{ball}} = \frac{1}{3}\pi\left(45h^2 - h^3 - 500\right) \text{ cm}^3$.

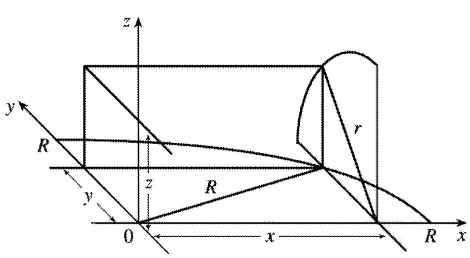




67. Take the *x*– axis to be the axis of the cylindrical hole of radius r . A quarter of the cross–section through y, perpendicular to the *y*– axis, is the rectangle shown. Using the Pythagorean Theorem twice, we see that the dimensions of this rectangle are $x=\sqrt{R^2-y^2}$ and $z=\sqrt{r^2-y^2}$, so $\frac{1}{4}A(y)=xz=\sqrt{r^2-y^2}\sqrt{R^2-y^2}$, and

$$V = \int_{-r}^{r} A(y)\,dy=\int_{-r}^{r} 4\sqrt{r^2-y^2}\sqrt{R^2-y^2}\,dy$$

$$= 8\int_{0}^{r}\sqrt{r^2-y^2}\sqrt{R^2-y^2}\,dy$$



68. The line $y=r$ intersects the semicircle $y=\sqrt{R^2-x^2}$ when $r=\sqrt{R^2-x^2} \Rightarrow r^2=R^2-x^2 \Rightarrow x^2=R^2-r^2 \Rightarrow x=\pm\sqrt{R^2-r^2}$. Rotating the shaded region about the x –axis gives us

$$V = \int_{-\sqrt{R^2-r^2}}^{\sqrt{R^2-r^2}} \pi\left[\left(\sqrt{R^2-x^2}\right)^2-r^2\right]dx$$

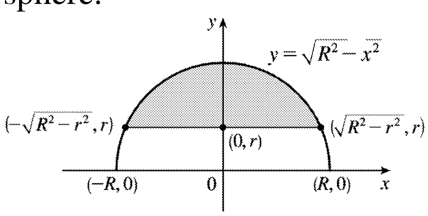
$$= 2\pi \int_0^{\sqrt{R^2-r^2}} \left(R^2-x^2-r^2\right) dx \text{ [by symmetry]}$$

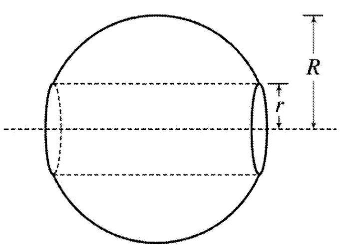
$$= 2\pi \int_0^{\sqrt{R^2-r^2}} \left[\left(R^2-r^2\right)-x^2\right] dx = 2\pi \left[\left(R^2-r^2\right)x-\frac{1}{3}x^3\right]_0^{\sqrt{R^2-r^2}}$$

$$= 2\pi \left[\left(R^2-r^2\right)^{3/2}-\frac{1}{3}\left(R^2-r^2\right)^{3/2}\right]$$

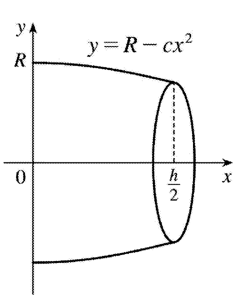
$$= 2\pi \cdot \frac{2}{3}\left(R^2-r^2\right)^{3/2} = \frac{4\pi}{3}\left(R^2-r^2\right)^{3/2}$$

Our answer makes sense in limiting cases. As $r \to 0$, $V \to \frac{4}{3}\pi R^3$, which is the volume of the full sphere. As $r \to R$, $V \to 0$, which makes sense because the hole's radius is approaching that of the sphere.





69. **(a)** The radius of the barrel is the same at each end by symmetry, since the function $y=R-cx^2$ is even. Since the barrel is obtained by rotating the graph of the function y about the x– axis, this radius is equal to the value of y at $x=\frac{1}{2}h$, which is $R-c\left(\frac{1}{2}h\right)^2=R-d=r$.



(b) The barrel is symmetric about the $y-$ axis, so its volume is twice the volume of that part of the barrel for $x>0$. Also, the barrel is a volume of rotation, so

$$V = 2\int_0^{h/2} \pi y^2 dx = 2\pi \int_0^{h/2} \left(R-cx^2\right)^2 dx = 2\pi \left[R^2 x - \frac{2}{3} Rcx^3 + \frac{1}{5} c^2 x^5 \right]_0^{h/2}$$

$$= 2\pi \left(\frac{1}{2} R^2 h - \frac{1}{12} Rch^3 + \frac{1}{160} c^2 h^5 \right)$$

Trying to make this look more like the expression we want, we rewrite it as

$V = \frac{1}{3} \pi h \left[2R^2 + \left(R^2 - \frac{1}{2} Rch^2 + \frac{3}{80} c^2 h^4 \right) \right]$. But

$R^2 - \frac{1}{2} Rch^2 + \frac{3}{80} c^2 h^4 = \left(R - \frac{1}{4} ch^2 \right)^2 - \frac{1}{40} c^2 h^4 = (R-d)^2 - \frac{2}{5} \left(\frac{1}{4} ch^2 \right)^2 = r^2 - \frac{2}{5} d^2$.

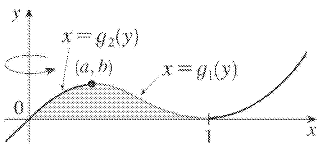
Substituting this back into V , we see that $V = \frac{1}{3} \pi h \left(2R^2 + r^2 - \frac{2}{5} d^2 \right)$, as required.

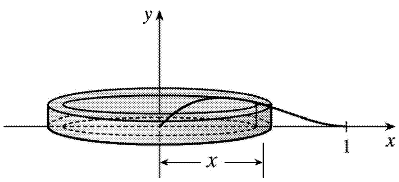
70. It suffices to consider the case where is bounded by the curves $y=f(x)$ and $y=g(x)$ for $a \le x \le b$, where $g(x) \le f(x)$ for all x in $[a,b]$, since other regions can be decomposed into subregions of this type. We are concerned with the volume obtained when is rotated about the line $y=-k$, which is equal to

$$V_2 = \pi \int_a^b \left([f(x)+k]^2 - [g(x)+k]^2 \right) dx = \pi \int_a^b \left([f(x)]^2 - [g(x)]^2 \right) dx + 2\pi k \int_a^b [f(x) - g(x)] dx$$

$$= V_1 + 2\pi kA$$

1.



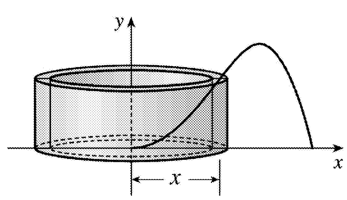


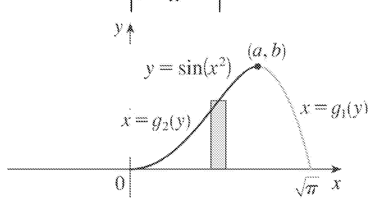
If we were to use the "washer'' method, we would first have to locate the local maximum point (a,b) *of* $y=x(x-1)^2$ using the methods of Chapter 4. Then we would have to solve the equation $y=x(x-1)^2$ for x in terms of y to obtain the functions $x=g_1(y)$ and $x=g_2(y)$ shown in the first figure. This step would be difficult because it involves the cubic formula. Finally we would find the volume using $V=\pi\int_0^b \left\{ \left[g_1(y)\right]^2-\left[g_2(y)\right]^2 \right\} dy$.

Using shells, we find that a typical approximating shell has radius x , so its circumference is $2\pi x$. Its height is y , that is, $x(x-1)^2$. So the total volume is

$$V=\int_0^1 2\pi x\left[x(x-1)^2\right]dx=2\pi\int_0^1 \left(x^4-2x^3+x^2\right)dx=2\pi\left[\frac{x^5}{5}-2\frac{x^4}{4}+\frac{x^3}{3}\right]_0^1=\frac{\pi}{15}$$

2.





A typical cylindrical shell has circumference $2\pi x$ and height $\sin\left(x^2\right)$. $V=\int_0^{\sqrt{\pi}} 2\pi x\sin\left(x^2\right)dx$. Let

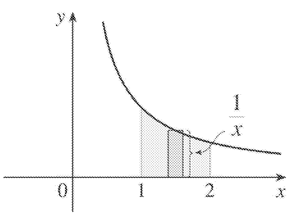
$u=x^2$. Then $du=2x\,dx$, so $V=\pi\int_0^{\pi}\sin u\,du=\pi\left[-\cos u\right]_0^{\pi}=\pi[1-(-1)]=2\pi$.

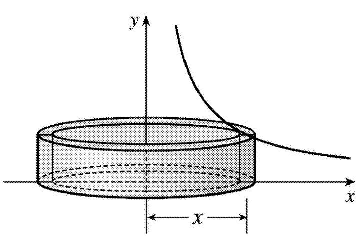
For slicing, we would first have to locate the local maximum point (a,b) of $y=\sin\left(x^2\right)$ using the methods of Chapter 4. Then we would have to solve the equation $y=\sin\left(x^2\right)$ for x in terms of y to obtain the functions $x=g_1(y)$ and $x=g_2(y)$ shown in the second figure. Finally we would find the volume using □ . Using shells is definitely preferable to slicing.

3.

$$V=\int_1^2 2\pi x\cdot\frac{1}{x}\,dx=2\pi\int_1^2 1\,dx$$

$$=2\pi\left[x\right]_1^2=2\pi(2-1)=2\pi$$

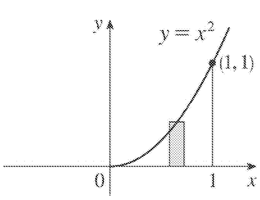


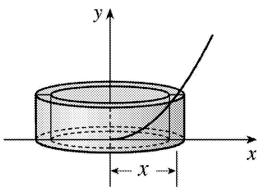


4.

$$V=\int_0^1 2\pi x\cdot x^2\,dx=2\pi\int_0^1 x^3\,dx$$

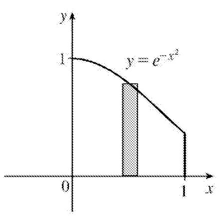
$$=2\pi\left[\frac{1}{4}x^4\right]_0^1=2\pi\cdot\frac{1}{4}=\frac{\pi}{2}$$

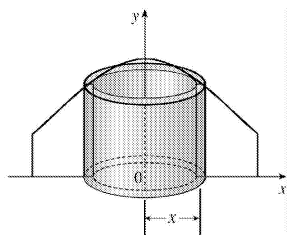




5. $V=\int_0^1 2\pi xe^{-x^2}\,dx$. Let $u=x^2$. Thus, $du=2x\,dx$, so

$$V=\pi\int_0^1 e^{-u}\,du=\pi\left[-e^{-u}\right]_0^1=\pi(1-1/e)$$

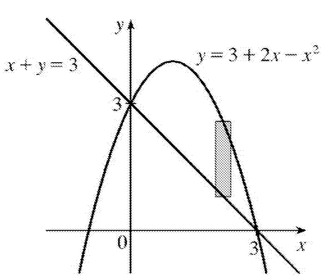


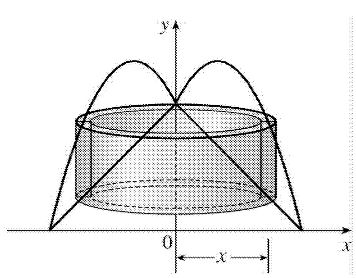


6.

$$V=2\pi\int_0^3 \left\{x[(3+2x-x^2)-(3-x)]\right\}dx=2\pi\int_0^3\left[x(3x-x^2)\right]dx$$

$$=2\pi\int_0^3(3x^2-x^3)dx=2\pi\left[x^3-\frac{1}{4}x^4\right]_0^3=2\pi\left(27-\frac{81}{4}\right)=2\pi\left(\frac{27}{4}\right)=\frac{27\pi}{2}$$



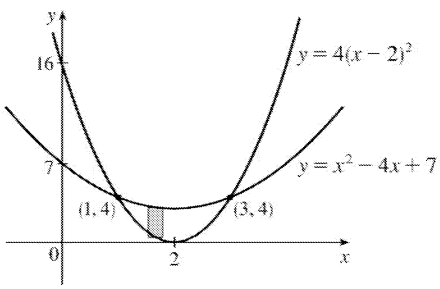


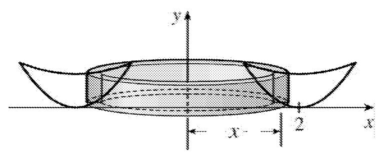
7. The curves intersect when $4(x-2)^2=x^2-4x+7\Leftrightarrow 4x^2-16x+16=x^2-4x+7\Leftrightarrow 3x^2-12x+9=0\Leftrightarrow$ $3(x^2-4x+3)=0\Leftrightarrow 3(x-1)(x-3)=0$, so $x=1$ or 3 .

$$V=2\pi\int_1^3\left\{x\left[\left(x^2-4x+7\right)-4(x-2)^2\right]\right\}dx=2\pi\int_1^3\left[x(x^2-4x+7-4x^2+16x-16)\right]dx$$

$$=2\pi\int_1^3\left[x(-3x^2+12x-9)\right]dx=2\pi(-3)\int_1^3(x^3-4x^2+3x)\,dx=-6\pi\left[\frac{1}{4}x^4-\frac{4}{3}x^3+\frac{3}{2}x^2\right]_1^3$$

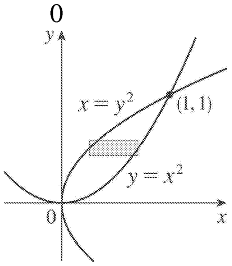
$$=-6\pi\left[\left(\frac{81}{4}-36+\frac{27}{2}\right)-\left(\frac{1}{4}-\frac{4}{3}+\frac{3}{2}\right)\right]=-6\pi\left(20-36+12+\frac{4}{3}\right)=-6\pi\left(-\frac{8}{3}\right)=16\pi$$

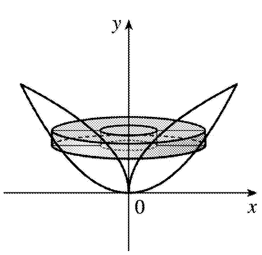




8. By slicing:

$$V=\int_0^1\pi\left[(\sqrt{y})^2-(y^2)^2\right]dy=\pi\int_0^1\left(y-y^4\right)dy=\pi\left[\frac{1}{2}y^2-\frac{1}{5}y^5\right]_0^1=\pi\left(\frac{1}{2}-\frac{1}{5}\right)=\frac{3\pi}{10}$$



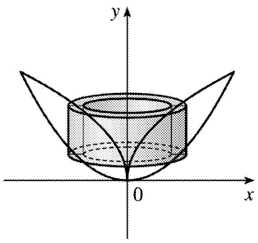


By cylindrical shells:

$$V=\int_0^1 2\pi x\left(\sqrt{x}-x^2\right)dx=2\pi\int_0^1(x^{3/2}-x^3)\,dx$$

$$=2\pi\left[\frac{2}{5}x^{5/2}-\frac{1}{4}x^4\right]_0^1=2\pi\left(\frac{2}{5}-\frac{1}{4}\right)$$

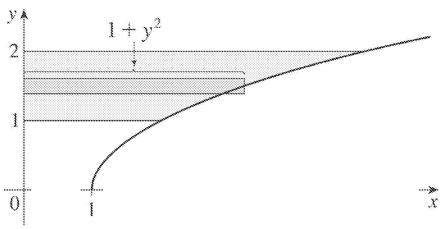
$$=2\pi\left(\frac{3}{20}\right)=\frac{3\pi}{10}$$

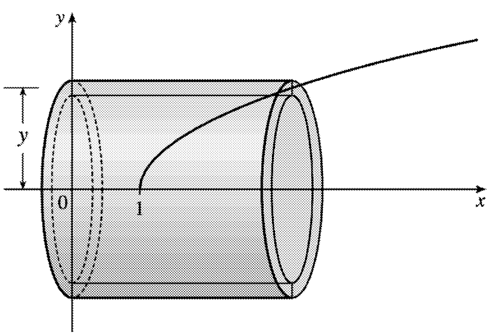


9.

$$V=\int_1^2 2\pi y\left(1+y^2\right)dy=2\pi\int_1^2\left(y+y^3\right)dy=2\pi\left[\frac{1}{2}y^2+\frac{1}{4}y^4\right]_1^2$$

$$=2\pi\left[(2+4)-\left(\frac{1}{2}+\frac{1}{4}\right)\right]=2\pi\left(\frac{21}{4}\right)=\frac{21\pi}{2}$$

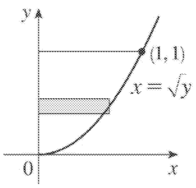


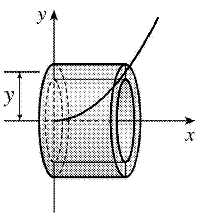


10.

$$V=\int_0^1 2\pi y\sqrt{y}\,dy=2\pi\int_0^1 y^{3/2}dy$$

$$=2\pi\left[\frac{2}{5}y^{5/2}\right]_0^1=\frac{4\pi}{5}$$



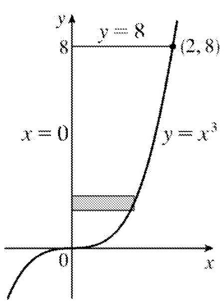


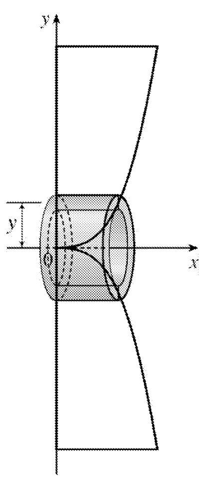
11.

$$V = 2\pi \int_0^8 \left[y(\sqrt[3]{y} - 0) \right] dy$$

$$= 2\pi \int_0^8 y^{4/3} dy = 2\pi \left[\frac{3}{7} y^{7/3} \right]_0^8$$

$$= \frac{6\pi}{7} (8^{7/3}) = \frac{6\pi}{7} (2^7) = \frac{768\pi}{7}$$



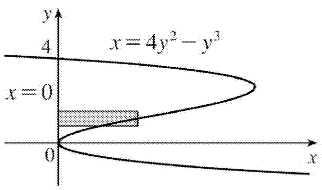


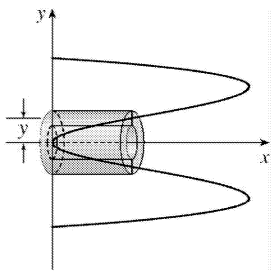
12.

$$V = 2\pi \int_0^4 \left[y(4y^2 - y^3) \right] dy$$

$$= 2\pi \int_0^4 (4y^3 - y^4) dy$$

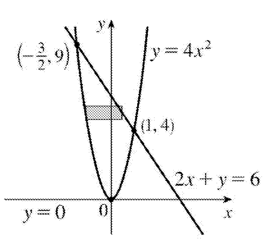
$$=2\pi\left[y^4-\frac{1}{5}y^5\right]_0^4=2\pi\left(256-\frac{1024}{5}\right)$$
$$=2\pi\left(\frac{256}{5}\right)=\frac{512\pi}{5}$$

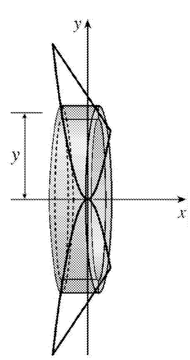




13. The curves intersect when $4x^2=6-2x \Leftrightarrow 2x^2+x-3=0 \Leftrightarrow (2x+3)(x-1)=0 \Leftrightarrow x=-\frac{3}{2}$ or 1 . Solving the equations for x gives us $y=4x^2 \Rightarrow x=\pm\frac{1}{2}\sqrt{y}$ and $2x+y=6 \Rightarrow x=-\frac{1}{2}y+3$.

$$V=2\pi\int_0^4\left\{y\left[\left(\frac{1}{2}\sqrt{y}\right)-\left(-\frac{1}{2}\sqrt{y}\right)\right]\right\}dy+2\pi\int_4^9\left\{y\left[\left(-\frac{1}{2}y+3\right)-\left(-\frac{1}{2}\sqrt{y}\right)\right]\right\}dy$$
$$=2\pi\int_0^4\left(y\sqrt{y}\right)dy+2\pi\int_4^9\left(-\frac{1}{2}y^2+3y+\frac{1}{2}y^{3/2}\right)dy=2\pi\left[\frac{2}{5}y^{5/2}\right]_0^4+2\pi\left[-\frac{1}{6}y^3+\frac{3}{2}y^2+\frac{1}{5}y^{5/2}\right]_4^9$$
$$=2\pi\left(\frac{2}{5}\cdot 32\right)+2\pi\left[\left(-\frac{243}{2}+\frac{243}{2}+\frac{243}{5}\right)-\left(-\frac{32}{3}+24+\frac{32}{5}\right)\right]$$
$$=\frac{128}{5}\pi+2\pi\left(\frac{433}{15}\right)=\frac{1250}{15}\pi=\frac{250}{3}\pi$$





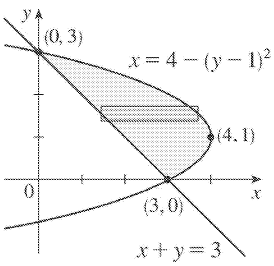
14.

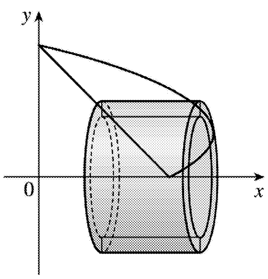
$$V = \int_0^3 2\pi y\left[4-(y-1)^2-(3-y)\right]dy$$

$$=2\pi\int_0^3 y\left(-y^2+3y\right)dy$$

$$=2\pi\int_0^3 \left(-y^3+3y^2\right)dy=2\pi\left[-\frac{1}{4}y^4+y^3\right]_0^3$$

$$=2\pi\left(-\frac{81}{4}+27\right)=2\pi\left(\frac{27}{4}\right)=\frac{27\pi}{2}$$

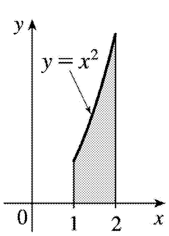


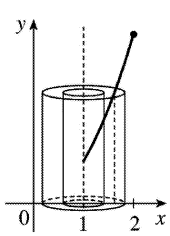


15.

$$V = \int_1^2 2\pi(x-1)x^2\,dx=2\pi\left[\frac{1}{4}x^4-\frac{1}{3}x^3\right]_1^2$$

$$=2\pi\left[\left(4-\frac{8}{3}\right)-\left(\frac{1}{4}-\frac{1}{3}\right)\right]=\frac{17}{6}\pi$$

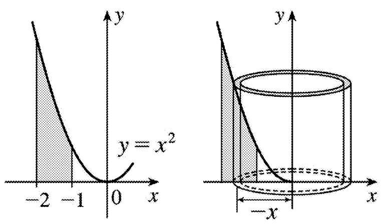




16.

$$V = \int_{-2}^{-1} 2\pi(-x)\cdot x^2\,dx = 2\pi\left[-\frac{1}{4}x^4\right]_{-2}^{-1}$$

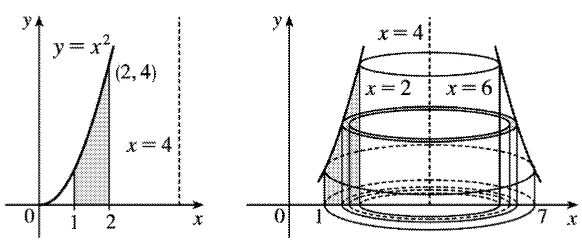
$$= 2\pi\left[\left(-\frac{1}{4}\right)-(-4)\right] = \frac{15}{2}\pi$$



17.

$$V = \int_{1}^{2} 2\pi(4-x)x^2\,dx = 2\pi\left[\frac{4}{3}x^3-\frac{1}{4}x^4\right]_{1}^{2}$$

$$= 2\pi\left[\left(\frac{32}{3}-4\right)-\left(\frac{4}{3}-\frac{1}{4}\right)\right] = \frac{67}{6}\pi$$



18.

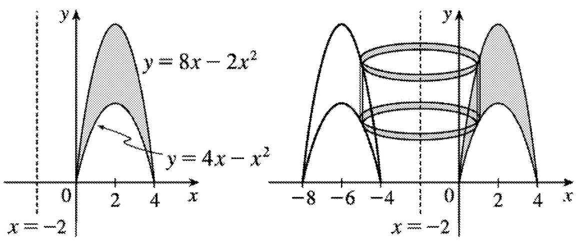
$$V = \int_{0}^{4} 2\pi[x-(-2)][(8x-2x^2)-(4x-x^2)]\,dx$$

$$= \int_{0}^{4} 2\pi(2+x)(4x-x^2)\,dx$$

$$= 2\pi\int_{0}^{4}(8x+2x^2-x^3)\,dx$$

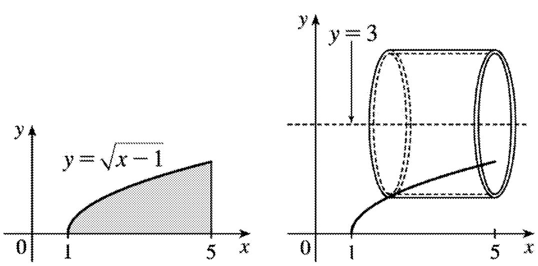
$$= 2\pi\left[4x^2+\frac{2}{3}x^3-\frac{1}{4}x^4\right]_{0}^{4}$$

$$= 2\pi\left(64+\frac{128}{3}-64\right) = \frac{256}{3}\pi$$



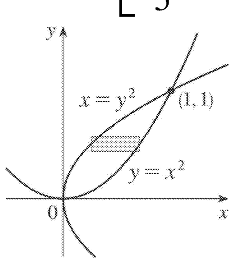
19.

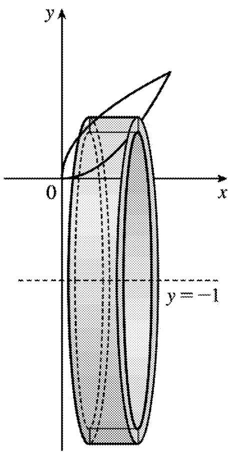
$$V = \int_0^2 2\pi (3-y)(5-x)dy$$
$$= \int_0^2 2\pi\ (3-y)\ \left(5-y^2-1\right) dy$$
$$= \int_0^2 2\pi\ \left(12-4y-3y^2+y^3\right) dy$$
$$= 2\pi \left[\ 12y-2y^2-y^3+\frac{1}{4}\ y^4\right]_0^2$$
$$= 2\pi (24-8-8+4)=24\pi$$



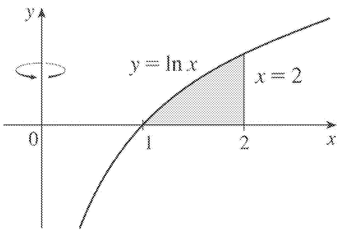
20.

$$V = \int_0^1 2\pi\ (y+1)\ \left(\sqrt{y}-y^2\right) dy = 2\pi \int_0^1 (y^{3/2}+y^{1/2}-y^3-y^2)dy$$
$$= 2\pi \left[\ \frac{2}{5}\ y^{5/2}+\frac{2}{3}\ y^{3/2}-\frac{1}{4}\ y^4-\frac{1}{3}\ y^3\right]_0^1 = 2\pi \left(\ \frac{2}{5}+\frac{2}{3}-\frac{1}{4}-\frac{1}{3}\ \right) = 2\pi \left(\ \frac{29}{60}\ \right) = \frac{29\pi}{30}$$

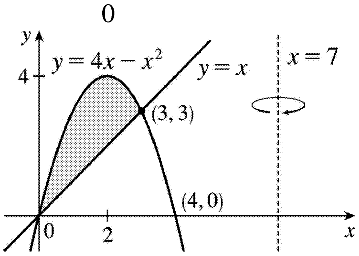




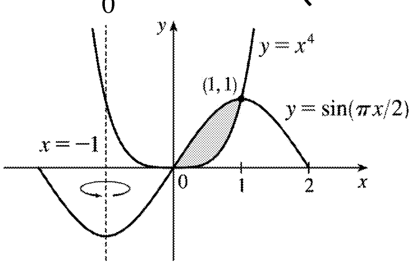
21. $V = \int_1^2 2\pi x \ln x \, dx$



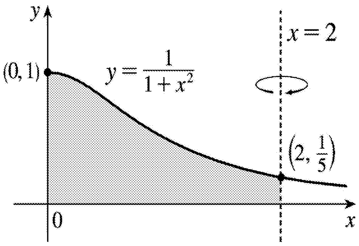
22. $V = \int_0^3 2\pi\,(7-x)\left[\left(4x-x^2\right)-x\right]dx$



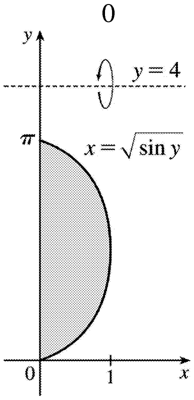
23. $V = \int_0^1 2\pi\,[x-(-1)]\left(\sin\frac{\pi}{2}x - x^4\right)dx$



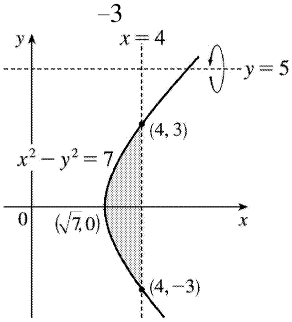
24. $V = \int_0^2 2\pi(2-x)\left(\frac{1}{1+x^2}\right)dx$



25. $V=\int_{0}^{\pi} 2\pi(4-y)\sqrt{\sin y}\,dy$



26. $V=\int_{-3}^{3} 2\pi(5-y)\left(4-\sqrt{y^2+7}\right)dy$



27. $\Delta x=\dfrac{\pi/4-0}{4}=\dfrac{\pi}{16}$.

$$V=\int_{0}^{\pi/4} 2\pi x\tan x\,dx\approx 2\pi\cdot\frac{\pi}{16}\left(\frac{\pi}{32}\tan\frac{\pi}{32}+\frac{3\pi}{32}\tan\frac{3\pi}{32}+\frac{5\pi}{32}\tan\frac{5\pi}{32}+\frac{7\pi}{32}\tan\frac{7\pi}{32}\right)\approx 1.142$$

28. $\Delta x=\dfrac{12-2}{5}=2$, $n=5$ and $x_i^*=2+(2i+1)$, where $i=0$, 1 , 2 , 3 , 4 . The values of $f(x)$ are taken directly from the diagram.

$$V=\int_{2}^{12} 2\pi x f(x)\,dx\approx 2\pi\left[3f(3)+5f(5)+7f(7)+9f(9)+11f(11)\right]\cdot 2$$

$$\approx 2\pi\left[3(2)+5(4)+7(4)+9(2)+11(1)\right]2=332\pi$$

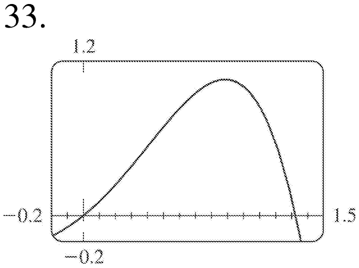
29. $\int_0^3 2\pi x^5 dx = 2\pi \int_0^3 x(x^4) dx$. The solid is obtained by rotating the region $0 \le y \le x^4$, $0 \le x \le 3$ about the y –axis using cylindrical shells.

30. $2\pi \int_0^2 \frac{y}{1+y^2} dy = 2\pi \int_0^2 y \left(\frac{1}{1+y^2} \right) dy$. The solid is obtained by rotating the region $0 \le x \le \frac{1}{1+y^2}$, $0 \le y \le 2$ about the x –axis using cylindrical shells.

31. $\int_0^1 2\pi(3-y)(1-y^2) dy$. The solid is obtained by rotating the region bounded by (i) $x=1-y^2$, $x=0$, and $y=0$ or (ii) $x=y^2$, $x=1$, and $y=0$ about the line $y=3$ using cylindrical shells.

32. $\int_0^{\pi/4} 2\pi(\pi-x)(\cos x - \sin x) dx$. The solid is obtained by rotating the region bounded by (i) $0 \le y \le \cos x - \sin x$, $0 \le x \le \frac{\pi}{4}$ or (ii) $\sin x \le y \le \cos x$, $0 \le x \le \frac{\pi}{4}$ about the line $x=\pi$ using cylindrical shells.

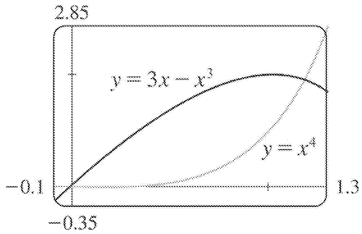
33.



From the graph, the curves intersect at $x=0$ and at $x=a \approx 1.32$, with $x+x^2-x^4>0$ on the interval $(0,a)$. So the volume of the solid obtained by rotating the region about the y– axis is

$$V = 2\pi \int_0^a \left[x(x+x^2-x^4) \right] dx = 2\pi \int_0^a (x^2+x^3-x^5) dx$$
$$= 2\pi \left[\frac{1}{3}x^3 + \frac{1}{4}x^4 - \frac{1}{6}x^6 \right]_0^a \approx 4.05$$

34.



From the graph, the curves intersect at $x=0$ and at $x=a\approx 1.17$, with $3x-x^3>x^4$ on the interval $(0,a)$. So the volume of the solid obtained by rotating the region about the $y-$ axis is

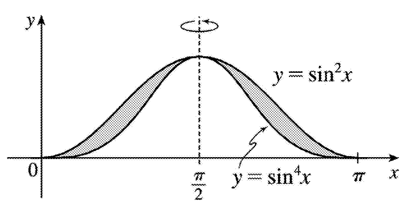
$$V = 2\pi\int_0^a \left\{ x[(3x-x^3)-x^4]\right\} dx=2\pi\int_0^a(3x^2-x^4-x^5)dx$$

$$= 2\pi\left[x^3-\frac{1}{5}x^5-\frac{1}{6}x^6\right]_0^a\approx 4.62$$

35.

$$V =2\pi\int_0^{\pi/2}\left[\left(\frac{\pi}{2}-x\right)\left(\sin^2 x-\sin^4 x\right)\right]dx$$

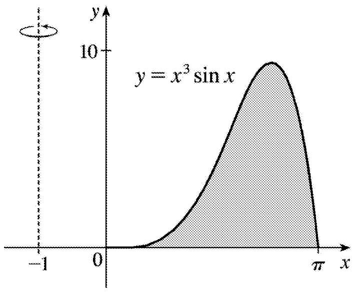
$$=\frac{1}{32}\pi^3$$



36.

$$V =2\pi\int_0^{\pi}\left\{[x-(-1)](x^3\sin x)\right\} dx=2\pi(\pi^4+\pi^3-12\pi^2-6\pi+48)$$

$$=2\pi^5+2\pi^4-24\pi^3-12\pi^2+96\pi$$



37. Use disks:

$$V = \int_{-2}^{1} \pi \left(x^2+x-2\right)^2 dx = \pi \int_{-2}^{1} \left(x^4+2x^3-3x^2-4x+4\right) dx$$

$$= \pi \left[\frac{1}{5}x^5 + \frac{1}{2}x^4 - x^3 - 2x^2 + 4x \right]_{-2}^{1} = \pi \left[\left(\frac{1}{5} + \frac{1}{2} - 1 - 2 + 4 \right) - \left(-\frac{32}{5} + 8 + 8 - 8 - 8 \right) \right]$$

$$= \pi \left(\frac{33}{5} + \frac{3}{2} \right) = \frac{81}{10}\pi$$

38. Use shells:

$$V = \int_{1}^{2} 2\pi x \left(-x^2+3x-2\right) dx = 2\pi \int_{1}^{2} \left(-x^3+3x^2-2x\right) dx$$

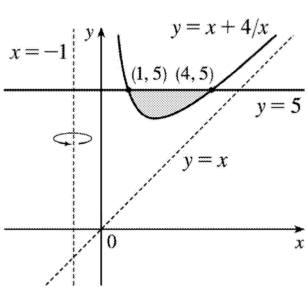
$$= 2\pi \left[-\frac{1}{4}x^4 + x^3 - x^2 \right]_{1}^{2} = 2\pi \left[(-4+8-4) - \left(-\frac{1}{4} + 1 - 1 \right) \right] = \frac{\pi}{2}$$

39. Use shells:

$$V = \int_1^4 2\pi \left[x-(-1)\right]\left[5-(x+4/x)\right] dx = 2\pi \int_1^4 (x+1)(5-x-4/x)\, dx = 2\pi \int_1^4 \left(5x-x^2-4+5-x-4/x\right) dx =$$

$$2\pi \int_1^4 \left(-x^2+4x+1-4/x\right) dx = 2\pi \left[-\frac{1}{3}x^3 + 2x^2 + x - 4\ln x \right]_1^4 =$$

$$2\pi \left[\left(-\frac{64}{3} + 32 + 4 - 4\ln 4 \right) - \left(-\frac{1}{3} + 2 + 1 - 0 \right) \right] = 2\pi(12-4\ln 4) = 8\pi(3-\ln 4)$$



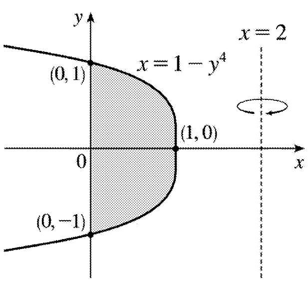
40. Use washers:

$$V=\int_{-1}^{1}\pi\left\{[2-0]^2-\left[2-\left(1-y^4\right)\right]^2\right\}dy$$

$$=2\pi\int_0^1\left[4-\left(1+y^4\right)^2\right]dy \text{ [by symmetry]}$$

$$=2\pi\int_0^1\left[4-\left(1+2y^4+y^8\right)\right]dy=2\pi\int_0^1\left(3-2y^4-y^8\right)dy$$

$$=2\pi\left[3y-\frac{2}{5}y^5-\frac{1}{9}y^9\right]_0^1=2\pi\left(3-\frac{2}{5}-\frac{1}{9}\right)=2\pi\left(\frac{112}{45}\right)=\frac{224\pi}{45}$$



41. Use disks: $V=\pi\int_0^2\left[\sqrt{1-(y-1)^2}\right]^2dy=\pi\int_0^2\left(2y-y^2\right)dy=\pi\left[y^2-\frac{1}{3}y^3\right]_0^2=\pi\left(4-\frac{8}{3}\right)=\frac{4}{3}\pi$

42. Using shells, we have

$$V=\int_0^2 2\pi y\left[\sqrt{1-(y-1)^2}-\left(-\sqrt{1-(y-1)^2}\right)\right]dy$$

$$=2\pi\int_0^2 y\cdot 2\sqrt{1-(y-1)^2}\,dy=4\pi\int_{-1}^{1}(u+1)\sqrt{1-u^2}\,du \quad [\text{let } u=y-1]$$

$$=4\pi\int_{-1}^{1}u\sqrt{1-u^2}\,du+4\pi\int_{-1}^{1}\sqrt{1-u^2}\,du$$

The first definite integral equals zero because its integrand is an odd function. The second is the area of a semicircle of radius 1 , that is, $\frac{\pi}{2}$. Thus, $V=4\pi\cdot 0+4\pi\cdot\frac{\pi}{2}=2\pi^2$.

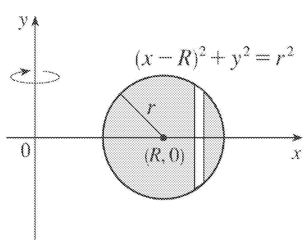
43.

$$V=2\int_0^r 2\pi x\sqrt{r^2-x^2}\,dx=-2\pi\int_0^r \left(r^2-x^2\right)^{1/2}(-2x)\,dx=\left[-2\pi\cdot\frac{2}{3}\left(r^2-x^2\right)^{3/2}\right]_0^r$$
$$=-\frac{4}{3}\pi\left(0-r^3\right)=\frac{4}{3}\pi r^3$$

44.

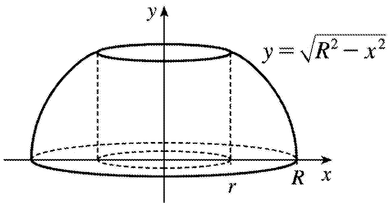
$$V=\int_{R-r}^{R+r} 2\pi x\cdot 2\sqrt{r^2-(x-R)^2}\,dx$$
$$=\int_{-r}^{r} 4\pi(u+R)\sqrt{r^2-u^2}\,du\ \ [u=x-R\ ]$$
$$=4\pi R\int_{-r}^{r}\sqrt{r^2-u^2}\,du+4\pi\int_{-r}^{r}u\sqrt{r^2-u^2}\,du$$

The first integral is the area of a semicircle of radius r , that is, $\frac{1}{2}\pi r^2$, and the second is zero since the integrand is an odd function. Thus, $V=4\pi R\left(\frac{1}{2}\pi r^2\right)+4\pi\cdot 0=2\pi Rr^2$.



45. $V=2\pi\int_0^r x\left(-\frac{h}{r}x+h\right)dx=2\pi h\int_0^r\left(-\frac{x^2}{r}+x\right)dx=2\pi h\left[-\frac{x^3}{3r}+\frac{x^2}{2}\right]_0^r=2\pi h\,\frac{r^2}{6}=\frac{\pi r^2 h}{3}$

46. By symmetry, the volume of a napkin ring obtained by drilling a hole of radius r through a sphere with radius R is twice the volume obtained by rotating the area above the x– axis and below the curve $y=\sqrt{R^2-x^2}$ (the equation of the top half of the cross–section of the sphere), between $x=r$ and $x=R$, about the y– axis.
This volume is equal to



$$2\int_{\text{innerradius}}^{\text{outerradius}} 2\pi rh\,dx = 2\cdot 2\pi\int_r^R x\sqrt{R^2-x^2}\,dx = 4\pi\left[-\frac{1}{3}\left(R^2-x^2\right)^{3/2}\right]_r^R = \frac{4}{3}\pi\left(R^2-r^2\right)^{3/2}$$

But by the Pythagorean Theorem, $R^2-r^2=\left(\frac{1}{2}h\right)^2$, so the volume of the napkin ring is $\frac{4}{3}\pi\left(\frac{1}{2}h\right)^3=\frac{1}{6}\pi h^3$, which is independent of both R and r ; that is, the amount of wood in a napkin ring of height h is the same regardless of the size of the sphere used. Note that most of this calculation has been done already, but with more difficulty, in Exercise 6.2.68.

Another solution: The height of the missing cap is the radius of the sphere minus half the height of the cut–out cylinder, that is, $R-\frac{1}{2}h$. Using Exercise 6.2.49,

$$V_{\text{napkinring}}=V_{\text{sphere}}-V_{\text{cylinder}}-2V_{\text{cap}}=\frac{4}{3}\pi R^3-\pi r^2h-2\cdot\frac{\pi}{3}\left(R-\frac{1}{2}h\right)^2\left[3R-\left(R-\frac{1}{2}h\right)\right]=\frac{1}{6}\pi h^3$$

1. By Equation 2, $W=Fd=(900)(8)=7200$ J.
2. $F=mg=(60)(9.8)=588$ N; $W=Fd=588\cdot 2=1176$ J

3.

$$W=\int_a^b f(x)\,dx=\int_0^9 \frac{10}{(1+x)^2}\,dx=10\int_1^{10}\frac{1}{u^2}\,du\ [u=1+x, du=dx]$$

$$=10\left[-\frac{1}{u}\right]_1^{10}=10\left(-\frac{1}{10}+1\right)=9\text{ft–lb}$$

4. $W=\int_1^2 \cos\left(\frac{1}{3}\pi x\right)dx=\frac{3}{\pi}\left[\sin\left(\frac{1}{3}\pi x\right)\right]_1^2=\frac{3}{\pi}\left(\frac{\sqrt{3}}{2}-\frac{\sqrt{3}}{2}\right)=0$ N–m $=0$ J.

Interpretation: From $x=1$ to $x=\frac{3}{2}$, the force does work equal to $\int_1^{3/2}\cos\left(\frac{1}{3}\pi x\right)dx=\frac{3}{\pi}\left(1-\frac{\sqrt{3}}{2}\right)$ J in accelerating the particle and increasing its kinetic energy. From $x=\frac{3}{2}$ to $x=2$, the force opposes the motion of the particle, decreasing its kinetic energy. This is negative work, equal in magnitude but opposite in sign to the work done from $x=1$ to $x=\frac{3}{2}$.

5. The force function is given by $F(x)$ (in newtons) and the work (in joules) is the area under the curve, given by $\int_0^8 F(x)\,dx=\int_0^4 F(x)\,dx+\int_4^8 F(x)\,dx=\frac{1}{2}(4)(30)+(4)(30)=180$ J.

6. $W=\int_4^{20} f(x)\,dx\approx M_4=\Delta x[f(6)+f(10)+f(14)+f(18)]=\frac{20-4}{4}[5.8+8.8+8.2+5.2]=4(28)=112$ J

7. $10=f(x)=kx=\frac{1}{3}k$ [4 inches $=\frac{1}{3}$ foot], so $k=30$ lb / ft and $f(x)=30x$. Now 6 inches $=\frac{1}{2}$ foot, so $W=\int_0^{1/2}30x\,dx=\left[15x^2\right]_0^{1/2}=\frac{15}{4}$ ft–lb.

8. $25=f(x)=kx=k(0.1)$ [10 cm $=0.1$ m], so $k=250$ N / m and $f(x)=250x$. Now 5 cm $=0.05$ m, so $W=\int_0^{0.05}250x\,dx=\left[125x^2\right]_0^{0.05}=125(0.0025)=0.3125\approx 0.31$ J.

9. If $\int_0^{0.12} kx\,dx = 2$ J, then $2 = \left[\frac{1}{2}kx^2\right]_0^{0.12} = \frac{1}{2}k(0.0144) = 0.0072k$ and $k = \frac{2}{0.0072} = \frac{2500}{9} \approx 277.78$ N / m. Thus, the work needed to stretch the spring from 35 cm to 40 cm is $\int_{0.05}^{0.10} \frac{2500}{9}x\,dx = \left[\frac{1250}{9}x^2\right]_{1/20}^{1/10} = \frac{1250}{9}\left(\frac{1}{100} - \frac{1}{400}\right) = \frac{25}{24} \approx 1.04$ J.

10. If $12 = \int_0^1 kx\,dx = \left[\frac{1}{2}kx^2\right]_0^1 = \frac{1}{2}k$, then $k = 24$ lb / ft and the work required is $\int_0^{3/4} 24x\,dx = \left[12x^2\right]_0^{3/4} = 12 \cdot \frac{9}{16} = \frac{27}{4} = 6.75$ ft–lb.

11. $f(x) = kx$, so $30 = \frac{2500}{9}x$ and $x = \frac{270}{2500}$ m $= 10.8$ cm

12. Let L be the natural length of the spring in meters. Then $6 = \int_{0.10-L}^{0.12-L} kx\,dx = \left[\frac{1}{2}kx^2\right]_{0.10-L}^{0.12-L} = \frac{1}{2}k\left[(0.12-L)^2 - (0.10-L)^2\right]$ and $10 = \int_{0.12-L}^{0.14-L} kx\,dx = \left[\frac{1}{2}kx^2\right]_{0.12-L}^{0.14-L} = \frac{1}{2}k\left[(0.14-L)^2 - (0.12-L)^2\right]$. Simplifying gives us $12 = k(0.0044 - 0.04L)$ and $20 = k(0.0052 - 0.04L)$. Subtracting the first equation from the second gives $8 = 0.0008k$, so $k = 10\,,000$. Now the second equation becomes $20 = 52 - 400L$, so $L = \frac{32}{400}$ m $= 8$ cm.

13. **(a)** The portion of the rope from x ft to $(x + \Delta x)$ ft below the top of the building weighs $\frac{1}{2}\Delta x$ lb and must be lifted x_i^* ft, so its contribution to the total work is $\frac{1}{2}x_i^*\Delta x$ ft–lb. The total work is $W = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{2}x_i^*\Delta x = \int_0^{50} \frac{1}{2}x\,dx = \left[\frac{1}{4}x^2\right]_0^{50} = \frac{2500}{4} = 625$ ft–lb Notice that the exact height of the building does not matter (as long as it is more than 50 ft).

(b) When half the rope is pulled to the top of the building, the work to lift the top half of the rope is $W_1 = \int_0^{25} \frac{1}{2}x\,dx = \left[\frac{1}{4}x^2\right]_0^{25} = \frac{625}{4}$ ft–lb. The bottom half of the rope is lifted 25 ft and the work needed to accomplish that is $W_2 = \int_{25}^{50} \frac{1}{2} \cdot 25\,dx = \frac{25}{2}\left[x\right]_{25}^{50} = \frac{625}{2}$ ft–lb. The total work done in pulling half the rope to the top of the building is $W=W_1+W_2=\frac{625}{2}+\frac{625}{4}=\frac{3}{4}\cdot 625=\frac{1875}{4}$ ft–lb.

14. *Assumptions* : 1. After lifting, the chain is L–shaped, with 4 m of the chain lying along the ground.
2. The chain slides effortlessly and without friction along the ground while its end is lifted.
3. The weight density of the chain is constant throughout its length and therefore equals $(8\text{kg/m})(9.8\text{m/s}^2)=78.4\text{N/m}$.
The part of the chain xm from the lifted end is raised $6-x$m if $0\le x\le 6$ m, and it is lifted 0 m if $x>6$m .
Thus, the work needed is

$$W=\lim_{n\to\infty}\sum_{i=1}^{n}(6-x_i^*)\cdot 78.4\,\Delta x=\int_0^6(6-x)78.4\,dx=78.4\left[6x-\frac{1}{2}x^2\right]_0^6=(78.4)(18)=1411.2\text{ J.}$$

15. The work needed to lift the cable is $\lim_{n\to\infty}\sum_{i=1}^{n}2x_i^*\,\Delta x=\int_0^{500}2x\,dx=\left[x^2\right]_0^{500}=250,000$ ft–lb. The work needed to lift the coal is 800 lb · 500 ft =400 , 000 ft–lb. Thus, the total work required is 250 , 000+400 , 000=650 , 000 ft–lb.

16. The work needed to lift the bucket itself is 4 lb · 80 ft =320 ft–lb. At time t (in seconds) the bucket is $x_i^*=2t$ ft above its original 80 ft depth, but it now holds only $(40-0.2t)$ lb of water. In terms of distance, the bucket holds $\left[40-0.2\left(\frac{1}{2}x_i^*\right)\right]$ lb of water when it is x_i^* ft above its original 80 ft depth. Moving this amount of water a distance Δx requires $\left(40-\frac{1}{10}x_i^*\right)\Delta x$ ft–lb of work. Thus, the work needed to lift the water is

$$W=\lim_{n\to\infty}\sum_{i=1}^{n}\left(40-\frac{1}{10}x_i^*\right)\Delta x=\int_0^{80}\left(40-\frac{1}{10}x\right)dx=\left[40x-\frac{1}{20}x^2\right]_0^{80}=(3200-320)\text{ ft–lb}$$

Adding the work of lifting the bucket gives a total of 3200 ft–lb of work.

17. At a height of x meters ($0\le x\le 12$), the mass of the rope is (0.8 kg / m)($12-x$ m)=$(9.6-0.8x)$ kg and the mass of the water is $\left(\frac{36}{12}\text{ kg/m}\right)(12-x\text{m})=(36-3x)$ kg. The mass of the bucket is 10 kg, so the total mass is $(9.6-0.8x)+(36-3x)+10=(55.6-3.8x)$ kg, and hence, the total force is $9.8(55.6-3.8x)$ N.
The work needed to lift the bucket Δx m through the i th subinterval of $[0,12]$ is $9.8(55.6-3.8x_i^*)\Delta x$, so the total work is

$$W=\lim_{n\to\infty}\sum_{i=1}^{n}9.8(55.6-3.8x_i^*)\Delta x=\int_0^{12}(9.8)(55.6-3.8x)dx=9.8\left[55.6x-1.9x^2\right]_0^{12}$$
$$=9.8(393.6)\approx 3857\text{ J}$$

18. The chain's weight density is $\frac{25\text{lb}}{10\text{ft}}=2.5$ lb / ft. The part of the chain x ft below the ceiling (for $5\le x\le 10$) has to be lifted $2(x-5)$ ft, so the work needed to lift the i th subinterval of the chain is 2 ($x_i^*-5)(2.5\,\Delta x)$. The total work needed is

$$W=\lim_{n\to\infty}\sum_{i=1}^{n}2(x_i^*-5)(2.5)\Delta x=\int_5^{10}[2(x-5)(2.5)]dx=5\int_5^{10}(x-5)dx$$
$$=5\left[\frac{1}{2}x^2-5x\right]_5^{10}=5\left[(50-50)-\left(\frac{25}{2}-25\right)\right]=5\left(\frac{25}{2}\right)=62.5\text{ ft–lb}$$

19. A ''slice'' of water Δx m thick and lying at a depth of x_i^* m (where $0\le x_i^*\le\frac{1}{2}$) has volume $(2\times 1\times\Delta x)$ m ^{3} , a mass of $2000\,\Delta x$ kg, weighs about $(9.8)(2000\,\Delta x)=19\,,600\,\Delta x$ N, and thus requires about $19\,,600x_i^*\,\Delta x$ J of work for its removal. So $W=\lim_{n\to\infty}\sum_{i=1}^{n}19\,,600x_i^*\,\Delta x=\int_0^{1/2}19\,,600x\,dx=\left[9800x^2\right]_0^{1/2}=2450$ J.

20. A horizontal cylindrical slice of water Δx ft thick has a volume of $\pi r^2h=\pi\cdot 12^2\cdot\Delta x$ ft ^{3} and weighs about $\left(62.5\text{lb/ft}^3\right)\left(144\pi\,\Delta x\text{ft}^3\right)=9000\pi\,\Delta x$ lb. If the slice lies x_i^* ft below the edge of the pool (where $1\le x_i^*\le 5$), then the work needed to pump it out is about $9000\pi x_i^*\,\Delta x$. Thus,

$$W=\lim_{n\to\infty}\sum_{i=1}^{n}9000\pi x_i^*\,\Delta x=\int_1^5 9000\pi x\,dx=\left[4500\pi x^2\right]_1^5=4500\pi(25-1)=108{,}000\pi\,\text{ft–lb}$$

21. A rectangular ''slice'' of water Δx m thick and lying x ft above the bottom has width x ft and volume $8x\,\Delta x$ m ^{3} . It weighs about $(9.8\times 1000)\,(8x\,\Delta x)$ N, and must be lifted $(5-x)$ m by the pump, so the work needed is about $\left(9.8\times 10^3\right)(5-x)(8x\,\Delta x)$ J. The total work required is

$$W \approx \int_0^3 \left(9.8\times 10^3\right)(5-x)8x\,dx=\left(9.8\times 10^3\right)\int_0^3 \left(40x-8x^2\right)dx=\left(9.8\times 10^3\right)\left[20x^2-\frac{8}{3}x^3\right]_0^3$$
$$=\left(9.8\times 10^3\right)(180-72)=\left(9.8\times 10^3\right)(108)=1058.4\times 10^3\approx 1.06\times 10^6\,\mathrm{J}$$

22. For convenience, measure depth x from the middle of the tank, so that $-1.5\le x\le 1.5$ m. Lifting a slice of water of thickness Δx at depth x requires a work contribution of
$\Delta W\approx\left(9.8\times 10^3\right)\left(2\sqrt{(1.5)^2-x^2}\right)(6\,\Delta x)(2.5+x)$, so
$$W \approx \int_{-1.5}^{1.5}\left(9.8\times 10^3\right)12\sqrt{2.25-x^2}\,(2.5+x)\,dx$$
$$=\left(9.8\times 10^3\right)\left[60\int_0^{3/2}\sqrt{\frac{9}{4}-x^2}\,dx+12\int_{-3/2}^{3/2}x\sqrt{\frac{9}{4}-x^2}\,dx\right]$$
The second integral is 0 because its integrand is an odd function, and the first integral represents the area of a quarter–circle of radius
$\frac{3}{2}$ .Therefore,[$W\approx\left(9.8\times 10^3\right)60\int_0^{3/2}\sqrt{\frac{9}{4}-x^2}\,dx=\left(9.8\times 10^3\right)(60)\left(\frac{1}{4}\pi\right)\left(\frac{3}{2}\right)^2=330{,}750\pi\approx 1.04\times 10^6\,\mathrm{J}$]

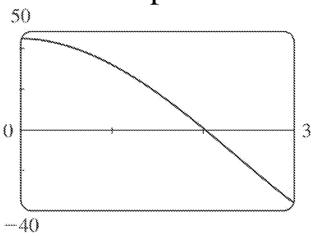
23. Measure depth x downward from the flat top of the tank, so that $0\le x\le 2$ ft. Then
$\Delta W=(62.5)\left(2\sqrt{4-x^2}\right)(8\,\Delta x)(x+1)$ ft–lb, so
$$W \approx (62.5)(16)\int_0^2(x+1)\sqrt{4-x^2}\,dx=1000\left(\int_0^2 x\sqrt{4-x^2}\,dx+\int_0^2\sqrt{4-x^2}\,dx\right)$$
$$=1000\left[\int_0^4 u^{1/2}\left(\frac{1}{2}\right)du+\frac{1}{4}\pi\left(2^2\right)\right]\quad[\text{Put } u=4-x^2,\text{so } du=-2x\,dx]$$
$$=1000\left(\left[\frac{1}{2}\cdot\frac{2}{3}u^{3/2}\right]_0^4+\pi\right)=1000\left(\frac{8}{3}+\pi\right)\approx 5.8\times 10^3\,\text{ft–lb}$$
Note: The second integral represents the area of a quarter–circle of radius 2 .

24. Let x be depth in feet, so that $0\le x\le 5$. Then $\Delta W=(62.5)\pi\left(\sqrt{5^2-x^2}\right)^2\Delta x\cdot x$ ft–lb and
$$W \approx 62.5\pi\int_0^5 x\left(25-x^2\right)dx=62.5\pi\left[\frac{25}{2}x^2-\frac{1}{4}x^4\right]_0^5=62.5\pi\left(\frac{625}{2}-\frac{625}{4}\right)=62.5\pi\left(\frac{625}{4}\right)$$
$$\approx 3.07\times 10^4\,\text{ft–lb}$$

25. If only 4.7×10^5 J of work is done, then only the water above a certain level (call it h) will be pumped out. So we use the same formula as in Exercise 21 , except that the work is fixed, and we are trying to find the lower limit of integration:

$$4.7\times 10^5\approx\int_h^3 \left(9.8\times 10^3\right)(5-x)8x\,dx=\left(9.8\times 10^3\right)\left[20x^2-\frac{8}{3}x^3\right]_h^3\Leftrightarrow$$

$\frac{4.7}{9.8}\times 10^2\approx 48=\left(20\cdot 3^2-\frac{8}{3}\cdot 3^3\right)-\left(20h^2-\frac{8}{3}h^3\right)\Leftrightarrow 2h^3-15h^2+45=0$. To find the solution of this equation, we plot $2h^3-15h^2+45$ between $h=0$ and $h=3$. We see that the equation is satisfied for $h\approx 2.0$. So the depth of water remaining in the tank is about 2.0 m.



26. $W\approx(9.8\times 920)\int_0^{3/2} 12\sqrt{\frac{9}{4}-x^2}\left(\frac{5}{2}+x\right)dx=9016\left[30\int_0^{3/2}\sqrt{\frac{9}{4}-x^2}\,dx+12\int_0^{3/2}x\sqrt{\frac{9}{4}-x^2}\,dx\right]$.

Here $\int_0^{3/2}\sqrt{\frac{9}{4}-x^2}\,dx=\frac{1}{4}\pi\left(\frac{3}{2}\right)^2=\frac{9\pi}{16}$ and $\int_0^{3/2}x\sqrt{\frac{9}{4}-x^2}\,dx=\int_0^{9/4}\frac{1}{2}u^{1/2}du$ [where $u=\frac{9}{4}-x^2$, so $du=-2x\,dx$] $=\left[\frac{1}{3}u^{3/2}\right]_0^{9/4}=\frac{1}{3}\left(\frac{27}{8}\right)=\frac{9}{8}$, so

$$W\approx 9016\left[30\cdot\frac{9}{16}\pi+12\cdot\frac{9}{8}\right]=9016\left(\frac{135}{8}\pi+\frac{27}{2}\right)\approx 6.00\times 10^5\text{ J.}$$

27. $V=\pi r^2x$, so V is a function of x and P can also be regarded as a function of x . If $V_1=\pi r^2x_1$ and $V_2=\pi r^2x_2$, then

$$W=\int_{x_1}^{x_2}F(x)dx=\int_{x_1}^{x_2}\pi r^2P(V(x))\,dx$$

$$=\int_{x_1}^{x_2}P(V(x))\,dV(x)\;[\text{Let}V(x)=\pi r^2x,\text{so}dV(x)=\pi r^2dx.]$$

$$= \int_{V_1}^{V_2} P(V)\,dV \text{ by the Substitution Rule.}$$

28. $160 \text{ lb / in}^2 = 160 \cdot 144 \text{ lb / ft}^2$, $100 \text{ in}^3 = \frac{100}{1728} \text{ ft}^3$, and $800 \text{ in}^3 = \frac{800}{1728} \text{ ft}^3$.

$k = PV^{1.4} = (160 \cdot 144)\left(\frac{100}{1728}\right)^{1.4} = 23{,}040\left(\frac{25}{432}\right)^{1.4} \approx 426.5$. Therefore, $P \approx 426.5V^{-1.4}$ and

$$W = \int_{100/1728}^{800/1728} 426.5V^{-1.4}\,dV = 426.5\left[\frac{1}{-0.4}V^{-0.4}\right]_{25/432}^{25/54}$$

$$= (426.5)(2.5)\left[\left(\frac{432}{25}\right)^{0.4} - \left(\frac{54}{25}\right)^{0.4}\right]$$

$$\approx 1.88 \times 10^3 \text{ ft–lb}$$

29. $W = \int_a^b F(r)\,dr = \int_a^b G\frac{m_1 m_2}{r^2}\,dr = Gm_1 m_2\left[\frac{-1}{r}\right]_a^b = Gm_1 m_2\left(\frac{1}{a} - \frac{1}{b}\right)$

30. By Exercise 29, $W = GMm\left(\frac{1}{R} - \frac{1}{R + 1{,}000{,}000}\right)$ where M= mass of Earth in kg, R= radius of Earth in m, and m= mass of satellite in kg. (Note that 1000 km = 1,000,000 m.) Thus,

$$W = \left(6.67 \times 10^{-11}\right)\left(5.98 \times 10^{24}\right)(1000) \times \left(\frac{1}{6.37 \times 10^6} - \frac{1}{7.37 \times 10^6}\right) \approx 8.50 \times 10^9 \text{J}$$

1. $f_{\text{ave}}=\dfrac{1}{b-a}\displaystyle\int_a^b f(x)dx=\dfrac{1}{1-(-1)}\int_{-1}^{1}x^2dx=\dfrac{1}{2}\cdot 2\int_0^1 x^2dx=\left[\dfrac{1}{3}x^3\right]_0^1=\dfrac{1}{3}$

2. $f_{\text{ave}}=\dfrac{1}{4-1}\displaystyle\int_1^4(1/x)dx=\dfrac{1}{3}\left[\ln x\right]_1^4=\dfrac{1}{3}\ln 4\approx 0.46$

3. $g_{\text{ave}}=\dfrac{1}{\dfrac{\pi}{2}-0}\displaystyle\int_0^{\pi/2}\cos x\,dx=\dfrac{2}{\pi}\left[\sin x\right]_0^{\pi/2}=\dfrac{2}{\pi}(1-0)=\dfrac{2}{\pi}$

4.

$$g_{\text{ave}}=\frac{1}{2-0}\int_0^2 x^2\sqrt{1+x^3}\,dx=\frac{1}{2}\int_1^9\sqrt{u}\cdot\frac{1}{3}\,du\ [\,u=1+x^3\,,\,du=3x^2dx\,]$$

$$=\frac{1}{6}\left[\frac{2}{3}u^{3/2}\right]_1^9=\frac{1}{9}(27-1)=\frac{26}{9}$$

5. $f_{\text{ave}}=\dfrac{1}{5-0}\displaystyle\int_0^5 te^{-t^2}dt=\dfrac{1}{5}\int_0^{-25}e^u\left(-\dfrac{1}{2}du\right)\ [\,u=-t^2\,,\,du=-2t\,dt\,,\,t\,dt=-\dfrac{1}{2}du\,]$

$=-\dfrac{1}{10}\left[e^u\right]_0^{-25}=-\dfrac{1}{10}\left(e^{-25}-1\right)=\dfrac{1}{10}\left(1-e^{-25}\right)$

6. $f_{\text{ave}}=\dfrac{1}{\dfrac{\pi}{4}-0}\displaystyle\int_0^{\pi/4}\sec\theta\tan\theta\,d\theta=\dfrac{4}{\pi}\left[\sec\theta\right]_0^{\pi/4}=\dfrac{4}{\pi}\left(\sqrt{2}-1\right)$

7.

$$h_{\text{ave}}=\frac{1}{\pi-0}\int_0^{\pi}\cos^4x\sin x\,dx=\frac{1}{\pi}\int_1^{-1}u^4(-du)\ [\,u=\cos x\,,\,du=-\sin x\,dx\,]$$

$$=\frac{1}{\pi}\int_{-1}^{1}u^4du=\frac{1}{\pi}\cdot 2\int_0^1u^4du=\frac{2}{\pi}\left[\frac{1}{5}u^5\right]_0^1=\frac{2}{5\pi}$$

8.

$$h_{\text{ave}}=\frac{1}{6-1}\int_1^6\frac{3}{(1+r)^2}\,dr=\frac{1}{5}\int_2^7 3u^{-2}du\ [\,u=1+r\,,\,du=dr\,]$$

$$=-\frac{3}{5}\left[u^{-1}\right]_2^7=-\frac{3}{5}\left(\frac{1}{7}-\frac{1}{2}\right)=\frac{3}{5}\left(\frac{1}{2}-\frac{1}{7}\right)=\frac{3}{5}\cdot\frac{5}{14}=\frac{3}{14}$$

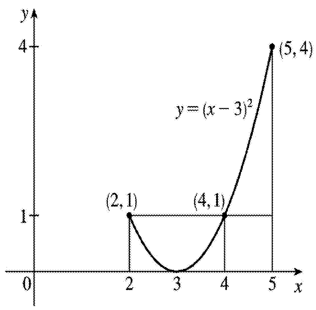
9.

(a) $$f_{\text{ave}}=\frac{1}{5-2}\int_2^5(x-3)^2\,dx=\frac{1}{3}\left[\frac{1}{3}(x-3)^3\right]_2^5$$

$$=\frac{1}{9}\left[2^3-(-1)^3\right]=\frac{1}{9}(8+1)=1$$

(b) $f(c)=f_{\text{ave}}\Leftrightarrow(c-3)^2=1\Leftrightarrow c-3=\pm 1$
$\Leftrightarrow c=2$ or 4

(c)



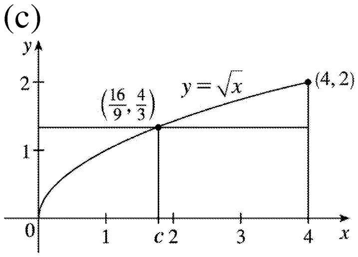
10.

(a) $$f_{\text{ave}}=\frac{1}{4-0}\int_0^4\sqrt{x}\,dx=\frac{1}{4}\left[\frac{2}{3}x^{3/2}\right]_0^4$$

$$=\frac{1}{6}\left[x^{3/2}\right]_0^4=\frac{1}{6}[8-0]=\frac{4}{3}$$

(b) $f(c)=f_{\text{ave}}\Leftrightarrow\sqrt{c}=\frac{4}{3}\Leftrightarrow c=\frac{16}{9}$

(c)



11.

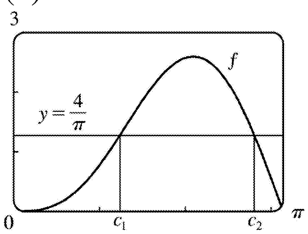
(a)

$$f_{\text{ave}}=\frac{1}{\pi-0}\int_0^{\pi}(2\sin x-\sin 2x)\,dx$$
$$=\frac{1}{\pi}\left[-2\cos x+\frac{1}{2}\cos 2x\right]_0^{\pi}$$
$$=\frac{1}{\pi}\left[\left(2+\frac{1}{2}\right)-\left(-2+\frac{1}{2}\right)\right]=\frac{4}{\pi}$$

(b) $f(c)=f_{\text{ave}}\Leftrightarrow 2\sin c-\sin 2c=\frac{4}{\pi}\Leftrightarrow$

$c_1\approx 1.238$ or $c_2\approx 2.808$

(c)



12.

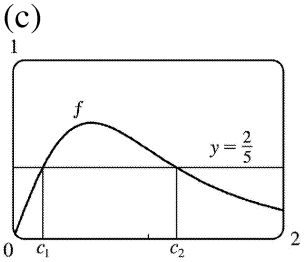
(a)

$$f_{\text{ave}}=\frac{1}{2-0}\int_0^{2}\frac{2x}{(1+x^2)^2}\,dx$$
$$=\frac{1}{2}\int_1^{5}\frac{1}{u^2}\,du\ [\,u=1+x^2,\ du=2x\,dx\,]$$
$$=\frac{1}{2}\left[-\frac{1}{u}\right]_1^5=-\frac{1}{2}\left(\frac{1}{5}-1\right)=\frac{2}{5}$$

(b) $f(c)=f_{\text{ave}}\Leftrightarrow\frac{2c}{(1+c^2)^2}=\frac{2}{5}\Leftrightarrow 5c=(1+c^2)^2$

$\Leftrightarrow c_1\approx 0.220$ or $c_2\approx 1.207$

(c)



13. f is continuous on $[1,3]$, so by the Mean Value Theorem for Integrals there exists a number c in $[1,3]$ such that $\int_1^3 f(x)\,dx=f(c)(3-1)\Rightarrow 8=2f(c)$; that is, there is a number c such that $f(c)=\frac{8}{2}=4$.

14. The requirement is that

$\frac{1}{b-0}\int_0^b f(x)dx=3$. The LHS of this equation is equal to

$\frac{1}{b}\int_0^b \left(2+6x-3x^2\right)dx=\frac{1}{b}\left[2x+3x^2-x^3\right]_0^b=2+3b-b^2$, so we solve the equation $2+3b-b^2=3\Leftrightarrow b^2-3b+1=0\Leftrightarrow$

$b=\frac{3\pm\sqrt{(-3)^2-4\cdot 1\cdot 1}}{2\cdot 1}=\frac{3\pm\sqrt{5}}{2}$. Both roots are valid since they are positive.

15.

$$f_{\text{ave}}=\frac{1}{50-20}\int_{20}^{50}f(x)dx\approx\frac{1}{30}M_3=\frac{1}{30}\cdot\frac{50-20}{3}[f(25)+f(35)+f(45)]$$
$$=\frac{1}{3}(38+29+48)=\frac{115}{3}=38\frac{1}{3}$$

16. **(a)** $v_{\text{ave}}=\frac{1}{12-0}\int_0^{12}v(t)dt=\frac{1}{12}I$. Use the Midpoint Rule with $n=3$ and $\Delta t=\frac{12-0}{3}=4$ to estimate I .

$I\approx M_3=4[v(2)+v(6)+v(10)]=4[21+50+66]=4(137)=548$. Thus, $v_{\text{ave}}\approx\frac{1}{12}(548)=45\frac{2}{3}$ km / h.

(b) Estimating from the graph, $v(t)=45\frac{2}{3}$ when $t\approx 5.2$ s.

17. Let $t=0$ and $t=12$ correspond to 9 A.M. and 9 P.M., respectively.

$$T_{\text{ave}}=\frac{1}{12-0}\int_0^{12}\left[50+14\sin\frac{1}{12}\pi t\right]dt=\frac{1}{12}\left[50t-14\cdot\frac{12}{\pi}\cos\frac{1}{12}\pi t\right]_0^{12}$$
$$=\frac{1}{12}\left[50\cdot 12+14\cdot\frac{12}{\pi}+14\cdot\frac{12}{\pi}\right]=\left(50+\frac{28}{\pi}\right)\text{ F}\approx 59^\circ\text{ F}$$

18.

$$T_{\text{ave}}=\frac{1}{30-0}\int_0^{30}\left(20+75e^{-t/50}\right)dt=\frac{1}{30}\left[20t-50\cdot 75e^{-t/50}\right]_0^{30}=\frac{1}{30}\left[\left(600-3750e^{-3/5}\right)-(-3750)\right]$$
$$=\frac{1}{30}\left(4350-3750e^{-3/5}\right)=145-125e^{-3/5}\approx 76.4^\circ\text{ C}$$

19. $\rho_{\text{ave}}=\frac{1}{8}\int_0^8\frac{12}{\sqrt{x+1}}dx=\frac{3}{2}\int_0^8(x+1)^{-1/2}dx=\left[3\sqrt{x+1}\right]_0^8=9-3=6$ kg / m

20. $s=\frac{1}{2}gt^2 \Rightarrow t=\sqrt{2s/g}$ [since $t\geq 0$]. Now $v=\frac{ds}{dt}=gt=g\sqrt{2s/g}=\sqrt{2gs} \Rightarrow v^2=2gs \Rightarrow s=\frac{v^2}{2g}$. We see that v can be regarded as a function of t or of s : $v=F(t)=gt$ and $v=G(s)=\sqrt{2gs}$. Note that $v_T=F(T)=gT$.

Displacement can be viewed as a function of t : $s=s(t)=\frac{1}{2}gt^2$; also $s(t)=\frac{v^2}{2g}=\frac{[F(t)]^2}{2g}$.

When $t=T$, these two formulas for $s(t)$ imply that

$\sqrt{2gs(T)}=F(T)=v_T=gT=2\left(\frac{1}{2}gT^2\right)/T=2s(T)/T\,(*)$

The average of the velocities with respect to time t during the interval $[0,T]$ is

$$v_{t\text{-ave}}=F_{\text{ave}}=\frac{1}{T-0}\int_0^T F(t)\,dt=\frac{1}{T}\,[s(T)-s(0)]\text{ [by FTC]}$$

$$=\frac{s(T)}{T}\text{ [since }s(0)=0\text{]}=\frac{1}{2}v_T\text{ [by (*)]}$$

But the average of the velocities with respect to displacement s during the corresponding displacement interval $[s(0),s(T)]=[0,s(T)]$ is

$$v_{s\text{-ave}}=G_{\text{ave}}=\frac{1}{s(T)-0}\int_0^{s(T)} G(s)\,ds=\frac{1}{s(T)}\int_0^{s(T)}\sqrt{2gs}\,ds=\frac{\sqrt{2g}}{s(T)}\int_0^{s(T)} s^{1/2}\,ds$$

$$=\frac{\sqrt{2g}}{s(T)}\cdot\frac{2}{3}\left[s^{3/2}\right]_0^{s(T)}=\frac{2}{3}\cdot\frac{\sqrt{2g}}{s(T)}\cdot[s(T)]^{3/2}=\frac{2}{3}\sqrt{2gs(T)}=\frac{2}{3}v_T\text{[by(*)]}$$

21.

$$V_{\text{ave}}=\frac{1}{5}\int_0^5 V(t)\,dt=\frac{1}{5}\int_0^5\frac{5}{4\pi}\left[1-\cos\left(\frac{2}{5}\pi t\right)\right]dt=\frac{1}{4\pi}\int_0^5\left[1-\cos\left(\frac{2}{5}\pi t\right)\right]dt$$

$$=\frac{1}{4\pi}\left[t-\frac{5}{2\pi}\sin\left(\frac{2}{5}\pi t\right)\right]_0^5=\frac{1}{4\pi}\,[(5-0)-0]=\frac{5}{4\pi}\approx 0.4\text{ L}$$

22. $v_{\text{ave}}=\frac{1}{R-0}\int_0^R v(r)\,dr=\frac{1}{R}\int_0^R\frac{P}{4\eta l}\left(R^2-r^2\right)dr=\frac{P}{4\eta lR}\left[R^2r-\frac{1}{3}r^3\right]_0^R=\frac{P}{4\eta lR}\left(\frac{2}{3}\right)R^3=\frac{PR^2}{6\eta l}$.

Since $v(r)$ is decreasing on $(0,R]$, $v_{\max}=v(0)=\frac{PR^2}{4\eta l}$. Thus, $v_{\text{ave}}=\frac{2}{3}v_{\max}$.

23. Let

$F(x)=\int_a^x f(t)\,dt$ for x in $[a,b]$. Then F is continuous on $[a,b]$ and differentiable on (a,b) , so by the Mean Value Theorem there is a number c in (a,b) such that $F(b)-F(a)=F'(c)(b-a)$. But $F'(x)=f(x)$ by the Fundamental Theorem of Calculus. Therefore, $\int_a^b f(t)\,dt-0=f(c)(b-a)$.

24.

$$f_{\text{ave}}[a,b] = \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{b-a}\int_a^c f(x)\,dx + \frac{1}{b-a}\int_c^b f(x)\,dx$$

$$= \frac{c-a}{b-a}\left[\frac{1}{c-a}\int_a^c f(x)\,dx\right] + \frac{b-c}{b-a}\left[\frac{1}{b-c}\int_c^b f(x)\,dx\right] = \frac{c-a}{b-a}\, f_{\text{ave}}[a,c] + \frac{b-c}{b-a}\, f_{\text{ave}}[c,b]$$

1. Let $u=\ln x$, $dv=x\,dx \Rightarrow du=dx/x$, $v=\frac{1}{2}x^2$. Then by Equation 2, $\int u\,dv=uv-\int v\,du$,

$$\int x\ln x\,dx = \frac{1}{2}x^2\ln x-\int \frac{1}{2}x^2(dx/x)=\frac{1}{2}x^2\ln x-\frac{1}{2}\int x\,dx=\frac{1}{2}x^2\ln x-\frac{1}{2}\cdot\frac{1}{2}x^2+C$$
$$= \frac{1}{2}x^2\ln x-\frac{1}{4}x^2+C$$

2. Let $u=\theta$, $dv=\sec^2\theta\,d\theta \Rightarrow du=d\theta$, $v=\tan\theta$. Then
$\int \theta\sec^2\theta\,d\theta=\theta\tan\theta-\int\tan\theta\,d\theta=\theta\tan\theta-\ln|\sec\theta|+C$.

3. Let $u=x$, $dv=\cos 5x\,dx \Rightarrow du=dx$, $v=\frac{1}{5}\sin 5x$. Then by Equation 2,
$\int x\cos 5x\,dx=\frac{1}{5}x\sin 5x-\int\frac{1}{5}\sin 5x\,dx=\frac{1}{5}x\sin 5x+\frac{1}{25}\cos 5x+C$.

4. Let $u=x$, $dv=e^{-x}dx \Rightarrow du=dx$, $v=-e^{-x}$. Then $\int xe^{-x}dx=-xe^{-x}+\int e^{-x}dx=-xe^{-x}-e^{-x}+C$.

5. Let $u=r$, $dv=e^{r/2}dr \Rightarrow du=dr$, $v=2e^{r/2}$. Then $\int re^{r/2}dr=2re^{r/2}-\int 2e^{r/2}dr=2re^{r/2}-4e^{r/2}+C$.

6. Let $u=t$, $dv=\sin 2t\,dt \Rightarrow du=dt$, $v=-\frac{1}{2}\cos 2t$. Then $\int t\sin 2t\,dt=-\frac{1}{2}t\cos 2t+\frac{1}{2}\int\cos 2t$
$dt=-\frac{1}{2}t\cos 2t+\frac{1}{4}\sin 2t+C$.

7. Let $u=x^2$, $dv=\sin\pi x\,dx \Rightarrow du=2x\,dx$ and $v=-\frac{1}{\pi}\cos\pi x$. Then
$I=\int x^2\sin\pi x\,dx=-\frac{1}{\pi}x^2\cos\pi x+\frac{2}{\pi}\int x\cos\pi x\,dx$ (*).
Next let $U=x$, $dV=\cos\pi x\,dx \Rightarrow dU=dx$, $V=\frac{1}{\pi}\sin\pi x$, so
$\int x\cos\pi x\,dx=\frac{1}{\pi}x\sin\pi x-\frac{1}{\pi}\int\sin\pi x\,dx=\frac{1}{\pi}x\sin\pi x+\frac{1}{\pi^2}\cos\pi x+C_1$. Substituting for $\int x\cos\pi x\,dx$ in (*), we get
$I=-\frac{1}{\pi}x^2\cos\pi x+\frac{2}{\pi}\left(\frac{1}{\pi}x\sin\pi x+\frac{1}{\pi^2}\cos\pi x+C_1\right)=-\frac{1}{\pi}x^2\cos\pi x+\frac{2}{\pi^2}x\sin\pi x+\frac{2}{\pi^3}\cos\pi x+C$,
where $C=\frac{2}{\pi}C_1$.

8. Let $u=x^2$, $dv=\cos mx\,dx \Rightarrow du=2x\,dx$, $v=\frac{1}{m}\sin mx$. Then

$I=\int x^2\cos mx\,dx=\frac{1}{m}x^2\sin mx-\frac{2}{m}\int x\sin mx\,dx$ (*). Next let $U=x$, $dV=\sin mx\,dx \Rightarrow dU=dx$,

$V=-\frac{1}{m}\cos mx$, so $\int x\sin mx\,dx=-\frac{1}{m}x\cos mx+\frac{1}{m}\int\cos mx\,dx=-\frac{1}{m}x\cos mx+\frac{1}{m^2}\sin mx+C_1$.

Substituting for $\int x\sin mx\,dx$ in (*), we get

$I=\frac{1}{m}x^2\sin mx-\frac{2}{m}\left(-\frac{1}{m}x\cos mx+\frac{1}{m^2}\sin mx+C_1\right)=\frac{1}{m}x^2\sin mx+\frac{2}{m^2}x\cos mx-\frac{2}{m^3}\sin mx+C$,

where $C=-\frac{2}{m}C_1$.

9. Let $u=\ln(2x+1)$, $dv=dx \Rightarrow du=\frac{2}{2x+1}dx$, $v=x$. Then

$$\begin{aligned}\int\ln(2x+1)dx &= x\ln(2x+1)-\int\frac{2x}{2x+1}dx=x\ln(2x+1)-\int\frac{(2x+1)-1}{2x+1}dx\\ &= x\ln(2x+1)-\int\left(1-\frac{1}{2x+1}\right)dx=x\ln(2x+1)-x+\frac{1}{2}\ln(2x+1)+C\\ &= \frac{1}{2}(2x+1)\ln(2x+1)-x+C\end{aligned}$$

10. Let $u=\sin^{-1}x$, $dv=dx \Rightarrow du=\frac{dx}{\sqrt{1-x^2}}$, $v=x$. Then $\int\sin^{-1}x\,dx=x\sin^{-1}x-\int\frac{x}{\sqrt{1-x^2}}dx$. Setting $t=1-x^2$, we get $dt=-2x\,dx$, so $-\int\frac{x\,dx}{\sqrt{1-x^2}}=-\int t^{-1/2}\left(-\frac{1}{2}dt\right)=\frac{1}{2}\left(2t^{1/2}\right)+C=t^{1/2}+C=\sqrt{1-x^2}+C$.

Hence, $\int\sin^{-1}x\,dx=x\sin^{-1}x+\sqrt{1-x^2}+C$.

11. Let $u=\arctan 4t$, $dv=dt \Rightarrow du=\frac{4}{1+(4t)^2}dt=\frac{4}{1+16t^2}dt$, $v=t$. Then

$$\begin{aligned}\int\arctan 4t\,dt &= t\arctan 4t-\int\frac{4t}{1+16t^2}dt=t\arctan 4t-\frac{1}{8}\int\frac{32t}{1+16t^2}dt\\ &= t\arctan 4t-\frac{1}{8}\ln(1+16t^2)+C\end{aligned}$$

12. Let $u=\ln p$, $dv=p^5dp \Rightarrow$

$du=\frac{1}{p}\,dp$, $v=\frac{1}{6}p^6$. Then $\int p^5\ln p\,dp=\frac{1}{6}p^6\ln p-\frac{1}{6}\int p^5\,dp=\frac{1}{6}p^6\ln p-\frac{1}{36}p^6+C$.

13. First let $u=(\ln x)^2$, $dv=dx\Rightarrow du=2\ln x\cdot\frac{1}{x}\,dx$, $v=x$. Then by Equation 2,
$I=\int(\ln x)^2dx=x(\ln x)^2-2\int x\ln x\cdot\frac{1}{x}\,dx=x(\ln x)^2-2\int\ln x\,dx$. Next let $U=\ln x$, $dV=dx\Rightarrow dU=1/x\,dx$, $V=x$ to get $\int\ln x\,dx=x\ln x-\int x\cdot(1/x)\,dx=x\ln x-\int dx=x\ln x-x+C_1$. Thus,
$I=x(\ln x)^2-2\left(x\ln x-x+C_1\right)=x(\ln x)^2-2x\ln x+2x+C$, where $C=-2C_1$.

14. Let $u=t^3$, $dv=e^t\,dt\Rightarrow du=3t^2\,dt$, $v=e^t$. Then $I=\int t^3e^t\,dt=t^3e^t-\int 3t^2e^t\,dt$. Integrate by parts twice more with $dv=e^t\,dt$.

$$\begin{aligned} I &= t^3e^t-\left(3t^2e^t-\int 6te^t\,dt\right)=t^3e^t-3t^2e^t+6te^t-\int 6e^t\,dt \\ &= t^3e^t-3t^2e^t+6te^t-6e^t+C=\left(t^3-3t^2+6t-6\right)e^t+C \end{aligned}$$

More generally, if $p(t)$ is a polynomial of degree n in t , then repeated integration by parts shows that
$\int p(t)e^t\,dt=\left[p(t)-p'(t)+p''(t)-p'''(t)+\cdots+(-1)^np^{(n)}(t)\right]e^t+C$.

15. First let $u=\sin 3\theta$, $dv=e^{2\theta}\,d\theta\Rightarrow du=3\cos 3\theta\,d\theta$, $v=\frac{1}{2}e^{2\theta}$. Then
$I=\int e^{2\theta}\sin 3\theta\,d\theta=\frac{1}{2}e^{2\theta}\sin 3\theta-\frac{3}{2}\int e^{2\theta}\cos 3\theta\,d\theta$. Next let $U=\cos 3\theta$, $dV=e^{2\theta}\,d\theta$
$\Rightarrow dU=-3\sin 3\theta\,d\theta$, $V=\frac{1}{2}e^{2\theta}$ to get
$\int e^{2\theta}\cos 3\theta\,d\theta=\frac{1}{2}e^{2\theta}\cos 3\theta+\frac{3}{2}\int e^{2\theta}\sin 3\theta\,d\theta$. Substituting in the previous formula gives
$I=\frac{1}{2}e^{2\theta}\sin 3\theta-\frac{3}{4}e^{2\theta}\cos 3\theta-\frac{9}{4}\int e^{2\theta}\sin 3\theta\,d\theta=\frac{1}{2}e^{2\theta}\sin 3\theta-\frac{3}{4}e^{2\theta}\cos 3\theta-\frac{9}{4}I\Rightarrow$
$\frac{13}{4}I=\frac{1}{2}e^{2\theta}\sin 3\theta-\frac{3}{4}e^{2\theta}\cos 3\theta+C_1$. Hence, $I=\frac{1}{13}e^{2\theta}(2\sin 3\theta-3\cos 3\theta)+C$, where $C=\frac{4}{13}C_1$.

16. First let $u=e^{-\theta}$, $dv=\cos 2\theta\,d\theta\Rightarrow du=-e^{-\theta}\,d\theta$, $v=\frac{1}{2}\sin 2\theta$. Then
$I=\int e^{-\theta}\cos 2\theta\,d\theta=\frac{1}{2}e^{-\theta}\sin 2\theta-\int\frac{1}{2}\sin 2\theta\left(-e^{-\theta}\,d\theta\right)=\frac{1}{2}e^{-\theta}\sin 2\theta+\frac{1}{2}\int e^{-\theta}\sin 2\theta\,d\theta$.
Next let $U=e^{-\theta}$, $dV=\sin 2\theta\,d\theta\Rightarrow dU=-e^{-\theta}\,d\theta$, $V=-\frac{1}{2}\cos 2\theta$, so

$\int e^{-\theta}\sin 2\theta\, d\theta = -\frac{1}{2} e^{-\theta}\cos 2\theta - \int \left(-\frac{1}{2}\right)\cos 2\theta \left(-e^{-\theta}\, d\theta\right) = -\frac{1}{2} e^{-\theta}\cos 2\theta - \frac{1}{2}\int e^{-\theta}\cos 2\theta\, d\theta$.

So $I = \frac{1}{2} e^{-\theta}\sin 2\theta + \frac{1}{2}\left[\left(-\frac{1}{2} e^{-\theta}\cos 2\theta\right) - \frac{1}{2} I\right] = \frac{1}{2} e^{-\theta}\sin 2\theta - \frac{1}{4} e^{-\theta}\cos 2\theta - \frac{1}{4} I \Rightarrow$

$\frac{5}{4} I = \frac{1}{2} e^{-\theta}\sin 2\theta - \frac{1}{4} e^{-\theta}\cos 2\theta + C_1 \Rightarrow$

$I = \frac{4}{5}\left(\frac{1}{2} e^{-\theta}\sin 2\theta - \frac{1}{4} e^{-\theta}\cos 2\theta + C_1\right) = \frac{2}{5} e^{-\theta}\sin 2\theta - \frac{1}{5} e^{-\theta}\cos 2\theta + C$.

17. Let $u=y$, $dv=\sinh y\,dy \Rightarrow du=dy$, $v=\cosh y$. Then
$\int y\sinh y\,dy = y\cosh y - \int \cosh y\,dy = y\cosh y - \sinh y + C$.

18. Let $u=y$, $dv=\cosh ay\,dy \Rightarrow du=dy$, $v=\frac{\sinh ay}{a}$. Then
$\int y\cosh ay\,dy = \frac{y\sinh ay}{a} - \frac{1}{a}\int \sinh ay\,dy = \frac{y\sinh ay}{a} - \frac{\cosh ay}{a^2} + C$.

19. Let $u=t$, $dv=\sin 3t\,dt \Rightarrow du=dt$, $v=-\frac{1}{3}\cos 3t$. Then
$\int_0^{\pi} t\sin 3t\,dt = \left[-\frac{1}{3} t\cos 3t\right]_0^{\pi} + \frac{1}{3}\int_0^{\pi}\cos 3t\,dt = \left(\frac{1}{3}\pi - 0\right) + \frac{1}{9}\left[\sin 3t\right]_0^{\pi} = \frac{\pi}{3}$.

20. First let $u=x^2+1$, $dv=e^{-x}dx \Rightarrow du=2x\,dx$, $v=-e^{-x}$. By (6),
$\int_0^1 \left(x^2+1\right) e^{-x}dx = \left[-\left(x^2+1\right) e^{-x}\right]_0^1 + \int_0^1 2xe^{-x}dx = -2e^{-1}+1+2\int_0^1 xe^{-x}dx$. Next let $U=x$, $dV=e^{-x}dx \Rightarrow dU=dx$, $V=-e^{-x}$. By (6) again, $\int_0^1 xe^{-x}dx = \left[-xe^{-x}\right]_0^1 + \int_0^1 e^{-x}dx = -e^{-1} + \left[-e^{-x}\right]_0^1 = -e^{-1} - e^{-1} + 1 = -2e^{-1}+1$. So
$\int_0^1 \left(x^2+1\right) e^{-x}dx = -2e^{-1}+1+2\left(-2e^{-1}+1\right) = -2e^{-1}+1-4e^{-1}+2 = -6e^{-1}+3$.

21. Let $u=\ln x$, $dv=x^{-2}dx \Rightarrow du=\frac{1}{x}\,dx$, $v=-x^{-1}$. By (6),

$$\int_1^2 \frac{\ln x}{x^2}\,dx = \left[-\frac{\ln x}{x}\right]_1^2 + \int_1^2 x^{-2}dx = -\frac{1}{2}\ln 2 + \ln 1 + \left[-\frac{1}{x}\right]_1^2 = -\frac{1}{2}\ln 2 + 0 - \frac{1}{2} + 1 = \frac{1}{2} - \frac{1}{2}\ln 2 .$$

22. Let $u=\ln t$, $dv=\sqrt{t}\,dt \Rightarrow du=dt/t$, $v=\frac{2}{3}t^{3/2}$. By Formula 6,

$$\int_1^4 \sqrt{t}\ln t\,dt = \left[\frac{2}{3}t^{3/2}\ln t\right]_1^4 - \frac{2}{3}\int_1^4 \sqrt{t}\,dt = \frac{2}{3}\cdot 8\cdot \ln 4 - 0 - \left[\frac{2}{3}\cdot\frac{2}{3}t^{3/2}\right]_1^4 = \frac{16}{3}\ln 4 - \frac{4}{9}(8-1) = \frac{16}{3}\ln 4 - \frac{28}{9}$$
.

23. Let $u=y$, $dv=\dfrac{dy}{e^{2y}}=e^{-2y}dy \Rightarrow du=dy$, $v=-\dfrac{1}{2}e^{-2y}$. Then

$$\int_0^1 \frac{y}{e^{2y}}dy = \left[-\frac{1}{2}ye^{-2y}\right]_0^1 + \frac{1}{2}\int_0^1 e^{-2y}dy = \left(-\frac{1}{2}e^{-2}+0\right) - \frac{1}{4}\left[e^{-2y}\right]_0^1 = -\frac{1}{2}e^{-2} - \frac{1}{4}e^{-2} + \frac{1}{4} = \frac{1}{4} - \frac{3}{4}e^{-2}.$$

24. Let $u=x$, $dv=\csc^2 x\,dx \Rightarrow du=dx$, $v=-\cot x$. Then

$$\int_{\pi/4}^{\pi/2} x\csc^2 x\,dx = \left[-x\cot x\right]_{\pi/4}^{\pi/2} + \int_{\pi/4}^{\pi/2}\cot x\,dx = -\frac{\pi}{2}\cdot 0 + \frac{\pi}{4}\cdot 1 + \left[\ln|\sin x|\right]_{\pi/4}^{\pi/2} \qquad \text{[see Exercise 5.5.]}$$

$$= \frac{\pi}{4} + \ln 1 - \ln\frac{1}{\sqrt{2}} = \frac{\pi}{4} + 0 - \ln 2^{-1/2} = \frac{\pi}{4} + \frac{1}{2}\ln 2$$

25. Let $u=\cos^{-1}x$, $dv=dx \Rightarrow du=-\dfrac{dx}{\sqrt{1-x^2}}$, $v=x$. Then

$$I=\int_0^{1/2}\cos^{-1}x\,dx = \left[x\cos^{-1}x\right]_0^{1/2} + \int_0^{1/2}\frac{x\,dx}{\sqrt{1-x^2}} = \frac{1}{2}\cdot\frac{\pi}{3} + \int_1^{3/4} t^{-1/2}\left[-\frac{1}{2}\right]dt \text{ , where } t=1-x^2 \Rightarrow dt=-2x\,dx .$$

Thus, $I=\dfrac{\pi}{6}+\dfrac{1}{2}\displaystyle\int_{3/4}^{1} t^{-1/2}dt = \dfrac{\pi}{6} + \left[\sqrt{t}\right]_{3/4}^{1} = \dfrac{\pi}{6} + 1 - \dfrac{\sqrt{3}}{2} = \dfrac{1}{6}\left(\pi + 6 - 3\sqrt{3}\right)$.

26. Let $u=x$, $dv=5^x dx \Rightarrow du=dx$, $v=\left(5^x/\ln 5\right)$. Then

$$\int_0^1 x5^x dx = \left[\frac{x5^x}{\ln 5}\right]_0^1 - \int_0^1 \frac{5^x}{\ln 5}dx = \frac{5}{\ln 5} - 0 - \frac{1}{\ln 5}\left[\frac{5^x}{\ln 5}\right]_0^1 = \frac{5}{\ln 5} - \frac{5}{(\ln 5)^2} + \frac{1}{(\ln 5)^2}$$

$$= \frac{5}{\ln 5} - \frac{4}{(\ln 5)^2}$$

27. Let $u=\ln(\sin x)$, $dv=\cos x\,dx \Rightarrow du=\dfrac{\cos x}{\sin x}dx$, $v=\sin x$. Then

$I=\int\cos x\ln(\sin x)dx = \sin x\ln(\sin x) - \int\cos x\,dx = \sin x\ln(\sin x) - \sin x + C$.

Another method: Substitute $t=\sin x$, so $dt=\cos x\,dx$. Then $I=\int \ln t\,dt=t\ln t-t+C$ (see Example 2) and so $I=\sin x(\ln \sin x-1)+C$.

28. Let $u=\arctan(1/x)$, $dv=dx \Rightarrow du=\frac{1}{1+(1/x)^2}\cdot\frac{-1}{x^2}\,dx=\frac{-dx}{x^2+1}$, $v=x$. Then

$$\int_1^{\sqrt{3}} \arctan(1/x)\,dx = \left[x\arctan\left(\frac{1}{x}\right)\right]_1^{\sqrt{3}}+\int_1^{\sqrt{3}}\frac{x}{dx}\,x^2+1=\sqrt{3}\,\frac{\pi}{6}-1\cdot\frac{\pi}{4}+\frac{1}{2}\left[\ln(x^2+1)\right]_1^{\sqrt{3}}$$

$$=\frac{\pi\sqrt{3}}{6}-\frac{\pi}{4}+\frac{1}{2}(\ln 4-\ln 2)=\frac{\pi\sqrt{3}}{6}-\frac{\pi}{2}+\frac{1}{2}\ln\frac{4}{2}=\frac{\pi\sqrt{3}}{6}-\frac{\pi}{2}+\frac{1}{2}\ln 2$$

29. Let $w=\ln x \Rightarrow dw=dx/x$. Then $x=e^w$ and $dx=e^w dw$, so

$$\int \cos(\ln x)\,dx = \int e^w\cos w\,dw=\frac{1}{2}e^w(\sin w+\cos w)+C \qquad \text{[by the method of Example 4]}$$

$$= \frac{1}{2}x[\sin(\ln x)+\cos(\ln x)]+C$$

30. Let $u=r^2$, $dv=\frac{r}{\sqrt{4+r^2}}\,dr \Rightarrow du=2r\,dr$, $v=\sqrt{4+r^2}$. By (6),

$$\int_0^1 \frac{r^3}{\sqrt{4+r^2}}\,dr = \left[r^2\sqrt{4+r^2}\right]_0^1-2\int_0^1 r\sqrt{4+r^2}\,dr=\sqrt{5}-\frac{2}{3}\left[\left(4+r^2\right)^{3/2}\right]_0^1$$

$$= \sqrt{5}-\frac{2}{3}(5)^{3/2}+\frac{2}{3}(8)=\sqrt{5}\left(1-\frac{10}{3}\right)+\frac{16}{3}=\frac{16}{3}-\frac{7}{3}\sqrt{5}$$

31. Let $u=(\ln x)^2$, $dv=x^4dx \Rightarrow du=2\frac{\ln x}{x}\,dx$, $v=\frac{x^5}{5}$. By (6),

$$\int_1^2 x^4(\ln x)^2dx=\left[\frac{x^5}{5}(\ln x)^2\right]_1^2-2\int_1^2\frac{x^4}{5}\ln x\,dx=\frac{32}{5}(\ln 2)^2-0-2\int_1^2\frac{x^4}{5}\ln x\,dx .$$

Let $U=\ln x$, $dV=\frac{x^4}{5}\,dx \Rightarrow dU=\frac{1}{x}\,dx$, $V=\frac{x^5}{25}$.

Then $\int_1^2\frac{x^4}{5}\ln x\,dx=\left[\frac{x^5}{25}\ln x\right]_1^2-\int_1^2\frac{x^4}{25}\,dx=\frac{32}{25}\ln 2-0-\left[\frac{x^5}{125}\right]_1^2=\frac{32}{25}\ln 2-\left(\frac{32}{125}-\frac{1}{125}\right)$.

So $\int_1^2 x^4(\ln x)^2\,dx=\frac{32}{5}(\ln 2)^2-2\left(\frac{32}{25}\ln 2-\frac{31}{125}\right)=\frac{32}{5}(\ln 2)^2-\frac{64}{25}\ln 2+\frac{62}{125}$.

32. Let $u=\sin(t-s)$, $dv=e^s ds\Rightarrow du=-\cos(t-s)ds$, $v=e^s$. Then $I=\int_0^t e^s\sin(t-s)ds=\left[e^s\sin(t-s)\right]_0^t+\int_0^t e^s\cos(t-s)ds=e^t\sin 0-e^0\sin t+I_1$. For I_1 , let $U=\cos(t-s)$, $dV=e^s ds$ $\Rightarrow dU=\sin(t-s)ds$, $V=e^s$. So $I_1=\left[e^s\cos(t-s)\right]_0^t-\int_0^t e^s\sin(t-s)ds=e^t\cos 0-e^0\cos t-I$. Thus, $I=-\sin t+e^t-\cos t-I\Rightarrow 2I=e^t-\cos t-\sin t\Rightarrow I=\frac{1}{2}\left(e^t-\cos t-\sin t\right)$.

33. Let $w=\sqrt{x}$, so that $x=w^2$ and $dx=2w\,dw$. Thus, $\int\sin\sqrt{x}\,dx=\int 2w\sin w\,dw$. Now use parts with $u=2w$, $dv=\sin w\,dw$, $du=2\,dw$, $v=-\cos w$ to get

$$\begin{aligned}\int 2w\sin w\,dw &= -2w\cos w+\int 2\cos w\,dw=-2w\cos w+2\sin w+C\\ &= -2\sqrt{x}\cos\sqrt{x}+2\sin\sqrt{x}+C=2\left(\sin\sqrt{x}-\sqrt{x}\cos\sqrt{x}\right)+C\end{aligned}$$

34. Let $w=\sqrt{x}$, so that $x=w^2$ and $dx=2w\,dw$. Thus, $\int_1^4 e^{\sqrt{x}}\,dx=\int_1^2 e^w 2w\,dw$. Now use parts with $u=2w$, $dv=e^w dw$, $du=2\,dw$, $v=e^w$ to get $\int_1^2 e^w 2w\,dw=\left[2we^w\right]_1^2-2\int_1^2 e^w dw=4e^2-2e-2\left(e^2-e\right)=2e^2$.

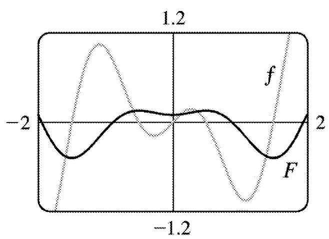
35. Let $x=\theta^2$, so that $dx=2\theta\,d\theta$. Thus, $\int_{\sqrt{\pi/2}}^{\sqrt{\pi}}\theta^3\cos\left(\theta^2\right)d\theta=\int_{\sqrt{\pi/2}}^{\sqrt{\pi}}\theta^2\cos\left(\theta^2\right)\cdot\frac{1}{2}(2\theta\,d\theta)=\frac{1}{2}\int_{\pi/2}^{\pi}x\cos x\}\{dx\}$. Now use parts with $u=x$, $dv=\cos x\,dx$, $du=dx$, $v=\sin x$ to get

$$\begin{aligned}\frac{1}{2}\int_{\pi/2}^{\pi}x\cos x\,dx &= \frac{1}{2}\left(\left[x\sin x\right]_{\pi/2}^{\pi}-\int_{\pi/2}^{\pi}\sin x\,dx\right)=\frac{1}{2}\left[x\sin x+\cos x\right]_{\pi/2}^{\pi}\\ &= \frac{1}{2}(\pi\sin\pi+\cos\pi)-\frac{1}{2}\left(\frac{\pi}{2}\sin\frac{\pi}{2}+\cos\frac{\pi}{2}\right)=\frac{1}{2}(\pi\cdot 0-1)-\frac{1}{2}\left(\frac{\pi}{2}\cdot 1+0\right)=-\frac{1}{2}-\frac{\pi}{4}\end{aligned}$$

36.

$$\begin{aligned}\int x^5e^{x^2}dx &=\int\left(x^2\right)^2e^{x^2}x\,dx=\int t^2e^t\frac{1}{2}\,dt\ \text{[where } t=x^2\Rightarrow\tfrac{1}{2}\,dt=x\,dx\text{]}\\ &=\frac{1}{2}\left(t^2-2t+2\right)e^t+C\ \text{[by Example 3]}=\frac{1}{2}\left(x^4-2x^2+2\right)e^{x^2}+C\end{aligned}$$

37. Let $u=x$, $dv=\cos \pi x dx \Rightarrow du=dx$, $v=(\sin \pi x)/\pi$. Then $\int x\cos \pi x dx = x\cdot \frac{\sin \pi x}{\pi} - \int \frac{\sin \pi x}{\pi} dx = \frac{x\sin \pi x}{\pi} + \frac{\cos \pi x}{\pi^2} + C$. We see from the graph that this is reasonable, since F has extreme values where f is 0 .

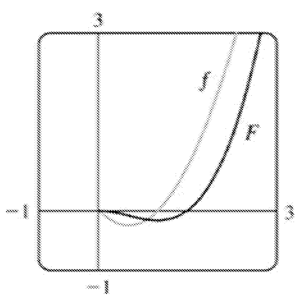


38. Let $u=\ln x$, $dv=x^{3/2}dx \Rightarrow du=\frac{1}{x}dx$, $v=\frac{2}{5}x^{5/2}$. Then

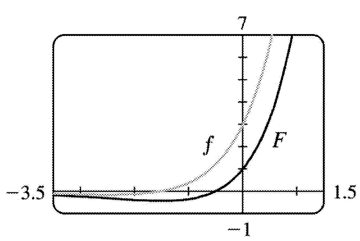
$$\int x^{3/2}\ln x dx = \frac{2}{5}x^{5/2}\ln x - \frac{2}{5}\int x^{3/2}dx = \frac{2}{5}x^{5/2}\ln x - \left(\frac{2}{5}\right)^2 x^{5/2} + C$$

$$= \frac{2}{5}x^{5/2}\ln x - \frac{4}{25}x^{5/2} + C.$$

We see from the graph that this is reasonable, since F has a minimum where f changes from negative to positive.



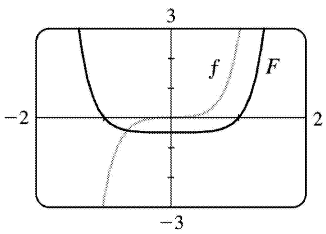
39. Let $u=2x+3$, $dv=e^x dx \Rightarrow du=2dx$, $v=e^x$. Then $\int (2x+3)e^x dx = (2x+3)e^x - 2\int e^x dx = (2x+3)e^x - 2e^x + C = (2x+1)e^x + C$. We see from the graph that this is reasonable, since F has a minimum where f changes from negative to positive.



40. $\int x^3 e^{x^2}\,dx = \int x^2 \cdot xe^{x^2}\,dx = I$.

Let $u=x^2$, $dv=xe^{x^2}\,dx \Rightarrow du=2x\,dx$, $v=\frac{1}{2}e^{x^2}$. Then

$I=\frac{1}{2}x^2e^{x^2}-\int xe^{x^2}\,dx=\frac{1}{2}x^2e^{x^2}-\frac{1}{2}e^{x^2}+C=\frac{1}{2}e^{x^2}\left(x^2-1\right)+C$. We see from the graph that this is reasonable, since F has a minimum where f changes from negative to positive.



41. **(a)** Take $n=2$ in Example 6 to get $\int \sin^2 x\,dx=-\frac{1}{2}\cos x\sin x+\frac{1}{2}\int 1\,dx=\frac{x}{2}-\frac{\sin 2x}{4}+C$.

(b) $\int \sin^4 x\,dx=-\frac{1}{4}\cos x\sin^3 x+\frac{3}{4}\int \sin^2 x\,dx=-\frac{1}{4}\cos x\sin^3 x+\frac{3}{8}x-\frac{3}{16}\sin 2x+C$.

42. **(a)** Let $u=\cos^{n-1}x$, $dv=\cos x\,dx \Rightarrow du=-(n-1)\cos^{n-2}x\sin x\,dx$, $v=\sin x$ in (2):

$$\begin{aligned}\int \cos^n x\,dx &= \cos^{n-1}x\sin x+(n-1)\int \cos^{n-2}x\sin^2 x\,dx\\ &= \cos^{n-1}x\sin x+(n-1)\int \cos^{n-2}x\left(1-\cos^2 x\right)dx\\ &= \cos^{n-1}x\sin x+(n-1)\int \cos^{n-2}x\,dx-(n-1)\int \cos^n x\,dx\end{aligned}$$

Rearranging terms gives $n\int \cos^n x\,dx=\cos^{n-1}x\sin x+(n-1)\int \cos^{n-2}x\,dx$ or

$$\int \cos^n x\,dx=\frac{1}{n}\cos^{n-1}x\sin x+\frac{n-1}{n}\int \cos^{n-2}x\,dx$$

(b) Take $n=2$ in part (a) to get $\int \cos^2 x\,dx=\frac{1}{2}\cos x\sin x+\frac{1}{2}\int 1\,dx=\frac{x}{2}+\frac{\sin 2x}{4}+C$.

(c) $\int \cos^4 x\,dx=\frac{1}{4}\cos^3 x\sin x+\frac{3}{4}\int \cos^2 x\,dx=\frac{1}{4}\cos^3 x\sin x+\frac{3}{8}x+\frac{3}{16}\sin 2x+C$

43. **(a)** From Example 6, $\int \sin^n x\,dx=-\frac{1}{n}\cos x\sin^{n-1}x+\frac{n-1}{n}\int \sin^{n-2}x\,dx$. Using (6),

$$\int_0^{\pi/2}\sin^n x\,dx = \left[-\frac{\cos x\sin^{n-1}x}{n}\right]_0^{\pi/2} + \frac{n-1}{n}\int_0^{\pi/2}\sin^{n-2}x\,dx$$
$$= (0-0)+\frac{n-1}{n}\int_0^{\pi/2}\sin^{n-2}x\,dx=\frac{n-1}{n}\int_0^{\pi/2}\sin^{n-2}x\,dx$$

(b) Using $n=3$ in part (a), we have $\int_0^{\pi/2}\sin^3 x\,dx=\frac{2}{3}\int_0^{\pi/2}\sin x\,dx=\left[-\frac{2}{3}\cos x\right]_0^{\pi/2}=\frac{2}{3}$.

Using $n=5$ in part (a), we have $\int_0^{\pi/2}\sin^5 x\,dx=\frac{4}{5}\int_0^{\pi/2}\sin^3 x\,dx=\frac{4}{5}\cdot\frac{2}{3}=\frac{8}{15}$.

(c) The formula holds for $n=1$ (that is, $2n+1=3$) by (b). Assume it holds for some $k\geq 1$. Then $\int_0^{\pi/2}\sin^{2k+1}x\,dx=\frac{2\cdot 4\cdot 6\cdot\cdots\cdot(2k)}{3\cdot 5\cdot 7\cdot\cdots\cdot(2k+1)}$. By Example 6,

$$\int_0^{\pi/2}\sin^{2k+3}x\,dx = \frac{2k+2}{2k+3}\int_0^{\pi/2}\sin^{2k+1}x\,dx=\frac{2k+2}{2k+3}\cdot\frac{2\cdot 4\cdot 6\cdot\cdots\cdot(2k)}{3\cdot 5\cdot 7\cdot\cdots\cdot(2k+1)}$$
$$= \frac{2\cdot 4\cdot 6\cdot\cdots\cdot(2k)[2(k+1)]}{3\cdot 5\cdot 7\cdot\cdots\cdot(2k+1)[2(k+1)+1]},$$

so the formula holds for $n=k+1$. By induction, the formula holds for all $n\geq 1$.

44. Using Exercise 43 (a), we see that the formula holds for $n=1$, because $\int_0^{\pi/2}\sin^2 x\,dx=\frac{1}{2}\int_0^{\pi/2}1\,dx=\frac{1}{2}\,[x]_0^{\pi/2}=\frac{1}{2}\cdot\frac{\pi}{2}$.

Now assume it holds for some $k\geq 1$. Then $\int_0^{\pi/2}\sin^{2k}x\,dx=\frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2k-1)}{2\cdot 4\cdot 6\cdot\cdots\cdot(2k)}\,\frac{\pi}{2}$. By Exercise 43 (a),

$$\int_0^{\pi/2}\sin^{2(k+1)}x\,dx = \frac{2k+1}{2k+2}\int_0^{\pi/2}\sin^{2k}x\,dx=\frac{2k+1}{2k+2}\cdot\frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2k-1)}{2\cdot 4\cdot 6\cdot\cdots\cdot(2k)}\,\frac{\pi}{2}$$
$$= \frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2k-1)(2k+1)}{2\cdot 4\cdot 6\cdot\cdots\cdot(2k)(2k+2)}\cdot\frac{\pi}{2},$$

so the formula holds for $n=k+1$. By induction, the formula holds for all $n\geq 1$.

45. Let $u=(\ln x)^n$, $dv=dx\Rightarrow du=n(\ln x)^{n-1}(dx/x)$, $v=x$. By Equation 2, $\int(\ln x)^n dx=x(\ln x)^n-\int nx(\ln x)^{n-1}(dx/x)=x(\ln x)^n-n\int(\ln x)^{n-1}dx$.

46. Let $u=x^n$, $dv=e^x dx\Rightarrow du=nx^{n-1}dx$, $v=e^x$. By Equation 2, $\int x^n e^x dx=x^n e^x-n\int x^{n-1}e^x dx$.

47. Let $u=\left(x^2+a^2\right)^n$, $dv=dx\Rightarrow du=n\left(x^2+a^2\right)^{n-1}2x\,dx$, $v=x$. Then

$$\int\left(x^2+a^2\right)^n dx = x\left(x^2+a^2\right)^n - 2n\int x^2\left(x^2+a^2\right)^{n-1}dx$$
$$= x\left(x^2+a^2\right)^n - 2n\left[\int\left(x^2+a^2\right)^n dx - a^2\int\left(x^2+a^2\right)^{n-1}dx\right] \quad \left[\text{since } x^2=\left(x^2+a^2\right)-a^2\right]$$

$\Rightarrow (2n+1)\int\left(x^2+a^2\right)^n dx = x\left(x^2+a^2\right)^n + 2na^2\int\left(x^2+a^2\right)^{n-1}dx$, and

$\int\left(x^2+a^2\right)^n dx = \dfrac{x\left(x^2+a^2\right)^n}{2n+1} + \dfrac{2na^2}{2n+1}\int\left(x^2+a^2\right)^{n-1}dx$ [provided $2n+1\neq 0$].

48. Let $u=\sec^{n-2}x$, $dv=\sec^2 x\,dx \Rightarrow du=(n-2)\sec^{n-3}x\sec x\tan x\,dx$, $v=\tan x$. Then by Equation 2,

$$\int\sec^n x\,dx = \tan x\sec^{n-2}x-(n-2)\int\sec^{n-2}x\tan^2 x\,dx$$
$$= \tan x\sec^{n-2}x-(n-2)\int\sec^{n-2}x\left(\sec^2 x-1\right)dx$$
$$= \tan x\sec^{n-2}x-(n-2)\int\sec^n x\,dx+(n-2)\int\sec^{n-2}x\,dx$$

so $(n-1)\int\sec^n x\,dx=\tan x\sec^{n-2}x+(n-2)\int\sec^{n-2}x\,dx$. If $n-1\neq 0$, then

$\int\sec^n x\,dx = \dfrac{\tan x\sec^{n-2}x}{n-1} + \dfrac{n-2}{n-1}\int\sec^{n-2}x\,dx$.

49. Take $n=3$ in Exercise 45 to get $\int(\ln x)^3 dx = x(\ln x)^3 - 3\int(\ln x)^2 dx = x(\ln x)^3 - 3x(\ln x)^2 + 6x\ln x - 6x + C$ [by Exercise 13].
Or : Instead of using Exercise 13 , apply Exercise 45 again with $n=2$.

50. Take $n=4$ in Exercise 46 to get

$$\int x^4e^x dx = x^4e^x-4\int x^3e^x dx = x^4e^x-4\left(x^3-3x^2+6x-6\right)e^x+C \qquad \text{[by Exercise 14]}$$
$$= e^x\left(x^4-4x^3+12x^2-24x+24\right)+C$$

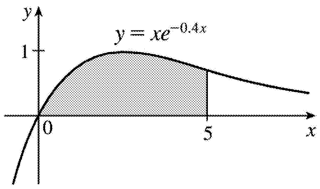
Or: Instead of using Exercise 14 , apply Exercise 46 with $n=3$, then $n=2$, then $n=1$.

51. Area $=\int_0^5 xe^{-0.4x}dx$. Let $u=x$, $dv=e^{-0.4x}dx\Rightarrow$

$du=dx$, $v=-2.5e^{-0.4x}$. Then

area $=\left[-2.5xe^{-0.4x}\right]_0^5+2.5\int_0^5 e^{-0.4x}dx$

$$=-12.5e^{-2}+0+2.5\left[-2.5e^{-0.4x}\right]_0^5$$
$$=-12.5e^{-2}-6.25(e^{-2}-1)=6.25-18.75e^{-2} \text{ or } \frac{25}{4}-\frac{75}{4}e^{-2}$$

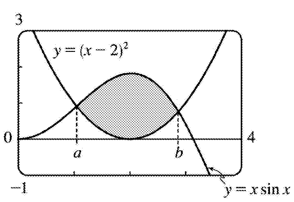


52. The curves $y=x\ln x$ and $y=5\ln x$ intersect when $x\ln x=5\ln x \Leftrightarrow x\ln x-5\ln x=0 \Leftrightarrow (x-5)\ln x=0$; that is, when $x=1$ or $x=5$. For $1<x<5$, we have $5\ln x>x\ln x$ since $\ln x>0$. Thus, area $=\int_1^5 (5\ln x-x\ln x)\,dx=\int_1^5 [(5-x)\ln x]\,dx$. Let $u=\ln x$, $dv=(5-x)\,dx \Rightarrow du=dx/x$, $v=5x-\frac{1}{2}x^2$. Then

$$\text{area} = \left[(\ln x)\left(5x-\frac{1}{2}x^2\right)\right]_1^5 - \int_1^5 \left[\left(5x-\frac{1}{2}x^2\right)\frac{1}{x}\right]dx = (\ln 5)\left(\frac{25}{2}\right) - 0 - \int_1^5 \left(5-\frac{1}{2}x\right)dx$$
$$= \frac{25}{2}\ln 5 - \left[5x-\frac{1}{4}x^2\right]_1^5 = \frac{25}{2}\ln 5 - \left[\left(25-\frac{25}{4}\right)-\left(5-\frac{1}{4}\right)\right] = \frac{25}{2}\ln 5 - 14$$

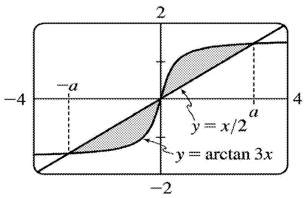
53. The curves $y=x\sin x$ and $y=(x-2)^2$ intersect at $a\approx 1.04748$ and $b\approx 2.87307$, so

$$\text{area} = \int_a^b \left[x\sin x-(x-2)^2\right]dx$$
$$= \left[-x\cos x+\sin x-\frac{1}{3}(x-2)^3\right]_a^b \text{ [by Example 1]}$$
$$\approx 2.81358-0.63075=2.18283$$



54. The curves $y=\arctan 3x$ and $y=x/2$ intersect at $x=\pm a\approx \pm 2.91379$, so

$$\text{area} = \int_{-a}^{a} \left|\arctan 3x-\frac{1}{2}x\right| dx = 2\int_0^a \left(\arctan 3x-\frac{1}{2}x\right)dx$$
$$= 2\left[x\arctan 3x-\frac{1}{6}\ln(1+9x^2)-\frac{1}{4}x^2\right]_0^a \text{ [see Example 5]}$$
$$\approx 2(1.39768)=2.79536 .$$



55. $V=\int_0^1 2\pi x\cos(\pi x/2)dx$. Let $u=x$, $dv=\cos(\pi x/2)dx \Rightarrow du=dx$, $v=\frac{2}{\pi}\sin(\pi x/2)$.

$$V=2\pi\left[\frac{2}{\pi}x\sin\left(\frac{\pi x}{2}\right)\right]_0^1-2\pi\cdot\frac{2}{\pi}\int_0^1\sin\left(\frac{\pi x}{2}\right)dx=2\pi\left(\frac{2}{\pi}-0\right)-4\left[-\frac{2}{\pi}\cos\left(\frac{\pi x}{2}\right)\right]_0^1$$

$$=4+\frac{8}{\pi}(0-1)=4-\frac{8}{\pi} .$$

56.

$$\text{Volume }=\int_0^1 2\pi x\left(e^x-e^{-x}\right)dx=2\pi\int_0^1\left(xe^x-xe^{-x}\right)dx$$

$$=2\pi\left[\int_0^1 xe^x dx-\int_0^1 xe^{-x}dx\right] \text{ [both integrals by parts]}$$

$$=2\pi\left[\left(xe^x-e^x\right)-\left(-xe^{-x}-e^{-x}\right)\right]_0^1=2\pi[2/e-0]=4\pi/e$$

57. Volume $=\int_{-1}^0 2\pi(1-x)e^{-x}dx$. Let $u=1-x$, $dv=e^{-x}dx \Rightarrow du=-dx$, $v=-e^{-x}$.

$$V=2\pi\left[(1-x)(-e^{-x})\right]_{-1}^0-2\pi\int_{-1}^0 e^{-x}dx=2\pi\left[(x-1)(e^{-x})+e^{-x}\right]_{-1}^0$$

$$=2\pi\left[xe^{-x}\right]_{-1}^0=2\pi(0+e)=2\pi e$$

58.

$$\text{Volume }=\int_1^\pi 2\pi y\cdot\ln y\,dy=2\pi\left[\frac{1}{2}y^2\ln y-\frac{1}{4}y^2\right]_1^\pi$$

$$=2\pi\left[\frac{1}{4}y^2(2\ln y-1)\right]_1^\pi=2\pi\left[\frac{\pi^2(2\ln\pi-1)}{4}-\frac{(0-1)}{4}\right]=\pi^3\ln\pi-\frac{\pi^3}{2}+\frac{\pi}{2}$$

59. The average value of $f(x)=x^2\ln x$ on the interval $[1,3]$ is $f_{\text{ave}}=\frac{1}{3-1}\int_1^3 x^2\ln x\,dx=\frac{1}{2}I$.

Let $u=\ln x$, $dv=x^2dx \Rightarrow du=(1/x)dx$, $v=\frac{1}{3}x^3$. So

$I=\left[\frac{1}{3}x^3\ln x\right]_1^3-\int_1^3\frac{1}{3}x^2dx=(9\ln 3-0)-\left[\frac{1}{9}x^3\right]_1^3=9\ln 3-\left(3-\frac{1}{9}\right)=9\ln 3-\frac{26}{9}$.

Thus, $f_{\text{ave}}=\frac{1}{2}I=\frac{1}{2}\left(9\ln 3-\frac{26}{9}\right)=\frac{9}{2}\ln 3-\frac{13}{9}$.

60. The rocket will have height $H=\int_0^{60}v(t)dt$ after 60 seconds.

$$\begin{aligned}H &= \int_0^{60}\left[-gt-v_e\ln\left(\frac{m-rt}{m}\right)\right]dt=-g\left[\frac{1}{2}t^2\right]_0^{60}-v_e\left[\int_0^{60}\ln(m-rt)dt-\int_0^{60}\ln m\,dt\right]\\ &= -g(1800)+v_e(\ln m)(60)-v_e\int_0^{60}\ln(m-rt)dt\end{aligned}$$

Let $u=\ln(m-rt)$, $dv=dt\Rightarrow du=\frac{1}{m-rt}(-r)dt$, $v=t$. Then

$$\begin{aligned}\int_0^{60}\ln(m-rt)dt &= [t\ln(m-rt)]_0^{60}+\int_0^{60}\frac{rt}{m-rt}dt=60\ln(m-60r)+\int_0^{60}\left(-1+\frac{m}{m-rt}\right)dt\\ &= 60\ln(m-60r)+\left[-t-\frac{m}{r}\ln(m-rt)\right]_0^{60}\\ &= 60\ln(m-60r)-60-\frac{m}{r}\ln(m-60r)+\frac{m}{r}\ln m\end{aligned}$$

So $H=-1800g+60v_e\ln m-60v_e\ln(m-60r)+60v_e+\frac{m}{r}v_e\ln(m-60r)-\frac{m}{r}v_e\ln m$. Substituting $g=9.8$, $m=30,000$, $r=160$, and $v_e=3000$ gives us $H\approx 14,844$ m.

61. Since $v(t)>0$ for all t , the desired distance is $s(t)=\int_0^t v(w)dw=\int_0^t w^2e^{-w}dw$.

First let $u=w^2$, $dv=e^{-w}dw\Rightarrow du=2w\,dw$, $v=-e^{-w}$. Then $s(t)=\left[-w^2e^{-w}\right]_0^t+2\int_0^t we^{-w}dw$.

Next let $U=w$, $dV=e^{-w}dw\Rightarrow dU=dw$, $V=-e^{-w}$. Then

$$\begin{aligned}s(t) &= -t^2e^{-t}+2\left(\left[-we^{-w}\right]_0^t+\int_0^t e^{-w}dw\right)=-t^2e^{-t}+2\left(-te^{-t}+0+\left[-e^{-w}\right]_0^t\right)\\ &= -t^2e^{-t}+2\left(-te^{-t}-e^{-t}+1\right)=-t^2e^{-t}-2te^{-t}-2e^{-t}+2\\ &= 2-e^{-t}\left(t^2+2t+2\right)\text{ meters}\end{aligned}$$

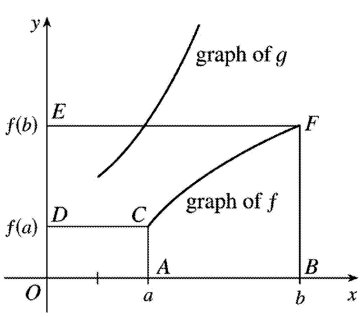
62. Suppose $f(0)=g(0)=0$ and let $u=f(x)$, $dv=g''(x)dx\Rightarrow du=f'(x)dx$, $v=g'(x)$. Then $\int_0^a f(x)g''(x)dx=\left[f(x)g'(x)\right]_0^a-\int_0^a f'(x)g'(x)dx=f(a)g'(a)-\int_0^a f'(x)g'(x)dx$. Now let $U=f'(x)$,

$dV=g'(x)dx \Rightarrow dU=f''(x)dx$ and $V=g(x)$, so
$\int_0^a f'(x)g'(x)dx=\Big[f'(x)g(x)\Big]_0^a-\int_0^a f''(x)g(x)dx=f'(a)g(a)-\int_0^a f''(x)g(x)dx$.
Combining the two results, we get $\int_0^a f(x)g''(x)dx=f(a)g'(a)-f'(a)g(a)+\int_0^a f''(x)g(x)dx$.

63. For $I=\int_1^4 xf''(x)dx$, let $u=x$, $dv=f''(x)dx \Rightarrow du=dx$, $v=f'(x)$. Then
$I=\Big[xf'(x)\Big]_1^4-\int_1^4 f'(x)dx=4f'(4)-1\cdot f'(1)-[f(4)-f(1)]=4\cdot 3-1\cdot 5-(7-2)=12-5-5=2$.
We used the fact that f'' is continuous to guarantee that I exists.

64. **(a)** Take $g(x)=x$ and $g'(x)=1$ in Equation 1.
(b) By part (a), $\int_a^b f(x)dx=bf(b)-af(a)-\int_a^b xf'(x)dx$. Now let $y=f(x)$, so that $x=g(y)$ and $dy=f'(x)dx$. Then $\int_a^b xf'(x)dx=\int_{f(a)}^{f(b)} g(y)dy$. The result follows.
(c) Part (b) says that the area of region $ABFC$ is

$= bf(b)$	$-\ af(a)$	$-\ \int_{f(a)}^{f(b)} g(y)dy$
= (area of rectangle $OBFE$)	− (area of rectangle $OACD$)	− (area of region $DCFE$)



(d) We have $f(x)=\ln x$, so $f^{-1}(x)=e^x$, and since $g=f^{-1}$, we have $g(y)=e^y$. By part (b),
$\int_1^e \ln x\,dx=e\ln e-1\ln 1-\int_{\ln 1}^{\ln e} e^y dy=e-\int_0^1 e^y dy=e-\Big[e^y\Big]_0^1=e-(e-1)=1$.

65. Using the formula for volumes of rotation and the figure, we see that Volume
$=\int_0^d \pi b^2 dy-\int_0^c \pi a^2 dy-\int_c^d \pi\,[g(y)]^2 dy=\pi b^2 d-\pi a^2 c-\int_c^d \pi\,[g(y)]^2 dy$. Let $y=f(x)$, which gives $dy=f'(x)dx$ and $g(y)=x$, so that $V=\pi b^2 d-\pi a^2 c-\pi\int_a^b x^2 f'(x)dx$. Now integrate by parts with $u=x^2$, and
$dv=f'(x)dx \Rightarrow du=2x\,dx$, $v=f(x)$, and

$\int_a^b x^2 f'(x)dx = \left[x^2 f(x)\right]_a^b - \int_a^b 2x f(x)dx = b^2 f(b) - a^2 f(a) - \int_a^b 2x f(x)dx$, but $f(a)=c$ and $f(b)=d \Rightarrow$
$V = \pi b^2 d - \pi a^2 c - \pi \left[b^2 d - a^2 c - \int_a^b 2x f(x)dx \right] = \int_a^b 2\pi x f(x)dx$.

66. **(a)** We note that for $0 \le x \le \frac{\pi}{2}$, $0 \le \sin x \le 1$, so $\sin^{2n+2} x \le \sin^{2n+1} x \le \sin^{2n} x$. So by the second Comparison Property of the Integral, $I_{2n+2} \le I_{2n+1} \le I_{2n}$.
(b) Substituting directly into the result from Exercise 44 , we get

$$\frac{I_{2n+2}}{I_{2n}} = \frac{\dfrac{1\cdot 3\cdot 5\cdot \cdots \cdot [2(n+1)-1]}{2\cdot 4\cdot 6\cdot \cdots \cdot [2(n+1)]}\,\dfrac{\pi}{2}}{\dfrac{1\cdot 3\cdot 5\cdot \cdots \cdot (2n-1)}{2\cdot 4\cdot 6\cdot \cdots \cdot (2n)}\,\dfrac{\pi}{2}} = \frac{2(n+1)-1}{2(n+1)} = \frac{2n+1}{2n+2}$$

(c) We divide the result from part (a) by I_{2n} . The inequalities are preserved since I_{2n} is positive:
$\frac{I_{2n+2}}{I_{2n}} \le \frac{I_{2n+1}}{I_{2n}} \le \frac{I_{2n}}{I_{2n}}$. Now from part (b), the left term is equal to $\frac{2n+1}{2n+2}$, so the expression becomes
$\frac{2n+1}{2n+2} \le \frac{I_{2n+1}}{I_{2n}} \le 1$. Now $\lim_{n\to\infty} \frac{2n+1}{2n+2} = \lim_{n\to\infty} 1 = 1$, so by the Squeeze Theorem, $\lim_{n\to\infty} \frac{I_{2n+1}}{I_{2n}} = 1$.
(d) We substitute the results from Exercises 43 and 44 into the result from part (c):

$$1 = \lim_{n\to\infty} \frac{I_{2n+1}}{I_{2n}} = \lim_{n\to\infty} \frac{\dfrac{2\cdot 4\cdot 6\cdot \cdots \cdot (2n)}{3\cdot 5\cdot 7\cdot \cdots \cdot (2n+1)}}{\dfrac{1\cdot 3\cdot 5\cdot \cdots \cdot (2n-1)}{2\cdot 4\cdot 6\cdot \cdots \cdot (2n)}\,\dfrac{\pi}{2}}$$
$$= \lim_{n\to\infty} \left[\frac{2\cdot 4\cdot 6\cdot \cdots \cdot (2n)}{3\cdot 5\cdot 7\cdot \cdots \cdot (2n+1)} \right] \left[\frac{2\cdot 4\cdot 6\cdot \cdots \cdot (2n)}{1\cdot 3\cdot 5\cdot \cdots \cdot (2n-1)} \left(\frac{2}{\pi} \right) \right]$$
$$= \lim_{n\to\infty} \frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdot \cdots \cdot \frac{2n}{2n-1}\cdot\frac{2n}{2n+1}\cdot\frac{2}{\pi} \qquad \text{[rearrange terms]}$$

Multiplying both sides by $\frac{\pi}{2}$ gives us the *Wallis product* :

$$\frac{\pi}{2} = \frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdot \cdots$$

(e) The area of the k th rectangle is k . At the $2n$ th step, the area is increased from $2n-1$ to $2n$ by multiplying the width by $\dfrac{2n}{2n-1}$, and at the $(2n+1)$ th step, the area is increased from $2n$ to $2n+1$ by multiplying the height by $\dfrac{2n+1}{2n}$. These two steps multiply the ratio of width to height by $\dfrac{2n}{2n-1}$ and $\dfrac{1}{(2n+1)/(2n)}=\dfrac{2n}{2n+1}$ respectively. So, by part (d), the limiting ratio is $\dfrac{2}{1}\cdot\dfrac{2}{3}\cdot\dfrac{4}{3}\cdot\dfrac{4}{5}\cdot\dfrac{6}{5}\cdot\dfrac{6}{7}\cdot\cdots=\dfrac{\pi}{2}$.

1.

$$\int \sin^3 x\cos^2 x\,dx = \int \sin^2 x\cos^2 x\sin x\,dx = \int \left(1-\cos^2 x\right)\cos^2 x\sin x\,dx = \int \left(1-u^2\right)u^2(-du)$$
$$= \int \left(u^2-1\right)u^2\,du = \int \left(u^4-u^2\right)du = \frac{1}{5}u^5 - \frac{1}{3}u^3 + C = \frac{1}{5}\cos^5 x - \frac{1}{3}\cos^3 x + C$$

2.

$$\int \sin^6 x\cos^3 x\,dx = \int \sin^6 x\cos^2 x\cos x\,dx = \int \sin^6 x\left(1-\sin^2 x\right)\cos x\,dx = \int u^6\left(1-u^2\right)du$$
$$= \int \left(u^6-u^8\right)du = \frac{1}{7}u^7 - \frac{1}{9}u^9 + C = \frac{1}{7}\sin^7 x - \frac{1}{9}\sin^9 x + C$$

3.

$$\int_{\pi/2}^{3\pi/4} \sin^5 x\cos^3 x\,dx = \int_{\pi/2}^{3\pi/4} \sin^5 x\cos^2 x\cos x\,dx = \int_{\pi/2}^{3\pi/4} \sin^5 x\left(1-\sin^2 x\right)\cos x\,dx$$
$$\overset{s}{=} \int_1^{\sqrt{2}/2} u^5\left(1-u^2\right)du = \int_1^{\sqrt{2}/2} \left(u^5-u^7\right)du = \left[\frac{1}{6}u^6 - \frac{1}{8}u^8\right]_1^{\sqrt{2}/2}$$
$$= \left(\frac{1/8}{6} - \frac{1/16}{8}\right) - \left(\frac{1}{6} - \frac{1}{8}\right) = -\frac{11}{384}$$

4.

$$\int_0^{\pi/2} \cos^5 x\,dx = \int_0^{\pi/2} \left(\cos^2 x\right)^2\cos x\,dx = \int_0^{\pi/2} \left(1-\sin^2 x\right)^2\cos x\,dx = \int_0^1 \left(1-u^2\right)^2 du$$
$$= \int_0^1 \left(1-2u^2+u^4\right)du = \left[u - \frac{2}{3}u^3 + \frac{1}{5}u^5\right]_0^1 = \left(1 - \frac{2}{3} + \frac{1}{5}\right) - 0 = \frac{8}{15}$$

5.

$$\int \cos^5 x\sin^4 x\,dx = \int \cos^4 x\sin^4 x\cos x\,dx = \int \left(1-\sin^2 x\right)^2\sin^4 x\cos x\,dx = \int \left(1-u^2\right)^2 u^4\,du$$
$$= \int \left(1-2u^2+u^4\right)u^4\,du = \int \left(u^4-2u^6+u^8\right)du = \frac{1}{5}u^5 - \frac{2}{7}u^7 + \frac{1}{9}u^9 + C$$
$$= \frac{1}{5}\sin^5 x - \frac{2}{7}\sin^7 x + \frac{1}{9}\sin^9 x + C$$

6.

$$\int \sin^3(mx)dx = \int \left(1-\cos^2 mx\right)\sin mx\,dx = -\frac{1}{m}\int\left(1-u^2\right)du \;[\,u=\cos mx\,,\;du=-m\sin mx\,dx\,]$$
$$=-\frac{1}{m}\left(u-\frac{1}{3}u^3\right)+C=-\frac{1}{m}\left(\cos mx-\frac{1}{3}\cos^3 mx\right)+C$$
$$=\frac{1}{3m}\cos^3 mx-\frac{1}{m}\cos mx+C$$

7.

$$\int_0^{\pi/2}\cos^2\theta\,d\theta = \int_0^{\pi/2}\frac{1}{2}(1+\cos 2\theta)d\theta \;\text{[half–angle identity]}$$
$$=\frac{1}{2}\left[\theta+\frac{1}{2}\sin 2\theta\right]_0^{\pi/2}=\frac{1}{2}\left[\left(\frac{\pi}{2}+0\right)-(0+0)\right]=\frac{\pi}{4}$$

8. $\int_0^{\pi/2}\sin^2(2\theta)d\theta=\int_0^{\pi/2}\frac{1}{2}(1-\cos 4\theta)d\theta=\frac{1}{2}\left[\theta-\frac{1}{4}\sin 4\theta\right]_0^{\pi/2}=\frac{1}{2}\left[\left(\frac{\pi}{2}-0\right)-(0-0)\right]=\frac{\pi}{4}$

9.

$$\int_0^{\pi}\sin^4(3t)dt = \int_0^{\pi}\left[\sin^2(3t)\right]^2dt=\int_0^{\pi}\left[\frac{1}{2}(1-\cos 6t)\right]^2dt=\frac{1}{4}\int_0^{\pi}(1-2\cos 6t+\cos^2 6t)dt$$
$$=\frac{1}{4}\int_0^{\pi}\left[1-2\cos 6t+\frac{1}{2}(1+\cos 12t)\right]dt=\frac{1}{4}\int_0^{\pi}\left(\frac{3}{2}-2\cos 6t+\frac{1}{2}\cos 12t\right)dt$$
$$=\frac{1}{4}\left[\frac{3}{2}t-\frac{1}{3}\sin 6t+\frac{1}{24}\sin 12t\right]_0^{\pi}=\frac{1}{4}\left[\left(\frac{3\pi}{2}-0+0\right)-(0-0+0)\right]=\frac{3\pi}{8}$$

10.

$$\int_0^{\pi}\cos^6\theta\,d\theta = \int_0^{\pi}(\cos^2\theta)^3d\theta=\int_0^{\pi}\left[\frac{1}{2}(1+\cos 2\theta)\right]^3d\theta=\frac{1}{8}\int_0^{\pi}(1+3\cos 2\theta+3\cos^2 2\theta+\cos^3 2\theta)d\theta$$
$$=\frac{1}{8}\left[\theta+\frac{3}{2}\sin 2\theta\right]_0^{\pi}+\frac{1}{8}\int_0^{\pi}\left[\frac{3}{2}(1+\cos 4\theta)\right]d\theta+\frac{1}{8}\int_0^{\pi}\left[(1-\sin^2 2\theta)\cos 2\theta\right]d\theta$$
$$=\frac{1}{8}\pi+\frac{3}{16}\left[\theta+\frac{1}{4}\sin 4\theta\right]_0^{\pi}+\frac{1}{8}\int_0^{0}(1-u^2)\left(\frac{1}{2}du\right)\;[\,u=\sin 2\theta\,,\;du=2\cos 2\theta\,d\theta\,]$$
$$=\frac{\pi}{8}+\frac{3\pi}{16}+0=\frac{5\pi}{16}$$

11.

$$\int(1+\cos\theta)^2 d\theta=\int(1+2\cos\theta+\cos^2\theta)d\theta=\theta+2\sin\theta+\frac{1}{2}\int(1+\cos 2\theta)d\theta$$

$$=\theta+2\sin\theta+\frac{1}{2}\theta+\frac{1}{4}\sin 2\theta+C=\frac{3}{2}\theta+2\sin\theta+\frac{1}{4}\sin 2\theta+C$$

12. Let $u=x$, $dv=\cos^2 x\,dx\Rightarrow du=dx$, $v=\int\cos^2 x\,dx=\int\frac{1}{2}(1+\cos 2x)dx=\frac{1}{2}x+\frac{1}{4}\sin 2x$, so

$$\int x\cos^2 x\,dx=x\left(\frac{1}{2}x+\frac{1}{4}\sin 2x\right)-\int\left(\frac{1}{2}x+\frac{1}{4}\sin 2x\right)dx=\frac{1}{2}x^2+\frac{1}{4}x\sin 2x-\frac{1}{4}x^2+\frac{1}{8}\cos 2x+C$$

$$=\frac{1}{4}x^2+\frac{1}{4}x\sin 2x+\frac{1}{8}\cos 2x+C$$

13.

$$\int_0^{\pi/4}\sin^4 x\cos^2 x\,dx=\int_0^{\pi/4}\sin^2 x(\sin x\cos x)^2 dx=\int_0^{\pi/4}\frac{1}{2}(1-\cos 2x)\left(\frac{1}{2}\sin 2x\right)^2 dx$$

$$=\frac{1}{8}\int_0^{\pi/4}(1-\cos 2x)\sin^2 2x\,dx=\frac{1}{8}\int_0^{\pi/4}\sin^2 2x\,dx-\frac{1}{8}\int_0^{\pi/4}\sin^2 2x\cos 2x\,dx$$

$$=\frac{1}{16}\int_0^{\pi/4}(1-\cos 4x)dx-\frac{1}{16}\left[\frac{1}{3}\sin^3 2x\right]_0^{\pi/4}=\frac{1}{16}\left[x-\frac{1}{4}\sin 4x-\frac{1}{3}\sin^3 2x\right]_0^{\pi/4}$$

$$=\frac{1}{16}\left(\frac{\pi}{4}-0-\frac{1}{3}\right)=\frac{1}{192}(3\pi-4)$$

14.

$$\int_0^{\pi/2}\sin^2 x\cos^2 x\,dx=\int_0^{\pi/2}\frac{1}{4}\left(4\sin^2 x\cos^2 x\right)dx=\int_0^{\pi/2}\frac{1}{4}(2\sin x\cos x)^2 dx=\frac{1}{4}\int_0^{\pi/2}\sin^2 2x\,dx$$

$$=\frac{1}{4}\int_0^{\pi/2}\frac{1}{2}(1-\cos 4x)dx=\frac{1}{8}\int_0^{\pi/2}(1-\cos 4x)dx=\frac{1}{8}\left[x-\frac{1}{4}\sin 4x\right]_0^{\pi/2}$$

$$=\frac{1}{8}\left(\frac{\pi}{2}\right)=\frac{\pi}{16}$$

15.

$$\int\sin^3 x\sqrt{\cos x}\,dx=\int\left(1-\cos^2 x\right)\sqrt{\cos x}\sin x\,dx=\int\left(1-u^2\right)u^{1/2}(-du)=\int\left(u^{5/2}-u^{1/2}\right)du$$

$$=\frac{2}{7}u^{7/2}-\frac{2}{3}u^{3/2}+C=\frac{2}{7}(\cos x)^{7/2}-\frac{2}{3}(\cos x)^{3/2}+C$$

$$=\left(\frac{2}{7}\cos^3 x-\frac{2}{3}\cos x\right)\sqrt{\cos x}+C$$

16. Let $u=\sin\theta$. Then $du=\cos\theta\, d\theta$ and

$$\int \cos\theta \cos^5(\sin\theta)\,d\theta = \int \cos^5 u\, du = \int (\cos^2 u)^2 \cos u\, du = \int (1-\sin^2 u)^2 \cos u\, du$$
$$= \int (1-2\sin^2 u+\sin^4 u)\cos u\, du = I$$

Now let $x=\sin u$. Then $dx=\cos u\,du$ and

$$I = \int (1-2x^2+x^4)\,dx = x-\frac{2}{3}x^3+\frac{1}{5}x^5+C = \sin u-\frac{2}{3}\sin^3 u+\frac{1}{5}\sin^5 u+C$$
$$= \sin(\sin\theta)-\frac{2}{3}\sin^3(\sin\theta)+\frac{1}{5}\sin^5(\sin\theta)+C$$

17.

$$\int \cos^2 x\tan^3 x\,dx = \int \frac{\sin^3 x}{\cos x}\,dx = \int \frac{\left(1-u^2\right)(-du)}{u} = \int \left[\frac{-1}{u}+u\right] du$$
$$= -\ln|u|+\frac{1}{2}u^2+C = \frac{1}{2}\cos^2 x-\ln|\cos x|+C$$

18.

$$\int \cot^5\theta \sin^4\theta\, d\theta = \int \frac{\cos^5\theta}{\sin^5\theta}\sin^4\theta\, d\theta = \int \frac{\cos^5\theta}{\sin\theta}\,d\theta = \int \frac{\cos^4\theta}{\sin\theta}\cos\theta\, d\theta = \int \frac{\left(1-\sin^2\theta\right)^2}{\sin\theta}\cos\theta\, d\theta$$
$$= \int \frac{\left(1-u^2\right)^2}{u}\,du = \int \frac{1-2u^2+u^4}{u}\,du = \int \left(\frac{1}{u}-2u+u^3\right) du$$
$$= \ln|u|-u^2+\frac{1}{4}u^4+C = \ln|\sin\theta|-\sin^2\theta+\frac{1}{4}\sin^4\theta+C$$

19.

$$\int \frac{1-\sin x}{\cos x}\,dx = \int (\sec x-\tan x)\,dx = \ln|\sec x+\tan x|-\ln|\sec x|+C \quad \left[\begin{array}{c}\text{by (1) and the boxed}\\ \text{formula above it}\end{array}\right]$$
$$= \ln|(\sec x+\tan x)\cos x|+C = \ln|1+\sin x|+C$$
$$= \ln(1+\sin x)+C \text{ since } 1+\sin x \geq 0$$

Or:

$$\int \frac{1-\sin x}{\cos x}\,dx = \int \frac{1-\sin x}{\cos x}\cdot\frac{1+\sin x}{1+\sin x}\,dx = \int \frac{\left(1-\sin^2 x\right)dx}{\cos x(1+\sin x)} = \int \frac{\cos x\,dx}{1+\sin x}$$

$$=\int \frac{dw}{w} \text{ [where } w=1+\sin x \text{ , } dw=\cos x\,dx \text{]}$$

$$=\ln |w|+C=\ln |1+\sin x|+C=\ln (1+\sin x)+C$$

20. $\int \cos^2 x \sin 2x\,dx=2\int \cos^3 x \sin x\,dx=-2\int u^3 du=-\frac{1}{2}u^4+C=-\frac{1}{2}\cos^4 x+C$

21. Let $u=\tan x$, $du=\sec^2 x\,dx$. Then $\int \sec^2 x \tan x\,dx=\int u\,du=\frac{1}{2}u^2+C=\frac{1}{2}\tan^2 x+C$.

Or: Let $v=\sec x$, $dv=\sec x \tan x\,dx$. Then $\int \sec^2 x \tan x\,dx=\int v\,dv=\frac{1}{2}v^2+C=\frac{1}{2}\sec^2 x+C$.

22.

$$\int_0^{\pi/2}\sec^4(t/2)\,dt=\int_0^{\pi/4}\sec^4 x(2\,dx) \text{ [} x=t/2 \text{ , } dx=\frac{1}{2}\,dt \text{] }=2\int_0^{\pi/4}\sec^2 x(1+\tan^2 x)\,dx$$

$$=2\int_0^1(1+u^2)\,du \text{ [} u=\tan x \text{ , } du=\sec^2 x\,dx \text{] }=2\left[u+\frac{1}{3}u^3\right]_0^1=2\left(1+\frac{1}{3}\right)=\frac{8}{3}$$

23. $\int \tan^2 x\,dx=\int \left(\sec^2 x-1\right)dx=\tan x-x+C$

24. $\int \tan^4 x\,dx=\int \tan^2 x\left(\sec^2 x-1\right)dx=\int \tan^2 x \sec^2 x\,dx-\int \tan^2 x\,dx=\frac{1}{3}\tan^3 x-\tan x+x+C$

(Set $u=\tan x$ in the first integral and use Exercise 23 for the second.)

25.

$$\int \sec^6 t\,dt=\int \sec^4 t\cdot \sec^2 t\,dt=\int (\tan^2 t+1)^2\sec^2 t\,dt=\int (u^2+1)^2\,du$$

$$=\int (u^4+2u^2+1)\,du=\frac{1}{5}u^5+\frac{2}{3}u^3+u+C=\frac{1}{5}\tan^5 t+\frac{2}{3}\tan^3 t+\tan t+C$$

26.

$$\int_0^{\pi/4}\sec^4\theta \tan^4\theta\,d\theta=\int_0^{\pi/4}(\tan^2\theta+1)\tan^4\theta \sec^2\theta\,d\theta=\int_0^1(u^2+1)u^4\,du$$

$$=\int_0^1(u^6+u^4)\,du=\left[\frac{1}{7}u^7+\frac{1}{5}u^5\right]_0^1=\frac{1}{7}+\frac{1}{5}=\frac{12}{35}$$

27.

$$\int_0^{\pi/3}\tan^5 x \sec^4 x\,dx=\int_0^{\pi/3}\tan^5 x(\tan^2 x+1)\sec^2 x\,dx$$

$$=\int_0^{\sqrt{3}} u^5(u^2+1)\,du$$

$$=\int_0^{\sqrt{3}}(u^7+u^5)\,du=\left[\frac{1}{8}u^8+\frac{1}{6}u^6\right]_0^{\sqrt{3}}=\frac{81}{8}+\frac{27}{6}=\frac{81}{8}+\frac{9}{2}=\frac{81}{8}+\frac{36}{8}=\frac{117}{8}$$

Alternate solution:

$$\int_0^{\pi/3}\tan^5 x\sec^4 x\,dx=\int_0^{\pi/3}\tan^4 x\sec^3 x\sec x\tan x\,dx=\int_0^{\pi/3}(\sec^2 x-1)^2\sec^3 x\sec x\tan x\,dx$$

$$=\int_1^2(u^2-1)^2u^3\,du$$

$$=\int_1^2(u^4-2u^2+1)u^3\,du=\int_1^2(u^7-2u^5+u^3)\,du$$

$$=\left[\frac{1}{8}u^8-\frac{1}{3}u^6+\frac{1}{4}u^4\right]_1^2=\left(32-\frac{64}{3}+4\right)-\left(\frac{1}{8}-\frac{1}{3}+\frac{1}{4}\right)=\frac{117}{8}$$

28.

$$\int\tan^3(2x)\sec^5(2x)\,dx=\int\tan^2(2x)\sec^4(2x)\cdot\sec(2x)\tan(2x)\,dx$$

$$=\int(u^2-1)u^4\left(\frac{1}{2}\,du\right)\ [\,u=\sec(2x), du=2\sec(2x)\tan(2x)\,dx\,]$$

$$=\frac{1}{2}\int(u^6-u^4)\,du=\frac{1}{14}u^7-\frac{1}{10}u^5+C=\frac{1}{14}\sec^7(2x)-\frac{1}{10}\sec^5(2x)+C$$

29.

$$\int\tan^3 x\sec x\,dx=\int\tan^2 x\sec x\tan x\,dx=\int\left(\sec^2 x-1\right)\sec x\tan x\,dx$$

$$=\int(u^2-1)\,du$$

$$=\frac{1}{3}u^3-u+C=\frac{1}{3}\sec^3 x-\sec x+C$$

30.

$$\int_0^{\pi/3}\tan^5 x\sec^6 x\,dx=\int_0^{\pi/3}\tan^5 x\sec^4 x\sec^2 x\,dx=\int_0^{\pi/3}\tan^5 x\left(1+\tan^2 x\right)^2\sec^2 x\,dx$$

$$=\int_0^{\sqrt{3}}u^5\left(1+u^2\right)^2du\ [\,u=\tan x\,,\ du=\sec^2 x\,dx\,]=\int_0^{\sqrt{3}}u^5\left(1+2u^2+u^4\right)du$$

$$=\int_0^{\sqrt{3}}\left(u^5+2u^7+u^9\right)du=\left[\frac{1}{6}u^6+\frac{1}{4}u^8+\frac{1}{10}u^{10}\right]_0^{\sqrt{3}}=\frac{27}{6}+\frac{81}{4}+\frac{243}{10}=\frac{981}{20}$$

Alternate solution:

$$\int_0^{\pi/3} \tan^5 x \sec^6 x\,dx = \int_0^{\pi/3} \tan^4 x \sec^5 x \sec x \tan x\,dx = \int_0^{\pi/3} \left(\sec^2 x - 1\right)^2 \sec^5 x \sec x \tan x\,dx$$

$$= \int_1^2 \left(u^2 - 1\right)^2 u^5\,du \ [\ u = \sec x\ ,\ du = \sec x \tan x\,dx\ ]$$

$$= \int_1^2 \left(u^4 - 2u^2 + 1\right) u^5\,du = \int_1^2 \left(u^9 - 2u^7 + u^5\right) du$$

$$= \left[\ \frac{1}{10} u^{10} - \frac{1}{4} u^8 + \frac{1}{6} u^6\right]_1^2 = \left(\ \frac{512}{5} - 64 + \frac{32}{3}\ \right) - \left(\ \frac{1}{10} - \frac{1}{4} + \frac{1}{6}\ \right) = \frac{981}{20}$$

31.

$$\int \tan^5 x\,dx = \int \left(\sec^2 x - 1\right)^2 \tan x\,dx = \int \sec^4 x \tan x\,dx - 2\int \sec^2 x \tan x\,dx + \int \tan x\,dx$$

$$= \int \sec^3 x \sec x \tan x\,dx - 2\int \tan x \sec^2 x\,dx + \int \tan x\,dx$$

$$= \frac{1}{4} \sec^4 x - \tan^2 x + \ln\,|\sec x| + C\ [\ \text{or}\ \ \frac{1}{4} \sec^4 x - \sec^2 x + \ln\,|\sec x| + C\ ]$$

32.

$$\int \tan^6 ay\,dy = \int \tan^4 ay \left(\sec^2 ay - 1\right) dy = \int \tan^4 ay \sec^2 ay\,dy - \int \tan^4 ay\,dy$$

$$= \frac{1}{5a} \tan^5 ay - \int \tan^2 ay \left(\sec^2 ay - 1\right) dy$$

$$= \frac{1}{5a} \tan^5 ay - \int \tan^2 ay \sec^2 ay\,dy + \int \left(\sec^2 ay - 1\right) dy$$

$$= \frac{1}{5a} \tan^5 ay - \frac{1}{3a} \tan^3 ay + \frac{1}{a} \tan ay - y + C$$

33.

$$\int \frac{\tan^3 \theta}{\cos^4 \theta}\,d\theta = \int \tan^3 \theta \sec^4 \theta\,d\theta = \int \tan^3 \theta \cdot (\tan^2 \theta + 1) \cdot \sec^2 \theta\,d\theta$$

$$= \int u^3 (u^2 + 1)\,du u = \tan \theta\ ,\ du = \sec^2 \theta\,d\theta\ \ ]$$

$$= \int (u^5 + u^3)\,du = \frac{1}{6} u^6 + \frac{1}{4} u^4 + C = \frac{1}{6} \tan^6 \theta + \frac{1}{4} \tan^4 \theta + C$$

34.

$$\int \tan^2 x \sec x\,dx = \int \left(\sec^2 x - 1\right) \sec x\,dx = \int \sec^3 x\,dx - \int \sec x\,dx$$

$$=\frac{1}{2}\left(\sec x\tan x+\ln|\sec x+\tan x|\right)-\ln|\sec x+\tan x|+C \text{ [by Example 8 and (1)]}$$

$$=\frac{1}{2}\left(\sec x\tan x-\ln|\sec x+\tan x|\right)+C$$

35. $\int_{\pi/6}^{\pi/2}\cot^2 x\,dx=\int_{\pi/6}^{\pi/2}\left(\csc^2 x-1\right)dx=[-\cot x-x]_{\pi/6}^{\pi/2}=\left(0-\frac{\pi}{2}\right)-\left(-\sqrt{3}-\frac{\pi}{6}\right)=\sqrt{3}-\frac{\pi}{3}$

36.

$$\int_{\pi/4}^{\pi/2}\cot^3 x\,dx=\int_{\pi/4}^{\pi/2}\cot x\left(\csc^2 x-1\right)dx=\int_{\pi/4}^{\pi/2}\cot x\csc^2 x\,dx-\int_{\pi/4}^{\pi/2}\frac{\cos x}{\sin x}\,dx$$

$$=\left[-\frac{1}{2}\cot^2 x-\ln|\sin x|\right]_{\pi/4}^{\pi/2}=(0-\ln 1)-\left[-\frac{1}{2}-\ln\frac{1}{\sqrt{2}}\right]=\frac{1}{2}+\ln\frac{1}{\sqrt{2}}=\frac{1}{2}(1-\ln 2)$$

37.

$$\int\cot^3\alpha\csc^3\alpha\,d\alpha=\int\cot^2\alpha\csc^2\alpha\cdot\csc\alpha\cot\alpha\,d\alpha=\int(\csc^2\alpha-1)\csc^2\alpha\cdot\csc\alpha\cot\alpha\,d\alpha$$

$$=\int(u^2-1)u^2\cdot(-du)\text{ [}u=\csc\alpha\text{ , }du=-\csc\alpha\cot\alpha\,d\alpha\text{]}$$

$$=\int(u^2-u^4)du=\frac{1}{3}u^3-\frac{1}{5}u^5+C=\frac{1}{3}\csc^3\alpha-\frac{1}{5}\csc^5\alpha+C$$

38.

$$\int\csc^4 x\cot^6 x\,dx=\int\cot^6 x(\cot^2 x+1)\csc^2 x\,dx$$

$$=\int u^6(u^2+1)\cdot(-du)\text{ [}u=\cot x\text{ , }du=-\csc^2 x\,dx\text{]}$$

$$=\int(-u^8-u^6)du=-\frac{1}{9}u^9-\frac{1}{7}u^7+C=-\frac{1}{9}\cot^9 x-\frac{1}{7}\cot^7 x+C$$

39. $I=\int\csc x\,dx=\int\frac{\csc x(\csc x-\cot x)}{\csc x-\cot x}\,dx=\int\frac{-\csc x\cot x+\csc^2 x}{\csc x-\cot x}\,dx$. Let $u=\csc x-\cot x\Rightarrow$ $du=\left(-\csc x\cot x+\csc^2 x\right)dx$. Then $I=\int du/u=\ln|u|=\ln|\csc x-\cot x|+C$.

40.

$$\int\csc^4 x\cot^6 x\,dx=\int\cot^6 x(\cot^2 x+1)\csc^2 x\,dx$$

$$=\int u^6(u^2+1)\cdot(-du)\text{ [}u=\cot x\text{ , }du=-\csc^2 x\,dx\text{]}$$

$$=\int (-u^8-u^6)du=-\frac{1}{9}u^9-\frac{1}{7}u^7+C=-\frac{1}{9}\cot^9 x-\frac{1}{7}\cot^7 x+C$$

41. Use Equation 2(b):

$$\int \sin 5x\sin 2x\,dx=\int \frac{1}{2}\left[\cos(5x-2x)-\cos(5x+2x)\right]dx=\frac{1}{2}\int(\cos 3x-\cos 7x)dx$$

$$=\frac{1}{6}\sin 3x-\frac{1}{14}\sin 7x+C$$

42. Use Equation 2(a):

$$\int \sin 3x\cos x\,dx=\int \frac{1}{2}\left[\sin(3x+x)+\sin(3x-x)\right]dx=\frac{1}{2}\int(\sin 4x+\sin 2x)dx$$

$$=-\frac{1}{8}\cos 4x-\frac{1}{4}\cos 2x+C$$

43. Use Equation 2(c):

$$\int \cos 7\theta\cos 5\theta\,d\theta=\int \frac{1}{2}\left[\cos(7\theta-5\theta)+\cos(7\theta+5\theta)\right]d\theta=\frac{1}{2}\int(\cos 2\theta+\cos 12\theta)d\theta$$

$$=\frac{1}{2}\left(\frac{1}{2}\sin 2\theta+\frac{1}{12}\sin 12\theta\right)+C=\frac{1}{4}\sin 2\theta+\frac{1}{24}\sin 12\theta+C$$

44.

$$\int \frac{\cos x+\sin x}{\sin 2x}dx=\frac{1}{2}\int \frac{\cos x+\sin x}{\sin x\cos x}dx=\frac{1}{2}\int(\csc x+\sec x)dx$$

$$=\frac{1}{2}\left(\ln|\csc x-\cot x|+\ln|\sec x+\tan x|\right)+C\ \text{[by Exercise 39 and (1)]}$$

45. $\displaystyle\int \frac{1-\tan^2 x}{\sec^2 x}dx=\int\left(\cos^2 x-\sin^2 x\right)dx=\int \cos 2x\,dx=\frac{1}{2}\sin 2x+C$

46.

$$\int \frac{dx}{\cos x-1}=\int \frac{1}{\cos x-1}\cdot\frac{\cos x+1}{\cos x+1}dx=\int \frac{\cos x+1}{\cos^2 x-1}dx=\int \frac{\cos x+1}{-\sin^2 x}dx$$

$$=\int\left(-\cot x\csc x-\csc^2 x\right)dx=\csc x+\cot x+C$$

47. Let $u=\tan(t^2)\Rightarrow du=2t\sec^2(t^2)dt$. Then

$\int t\sec^2\left(t^2\right)\tan^4\left(t^2\right)dt=\int u^4\left(\frac{1}{2}\,du\right)=\frac{1}{10}u^5+C=\frac{1}{10}\tan^5(t^2)+C$.

48. Let $u=\tan^7 x$, $dv=\sec x\tan x dx \Rightarrow du=7\tan^6 x\sec^2 x dx$, $v=\sec x$. Then

$$\begin{aligned}\int \tan^8 x\sec x dx &=\int \tan^7 x\cdot \sec x\tan x dx=\tan^7 x\sec x-\int 7\tan^6 x\sec^2 x\sec x dx\\ &=\tan^7 x\sec x-7\int \tan^6 x\left(\tan^2 x+1\right)\sec x dx\\ &=\tan^7 x\sec x-7\int \tan^8 x\sec x dx-7\int \tan^6 x\sec x dx .\end{aligned}$$

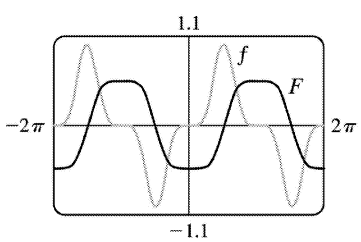
Thus, $8\int \tan^8 x\sec x dx=\tan^7 x\sec x-7\int \tan^6 x\sec x dx$ and $\int_0^{\pi/4}\tan^8 x\sec x$

$dx=\frac{1}{8}\left[\tan^7 x\sec x\right]_0^{\pi/4}-\frac{7}{8}\int_0^{\pi/4}\tan^6 x\sec x dx=\frac{\sqrt{2}}{8}-\frac{7}{8}I$.

49. Let $u=\cos x \Rightarrow du=-\sin x dx$. Then

$$\begin{aligned}\int \sin^5 x dx &= \int \left(1-\cos^2 x\right)^2\sin x dx=\int \left(1-u^2\right)^2(-du)\\ &= \int \left(-1+2u^2-u^4\right)du=-\frac{1}{5}u^5+\frac{2}{3}u^3-u+C\\ &= -\frac{1}{5}\cos^5 x+\frac{2}{3}\cos^3 x-\cos x+C\end{aligned}$$

Notice that F is increasing when $f(x)>0$, so the graphs serve as a check on our work.

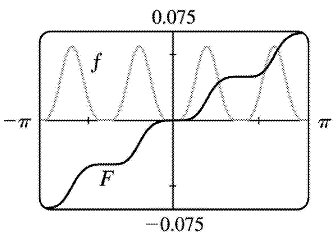


50.

$$\begin{aligned}\int \sin^4 x\cos^4 x dx &=\int \left(\frac{1}{2}\sin 2x\right)^4 dx=\frac{1}{16}\int \sin^4 2x dx=\frac{1}{16}\int \left[\frac{1}{2}(1-\cos 4x)\right]^2 dx\\ &=\frac{1}{64}\int \left(1-2\cos 4x+\cos^2 4x\right)dx\\ &=\frac{1}{64}\left(x-\frac{1}{2}\sin 4x\right)+\frac{1}{128}\int (1+\cos 8x)dx\end{aligned}$$

$$=\frac{1}{64}\left(x-\frac{1}{2}\sin 4x\right)+\frac{1}{128}\left(x+\frac{1}{8}\sin 8x\right)+C$$
$$=\frac{3}{128}x-\frac{1}{128}\sin 4x+\frac{1}{1024}\sin 8x+C$$

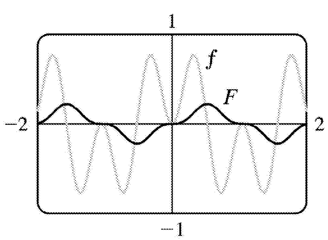
Notice that $f(x)=0$ whenever F has a horizontal tangent.



51.

$$\int \sin 3x\sin 6x\,dx=\int \frac{1}{2}\left[\cos(3x-6x)-\cos(3x+6x)\right]dx$$
$$=\frac{1}{2}\int(\cos 3x-\cos 9x)\,dx$$
$$=\frac{1}{6}\sin 3x-\frac{1}{18}\sin 9x+C$$

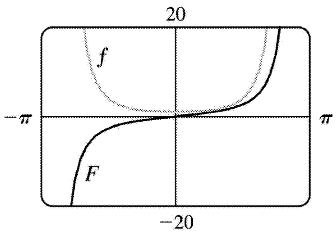
Notice that $f(x)=0$ whenever F has a horizontal tangent.



52.

$$\int \sec^4\frac{x}{2}\,dx=\int\left(\tan^2\frac{x}{2}+1\right)\sec^2\frac{x}{2}\,dx$$
$$=\int\left(u^2+1\right)2\,du\ \left[u=\tan\frac{x}{2},\ du=\frac{1}{2}\sec^2\frac{x}{2}\,dx\right]$$
$$=\frac{2}{3}u^3+2u+C=\frac{2}{3}\tan^3\frac{x}{2}+2\tan\frac{x}{2}+C$$

Notice that F is increasing and f is positive on the intervals on which they are defined. Also, F has no horizontal tangent and f is never zero.



53.

$$f_{\text{ave}} = \frac{1}{2\pi}\int_{-\pi}^{\pi}\sin^2 x\cos^3 x\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi}\sin^2 x\left(1-\sin^2 x\right)\cos x\,dx$$
$$= \frac{1}{2\pi}\int_0^0 u^2\left(1-u^2\right)du \text{ [where } u=\sin x \text{]}$$
$$=0$$

54. **(a)** Let $u=\cos x$. Then $du=-\sin x\,dx \Rightarrow \int \sin x\cos x\,dx = \int u(-du) = -\frac{1}{2}u^2+C = -\frac{1}{2}\cos^2 x+C_1$.

(b) Let $u=\sin x$. Then $du=\cos x\,dx \Rightarrow \int \sin x\cos x\,dx = \int u\,du = \frac{1}{2}u^2+C = \frac{1}{2}\sin^2 x+C_2$.

(c) $\int \sin x\cos x\,dx = \int \frac{1}{2}\sin 2x\,dx = -\frac{1}{4}\cos 2x+C_3$

(d) Let $u=\sin x$, $dv=\cos x\,dx$. Then $du=\cos x\,dx$, $v=\sin x$,

so $\int \sin x\cos x\,dx = \sin^2 x - \int \sin x\cos x\,dx$, by Equation .1.2, so $\int \sin x\cos x\,dx = \frac{1}{2}\sin^2 x+C_4$.

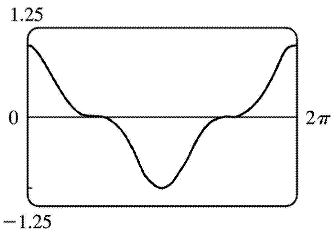
The answers differ from one another by constants. Since
$\cos 2x = 1-2\sin^2 x = 2\cos^2 x-1$, we find that $-\frac{1}{4}\cos 2x = \frac{1}{2}\sin^2 x - \frac{1}{4} = -\frac{1}{2}\cos^2 x + \frac{1}{4}$.

55. For $0<x<\frac{\pi}{2}$, we have $0<\sin x<1$, so $\sin^3 x<\sin x$. Hence the area is
$\int_0^{\pi/2}\left(\sin x-\sin^3 x\right)dx = \int_0^{\pi/2}\sin x\left(1-\sin^2 x\right)dx = \int_0^{\pi/2}\cos^2 x\sin x\,dx$. Now let $u=\cos x \Rightarrow du=-\sin x\,dx$.
Then area $=\int_1^0 u^2(-du) = \int_0^1 u^2\,du = \left[\frac{1}{3}u^3\right]_0^1 = \frac{1}{3}$.

56. $\sin x>0$ for $0<x<\frac{\pi}{2}$, so the sign of $2\sin^2 x-\sin x$ [which equals $2\sin x\left(\sin x-\frac{1}{2}\right)$] is the same as that of $\sin x-\frac{1}{2}$. Thus $2\sin^2 x-\sin x$ is positive on $\left(\frac{\pi}{6},\frac{\pi}{2}\right)$ and negative on $\left(0,\frac{\pi}{6}\right)$. The desired area is

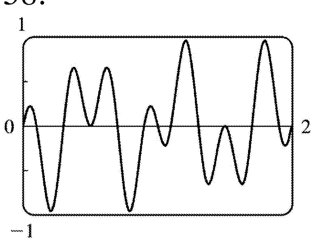
$$\int_0^{\pi/6}\left(\sin x - 2\sin^2 x\right)dx + \int_{\pi/6}^{\pi/2}\left(2\sin^2 x - \sin x\right)dx$$
$$= \int_0^{\pi/6}(\sin x - 1 + \cos 2x)\,dx + \int_{\pi/6}^{\pi/2}(1 - \cos 2x - \sin x)\,dx$$
$$= \left[-\cos x - x + \frac{1}{2}\sin 2x\right]_0^{\pi/6} + \left[x - \frac{1}{2}\sin 2x + \cos x\right]_{\pi/6}^{\pi/2}$$
$$= -\frac{\sqrt{3}}{2} - \frac{\pi}{6} + \frac{\sqrt{3}}{4} - (-1) + \frac{\pi}{2} - \left(\frac{\pi}{6} - \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{2}\right)$$
$$= 1 + \frac{\pi}{6} - \frac{\sqrt{3}}{2}$$

57.



It seems from the graph that $\int_0^{2\pi}\cos^3 x\,dx = 0$, since the area below the x–axis and above the graph looks about equal to the area above the axis and below the graph. By Example 1, the integral is $\left[\sin x - \frac{1}{3}\sin^3 x\right]_0^{2\pi} = 0$. Note that due to symmetry, the integral of any odd power of $\sin x$ or $\cos x$ between limits which differ by $2n\pi$ (n any integer) is 0.

58.



It seems from the graph that $\int_0^2 \sin 2\pi x \cos 5\pi x\,dx = 0$, since each bulge above the x– axis seems to have a corresponding depression below the x– axis. To evaluate the integral, we use a trigonometric identity:

$$\int_0^1 \sin 2\pi x \cos 5\pi x\,dx = \frac{1}{2}\int_0^2\left[\sin(2\pi x - 5\pi x) + \sin(2\pi x + 5\pi x)\right]dx$$
$$= \frac{1}{2}\int_0^2\left[\sin(-3\pi x) + \sin 7\pi x\right]dx$$

$$= \frac{1}{2}\left[\frac{1}{3\pi}\cos(-3\pi x) - \frac{1}{7\pi}\cos 7\pi x\right]_0^2$$
$$= \frac{1}{2}\left[\frac{1}{3\pi}(1-1) - \frac{1}{7\pi}(1-1)\right] = 0$$

59. $V = \int_{\pi/2}^{\pi} \pi \sin^2 x\,dx = \pi \int_{\pi/2}^{\pi} \frac{1}{2}(1-\cos 2x)\,dx = \pi\left[\frac{1}{2}x - \frac{1}{4}\sin 2x\right]_{\pi/2}^{\pi} = \pi\left(\frac{\pi}{2} - 0 - \frac{\pi}{4} + 0\right) = \frac{\pi^2}{4}$

60.

Volume $= \int_0^{\pi/4} \pi\left(\tan^2 x\right)^2 dx = \pi \int_0^{\pi/4} \tan^2 x\left(\sec^2 x - 1\right) dx = \pi \int_0^{\pi/4} \tan^2 x \sec^2 x\,dx - \pi \int_0^{\pi/4} \tan^2 x\,dx$

$$= \pi \int_0^{\pi/4} u^2\,du - \pi \int_0^{\pi/4} \left(\sec^2 x - 1\right) dx$$

$$= \pi\left[\frac{1}{3}u^3\right]_{x=0}^{\pi/4} - \pi\left[\tan x - x\right]_0^{\pi/4} = \pi\left[\frac{1}{3}\tan^3 x - \tan x + x\right]_0^{\pi/4} = \pi\left[\frac{1}{3} - 1 + \frac{\pi}{4}\right] = \pi\left(\frac{\pi}{4} - \frac{2}{3}\right)$$

61.

Volume $= \pi \int_0^{\pi/2} \left[(1+\cos x)^2 - 1^2\right] dx = \pi \int_0^{\pi/2} \left(2\cos x + \cos^2 x\right) dx$

$$= \pi\left[2\sin x + \frac{1}{2}x + \frac{1}{4}\sin 2x\right]_0^{\pi/2} = \pi\left(2 + \frac{\pi}{4}\right) = 2\pi + \frac{\pi^2}{4}$$

62.

Volume $= \pi \int_0^{\pi/2} \left[1^2 - (1-\cos x)^2\right] dx = \pi \int_0^{\pi/2} \left(2\cos x - \cos^2 x\right) dx$

$$= \pi\left[2\sin x - \frac{1}{2}x - \frac{1}{4}\sin 2x\right]_0^{\pi/2} = \pi\left[\left(2 - \frac{\pi}{4} - 0\right) - 0\right] = 2\pi - \frac{\pi^2}{4}$$

63. $s = f(t) = \int_0^t \sin \omega u \cos^2 \omega u\,du$. Let $y = \cos \omega u \Rightarrow dy = -\omega \sin \omega u\,du$. Then

$s = -\frac{1}{\omega}\int_1^{\cos \omega t} y^2\,dy = -\frac{1}{\omega}\left[\frac{1}{3}y^3\right]_1^{\cos \omega t} = \frac{1}{3\omega}\left(1 - \cos^3 \omega t\right)$.

64. **(a)** We want to calculate the square root of the average value of $[E(t)]^2 = [155\sin(120\pi t)]^2 = 155^2 \sin^2(120\pi t)$. First, we calculate the average value itself, by integrating $[E(t)]^2$ over one cycle (between $t=0$ and $t = \frac{1}{60}$, since there are 60 cycles per second) and

dividing by $\left(\frac{1}{60}-0\right)$:

$$[E(t)]^2_{\text{ave}} = \frac{1}{1/60}\int_0^{1/60}\left[155^2\sin^2(120\pi t)\right]dt=60\cdot 155^2\int_0^{1/60}\frac{1}{2}\left[1-\cos(240\pi t)\right]dt$$

$$= 60\cdot 155^2\left(\frac{1}{2}\right)\left[t-\frac{1}{240\pi}\sin(240\pi t)\right]_0^{1/60}=60\cdot 155^2\left(\frac{1}{2}\right)\left[\left(\frac{1}{60}-0\right)-(0-0)\right]=\frac{155^2}{2}$$

The RMS value is just the square root of this quantity, which is $\frac{155}{\sqrt{2}}\approx 110$ V.

(b) $220=\sqrt{[E(t)]^2_{\text{ave}}}\Rightarrow$

$$220^2 = [E(t)]^2_{\text{ave}}=\frac{1}{1/60}\int_0^{1/60}A^2\sin^2(120\pi t)\,dt=60A^2\int_0^{1/60}\frac{1}{2}\left[1-\cos(240\pi t)\right]dt$$

$$= 30A^2\left[t-\frac{1}{240\pi}\sin(240\pi t)\right]_0^{1/60}=30A^2\left[\left(\frac{1}{60}-0\right)-(0-0)\right]=\frac{1}{2}A^2$$

Thus, $220^2=\frac{1}{2}A^2\Rightarrow A=220\sqrt{2}\approx 311$ V.

65. Just note that the integrand is odd .
Or: If $m\neq n$, calculate

$$\int_{-\pi}^{\pi}\sin mx\cos nx\,dx = \int_{-\pi}^{\pi}\frac{1}{2}\left[\sin(m-n)x+\sin(m+n)x\right]dx$$

$$= \frac{1}{2}\left[-\frac{\cos(m-n)x}{m-n}-\frac{\cos(m+n)x}{m+n}\right]_{-\pi}^{\pi}=0$$

If $m=n$, then the first term in each set of brackets is zero.

66. $\int_{-\pi}^{\pi}\sin mx\sin nx\,dx=\int_{-\pi}^{\pi}\frac{1}{2}\left[\cos(m-n)x-\cos(m+n)x\right]dx$. If $m\neq n$, this is equal to $\frac{1}{2}\left[\frac{\sin(m-n)x}{m-n}-\frac{\sin(m+n)x}{m+n}\right]_{-\pi}^{\pi}=0$. If $m=n$, we get

$$\int_{-\pi}^{\pi}\frac{1}{2}\left[1-\cos(m+n)x\right]dx=\left[\frac{1}{2}x\right]_{-\pi}^{\pi}-\left[\frac{\sin(m+n)x}{2(m+n)}\right]_{-\pi}^{\pi}=\pi-0=\pi .$$

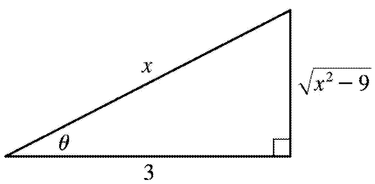
67. $\int_{-\pi}^{\pi}\cos mx\cos nx\,dx=\int_{-\pi}^{\pi}\frac{1}{2}\left[\cos(m-n)x+\cos(m+n)x\right]dx$. If $m\neq n$, this is equal to $\frac{1}{2}\left[\frac{\sin(m-n)x}{m-n}+\frac{\sin(m+n)x}{m+n}\right]_{-\pi}^{\pi}=0$. If $m=n$, we get

$$\int_{-\pi}^{\pi}\frac{1}{2}\left[1+\cos(m+n)x\right]dx=\left[\frac{1}{2}x\right]_{-\pi}^{\pi}+\left[\frac{\sin(m+n)x}{2(m+n)}\right]_{-\pi}^{\pi}=\pi+0=\pi .$$

68. $\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin mx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}\left[\left(\sum_{n=1}^{m} a_n \sin nx\right)\sin mx\right]dx = \sum_{n=1}^{m}\frac{a_n}{\pi}\int_{-\pi}^{\pi}\sin mx \sin nx\,dx$. By Exercise 66 , every term is zero except the m th one, and that term is $\frac{a_m}{\pi}\cdot\pi = a_m$.

1. Let $x=3\sec\theta$, where $0\le\theta<\dfrac{\pi}{2}$ or $\pi\le\theta<\dfrac{3\pi}{2}$. Then $dx=3\sec\theta\,\tan\theta\,d\theta$ and

$$\begin{aligned}\sqrt{x^2-9} &= \sqrt{9\sec^2\theta-9}=\sqrt{9\left(\sec^2\theta-1\right)}=\sqrt{9\tan^2\theta}\\ &= 3\left|\tan\theta\right|=3\tan\theta \text{ for the relevant values of } \theta\end{aligned}$$



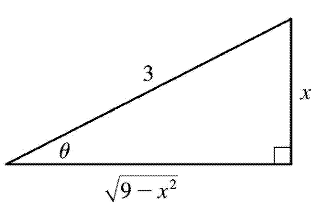
$$\int\frac{1}{x^2\sqrt{x^2-9}}\,dx=\int\frac{1}{9\sec^2\theta\cdot 3\tan\theta}\,3\sec\theta\,\tan\theta\,d\theta=\frac{1}{9}\int\cos\theta\,d\theta=\frac{1}{9}\sin\theta+C=\frac{1}{9}\,\frac{\sqrt{x^2-9}}{x}+C$$

Note that $-\sec(\theta+\pi)=\sec\theta$, so the figure is sufficient for the case $\pi\le\theta<\dfrac{3\pi}{2}$.

2. Let $x=3\sin\theta$, where $-\dfrac{\pi}{2}\le\theta\le\dfrac{\pi}{2}$. Then $dx=3\cos\theta\,d\theta$ and

$$\sqrt{9-x^2} = \sqrt{9-9\sin^2\theta}=\sqrt{9\left(1-\sin^2\theta\right)}=\sqrt{9\cos^2\theta}$$

$= 3\left|\cos\theta\right|=3\cos\theta$ for the relevant values of \italic{\theta}\,\,.

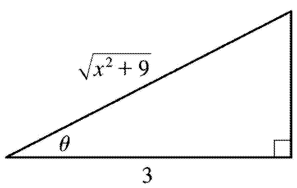


$$\begin{aligned}\int x^3\sqrt{9-x^2}\,dx &= \int 3^3\sin^3\theta\cdot 3\cos\theta\cdot 3\cos\theta\,d\theta=3^5\int\sin^3\theta\,\cos^2\theta\,d\theta\\ &= 3^5\int\sin^2\theta\,\cos^2\theta\,\sin\theta\,d\theta=3^5\int\left(1-\cos^2\theta\right)\cos^2\theta\,\sin\theta\,d\theta\\ &= 3^5\int\left(1-u^2\right)u^2(-du)[u=\cos\theta,du=-\sin\theta\,d\theta]\\ &= 3^5\int\left(u^4-u^2\right)du=3^5\left(\frac{1}{5}u^5-\frac{1}{3}u^3\right)+C=3^5\left(\frac{1}{5}\cos^5\theta-\frac{1}{3}\cos^3\theta\right)+C\\ &= 3^5\left[\frac{1}{5}\,\frac{\left(9-x^2\right)^{5/2}}{3^5}-\frac{1}{3}\,\frac{\left(9-x^2\right)^{3/2}}{3^3}\right]+C\end{aligned}$$

$$= \frac{1}{5}\left(9-x^2\right)^{5/2} - 3\left(9-x^2\right)^{3/2} + C \text{ or} - \frac{1}{5}\left(x^2+6\right)(9-x)^{3/2} + C$$

3. Let $x=3\tan\theta$, where $-\frac{\pi}{2}<\theta<\frac{\pi}{2}$. Then $dx=3\sec^2\theta\ d\theta$ and

$$\sqrt{x^2+9} = \sqrt{9\tan^2\theta+9} = \sqrt{9\left(\tan^2\theta+1\right)} = \sqrt{9\sec^2\theta}$$
$$= 3|\sec\theta| = 3\sec\theta \text{ for the relevant values of } \theta .$$



$$\int \frac{x^3}{\sqrt{x^2+9}}\,dx = \int \frac{3^3\tan^3\theta}{3\sec\theta}\,3\sec^2\theta\ d\theta = 3^3\int \tan^3\theta\ \sec\theta\ d\theta = 3^3\int \tan^2\theta\ \tan\theta\ \sec\theta\ d\theta$$
$$= 3^3\int\left(\sec^2\theta-1\right)\tan\theta\ \sec\theta\ d\theta = 3^3\int\left(u^2-1\right)du\,[u=\sec\theta\ ,du=\sec\theta\ \tan\theta\ d\theta\ ]$$
$$= 3^3\left(\frac{1}{3}u^3-u\right)+C = 3^3\left(\frac{1}{3}\sec^3\theta-\sec\theta\right)+C = 3^3\left[\frac{1}{3}\frac{\left(x^2+9\right)^{3/2}}{3^3}-\frac{\sqrt{x^2+9}}{3}\right]+C$$
$$= \frac{1}{3}\left(x^2+9\right)^{3/2} - 9\sqrt{x^2+9}+C \text{ or } \frac{1}{3}\left(x^2-18\right)\sqrt{x^2+9}+C$$

4. Let $x=4\sin\theta$, where $-\pi/2\le\theta\le\pi/2$. Then $dx=4\cos\theta\,d\theta$ and $\sqrt{16-x^2}=\sqrt{16-16\sin^2\theta}=\sqrt{16\cos^2\theta}=4|\cos\theta|=4\cos\theta$. When $x=0$, $4\sin\theta=0\Rightarrow\theta=0$, and when $x=2\sqrt{3}$, $4\sin\theta=2\sqrt{3}\Rightarrow\sin\theta=\frac{\sqrt{3}}{2}\Rightarrow\theta=\frac{\pi}{3}$. Thus, substitution gives

$$\int_0^{2\sqrt{3}} \frac{x^3}{\sqrt{16-x^2}}\,dx = \int_0^{\pi/3} \frac{4^3\sin^3\theta}{4\cos\theta}\,4\cos\theta\,d\theta = 4^3\int_0^{\pi/3}\sin^3\theta\,d\theta$$

$$=4^3\int_0^{\pi/3}\left(1-\cos^2\theta\right)\sin\theta\,d\theta$$

$$=-4^3\int_1^{1/2}\left(1-u^2\right)du=-64\left[u-\frac{1}{3}u^3\right]_1^{1/2}$$

$$=-64\left[\left(\frac{1}{2}-\frac{1}{24}\right)-\left(1-\frac{1}{3}\right)\right]=-64\left(-\frac{5}{24}\right)=\frac{40}{3}$$

Or: Let $u=16-x^2$, $x^2=16-u$, $du=-2x\,dx$.

5. Let $t=\sec\theta$, so $dt=\sec\theta\tan\theta\,d\theta$, $t=\sqrt{2}\Rightarrow\theta=\frac{\pi}{4}$, and $t=2\Rightarrow\theta=\frac{\pi}{3}$. Then

$$\int_{\sqrt{2}}^{2}\frac{1}{t^3\sqrt{t^2-1}}\,dt = \int_{\pi/4}^{\pi/3}\frac{1}{\sec^3\theta\tan\theta}\sec\theta\tan\theta\,d\theta=\int_{\pi/4}^{\pi/3}\frac{1}{\sec^2\theta}\,d\theta=\int_{\pi/4}^{\pi/3}\cos^2\theta\,d\theta$$

$$=\int_{\pi/4}^{\pi/3}\frac{1}{2}(1+\cos 2\theta)\,d\theta=\frac{1}{2}\left[\theta+\frac{1}{2}\sin 2\theta\right]_{\pi/4}^{\pi/3}$$

$$=\frac{1}{2}\left[\left(\frac{\pi}{3}+\frac{1}{2}\cdot\frac{\sqrt{3}}{2}\right)-\left(\frac{\pi}{4}+\frac{1}{2}\cdot 1\right)\right]=\frac{1}{2}\left(\frac{\pi}{12}+\frac{\sqrt{3}}{4}-\frac{1}{2}\right)=\frac{\pi}{24}+\frac{\sqrt{3}}{8}-\frac{1}{4}$$

6. Let $x=2\tan\theta$, so $dx=2\sec^2\theta\,d\theta$, $x=0\Rightarrow\theta=0$, and $x=2\Rightarrow\theta=\frac{\pi}{4}$. Then

$$\int_0^2 x^3\sqrt{x^2+4}\,dx=\int_0^{\pi/4}2^3\tan^3\theta\cdot 2\sec\theta\cdot 2\sec^2\theta\,d\theta=2^5\int_0^{\pi/4}\tan^2\theta\,\sec^2\theta\,\sec\theta\,\tan\theta\,d\theta$$

$$=2^5\int_0^{\pi/4}(\sec^2\theta-1)\sec^2\theta\,\sec\theta\,\tan\theta\,d\theta$$

$$=2^5\int_1^{\sqrt{2}}(u^2-1)u^2\,du\ [u=\sec\theta,\ du=\sec\theta\,\tan\theta\,d\theta]$$

$$=2^5\int_1^{\sqrt{2}}(u^4-u^2)\,du=2^5\left[\frac{1}{5}u^5-\frac{1}{3}u^3\right]_1^{\sqrt{2}}=2^5\left[\left(\frac{1}{5}\cdot 4\sqrt{2}-\frac{1}{3}\cdot 2\sqrt{2}\right)-\left(\frac{1}{5}-\frac{1}{3}\right)\right]$$

$$=32\left(\frac{2}{15}\sqrt{2}+\frac{2}{15}\right)=\frac{64}{15}\left(\sqrt{2}+1\right)$$

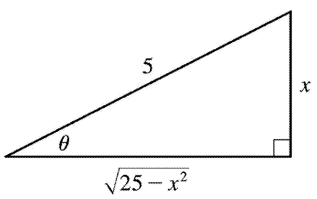
Or: Let $u=x^2+4$, $x^2=u-4$, $du=2x\,dx$.

7. Let $x=5\sin\theta$, so $dx=5\cos\theta\ d\theta$. Then

$$\int \frac{1}{x^2\sqrt{25-x^2}}\,dx = \int \frac{1}{5^2\sin^2\theta\cdot 5\cos\theta}\,5\cos\theta\ d\theta$$

$$= \frac{1}{25}\int \csc^2\theta\ d\theta = -\frac{1}{25}\cot\theta + C$$

$$= -\frac{1}{25}\,\frac{\sqrt{25-x^2}}{x} + C$$

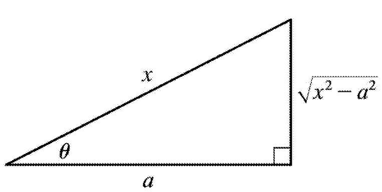


8. Let $x=a\sec\theta$, where $0\le\theta<\frac{\pi}{2}$ or $\pi\le\theta<\frac{3\pi}{2}$. Then $dx=a\sec\theta\ \tan\theta\ d\theta$ and $\sqrt{x^2-a^2}=a\tan\theta$, so

$$\int \frac{\sqrt{x^2-a^2}}{x^4}\,dx = \int \frac{a\tan\theta}{a^4\sec^4\theta}\,a\sec\theta\ \tan\theta\ d\theta$$

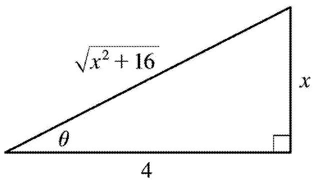
$$= \frac{1}{a^2}\int \sin^2\theta\ \cos\theta\ d\theta$$

$$= \frac{1}{3a^2}\sin^3\theta + C = \frac{\left(x^2-a^2\right)^{3/2}}{3a^2x^3} + C$$



9. Let $x=4\tan\theta$, where $-\frac{\pi}{2}<\theta<\frac{\pi}{2}$. Then $dx=4\sec^2\theta\ d\theta$ and

$$\sqrt{x^2+16}=\sqrt{16\tan^2\theta+16}=\sqrt{16(\tan^2\theta+1)}$$
$$=\sqrt{16\sec^2\theta}=4|\sec\theta|$$
$$=4\sec\theta \text{ for the relevant values of } \theta .$$

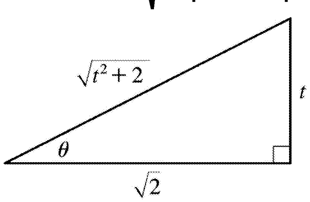


$$\int\frac{dx}{\sqrt{x^2+16}}=\int\frac{4\sec^2\theta\ d\theta}{4\sec\theta}=\int\sec\theta\ d\theta=\ln|\sec\theta+\tan\theta|+C_1$$
$$=\ln\left|\frac{\sqrt{x^2+16}}{4}+\frac{x}{4}\right|+C_1=\ln\left|\sqrt{x^2+16}+x\right|-\ln|4|+C_1$$
$$=\ln\left(\sqrt{x^2+16}+x\right)+C \text{ , where } C=C_1-\ln 4 .$$

(Since $\sqrt{x^2+16}+x>0$, we don’t need the absolute value.)

10. Let $t=\sqrt{2}\tan\theta$, where $-\frac{\pi}{2}<\theta<\frac{\pi}{2}$. Then $dt=\sqrt{2}\sec^2\theta\ d\theta$ and

$$\sqrt{t^2+2}=\sqrt{2\tan^2\theta+2}=\sqrt{2(\tan^2\theta+1)}=\sqrt{2\sec^2\theta}$$
$$=\sqrt{2}|\sec\theta|=\sqrt{2}\sec\theta \text{ for the relevant values of } \theta .$$



$$\int\frac{t^5}{\sqrt{t^2+2}}\,dt=\int\frac{4\sqrt{2}\tan^5\theta}{\sqrt{2}\sec\theta}\sqrt{2}\sec^2\theta\ d\theta=4\sqrt{2}\int\tan^5\theta\ \sec\theta\ d\theta=4\sqrt{2}\int\left(\sec^2\theta-1\right)^2\sec\theta\ \tan\theta\ d\theta$$
$$=4\sqrt{2}\int\left(u^2-1\right)^2du\ [\ u=\sec\theta\ ,\ du=\sec\theta\ \tan\theta\ d\theta\ ]=4\sqrt{2}\int\left(u^4-2u^2+1\right)du$$

$$=4\sqrt{2}\left(\frac{1}{5}u^5-\frac{2}{3}u^3+u\right)+C=\frac{4\sqrt{2}}{15}u(3u^4-10u^2+15)+C$$

$$=\frac{4\sqrt{2}}{15}\cdot\frac{\sqrt{t^2+2}}{\sqrt{2}}\left[3\cdot\frac{(t^2+2)^2}{2^2}-10\,\frac{t^2+2}{2}+15\right]+C$$

$$=\frac{4}{15}\sqrt{t^2+2}\cdot\frac{1}{4}\left[3(t^4+4t^2+4)-20(t^2+2)+60\right]+C$$

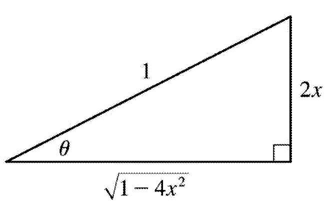
$$=\frac{1}{15}\sqrt{t^2+2}\,(3t^4-8t^2+32)+C$$

11. Let $2x=\sin\theta$, where $-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$. Then $x=\frac{1}{2}\sin\theta$, $dx=\frac{1}{2}\cos\theta\;d\theta$, and $\sqrt{1-4x^2}=\sqrt{1-(2x)^2}=\cos\theta$.

$$\int\sqrt{1-4x^2}\,dx=\int\cos\theta\left(\frac{1}{2}\cos\theta\right)d\theta=\frac{1}{4}\int(1+\cos 2\theta)\,d\theta$$

$$=\frac{1}{4}\left(\theta+\frac{1}{2}\sin 2\theta\right)+C=\frac{1}{4}(\theta+\sin\theta\cos\theta)+C$$

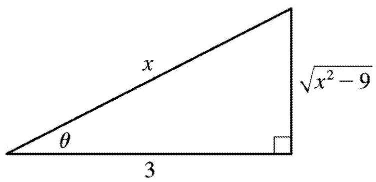
$$=\frac{1}{4}\left[\sin^{-1}(2x)+2x\sqrt{1-4x^2}\right]+C$$



12. $\int_0^1 x\sqrt{x^2+4}\,dx=\int_4^5\sqrt{u}\left(\frac{1}{2}\,du\right)$ [$u=x^2+4$, $du=2x\,dx$] $=\frac{1}{2}\cdot\frac{2}{3}\left[u^{3/2}\right]_4^5=\frac{1}{3}\left(5\sqrt{5}-8\right)$

13. Let $x=3\sec\theta$, where $0\le\theta<\frac{\pi}{2}$ or $\pi\le\theta<\frac{3\pi}{2}$. Then $dx=3\sec\theta\tan\theta\;d\theta$ and $\sqrt{x^2-9}=3\tan\theta$, so

$$\int\frac{\sqrt{x^2-9}}{x^3}\,dx=\int\frac{3\tan\theta}{27\sec^3\theta}\,3\sec\theta\tan\theta\;d\theta=\frac{1}{3}\int\frac{\tan^2\theta}{\sec^2\theta}\,d\theta$$



$$=\frac{1}{3}\int \sin^2\theta\ d\theta=\frac{1}{3}\int\frac{1}{2}(1-\cos 2\theta)d\theta=\frac{1}{6}\theta-\frac{1}{12}\sin 2\theta+C=\frac{1}{6}\theta-\frac{1}{6}\sin\theta\ \cos\theta+C$$

$$=\frac{1}{6}\sec^{-1}\left(\frac{x}{3}\right)-\frac{1}{6}\ \frac{\sqrt{x^2-9}}{x}\ \frac{3}{x}+C=\frac{1}{6}\sec^{-1}\left(\frac{x}{3}\right)-\frac{\sqrt{x^2-9}}{2x^2}+C$$

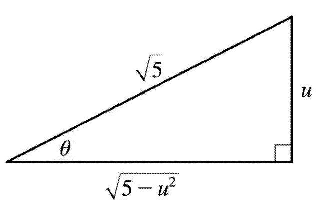
14. Let $u=\sqrt{5}\sin\theta$, so $du=\sqrt{5}\cos\theta\ d\theta$. Then

$$\int\frac{du}{u\sqrt{5-u^2}}=\int\frac{1}{\sqrt{5}\sin\theta\cdot\sqrt{5}\cos\theta}\sqrt{5}\cos\theta\ d\theta=\frac{1}{\sqrt{5}}\int\csc\theta\ d\theta$$

$$=\frac{1}{\sqrt{5}}\ln\left|\csc\theta-\cot\theta\right|+C \text{ [byExercise2.39}$$

$$=\frac{1}{\sqrt{5}}\ln\left|\frac{\sqrt{5}}{u}-\frac{\sqrt{5-u^2}}{u}\right|+C$$

$$=\frac{1}{\sqrt{5}}\ln\left|\frac{\sqrt{5}-\sqrt{5-u^2}}{u}\right|+C$$

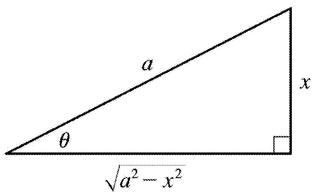


15. Let $x=a\sin\theta$, where $-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$. Then $dx=a\cos\theta\ d\theta$ and

$$\frac{x^2dx}{\left(a^2-x^2\right)^{3/2}}=\int\frac{a^2\sin^2\theta\ a\cos\theta\ d\theta}{a^3\cos^3\theta}=\int\tan^2\theta\ d\theta$$

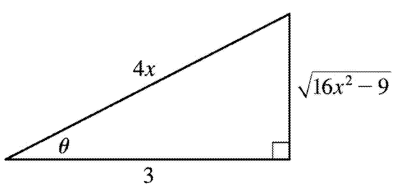
$$=\int\left(\sec^2\theta-1\right)d\theta=\tan\theta-\theta+C$$

$$= \frac{x}{\sqrt{a^2 - x^2}} - \sin^{-1}\frac{x}{a} + C$$



16. Let $4x = 3\sec\theta$, where $0 \le \theta < \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$. Then $dx = \frac{3}{4}\sec\theta\tan\theta\,d\theta$ and $\sqrt{16x^2 - 9} = 3\tan\theta$, so

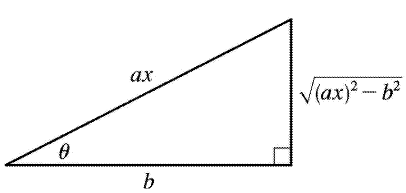
$$\int \frac{dx}{x^2\sqrt{16x^2-9}} = \int \frac{\frac{3}{4}\sec\theta\tan\theta\,d\theta}{\left(\frac{3}{4}\right)^2\sec^2\theta\;3\tan\theta}$$



$$= \frac{4}{9}\int\cos\theta\,d\theta = \frac{4}{9}\sin\theta + C = \frac{4}{9}\,\frac{\sqrt{16x^2-9}}{4x} + C = \frac{\sqrt{16x^2-9}}{9x} + C$$

17. Let $u = x^2 - 7$, so $du = 2x\,dx$. Then $\int \frac{x}{\sqrt{x^2-7}}\,dx = \frac{1}{2}\int\frac{1}{\sqrt{u}}\,du = \frac{1}{2}\cdot 2\sqrt{u} + C = \sqrt{x^2-7} + C$.

18. Let $ax = b\sec\theta$, so $(ax)^2 = b^2\sec^2\theta \Rightarrow (ax)^2 - b^2 = b^2\sec^2\theta - b^2 = b^2\left(\sec^2\theta - 1\right) = b^2\tan^2\theta$. So $\sqrt{\left(ax^2\right) - b^2} = b\tan\theta$, $dx = \frac{b}{a}\sec\theta\tan\theta\,d\theta$, and



$$\int \frac{dx}{\left[(ax)^2-b^2\right]^{3/2}} = \int \frac{\frac{b}{a}\sec\theta\,\tan\theta}{b^3\tan^3\theta}\,d\theta = \frac{1}{ab^2}\int \frac{\sec\theta}{\tan^2\theta}\,d\theta = \frac{1}{ab^2}\int \frac{\cos\theta}{\sin^2\theta}\,d\theta = \frac{1}{ab^2}\int \csc\theta\,\cot\theta\,d\theta$$

$$= -\frac{1}{ab^2}\csc\theta + C = -\frac{1}{ab^2}\,\frac{ax}{\sqrt{(ax)^2-b^2}} + C = -\frac{x}{b^2\sqrt{(ax)^2-b^2}} + C$$

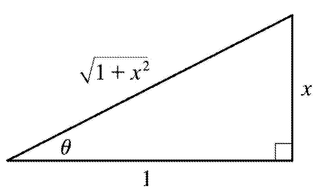
19. Let $x=\tan\theta$, where $-\frac{\pi}{2}<\theta<\frac{\pi}{2}$. Then $dx=\sec^2\theta\,d\theta$ and $\sqrt{1+x^2}=\sec\theta$, so

$$\int \frac{\sqrt{1+x^2}}{x}\,dx = \int \frac{\sec\theta}{\tan\theta}\sec^2\theta\,d\theta = \int \frac{\sec\theta}{\tan\theta}(1+\tan^2\theta)\,d\theta$$

$$= \int (\csc\theta + \sec\theta\,\tan\theta)\,d\theta$$

$$= \ln|\csc\theta - \cot\theta| + \sec\theta + C \quad \text{[by Exercise 8.2.39]}$$

$$= \ln\left|\frac{\sqrt{1+x^2}}{x} - \frac{1}{x}\right| + \frac{\sqrt{1+x^2}}{1} + C = \ln\left|\frac{\sqrt{1+x^2}-1}{x}\right| + \sqrt{1+x^2} + C$$

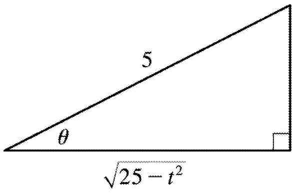


20. Let $t=5\sin\theta$, where $-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$. Then $dt=5\cos\theta\,d\theta$ and $\sqrt{25-t^2}=5\cos\theta$, so

$$\int \frac{t}{\sqrt{25-t^2}}\,dt = \int \frac{5\sin\theta}{5\cos\theta}\,5\cos\theta\,d\theta = 5\int \sin\theta\,d\theta$$

$$= -5\cos\theta + C = -5\cdot\frac{\sqrt{25-t^2}}{5} + C = -\sqrt{25-t^2} + C$$

Or: Let $u=25-t^2$, so $du=-2t\,dt$.



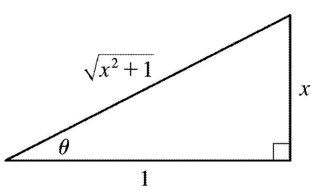
21. Let $u=4-9x^2 \Rightarrow du=-18x\,dx$. Then $x^2=\frac{1}{9}\,(4-u)$ and

$$\int_0^{2/3} x^3\sqrt{4-9x^2}\,dx = \int_4^0 \frac{1}{9}\,(4-u)u^{1/2}\left(-\frac{1}{18}\right)du=\frac{1}{162}\int_0^4\left(4u^{1/2}-u^{3/2}\right)du$$
$$=\frac{1}{162}\left[\frac{8}{3}u^{3/2}-\frac{2}{5}u^{5/2}\right]_0^4=\frac{1}{162}\left[\frac{64}{3}-\frac{64}{5}\right]=\frac{64}{1215}$$

Or: Let $3x=2\sin\theta$, where $-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$.

22. Let $x=\tan\theta$, where $-\frac{\pi}{2}<\theta<\frac{\pi}{2}$. Then $dx=\sec^2\theta\,d\theta$, $\sqrt{x^2+1}=\sec\theta$ and $x=0\Rightarrow\theta=0$, $x=1\Rightarrow$ $\theta=\frac{\pi}{4}$, so

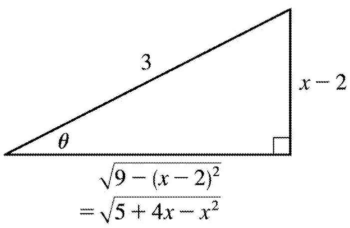
$$\int_0^1\sqrt{x^2+1}\,dx = \int_0^{\pi/4}\sec\theta\sec^2\theta\,d\theta=\int_0^{\pi/4}\sec^3\theta\,d\theta$$
$$=\frac{1}{2}\left[\sec\theta\tan\theta+\ln|\sec\theta+\tan\theta|\right]_0^{\pi/4}\quad\text{[by Example 8.2.8]}$$
$$=\frac{1}{2}\left[\sqrt{2}\cdot 1+\ln\left(1+\sqrt{2}\right)-0-\ln(1+0)\right]=\frac{1}{2}\left[\sqrt{2}+\ln\left(1+\sqrt{2}\right)\right]$$



23. $5+4x-x^2=-(x^2-4x+4)+9=-(x-2)^2+9$. Let $x-2=3\sin\theta$, $-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$, so $dx=3\cos\theta\,d\theta$. Then

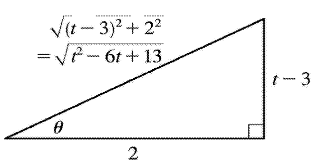
$$\int\sqrt{5+4x-x^2}\,dx=\int\sqrt{9-(x-2)^2}\,dx=\int\sqrt{9-9\sin^2\theta}\;3\cos\theta\,d\theta$$
$$=\int\sqrt{9\cos^2\theta}\;3\cos\theta\,d\theta=\int 9\cos^2\theta\,d\theta$$

$$=\frac{9}{2}\int(1+\cos 2\theta)\,d\theta=\frac{9}{2}\left(\theta+\frac{1}{2}\sin 2\theta\right)+C$$
$$=\frac{9}{2}\theta+\frac{9}{4}\sin 2\theta+C=\frac{9}{2}\theta+\frac{9}{4}(2\sin\theta\cos\theta)+C$$
$$=\frac{9}{2}\sin^{-1}\left(\frac{x-2}{3}\right)+\frac{9}{2}\cdot\frac{x-2}{3}\cdot\frac{\sqrt{5+4x-x^2}}{3}+C$$
$$=\frac{9}{2}\sin^{-1}\left(\frac{x-2}{3}\right)+\frac{1}{2}(x-2)\sqrt{5+4x-x^2}+C$$



24. $t^2-6t+13=\left(t^2-6t+9\right)+4=(t-3)^2+2^2$. Let $t-3=2\tan\theta$, so $dt=2\sec^2\theta\,d\theta$. Then

$$\int\frac{dt}{\sqrt{t^2-6t+13}}=\int\frac{1}{\sqrt{(2\tan\theta)^2+2^2}}\,2\sec^2\theta\,d\theta=\int\frac{2\sec^2\theta}{2\sec\theta}\,d\theta$$
$$=\int\sec\theta\,d\theta=\ln|\sec\theta+\tan\theta|+C_1\qquad\text{[by Formula 8.2.1]}$$
$$=\ln\left|\frac{\sqrt{t^2-6t+13}}{2}+\frac{t-3}{2}\right|+C_1$$
$$=\ln\left|\sqrt{t^2-6t+13}+t-3\right|+C\text{ where }C=C_1-\ln 2$$



25. $9x^2+6x-8=(3x+1)^2-9$, so let $u=3x+1$, $du=3dx$. Then $\int\frac{dx}{\sqrt{9x^2+6x-8}}=\int\frac{\frac{1}{3}du}{\sqrt{u^2-9}}$. Now let $u=3\sec\theta$, where

$0 \le \theta < \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$. Then $du = 3\sec\theta\,\tan\theta\;d\theta$ and $\sqrt{u^2-9} = 3\tan\theta$, so

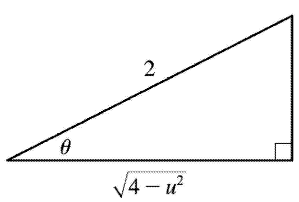
$$\int \frac{\frac{1}{3}\,du}{\sqrt{u^2-9}} = \int \frac{\sec\theta\,\tan\theta\;d\theta}{3\tan\theta} = \frac{1}{3}\int \sec\theta\;d\theta = \frac{1}{3}\ln\left|\sec\theta+\tan\theta\right| + C_1 = \frac{1}{3}\ln\left|\frac{u+\sqrt{u^2-9}}{3}\right| + C_1$$

$$= \frac{1}{3}\ln\left|u+\sqrt{u^2-9}\right| + C = \frac{1}{3}\ln\left|3x+1+\sqrt{9x^2+6x-8}\right| + C$$

26. $4x-x^2 = -\left(x^2-4x+4\right)+4 = 4-(x-2)^2$, so let $u = x-2$. Then $x = u+2$ and $dx = du$, so

$$\int \frac{x^2\,dx}{\sqrt{4x-x^2}} = \int \frac{(u+2)^2\,du}{\sqrt{4-u^2}} = \int \frac{(2\sin\theta+2)^2}{2\cos\theta}\,2\cos\theta\;d\theta$$

$$=4\int\left(\sin^2\theta+2\sin\theta+1\right)d\theta$$
$$=2\int(1-\cos 2\theta)\,d\theta+8\int\sin\theta\;d\theta+4\int d\theta$$
$$=2\theta-\sin 2\theta-8\cos\theta+4\theta+C$$
$$=6\theta-8\cos\theta-2\sin\theta\,\cos\theta+C$$
$$=6\sin^{-1}\left(\frac{1}{2}u\right)-4\sqrt{4-u^2}-\frac{1}{2}u\sqrt{4-u^2}+C$$
$$=6\sin^{-1}\left(\frac{x-2}{2}\right)-4\sqrt{4x-x^2}-\left(\frac{x-2}{2}\right)\sqrt{4x-x^2}+C$$



27. $x^2+2x+2 = (x+1)^2+1$. Let $u = x+1$, $du = dx$. Then

$$\int \frac{dx}{\left(x^2+2x+2\right)^2} = \int \frac{du}{\left(u^2+1\right)^2} = \int \frac{\sec^2\theta\;d\theta}{\sec^4\theta} \quad \left[\begin{array}{c}\text{where } u=\tan\theta\ ,du=\sec^2\theta\;d\theta\ ,\\ \text{and } u^2+1=\sec^2\theta \text{ x}\end{array}\right]$$

$$= \int \cos^2\theta\;d\theta = \frac{1}{2}\int(1+\cos 2\theta)\,d\theta = \frac{1}{2}(\theta+\sin\theta\,\cos\theta)+C$$

$$=\frac{1}{2}\left[\tan^{-1}u+\frac{u}{1+u^2}\right]+C=\frac{1}{2}\left[\tan^{-1}(x+1)+\frac{x+1}{x^2+2x+2}\right]+C$$

28. $5-4x-x^2=-\left(x^2+4x+4\right)+9=9-(x+2)^2$. Let $u=x+2 \Rightarrow du=dx$. Then

$$\int\frac{dx}{\left(5-4x-x^2\right)^{5/2}}=\int\frac{du}{\left(9-u^2\right)^{5/2}}=\int\frac{3\cos\theta\,d\theta}{(3\cos\theta)^5}\quad\left[\begin{array}{c}\text{where } u=3\sin\theta,\ du=3\cos\theta\,d\theta,\\ \text{and } \sqrt{9-u^2}=3\cos\theta\end{array}\right]$$

$$=\frac{1}{81}\int\sec^4\theta\,d\theta=\frac{1}{81}\int\left(\tan^2\theta+1\right)\sec^2\theta\,d\theta=\frac{1}{81}\left[\frac{1}{3}\tan^3\theta+\tan\theta\right]+C$$

$$=\frac{1}{243}\left[\frac{u^3}{\left(9-u^2\right)^{3/2}}+\frac{3u}{\sqrt{9-u^2}}\right]+C=\frac{1}{243}\left[\frac{(x+2)^3}{\left(5-4x-x^2\right)^{3/2}}+\frac{3(x+2)}{\sqrt{5-4x-x^2}}\right]+C$$

29. Let $u=x^2$, $du=2x\,dx$. Then

$$\int x\sqrt{1-x^4}\,dx=\int\sqrt{1-u^2}\left(\frac{1}{2}du\right)=\frac{1}{2}\int\cos\theta\cdot\cos\theta\,d\theta\quad\left[\begin{array}{c}\text{whereu}=\sin\theta,\text{du}=\cos\theta\ \text{d}\theta,\\ \text{and}\sqrt{1-u^2}=\cos\theta\end{array}\right]$$

$$=\frac{1}{2}\int\frac{1}{2}(1+\cos 2\theta)d\theta=\frac{1}{4}\theta+\frac{1}{8}\sin 2\theta+C=\frac{1}{4}\theta+\frac{1}{4}\sin\theta\cos\theta+C$$

$$=\frac{1}{4}\sin^{-1}u+\frac{1}{4}u\sqrt{1-u^2}+C=\frac{1}{4}\sin^{-1}(x^2)+\frac{1}{4}x^2\sqrt{1-x^4}+C$$

30. Let $u=\sin t$, $du=\cos t\,dt$. Then

$$\int_0^{\pi/2}\frac{\cos t}{\sqrt{1+\sin^2 t}}\,dt=\int_0^1\frac{1}{\sqrt{1+u^2}}\,dt=\int_0^{\pi/4}\frac{1}{\sec\theta}\sec^2\theta\,d\theta\quad\left[\begin{array}{c}\text{where u}=\tan\theta,\ \text{du}=\sec^2\theta\ \text{d}\theta,\\ \text{and}\sqrt{1+u^2}=\sec\theta\end{array}\right]$$

$$=\int_0^{\pi/4}\sec\theta\,d\theta=\left[\ln|\sec\theta+\tan\theta|\right]_0^{\pi/4}\quad\text{[by (1) in Section 8.2]}$$

$$=\ln\left(\sqrt{2}+1\right)-\ln(1+0)=\ln\left(\sqrt{2}+1\right)$$

31. **(a)** Let $x=a\tan\theta$, where $-\frac{\pi}{2}<\theta<\frac{\pi}{2}$. Then $\sqrt{x^2+a^2}=a\sec\theta$ and

$$\int\frac{dx}{\sqrt{x^2+a^2}}=\int\frac{a\sec^2\theta\,d\theta}{a\sec\theta}=\int\sec\theta\,d\theta=\ln|\sec\theta+\tan\theta|+C_1=\ln\left|\frac{\sqrt{x^2+a^2}}{a}+\frac{x}{a}\right|+C_1$$

$$= \ln\left(x+\sqrt{x^2+a^2}\right)+C \text{ where } C=C_1-\ln|a|$$

(b) Let $x=a\sinh t$, so that $dx=a\cosh t\,dt$ and $\sqrt{x^2+a^2}=a\cosh t$. Then

$$\int \frac{dx}{\sqrt{x^2+a^2}}=\int \frac{a\cosh t\,dt}{a\cosh t}=t+C=\sinh^{-1}\frac{x}{a}+C.$$

32. **(a)** Let $x=a\tan\theta$, $-\frac{\pi}{2}<\theta<\frac{\pi}{2}$. Then

$$I=\int \frac{x^2}{\left(x^2+a^2\right)^{3/2}}\,dx=\int \frac{a^2\tan^2\theta}{a^3\sec^3\theta}\,a\sec^2\theta\,d\theta=\int \frac{\tan^2\theta}{\sec\theta}\,d\theta=\int \frac{\sec^2\theta-1}{\sec\theta}\,d\theta$$

$$=\int(\sec\theta-\cos\theta)\,d\theta=\ln|\sec\theta+\tan\theta|-\sin\theta+C$$

$$=\ln\left|\frac{\sqrt{x^2+a^2}}{a}+\frac{x}{a}\right|-\frac{x}{\sqrt{x^2+a^2}}+C=\ln\left(x+\sqrt{x^2+a^2}\right)-\frac{x}{\sqrt{x^2+a^2}}+C_1$$

(b) Let $x=a\sinh t$. Then

$$I=\int \frac{a^2\sinh^2 t}{a^3\cosh^3 t}\,a\cosh t\,dt=\int \tanh^2 t\,dt=\int\left(1-\operatorname{sech}^2 t\right)dt=t-\tanh t+C$$

$$=\sinh^{-1}\frac{x}{a}-\frac{x}{\sqrt{a^2+x^2}}+C$$

33. The average value of $f(x)=\sqrt{x^2-1}/x$ on the interval $[1,7]$ is

$$\frac{1}{7-1}\int_1^7 \frac{\sqrt{x^2-1}}{x}\,dx=\frac{1}{6}\int_0^{\alpha}\frac{\tan\theta}{\sec\theta}\cdot\sec\theta\tan\theta\,d\theta\quad\left[\begin{array}{c}\text{where } x=\sec\theta,\ dx=\sec\theta\tan\theta\,d\theta,\\ \sqrt{x^2-1}=\tan\theta,\text{ and }\alpha=\sec^{-1}7\end{array}\right]$$

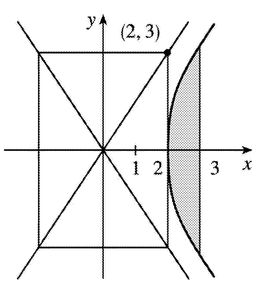
$$=\frac{1}{6}\int_0^{\alpha}\tan^2\theta\,d\theta=\frac{1}{6}\int_0^{\alpha}(\sec^2\theta-1)\,d\theta$$

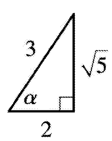
$$=\frac{1}{6}\left[\tan\theta-\theta\right]_0^{\alpha}=\frac{1}{6}(\tan\alpha-\alpha)$$

$$=\frac{1}{6}\left(\sqrt{48}-\sec^{-1}7\right)$$

34. $9x^2-4y^2=36 \Rightarrow y=\pm\frac{3}{2}\sqrt{x^2-4} \Rightarrow$

$$\begin{aligned}
\text{area} &= 2\int_2^3 \frac{3}{2}\sqrt{x^2-4}\,dx=3\int_2^3\sqrt{x^2-4}\,dx \\
&= 3\int_0^{\alpha} 2\tan\theta\, 2\sec\theta\tan\theta\, d\theta \quad \begin{bmatrix} \text{where } x=2\sec\theta, \\ dx=2\sec\theta\tan\theta\, d\theta, \\ \alpha=\sec^{-1}\frac{3}{2} \end{bmatrix} \\
&= 12\int_0^{\alpha}\left(\sec^2\theta-1\right)\sec\theta\, d\theta=12\int_0^{\alpha}\left(\sec^3\theta-\sec\theta\right)d\theta \\
&= 12\left[\frac{1}{2}\left(\sec\theta\tan\theta+\ln|\sec\theta+\tan\theta|\right)-\ln|\sec\theta+\tan\theta|\right]_0^{\alpha} \\
&= 6\left[\sec\theta\tan\theta-\ln|\sec\theta+\tan\theta|\right]_0^{\alpha} \\
&= 6\left[\frac{3\sqrt{5}}{4}-\ln\left(\frac{3}{2}+\frac{\sqrt{5}}{2}\right)\right]=\frac{9\sqrt{5}}{2}-6\ln\left(\frac{3+\sqrt{5}}{2}\right)
\end{aligned}$$





35. Area of $\triangle POQ=\frac{1}{2}(r\cos\theta)(r\sin\theta)=\frac{1}{2}r^2\sin\theta\cos\theta$. Area of region $PQR=\int_{r\cos\theta}^{r}\sqrt{r^2-x^2}\,dx$. Let $x=r\cos u \Rightarrow dx=-r\sin u\,du$ for $\theta\le u\le\frac{\pi}{2}$. Then we obtain

$$\begin{aligned}
\int\sqrt{r^2-x^2}\,dx &= \int r\sin u(-r\sin u)du=-r^2\int\sin^2 u\,du=-\frac{1}{2}r^2(u-\sin u\cos u)+C \\
&= -\frac{1}{2}r^2\cos^{-1}(x/r)+\frac{1}{2}x\sqrt{r^2-x^2}+C
\end{aligned}$$

so

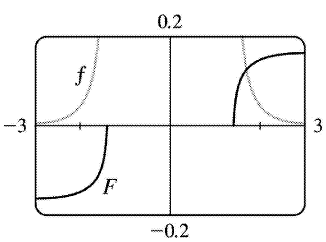
$$\begin{aligned}
\text{area of region} &= \frac{1}{2}\left[-r^2\cos^{-1}(x/r)+x\sqrt{r^2-x^2}\right]_{r\cos\theta}^{r} \\
&= \frac{1}{2}\left[0-\left(-r^2\theta+r\cos\theta\, r\sin\theta\right)\right]
\end{aligned}$$

$$= \frac{1}{2}r^2\theta - \frac{1}{2}r^2\sin\theta\cos\theta$$

and thus, (area of sector POR)=(area of $\triangle POQ$)+(area of region PQR)=$\frac{1}{2}r^2\theta$.

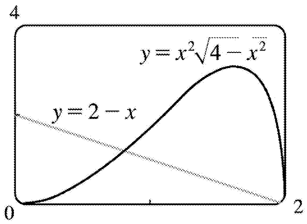
36. Let $x=\sqrt{2}\sec\theta$, where $0\le\theta<\frac{\pi}{2}$ or $\pi\le\theta<\frac{3\pi}{2}$, so $dx=\sqrt{2}\sec\theta\tan\theta\, d\theta$. Then

$$\int\frac{dx}{x^4\sqrt{x^2-2}}=\int\frac{\sqrt{2}\sec\theta\tan\theta\, d\theta}{4\sec^4\theta\sqrt{2}\tan\theta}$$
$$=\frac{1}{4}\int\cos^3\theta\, d\theta=\frac{1}{4}\int\left(1-\sin^2\theta\right)\cos\theta\, d\theta$$
$$=\frac{1}{4}\left[\sin\theta-\frac{1}{3}\sin^3\theta\right]+C \quad [\text{substitute } u=\sin\theta]$$
$$=\frac{1}{4}\left[\frac{\sqrt{x^2-2}}{x}-\frac{\left(x^2-2\right)^{3/2}}{3x^3}\right]+C$$



From the graph, it appears that our answer is reasonable.

37. From the graph, it appears that the curve $y=x^2\sqrt{4-x^2}$ and the line $y=2-x$ intersect at about $x=0.81$ and $x=2$, with $x^2\sqrt{4-x^2}>2-x$ on $(0.81,2)$. So the area bounded by the curve and the line is $A\approx\int_{0.81}^{2}\left[x^2\sqrt{4-x^2}-(2-x)\right]dx=\int_{0.81}^{2}x^2\sqrt{4-x^2}\,dx-\left[2x-\frac{1}{2}x^2\right]_{0.81}^{2}$. To evaluate the integral, we put $x=2\sin\theta$, where $-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$. Then



$dx=2\cos\theta\ d\theta$, $x=2\Rightarrow\theta=\sin^{-1}1=\dfrac{\pi}{2}$, and $x=0.81\Rightarrow\theta=\sin^{-1}0.405\approx0.417$. So

$$\int_{0.81}^{2}x^2\sqrt{4-x^2}\,dx\approx\int_{0.417}^{\pi/2}4\sin^2\theta\,(2\cos\theta)(2\cos\theta\ d\theta)=4\int_{0.417}^{\pi/2}\sin^2 2\theta\ d\theta=4\int_{0.417}^{\pi/2}\frac{1}{2}(1-\cos 4\theta)d\theta$$

$$=2\left[\theta-\frac{1}{4}\sin 4\theta\right]_{0.417}^{\pi/2}=2\left[\left(\frac{\pi}{2}-0\right)-\left(0.417-\frac{1}{4}(0.995)\right)\right]\approx2.81$$

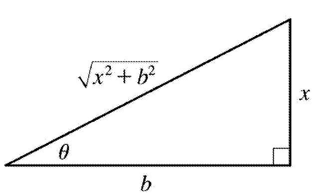
Thus, $A\approx2.81-\left[\left(2\cdot2-\frac{1}{2}\cdot2^2\right)-\left(2\cdot0.81-\frac{1}{2}\cdot0.81^2\right)\right]\approx2.10$.

38. Let $x=b\tan\theta$, so that $dx=b\sec^2\theta\,d\theta$ and $\sqrt{x^2+b^2}=b\sec\theta$.

$$E(P)=\int_{-a}^{L-a}\frac{\lambda b}{4\pi\varepsilon_0\left(x^2+b^2\right)^{3/2}}\,dx=\frac{\lambda b}{4\pi\varepsilon_0}\int_{\theta_1}^{\theta_2}\frac{1}{(b\sec\theta)^3}\,b\sec^2\theta\,d\theta$$

$$=\frac{\lambda}{4\pi\varepsilon_0 b}\int_{\theta_1}^{\theta_2}\frac{1}{\sec\theta}\,d\theta=\frac{\lambda}{4\pi\varepsilon_0 b}\int_{\theta_1}^{\theta_2}\cos\theta\,d\theta=\frac{\lambda}{4\pi\varepsilon_0 b}\left[\sin\theta\right]_{\theta_1}^{\theta_2}$$

$$=\frac{\lambda}{4\pi\varepsilon_0 b}\left[\frac{x}{\sqrt{x^2+b^2}}\right]_{-a}^{L-a}=\frac{\lambda}{4\pi\varepsilon_0 b}\left(\frac{L-a}{\sqrt{(L-a)^2+b^2}}+\frac{a}{\sqrt{a^2+b^2}}\right)$$



39. Let the equation of the large circle be $x^2+y^2=R^2$. Then the equation of the small circle is $x^2+(y-b)^2=r^2$, where $b=\sqrt{R^2-r^2}$ is the distance between the centers of the circles. The desired area is

$$A = \int_{-r}^{r}\left[\left(b+\sqrt{r^2-x^2}\right)-\sqrt{R^2-x^2}\right]dx=2\int_0^r\left(b+\sqrt{r^2-x^2}-\sqrt{R^2-x^2}\right)dx$$
$$= 2\int_0^r b\,dx+2\int_0^r\sqrt{r^2-x^2}\,dx-2\int_0^r\sqrt{R^2-x^2}\,dx$$

The first integral is just $2br=2r\sqrt{R^2-r^2}$. To evaluate the other two integrals, note that

$$\int\sqrt{a^2-x^2}\,dx = \int a^2\cos^2\theta\,d\theta\ \ [x=a\sin\theta\ ,dx=a\cos\theta\ d\theta\ ]=\frac{1}{2}a^2\int(1+\cos 2\theta\ )d\theta$$
$$= \frac{1}{2}a^2\left(\theta+\frac{1}{2}\sin 2\theta\right)+C=\frac{1}{2}a^2(\theta\ +\sin\theta\ \cos\theta\ )+C$$
$$= \frac{a^2}{2}\arcsin\left(\frac{x}{a}\right)+\frac{a^2}{2}\left(\frac{x}{a}\right)\frac{\sqrt{a^2-x^2}}{a}+C=\frac{a^2}{2}\arcsin\left(\frac{x}{a}\right)+\frac{x}{2}\sqrt{a^2-x^2}+C$$

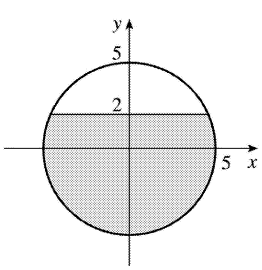
so the desired area is

$$A = 2r\sqrt{R^2-r^2}+\left[r^2\arcsin(x/r)+x\sqrt{r^2-x^2}\right]_0^r-\left[R^2\arcsin(x/R)+x\sqrt{R^2-x^2}\right]_0^r$$
$$= 2r\sqrt{R^2-r^2}+r^2\left(\frac{\pi}{2}\right)-\left[R^2\arcsin(r/R)+r\sqrt{R^2-r^2}\right]=r\sqrt{R^2-r^2}+\frac{\pi}{2}r^2-R^2\arcsin(r/R)$$

40. Note that the circular cross–sections of the tank are the same everywhere, so the percentage of the total capacity that is being used is equal to the percentage of any cross–section that is under water. The underwater area is

$$A = 2\int_{-5}^{2}\sqrt{25-y^2}\,dy$$
$$= \left[25\arcsin(y/5)+y\sqrt{25-y^2}\right]_{-5}^{2}\ y=5\sin\theta\ ]$$
$$= 25\arcsin\frac{2}{5}+2\sqrt{21}+\frac{25}{2}\pi\approx 58.72\text{ft}^2$$

so the fraction of the total capacity in use is $\dfrac{A}{\pi(5)^2}\approx\dfrac{58.72}{25\pi}\approx 0.748$ or 74.8% .



41. We use cylindrical shells and assume that $R>r$. $x^2=r^2-(y-R)^2 \Rightarrow x=\pm\sqrt{r^2-(y-R)^2}$, so $g(y)=2\sqrt{r^2-(y-R)^2}$ and

$$V = \int_{R-r}^{R+r} 2\pi y \cdot 2\sqrt{r^2-(y-R)^2}\,dy=\int_{-r}^{r} 4\pi(u+R)\sqrt{r^2-u^2}\,du \quad [\text{where } u=y-R]$$

$$= 4\pi\int_{-r}^{r} u\sqrt{r^2-u^2}\,du+4\pi R\int_{-r}^{r}\sqrt{r^2-u^2}\,du \quad \begin{bmatrix}\text{where } u=r\sin\theta\ ,du=r\cos\theta\ d\theta \\ \text{in the second tegral}\end{bmatrix}$$

$$= 4\pi\left[-\frac{1}{3}\left(r^2-u^2\right)^{3/2}\right]_{-r}^{r}+4\pi R\int_{-\pi/2}^{\pi/2} r^2\cos^2\theta\ d\theta =-\frac{4\pi}{3}(0-0)+4\pi Rr^2\int_{-\pi/2}^{\pi/2}\cos^2\theta\ d\theta$$

$$= 2\pi Rr^2\int_{-\pi/2}^{\pi/2}(1+\cos 2\theta\ )d\theta =2\pi Rr^2\left[\theta+\frac{1}{2}\sin 2\theta\ \right]_{-\pi/2}^{\pi/2}=2\pi^2Rr^2$$

Another method: Use washers instead of shells, so $V=8\pi R\int_0^r\sqrt{r^2-y^2}\,dy$ as in Exercise 6.2.61(a) , but evaluate the integral using $y=r\sin\theta$.

1. **(a)** $\dfrac{2x}{(x+3)(3x+1)}=\dfrac{A}{x+3}+\dfrac{B}{3x+1}$

(b) $\dfrac{1}{x^3+2x^2+x}=\dfrac{1}{x(x^2+2x+1)}=\dfrac{1}{x(x+1)^2}=\dfrac{A}{x}+\dfrac{B}{x+1}+\dfrac{C}{(x+1)^2}$

2. **(a)** $\dfrac{x-1}{x^3+x^2}=\dfrac{x-1}{x^2(x+1)}=\dfrac{A}{x}+\dfrac{B}{x^2}+\dfrac{C}{x+1}$

(b) $\dfrac{x-1}{x^3+x}=\dfrac{x-1}{x\left(x^2+1\right)}=\dfrac{A}{x}+\dfrac{Bx+C}{x^2+1}$

3. **(a)** $\dfrac{2}{x^2+3x-4}=\dfrac{2}{(x+4)(x-1)}=\dfrac{A}{x+4}+\dfrac{B}{x-1}$

(b) x^2+x+1 is irreducible, so $\dfrac{x^2}{(x-1)\left(x^2+x+1\right)}=\dfrac{A}{x-1}+\dfrac{Bx+C}{x^2+x+1}$.

4. **(a)** $\dfrac{x^3}{x^2+4x+3}=x-4+\dfrac{13x+12}{x^2+4x+3}=x-4+\dfrac{13x+12}{(x+1)(x+3)}=x-4+\dfrac{A}{x+1}+\dfrac{B}{x+3}$

(b) $\dfrac{2x+1}{(x+1)^3(x^2+4)^2}=\dfrac{A}{x+1}+\dfrac{B}{(x+1)^2}+\dfrac{C}{(x+1)^3}+\dfrac{Dx+E}{x^2+4}+\dfrac{Fx+G}{(x^2+4)^2}$

5. **(a)**

$\dfrac{x^4}{x^4-1}=\dfrac{(x^4-1)+1}{x^4-1}=1+\dfrac{1}{x^4-1}$ [or use long division] $=1+\dfrac{1}{(x^2-1)(x^2+1)}$

$=1+\dfrac{1}{(x-1)(x+1)(x^2+1)}=1+\dfrac{A}{x-1}+\dfrac{B}{x+1}+\dfrac{Cx+D}{x^2+1}$

(b) $\dfrac{t^4+t^2+1}{\left(t^2+1\right)\left(t^2+4\right)^2}=\dfrac{At+B}{t^2+1}+\dfrac{Ct+D}{t^2+4}+\dfrac{Et+F}{\left(t^2+4\right)^2}$

6. **(a)** $\dfrac{x^4}{(x^3+x)(x^2-x+3)}=\dfrac{x^4}{x(x^2+1)(x^2-x+3)}=\dfrac{x^3}{(x^2+1)(x^2-x+3)}=\dfrac{Ax+B}{x^2+1}+\dfrac{Cx+D}{x^2-x+3}$

(b)

$$\frac{1}{x^6-x^3}=\frac{1}{x^3\left(x^3-1\right)}=\frac{1}{x^3(x-1)\left(x^2+x+1\right)}=\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x^3}+\frac{D}{x-1}+\frac{Ex+F}{x^2+x+1}$$

7. $\int \frac{x}{x-6}\,dx=\int \frac{(x-6)+6}{x-6}\,dx=\int \left(1+\frac{6}{x-6}\right)dx=x+6\ln|x-6|+C$

8.

$$\int \frac{r^2}{r+4}\,dr=\int\left(\frac{r^2-16}{r+4}+\frac{16}{r+4}\right)dr=\int\left(r-4+\frac{16}{r+4}\right)dr \text{ [or use long division]}$$

$$=\frac{1}{2}r^2-4r+16\ln|r+4|+C$$

9. $\frac{x-9}{(x+5)(x-2)}=\frac{A}{x+5}+\frac{B}{x-2}$. Multiply both sides by $(x+5)(x-2)$ to get $x-9=A(x-2)+B(x+5)$. Substituting 2 for x gives $-7=7B\Leftrightarrow B=-1$. Substituting -5 for x gives $-14=-7A\Leftrightarrow A=2$. Thus,

$$\int \frac{x-9}{(x+5)(x-2)}\,dx=\int\left(\frac{2}{x+5}+\frac{-1}{x-2}\right)dx=2\ln|x+5|-\ln|x-2|+C$$

10. $\frac{1}{(t+4)(t-1)}=\frac{A}{t+4}+\frac{B}{t-1}\Rightarrow 1=A(t-1)+B(t+4)$.

$t=1\Rightarrow 1=5B\Rightarrow B=\frac{1}{5}$. $t=-4\Rightarrow 1=-5A\Rightarrow A=-\frac{1}{5}$. Thus,

$$\int \frac{1}{(t+4)(t-1)}\,dt=\int\left(\frac{-1/5}{t+4}+\frac{1/5}{t-1}\right)dt=-\frac{1}{5}\ln|t+4|+\frac{1}{5}\ln|t-1|+C \text{ or } \frac{1}{5}\ln\left|\frac{t-1}{t+4}\right|+C$$

11. $\frac{1}{x^2-1}=\frac{1}{(x+1)(x-1)}=\frac{A}{x+1}+\frac{B}{x-1}$. Multiply both sides by $(x+1)(x-1)$ to get $1=A(x-1)+B(x+1)$.

Substituting 1 for x gives $1=2B\Leftrightarrow B=\frac{1}{2}$. Substituting -1 for x gives $1=-2A\Leftrightarrow A=-\frac{1}{2}$. Thus,

$$\int_2^3 \frac{1}{x^2-1}\,dx=\int_2^3\left(\frac{-1/2}{x+1}+\frac{1/2}{x-1}\right)dx=\left[-\frac{1}{2}\ln|x+1|+\frac{1}{2}\ln|x-1|\right]_2^3$$

$$=\left(-\frac{1}{2}\ln 4+\frac{1}{2}\ln 2\right)-\left(-\frac{1}{2}\ln 3+\frac{1}{2}\ln 1\right)=\frac{1}{2}(\ln 2+\ln 3-\ln 4)\left[\text{ or }\frac{1}{2}\ln\frac{3}{2}\right]$$

12. $\frac{x-1}{x^2+3x+2}=\frac{A}{x+1}+\frac{B}{x+2}$. Multiply both sides by $(x+1)(x+2)$ to get $x-1=A(x+2)+B(x+1)$.

Substituting -2 for x gives $-3=-B\Leftrightarrow B=3$. Substituting -1 for x gives $-2=A$. Thus,

$$\int_0^1 \frac{x-1}{x^2+3x+2}\,dx = \int_0^1 \left(\frac{-2}{x+1} + \frac{3}{x+2} \right) dx = \left[-2\ln|x+1| + 3\ln|x+2| \right]_0^1$$

$$= (-2\ln 2 + 3\ln 3) - (-2\ln 1 + 3\ln 2) = 3\ln 3 - 5\ln 2 \left[\text{or} \ln \frac{27}{32} \right]$$

13. $\int \frac{ax}{x^2-bx}\,dx = \int \frac{ax}{x(x-b)}\,dx = \int \frac{a}{x-b}\,dx = a\ln|x-b| + C$

14. If $a \neq b$, $\frac{1}{(x+a)(x+b)} = \frac{1}{b-a} \left(\frac{1}{x+a} - \frac{1}{x+b} \right)$, so if $a \neq b$, then

$$\int \frac{dx}{(x+a)(x+b)} = \frac{1}{b-a} (\ln|x+a| - \ln|x+b|) + C = \frac{1}{b-a} \ln \left| \frac{x+a}{x+b} \right| + C$$

If $a=b$, then $\int \frac{dx}{(x+a)^2} = -\frac{1}{x+a} + C$.

15. $\frac{2x+3}{(x+1)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2} \Rightarrow 2x+3 = A(x+1) + B$. Take $x=-1$ to get $B=1$, and equate coefficients of x to get $A=2$. Now

$$\int_0^1 \frac{2x+3}{(x+1)^2}\,dx = \int_0^1 \left[\frac{2}{x+1} + \frac{1}{(x+1)^2} \right] dx = \left[2\ln(x+1) - \frac{1}{x+1} \right]_0^1$$

$$= 2\ln 2 - \frac{1}{2} - (2\ln 1 - 1) = 2\ln 2 + \frac{1}{2}$$

16. $\frac{x^3-4x-10}{x^2-x-6} = x+1 + \frac{3x-4}{(x-3)(x+2)}$. Write $\frac{3x-4}{(x-3)(x+2)} = \frac{A}{x-3} + \frac{B}{x+2}$. Then $3x-4 = A(x+2) + B(x-3)$. Taking $x=3$ and $x=-2$, we get $5=5A \Leftrightarrow A=1$ and $-10=-5B \Leftrightarrow B=2$, so

$$\int_0^1 \frac{x^3-4x-10}{x^2-x-6}\,dx = \int_0^1 \left(x+1 + \frac{1}{x-3} + \frac{2}{x+2} \right) dx = \left[\frac{1}{2}x^2 + x + \ln|x-3| + 2\ln(x+2) \right]_0^1$$

$$= \left(\frac{1}{2} + 1 + \ln 2 + 2\ln 3 \right) - (0 + 0 + \ln 3 + 2\ln 2) = \frac{3}{2} + \ln 3 - \ln 2 = \frac{3}{2} + \ln \frac{3}{2}$$

17. $\dfrac{4y^2-7y-12}{y(y+2)(y-3)}=\dfrac{A}{y}+\dfrac{B}{y+2}+\dfrac{C}{y-3}\Rightarrow 4y^2-7y-12=A(y+2)(y-3)+By(y-3)+Cy(y+2)$. Setting $y=0$ gives $-12=-6A$, so $A=2$. Setting $y=-2$ gives $18=10B$, so $B=\dfrac{9}{5}$. Setting $y=3$ gives $3=15C$, so $C=\dfrac{1}{5}$. Now

$$\int_1^2 \frac{4y^2-7y-12}{y(y+2)(y-3)}\,dy=\int_1^2\left(\frac{2}{y}+\frac{9/5}{y+2}+\frac{1/5}{y-3}\right)dy=\left[2\ln|y|+\frac{9}{5}\ln|y+2|+\frac{1}{5}\ln|y-3|\right]_1^2$$

$$=2\ln 2+\frac{9}{5}\ln 4+\frac{1}{5}\ln 1-2\ln 1-\frac{9}{5}\ln 3-\frac{1}{5}\ln 2$$

$$=2\ln 2+\frac{18}{5}\ln 2-\frac{1}{5}\ln 2-\frac{9}{5}\ln 3=\frac{27}{5}\ln 2-\frac{9}{5}\ln 3=\frac{9}{5}(3\ln 2-\ln 3)=\frac{9}{5}\ln\frac{8}{3}$$

18. $\dfrac{x^2+2x-1}{x^3-x}=\dfrac{x^2+2x-1}{x(x+1)(x-1)}=\dfrac{A}{x}+\dfrac{B}{x+1}+\dfrac{C}{x-1}$. Multiply both sides by $x(x+1)(x-1)$ to get $x^2+2x-1=A(x+1)(x-1)+Bx(x-1)+Cx(x+1)$. Substituting 0 for x gives $-1=-A\Leftrightarrow A=1$. Substituting -1 for x gives $-2=2B\Leftrightarrow B=-1$. Substituting 1 for x gives $2=2C\Leftrightarrow C=1$. Thus,

$$\int\frac{x^2+2x-1}{x^3-x}\,dx=\int\left(\frac{1}{x}-\frac{1}{x+1}+\frac{1}{x-1}\right)dx=\ln|x|-\ln|x+1|+\ln|x-1|+C=\ln\left|\frac{x(x-1)}{x+1}\right|+C\ .$$

19. $\dfrac{1}{(x+5)^2(x-1)}=\dfrac{A}{x+5}+\dfrac{B}{(x+5)^2}+\dfrac{C}{x-1}\Rightarrow 1=A(x+5)(x-1)+B(x-1)+C(x+5)^2$. Setting $x=-5$ gives $1=-6B$, so $B=-\dfrac{1}{6}$. Setting $x=1$ gives $1=36C$, so $C=\dfrac{1}{36}$. Setting $x=-2$ gives $1=A(3)(-3)+B(-3)+C\left(3^2\right)=-9A-3B+9C=-9A+\dfrac{1}{2}+\dfrac{1}{4}=-9A+\dfrac{3}{4}$, so $9A=-\dfrac{1}{4}$ and $A=-\dfrac{1}{36}$. Now

$$\int\frac{1}{(x+5)^2(x-1)}\,dx=\int\left[\frac{-1/36}{x+5}-\frac{1/6}{(x+5)^2}+\frac{1/36}{x-1}\right]dx$$

$$=-\frac{1}{36}\ln|x+5|+\frac{1}{6(x+5)}+\frac{1}{36}\ln|x-1|+C$$

20. $\dfrac{x^2}{(x-3)(x+2)^2}=\dfrac{A}{x-3}+\dfrac{B}{x+2}+\dfrac{C}{(x+2)^2}\Rightarrow x^2=A(x+2)^2+B(x-3)(x+2)+C(x-3)$.

Setting $x=3$ gives $A=\dfrac{9}{25}$. Take $x=-2$ to get $C=-\dfrac{4}{5}$, and equate the coefficients of x^2 to get $1=A+B$

$\Rightarrow B=\frac{16}{25}$. Then

$$\int \frac{x^2}{(x-3)(x+2)^2}\,dx = \int\left[\frac{9/25}{x-3}+\frac{16/25}{x+2}-\frac{4/5}{(x+2)^2}\right]dx$$
$$= \frac{9}{25}\ln|x-3|+\frac{16}{25}\ln|x+2|+\frac{4}{5(x+2)}+C$$

21. $\frac{5x^2+3x-2}{x^3+2x^2}=\frac{5x^2+3x-2}{x^2(x+2)}=\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x+2}$. Multiply by $x^2(x+2)$ to get $5x^2+3x-2=Ax(x+2)+B(x+2)+Cx^2$. Set $x=-2$ to get $C=3$, and take $x=0$ to get $B=-1$. Equating the coefficients of x^2 gives $5=A+C\Rightarrow A=2$. So
$\int\frac{5x^2+3x-2}{x^3+2x^2}\,dx=\int\left(\frac{2}{x}-\frac{1}{x^2}+\frac{3}{x+2}\right)dx=2\ln|x|+\frac{1}{x}+3\ln|x+2|+C$.

22. $\frac{1}{s^2(s-1)^2}=\frac{A}{s}+\frac{B}{s^2}+\frac{C}{s-1}+\frac{D}{(s-1)^2}\Rightarrow 1=As(s-1)^2+B(s-1)^2+Cs^2(s-1)+Ds^2$. Set $s=0$, giving $B=1$. Then set $s=1$ to get $D=1$. Equate the coefficients of s^3 to get $0=A+C$ or $A=-C$, and finally set $s=2$ to get $1=2A+1-4A+4$ or $A=2$. Now
$\int\frac{ds}{s^2(s-1)^2}=\int\left[\frac{2}{s}+\frac{1}{s^2}-\frac{2}{s-1}+\frac{1}{(s-1)^2}\right]ds=2\ln|s|-\frac{1}{s}-2\ln|s-1|-\frac{1}{s-1}+C$.

23. $\frac{x^2}{(x+1)^3}=\frac{A}{x+1}+\frac{B}{(x+1)^2}+\frac{C}{(x+1)^3}$. Multiply by $(x+1)^3$ to get $x^2=A(x+1)^2+B(x+1)+C$. Setting $x=-1$ gives $C=1$. Equating the coefficients of x^2 gives $A=1$, and setting $x=0$ gives $B=-2$. Now
$\int\frac{x^2\,dx}{(x+1)^3}=\int\left[\frac{1}{x+1}-\frac{2}{(x+1)^2}+\frac{1}{(x+1)^3}\right]dx=\ln|x+1|+\frac{2}{x+1}-\frac{1}{2(x+1)^2}+C$.

24. $\frac{x}{x+1}=\frac{(x+1)-1}{x+1}=1-\frac{1}{x+1}$, so $\frac{x^3}{(x+1)^3}=\left[1-\frac{1}{x+1}\right]^3=1-\frac{3}{x+1}+\frac{3}{(x+1)^2}-\frac{1}{(x+1)^3}$. Thus,
$\int\frac{x^3}{(x+1)^3}\,dx=\int\left[1-\frac{3}{x+1}+\frac{3}{(x+1)^2}-\frac{1}{(x+1)^3}\right]dx=x-3\ln|x+1|-\frac{3}{x+1}+\frac{1}{2(x+1)^2}+C$.

25. $\dfrac{10}{(x-1)\left(x^2+9\right)}=\dfrac{A}{x-1}+\dfrac{Bx+C}{x^2+9}$. Multiply both sides by $(x-1)\left(x^2+9\right)$ to get $10=A\left(x^2+9\right)+(Bx+C)(x-1)$ (*). Substituting 1 for x gives $10=10A \Leftrightarrow A=1$. Substituting 0 for x gives $10=9A-C \Rightarrow C=9(1)-10=-1$. The coefficients of the x^2 –terms in (*) must be equal, so $0=A+B$ $\Rightarrow$ $B=-1$. Thus,

$$\int \frac{10}{(x-1)(x^2+9)}\,dx=\int \left(\frac{1}{x-1}+\frac{-x-1}{x^2+9} \right)dx=\int \left(\frac{1}{x-1}-\frac{x}{x^2+9}-\frac{1}{x^2+9} \right)dx$$

$$=\ln |x-1|-\frac{1}{2}\ln \left(x^2+9\right)\ [\text{let } u=x^2+9]\ -\frac{1}{3}\tan^{-1}\left(\frac{x}{3} \right)\ [\textit{Formula}\,10]\ +C$$

26. $\dfrac{x^2-x+6}{x^3+3x}=\dfrac{x^2-x+6}{x\left(x^2+3\right)}=\dfrac{A}{x}+\dfrac{Bx+C}{x^2+3}$. Multiply by $x\left(x^2+3\right)$ to get $x^2-x+6=A\left(x^2+3\right)+(Bx+C)x$. Substituting 0 for x gives $6=3A \Leftrightarrow A=2$. The coefficients of the x^2 –terms must be equal, so $1=A+B \Rightarrow$ $B=1-2=-1$. The coefficients of the x –terms must be equal, so $-1=C$. Thus,

$$\int \frac{x^2-x+6}{x^3+3x}\,dx=\int \left(\frac{2}{x}+\frac{-x-1}{x^2+3} \right)dx=\int \left(\frac{2}{x}-\frac{x}{x^2+3}-\frac{1}{x^2+3} \right)dx$$

$$=2\ln |x|-\frac{1}{2}\ln \left(x^2+3\right)-\frac{1}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}}+C$$

27. $\dfrac{x^3+x^2+2x+1}{\left(x^2+1\right)\left(x^2+2\right)}=\dfrac{Ax+B}{x^2+1}+\dfrac{Cx+D}{x^2+2}$. Multiply both sides by $\left(x^2+1\right)\left(x^2+2\right)$ to get

$x^3+x^2+2x+1=(Ax+B)\left(x^2+2\right)+(Cx+D)\left(x^2+1\right) \Leftrightarrow$

$x^3+x^2+2x+1=\left(Ax^3+Bx^2+2Ax+2B\right)+\left(Cx^3+Dx^2+Cx+D\right) \Leftrightarrow$

$x^3+x^2+2x+1=(A+C)x^3+(B+D)x^2+(2A+C)x+(2B+D)$. Comparing coefficients gives us the following system of equations:

$A+C=1$	**(1)**	$B+D=1$	**(2)**
$2A+C=2$	**(3)**	$2B+D=1$	**(4)**

Subtracting equation **(1)** from equation **(3)** gives us $A=1$, so $C=0$. Subtracting equation **(2)** from equation **(4)** gives us $B=0$, so $D=1$. Thus, $I=\displaystyle\int \frac{x^3+x^2+2x+1}{\left(x^2+1\right)\left(x^2+2\right)}\,dx=\int \left(\frac{x}{x^2+1}+\frac{1}{x^2+2} \right)dx$. For

$\int \frac{x}{x^2+1}\,dx$, let $u=x^2+1$ so $du=2x\,dx$ and then $\int \frac{x}{x^2+1}\,dx=\frac{1}{2}\int \frac{1}{u}\,du=\frac{1}{2}\ln|u|+C=\frac{1}{2}\ln\left(x^2+1\right)+C$. For $\int \frac{1}{x^2+2}\,dx$, use Formula 10 with $a=\sqrt{2}$. So $\int \frac{1}{x^2+2}\,dx=\int \frac{1}{x^2+\left(\sqrt{2}\right)^2}\,dx=\frac{1}{\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}}+C$. Thus, $I=\frac{1}{2}\ln\left(x^2+1\right)+\frac{1}{\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}}+C$.

28. $\frac{x^2-2x-1}{(x-1)^2\left(x^2+1\right)}=\frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{Cx+D}{x^2+1}\Rightarrow x^2-2x-1=A(x-1)\left(x^2+1\right)+B\left(x^2+1\right)+(Cx+D)(x-1)^2$.

Setting $x=1$ gives $B=-1$. Equating the coefficients of x^3 gives $A=-C$. Equating the constant terms gives $-1=-A-1+D$, so $D=A$,
and setting $x=2$ gives $-1=5A-5-2A+A$ or $A=1$. We have

$$\int \frac{x^2-2x-1}{(x-1)^2\left(x^2+1\right)}\,dx=\int\left[\frac{1}{x-1}-\frac{1}{(x-1)^2}-\frac{x-1}{x^2+1}\right]dx$$

$$=\ln|x-1|+\frac{1}{x-1}-\frac{1}{2}\ln\left(x^2+1\right)+\tan^{-1}x+C$$

29.

$$\int \frac{x+4}{x^2+2x+5}\,dx=\int \frac{x+1}{x^2+2x+5}\,dx+\int \frac{3}{x^2+2x+5}\,dx=\frac{1}{2}\int \frac{(2x+2)\,dx}{x^2+2x+5}+\int \frac{3\,dx}{(x+1)^2+4}$$

$$=\frac{1}{2}\ln\left|x^2+2x+5\right|+3\int \frac{2\,du}{4(u^2+1)}\quad\left[\begin{array}{c}\text{where } x+1=2u,\\ \text{and } dx=2\,du\end{array}\right]$$

$$=\frac{1}{2}\ln(x^2+2x+5)+\frac{3}{2}\tan^{-1}u+C=\frac{1}{2}\ln(x^2+2x+5)+\frac{3}{2}\tan^{-1}\left(\frac{x+1}{2}\right)+C$$

30. $\frac{x^3-2x^2+x+1}{x^4+5x^2+4}=\frac{x^3-2x^2+x+1}{\left(x^2+1\right)\left(x^2+4\right)}=\frac{Ax+B}{x^2+1}+\frac{Cx+D}{x^2+4}\Rightarrow x^3-2x^2+x+1=(Ax+B)\left(x^2+4\right)+(Cx+D)\left(x^2+1\right)$. Equating coefficients gives $A+C=1$, $B+D=-2$, $4A+C=1$, $4B+D=1\Rightarrow A=0$, $C=1$, $B=1$, $D=-3$. Now $\int \frac{x^3-2x^2+x+1}{x^4+5x^2+4}\,dx=\int \frac{dx}{x^2+1}+\int \frac{x-3}{x^2+4}\,dx=\tan^{-1}x+\frac{1}{2}\ln\left(x^2+4\right)-\frac{3}{2}\tan^{-1}(x/2)+C$.

31. $\frac{1}{x^3-1}=\frac{1}{(x-1)\left(x^2+x+1\right)}=\frac{A}{x-1}+\frac{Bx+C}{x^2+x+1}\Rightarrow 1=A\left(x^2+x+1\right)+(Bx+C)(x-1)$.

Take $x=1$ to get $A=\frac{1}{3}$. Equating coefficients of x^2 and then comparing the constant terms, we get $0=\frac{1}{3}+B$, $1=\frac{1}{3}-C$, so $B=-\frac{1}{3}$, $C=-\frac{2}{3}$ $\Rightarrow$

$$\int \frac{1}{x^3-1}\,dx = \int \frac{\frac{1}{3}}{x-1}\,dx+\int \frac{-\frac{1}{3}x-\frac{2}{3}}{x^2+x+1}\,dx=\frac{1}{3}\ln|x-1|-\frac{1}{3}\int \frac{x+2}{x^2+x+1}\,dx$$

$$= \frac{1}{3}\ln|x-1|-\frac{1}{3}\int \frac{x+1/2}{x^2+x+1}\,dx-\frac{1}{3}\int \frac{(3/2)\,dx}{(x+1/2)^2+3/4}$$

$$= \frac{1}{3}\ln|x-1|-\frac{1}{6}\ln\left(x^2+x+1\right)-\frac{1}{2}\left(\frac{2}{\sqrt{3}}\right)\tan^{-1}\left(\frac{x+\frac{1}{2}}{\sqrt{3}/2}\right)+K$$

$$= \frac{1}{3}\ln|x-1|-\frac{1}{6}\ln\left(x^2+x+1\right)-\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{1}{\sqrt{3}}(2x+1)\right)+K$$

32.

$$\int_0^1 \frac{x}{x^2+4x+13}\,dx =\int_0^1 \frac{\frac{1}{2}(2x+4)}{x^2+4x+13}\,dx-2\int_0^1 \frac{dx}{(x+2)^2+9}$$

$$=\frac{1}{2}\int_{13}^{18}\frac{dy}{y}-2\int_{2/3}^{1}\frac{3}{du}\,9u^2+9\left[\begin{array}{c}\text{where } y=x^2+4x+13, dy=(2x+4)\,dx,\\ x+2=3u, \text{ and } dx=3\,du\end{array}\right]$$

$$=\frac{1}{2}\left[\ln y\right]_{13}^{18}-\frac{2}{3}\left[\tan^{-1}u\right]_{2/3}^{1}=\frac{1}{2}\ln\frac{18}{13}-\frac{2}{3}\left(\frac{\pi}{4}-\tan^{-1}\left(\frac{2}{3}\right)\right)$$

$$=\frac{1}{2}\ln\frac{18}{13}-\frac{\pi}{6}+\frac{2}{3}\tan^{-1}\left(\frac{2}{3}\right)$$

33. Let $u=x^3+3x^2+4$. Then $du=3\left(x^2+2x\right)dx\Rightarrow$

$$\int_2^5 \frac{x^2+2x}{x^3+3x^2+4}\,dx=\frac{1}{3}\int_{24}^{204}\frac{}{du}\,u=\frac{1}{3}\left[\ln u\right]_{24}^{204}=\frac{1}{3}(\ln 204-\ln 24)=\frac{1}{3}\ln\frac{204}{24}=\frac{1}{3}\ln\frac{17}{2}\ .$$

34. $\frac{x^3}{x^3+1}=\frac{\left(x^3+1\right)-1}{x^3+1}=1-\frac{1}{x^3+1}=1-\left(\frac{A}{x+1}+\frac{Bx+C}{x^2-x+1}\right)\Rightarrow 1=A\left(x^2-x+1\right)+(Bx+C)(x+1)$. Equate the terms of degree 2 , 1 and 0 to get $0=A+B$, $0=-A+B+C$, $1=A+C$. Solve the three equations to get

$A=\frac{1}{3}$, $B=-\frac{1}{3}$, and $C=\frac{2}{3}$. So

$$\int \frac{x^3}{x^3+1}\,dx = \int \left[1-\frac{\frac{1}{3}}{x+1}+\frac{\frac{1}{3}x-\frac{2}{3}}{x^2-x+1} \right] dx$$

$$= x-\frac{1}{3}\ln|x+1|+\frac{1}{6}\int \frac{2x-1}{x^2-x+1}\,dx-\frac{1}{2}\int \frac{dx}{\left(x-\frac{1}{2}\right)^2+\frac{3}{4}}$$

$$= x-\frac{1}{3}\ln|x+1|+\frac{1}{6}\ln\left(x^2-x+1\right)-\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{1}{\sqrt{3}}(2x-1)\right)+K$$

35. $\frac{1}{x^4-x^2}=\frac{1}{x^2(x-1)(x+1)}=\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x-1}+\frac{D}{x+1}$. Multiply by $x^2(x-1)(x+1)$ to get $1=Ax(x-1)(x+1)+B(x-1)(x+1)+Cx^2(x+1)+Dx^2(x-1)$. Setting $x=1$ gives $C=\frac{1}{2}$, taking $x=-1$ gives $D=-\frac{1}{2}$. Equating the coefficients of x^3 gives $0=A+C+D=A$. Finally, setting $x=0$ yields $B=-1$. Now

$$\int \frac{dx}{x^4-x^2}=\int \left[\frac{-1}{x^2}+\frac{1/2}{x-1}-\frac{1/2}{x+1} \right] dx=\frac{1}{x}+\frac{1}{2}\ln\left|\frac{x-1}{x+1}\right|+C .$$

36. Let $u=x^4+5x^2+4 \Rightarrow du=\left(4x^3+10x\right)dx=2\left(2x^3+5x\right)dx$, so

$$\int_0^1 \frac{2x^3+5x}{x^4+5x^2+4}\,dx=\frac{1}{2}\int_4^{10}\frac{du}{u}=\frac{1}{2}\left[\ln|u|\right]_4^{10}=\frac{1}{2}(\ln 10-\ln 4)=\frac{1}{2}\ln\frac{5}{2} .$$

37. $\int \frac{x-3}{\left(x^2+2x+4\right)^2}\,dx=\int \frac{x-3}{\left[(x+1)^2+3\right]^2}\,dx=\int \frac{u-4}{\left(u^2+3\right)^2}\,du$ [with $u=x+1$]

$$=\int \frac{u\,du}{\left(u^2+3\right)^2}-4\int \frac{du}{\left(u^2+3\right)^2}=\frac{1}{2}\int \frac{dv}{v^2}-4\int \frac{\sqrt{3}\sec^2\theta\,d\theta}{9\sec^4\theta} \quad \left[\begin{array}{c} v=u^2+3 \text{ in the first integral;} \\ u=\sqrt{3}\tan\theta \text{ in the second} \end{array} \right]$$

$$=\frac{-1}{(2v)}-\frac{4\sqrt{3}}{9}\int \cos^2\theta\,d\theta=\frac{-1}{2\left(u^2+3\right)}-\frac{2\sqrt{3}}{9}(\theta+\sin\theta\cos\theta)+C$$

$$=\frac{-1}{2\left(x^2+2x+4\right)}-\frac{2\sqrt{3}}{9}\left[\tan^{-1}\left(\frac{x+1}{\sqrt{3}}\right)+\frac{\sqrt{3}(x+1)}{x^2+2x+4}\right]+C$$

$=\frac{-1}{2\left(x^2+2x+4\right)}-\frac{2\sqrt{3}}{9}\tan^{-1}\left(\frac{x+1}{\sqrt{3}}\right)-\frac{2(x+1)}{3\left(x^2+2x+4\right)}+C$

38. $\frac{x^4+1}{x\left(x^2+1\right)^2}=\frac{A}{x}+\frac{Bx+C}{x^2+1}+\frac{Dx+E}{\left(x^2+1\right)^2}\Rightarrow x^4+1=A\left(x^2+1\right)^2+(Bx+C)x\left(x^2+1\right)+(Dx+E)x$. Setting $x=0$ gives $A=1$, and equating the coefficients of x^4 gives $1=A+B$, so $B=0$. Now $\frac{C}{x^2+1}+\frac{Dx+E}{\left(x^2+1\right)^2}=\frac{x^4+1}{x\left(x^2+1\right)^2}-\frac{1}{x}=\frac{1}{x}\left[\frac{x^4+1-\left(x^4+2x^2+1\right)}{\left(x^2+1\right)^2}\right]=\frac{-2x}{\left(x^2+1\right)^2}$, so we can take $C=0$, $D=-2$, and $E=0$. Hence, $\int\frac{x^4+1}{x\left(x^2+1\right)^2}dx=\int\left[\frac{1}{x}-\frac{2x}{\left(x^2+1\right)^2}\right]dx=\ln|x|+\frac{1}{x^2+1}+C$.

39. Let $u=\sqrt{x+1}$. Then $x=u^2-1$, $dx=2u\,du\Rightarrow$
$\int\frac{dx}{x\sqrt{x+1}}=\int\frac{2u\,du}{\left(u^2-1\right)u}=2\int\frac{du}{u^2-1}=\ln\left|\frac{u-1}{u+1}\right|+C=\ln\left|\frac{\sqrt{x+1}-1}{\sqrt{x+1}+1}\right|+C$.

40. Let $u=\sqrt{x+2}$. Then $x=u^2-2$, $dx=2u\,du\Rightarrow I=\int\frac{dx}{x-\sqrt{x+2}}=\int\frac{2u\,du}{u^2-2-u}=2\int\frac{u\,du}{u^2-u-2}$ and $\frac{u}{u^2-u-2}=\frac{A}{u-2}+\frac{B}{u+1}\Rightarrow u=A(u+1)+B(u-2)$. Substituting -1 for u gives $-1=-3B\Leftrightarrow B=\frac{1}{3}$ and substituting 2 for u gives $2=3A\Leftrightarrow A=\frac{2}{3}$. Thus,

$$I=\frac{2}{3}\int\left[\frac{2}{u-2}+\frac{1}{u+1}\right]du=\frac{2}{3}(2\ln|u-2|+\ln|u+1|)+C$$
$$=\frac{2}{3}\left[2\ln\left|\sqrt{x+2}-2\right|+\ln\left(\sqrt{x+2}+1\right)\right]+C$$

41. Let $u=\sqrt{x}$, so $u^2=x$ and $dx=2u\,du$. Thus,

$$\int_9^{16}\frac{\sqrt{x}}{x-4}dx=\int_3^4\frac{u}{u^2-4}2u\,du=2\int_3^4\frac{u^2}{u^2-4}du=2\int_3^4\left(1+\frac{4}{u^2-4}\right)du\quad\text{[by long division]}$$
$$=2+8\int_3^4\frac{}{du}(u+2)(u-2).\ (\ *\ )$$

Multiply $\frac{1}{(u+2)(u-2)}=\frac{A}{u+2}+\frac{B}{u-2}$ by $(u+2)(u-2)$ to get $1=A(u-2)+B(u+2)$. Equating coefficients we get $A+B=0$ and $-2A+2B=1$. Solving gives us $B=\frac{1}{4}$ and $A=-\frac{1}{4}$, so $\frac{1}{(u+2)(u-2)}=\frac{-1/4}{u+2}+\frac{1/4}{u-2}$ and (*) is

$$2+8\int_3^4\left(\frac{-1/4}{u+2}+\frac{1/4}{u-2}\right)du = 2+8\left[-\frac{1}{4}\ln|u+2|+\frac{1}{4}\ln|u-2|\right]_3^4$$

$$= 2+[2\ln|u-2|-2\ln|u+2|]_3^4=2+2\left[\ln\left|\frac{u-2}{u+2}\right|\right]_3^4$$

$$= 2+2\left(\ln\frac{2}{6}-\ln\frac{1}{5}\right)=2+2\ln\frac{2/6}{1/5}$$

$$= 2+2\ln\frac{5}{3}\text{ or}2+\ln\left(\frac{5}{3}\right)^2=2+\ln\frac{25}{9}$$

42. Let $u=\sqrt[3]{x}$. Then $x=u^3$, $dx=3u^2du\Rightarrow$

$$\int_0^1\frac{1}{1+\sqrt[3]{x}}dx = \int_0^1\frac{3u^2}{du}1+u=\int_0^1\left(3u-3+\frac{3}{1+u}\right)du=\left[\frac{3}{2}u^2-3u+3\ln(1+u)\right]_0^1$$

$$= 3\left(\ln 2-\frac{1}{2}\right)$$

43. Let $u=\sqrt[3]{x^2+1}$. Then $x^2=u^3-1$, $2xdx=3u^2du\Rightarrow$

$$\int\frac{x^3dx}{\sqrt[3]{x^2+1}} = \int\frac{\left(u^3-1\right)\frac{3}{2}u^2du}{u}=\frac{3}{2}\int\left(u^4-u\right)du=\frac{3}{10}u^5-\frac{3}{4}u^2+C$$

$$= \frac{3}{10}\left(x^2+1\right)^{5/3}-\frac{3}{4}\left(x^2+1\right)^{2/3}+C$$

44. Let $u=\sqrt{x}$. Then $x=u^2$, $dx=2udu\Rightarrow\int_{1/3}^{3}\frac{\sqrt{x}}{x^2+x}dx=\int_{1/\sqrt{3}}^{\sqrt{3}}\frac{u\cdot 2udu}{u^4+u^2}=2\int_{1/\sqrt{3}}^{\sqrt{3}}\frac{du}{u^2+1}$

$$=2\left[\tan^{-1}u\right]_{1/\sqrt{3}}^{\sqrt{3}}=2\left(\frac{\pi}{3}-\frac{\pi}{6}\right)=\frac{\pi}{3}$$

45. If we were to substitute $u=\sqrt{x}$, then the square root would disappear but a cube root would remain. On the other hand, the substitution

$u=\sqrt[3]{x}$ would eliminate the cube root but leave a square root. We can eliminate both roots by means of the substitution $u=\sqrt[6]{x}$. (Note that 6 is the least common multiple of 2 and 3 .)

Let $u=\sqrt[6]{x}$. Then $x=u^6$, so $dx=6u^5\,du$ and $\sqrt{x}=u^3$, $\sqrt[3]{x}=u^2$. Thus,

$$\int \frac{dx}{\sqrt{x}-\sqrt[3]{x}} = \int \frac{6u^5\,du}{u^3-u^2} = 6\int \frac{u^5}{u^2(u-1)}\,du = 6\int \frac{u^3}{u-1}\,du$$

$$= 6\int \left(u^2+u+1+\frac{1}{u-1}\right)du \quad \text{[by long division]}$$

$$= 6\left(\frac{1}{3}u^3+\frac{1}{2}u^2+u+\ln|u-1|\right)+C = 2\sqrt{x}+3\sqrt[3]{x}+6\sqrt[6]{x}+6\ln\left|\sqrt[6]{x}-1\right|+C$$

46. Let $u=\sqrt[12]{x}$. Then $x=u^{12}$, $dx=12u^{11}\,du \Rightarrow$

$$\int \frac{dx}{\sqrt[3]{x}+\sqrt[4]{x}} = \int \frac{12u^{11}\,du}{u^4+u^3} = 12\int \frac{u^8\,du}{u+1} = 12\int \left(u^7-u^6+u^5-u^4+u^3-u^2+u-1+\frac{1}{u+1}\right)du$$

$$= \frac{3}{2}u^8-\frac{12}{7}u^7+2u^6-\frac{12}{5}u^5+3u^4-4u^3+6u^2-12u+12\ln|u+1|+C$$

$$= \frac{3}{2}x^{2/3}-\frac{12}{7}x^{7/12}+2\sqrt{x}-\frac{12}{5}x^{5/12}+3\sqrt[3]{x}-4\sqrt[4]{x}+6\sqrt[6]{x}-12\sqrt[12]{x}+12\ln\left(\sqrt[12]{x}+1\right)+C$$

47. Let $u=e^x$. Then $x=\ln u$, $dx=\frac{du}{u} \Rightarrow$

$$\int \frac{e^{2x}\,dx}{e^{2x}+3e^x+2} = \int \frac{u^2\,(du/u)}{u^2+3u+2} = \int \frac{u\,du}{(u+1)(u+2)} = \int \left[\frac{-1}{u+1}+\frac{2}{u+2}\right]du$$

$$= 2\ln|u+2|-\ln|u+1|+C = \ln\left[(e^x+2)^2/(e^x+1)\right]+C$$

48. Let $u=\sin x$. Then $du=\cos x\,dx \Rightarrow$

$$\int \frac{\cos x\,dx}{\sin^2 x+\sin x} = \int \frac{du}{u^2+u} = \int \frac{du}{u(u+1)} = \int \left[\frac{1}{u}-\frac{1}{u+1}\right]du$$

$$= \ln\left|\frac{u}{u+1}\right|+C = \ln\left|\frac{\sin x}{1+\sin x}\right|+C$$

49. Let $u=\ln\left(x^2-x+2\right)$, $dv=dx$. Then $du=\frac{2x-1}{x^2-x+2}\,dx$, $v=x$, and (by integration by parts)

$$\int \ln\left(x^2-x+2\right)dx = x\ln\left(x^2-x+2\right)-\int \frac{2x^2-x}{x^2-x+2}\,dx = x\ln\left(x^2-x+2\right)-\int\left(2+\frac{x-4}{x^2-x+2}\right)dx$$

$$= x\ln\left(x^2-x+2\right)-2x-\int \frac{\frac{1}{2}(2x-1)}{x^2-x+2}\,dx+\frac{7}{2}\int \frac{dx}{\left(x-\frac{1}{2}\right)^2+\frac{7}{4}}$$

$$= x\ln\left(x^2-x+2\right)-2x-\frac{1}{2}\ln\left(x^2-x+2\right)+\frac{7}{2}\int \frac{\frac{\sqrt{7}}{2}\,du}{\frac{7}{4}(u^2+1)}$$

$$\left[\begin{array}{c} \text{where } x-\frac{1}{2}=\frac{\sqrt{7}}{2}u, \\ dx=\frac{\sqrt{7}}{2}du, \\ \left(x-\frac{1}{2}\right)^2+\frac{7}{4}=\frac{7}{4}(u^2+1) \end{array}\right]$$

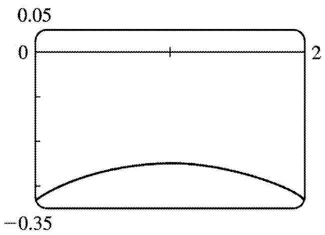
$$=\left(x-\frac{1}{2}\right)\ln\left(x^2-x+2\right)-2x+\sqrt{7}\tan^{-1}u+C$$

$$=\left(x-\frac{1}{2}\right)\ln\left(x^2-x+2\right)-2x+\sqrt{7}\tan^{-1}\frac{2x-1}{\sqrt{7}}+C$$

50. Let $u=\tan^{-1}x$, $dv=x\,dx \Rightarrow du=dx/\left(1+x^2\right)$, $v=\frac{1}{2}x^2$.

Then $\int x\tan^{-1}x\,dx=\frac{1}{2}x^2\tan^{-1}x-\frac{1}{2}\int \frac{x^2}{1+x^2}\,dx$. To evaluate the last integral, use long division or observe that $\int \frac{x^2}{1+x^2}\,dx=\int \frac{\left(1+x^2\right)-1}{1+x^2}\,dx=\int 1\,dx-\int \frac{1}{1+x^2}\,dx=x-\tan^{-1}x+C_1$. So $\int x\tan^{-1}x\,dx=\frac{1}{2}x^2\tan^{-1}x-\frac{1}{2}\left(x-\tan^{-1}x+C_1\right)=\frac{1}{2}\left(x^2\tan^{-1}x+\tan^{-1}x-x\right)+C$.

51.



From the graph, we see that the integral will be negative, and we guess that the area is about the same as that of a rectangle with width 2 and height 0.3 , so we estimate the integral to be $-(2\cdot 0.3)=-0.6$.

Now $\frac{1}{x^2-2x-3}=\frac{1}{(x-3)(x+1)}=\frac{A}{x-3}+\frac{B}{x+1}\Leftrightarrow 1=(A+B)x+A-3B$, so $A=-B$ and $A-3B=1\Leftrightarrow A=\frac{1}{4}$ and $B=-\frac{1}{4}$, so the integral becomes

$$\int_0^2 \frac{dx}{x^2-2x-3}=\frac{1}{4}\int_0^2\frac{dx}{x-3}-\frac{1}{4}\int_0^2\frac{dx}{x+1}=\frac{1}{4}\left[\ln|x-3|-\ln|x+1|\right]_0^2$$

$$=\frac{1}{4}\left[\ln\left|\frac{x-3}{x+1}\right|\right]_0^2=\frac{1}{4}\left(\ln\frac{1}{3}-\ln 3\right)=-\frac{1}{2}\ln 3\approx -0.55$$

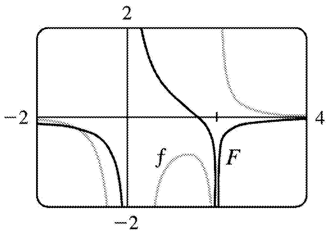
52. $\frac{1}{x^3-2x^2}=\frac{1}{x^2(x-2)}=\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x-2}\Rightarrow 1=(A+C)x^2+(B-2A)x-2B$, so $A+C=B-2A=0$ and $-2B=1\Rightarrow B=-\frac{1}{2}$, $A=-\frac{1}{4}$, and $C=\frac{1}{4}$. So the general antiderivative of $\frac{1}{x^3-2x^2}$ is

$$\int\frac{dx}{x^3-2x^2}=-\frac{1}{4}\int\frac{dx}{x}-\frac{1}{2}\int\frac{dx}{x^2}+\frac{1}{4}\int\frac{dx}{x-2}$$

$$=-\frac{1}{4}\ln|x|-\frac{1}{2}(-1/x)+\frac{1}{4}\ln|x-2|+C$$

$$=\frac{1}{4}\ln\left|\frac{x-2}{x}\right|+\frac{1}{2x}+C$$

We plot this function with $C=0$ on the same screen as $y=\frac{1}{x^3-2x^2}$.



53.

$$\int \frac{dx}{x^2-2x} = \int \frac{dx}{(x-1)^2-1} = \int \frac{du}{u^2-1}$$

$$= \frac{1}{2} \ln \left| \frac{u-1}{u+1} \right| + C \text{ [by Equation 6]} = \frac{1}{2} \ln \left| \frac{x-2}{x} \right| + C$$

54.

$$\int \frac{(2x+1)dx}{4x^2+12x-7} = \frac{1}{4} \int \frac{(8x+12)dx}{4x^2+12x-7} - \int \frac{2dx}{(2x+3)^2-16}$$

$$= \frac{1}{4} \ln \left|4x^2+12x-7\right| - \int \frac{du}{u^2-16}$$

$$= \frac{1}{4} \ln \left|4x^2+12x-7\right| - \frac{1}{8} \ln \left|(u-4)/(u+4)\right| + C$$

$$= \frac{1}{4} \ln \left|4x^2+12x-7\right| - \frac{1}{8} \ln \left|(2x-1)/(2x+7)\right| + C$$

55. **(a)** If $t=\tan\left(\frac{x}{2}\right)$, then $\frac{x}{2}=\tan^{-1}t$. The figure gives $\cos\left(\frac{x}{2}\right)=\frac{1}{\sqrt{1+t^2}}$ and $\sin\left(\frac{x}{2}\right)=\frac{t}{\sqrt{1+t^2}}$.

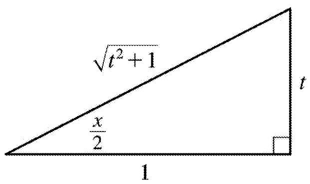
(b) $\cos x=\cos\left(2\cdot\frac{x}{2}\right)=2\cos^2\left(\frac{x}{2}\right)-1$

$$=2\left(\frac{1}{\sqrt{1+t^2}}\right)^2-1=\frac{2}{1+t^2}-1=\frac{1-t^2}{1+t^2}$$

$$\sin x=\sin\left(2\cdot\frac{x}{2}\right)=2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)=2\frac{t}{\sqrt{1+t^2}}\frac{1}{\sqrt{1+t^2}}=\frac{2t}{1+t^2}$$

(c)

$$\frac{x}{2}=\arctan t \Rightarrow x=2\arctan t \Rightarrow dx=\frac{2}{1+t^2}\,dt$$



56. Let $t=\tan(x/2)$. Then, using Exercise 55 , $dx=\frac{2}{1+t^2}\,dt$, $\sin x=\frac{2t}{1+t^2} \Rightarrow$

$$\int\frac{dx}{3-5\sin x}=\int\frac{2\,dt/\left(1+t^2\right)}{3-10t/\left(1+t^2\right)}=\int\frac{2\,dt}{3\left(1+t^2\right)-10t}=2\int\frac{dt}{3t^2-10t+3}$$

$$=\frac{1}{4}\int\left[\frac{1}{t-3}-\frac{3}{3t-1}\right]dt=\frac{1}{4}\left(\ln|t-3|-\ln|3t-1|\right)+C=\frac{1}{4}\ln\left|\frac{\tan(x/2)-3}{3\tan(x/2)-1}\right|+C$$

57. Let $t=\tan(x/2)$. Then, the expressions in Exercise 55 , we have

$$\int\frac{1}{3\sin x-4\cos x}\,dx=\int\frac{1}{3\left(\frac{2t}{1+t^2}\right)-4\left(\frac{1-t^2}{1+t^2}\right)}\,\frac{2\,dt}{1+t^2}=2\int\frac{dt}{3(2t)-4\left(1-t^2\right)}=\int\frac{dt}{2t^2+3t-2}$$

$$=\int\frac{dt}{(2t-1)(t+2)}=\int\left[\frac{2}{5}\,\frac{1}{2t-1}-\frac{1}{5}\,\frac{1}{t+2}\right]dt\quad\text{[using partial fractions]}$$

$$=\frac{1}{5}\left[\ln|2t-1|-\ln|t+2|\right]+C=\frac{1}{5}\ln\left|\frac{2t-1}{t+2}\right|+C=\frac{1}{5}\ln\left|\frac{2\tan(x/2)-1}{\tan(x/2)+2}\right|+C$$

58. Let $t=\tan(x/2)$. Then, by Exercise 55 ,

$$\int_{\pi/3}^{\pi/2}\frac{dx}{1+\sin x-\cos x}=\int_{1/\sqrt{3}}^{1}\frac{2dt/\left(1+t^2\right)}{1+2t/\left(1+t^2\right)-\left(1-t^2\right)/\left(1+t^2\right)}=\int_{1/\sqrt{3}}^{1}\frac{2dt}{1+t^2+2t-1+t^2}$$

$$=\int_{1/\sqrt{3}}^{1}\left[\frac{1}{t}-\frac{1}{t+1}\right]dt=\left[\ln t-\ln(t+1)\right]_{1/\sqrt{3}}^{1}=\ln\frac{1}{2}-\ln\frac{1}{\sqrt{3}+1}=\ln\frac{\sqrt{3}+1}{2}$$

59. Let $t=\tan(x/2)$. Then, by Exercise 55 ,

$$\begin{aligned}\int \frac{dx}{2\sin x+\sin 2x} &= \frac{1}{2}\int \frac{dx}{\sin x+\sin x\cos x}=\frac{1}{2}\int \frac{2\,dt/\left(1+t^2\right)}{2t/\left(1+t^2\right)+2t\left(1-t^2\right)/\left(1+t^2\right)^2}\\ &= \frac{1}{2}\int \frac{\left(1+t^2\right)dt}{t\left(1+t^2\right)+t\left(1-t^2\right)}=\frac{1}{4}\int \frac{\left(1+t^2\right)dt}{t}=\frac{1}{4}\int \left(\frac{1}{t}+t\right)dt\\ &= \frac{1}{4}\ln|t|+\frac{1}{8}t^2+C=\frac{1}{4}\ln\left|\tan\left(\frac{1}{2}x\right)\right|+\frac{1}{8}\tan^2\left(\frac{1}{2}x\right)+C\end{aligned}$$

60. $x^2-6x+8=(x-3)^2-1$ is positive for $5\le x\le 10$, so

$$\begin{aligned}\text{area} &= \int_5^{10}\frac{dx}{(x-3)^2-1}=\int_2^7\frac{du}{u^2-1}\ [\text{put}\,u=x-3]=\left[\frac{1}{2}\ln\left|\frac{u-1}{u+1}\right|\right]_2^7\\ &= \frac{1}{2}\ln\frac{3}{4}-\frac{1}{2}\ln\frac{1}{3}=\frac{1}{2}(\ln 3-2\ln 2+\ln 3)=\ln 3-\ln 2=\ln\frac{3}{2}\end{aligned}$$

61. $\dfrac{x+1}{x-1}=1+\dfrac{2}{x-1}>0$ for $2\le x\le 3$, so area

$$=\int_2^3\left[1+\frac{2}{x-1}\right]dx=\left[x+2\ln|x-1|\right]_2^3=(3+2\ln 2)-(2+2\ln 1)=1+2\ln 2\,.$$

62. **(a)** We use disks, so the volume is $V=\pi\int_0^1\left[\dfrac{1}{x^2+3x+2}\right]^2dx=\pi\int_0^1\dfrac{dx}{(x+1)^2(x+2)^2}$. To evaluate the integral, we use partial fractions: $\dfrac{1}{(x+1)^2(x+2)^2}=\dfrac{A}{x+1}+\dfrac{B}{(x+1)^2}+\dfrac{C}{x+2}+\dfrac{D}{(x+2)^2}\Rightarrow$ $1=A(x+1)(x+2)^2+B(x+2)^2+C(x+1)^2(x+2)+D(x+1)^2$. We set $x=-1$, giving $B=1$, then set $x=-2$, giving $D=1$. Now equating coefficients of x^3 gives $A=-C$, and then equating constants gives $1=4A+4+2(-A)+1\Rightarrow A=-2\Rightarrow C=2$. So the expression becomes

$$\begin{aligned}V &= \pi\int_0^1\left[\frac{-2}{x+1}+\frac{1}{(x+1)^2}+\frac{2}{(x+2)}+\frac{1}{(x+2)^2}\right]dx=\pi\left[2\ln\left|\frac{x+2}{x+1}\right|-\frac{1}{x+1}-\frac{1}{x+2}\right]_0^1\\ &= \pi\left[\left(2\ln\frac{3}{2}-\frac{1}{2}-\frac{1}{3}\right)-\left(2\ln 2-1-\frac{1}{2}\right)\right]=\pi\left(2\ln\frac{3/2}{2}+\frac{2}{3}\right)=\pi\left(\frac{2}{3}+\ln\frac{9}{16}\right)\end{aligned}$$

(b) In this case, we use cylindrical shells, so the volume is $V=2\pi\int_0^1\dfrac{x\,dx}{x^2+3x+2}=2\pi\int_0^1\dfrac{x\,dx}{(x+1)(x+2)}$. We use partial fractions to simplify the integrand: $\dfrac{x}{(x+1)(x+2)} = \dfrac{A}{x+1} + \dfrac{B}{x+2} \Rightarrow x=(A+B)x+2A+B$. So $A+B=1$ and $2A+B=0 \Rightarrow A=-1$ and $B=2$. So the volume is

$$2\pi\int_0^1 \left[\frac{-1}{x+1} + \frac{2}{x+2} \right] dx = 2\pi \left[-\ln |x+1| + 2\ln |x+2| \right]_0^1$$

$$= 2\pi(-\ln 2+2\ln 3+\ln 1-2\ln 2)=2\pi(2\ln 3-3\ln 2)=2\pi\ln \frac{9}{8}$$

63. $\dfrac{P+S}{P[(r-1)P-S]} = \dfrac{A}{P} + \dfrac{B}{(r-1)P-S} \Rightarrow P+S=A[(r-1)P-S]+BP=[(r-1)A+B]P-AS \Rightarrow (r-1)A+B=1$, $-A=1$ $\Rightarrow A=-1$, $B=r$. Now

$$t=\int \frac{P+S}{P[(r-1)P-S]}\, dP=\int \left[\frac{-1}{P} + \frac{r}{(r-1)P-S} \right] dP=-\int \frac{dP}{P} + \frac{r}{r-1} \int \frac{r-1}{(r-1)P-S}\, dP$$

so $t=-\ln P+ \dfrac{r}{r-1} \ln |(r-1)P-S|+C$. Here $r=0.10$ and $S=900$, so

$$t = -\ln P+ \frac{0.1}{-0.9} \ln |-0.9P-900|+C=-\ln P- \frac{1}{9} \ln (|-1|\,|0.9P+900|)$$

$$=-\ln P- \frac{1}{9} \ln (0.9P+900)+C$$

When $t=0$, $P=10,000$, so $0=-\ln 10,000- \dfrac{1}{9} \ln (9900)+C$. Thus, $C=\ln 10,000+ \dfrac{1}{9} \ln 9900$, so our equation becomes

$$t = \ln 10{,}000-\ln P+ \frac{1}{9} \ln 9900- \frac{1}{9} \ln (0.9P+900)=\ln \frac{10{,}000}{P} + \frac{1}{9} \ln \frac{9900}{0.9P+900}$$

$$= \ln \frac{10{,}000}{P} + \frac{1}{9} \ln \frac{1100}{0.1P+100} =\ln \frac{10{,}000}{P} + \frac{1}{9} \ln \frac{11{,}000}{P+1000}$$

64. If we subtract and add $2x^2$, we get

$$x^4+1 = x^4+2x^2+1-2x^2=\left(x^2+1\right)^2-2x^2=\left(x^2+1\right)^2-\left(\sqrt{2}\,x\right)^2$$

$$= \left[\left(x^2+1\right)-\sqrt{2}\,x\right]\left[\left(x^2+1\right)+\sqrt{2}\,x\right]=\left(x^2-\sqrt{2}\,x+1\right)\left(x^2+\sqrt{2}\,x+1\right)$$

So we can decompose $\dfrac{1}{x^4+1} = \dfrac{Ax+B}{x^2+\sqrt{2}\,x+1} + \dfrac{Cx+D}{x^2-\sqrt{2}\,x+1} \Rightarrow$

$1=(Ax+B)\left(x^2-\sqrt{2}\,x+1\right)+(Cx+D)\left(x^2+\sqrt{2}\,x+1\right)$. Setting the constant terms equal gives $B+D=1$, then from the coefficients of x^3 we get $A+C=0$. Now from the coefficients of x we get $A+C+(B-D)\sqrt{2}=0$ $\Leftrightarrow [(1-D)-D]\sqrt{2}=0 \Rightarrow D=\frac{1}{2} \Rightarrow B=\frac{1}{2}$, and finally, from the coefficients of x^2 we get $\sqrt{2}\,(C-A)+B+D=0 \Rightarrow C-A=-\frac{1}{\sqrt{2}} \Rightarrow C=-\frac{\sqrt{2}}{4}$ and $A=\frac{\sqrt{2}}{4}$.

So we rewrite the integrand, splitting the terms into forms which we know how to integrate:

$$\frac{1}{x^4+1} = \frac{\frac{\sqrt{2}}{4}x+\frac{1}{2}}{x^2+\sqrt{2}\,x+1} + \frac{-\frac{\sqrt{2}}{4}x+\frac{1}{2}}{x^2-\sqrt{2}\,x+1} = \frac{1}{4\sqrt{2}}\left[\frac{2x+2\sqrt{2}}{x^2+\sqrt{2}\,x+1} - \frac{2x-2\sqrt{2}}{x^2-\sqrt{2}\,x+1}\right]$$

$$= \frac{\sqrt{2}}{8}\left[\frac{2x+\sqrt{2}}{x^2+\sqrt{2}\,x+1} - \frac{2x-\sqrt{2}}{x^2-\sqrt{2}\,x+1}\right] + \frac{1}{4}\left[\frac{1}{\left(x+\frac{1}{\sqrt{2}}\right)^2+\frac{1}{2}} + \frac{1}{\left(x-\frac{1}{\sqrt{2}}\right)^2+\frac{1}{2}}\right]$$

Now we integrate: $\int \frac{dx}{x^4+1} = \frac{\sqrt{2}}{8}\ln\left(\frac{x^2+\sqrt{2}\,x+1}{x^2-\sqrt{2}\,x+1}\right) + \frac{\sqrt{2}}{4}\left[\tan^{-1}(\sqrt{2}\,x+1)+\tan^{-1}(\sqrt{2}\,x-1)\right]+C$.

65. **(a)** In Maple, we define $f(x)$, and then use `convert(f,parfrac,x)` ; to obtain

$$f(x)=\frac{24{,}110/4879}{5x+2} - \frac{668/323}{2x+1} - \frac{9438/80{,}155}{3x-7} + \frac{(22{,}098x+48{,}935)/260{,}015}{x^2+x+5} .$$

In Mathematica, we use the command Apart, and in Derive, we use Expand.

(b)

$$\int f(x)dx = \frac{24{,}110}{4879}\cdot\frac{1}{5}\ln|5x+2| - \frac{668}{323}\cdot\frac{1}{2}\ln|2x+1| - \frac{9438}{80{,}155}\cdot\frac{1}{3}\ln|3x-7|$$

$$+\frac{1}{260{,}015}\int \frac{22{,}098\left(x+\frac{1}{2}\right)+37{,}886}{\left(x+\frac{1}{2}\right)^2+\frac{19}{4}}\,dx+C$$

$$=\frac{24{,}110}{4879}\cdot\frac{1}{5}\ln|5x+2| - \frac{668}{323}\cdot\frac{1}{2}\ln|2x+1| - \frac{9438}{80{,}155}\cdot\frac{1}{3}\ln|3x-7|$$

$$+\frac{1}{260{,}015}\left[22{,}098\cdot\frac{1}{2}\ln\left(x^2+x+5\right)+37{,}886\cdot\sqrt{\frac{4}{19}}\tan^{-1}\left(\frac{1}{\sqrt{19/4}}\left(x+\frac{1}{2}\right)\right)\right]+C$$

$$=\frac{4822}{4879}\ln|5x+2| - \frac{334}{323}\ln|2x+1| - \frac{3146}{80{,}155}\ln|3x-7| + \frac{11{,}049}{260{,}015}\ln\left(x^2+x+5\right)$$

$$+\frac{75{,}772}{260{,}015\sqrt{19}}\tan^{-1}\left[\frac{1}{\sqrt{19}}(2x+1)\right]+C$$

Using a CAS, we get

$$\frac{4822\ln(5x+2)}{4879} - \frac{334\ln(2x+1)}{323} - \frac{3146\ln(3x-7)}{80{,}155} + \frac{11{,}049\ln\left(x^2+x+5\right)}{260{,}015} + \frac{3988\sqrt{19}}{260{,}015}\tan^{-1}\left[\frac{\sqrt{19}}{19}(2x+1)\right]$$

The main difference in this answer is that the absolute value signs and the constant of integration have been omitted. Also, the fractions have been reduced and the denominators rationalized.

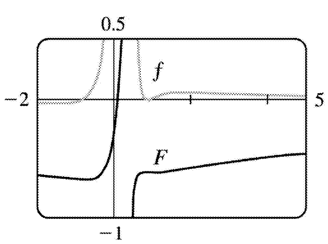
66. **(a)** In Maple, we define $f(x)$, and then use `convert(f,parfrac,x);` to get

$$f(x)=\frac{5828/1815}{(5x-2)^2}-\frac{59{,}096/19{,}965}{5x-2}+\frac{2(2843x+816)/3993}{2x^2+1}+\frac{(313x-251)/363}{\left(2x^2+1\right)^2}\ .$$

In Mathematica, we use the command Apart, and in Derive, we use Expand.

(b) As we saw in Exercise 65, computer algebra systems omit the absolute value signs in $\int(1/y)\,dy=\ln|y|$. So we use the CAS to integrate the expression in part (a) and add the necessary absolute value signs and constant of integration to get

$$\int f(x)dx = -\frac{5828}{9075(5x-2)}-\frac{59{,}096\ln|5x-2|}{99{,}825}+\frac{2843\ln\left(2x^2+1\right)}{7986} + \frac{503}{15{,}972}\sqrt{2}\tan^{-1}\left(\sqrt{2}\,x\right)-\frac{1}{2904}\,\frac{1004x+626}{2x^2+1}+C$$



(c) From the graph, we see that f goes from negative to positive at $x\approx-0.78$, then back to negative at $x\approx0.8$, and finally back to positive at $x=1$. Also, $\lim_{x\to0.4}f(x)=\infty$. So we see (by the First Derivative Test) that $\int f(x)dx$ has minima at $x\approx-0.78$ and $x=1$, and a maximum at $x\approx0.80$, and that $\int f(x)dx$ is unbounded as $x\to0.4$. Note also that just to the right of $x=0.4$, f has large values, so $\int f(x)dx$ increases rapidly, but slows down as f drops toward 0 . $\int f(x)dx$ decreases from about 0.8 to 1 , then increases slowly since f stays small and positive.

67. There are only finitely many values of x where $Q(x)=0$ (assuming that Q is not the zero

polynomial). At all other values of x , $F(x)/Q(x) = G(x)/Q(x)$, so $F(x)=G(x)$. In other words, the values of F and G agree at all except perhaps finitely many values of x . By continuity of F and G , the polynomials F and G must agree at those values of x too.
More explicitly: if a is a value of x such that $Q(a)=0$, then $Q(x)\neq 0$ for all x sufficiently close to a . Thus,

$$F(a)=\lim_{x\to a} F(x)\,[\text{by continuity of } F] = \lim_{x\to a} G(x)\,[\text{whenever } Q(x)\neq 0]$$
$$= G(a)\,[\text{by continuity of } G]$$

68. Let $f(x)=ax^2+bx+c$. We calculate the partial fraction decomposition of $\dfrac{f(x)}{x^2(x+1)^3}$. Since $f(0)=1$, we must have $c=1$, so $\dfrac{f(x)}{x^2(x+1)^3} = \dfrac{ax^2+bx+1}{x^2(x+1)^3} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x+1} + \dfrac{D}{(x+1)^2} + \dfrac{E}{(x+1)^3}$. Now in order for the integral not to contain any logarithms (that is, in order for it to be a rational function), we must have $A=C=0$, so $ax^2+bx+1=B(x+1)^3+Dx^2(x+1)+Ex^2$. Equating constant terms gives $B=1$, then equating coefficients of x gives $3B=b \Rightarrow b=3$. This is the quantity we are looking for, since $f'(0)=b$.

1. $\int \frac{\sin x+\sec x}{\tan x}\,dx=\int \left(\frac{\sin x}{\tan x}+\frac{\sec x}{\tan x}\right)dx=\int (\cos x+\csc x)dx=\sin x+\ln|\csc x-\cot x|+C$

2.

$$\int \tan^3\theta\,d\theta=\int(\sec^2\theta-1)\tan\theta\,d\theta=\int\tan\theta\sec^2\theta\,d\theta-\int\frac{\sin\theta}{\cos\theta}\,d\theta$$

$$=\int u\,du+\int\frac{dv}{v}\quad\left[\begin{array}{ll}\text{u}=\tan\theta, & \text{v}=\cos\theta,\\ \text{du}=\sec^2\theta\,\text{d}\theta & \text{dv}=-\sin\theta\,\text{d}\theta\end{array}\right]$$

$$=\frac{1}{2}u^2+\ln|v|+C=\frac{1}{2}\tan^2\theta+\ln|\cos\theta|+C$$

3.

$$\int_0^2\frac{2t}{(t-3)^2}\,dt=\int_{-3}^{-1}\frac{2(u+3)}{u^2}\,du=\int_{-3}^{-1}\left(\frac{2}{u}+\frac{6}{u^2}\right)du=\left[2\ln|u|-\frac{6}{u}\right]_{-3}^{-1}$$

$$=(2\ln 1+6)-(2\ln 3+2)=4-2\ln 3\text{ or }4-\ln 9$$

4. Let $u=x^2$. Then $du=2x\,dx\Rightarrow\int\frac{x\,dx}{\sqrt{3-x^4}}=\frac{1}{2}\int\frac{du}{\sqrt{3-u^2}}=\frac{1}{2}\sin^{-1}\frac{u}{\sqrt{3}}+C=\frac{1}{2}\sin^{-1}\frac{x^2}{\sqrt{3}}+C$.

5. Let $u=\arctan y$. Then $du=\frac{dy}{1+y^2}\Rightarrow\int_{-1}^{1}\frac{e^{\arctan y}}{1+y^2}\,dy=\int_{-\pi/4}^{\pi/4}e^u\,du=\left[e^u\right]_{-\pi/4}^{\pi/4}=e^{\pi/4}-e^{-\pi/4}$.

6.

$$\int x\csc x\cot x\,dx\quad\left[\begin{array}{ll}\text{u}=\text{x}, & \text{dv}=\csc\text{x}\cot\text{x}\,\text{dx},\\ \text{du}=\text{dx} & \text{v}=-\csc\text{x}\end{array}\right]=-x\csc x-\int(-\csc x)dx$$

$$=-x\csc x+\ln|\csc x-\cot x|+C$$

7.

$$\int_1^3 r^4\ln r\,dr\quad\left[\begin{array}{ll}\text{u}=\ln\text{r}, & \text{dv}=\text{r}^4\,\text{dr}\\ \text{du}=\frac{dr}{r} & \text{v}=\frac{1}{5}\text{r}^5\end{array}\right]=\left[\frac{1}{5}r^5\ln r\right]_1^3-\int_1^3\frac{1}{5}r^4\,dr=\frac{243}{5}\ln 3-0-\left[\frac{1}{25}r^5\right]_1^3$$

$$=\frac{243}{5}\ln 3-\left(\frac{243}{25}-\frac{1}{25}\right)=\frac{243}{5}\ln 3-\frac{242}{25}$$

8. $\dfrac{x-1}{x^2-4x-5}=\dfrac{x-1}{(x-5)(x+1)}=\dfrac{A}{x-5}+\dfrac{B}{x+1}\Rightarrow x-1=A(x+1)+B(x-5)$. Setting $x=-1$ gives $-2=-6B$, so $B=\dfrac{1}{3}$. Setting $x=5$ gives $4=6A$, so $A=\dfrac{2}{3}$. Now

$$\int_0^4 \frac{x-1}{x^2-4x-5}\,dx=\int_0^4\left(\frac{2/3}{x-5}+\frac{1/3}{x+1}\right)dx=\left[\frac{2}{3}\ln|x-5|+\frac{1}{3}\ln|x+1|\right]_0^4$$
$$=\frac{2}{3}\ln 1+\frac{1}{3}\ln 5-\frac{2}{3}\ln 5-\frac{1}{3}\ln 1=-\frac{1}{3}\ln 5$$

9.

$$\int\frac{x-1}{x^2-4x+5}\,dx=\int\frac{(x-2)+1}{(x-2)^2+1}\,dx=\int\left(\frac{u}{u^2+1}+\frac{1}{u^2+1}\right)du\ [\,u=x-2\,,\,du=dx\,]$$
$$=\frac{1}{2}\ln\left(u^2+1\right)+\tan^{-1}u+C=\frac{1}{2}\ln\left(x^2-4x+5\right)+\tan^{-1}(x-2)+C$$

10.

$$\int\frac{x}{x^4+x^2+1}\,dx=\int\frac{\frac{1}{2}\,du}{u^2+u+1}\ \ [\,u=x^2\,,\,du=2x\,dx\,]=\frac{1}{2}\int\frac{du}{\left(u+\frac{1}{2}\right)^2+\frac{3}{4}}$$
$$=\frac{1}{2}\int\frac{\frac{\sqrt{3}}{2}\,dv}{\frac{3}{4}\left(v^2+1\right)}\ \ \left[\,u+\frac{1}{2}=\frac{\sqrt{3}}{2}v\,,\,du=\frac{\sqrt{3}}{2}\,dv\,\right]=\frac{\sqrt{3}}{4}\cdot\frac{4}{3}\int\frac{dv}{v^2+1}$$
$$=\frac{1}{\sqrt{3}}\tan^{-1}v+C=\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2}{\sqrt{3}}\left(x^2+\frac{1}{2}\right)\right)+C$$

11.

$$\int\sin^3\theta\,\cos^5\theta\,d\theta=\int\cos^5\theta\,\sin^2\theta\,\sin\theta\,d\theta=-\int\cos^5\theta\,(1-\cos^2\theta)(-\sin\theta)\,d\theta$$
$$=-\int u^5(1-u^2)\,du\left[\begin{array}{l}\mathrm{u}=\cos\theta\,,\\ \mathrm{du}=-\sin\theta\,\mathrm{d}\theta\end{array}\right]$$
$$=\int(u^7-u^5)\,du=\frac{1}{8}u^8-\frac{1}{6}u^6+C=\frac{1}{8}\cos^8\theta-\frac{1}{6}\cos^6\theta+C$$

Another solution:

$$\int \sin^3\theta\ \cos^5\theta\ d\theta = \int \sin^3\theta\ (\cos^2\theta)^2 \cos\theta\ d\theta = \int \sin^3\theta\ (1-\sin^2\theta)^2 \cos\theta\ d\theta$$

$$= \int u^3 (1-u^2)^2\, du \left[\begin{array}{l} u=\sin\theta\ , \\ du=\cos\theta\ d\theta \end{array}\right] = \int u^3 (1-2u^2+u^4)\, du$$

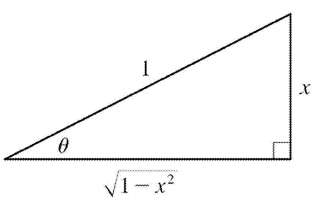
$$= \int (u^3-2u^5+u^7)\, du = \frac{1}{4}u^4 - \frac{1}{3}u^6 + \frac{1}{8}u^8 + C = \frac{1}{4}\sin^4\theta - \frac{1}{3}\sin^6\theta + \frac{1}{8}\sin^8\theta + C$$

12. Let $u=\cos x$. Then $du=-\sin x\,dx \Rightarrow \int \sin x \cos(\cos x)\,dx = -\int \cos u\,du = -\sin u + C = -\sin(\cos x) + C$.

13. Let $x=\sin\theta$, where $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$. Then $dx=\cos\theta\ d\theta$ and $(1-x^2)^{1/2}=\cos\theta$, so

$$\int \frac{dx}{(1-x^2)^{3/2}} = \int \frac{\cos\theta\ d\theta}{(\cos\theta)^3} = \int \sec^2\theta\ d\theta$$

$$= \tan\theta + C = \frac{x}{\sqrt{1-x^2}} + C$$



14. Let $u=\ln x$. Then $du=dx/x \Rightarrow$

$$\int \frac{\sqrt{1+\ln x}}{x\ln x}\,dx = \int \frac{\sqrt{1+u}}{u}\,du = \int \frac{v}{v^2-1}\,2v\,dv \quad [\text{put } v=\sqrt{1+u}, u=v^2-1, du=2v\,dv]$$

$$= 2\int \left(1+\frac{1}{v^2-1}\right)dv = 2v + \ln\left|\frac{v-1}{v+1}\right| + C = 2\sqrt{1+\ln x} + \ln\left(\frac{\sqrt{1+\ln x}-1}{\sqrt{1+\ln x}+1}\right) + C$$

15. Let $u=1-x^2 \Rightarrow du=-2x\,dx$. Then

$$\int_0^{1/2} \frac{x}{\sqrt{1-x^2}}\,dx = -\frac{1}{2}\int_1^{3/4} \frac{1}{\sqrt{u}}\,du = \frac{1}{2}\int_{3/4}^{1} u^{-1/2}\,du = \frac{1}{2}\left[2u^{1/2}\right]_{3/4}^{1} = \left[\sqrt{u}\,\right]_{3/4}^{1} = 1 - \frac{\sqrt{3}}{2}$$

16.

$$\int_0^{\sqrt{2}/2} \frac{x^2}{\sqrt{1-x^2}}\,dx = \int_0^{\pi/4} \frac{\sin^2\theta}{\cos\theta}\cos\theta\,d\theta \quad [\,x=\sin\theta\ ,\ \}\{dx\}=\cos\theta\,d\theta\,]$$

$$=\int_0^{\pi/4}\frac{1}{2}(1-\cos 2\theta)\,d\theta=\frac{1}{2}\left[\theta-\frac{1}{2}\sin 2\theta\right]_0^{\pi/4}=\frac{1}{2}\left[\left(\frac{\pi}{4}-\frac{1}{2}\right)-(0-0)\right]=\frac{\pi}{8}-\frac{1}{4}$$

17.

$$\int x\sin^2 x\,dx \left[\begin{array}{ll} \mathrm{u=x,} & \mathrm{dv=\sin^2 xdx} \\ \mathrm{du=dx} & \mathrm{v}=\int \sin^2\mathrm{xdx}=\int\frac{1}{2}(1-\cos 2\mathrm{x})\mathrm{dx}=\frac{1}{2}\mathrm{x}-\frac{1}{2}\sin\mathrm{x}\cos\mathrm{x}\end{array}\right]$$

$$=\frac{1}{2}x^2-\frac{1}{2}x\sin x\cos x-\int\left(\frac{1}{2}x-\frac{1}{2}\sin x\cos x\right)dx$$

$$=\frac{1}{2}x^2-\frac{1}{2}x\sin x\cos x-\frac{1}{4}x^2+\frac{1}{4}\sin^2 x+C=\frac{1}{4}x^2-\frac{1}{2}x\sin x\cos x+\frac{1}{4}\sin^2 x+C$$

Note: $\int\sin x\cos x\,dx=\int s\,ds=\frac{1}{2}s^2+C$ [where $s=\sin x, ds=\cos x\,dx$] .

A slightly different method is to write $\int x\sin^2 x\,dx=\int x\cdot\frac{1}{2}(1-\cos 2x)\,dx=\frac{1}{2}\int x\,dx-\frac{1}{2}\int x\cos 2x\,dx$. If we evaluate the second integral by parts, we arrive at the equivalent answer $\frac{1}{4}x^2-\frac{1}{4}x\sin 2x-\frac{1}{8}\cos 2x+C$.

18. Let $u=e^{2t}$, $du=2e^{2t}\,dt$. Then

$$\int\frac{e^{2t}}{1+e^{4t}}\,dt=\int\frac{\frac{1}{2}(2e^{2t})\,dt}{1+(e^{2t})^2}=\int\frac{\frac{1}{2}du}{1+u^2}=\frac{1}{2}\tan^{-1}u+C=\frac{1}{2}\tan^{-1}(e^{2t})+C\ .$$

19. Let $u=e^x$. Then $\int e^{x+e^x}dx=\int e^{e^x}e^x\,dx=\int e^u\,du=e^u+C=e^{e^x}+C$.

20. Let $u=\sqrt[3]{x}$. Then $x=u^3\Rightarrow\int e^{\sqrt{[3]x}}dx=\int e^u\cdot 3u^2\,du$. Now use parts: let $w=u^2$, $dv=e^u\,du\Rightarrow dw=2u\,du$, $v=e^u\Rightarrow 3\int e^u u^2\,du=3\left(u^2e^u-2\int ue^u\,du\right)$. Now use parts again with $W=u$, $dV=e^u\,du$ to get

$\int e^u 3u^2\,du=e^u\left(3u^2-6u+6\right)+C=3e^{\sqrt[3]{x}}\left(x^{2/3}-2\sqrt[3]{x}+2\right)+C$.

21. Integrate by parts three times, first with $u=t^3$, $dv=e^{-2t}\,dt$:

$$\begin{aligned}\int t^3e^{-2t}\,dt &= -\frac{1}{2}t^3e^{-2t}+\frac{1}{2}\int 3t^2e^{-2t}\,dt=-\frac{1}{2}t^3e^{-2t}-\frac{3}{4}t^2e^{-2t}+\frac{1}{2}\int 3te^{-2t}\,dt\\ &= -e^{-2t}\left[\frac{1}{2}t^3+\frac{3}{4}t^2\right]-\frac{3}{4}te^{-2t}+\frac{3}{4}\int e^{-2t}\,dt=-e^{-2t}\left[\frac{1}{2}t^3+\frac{3}{4}t^2+\frac{3}{4}t+\frac{3}{8}\right]+C\\ &= -\frac{1}{8}e^{-2t}\left(4t^3+6t^2+6t+3\right)+C\end{aligned}$$

22. Integrate by parts: $u=\sin^{-1}x$, $dv=x\,dx\Rightarrow du=\left(1\big/\sqrt{1-x^2}\right)dx$, $v=\frac{1}{2}x^2$, so

$$\begin{aligned}\int x\sin^{-1}x\,dx &= \frac{1}{2}x^2\sin^{-1}x-\frac{1}{2}\int\frac{x^2\,dx}{\sqrt{1-x^2}}=\frac{1}{2}x^2\sin^{-1}x-\frac{1}{2}\int\frac{\sin^2\theta\,\cos\theta\,d\theta}{\cos\theta}\quad\text{where } x=\sin\theta\ \text{ for } -\frac{\pi}{2}\le\theta\le\frac{\pi}{2}\\ &= \frac{1}{2}x^2\sin^{-1}x-\frac{1}{4}\int(1-\cos 2\theta)\,d\theta=\frac{1}{2}x^2\sin^{-1}x-\frac{1}{4}(\theta-\sin\theta\,\cos\theta)+C\\ &= \frac{1}{2}x^2\sin^{-1}x-\frac{1}{4}\left[\sin^{-1}x-x\sqrt{1-x^2}\right]+C=\frac{1}{4}\left[\left(2x^2-1\right)\sin^{-1}x+x\sqrt{1-x^2}\right]+C\end{aligned}$$

23. Let $u=1+\sqrt{x}$. Then $x=(u-1)^2$, $dx=2(u-1)\,du\Rightarrow$

$$\begin{aligned}\int_0^1\left(1+\sqrt{x}\right)^8dx &= \int_1^2 u^8\cdot 2(u-1)\,du=2\int_1^2\left(u^9-u^8\right)du=\left[\frac{1}{5}u^{10}-2\cdot\frac{1}{9}u^9\right]_1^2\\ &= \frac{1024}{5}-\frac{1024}{9}-\frac{1}{5}+\frac{2}{9}=\frac{4097}{45}\end{aligned}$$

24. Let $u=\ln\left(x^2-1\right)$, $dv=dx\Leftrightarrow du=\dfrac{2x}{x^2-1}$, $v=x$. Then

$$\begin{aligned}\int\ln\left(x^2-1\right)dx &= x\ln\left(x^2-1\right)-\int\frac{2x^2}{x^2-1}\,dx=x\ln\left(x^2-1\right)-\int\left[2+\frac{2}{(x-1)(x+1)}\right]dx\\ &= x\ln\left(x^2-1\right)-\int\left[2+\frac{1}{x-1}-\frac{1}{x+1}\right]dx\end{aligned}$$

$$=x\ln\left(x^2-1\right)-2x-\ln|x-1|+\ln|x+1|+C$$

25. $\dfrac{3x^2-2}{x^2-2x-8}=3+\dfrac{6x+22}{(x-4)(x+2)}=3+\dfrac{A}{x-4}+\dfrac{B}{x+2}\Rightarrow 6x+22=A(x+2)+B(x-4)$. Setting $x=4$ gives $46=6A$, so $A=\dfrac{23}{3}$. Setting $x=-2$ gives $10=-6B$, so $B=-\dfrac{5}{3}$. Now

$$\int\frac{3x^2-2}{x^2-2x-8}\,dx=\int\left(3+\frac{23/3}{x-4}-\frac{5/3}{x+2}\right)dx=3x+\frac{23}{3}\ln|x-4|-\frac{5}{3}\ln|x+2|+C\ .$$

26. $\displaystyle\int\frac{3x^2-2}{x^3-2x-8}\,dx=\int\frac{du}{u}\left[\begin{array}{l}\mathrm{u=x^3-2x-8,}\\ \mathrm{du=}\left(3x^2-2\right)\mathrm{dx}\end{array}\right]=\ln|u|+C=\ln\left|x^3-2x-8\right|+C$

27. Let $u=\ln(\sin x)$. Then $du=\cot x\,dx\Rightarrow\int\cot x\ln(\sin x)\,dx=\int u\,du=\frac{1}{2}u^2+C=\frac{1}{2}[\ln(\sin x)]^2+C$.

28.

$$\int\sin\sqrt{at}\,dt=\int\sin u\cdot\frac{2}{a}u\,du\ [\ u=\sqrt{at}\ ,\ u^2=at\ ,\ 2u\,du=a\,dt\ ]=\frac{2}{a}\int u\sin u\,du$$

$$=\frac{2}{a}[-u\cos u+\sin u]+C\ [\text{ integration by parts}]=-\frac{2}{a}\sqrt{at}\cos\sqrt{at}+\frac{2}{a}\sin\sqrt{at}+C$$

$$=-2\sqrt{\frac{t}{a}}\cos\sqrt{at}+\frac{2}{a}\sin\sqrt{at}+C$$

29.

$$\int_0^5\frac{3w-1}{w+2}\,dw=\int_0^5\left(3-\frac{7}{w+2}\right)dw=[3w-7\ln|w+2|]_0^5$$

$$=15-7\ln 7+7\ln 2=15+7(\ln 2-\ln 7)=15+7\ln\frac{2}{7}$$

30. $x^2-4x<0$ on $[0,4]$, so

$$\int_{-2}^{2}\left|x^2-4x\right|dx=\int_{-2}^{0}(x^2-4x)\,dx+\int_0^2\left(4x-x^2\right)dx=\left[\frac{1}{3}x^3-2x^2\right]_{-2}^{0}+\left[2x^2-\frac{1}{3}x^3\right]_0^2$$

$$=0-\left(-\frac{8}{3}-8\right)+\left(8-\frac{8}{3}\right)-0=16$$

31. As in Example 5,

$$\int\sqrt{\frac{1+x}{1-x}}\,dx=\int\frac{\sqrt{1+x}}{\sqrt{1-x}}\cdot\frac{\sqrt{1+x}}{\sqrt{1+x}}\,dx=\int\frac{1+x}{\sqrt{1-x^2}}\,dx=\int\frac{dx}{\sqrt{1-x^2}}+\int\frac{x\,dx}{\sqrt{1-x^2}}$$

$$=\sin^{-1}x-\sqrt{1-x^2}+C$$

Another method: Substitute $u=\sqrt{(1+x)/(1-x)}$.

32.

$$\int\frac{\sqrt{2x-1}}{2x+3}\,dx=\int\frac{u\cdot u\,du}{u^2+4}\quad u=\sqrt{2x-1},2x+3=u^2+4,u^2=2x-1,u\,du=dx=\int\left(1-\frac{4}{u^2+4}\right)du$$

$$=u-4\cdot\frac{1}{2}\tan^{-1}\left(\frac{1}{2}u\right)+C=\sqrt{2x-1}-2\tan^{-1}\left(\frac{1}{2}\sqrt{2x-1}\right)+C$$

33. $3-2x-x^2=-(x^2+2x+1)+4=4-(x+1)^2$. Let $x+1=2\sin\theta$, where $-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$. Then $dx=2\cos\theta\ d\theta$ and

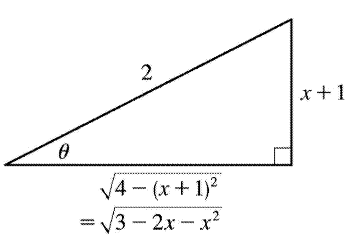
$$\int\sqrt{3-2x-x^2}\,dx=\int\sqrt{4-(x+1)^2}\,dx=\int\sqrt{4-4\sin^2\theta}\ 2\cos\theta\ d\theta$$

$$=4\int\cos^2\theta\ d\theta=2\int(1+\cos 2\theta)\,d\theta$$

$$=2\theta+\sin 2\theta+C=2\theta+2\sin\theta\cos\theta+C$$

$$=2\sin^{-1}\left(\frac{x+1}{2}\right)+2\cdot\frac{x+1}{2}\cdot\frac{\sqrt{3-2x-x^2}}{2}+C$$

$$=2\sin^{-1}\left(\frac{x+1}{2}\right)+\frac{x+1}{2}\sqrt{3-2x-x^2}+C$$



34.

$$\int_{\pi/4}^{\pi/2}\frac{1+4\cot x}{4-\cot x}\,dx=\int_{\pi/4}^{\pi/2}\left[\frac{(1+4\cos x/\sin x)}{(4-\cos x/\sin x)}\cdot\frac{\sin x}{\sin x}\right]dx=\int_{\pi/4}^{\pi/2}\frac{\sin x+4\cos x}{4\sin x-\cos x}\,dx$$

$$=\int_{3/\sqrt{2}}^{4}\frac{1}{u}\,du\left[\begin{array}{l}u=4\sin x-\cos x,\\ du=(4\cos x+\sin x)\,dx\end{array}\right]=[\ln|u|]_{3/\sqrt{2}}^{4}$$

$$=\ln 4-\ln\frac{3}{\sqrt{2}}=\ln\frac{4}{3/\sqrt{2}}=\ln\left(\frac{4}{3}\sqrt{2}\right)$$

35. Because $f(x)=x^{8}\sin x$ is the product of an even function and an odd function, it is odd. Therefore, $\int_{-1}^{1}x^{8}\sin x\,dx=0$ [by (5.5.)(b)].

36. $\sin 4x\cos 3x=\frac{1}{2}(\sin x+\sin 7x)$ by Formula .2.2(a), so

$\int\sin 4x\cos 3x\,dx=\frac{1}{2}\int(\sin x+\sin 7x)\,dx=\frac{1}{2}\left[-\cos x-\frac{1}{7}\cos 7x\right]+C=-\frac{1}{2}\cos x-\frac{1}{14}\cos 7x+C$.

37.

$$\int_{0}^{\pi/4}\cos^{2}\theta\,\tan^{2}\theta\,d\theta=\int_{0}^{\pi/4}\sin^{2}\theta\,d\theta=\int_{0}^{\pi/4}\frac{1}{2}(1-\cos 2\theta)\,d\theta=\left[\frac{1}{2}\theta-\frac{1}{4}\sin 2\theta\right]_{0}^{\pi/4}$$

$$=\left(\frac{\pi}{8}-\frac{1}{4}\right)-(0-0)=\frac{\pi}{8}-\frac{1}{4}$$

38.

$$\int_{0}^{\pi/4}\tan^{5}\theta\,\sec^{3}\theta\,d\theta=\int_{0}^{\pi/4}\left(\tan^{2}\theta\right)^{2}\sec^{2}\theta\cdot\sec\theta\,\tan\theta\,d\theta=\int_{1}^{\sqrt{2}}(u^{2}-1)^{2}u^{2}\,du\left[\begin{array}{l}u=\sec\theta\\ du=\sec\theta\,\tan\theta\,d\theta\end{array}\right]$$

$$=\int_{1}^{\sqrt{2}}(u^{6}-2u^{4}+u^{2})\,du=\left[\frac{1}{7}u^{7}-\frac{2}{5}u^{5}+\frac{1}{3}u^{3}\right]_{1}^{\sqrt{2}}$$

$$=\left(\frac{8}{7}\sqrt{2}-\frac{8}{5}\sqrt{2}+\frac{2}{3}\sqrt{2}\right)-\left(\frac{1}{7}-\frac{2}{5}+\frac{1}{3}\right)=\frac{22}{105}\sqrt{2}-\frac{8}{105}=\frac{2}{105}(11\sqrt{2}-4)$$

39. Let $u=1-x^{2}$. Then $du=-2x\,dx\Rightarrow$

$$\int\frac{x\,dx}{1-x^{2}+\sqrt{1-x^{2}}}=-\frac{1}{2}\int\frac{du}{u+\sqrt{u}}=-\int\frac{v\,dv}{v^{2}+v}\ [v=\sqrt{u},u=v^{2},du=2v\,dv]$$

$$=-\int\frac{dv}{v+1}=-\ln|v+1|+C=-\ln\left(\sqrt{1-x^{2}}+1\right)+C$$

40. $4y^2-4y-3=(2y-1)^2-2^2$, so let $u=2y-1\Rightarrow du=2dy$. Thus,

$$\int \frac{dy}{\sqrt{4y^2-4y-3}} = \int \frac{dy}{\sqrt{(2y-1)^2-2^2}} = \frac{1}{2}\int \frac{du}{\sqrt{u^2-2^2}}$$
$$= \frac{1}{2}\ln\left|u+\sqrt{u^2-2^2}\right|$$
$$= \frac{1}{2}\ln\left|2y-1+\sqrt{4y^2-4y-3}\right|+C$$

41. Let $u=\theta$, $dv=\tan^2\theta\ d\theta=\left(\sec^2\theta-1\right)d\theta\Rightarrow du=d\theta$ and $v=\tan\theta-\theta$. So

$$\int\theta\tan^2\theta\ d\theta = \theta(\tan\theta-\theta)-\int(\tan\theta-\theta)d\theta=\theta\tan\theta-\theta^2-\ln|\sec\theta|+\frac{1}{2}\theta^2+C$$
$$= \theta\tan\theta-\frac{1}{2}\theta^2-\ln|\sec\theta|+C$$

42. Integrate by parts with $u=\tan^{-1}x$, $dv=x^2dx\Rightarrow du=dx/\left(1+x^2\right)$, $v=\frac{1}{3}x^3$:

$$\int x^2\tan^{-1}x\,dx = \frac{1}{3}x^3\tan^{-1}x-\int\frac{x^3}{3}\frac{dx}{1+x^2}=\frac{1}{3}x^3\tan^{-1}x-\frac{1}{3}\int\left[x-\frac{x}{x^2+1}\right]dx$$
$$= \frac{1}{3}x^3\tan^{-1}x-\frac{1}{6}x^2+\frac{1}{6}\ln\left(x^2+1\right)+C$$

43. Let $u=1+e^x$, so that $du=e^xdx$. Then

$$\int e^x\sqrt{1+e^x}\,dx=\int u^{1/2}du=\frac{2}{3}u^{3/2}+C=\frac{2}{3}(1+e^x)^{3/2}+C\ .$$

Or: Let $u=\sqrt{1+e^x}$, so that $u^2=1+e^x$ and $2udu=e^xdx$. Then

$$\int e^x\sqrt{1+e^x}\,dx=\int u\cdot 2udu=\int 2u^2du=\frac{2}{3}u^3+C=\frac{2}{3}(1+e^x)^{3/2}+C\ .$$

44. Let $u=\sqrt{1+e^x}$. Then $u^2=1+e^x$, $2udu=e^xdx=(u^2-1)dx$, and $dx=\dfrac{2u}{u^2-1}du$, so

$$\int \sqrt{1+e^{x}}\,dx = \int u\cdot \frac{2u}{u^{2}-1}\,du = \int \frac{2u^{2}}{u^{2}-1}\,du = \int \left(2+\frac{2}{u^{2}-1}\right)du = \int \left(2+\frac{1}{u-1}-\frac{1}{u+1}\right)du$$

$$=2u+\ln |u-1|-\ln |u+1|+C=2\sqrt{1+e^{x}}+\ln \left(\sqrt{1+e^{x}}-1\right)-\ln \left(\sqrt{1+e^{x}}+1\right)+C$$

45. Let $t=x^{3}$. Then $dt=3x^{2}dx \Rightarrow I=\int x^{5}e^{-x^{3}}dx=\frac{1}{3}\int te^{-t}dt$. Now integrate by parts with $u=t$, $dv=e^{-t}dt$:

$I=-\frac{1}{3}te^{-t}+\frac{1}{3}\int e^{-t}dt=-\frac{1}{3}te^{-t}-\frac{1}{3}e^{-t}+C=-\frac{1}{3}e^{-x^{3}}\left(x^{3}+1\right)+C$.

46. Let $u=e^{x}$. Then $x=\ln u$, $dx=du/u \Rightarrow$

$$\int \frac{1+e^{x}}{1-e^{x}}\,dx = \int \frac{(1+u)du}{(1-u)u} = -\int \frac{(u+1)du}{(u-1)u} = -\int \left(\frac{2}{u-1}-\frac{1}{u}\right)du$$

$$= \ln |u|-2\ln |u-1|+C=\ln e^{x}-2\ln \left|e^{x}-1\right|+C=x-2\ln \left|e^{x}-1\right|+C$$

47.

$$\int \frac{x+a}{x^{2}+a^{2}}\,dx = \frac{1}{2}\int \frac{2x\,dx}{x^{2}+a^{2}}+a\int \frac{dx}{x^{2}+a^{2}} = \frac{1}{2}\ln \left(x^{2}+a^{2}\right)+a\cdot \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+C$$

$$=\ln \sqrt{x^{2}+a^{2}}+\tan^{-1}(x/a)+C$$

48. Let $u=x^{2}$. Then $du=2x\,dx \Rightarrow$

$$\int \frac{x\,dx}{x^{4}-a^{4}} = \int \frac{\frac{1}{2}\,du}{u^{2}-\left(a^{2}\right)^{2}} = \frac{1}{4a^{2}}\ln \left|\frac{u-a^{2}}{u+a^{2}}\right|+C=\frac{1}{4a^{2}}\ln \left|\frac{x^{2}-a^{2}}{x^{2}+a^{2}}\right|+C .$$

49. Let $u=\sqrt{4x+1} \Rightarrow u^{2}=4x+1 \Rightarrow 2u\,du=4\,dx \Rightarrow dx=\frac{1}{2}u\,du$. So

$$\int \frac{1}{x\sqrt{4x+1}}\,dx = \int \frac{\frac{1}{2}u\,du}{\frac{1}{4}\left(u^{2}-1\right)u} = 2\int \frac{du}{u^{2}-1} = 2\left(\frac{1}{2}\right)\ln \left|\frac{u-1}{u+1}\right|+C$$

$$= \ln \left|\frac{\sqrt{4x+1}-1}{\sqrt{4x+1}+1}\right|+C$$

50. As in Exercise 49, let $u=\sqrt{4x+1}$. Then $\int \dfrac{dx}{x^2\sqrt{4x+1}} = \int \dfrac{\frac{1}{2}u\,du}{\left[\frac{1}{4}\left(u^2-1\right)\right]^2 u} = 8\int \dfrac{du}{\left(u^2-1\right)^2}$. Now

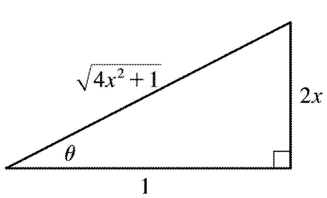
$\dfrac{1}{\left(u^2-1\right)^2} = \dfrac{1}{(u+1)^2(u-1)^2} = \dfrac{A}{u+1} + \dfrac{B}{(u+1)^2} + \dfrac{C}{u-1} + \dfrac{D}{(u-1)^2} \Rightarrow$

$1=A(u+1)(u-1)^2+B(u-1)^2+C(u-1)(u+1)^2+D(u+1)^2$. $u=1 \Rightarrow D=\dfrac{1}{4}$, $u=-1 \Rightarrow B=\dfrac{1}{4}$. Equating coefficients of u^3 gives $A+C=0$, and equating coefficients of 1 gives $1=A+B-C+D \Rightarrow 1=A+\dfrac{1}{4}-C+\dfrac{1}{4}$ $\Rightarrow \dfrac{1}{2}=A-C$. So $A=\dfrac{1}{4}$ and $C=-\dfrac{1}{4}$. Therefore,

$$\begin{aligned}
\int \frac{dx}{x^2\sqrt{4x+1}} &= 8\int\left[\frac{1/4}{u+1}+\frac{1/4}{(u+1)^2}+\frac{-1/4}{u-1}+\frac{1/4}{(u-1)^2}\right]du \\
&= \int\left[\frac{2}{u+1}+2(u+1)^{-2}-\frac{2}{u-1}+2(u-1)^{-2}\right]du \\
&= 2\ln|u+1|-\frac{2}{u+1}-2\ln|u-1|-\frac{2}{u-1}+C \\
&= 2\ln\left(\sqrt{4x+1}+1\right)-\frac{2}{\sqrt{4x+1}+1}-2\ln\left|\sqrt{4x+1}-1\right|-\frac{2}{\sqrt{4x+1}-1}+C
\end{aligned}$$

51. Let $2x=\tan\theta \Rightarrow x=\dfrac{1}{2}\tan\theta$, $dx=\dfrac{1}{2}\sec^2\theta\ d\theta$, $\sqrt{4x^2+1}=\sec\theta$, so

$$\begin{aligned}
&\int \frac{dx}{x\sqrt{4x^2+1}} = \int \frac{\frac{1}{2}\sec^2\theta\ d\theta}{\frac{1}{2}\tan\theta\ \sec\theta} = \int \frac{\sec\theta}{\tan\theta}\,d\theta = \int \csc\theta\ d\theta \\
&= -\ln|\csc\theta+\cot\theta|+C \\
&= -\ln\left|\frac{\sqrt{4x^2+1}}{2x}+\frac{1}{2x}\right|+C\left[\text{or} \ln\left|\frac{\sqrt{4x^2+1}}{2x}-\frac{1}{2x}\right|+C\right]
\end{aligned}$$



52. Let $u=x^{2}$. Then $du=2x\,dx\Rightarrow$

$$\int \frac{dx}{x\left(x^{4}+1\right)} = \int \frac{x\,dx}{x^{2}\left(x^{4}+1\right)}=\frac{1}{2}\int \frac{du}{u\left(u^{2}+1\right)}=\frac{1}{2}\int \left[\frac{1}{u}-\frac{u}{u^{2}+1}\right]du=\frac{1}{2}\ln|u|-\frac{1}{4}\ln\left(u^{2}+1\right)+C$$

$$=\frac{1}{2}\ln\left(x^{2}\right)-\frac{1}{4}\ln\left(x^{4}+1\right)+C=\frac{1}{4}\left[\ln\left(x^{4}\right)-\ln\left(x^{4}+1\right)\right]+C=\frac{1}{4}\ln\left(\frac{x^{4}}{x^{4}+1}\right)+C$$

Or: Write $I=\int \frac{x^{3}\,dx}{x^{4}\left(x^{4}+1\right)}$ and let $u=x^{4}$.

53. $\int x^{2}\sinh(mx)dx=\frac{1}{m}x^{2}\cosh(mx)-\frac{2}{m}\int x\cosh(mx)dx$ $\left[\begin{array}{ll} \mathrm{u=x^{2}}, & \mathrm{dv=\sinh(mx)\,dx} \\ \mathrm{du=2x\,dx} & \mathrm{v=\frac{1}{m}\cosh(mx)} \end{array}\right]$

$=\frac{1}{m}x^{2}\cosh(mx)-\frac{2}{m}\left(\frac{1}{m}x\sinh(mx)-\frac{1}{m}\int\sinh(mx)dx\right)$ $\left[\begin{array}{ll} U=x, & dV=\cosh(mx)dx \\ dU=dx & V=\frac{1}{m}\sinh(mx) \end{array}\right]$

$=\frac{1}{m}x^{2}\cosh(mx)-\frac{2}{m^{2}}x\sinh(mx)+\frac{2}{m^{3}}\cosh(mx)+C$

54.

$$\int(x+\sin x)^{2}dx=\int\left(x^{2}+2x\sin x+\sin^{2}x\right)dx=\frac{1}{3}x^{3}+2(\sin x-x\cos x)+\frac{1}{2}(x-\sin x\cos x)+C$$

$$=\frac{1}{3}x^{3}+\frac{1}{2}x+2\sin x-\frac{1}{2}\sin x\cos x-2x\cos x+C$$

55. Let $u=\sqrt{x+1}$. Then $x=u^{2}-1\Rightarrow$

$$\int \frac{dx}{x+4+4\sqrt{x+1}} = \int \frac{2u\,du}{u^{2}+3+4u}=\int\left[\frac{-1}{u+1}+\frac{3}{u+3}\right]du$$

$$=3\ln|u+3|-\ln|u+1|+C=3\ln\left(\sqrt{x+1}+3\right)-\ln\left(\sqrt{x+1}+1\right)+C$$

56. Let $t=\sqrt{x^{2}-1}$. Then $dt=\left(x\Big/\sqrt{x^{2}-1}\right)dx$, $x^{2}-1=t^{2}$, $x=\sqrt{t^{2}+1}$, so $I=\int \frac{x\ln x}{\sqrt{x^{2}-1}}dx=\int\ln\sqrt{t^{2}+1}\,dt=\frac{1}{2}\int\ln\left(t^{2}+1\right)dt$. Now use parts with $u=\ln\left(t^{2}+1\right)$, $dv=dt$:

$$I = \frac{1}{2} t\ln\left(t^2+1\right) - \int \frac{t^2}{t^2+1}\,dt = \frac{1}{2} t\ln\left(t^2+1\right) - \int \left[1 - \frac{1}{t^2+1}\right] dt$$
$$= \frac{1}{2} t\ln\left(t^2+1\right) - t + \tan^{-1} t + C = \sqrt{x^2-1}\,\ln x - \sqrt{x^2-1} + \tan^{-1}\sqrt{x^2-1} + C$$

Another method: First integrate by parts with $u=\ln x$, $dv=\left(x\Big/\sqrt{x^2-1}\right)dx$ and then use substitution $\left(x=\sec\theta \text{ or } u=\sqrt{x^2-1}\right)$.

57. Let $u=\sqrt[3]{x+c}$. Then $x=u^3-c\Rightarrow$

$$\int x\sqrt[3]{x+c}\,dx = \int \left(u^3-c\right) u\cdot 3u^2\,du = 3\int\left(u^6-cu^3\right)du = \frac{3}{7}u^7 - \frac{3}{4}cu^4 + C$$
$$= \frac{3}{7}(x+c)^{7/3} - \frac{3}{4}c(x+c)^{4/3} + C$$

58. Integrate by parts with $u=\ln(1+x)$, $dv=x^2dx\Rightarrow du=dx/(1+x)$, $v=\frac{1}{3}x^3$:

$$\int x^2\ln(1+x)dx = \frac{1}{3}x^3\ln(1+x) - \int\frac{x^3dx}{3(1+x)} = \frac{1}{3}x^3\ln(1+x) - \frac{1}{3}\int\left(x^2-x+1-\frac{1}{x+1}\right)dx$$
$$= \frac{1}{3}x^3\ln(1+x) - \frac{1}{9}x^3 + \frac{1}{6}x^2 - \frac{1}{3}x + \frac{1}{3}\ln(1+x) + C$$

59. Let $u=e^x$. Then $x=\ln u$, $dx=du/u\Rightarrow$

$$\int\frac{dx}{e^{3x}-e^x} = \int\frac{du/u}{u^3-u} = \int\frac{du}{(u-1)u^2(u+1)} = \int\left[\frac{1/2}{u-1} - \frac{1}{u^2} - \frac{1/2}{u+1}\right]du$$
$$= \frac{1}{u} + \frac{1}{2}\ln\left|\frac{u-1}{u+1}\right| + C = e^{-x} + \frac{1}{2}\ln\left|\frac{e^x-1}{e^x+1}\right| + C$$

60. Let $u=\sqrt[3]{x}$. Then $x=u^3$, $dx=3u^2du\Rightarrow$

$$\int\frac{dx}{x+\sqrt[3]{x}} = \int\frac{3u^2du}{u^3+u} = \frac{3}{2}\int\frac{2u\,du}{u^2+1} = \frac{3}{2}\ln\left(u^2+1\right) + C = \frac{3}{2}\ln\left(x^{2/3}+1\right) + C\ .$$

61. Let $u=x^5$. Then $du=5x^4dx\Rightarrow$

$$\int \frac{x^4\,dx}{x^{10}+16} = \int \frac{\frac{1}{5}\,du}{u^2+16} = \frac{1}{5}\cdot\frac{1}{4}\tan^{-1}\left(\frac{1}{4}u\right)+C = \frac{1}{20}\tan^{-1}\left(\frac{1}{4}x^5\right)+C\ .$$

62. Let $u=x+1$. Then $du=dx\Rightarrow$

$$\int \frac{x^3}{(x+1)^{10}}\,dx = \int \frac{(u-1)^3}{u^{10}}\,du = \int \left(u^{-7}-3u^{-8}+3u^{-9}-u^{-10}\right)du$$

$$= -\frac{1}{6}u^{-6}+\frac{3}{7}u^{-7}-\frac{3}{8}u^{-8}+\frac{1}{9}u^{-9}+C$$

$$= (x+1)^{-9}\left[-\frac{1}{6}(x+1)^3+\frac{3}{7}(x+1)^2-\frac{3}{8}(x+1)+\frac{1}{9}\right]+C$$

63. Let $y=\sqrt{x}$ so that $dy=\frac{1}{2\sqrt{x}}\,dx\Rightarrow dx=2\sqrt{x}\,dy=2y\,dy$. Then

$$\int \sqrt{x}\,e^{\sqrt{x}}\,dx = \int ye^y(2ydy) = \int 2y^2e^y\,dy \left[\begin{array}{ll} u=2y^2, & dv=e^y\,dy, \\ du=4ydy & v=e^y \end{array}\right]$$

$$=2y^2e^y-\int 4ye^y\,dy \left[\begin{array}{ll} U=4y, & dV=e^y\,dy, \\ dU=4\,dy & V=e^y \end{array}\right]$$

$$=2y^2e^y-\left(4ye^y-\int 4e^y\,dy\right)=2y^2e^y-4ye^y+4e^y+C$$

$$=2(y^2-2y+2)e^y+C=2\left(x-2\sqrt{x}+2\right)e^{\sqrt{x}}+C$$

64. Let $u=\tan x$. Then

$$\int_{\pi/4}^{\pi/3} \frac{\ln(\tan x)}{dx}\sin x\cos x = \int_{\pi/4}^{\pi/3} \frac{\ln(\tan x)}{\tan x}\sec^2 x\,dx = \int_1^{\sqrt{3}} \frac{\ln u}{u}\,du$$

$$= \left[\frac{1}{2}(\ln u)^2\right]_1^{\sqrt{3}} = \frac{1}{2}\left(\ln\sqrt{3}\right)^2 = \frac{1}{8}(\ln 3)^2$$

65.

$$\int \frac{dx}{\sqrt{x+1}+\sqrt{x}} = \int \left(\frac{1}{\sqrt{x+1}+\sqrt{x}}\cdot\frac{\sqrt{x+1}-\sqrt{x}}{\sqrt{x+1}-\sqrt{x}}\right)dx = \int \left(\sqrt{x+1}-\sqrt{x}\right)dx$$

$$=\frac{2}{3}\left[(x+1)^{3/2}-x^{3/2}\right]+C$$

66. $\int \frac{u^3+1}{u^3-u^2}\,du=\int \left[1+\frac{u^2+1}{(u-1)u^2}\right]du=u+\int \left[\frac{2}{u-1}-\frac{1}{u}-\frac{1}{u^2}\right]du=u+2\ln|u-1|-\ln|u|+\frac{1}{u}+C$. Thus,

$$\int_2^3 \frac{u^3+1}{u^3-u^2}\,du = \left[u+2\ln(u-1)-\ln u+\frac{1}{u}\right]_2^3=\left(3+2\ln 2-\ln 3+\frac{1}{3}\right)-\left(2+2\ln 1-\ln 2+\frac{1}{2}\right)$$

$$= 1+3\ln 2-\ln 3-\frac{1}{6}=\frac{5}{6}+\ln\frac{8}{3}$$

67. Let $u=\sqrt{t}$. Then $du=dt/\left(2\sqrt{t}\right)\Rightarrow$

$$\int_1^3 \frac{\arctan\sqrt{t}}{\sqrt{t}}\,dt=\int_1^{\sqrt{3}} \tan^{-1}u(2\,du)=2\left[u\tan^{-1}u-\frac{1}{2}\ln(1+u^2)\right]_1^{\sqrt{3}}$$ [Example 5 in Section 8.1]

$$=2\left[\left(\sqrt{3}\tan^{-1}\sqrt{3}-\frac{1}{2}\ln 4\right)-\left(\tan^{-1}1-\frac{1}{2}\ln 2\right)\right]$$

$$=2\left[\left(\sqrt{3}\cdot\frac{\pi}{3}-\ln 2\right)-\left(\frac{\pi}{4}-\frac{1}{2}\ln 2\right)\right]=\frac{2}{3}\sqrt{3}\,\pi-\frac{1}{2}\pi-\ln 2$$

68. Let $u=e^x$. Then $x=\ln u$, $dx=du/u\Rightarrow$

$$\int \frac{dx}{1+2e^x-e^{-x}} = \int \frac{du/u}{1+2u-1/u}=\int \frac{du}{2u^2+u-1}=\int \left[\frac{2/3}{2u-1}-\frac{1/3}{u+1}\right]du$$

$$= \frac{1}{3}\ln|2u-1|-\frac{1}{3}\ln|u+1|+C=\frac{1}{3}\ln\left|\left(2e^x-1\right)/\left(e^x+1\right)\right|+C$$

69. Let $u=e^x$. Then $x=\ln u$, $dx=du/u\Rightarrow$

$$\int \frac{e^{2x}}{1+e^x}\,dx = \int \frac{u^2}{1+u}\,\frac{du}{u}=\int \frac{u}{1+u}\,du=\int \left(1-\frac{1}{1+u}\right)du$$

$$= u-\ln|1+u|+C=e^x-\ln\left(1+e^x\right)+C$$

70. Use parts with $u=\ln(x+1)$, $dv=dx/x^2$:

$$\int \frac{\ln(x+1)}{x^2}\,dx = -\frac{1}{x}\ln(x+1)+\int \frac{dx}{x(x+1)}=-\frac{1}{x}\ln(x+1)+\int \left[\frac{1}{x}-\frac{1}{x+1}\right]dx$$

$$= -\frac{1}{x}\ln(x+1)+\ln|x|-\ln(x+1)+C=-\left(1+\frac{1}{x}\right)\ln(x+1)+\ln|x|+C$$

71. $\dfrac{x}{x^4+4x^2+3}=\dfrac{x}{\left(x^2+3\right)\left(x^2+1\right)}=\dfrac{Ax+B}{x^2+3}+\dfrac{Cx+D}{x^2+1}\Rightarrow$

$x=(Ax+B)\left(x^2+1\right)+(Cx+D)\left(x^2+3\right)=\left(Ax^3+Bx^2+Ax+B\right)+\left(Cx^3+Dx^2+3Cx+3D\right)$

$=(A+C)x^3+(B+D)x^2+(A+3C)x+(B+3D)\Rightarrow$

$A+C=0$, $B+D=0$, $A+3C=1$, $B+3D=0\Rightarrow A=-\dfrac{1}{2}$, $C=\dfrac{1}{2}$, $B=0$, $D=0$.Thus,

$$\int\frac{x}{x^4+4x^2+3}\,dx=\int\left(\frac{-\frac{1}{2}x}{x^2+3}+\frac{\frac{1}{2}x}{x^2+1}\right)dx$$

$$=-\frac{1}{4}\ln\left(x^2+3\right)+\frac{1}{4}\ln\left(x^2+1\right)+C\quad\text{or}\quad\frac{1}{4}\ln\left(\frac{x^2+1}{x^2+3}\right)+C$$

72. Let $u=\sqrt[6]{t}$. Then $t=u^6$, $dt=6u^5\,du\Rightarrow$

$$\int\frac{\sqrt{t}\,dt}{1+\sqrt[3]{t}}=\int\frac{u^3\cdot 6u^5\,du}{1+u^2}=6\int\frac{u^8}{u^2+1}\,du=6\int\left(u^6-u^4+u^2-1+\frac{1}{u^2+1}\right)du$$

$$=6\left(\frac{1}{7}u^7-\frac{1}{5}u^5+\frac{1}{3}u^3-u+\tan^{-1}u\right)+C$$

$$=6\left(\frac{1}{7}t^{7/6}-\frac{1}{5}t^{5/6}+\frac{1}{3}t^{1/2}-t^{1/6}+\tan^{-1}t^{1/6}\right)+C$$

73. $\dfrac{1}{(x-2)\left(x^2+4\right)}=\dfrac{A}{x-2}+\dfrac{Bx+C}{x^2+4}\Rightarrow 1=A\left(x^2+4\right)+(Bx+C)(x-2)=(A+B)x^2+(C-2B)x+(4A-2C)$. So

$0=A+B=C-2B$, $1=4A-2C$. Setting $x=2$ gives $A=\dfrac{1}{8}\Rightarrow B=-\dfrac{1}{8}$ and $C=-\dfrac{1}{4}$. So

$$\int\frac{1}{(x-2)\left(x^2+4\right)}\,dx=\int\left(\frac{\frac{1}{8}}{x-2}+\frac{-\frac{1}{8}x-\frac{1}{4}}{x^2+4}\right)dx=\frac{1}{8}\int\frac{dx}{x-2}-\frac{1}{16}\int\frac{2x\,dx}{x^2+4}-\frac{1}{4}\int\frac{dx}{x^2+4}$$

$$=\frac{1}{8}\ln|x-2|-\frac{1}{16}\ln\left(x^2+4\right)-\frac{1}{8}\tan^{-1}(x/2)+C$$

74. Let $u=e^x$. Then $x=\ln u$, $dx=du/u\Rightarrow$

$$\int\frac{dx}{e^x-e^{-x}}=\int\frac{e^x\,dx}{e^{2x}-1}=\int\frac{u}{u^2-1}\,\frac{du}{u}=\int\frac{du}{u^2-1}=\frac{1}{2}\ln\left|\frac{u-1}{u+1}\right|+C=\frac{1}{2}\ln\left(\frac{\left|e^x-1\right|}{e^x+1}\right)+C\ .$$

75.

$$\int \sin x \sin 2x \sin 3x\,dx = \int \sin x \cdot \frac{1}{2}\left[\cos(2x-3x)-\cos(2x+3x)\right]dx = \frac{1}{2}\int(\sin x\cos x - \sin x\cos 5x)\,dx$$

$$= \frac{1}{4}\int \sin 2x\,dx - \frac{1}{2}\int \frac{1}{2}\left[\sin(x+5x)+\sin(x-5x)\right]dx$$

$$= -\frac{1}{8}\cos 2x - \frac{1}{4}\int(\sin 6x - \sin 4x)\,dx = -\frac{1}{8}\cos 2x + \frac{1}{24}\cos 6x - \frac{1}{16}\cos 4x + C$$

76.

$$\int\left(x^2-bx\right)\sin 2x\,dx = -\frac{1}{2}\left(x^2-bx\right)\cos 2x + \frac{1}{2}\int(2x-b)\cos 2x\,dx$$

$$= -\frac{1}{2}\left(x^2-bx\right)\cos 2x + \frac{1}{2}\left[\frac{1}{2}(2x-b)\sin 2x - \int \sin 2x\,dx\right]$$

$$= -\frac{1}{2}\left(x^2-bx\right)\cos 2x + \frac{1}{4}(2x-b)\sin 2x + \frac{1}{4}\cos 2x + C$$

77. Let $u=x^{3/2}$ so that $u^2=x^3$ and $du=\frac{3}{2}x^{1/2}dx \Rightarrow \sqrt{x}\,dx=\frac{2}{3}\,du$. Then

$$\int \frac{\sqrt{x}}{1+x^3}\,dx = \int \frac{\frac{2}{3}}{1+u^2}\,du = \frac{2}{3}\tan^{-1}u + C = \frac{2}{3}\tan^{-1}(x^{3/2}) + C\ .$$

78.

$$\int \frac{\sec x\cos 2x}{\sin x+\sec x}\,dx = \int \frac{\sec x\cos 2x}{\sin x+\sec x}\cdot\frac{2\cos x}{2\cos x}\,dx = \int \frac{2\cos 2x}{2\sin x\cos x+2}\,dx$$

$$= \int \frac{2\cos 2x}{\sin 2x+2}\,dx = \int \frac{1}{u}\,du \left[\begin{array}{c} u=\sin 2x+2, \\ du=2\cos 2x dx \end{array}\right]$$

$$= \ln|u| + C = \ln|\sin 2x+2| + C = \ln(\sin 2x+2) + C$$

79. Let $u=x$, $dv=\sin^2 x\cos x\,dx \Rightarrow du=dx$, $v=\frac{1}{3}\sin^3 x$. Then

$$\int x\sin^2 x\cos x\,dx = \frac{1}{3}x\sin^3 x - \int \frac{1}{3}\sin^3 x\,dx = \frac{1}{3}x\sin^3 x - \frac{1}{3}\int(1-\cos^2 x)\sin x\,dx$$

$$= \frac{1}{3}x\sin^3 x + \frac{1}{3}\int(1-y^2)\,dy \left[\begin{array}{c} y=\cos x, \\ dy=-\sin x dx \end{array}\right]$$

$$=\frac{1}{3}x\sin^3 x+\frac{1}{3}y-\frac{1}{9}y^3+C=\frac{1}{3}x\sin^3 x+\frac{1}{3}\cos x-\frac{1}{9}\cos^3 x+C$$

80.

$$\int\frac{\sin x\cos x}{\sin^4 x+\cos^4 x}\,dx=\int\frac{\sin x\cos x}{(\sin^2 x)^2+(\cos^2 x)^2}\,dx=\int\frac{\sin x\cos x}{(\sin^2 x)^2+(1-\sin^2 x)^2}\,dx$$

$$=\int\frac{1}{u^2+(1-u)^2}\left(\frac{1}{2}\,du\right)\left[\begin{array}{l}\mathrm{u=\sin^2 x,}\\ \mathrm{du=2\sin x\cos x\,dx}\end{array}\right]$$

$$=\int\frac{1}{4u^2-4u+2}\,du=\int\frac{1}{(4u^2-4u+1)+1}\,du$$

$$=\int\frac{1}{(2u-1)^2+1}\,du=\frac{1}{2}\int\frac{1}{y^2+1}\,dy\left[\begin{array}{l}\mathrm{y=2u-1,}\\ \mathrm{dy=2\,du}\end{array}\right]$$

$$=\frac{1}{2}\tan^{-1}y+C=\frac{1}{2}\tan^{-1}(2u-1)+C=\frac{1}{2}\tan^{-1}(2\sin^2 x-1)+C$$

Another solution:

$$\int\frac{\sin x\cos x}{\sin^4 x+\cos^4 x}\,\mathrm{dx}=\int\frac{(\sin x\cos x)/\cos^4 x}{(\sin^4 x+\cos^4 x)/\cos^4 x}\,\mathrm{dx}=\int\frac{\tan x\sec^2 x}{\tan^4 x+1}\,\mathrm{dx}$$

$$=\int\frac{1}{u^2+1}\left(\frac{1}{2}\,du\right)\left[\begin{array}{l}\mathrm{u=\tan^2 x,}\\ \mathrm{du=2\tan x\sec^2 x\,dx}\end{array}\right]$$

$$=\frac{1}{2}\tan^{-1}\mathrm{u}+\mathrm{C}=\frac{1}{2}\tan^{-1}(\tan^2\mathrm{x})+\mathrm{C}$$

81. The function $y=2xe^{x^2}$ *does* have an elementary antiderivative, so we'll use this fact to help evaluate the integral.

$$\int(2x^2+1)e^{x^2}\,dx=\int 2x^2e^{x^2}\,dx+\int e^{x^2}\,dx=\int x(2xe^{x^2})\,dx+\int e^{x^2}\,dx$$

$$=xe^{x^2}-\int e^{x^2}\,dx+\int e^{x^2}\,dx\left[\begin{array}{ll}\mathrm{u=x,} & \mathrm{dv=2xe^{x^2}\,dx,}\\ \mathrm{du=dx} & \mathrm{v=e^{x^2}}\end{array}\right]=xe^{x^2}+C$$

1. We could make the substitution $u=\sqrt{2}\,x$ to obtain the radical $\sqrt{7-u^2}$ and then use Formula 33 with $a=\sqrt{7}$. Alternatively, we will factor $\sqrt{2}$ out of the radical and use $a=\sqrt{\frac{7}{2}}$.

$$\int \frac{\sqrt{7-2x^2}}{x^2}\,dx = \sqrt{2}\int \frac{\sqrt{\frac{7}{2}-x^2}}{x^2}\,dx=\sqrt{2}\left[-\frac{1}{x}\sqrt{\frac{7}{2}-x^2}-\sin^{-1}\frac{x}{\sqrt{\frac{7}{2}}}\right]+C$$

$$= -\frac{1}{x}\sqrt{7-2x^2}-\sqrt{2}\sin^{-1}\left(\sqrt{\frac{2}{7}}\,x\right)+C$$

2.

$$\int \frac{3x}{\sqrt{3-2x}}\,dx =3\int \frac{x}{\sqrt{3+(-2)x}}\,dx=3\left[\frac{2}{3(-2)^2}(-2x-2\cdot 3)\sqrt{3+(-2)x}\right]+C$$

$$=\frac{1}{2}(-2x-6)\sqrt{3-2x}+C=-(x+3)\sqrt{3-2x}+C$$

3. Let $u=\pi x\Rightarrow du=\pi\,dx$, so

$$\int \sec^3(\pi x)dx = \frac{1}{\pi}\int \sec^3 u\,du=\frac{1}{\pi}\left(\frac{1}{2}\sec u\tan u+\frac{1}{2}\ln|\sec u+\tan u|\right)+C$$

$$= \frac{1}{2\pi}\sec \pi x\tan \pi x+\frac{1}{2\pi}\ln|\sec \pi x+\tan \pi x|+C$$

4. $\int e^{2\theta}\sin 3\theta\,d\theta = \dfrac{e^{2\theta}}{2^2+3^2}(2\sin 3\theta-3\cos 3\theta)+C=\dfrac{2}{13}e^{2\theta}\sin 3\theta-\dfrac{3}{13}e^{2\theta}\cos 3\theta+C$

5. $\int_0^1 2x\cos^{-1}x\,dx=2\left[\dfrac{2x^2-1}{4}\cos^{-1}x-\dfrac{x\sqrt{1-x^2}}{4}\right]_0^1=2\left[\left(\dfrac{1}{4}\cdot 0-0\right)-\left(-\dfrac{1}{4}\cdot\dfrac{\pi}{2}-0\right)\right]=2\left(\dfrac{\pi}{8}\right)=\dfrac{\pi}{4}$

6.

$$\int_2^3 \frac{1}{x^2\sqrt{4x^2-7}}\,dx = \int_4^6 \frac{1}{\left(\frac{1}{2}u^2\sqrt{u^2-7}\right)}\left(\frac{1}{2}\,du\right) [u=2x,\ du=2\,dx]$$

$$=2\int_4^6 \frac{}{du}\,u^2\sqrt{u^2-7} = 2\left[\frac{\sqrt{u^2-7}}{7u}\right]_4^6$$

$$=2\left(\frac{\sqrt{29}}{42}-\frac{3}{28}\right)=\frac{\sqrt{29}}{21}-\frac{3}{14}$$

7. By Formula 99 with $a=-3$ and $b=4$,

$$\int e^{-3x}\cos 4x\,dx = \frac{e^{-3x}}{(-3)^2+4^2}\,(-3\cos 4x+4\sin 4x)+C=\frac{e^{-3x}}{25}\,(-3\cos 4x+4\sin 4x)+C\ .$$

8. Let $u=x/2$, so $dx=2\,du$, and we use Formula 72:

$$\int \csc^3(x/2)\,dx = 2\int \csc^3 u\ du = -\csc u\cot u + \ln|\csc u - \cot u| + C$$
$$= -\csc(x/2)\cot(x/2)+\ln|\csc(x/2)-\cot(x/2)|+C$$

9. Let $u=2x$ and $a=3$. Then $du=2\,dx$ and

$$\int \frac{dx}{x^2\sqrt{4x^2+9}} = \int \frac{\frac{1}{2}\,du}{\frac{u^2}{4}\sqrt{u^2+a^2}} = 2\int \frac{du}{u^2\sqrt{a^2+u^2}} = -2\,\frac{\sqrt{a^2+u^2}}{a^2u}+C$$

$$=-2\,\frac{\sqrt{4x^2+9}}{9\cdot 2x}+C=-\frac{\sqrt{4x^2+9}}{9x}+C$$

10. Let $u=\sqrt{2}\,y$ and $a=\sqrt{3}$. Then $du=\sqrt{2}\,dy$ and

$$\int \frac{\sqrt{2y^2-3}}{y^2}\,dy = \int \frac{\sqrt{u^2-a^2}}{\frac{1}{2}u^2}\,\frac{du}{\sqrt{2}} = \sqrt{2}\int \frac{\sqrt{u^2-a^2}}{u^2}\,du$$

$$=\sqrt{2}\left(-\frac{\sqrt{u^2-a^2}}{u}+\ln\left|u+\sqrt{u^2-a^2}\right|\right)+C$$

$$=\sqrt{2}\left(-\frac{\sqrt{2y^2-3}}{\sqrt{2}\,y}+\ln\left|\sqrt{2}\,y+\sqrt{2y^2-3}\right|\right)+C$$

$$=-\frac{\sqrt{2y^2-3}}{y}+\sqrt{2}\ln\left|\sqrt{2}\,y+\sqrt{2y^2-3}\right|+C$$

11.

$$\int_{-1}^{0} t^2e^{-t}\,dt = \left[\frac{1}{-1}t^2e^{-t}\right]_{-1}^{0} - \frac{2}{-1}\int_{-1}^{0} te^{-t}\,dt = e+2\int_{-1}^{0} te^{-t}\,dt = e+2\left[\frac{1}{(-1)^2}(-t-1)e^{-t}\right]_{-1}^{0}$$

$$=e+2\left[-e^0+0\right]=e-2$$

12. Let $u=3x$. Then $du=3\,dx$, so

$$\int x^2\cos 3x\,dx = \frac{1}{27}\int u^2\cos u\,du = \frac{1}{27}\left(u^2\sin u - 2\int u\sin u\,du\right)$$

$$=\frac{1}{3}x^2\sin 3x - \frac{2}{27}(\sin 3x - 3x\cos 3x)+C$$

$$=\frac{1}{27}\left[\left(9x^2-2\right)\sin 3x + 6x\cos 3x\right]+C$$

Thus, $\int_0^{\pi} x^2\cos 3x\,dx = \frac{1}{27}\left[\left(9x^2-2\right)\sin 3x+6x\cos 3x\right]_0^{\pi} = \frac{1}{27}\left[(0+6\pi(-1))-(0+0)\right] = -\frac{6\pi}{27} = -\frac{2\pi}{9}$.

13.

$$\int \frac{\tan^3(1/z)}{z^2}\,dz \left[\begin{array}{l} u=1/z, \\ du=-dz/z^2 \end{array}\right] = -\int \tan^3 u\,du = -\frac{1}{2}\tan^2 u - \ln|\cos u| + C$$

$$=-\frac{1}{2}\tan^2\left(\frac{1}{z}\right) - \ln\left|\cos\left(\frac{1}{z}\right)\right| + C$$

14. Let $u=\sqrt{x}$. Then $u^2=x$ and $2u\,du=dx$, so

$$\int \sin^{-1}\sqrt{x}\,dx = 2\int u\sin^{-1}u\,du = \frac{2u^2-1}{2}\sin^{-1}u + \frac{u\sqrt{1-u^2}}{2} + C$$
$$= \frac{2x-1}{2}\sin^{-1}\sqrt{x} + \frac{\sqrt{x(1-x)}}{2} + C$$

15. Let $u = e^x$. Then $du = e^x dx$, so
$$\int e^x \operatorname{sech}\left(e^x\right)dx = \int \operatorname{sech} u\, du \overset{107}{=} \tan^{-1}|\sinh u| + C = \tan^{-1}\left[\sinh\left(e^x\right)\right] + C$$

16. Let $u = x^2$, so that $du = 2x\,dx$. Then
$$\int x\sin\left(x^2\right)\cos\left(3x^2\right)dx = \frac{1}{2}\int \sin u\cos 3u\,du = -\frac{1}{2}\frac{\cos(1-3)u}{2(1-3)} - \frac{1}{2}\frac{\cos(1+3)u}{2(1+3)} + C$$
$$= \frac{1}{8}\cos 2u - \frac{1}{16}\cos 4u + C = \frac{1}{8}\cos(2x^2) - \frac{1}{16}\cos(4x^2) + C$$

17. Let $z = 6+4y-4y^2 = 6-(4y^2-4y+1)+1 = 7-(2y-1)^2$, $u = 2y-1$, and $a = \sqrt{7}$. Then $z = a^2-u^2$, $du = 2dy$, and

$$\int y\sqrt{6+4y-4y^2}\,dy = \int y\sqrt{z}\,dy = \int \frac{1}{2}(u+1)\sqrt{a^2-u^2}\,\frac{1}{2}\,du$$
$$= \frac{1}{4}\int u\sqrt{a^2-u^2}\,du + \frac{1}{4}\int\sqrt{a^2-u^2}\,du$$
$$= \frac{1}{4}\int\sqrt{a^2-u^2}\,du - \frac{1}{8}\int(-2u)\sqrt{a^2-u^2}\,du$$
$$= \frac{u}{8}\sqrt{a^2-u^2} + \frac{a^2}{8}\sin^{-1}\left(\frac{u}{a}\right) - \frac{1}{8}\int\sqrt{w}\,dw\quad\left[\begin{array}{l} w=a^2-u^2, \\ dw=-2u\,du \end{array}\right]$$
$$= \frac{2y-1}{8}\sqrt{6+4y-4y^2} + \frac{7}{8}\sin^{-1}\frac{2y-1}{\sqrt{7}} - \frac{1}{8}\cdot\frac{2}{3}w^{3/2} + C$$
$$= \frac{2y-1}{8}\sqrt{6+4y-4y^2} + \frac{7}{8}\sin^{-1}\frac{2y-1}{\sqrt{7}} - \frac{1}{12}(6+4y-4y^2)^{3/2} + C.$$

This can be rewritten as

$$\sqrt{6+4y-4y^2}\left[\frac{1}{8}(2y-1)-\frac{1}{12}(6+4y-4y^2)\right]+\frac{7}{8}\sin^{-1}\frac{2y-1}{\sqrt{7}}+C$$
$$=\left(\frac{1}{3}y^2-\frac{1}{12}y-\frac{5}{8}\right)\sqrt{6+4y-4y^2}+\frac{7}{8}\sin^{-1}\left(\frac{2y-1}{\sqrt{7}}\right)+C$$
$$=\frac{1}{24}(8y^2-2y-15)\sqrt{6+4y-4y^2}+\frac{7}{8}\sin^{-1}\left(\frac{2y-1}{\sqrt{7}}\right)+C$$

18. Let $u=x^2$. Then $du=2x\,dx$, so by Formula 48,

$$\int\frac{x^5\,dx}{x^2+\sqrt{2}}=\frac{1}{2}\int\frac{u^2}{u+\sqrt{2}}\,du=\frac{1}{2}\cdot\frac{1}{2}\left[\left(u+\sqrt{2}\right)^2-4\sqrt{2}\left(u+\sqrt{2}\right)+4\ln\left|u+\sqrt{2}\right|\right]+C$$
$$=\frac{1}{4}\left[\left(x^2+\sqrt{2}\right)^2-4\sqrt{2}\left(x^2+\sqrt{2}\right)+4\ln\left(x^2+\sqrt{2}\right)\right]+C$$
$$=\frac{1}{4}x^4-\frac{1}{\sqrt{2}}x^2+\ln\left(x^2+\sqrt{2}\right)+K$$

Or: Let $u=x^2+\sqrt{2}$.

19. Let $u=\sin x$. Then $du=\cos x\,dx$, so

$$\int\sin^2 x\cos x\ln(\sin x)\,dx=\int u^2\ln u\,du=\frac{u^{2+1}}{(2+1)^2}\left[(2+1)\ln u-1\right]+C=\frac{1}{9}u^3(3\ln u-1)+C$$
$$=\frac{1}{9}\sin^3 x\left[3\ln(\sin x)-1\right]+C$$

20. Let $u=e^x$. Then $x=\ln u$, $dx=du/u$, so

$$\int\frac{dx}{e^x\left(1+2e^x\right)}=\int\frac{du/u}{u(1+2u)}=\int\frac{du}{u^2(1+2u)}=-\frac{1}{u}+2\ln\left|\frac{1+2u}{u}\right|+C$$
$$=-e^{-x}+2\ln\left(e^{-x}+2\right)+C$$

21. Let $u=e^x$ and $a=\sqrt{3}$. Then $du=e^x\,dx$ and

$$\int\frac{e^x}{3-e^{2x}}\,dx=\int\frac{du}{a^2-u^2}=\frac{1}{2a}\ln\left|\frac{u+a}{u-a}\right|+C=\frac{1}{2\sqrt{3}}\ln\left|\frac{e^x+\sqrt{3}}{e^x-\sqrt{3}}\right|+C.$$

22. Let

$u=x^2$ and $a=2$. Then $du=2x\,dx$ and

$$\begin{aligned}\int_0^2 x^3\sqrt{4x^2-x^4}\,dx &= \frac{1}{2}\int_0^2 x^2\sqrt{2\cdot 2\cdot x^2-(x^2)^2}\cdot 2x\,dx=\frac{1}{2}\int_0^4 u\sqrt{2au-u^2}\,du\\ &=\left[\frac{2u^2-au-3a^2}{12}\sqrt{2au-u^2}+\frac{a^3}{4}\cos^{-1}\left(\frac{a-u}{a}\right)\right]_0^4\\ &=\left[\frac{2u^2-2u-12}{12}\sqrt{4u-u^2}+\frac{8}{4}\cos^{-1}\left(\frac{2-u}{2}\right)\right]_0^4\\ &=\left[\frac{u^2-u-6}{6}\sqrt{4u-u^2}+2\cos^{-1}\left(\frac{2-u}{2}\right)\right]_0^4\\ &=-(0+2\cos^{-1}1)=2\cdot\pi-2\cdot 0=2\pi\end{aligned}$$

23.

$$\begin{aligned}\int\sec^5 x\,dx &= \frac{1}{4}\tan x\sec^3 x+\frac{3}{4}\int\sec^3 x\,dx=\frac{1}{4}\tan x\sec^3 x+\frac{3}{4}\left(\frac{1}{2}\tan x\sec x+\frac{1}{2}\int\sec x\,dx\right)\\ &=\frac{1}{4}\tan x\sec^3 x+\frac{3}{8}\tan x\sec x+\frac{3}{8}\ln\left|\sec x+\tan x\right|+C\end{aligned}$$

24. Let $u=2x$. Then $du=2\,dx$, so

$$\begin{aligned}\int\sin^6 2x\,dx &= \frac{1}{2}\int\sin^6 u\,du=\frac{1}{2}\left(-\frac{1}{6}\sin^5 u\cos u+\frac{5}{6}\int\sin^4 u\,du\right)\\ &=-\frac{1}{12}\sin^5 u\cos u+\frac{5}{12}\left(-\frac{1}{4}\sin^3 u\cos u+\frac{3}{4}\int\sin^2 u\,du\right)\\ &=-\frac{1}{12}\sin^5 u\cos u-\frac{5}{48}\sin^3 u\cos u+\frac{5}{16}\left(\frac{1}{2}u-\frac{1}{4}\sin 2u\right)+C\\ &=-\frac{1}{12}\sin^5 2x\cos 2x-\frac{5}{48}\sin^3 2x\cos 2x-\frac{5}{64}\sin 4x+\frac{5}{16}x+C\end{aligned}$$

25. Let $u=\ln x$ and $a=2$. Then $du=\dfrac{dx}{x}$ and

$$\begin{aligned}\int\frac{\sqrt{4+(\ln x)^2}}{x}\,dx &=\int\sqrt{a^2+u^2}\,du=\frac{u}{2}\sqrt{a^2+u^2}+\frac{a^2}{2}\ln\left(u+\sqrt{a^2+u^2}\right)+C\\ &=\frac{1}{2}(\ln x)\sqrt{4+(\ln x)^2}+2\ln\left[\ln x+\sqrt{4+(\ln x)^2}\right]+C\end{aligned}$$

26.

$$\begin{aligned}\int x^4 e^{-x} dx &= -x^4 e^{-x} + 4\int x^3 e^{-x} dx = -x^4 e^{-x} + 4\left(-x^3 e^{-x} + 3\int x^2 e^{-x} dx\right) \\ &= -\left(x^4 + 4x^3\right) e^{-x} + 12\left(-x^2 e^{-x} + 2\int x e^{-x} dx\right) \\ &= -\left(x^4 + 4x^3 + 12x^2\right) e^{-x} + 24\left[(-x-1)e^{-x}\right] + C \\ &= -\left(x^4 + 4x^3 + 12x^2 + 24x + 24\right) e^{-x} + C\end{aligned}$$

$$\begin{aligned}\text{So } \int_0^1 x^4 e^{-x} dx &= \left[-\left(x^4 + 4x^3 + 12x^2 + 24x + 24\right) e^{-x}\right]_0^1 \\ &= -(1+4+12+24+24)e^{-1} + 24e^0 = 24 - 65e^{-1} .\end{aligned}$$

27. Let $u = e^x$. Then $x = \ln u$, $dx = du/u$, so

$$\int \sqrt{e^{2x} - 1}\, dx = \int \frac{\sqrt{u^2 - 1}}{u}\, du = \sqrt{u^2 - 1} - \cos^{-1}(1/u) + C = \sqrt{e^{2x} - 1} - \cos^{-1}\left(e^{-x}\right) + C .$$

28. Let $u = \alpha t - 3$ and assume that $\alpha \neq 0$. Then $du = \alpha\, dt$ and

$$\begin{aligned}\int e^t \sin(\alpha t - 3)\, dt &= \frac{1}{\alpha} \int e^{(u+3)/\alpha} \sin u\, du = \frac{1}{\alpha} e^{3/\alpha} \int e^{(1/\alpha)u} \sin u\, du \\ &= \frac{1}{\alpha} e^{3/\alpha} \frac{e^{(1/\alpha)u}}{(1/\alpha)^2 + 1^2} \left(\frac{1}{\alpha} \sin u - \cos u\right) + C \\ &= \frac{1}{\alpha} e^{3/\alpha} e^{(1/\alpha)u} \frac{\alpha^2}{1+\alpha^2} \left(\frac{1}{\alpha} \sin u - \cos u\right) + C \\ &= \frac{1}{1+\alpha^2} e^{(u+3)/\alpha} (\sin u - \alpha \cos u) + C \\ &= \frac{1}{1+\alpha^2} e^t [\sin(\alpha t - 3) - \alpha \cos(\alpha t - 3)] + C\end{aligned}$$

29.

$$\begin{aligned}\int \frac{x^4\, dx}{\sqrt{x^{10} - 2}} &= \int \frac{x^4\, dx}{\sqrt{(x^5)^2 - 2}} = \frac{1}{5} \int \frac{du}{\sqrt{u^2 - 2}} \quad [\, u = x^5 ,\ du = 5x^4\, dx \,] \\ &= \frac{1}{5} \ln \left| u + \sqrt{u^2 - 2} \right| + C = \frac{1}{5} \ln \left| x^5 + \sqrt{x^{10} - 2} \right| + C\end{aligned}$$

30. Let $u=\tan\theta$ and $a=3$. Then $du=\sec^2\theta\ d\theta$ and

$$\int \frac{\sec^2\theta\ \tan^2\theta}{\sqrt{9-\tan^2\theta}}\ d\theta = \int \frac{u^2}{\sqrt{a^2-u^2}}\ du = -\frac{u}{2}\sqrt{a^2-u^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{u}{a}\right)+C$$
$$=-\frac{1}{2}\tan\theta\ \sqrt{9-\tan^2\theta}+\frac{9}{2}\sin^{-1}\left(\frac{\tan\theta}{3}\right)+C$$

31. Using cylindrical shells, we get

$$V=2\pi\int_0^2 x\cdot x\sqrt{4-x^2}\ dx=2\pi\int_0^2 x^2\sqrt{4-x^2}\ dx=2\pi\left[\frac{x}{8}(2x^2-4)\sqrt{4-x^2}+\frac{16}{8}\sin^{-1}\frac{x}{2}\right]_0^2$$
$$=2\pi\left[(0+2\sin^{-1}1)-(0+2\sin^{-1}0\right]=2\pi\left(2\cdot\frac{\pi}{2}\right)=2\pi^2$$

32. Using disks, we get

$$\text{Volume}=\int_0^{\pi/4}\pi\tan^4 x dx=\pi\left(\left[\frac{1}{3}\tan^3 x\right]_0^{\pi/4}-\int_0^{\pi/4}\tan^2 x dx\right)=\pi\left[\frac{1}{3}\tan^3 x-\tan x+x\right]_0^{\pi/4}$$
$$=\pi\left(\frac{1}{3}-1+\frac{\pi}{4}\right)=\pi\left(\frac{\pi}{4}-\frac{2}{3}\right)$$

33. **(a)** $$\frac{d}{du}\left[\frac{1}{b^3}\left(a+bu-\frac{a^2}{a+bu}-2a\ln|a+bu|\right)+C\right]=\frac{1}{b^3}\left[b+\frac{ba^2}{(a+bu)^2}-\frac{2ab}{(a+bu)}\right]$$
$$=\frac{1}{b^3}\left[\frac{b(a+bu)^2+ba^2-(a+bu)2ab}{(a+bu)^2}\right]=\frac{1}{b^3}\left[\frac{b^3u^2}{(a+bu)^2}\right]=\frac{u^2}{(a+bu)^2}$$

(b) Let $t=a+bu \Rightarrow dt=b\,du$. Note that $u=\frac{t-a}{b}$ and $du=\frac{1}{b}\ dt$.

$$\int\frac{u^2 du}{(a+bu)^2}=\frac{1}{b^3}\int\frac{(t-a)^2}{t^2}\ dt=\frac{1}{b^3}\int\frac{t^2-2at+a^2}{t^2}\ dt$$
$$=\frac{1}{b^3}\int\left(1-\frac{2a}{t}+\frac{a^2}{t^2}\right)dt=\frac{1}{b^3}\left(t-2a\ln|t|-\frac{a^2}{t}\right)+C$$

$$= \frac{1}{b^3}\left(a+bu-\frac{a^2}{a+bu}-2a\ln|a+bu|\right)+C$$

34. **(a)** $\frac{d}{du}\left[\frac{u}{8}\left(2u^2-a^2\right)\sqrt{a^2-u^2}+\frac{a^4}{8}\sin^{-1}\frac{u}{a}+C\right]$

$$=\frac{u}{8}\left(2u^2-a^2\right)\frac{-u}{\sqrt{a^2-u^2}}+\sqrt{a^2-u^2}\left[\frac{u}{8}(4u)+\left(2u^2-a^2\right)\frac{1}{8}\right]+\frac{a^4}{8}\frac{1/a}{\sqrt{1-u^2/a^2}}$$

$$=-\frac{u^2\left(2u^2-a^2\right)}{8\sqrt{a^2-u^2}}+\sqrt{a^2-u^2}\left[\frac{u^2}{2}+\frac{2u^2-a^2}{8}\right]+\frac{a^4}{8\sqrt{a^2-u^2}}$$

$$=\frac{1}{2}\left(a^2-u^2\right)^{-1/2}\left[-\frac{u^2}{4}\left(2u^2-a^2\right)+u^2\left(a^2-u^2\right)+\frac{1}{4}\left(a^2-u^2\right)\left(2u^2-a^2\right)+\frac{a^4}{4}\right]$$

$$=\frac{1}{2}\left(a^2-u^2\right)^{-1/2}\left[2u^2a^2-2u^4\right]=\frac{u^2\left(a^2-u^2\right)}{\sqrt{a^2-u^2}}=u^2\sqrt{a^2-u^2}$$

(b) Let $u=a\sin\theta \Rightarrow du=a\cos\theta\ d\theta$. Then

$$\int u^2\sqrt{a^2-u^2}\,du=\int a^2\sin^2\theta\ a\sqrt{1-\sin^2\theta}\ a\cos\theta\ d\theta=a^4\int\sin^2\theta\cos^2\theta\ d\theta$$

$$=a^4\int\frac{1}{2}(1+\cos 2\theta)\frac{1}{2}(1-\cos 2\theta)d\theta=\frac{1}{4}a^4\int\left(1-\cos^2 2\theta\right)d\theta$$

$$=\frac{1}{4}a^4\int\left[1-\frac{1}{2}(1+\cos 4\theta)\right]d\theta=\frac{1}{4}a^4\left(\frac{1}{2}\theta-\frac{1}{8}\sin 4\theta\right)+C$$

$$=\frac{1}{4}a^4\left(\frac{1}{2}\theta-\frac{1}{8}2\sin 2\theta\cos 2\theta\right)+C=\frac{1}{4}a^4\left[\frac{1}{2}\theta-\frac{1}{2}\sin\theta\cos\theta\left(1-2\sin^2\theta\right)\right]+C$$

$$=\frac{a^4}{8}\left[\sin^{-1}\frac{u}{a}-\frac{u}{a}\frac{\sqrt{a^2-u^2}}{a}\left(1-\frac{2u^2}{a^2}\right)\right]+C$$

$$=\frac{a^4}{8}\left[\sin^{-1}\frac{u}{a}-\frac{u}{a}\frac{\sqrt{a^2-u^2}}{a}\frac{a^2-2u^2}{a^2}\right]+C$$

$$=\frac{u}{8}\left(2u^2-a^2\right)\sqrt{a^2-u^2}+\frac{a^4}{8}\sin^{-1}\frac{u}{a}+C$$

35. Maple, Mathematica and Derive all give

$\int x^2\sqrt{5-x^2}\,dx=-\frac{1}{4}x\left(5-x^2\right)^{3/2}+\frac{5}{8}x\sqrt{5-x^2}+\frac{25}{8}\sin^{-1}\left(\frac{1}{\sqrt{5}}x\right)$. Using Formula 31, we get $\int x^2\sqrt{5-x^2}\,dx=\frac{1}{8}x\left(2x^2-5\right)\sqrt{5-x^2}+\frac{1}{8}\left(5^2\right)\sin^{-1}\left(\frac{1}{\sqrt{5}}x\right)+C$. But $-\frac{1}{4}x\left(5-x^2\right)^{3/2}+\frac{5}{8}x\sqrt{5-x^2}=\frac{1}{8}x\sqrt{5-x^2}\left[5-2\left(5-x^2\right)\right]=\frac{1}{8}x\left(2x^2-5\right)\sqrt{5-x^2}$, and the $\sin^{-1}$ terms are the same in each expression, so the answers are equivalent.

36. Maple and Mathematica both give $\int x^2\left(1+x^3\right)^4dx=\frac{1}{15}x^{15}+\frac{1}{3}x^{12}+\frac{2}{3}x^9+\frac{2}{3}x^6+\frac{1}{3}x^3$, while Derive gives $\int x^2\left(1+x^3\right)^4dx=\frac{1}{15}\left(x^3+1\right)^5$. Using the substitution $u=1+x^3\Rightarrow du=3x^2dx$, we get $\int x^2\left(1+x^3\right)^4dx=\int u^4\left(\frac{1}{3}du\right)=\frac{1}{15}u^5+C=\frac{1}{15}\left(1+x^3\right)^5+C$. We can use the Binomial Theorem or a CAS to expand this expression, and we get $\frac{1}{15}\left(1+x^3\right)^5+C=\frac{1}{15}+\frac{1}{3}x^3+\frac{2}{3}x^6+\frac{2}{3}x^9+\frac{1}{3}x^{12}+\frac{1}{15}x^{15}+C$.

37. Maple and Derive both give $\int\sin^3x\cos^2x\,dx=-\frac{1}{5}\sin^2x\cos^3x-\frac{2}{15}\cos^3x$ (although Derive factors the expression), and Mathematica gives $\int\sin^3x\cos^2x\,dx=-\frac{1}{8}\cos x-\frac{1}{48}\cos 3x+\frac{1}{80}\cos 5x$. We can use a CAS to show that both of these expressions are equal to $-\frac{1}{3}\cos^3x+\frac{1}{5}\cos^5x$. Using Formula 86, we write

$$\begin{aligned}\int\sin^3x\cos^2x\,dx &= -\frac{1}{5}\sin^2x\cos^3x+\frac{2}{5}\int\sin x\cos^2x\,dx=-\frac{1}{5}\sin^2x\cos^3x+\frac{2}{5}\left(-\frac{1}{3}\cos^3x\right)+C\\ &= -\frac{1}{5}\sin^2x\cos^3x-\frac{2}{15}\cos^3x+C\end{aligned}$$

38. Maple gives $\int\tan^2x\sec^4dx=\frac{1}{5}\frac{\sin^3x}{\cos^5x}+\frac{2}{15}\frac{\sin^3x}{\cos^3x}$, Mathematica gives $\int\tan^2x\sec^4dx=-\frac{1}{120}\sec^5x(-20\sin x+5\sin 3x+\sin 5x)$, and Derive gives $\int\tan^2x\sec^4dx=-\frac{2}{15}\tan x-\frac{\sin x}{15\cos^3x}+\frac{\sin x}{5\cos^5x}$. All of these expressions can be ‘‘simplified’’ to $-\frac{1}{15}\frac{\sin x\left(\cos^2x-2\cos^4x-3\right)}{\cos^5x}$ using Maple. Using the identity $1+\tan^2x=\sec^2x$, we write

$\int \tan^2 x \sec^4 x dx = \int \tan^2 x \left(1+\tan^2 x\right)\sec^2 x dx = \int \left(\tan^2 x+\tan^4 x\right)\sec^2 x dx$. Now we substitute $u = \tan x$ $\Rightarrow du = \sec^2 x dx$, and the integral becomes $\int \left(u^2+u^4\right) du = \frac{1}{3} u^3 + \frac{1}{5} u^5 + C = \frac{1}{3} \tan^3 x + \frac{1}{5} \tan^5 x + C$. If we write $\sin^5 x = \sin^3 x\left(1-\cos^2 x\right)$ and substitute into the numerator of the $\tan^5 x$ term, this becomes $\frac{1}{3}\frac{\sin^3 x}{\cos^3 x}+\frac{1}{5}\frac{\sin^3 x\left(1-\cos^2 x\right)}{\cos^5 x}+C=\frac{1}{5}\frac{\sin^3 x}{\cos^5 x}+\left(\frac{1}{3}-\frac{1}{5}\right)\frac{\sin^3 x}{\cos^3 x}+C=\frac{1}{5}\frac{\sin^3 x}{\cos^5 x}+\frac{2}{15}\frac{\sin^3 x}{\cos^3 x}+C$, which is the same as Maple's expression.

39. Maple gives $\int x\sqrt{1+2x}\, dx = \frac{1}{10}(1+2x)^{5/2} - \frac{1}{6}(1+2x)^{3/2}$, Mathematica gives $\sqrt{1+2x}\left(\frac{2}{5}x^2+\frac{1}{15}x-\frac{1}{15}\right)$, and Derive gives $\frac{1}{15}(1+2x)^{3/2}(3x-1)$. The first two expressions can be simplified to Derive's result. If we use Formula 54, we get

$$\begin{aligned}\int x\sqrt{1+2x}\, dx &= \frac{2}{15(2)^2}(3\cdot 2x-2\cdot 1)(1+2x)^{3/2}+C=\frac{1}{30}(6x-2)(1+2x)^{3/2}+C\\ &= \frac{1}{15}(3x-1)(1+2x)^{3/2}\end{aligned}$$

40. Maple and Derive both give $\int \sin^4 x dx = -\frac{1}{4}\sin^3 x\cos x - \frac{3}{8}\cos x\sin x + \frac{3}{8}x$, while Mathematica gives $\frac{1}{32}(12x-8\sin 2x+\sin 4x)$, which can be expanded and simplified to give the other expression. Now

$$\begin{aligned}\int \sin^4 \mathrm{x}\,\mathrm{dx} &= -\frac{1}{4}\sin^3 \mathrm{x}\cos \mathrm{x}+\frac{3}{4}\int \sin^2 \mathrm{x}\,\mathrm{dx}=-\frac{1}{4}\sin^3 \mathrm{x}\cos \mathrm{x}+\frac{3}{4}\left(\frac{1}{2}x-\frac{1}{4}\sin 2x\right)+\mathrm{C}\\ &= -\frac{1}{4}\sin^3 \mathrm{x}\cos \mathrm{x}-\frac{3}{8}\sin \mathrm{x}\cos \mathrm{x}+\frac{3}{8}\mathrm{x}+\mathrm{C} \quad \text{since } \sin 2\mathrm{x}=2\sin \mathrm{x}\cos \mathrm{x}\end{aligned}$$

41. Maple gives $\int \tan^5 x dx = \frac{1}{4}\tan^4 x - \frac{1}{2}\tan^2 x + \frac{1}{2}\ln\left(1+\tan^2 x\right)$, Mathematica gives $\int \tan^5 x dx = \frac{1}{4}[-1-2\cos(2x)]\sec^4 x - \ln(\cos x)$, and Derive gives $\int \tan^5 x dx = \frac{1}{4}\tan^4 x - \frac{1}{2}\tan^2 x - \ln(\cos x)$. These expressions are equivalent, and none includes absolute value bars or a constant of integration. Note that Mathematica's and Derive's expressions suggest that the integral is undefined where $\cos x<0$, which is not the case.
Using Formula 75,

$\int \tan^5 x\,dx=\frac{1}{5-1}\tan^{5-1}x-\int \tan^{5-2}x\,dx=\frac{1}{4}\tan^4 x-\int \tan^3 x\,dx$. Using Formula 69,
$\int \tan^3 x\,dx=\frac{1}{2}\tan^2 x+\ln|\cos x|+C$, so $\int \tan^5 x\,dx=\frac{1}{4}\tan^4 x-\frac{1}{2}\tan^2 x-\ln|\cos x|+C$.

42. Maple gives $\int x^5\sqrt{x^2+1}\,dx=\frac{1}{35}x^4\sqrt{1+x^2}-\frac{4}{105}x^2\sqrt{1+x^2}+\frac{8}{105}\sqrt{1+x^2}+\frac{1}{7}x^6\sqrt{1+x^2}$. When we use the factor command on this expression, it becomes $\frac{1}{105}\left(1+x^2\right)^{3/2}\left(15x^4-12x^2+8\right)$. Mathematica gives $\sqrt{1+x^2}\left(\frac{8}{105}-\frac{4}{105}x^2+\frac{1}{35}x^4+\frac{1}{7}x^6\right)$, which again factors to give the above expression, and Derive gives the factored form immediately. If we substitute $u=\sqrt{x^2+1}\Rightarrow x^4=\left(u^2-1\right)^2$, $x\,dx=u\,du$, then the integral
becomes

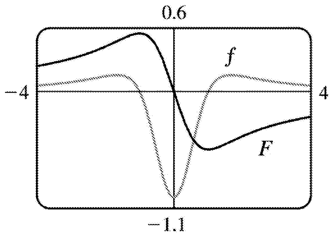
$$\begin{aligned}\int\left(u^2-1\right)^2 u\,(u\,du) &= \int\left(u^4-2u^2+1\right)u^2\,du=\frac{1}{7}u^7-\frac{2}{5}u^5+\frac{1}{3}u^3+C\\ &= \left(x^2+1\right)^{3/2}\left[\frac{1}{7}\left(x^2+1\right)^2-\frac{2}{5}\left(x^2+1\right)+\frac{1}{3}\right]+C\\ &= \frac{1}{105}\left(x^2+1\right)^{3/2}\left[15\left(x^2+1\right)^2-42\left(x^2+1\right)+35\right]+C\\ &= \frac{1}{105}\left(x^2+1\right)^{3/2}\left(15x^4-12x^2+8\right)+C\end{aligned}$$

43. Derive gives $I=\int 2^x\sqrt{4^x-1}\,dx=\frac{2^{x-1}\sqrt{2^{2x}-1}}{\ln 2}-\frac{\ln\left(\sqrt{2^{2x}-1}+2^x\right)}{2\ln 2}$ immediately. Neither Maple nor Mathematica is able to evaluate I in its given form. However, if we instead write I as $\int 2^x\sqrt{(2^x)^2-1}\,dx$, both systems give the same answer as Derive (after minor simplification). Our trick works because the CAS now recognizes 2^x as a promising substitution.

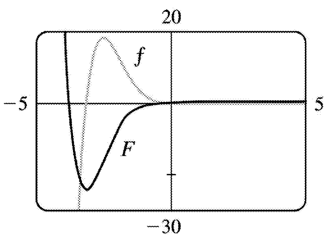
44. None of Maple, Mathematica and Derive is able to evaluate $\int(1+\ln x)\sqrt{1+(x\ln x)^2}\,dx$. However, if we let $u=x\ln x$, then $du=(1+\ln x)dx$ and the integral is simply $\int\sqrt{1+u^2}\,du$, which any CAS can evaluate. The antiderivative is $\frac{1}{2}\ln\left(x\ln x+\sqrt{1+(x\ln x)^2}\right)+\frac{1}{2}x\ln x\sqrt{1+(x\ln x)^2}+C$.

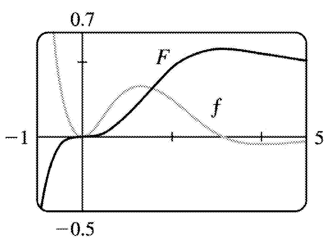
45. Maple gives the antiderivative $F(x)=\int\frac{x^2-1}{x^4+x^2+1}\,dx=-\frac{1}{2}\ln\left(x^2+x+1\right)+\frac{1}{2}\ln\left(x^2-x+1\right)$. We can see that at 0 , this antiderivative is 0 . From the graphs, it appears that F has a maximum at $x=-1$ and

a minimum at $x=1$, and that F has inflection points at $x\approx-1.7$, $x=0$, and $x\approx 1.7$.



46. Maple gives the antiderivative which, after we use the simplify command, becomes $\int xe^{-x}\sin x\,dx=-\frac{1}{2}e^{-x}(\cos x+x\cos x+x\sin x)$. At $x=0$, this antiderivative has the value $-\frac{1}{2}$, so we use $F(x)=-\frac{1}{2}e^{-x}(\cos x+x\cos x+x\sin x)+\frac{1}{2}$ to make $F(0)=0$.

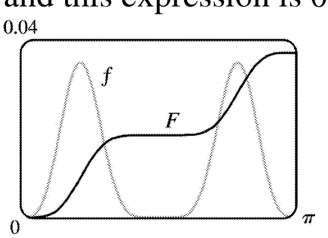




From the graphs, it appears that F has a minimum at $x\approx-3.1$ and a maximum at $x\approx 3.1$, and that F has inflection points where f' changes sign, at $x\approx-2.5$, $x=0$, $x\approx 1.3$ and $x\approx 4.1$.

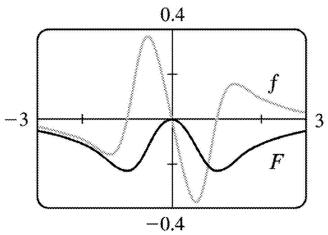
47. Since $f(x)=\sin^4 x\cos^6 x$ is everywhere positive, we know that its antiderivative F is increasing. Maple gives

$\int f(x)dx=-\frac{1}{10}\sin^3 x\cos^7 x-\frac{3}{80}\sin x\cos^7 x+\frac{1}{160}\cos^5 x\sin x+\frac{1}{128}\cos^3 x\sin x+\frac{3}{256}\cos x\sin x+\frac{3}{256}x$
and this expression is 0 at $x=0$.



F has a minimum at $x=0$ and a maximum at $x=\pi$. F has inflection points where f' changes sign, that is, at $x \approx 0.7$, $x = \pi/2$, and $x \approx 2.5$.

48. From the graph of $f(x) = \dfrac{x^3 - x}{x^6 + 1}$, we can see that F has a maximum at $x = 0$, and minima at $x \approx \pm 1$. The antiderivative given by Maple is $F(x) = -\dfrac{1}{3} \ln \left(x^2 + 1\right) + \dfrac{1}{6} \ln \left(x^4 - x^2 + 1\right)$, and $F(0) = 0$. Note that f is odd, and its antiderivative F is even.



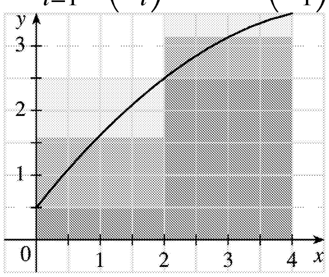
F has inflection points where f' changes sign, that is, at $x \approx \pm 0.5$ and $x \approx \pm 1.4$.

1. **(a)** $\Delta x=(b-a)/n=(4-0)/2=2$

$L_2=\sum_{i=1}^{2} f\left(x_{i-1}\right)\Delta x=f\left(x_0\right)\cdot 2+f\left(x_1\right)\cdot 2=2[f(0)+f(2)]=2(0.5+2.5)=6$

$R_2=\sum_{i=1}^{2} f\left(x_i\right)\Delta x=f\left(x_1\right)\cdot 2+f\left(x_2\right)\cdot 2=2[f(2)+f(4)]=2(2.5+3.5)=12$

$M_2=\sum_{i=1}^{2} f\left(\bar{x}_i\right)\Delta x=f\left(\bar{x}_1\right)\cdot 2+f\left(\bar{x}_2\right)\cdot 2=2[f(1)+f(3)]\approx 2(1.6+3.2)=9.6$



(b)

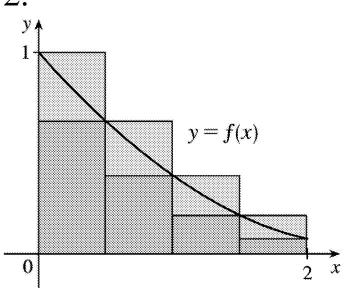
L_2 is an underestimate, since the area under the small rectangles is less than the area under the curve, and R_2 is an overestimate, since the area under the large rectangles is greater than the area under the curve. It appears that M_2 is an overestimate, though it is fairly close to I . See the solution to Exercise 45 for a proof of the fact that if f is concave down on $[a,b]$, then the Midpoint Rule is an overestimate of $\int_a^b f(x)dx$.

(c) $T_2=\left(\frac{1}{2}\Delta x\right)\left[f\left(x_0\right)+2f\left(x_1\right)+f\left(x_2\right)\right]=\frac{2}{2}[f(0)+2f(2)+f(4)]=0.5+2(2.5)+3.5=9$.

This approximation is an underestimate, since the graph is concave down. Thus, $T_2=9<I$. See the solution to Exercise 45 for a general proof of this conclusion.

(d) For any n , we will have $L_n<T_n<I<M_n<R_n$.

2.



The diagram shows that $L_4>T_4>\int_0^2 f(x)dx>R_4$, and it appears that M_4 is a bit less than $\int_0^2 f(x)dx$. In fact, for any function that is concave upward, it can be shown that $L_n>T_n>\int_0^2 f(x)dx>M_n>R_n$.

(a) Since $0.9540>0.8675>0.8632>0.7811$, it follows that $L_n=0.9540 T_n=0.8675$, $M_n=0.8632$, and

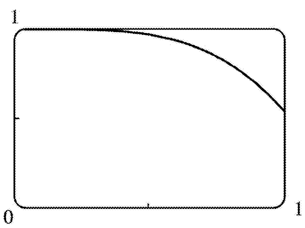
$R_n = 0.7811$.

(b) Since $M_n < \int_0^2 f(x)\,dx < T_n$, we have $0.8632 < \int_0^2 f(x)\,dx < 0.8675$.

3. $f(x) = \cos\left(x^2\right)$, $\Delta x = \frac{1-0}{4} = \frac{1}{4}$

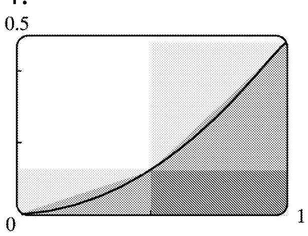
(a) $T_4 = \frac{1}{4\cdot 2}\left[f(0) + 2f\left(\frac{1}{4}\right) + 2f\left(\frac{2}{4}\right) + 2f\left(\frac{3}{4}\right) + f(1)\right] \approx 0.895759$

(b) $M_4 = \frac{1}{4}\left[f\left(\frac{1}{8}\right) + f\left(\frac{3}{8}\right) + f\left(\frac{5}{8}\right) + f\left(\frac{7}{8}\right)\right] \approx 0.908907$



The graph shows that f is concave down on $[0,1]$. So T_4 is an underestimate and M_4 is an overestimate. We can conclude that $0.895759 < \int_0^1 \cos\left(x^2\right) dx < 0.908907$.

4.



(a) Since f is increasing on $[0,1]$, L_2 will underestimate I (since the area of the darkest rectangle is less than the area under the curve), and R_2 will overestimate I . Since f is concave upward on $[0,1]$, M_2 will underestimate I and T_2 will overestimate I (the area under the straight line segments is greater than the area under the curve).

(b) For any n , we will have $L_n < M_n < I < T_n < R_n$.

(c) $L_5 = \sum_{i=1}^{5} f\left(x_{i-1}\right)\Delta x = \frac{1}{5}[f(0.0) + f(0.2) + f(0.4) + f(0.6) + f(0.8)] \approx 0.1187$

$R_5 = \sum_{i=1}^{5} f\left(x_i\right)\Delta x = \frac{1}{5}[f(0.2) + f(0.4) + f(0.6) + f(0.8) + f(1)] \approx 0.2146$

$M_5 = \sum_{i=1}^{5} f\left(\bar{x}_i\right)\Delta x = \frac{1}{5}[f(0.1) + f(0.3) + f(0.5) + f(0.7) + f(0.9)] \approx 0.1622$

$T_5=\left(\frac{1}{2}\,\Delta x\right)\approx 0.1666$

From the graph, it appears that the Midpoint Rule gives the best approximation. (This is in fact the case, since $I\approx 0.16371405$.)

5. $f(x)=x^2\sin x$, $\Delta x=\frac{b-a}{n}=\frac{\pi-0}{8}=\frac{\pi}{8}$

(a) $M_8=\frac{\pi}{8}\left[f\left(\frac{\pi}{16}\right)+f\left(\frac{3\pi}{16}\right)+f\left(\frac{5\pi}{16}\right)+\cdots+f\left(\frac{15\pi}{16}\right)\right]\approx 5.932957$

(b)

$$S_8=\frac{\pi}{8\cdot 3}\left[f(0)+4f\left(\frac{\pi}{8}\right)+2f\left(\frac{2\pi}{8}\right)+4f\left(\frac{3\pi}{8}\right)+2f\left(\frac{4\pi}{8}\right)+4f\left(\frac{5\pi}{8}\right)+2f\left(\frac{6\pi}{8}\right)+4f\left(\frac{7\pi}{8}\right)\right.$$

$$\approx 5.869247$$

Actual:

$$\int_0^{\pi}x^2\sin x\,dx=\left[-x^2\cos x\right]_0^{\pi}+2\int_0^{\pi}x\cos x\,dx=\left[-\pi^2(-1)-0\right]+2\left[\cos x+x\sin x\right]_0^{\pi}$$

$$=\pi^2+2[(-1+0)-(1+0)]=\pi^2-4\approx 5.869604$$

Errors: $E_M=$ actual $-M_8=\int_0^{\pi}x^2\sin x\,dx-M_8\approx -0.063353$

$E_S=$ actual $-S_8=\int_0^{\pi}x^2\sin x\,dx-S_8\approx 0.000357$

6. $f(x)=e^{-\sqrt{x}}$, $\Delta x=\frac{b-a}{n}=\frac{1-0}{6}=\frac{1}{6}$

(a) $M_6=\frac{1}{6}\left[f\left(\frac{1}{12}\right)+f\left(\frac{3}{12}\right)+f\left(\frac{5}{12}\right)+f\left(\frac{7}{12}\right)+f\left(\frac{9}{12}\right)+f\left(\frac{11}{12}\right)\right]\approx 0.525100$

(b) $S_6=\frac{1}{6\cdot 3}\left[f(0)+4f\left(\frac{1}{6}\right)+2f\left(\frac{2}{6}\right)+4f\left(\frac{3}{6}\right)+2f\left(\frac{4}{6}\right)+4f\left(\frac{5}{6}\right)+f(1)\right]\approx 0.533979$

Actual:

$$\int_0^1 e^{-\sqrt{x}}\,dx=\int_0^{-1}e^u\,2u\,du\quad u=-\sqrt{x}\ ,\ u^2=x\ ,\ 2u\,du=dx\ ]$$

$$=2\left[(u-1)e^u\right]_0^{-1}=2\left[-2e^{-1}-\left(-1e^0\right)\right]=2-4e^{-1}\approx 0.528482$$

Errors: $E_M=$ actual $-M_6=\int_0^1 e^{-\sqrt{x}}\,dx-M_6\approx 0.003382$

$E_S=$ actual $-S_6=\int_0^1 e^{-\sqrt{x}}\,dx-S_6\approx -0.005497$

7. $f(x)=\sqrt[4]{1+x^2}\;,\;\Delta x=\dfrac{2-0}{8}=\dfrac{1}{4}$

(a) $T_8=\dfrac{1}{4\cdot 2}\left[f(0)+2f\left(\dfrac{1}{4}\right)+2f\left(\dfrac{1}{2}\right)+\cdots+2f\left(\dfrac{3}{2}\right)+2f\left(\dfrac{7}{4}\right)+f(2)\right]\approx 2.413790$

(b) $M_8=\dfrac{1}{4}\left[f\left(\dfrac{1}{8}\right)+f\left(\dfrac{3}{8}\right)+\cdots+f\left(\dfrac{13}{8}\right)+f\left(\dfrac{15}{8}\right)\right]\approx 2.411453$

(c)

$S_8=\dfrac{1}{4\cdot 3}\left[f(0)+4f\left(\dfrac{1}{4}\right)+2f\left(\dfrac{1}{2}\right)+4f\left(\dfrac{3}{4}\right)+2f(1)+4f\left(\dfrac{5}{4}\right)+2f\left(\dfrac{3}{2}\right)+4f\left(\dfrac{7}{4}\right)+f(2)\right]\approx 2.4122$

8. $f(x)=\sin\left(x^2\right)\;,\;\Delta x=\dfrac{\frac{1}{2}-0}{4}=\dfrac{1}{8}$

(a) $T_4=\dfrac{1}{8\cdot 2}\left[f(0)+2f\left(\dfrac{1}{8}\right)+2f\left(\dfrac{2}{8}\right)+2f\left(\dfrac{3}{8}\right)+f\left(\dfrac{1}{2}\right)\right]\approx 0.042743$

(b) $M_4=\dfrac{1}{8}\left[f\left(\dfrac{1}{16}\right)+f\left(\dfrac{3}{16}\right)+f\left(\dfrac{5}{16}\right)+f\left(\dfrac{7}{16}\right)\right]\approx 0.040850$

(c) $S_4=\dfrac{1}{8\cdot 3}\left[f(0)+4f\left(\dfrac{1}{8}\right)+2f\left(\dfrac{2}{8}\right)+4f\left(\dfrac{3}{8}\right)+f\left(\dfrac{1}{2}\right)\right]\approx 0.041478$

9. $f(x)=\dfrac{\ln x}{1+x}\;,\;\Delta x=\dfrac{2-1}{10}=\dfrac{1}{10}$

(a) $T_{10}=\dfrac{1}{10\cdot 2}\,[f(1)+2f(1.1)+2f(1.2)+\cdots+2f(1.8)+2f(1.9)+f(2)]\approx 0.146879$

(b) $M_{10}=\dfrac{1}{10}\,[f(1.05)+f(1.15)+\cdots+f(1.85)+f(1.95)]\approx 0.147391$

(c)

$$\begin{aligned}S_{10}&=\frac{1}{10\cdot 3}[f(1)+4f(1.1)+2f(1.2)+4f(1.3)+2f(1.4)+4f(1.5)+2f(1.6)+4f(1.7)]\\&\quad+\;2f(1.8)+4f(1.9)+f(2)]\\&\quad\approx 0.147219\end{aligned}$$

10. $f(t)=\dfrac{1}{1+t^2+t^4}\;,\;\Delta t=\dfrac{3-0}{6}=\dfrac{1}{2}$

(a) $T_6=\dfrac{1}{2\cdot 2}\left[f(0)+2f\left(\dfrac{1}{2}\right)+2f(1)+2f\left(\dfrac{3}{2}\right)+2f(2)+2f\left(\dfrac{5}{2}\right)+f(3)\right]\approx 0.895122$

(b) $M_6=\dfrac{1}{2}\left[f\left(\dfrac{1}{4}\right)+f\left(\dfrac{3}{4}\right)+f\left(\dfrac{5}{4}\right)+f\left(\dfrac{7}{4}\right)+f\left(\dfrac{9}{4}\right)+f\left(\dfrac{11}{4}\right)\right]\approx 0.895478$

(c) $S_6=\dfrac{1}{2\cdot 3}\left[f(0)+4f\left(\dfrac{1}{2}\right)+2f(1)+4f\left(\dfrac{3}{2}\right)+2f(2)+4f\left(\dfrac{5}{2}\right)+f(3)\right]\approx 0.898014$

11.

$f(t)=\sin(e^{t/2})$, $\Delta t=\dfrac{\frac{1}{2}-0}{8}=\dfrac{1}{16}$

(a) $T_8=\dfrac{1}{16\cdot 2}\left[f(0)+2f\left(\dfrac{1}{16}\right)+2f\left(\dfrac{2}{16}\right)+\cdots+2f\left(\dfrac{7}{16}\right)+f\left(\dfrac{1}{2}\right)\right]\approx 0.451948$

(b) $M_8=\dfrac{1}{16}\left[f\left(\dfrac{1}{32}\right)+f\left(\dfrac{3}{32}\right)+f\left(\dfrac{5}{32}\right)+\cdots+f\left(\dfrac{13}{32}\right)+f\left(\dfrac{15}{32}\right)\right]\approx 0.451991$

(c) $S_8=\dfrac{1}{16\cdot 3}\left[f(0)+4f\left(\dfrac{1}{16}\right)+2f\left(\dfrac{2}{16}\right)+\cdots+4f\left(\dfrac{7}{16}\right)+f\left(\dfrac{1}{2}\right)\right]\approx 0.451976$

12. $f(x)=\sqrt{1+\sqrt{x}}$, $\Delta x=\dfrac{4-0}{8}=\dfrac{1}{2}$

(a) $T_8=\dfrac{1}{2\cdot 2}\left[f(0)+2f\left(\dfrac{1}{2}\right)+2f(1)+\cdots+2f(3)+2f\left(\dfrac{7}{2}\right)+f(4)\right]\approx 6.042985$

(b) $M_8=\dfrac{1}{2}\left[f\left(\dfrac{1}{4}\right)+f\left(\dfrac{3}{4}\right)+\cdots+f\left(\dfrac{13}{4}\right)+f\left(\dfrac{15}{4}\right)\right]\approx 6.084778$

(c)

$S_8=\dfrac{1}{2\cdot 3}\left[f(0)+4f\left(\dfrac{1}{2}\right)+2f(1)+4f\left(\dfrac{3}{2}\right)+2f(2)+4f\left(\dfrac{5}{2}\right)+2f(3)+4f\left(\dfrac{7}{2}\right)+f(4)\right]\approx 6.061678$

13. $f(x)=e^{1/x}$, $\Delta x=\dfrac{2-1}{4}=\dfrac{1}{4}$

(a) $T_4=\dfrac{1}{4\cdot 2}[f(1)+2f(1.25)+2f(1.5)+2f(1.75)+f(2)]\approx 2.031893$

(b) $M_4=\dfrac{1}{4}[f(1.125)+f(1.375)+f(1.625)+f(1.875)]\approx 2.014207$

(c) $S_4=\dfrac{1}{4\cdot 3}[f(1)+4f(1.25)+2f(1.5)+4f(1.75)+f(2)]\approx 2.020651$

14. $f(x)=\sqrt{x}\sin x$, $\Delta x=\dfrac{4-0}{8}=\dfrac{1}{2}$

(a)

$T_8=\dfrac{1}{2\cdot 2}\left\{f(0)+2\left[f\left(\dfrac{1}{2}\right)+f(1)+f\left(\dfrac{3}{2}\right)+f(2)+f\left(\dfrac{5}{2}\right)+f(3)+f\left(\dfrac{7}{2}\right)\right]+f(4)\right\}\approx 1.732865$

(b) $M_8=\dfrac{1}{2}\left[f\left(\dfrac{1}{4}\right)+f\left(\dfrac{3}{4}\right)+f\left(\dfrac{5}{4}\right)+f\left(\dfrac{7}{4}\right)+\cdots+f\left(\dfrac{13}{4}\right)+f\left(\dfrac{15}{4}\right)\right]\approx 1.787427$

(c)

$S_8=\dfrac{1}{2\cdot 3}\left[f(0)+4f\left(\dfrac{1}{2}\right)+2f(1)+4f\left(\dfrac{3}{2}\right)+2f(2)+4f\left(\dfrac{5}{2}\right)+2f(3)+4f\left(\dfrac{7}{2}\right)+f(4)\right]\approx 1.772142$

15. $f(x)=\dfrac{\cos x}{x}$, $\Delta x=\dfrac{5-1}{8}=\dfrac{1}{2}$

(a) $T_8=\frac{1}{2\cdot 2}\left[f(1)+2f\left(\frac{3}{2}\right)+2f(2)+\cdots+2f(4)+2f\left(\frac{9}{2}\right)+f(5)\right]\approx -0.495333$

(b)

$M_8=\frac{1}{2}\left[f\left(\frac{5}{4}\right)+f\left(\frac{7}{4}\right)+f\left(\frac{9}{4}\right)+f\left(\frac{11}{4}\right)+f\left(\frac{13}{4}\right)+f\left(\frac{15}{4}\right)+f\left(\frac{17}{4}\right)+f\left(\frac{19}{4}\right)\right]\approx -0.5433$

(c)

$S_8=\frac{1}{2\cdot 3}\left[f(1)+4f\left(\frac{3}{2}\right)+2f(2)+4f\left(\frac{5}{2}\right)+2f(3)+4f\left(\frac{7}{2}\right)+2f(4)+4f\left(\frac{9}{2}\right)+f(5)\right]$

≈ -0.526123

16. $f(x)=\ln\left(x^3+2\right)$, $\Delta x=\frac{6-4}{10}=\frac{1}{5}$

(a) $T_{10}=\frac{1}{5\cdot 2}[f(4)+2f(4.2)+2f(4.4)+\cdots+2f(5.6)+2f(5.8)+f(6)]\approx 9.649753$

(b) $M_{10}=\frac{1}{5}[f(4.1)+f(4.3)+\cdots+f(5.7)+f(5.9)]\approx 9.650912$

(c)

$S_{10}=\frac{1}{5\cdot 3}[f(4)+4f(4.2)+2f(4.4)+4f(4.6)+2f(4.8)+4f(5)]$

$+\ 2f(5.2)+4f(5.4)+2f(5.6)+4f(5.8)+f(6)]$

≈ 9.650526

17. $f(y)=\frac{1}{1+y^5}$, $\Delta y=\frac{3-0}{6}=\frac{1}{2}$

(a) $T_6=\frac{1}{2\cdot 2}\left[f(0)+2f\left(\frac{1}{2}\right)+2f\left(\frac{2}{2}\right)+2f\left(\frac{3}{2}\right)+2f\left(\frac{4}{2}\right)+2f\left(\frac{5}{2}\right)+f(3)\right]\approx 1.064275$

(b) $M_6=\frac{1}{2}\left[f\left(\frac{1}{4}\right)+f\left(\frac{3}{4}\right)+f\left(\frac{5}{4}\right)+f\left(\frac{7}{4}\right)+f\left(\frac{9}{4}\right)+f\left(\frac{11}{4}\right)\right]\approx 1.067416$

(c) $S_6=\frac{1}{2\cdot 3}\left[f(0)+4f\left(\frac{1}{2}\right)+2f\left(\frac{2}{2}\right)+4f\left(\frac{3}{2}\right)+2f\left(\frac{4}{2}\right)+4f\left(\frac{5}{2}\right)+f(3)\right]\approx 1.074915$

18. $f(x)=\frac{e^x}{x}$, $\Delta x=\frac{4-2}{10}=\frac{1}{5}$

(a) $T_{10}=\frac{1}{5\cdot 2}\{f(2)+2[f(2.2)+f(2.4)+f(2.6)+\cdots+f(3.8)]+f(4)\}\approx 14.704592$

(b) $M_{10}=\frac{1}{5}[f(2.1)+f(2.3)+f(2.5)+f(2.7)+\cdots+f(3.7)+f(3.9)]\approx 14.662669$

(c) $S_{10}=\frac{1}{5\cdot 3}[f(2)+4f(2.2)+2f(2.4)+4f(2.6)+\cdots+2f(3.6)+4f(3.8)+f(4)]\approx 14.676696$

19. $f(x)=e^{-x^2}$, $\Delta x=\frac{2-0}{10}=\frac{1}{5}$

(a) $T_{10}=\frac{1}{5\cdot 2}\{f(0)+2[f(0.2)+f(0.4)+\cdots+f(1.8)]+f(2)\}\approx 0.881839$

$M_{10}=\frac{1}{5}[f(0.1)+f(0.3)+f(0.5)+\cdots+f(1.7)+f(1.9)]\approx 0.882202$

(b) $f(x)=e^{-x^2}$, $f'(x)=-2xe^{-x^2}$, $f''(x)=\left(4x^2-2\right)e^{-x^2}$, $f'''(x)=4x\left(3-2x^2\right)e^{-x^2}$.

$f'''(x)=0\Leftrightarrow x=0$ or $x=\pm\sqrt{\frac{3}{2}}$. So to find the maximum value of $\left|f''(x)\right|$ on $[0,2]$, we need only consider its values at $x=0$, $x=2$, and $x=\sqrt{\frac{3}{2}}$. $\left|f''(0)\right|=2$, $\left|f''(2)\right|\approx 0.2564$ and $\left|f''\left(\sqrt{\frac{3}{2}}\right)\right|\approx 0.8925$. Thus, taking $K=2$, $a=0$, $b=2$, and $n=10$ in Theorem 3, we get $\left|E_T\right|\le 2\cdot 2^3/\left(12\cdot 10^2\right)=\frac{1}{75}=0.01\overline{3}$, and $\left|E_M\right|\le\left|E_T\right|/2\le 0.00\overline{6}$.

(c) Take $K=2$ in Theorem 3. $\left|E_T\right|\le\frac{K(b-a)^3}{12n^2}\le 10^{-5}\Leftrightarrow\frac{2(2-0)^3}{12n^2}\le 10^{-5}\Leftrightarrow\frac{3}{4}n^2\ge 10^5\Leftrightarrow n\ge 365.1\ldots\Leftrightarrow$ $n\ge 366$. Take $n=366$ for T_n . For E_M , again take $K=2$ in Theorem 3 to get $\left|E_M\right|\le 10^{-5}\Leftrightarrow\frac{3}{2}n^2\ge 10^5$ $\Leftrightarrow n\ge 258.2\Rightarrow n\ge 259$. Take $n=259$ for M_n .

20. **(a)** $T_8=\frac{1}{8\cdot 2}\left\{f(0)+2\left[f\left(\frac{1}{8}\right)+f\left(\frac{2}{8}\right)+\cdots+f\left(\frac{7}{8}\right)\right]+f(1)\right\}\approx 0.902333$

$M_8=\frac{1}{8}\left[f\left(\frac{1}{16}\right)+f\left(\frac{3}{16}\right)+f\left(\frac{5}{16}\right)+\cdots+f\left(\frac{15}{16}\right)\right]=0.905620$

(b) $f(x)=\cos\left(x^2\right)$, $f'(x)=-2x\sin\left(x^2\right)$, $f''(x)=-2\sin\left(x^2\right)-4x^2\cos\left(x^2\right)$. For $0\le x\le 1$, sin and cos are positive, so $\left|f''(x)\right|=2\sin\left(x^2\right)+4x^2\cos\left(x^2\right)\le 2\cdot 1+4\cdot 1\cdot 1=6$ since $\sin\left(x^2\right)\le 1$ and $\cos\left(x^2\right)\le 1$ for all x , and $x^2\le 1$ for $0\le x\le 1$. So for $n=8$, we take $K=6$, $a=0$, and $b=1$ in Theorem 3, to get $\left|E_T\right|\le 6\cdot 1^3/\left(12\cdot 8^2\right)=\frac{1}{128}=0.0078125$ and $\left|E_M\right|\le\frac{1}{256}=0.00390625$.

(c) Using $K=6$ as in part (b), we have $\left|E_T\right|\le 6\cdot 1^3/\left(12n^2\right)=1/\left(2n^2\right)\le 10^{-5}\Rightarrow 2n^2\ge 10^5\Rightarrow$ $n\ge\sqrt{\frac{1}{2}\cdot 10^5}$ or $n\ge 224$. To guarantee that $\left|E_M\right|\le 0.00001$, we need $6\cdot 1^3/\left(24n^2\right)\le 10^{-5}\Rightarrow$ $4n^2\ge 10^5\Rightarrow n\ge\sqrt{\frac{1}{4}\cdot 10^5}$ or $n\ge 159$.

21. **(a)** $T_{10}=\frac{1}{10\cdot 2}\{f(0)+2[f(0.1)+f(0.2)+\cdots+f(0.9)]+f(1)\}\approx 1.71971349$

$S_{10}=\frac{1}{10\cdot 3}[f(0)+4f(0.1)+2f(0.2)+4f(0.3)+\cdots+4f(0.9)+f(1)]\approx 1.71828278$

Since $I=\int_0^1 e^x dx=\left[e^x\right]_0^1=e-1\approx 1.71828183$, $E_T=I-T_{10}\approx -0.00143166$ and $E_S=I-S_{10}\approx -0.00000095$.

(b) $f(x)=e^x\Rightarrow f''(x)=e^x\leq e$ for $0\leq x\leq 1$. Taking $K=e$, $a=0$, $b=1$, and $n=10$ in Theorem 3, we get $\left|E_T\right|\leq e(1)^3/\left(12\cdot 10^2\right)\approx 0.002265>0.00143166$. $f^{(4)}(x)=e^x<e$ for $0\leq x\leq 1$. Using Theorem 4, we have $\left|E_S\right|\leq e(1)^5/\left(180\cdot 10^4\right)\approx 0.0000015>0.00000095$. We see that the actual errors are about two–thirds the size of the error estimates.

(c) From part (b), we take $K=e$ to get $\left|E_T\right|\leq\frac{K(b-a)^3}{12n^2}\leq 0.00001\Rightarrow n^2\geq\frac{e\left(1^3\right)}{12(0.00001)}\Rightarrow n\geq 150.5$.

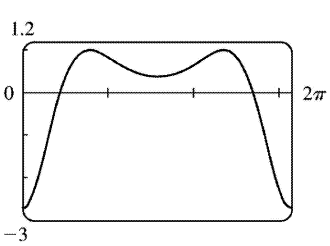
Take $n=151$ for T_n . Now $\left|E_M\right|\leq\frac{K(b-a)^3}{24n^2}\leq 0.00001\Rightarrow n\geq 106.4$. Take $n=107$ for M_n . Finally,

$\left|E_S\right|\leq\frac{K(b-a)^5}{180n^4}\leq 0.00001\Rightarrow n^4\geq\frac{e\left(1^5\right)}{180(0.00001)}\Rightarrow n\geq 6.23$. Take $n=8$ for S_n (since n has to be even for Simpson's Rule).

22. From Example 7(b), we take $K=76e$ to get $\left|E_S\right|\leq 76e(1)^5/\left(180n^4\right)\leq 0.00001\Rightarrow$

$n^4\geq 76e/[180(0.00001)]\Rightarrow n\geq 18.4$. Take $n=20$ (since n must be even).

23. **(a)** Using a CAS, we differentiate $f(x)=e^{\cos x}$ twice, and find that $f''(x)=e^{\cos x}\left(\sin^2 x-\cos x\right)$. From the graph, we see that the maximum value of $\left|f''(x)\right|$ occurs at the endpoints of the interval $[0,2\pi]$. Since $f''(0)=-e$, we can use $K=e$ or $K=2.8$.



(b) A CAS gives $M_{10}\approx 7.954926518$. (In Maple, use `student[middlesum]`.)

(c) Using Theorem 3 for the Midpoint Rule, with $K=e$, we get

$\left|E_M\right| \le \dfrac{e(2\pi-0)^3}{24\cdot 10^2} \approx 0.280945995$. With $K=2.8$, we get $\left|E_M\right| \le \dfrac{2.8(2\pi-0)^3}{24\cdot 10^2} = 0.289391916$.

(d) A CAS gives $I \approx 7.954926521$.

(e) The actual error is only about 3×10^{-9} , much less than the estimate in part (c).

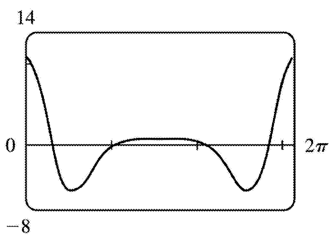
(f) We use the CAS to differentiate twice more, and then graph

$f^{(4)}(x) = e^{\cos x}\left(\sin^4 x - 6\sin^2 x\cos x + 3 - 7\sin^2 x + \cos x\right)$.

From the graph, we see that the maximum value of $\left|f^{(4)}(x)\right|$ occurs

at the endpoints of the interval $[0, 2\pi]$. Since $f^{(4)}(0) = 4e$, we can use

$K=4e$ or $K=10.9$.



(g) A CAS gives $S_{10} \approx 7.953789422$. (In Maple, use student[simpson]. .)

(h) Using Theorem 4 with $K=4e$, we get $\left|E_S\right| \le \dfrac{4e(2\pi-0)^5}{180\cdot 10^4} \approx 0.059153618$. With $K=10.9$, we get

$\left|E_S\right| \le \dfrac{10.9(2\pi-0)^5}{180\cdot 10^4} \approx 0.059299814$.

(i) The actual error is about $7.954926521 - 7.953789422 \approx 0.00114$. This is quite a bit smaller than the estimate in part (h), though the difference is not nearly as great as it was in the case of the Midpoint Rule.

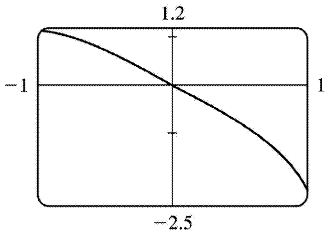
(j) To ensure that $\left|E_S\right| \le 0.0001$, we use Theorem 4: $\left|E_S\right| \le \dfrac{4e(2\pi)^5}{180\cdot n^4} \le 0.0001 \Rightarrow \dfrac{4e(2\pi)^5}{180\cdot 0.0001} \le n^4$

$\Rightarrow n^4 \ge 5,915,362 \Leftrightarrow n \ge 49.3$. So we must take $n \ge 50$ to ensure that $\left|I - S_n\right| \le 0.0001$. ($K=10.9$ leads to the same value of n .)

24. **(a)** Using the CAS, we differentiate $f(x) = \sqrt{4 - x^3}$ twice,

and find that $f''(x) = -\dfrac{9x^4}{4\left(4-x^3\right)^{3/2}} - \dfrac{3x}{\left(4-x^3\right)^{1/2}}$.

From the graph, we see that $\left|f''(x)\right|<2.2$ on $[-1,1]$.



(b) A CAS gives $M_{10}\approx 3.995804152$. (In Maple, use `student[middlesum]` .)

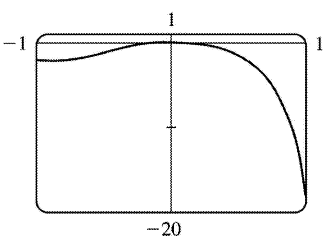
(c) Using Theorem 3 for the Midpoint Rule, with $K=2.2$, we get $\left|E_M\right|\leq \dfrac{2.2[1-(-1)]^3}{24\cdot 10^2}\approx 0.00733$.

(d) A CAS gives $I\approx 3.995487677$.

(e) The actual error is about -0.0003165 , much less than the estimate in part (c).

(f) We use the CAS to differentiate twice more, and then graph $f^{(4)}(x)=\dfrac{9}{16}\;\dfrac{x^2\left(x^6-224x^3-1280\right)}{\left(4-x^3\right)^{7/2}}$.

From the graph, we see that $\left|f^{(4)}(x)\right|<18.1$ on $[-1,1]$.



(g) A CAS gives $S_{10}\approx 3.995449790$. (In Maple, use `student[simpson]` .)

(h) Using Theorem 4 with $K=18.1$, we get $\left|E_S\right|\leq \dfrac{18.1[1-(-1)]^5}{180\cdot 10^4}\approx 0.000322$.

(i) The actual error is about $3.995487677-3.995449790\approx 0.0000379$. This is quite a bit smaller than the estimate in part (h).

(j) To ensure that $\left|E_S\right|\leq 0.0001$,we use Theorem 4: $\left|E_S\right|\leq \dfrac{18.1(2)^5}{180\cdot n^4}\leq 0.0001\Rightarrow \dfrac{18.1(2)^5}{180\cdot 0.0001}\leq n^4$

$\Rightarrow n^4\geq 32,178\Rightarrow n\geq 13.4$. So we must take $n\geq 14$ to ensure that $\left|I-S_n\right|\leq 0.0001$.

25. $I=\int_0^1 x^3\,dx=\left[\frac{1}{4}x^4\right]_0^1=0.25$. $f(x)=x^3$.

$n=4$:

$$L_4=\frac{1}{4}\left[0^3+\left(\frac{1}{4}\right)^3+\left(\frac{2}{4}\right)^3+\left(\frac{3}{4}\right)^3\right]=0.140625$$

$$R_4=\frac{1}{4}\left[\left(\frac{1}{4}\right)^3+\left(\frac{2}{4}\right)^3+\left(\frac{3}{4}\right)^3+1^3\right]=0.390625$$

$$T_4=\frac{1}{4\cdot 2}\left[0^3+2\left(\frac{1}{4}\right)^3+2\left(\frac{2}{4}\right)^3+2\left(\frac{3}{4}\right)^3+1^3\right]=0.265625,$$

$$M_4=\frac{1}{4}\left[\left(\frac{1}{8}\right)^3+\left(\frac{3}{8}\right)^3+\left(\frac{5}{8}\right)^3+\left(\frac{7}{8}\right)^3\right]=0.2421875,$$

$$E_L=I-L_4=\frac{1}{4}-0.140625=0.109375,\ E_R=\frac{1}{4}-0.390625=-0.140625,$$

$$E_T=\frac{1}{4}-0.265625=-0.015625,\ E_M=\frac{1}{4}-0.2421875=0.0078125$$

$n=8$:

$$L_8=\frac{1}{8}\left[f(0)+f\left(\frac{1}{8}\right)+f\left(\frac{2}{8}\right)+\cdots+f\left(\frac{7}{8}\right)\right]\approx 0.191406$$

$$R_8=\frac{1}{8}\left[f\left(\frac{1}{8}\right)+f\left(\frac{2}{8}\right)+\cdots+f\left(\frac{7}{8}\right)+f(1)\right]\approx 0.316406$$

$$T_8=\frac{1}{8\cdot 2}\left\{f(0)+2\left[f\left(\frac{1}{8}\right)+f\left(\frac{2}{8}\right)+\cdots+f\left(\frac{7}{8}\right)\right]+f(1)\right\}\approx 0.253906$$

$$M_8=\frac{1}{8}\left[f\left(\frac{1}{16}\right)+f\left(\frac{3}{16}\right)+\cdots+f\left(\frac{13}{16}\right)+f\left(\frac{15}{16}\right)\right]=0.248047$$

$$E_L\approx\frac{1}{4}-0.191406\approx 0.058594,\ E_R\approx\frac{1}{4}-0.316406\approx -0.066406,$$

$$E_T\approx\frac{1}{4}-0.253906\approx -0.003906,\ E_M\approx\frac{1}{4}-0.248047\approx 0.001953.$$

$n=16$:

$$L_{16}=\frac{1}{16}\left[f(0)+f\left(\frac{1}{16}\right)+f\left(\frac{2}{16}\right)+\cdots+f\left(\frac{15}{16}\right)\right]\approx 0.219727$$

$$R_{16}=\frac{1}{16}\left[f\left(\frac{1}{16}\right)+f\left(\frac{2}{16}\right)+\cdots+f\left(\frac{15}{16}\right)+f(1)\right]\approx 0.282227$$

$$T_{16}=\frac{1}{16\cdot 2}\left\{f(0)+2\left[f\left(\frac{1}{16}\right)+f\left(\frac{2}{16}\right)+\cdots+f\left(\frac{15}{16}\right)\right]+f(1)\right\}\approx 0.250977$$

$$M_{16}=\frac{1}{16}\left[f\left(\frac{1}{32}\right)+f\left(\frac{3}{32}\right)+\cdots+f\left(\frac{31}{32}\right)\right]\approx 0.249512$$

$$E_L\approx\frac{1}{4}-0.219727\approx 0.030273,\ E_R\approx\frac{1}{4}-0.282227\approx -0.032227,$$

$E_T \approx \frac{1}{4} - 0.250977 \approx -0.000977$, $E_M \approx \frac{1}{4} - 0.249512 \approx 0.000488$.

n	L_n	R_n	T_n	M_n
4	0.140625	0.390625	0.265625	0.242188
8	0.191406	0.316406	0.253906	0.248047
16	0.219727	0.282227	0.250977	0.249512

n	E_L	E_R	E_T	E_M
4	0.109375	−0.140625	−0.015625	0.007813
8	0.058594	−0.066406	−0.003906	0.001953
16	0.030273	−0.032227	−0.000977	0.000488

Observations:

(a) E_L and E_R are always opposite in sign, as are E_T and E_M .

(b) As n is doubled, E_L and E_R are decreased by about a factor of 2 , and E_T and E_M are decreased by a factor of about 4 .

(c) The Midpoint approximation is about twice as accurate as the Trapezoidal approximation.

(d) All the approximations become more accurate as the value of n increases.

(e) The Midpoint and Trapezoidal approximations are much more accurate than the endpoint approximations.

26. $\int_0^2 e^x dx = \left[e^x\right]_0^2 = e^2 - 1 \approx 6.389056$. $f(x) = e^x$

$n=4$:

$\Delta x = (2-0)/4 = \frac{1}{2}$

$L_4 = \frac{1}{2}\left[e^0 + e^{1/2} + e^1 + e^{3/2}\right] \approx 4.924346$

$R_4 = \frac{1}{2}\left[e^{1/2} + e^1 + e^{3/2} + e^2\right] \approx 8.118874$

$T_4 = \frac{1}{2 \cdot 2}\left[e^0 + 2e^{1/2} + 2e^1 + 2e^{3/2} + e^2\right] \approx 6.521610$

$M_4 = \frac{1}{2}\left[e^{1/4} + e^{3/4} + e^{5/4} + e^{7/4}\right] \approx 6.322986$.

$E_L \approx 6.389056 - 4.924346 \approx 1.464710$, $E_R \approx 6.389056 - 8.118874 = -1.729818$,

$E_T \approx 6.389056 - 6.521610 \approx -0.132554$, $E_M \approx 6.389056 - 6.322986 = 0.0660706$.

$n=8$:

$\Delta x = (2-0)/8 = \frac{1}{4}$

$L_8 = \frac{1}{4}\left[e^0 + e^{1/4} + e^{1/2} + e^{3/4} + e^1 + e^{5/4} + e^{3/2} + e^{7/4}\right] \approx 5.623666$

$R_8 = \frac{1}{4}\left[e^{1/4} + e^{1/2} + e^{3/4} + e^1 + e^{5/4} + e^{3/2} + e^{7/4} + e^2\right] \approx 7.220930$

$T_8 = \frac{1}{4 \cdot 2}\left[e^0 + 2e^{1/4} + 2e^{1/2} + 2e^{3/4} + 2e^1 + 2e^{5/4} + 2e^{3/2} + 2e^{7/4} + e^2\right] \approx 6.422298$

$M_8 = \frac{1}{4}\left[e^{1/8} + e^{3/8} + e^{5/8} + e^{7/8} + e^{9/8} + e^{11/8} + e^{13/8} + e^{15/8}\right] \approx 6.372448$

$E_L \approx 6.389056 - 5.623666 \approx 0.765390$, $E_R \approx 6.389056 - 7.220930 \approx -0.831874$,

$E_T \approx 6.389056 - 6.422298 \approx -0.033242$, $E_M \approx 6.389056 - 6.372448 \approx 0.016608$.

$n=16$:

$\Delta x = (2-0)/16 = \frac{1}{8}$

$L_{16} = \frac{1}{8}\left[f(0) + f\left(\frac{1}{8}\right) + f\left(\frac{2}{8}\right) + \cdots + f\left(\frac{14}{8}\right) + f\left(\frac{15}{8}\right)\right] \approx 5.998057$

$R_{16} = \frac{1}{8}\left[f\left(\frac{1}{8}\right) + f\left(\frac{2}{8}\right) + f\left(\frac{3}{8}\right) + \cdots + f\left(\frac{15}{8}\right) + f(2)\right] \approx 6.796689$

$T_{16} = \frac{1}{8 \cdot 2}\left(f(0) + 2\left[f\left(\frac{1}{8}\right) + f\left(\frac{2}{8}\right) + f\left(\frac{3}{8}\right) + \cdots + f\left(\frac{15}{8}\right)\right] + f(2)\right) \approx 6.397373$

$M_{16} = \frac{1}{8}\left[f\left(\frac{1}{16}\right) + f\left(\frac{3}{16}\right) + f\left(\frac{5}{16}\right) + \cdots + f\left(\frac{29}{16}\right) + f\left(\frac{31}{16}\right)\right] \approx 6.384899$

$E_L \approx 6.389056 - 5.998057 \approx 0.390999$, $E_R \approx 6.389056 - 6.796689 \approx -0.407633$,

$E_T \approx 6.389056 - 6.397373 \approx -0.008317$, $E_M \approx 6.389056 - 6.384899 \approx 0.004158$.

n	E_L	E_R	E_T	E_M
4	1.464710	−1.729818	−0.132554	0.066071
8	0.765390	−0.831874	−0.033242	0.016608
16	0.390999	−0.407633	−0.008317	0.004158

Observations:

(a) E_L and E_R are always opposite in sign, as are E_T and E_M .

(b) As n is doubled,

E_L and E_R are decreased by a factor of about 2 , and E_T and E_M are decreased by a factor of about 4 .
(c) The Midpoint approximation is about twice as accurate as the Trapezoidal approximation.
(d) All the approximations become more accurate as the value of n increases.
(e) The Midpoint and Trapezoidal approximations are much more accurate than the endpoint approximations.

27. $\int_1^4 \sqrt{x}\,dx = \left[\frac{2}{3} x^{3/2} \right]_1^4 = \frac{2}{3}(8-1) = \frac{14}{3} \approx 4.666667$

$n=6$:

$\Delta x = (4-1)/6 = \frac{1}{2}$

$T_6 = \frac{1}{2\cdot 2}\left[\sqrt{1}+2\sqrt{1.5}+2\sqrt{2}+2\sqrt{2.5}+2\sqrt{3}+2\sqrt{3.5}+\sqrt{4}\right] \approx 4.661488$

$M_6 = \frac{1}{2}\left[\sqrt{1.25}+\sqrt{1.75}+\sqrt{2.25}+\sqrt{2.75}+\sqrt{3.25}+\sqrt{3.75}\right] \approx 4.669245$

$S_6 = \frac{1}{2\cdot 3}\left[\sqrt{1}+4\sqrt{1.5}+2\sqrt{2}+4\sqrt{2.5}+2\sqrt{3}+4\sqrt{3.5}+\sqrt{4}\right] \approx 4.666563$

$E_T \approx \frac{14}{3} - 4.661488 \approx 0.005178$, $E_M \approx \frac{14}{3} - 4.669245 \approx -0.002578$,

$E_S \approx \frac{14}{3} - 4.666563 \approx 0.000104$.

$n=12$:

$\Delta x = (4-1)/12 = \frac{1}{4}$

$T_{12} = \frac{1}{4\cdot 2}\left(f(1)+2\left[f(1.25)+f(1.5)+\cdots+f(3.5)+f(3.75)\right]+f(4)\right) \approx 4.665367$

$M_{12} = \frac{1}{4}\left[f(1.125)+f(1.375)+f(1.625)+\cdots+f(3.875)\right] \approx 4.667316$

$S_{12} = \frac{1}{4\cdot 3} \approx 4.666659$

$E_T \approx \frac{14}{3} - 4.665367 \approx 0.001300$, $E_M \approx \frac{14}{3} - 4.667316 \approx -0.000649$,

$E_S \approx \frac{14}{3} - 4.666659 \approx 0.000007$.

Note: These errors were computed more precisely and then rounded to six places. That is, they were not computed by comparing the rounded values of T_n , M_n , and S_n with the rounded value of the actual integral.

n	T_n	M_n	S_n
6	4.661488	4.669245	4.666563
12	4.665367	4.667316	4.666659

n	E_T	E_M	E_S
6	0.005178	−0.002578	0.000104
12	0.001300	−0.000649	0.000007

Observations:

(a) E_T and E_M are opposite in sign and decrease by a factor of about 4 as n is doubled.

(b) The Simpson's approximation is much more accurate than the Midpoint and Trapezoidal approximations, and seems to decrease by a factor of about 16 as n is doubled.

28. $I=\int_{-1}^{2} xe^x dx=\left[xe^x-e^x\right]_{-1}^{2}=e^2+2/e\approx 8.124815$. $f(x)=xe^x$.

$n=6$:

$\Delta x=[2-(-1)]/6=\frac{1}{2}$

$T_6=\frac{1}{2\cdot 2}\ \{f(-1)+2\left[f(-0.5)+f(0)+\cdots+f(1.5)\right]+f(2)\}\approx 8.583514$

$M_6=\frac{1}{2}\ \left[f(-0.75)+f(-0.25)+\cdots+f(1.75)\right]\approx 7.896632$

$S_6=\frac{1}{2\cdot 3}\ \left[f(-1)+4f(-0.5)+2f(0)+4f(0.5)+2f(1)+4f(1.5)+f(2)\right]\approx 8.136885$

$E_T\approx I-8.583514\approx -0.458699$, $E_M\approx I-7.896632\approx 0.228183$,

$E_S\approx I-8.136885\approx -0.012070$.

$n=12$:

$\Delta x=[2-(-1)]/12=\frac{1}{4}$

$T_{12}=\frac{1}{4\cdot 2}\ \{f(-1)+2\left[f(-0.75)+f(-0.5)+\cdots+f(1.75)\right]+f(2)\}\approx 8.240073$

$M_{12}=\frac{1}{4}\ \left[f\left(-\frac{7}{8}\right)+f\left(-\frac{5}{8}\right)+\cdots+f\left(\frac{13}{8}\right)+f\left(\frac{15}{8}\right)\right]\approx 8.067259$

$S_{12}=\frac{1}{4\cdot 3}\ \left[f(-1)+4f(-0.75)+2f(-0.5)+\cdots+2f(1.5)+4f(1.75)+f(2)\right]\approx 8.125593$

$E_T\approx I-8.240073\approx -0.115258$, $E_M\approx I-8.067259\approx 0.057556$,

$E_S \approx I - 8.125593 \approx -0.000778$

n	T_n	M_n	S_n
6	8.583514	7.896632	8.136885
12	8.240073	8.067259	8.125593

n	E_T	E_M	E_S
6	−0.458699	0.228183	−0.012070
12	−0.115258	0.057556	−0.000778

Observations:

(a) E_T and E_M are opposite in sign and decrease by a factor of about 4 as n is doubled.

(b) The Simpson's approximation is much more accurate than the Midpoint and Trapezoidal approximations, and seems to decrease by a factor of about 16 as n is doubled.

29. $\Delta x = (4-0)/4 = 1$

(a) $T_4 = \frac{1}{2}\left[f(0)+2f(1)+2f(2)+2f(3)+f(4)\right] \approx \frac{1}{2}\left[0+2(3)+2(5)+2(3)+1\right] = 11.5$

(b) $M_4 = 1\cdot\left[f(0.5)+f(1.5)+f(2.5)+f(3.5)\right] \approx 1+4.5+4.5+2 = 12$

(c) $S_4 = \frac{1}{3}\left[f(0)+4f(1)+2f(2)+4f(3)+f(4)\right] \approx \frac{1}{3}\left[0+4(3)+2(5)+4(3)+1\right] = 11.\overline{6}$

30. If x= distance from left end of pool and $w=w(x)$= width at x , then Simpson's Rule with $n=8$ and $\Delta x=2$ gives Area $=\int_0^{16} w\,dx \approx \frac{2}{3}\left[0+4(6.2)+2(7.2)+4(6.8)+2(5.6)+4(5.0)+2(4.8)+4(4.8)+0\right] \approx 84\ \text{m}^2$.

31. **(a)** We are given the function values at the endpoints of 8 intervals of length 0.4, so we'll use the Midpoint Rule with $n=8/2=4$ and $\Delta x=(3.2-0)/4=0.8$.

$$\begin{aligned}\int_0^{3.2} f(x)\,dx \approx M_4 &= 0.8[f(0.4)+f(1.2)+f(2.0)+f(2.8)]\\ &= 0.8[6.5+6.4+7.6+8.8]\\ &= 0.8(29.3) = 23.44\end{aligned}$$

(b) $-4 \le f''(x) \le 1 \Rightarrow \left|f''(x)\right| \le 4$, so use $K=4$, $a=0$, $b=3.2$, and $n=4$ in Theorem 3. So

$$\left|E_M\right| \leq \frac{4(3.2-0)^3}{24(4)^2} = \frac{128}{375} = 0.341\overline{3}\ .$$

32. We use Simpson's Rule with $n=10$ and $\Delta x=\frac{1}{2}$:

$$\begin{aligned} \text{distance}=\int_0^5 \mathrm{v(t)\,dt} \approx \mathrm{S}_{10} &= \frac{1}{2\cdot 3}\left[f(0)+4f(0.5)+2f(1)+\cdots+4f(4.5)+f(5)\right] \\ &= \frac{1}{6} \\ &= \frac{1}{6}(268.41)=44.735\mathrm{m} \end{aligned}$$

33. By the Net Change Theorem, the increase in velocity is equal to $\int_0^6 a(t)\,dt$. We use Simpson's Rule with $n=6$ and $\Delta t=(6-0)/6=1$ to estimate this integral:

$$\begin{aligned} \int_0^6 a(t)\,dt \approx S_6 &= \frac{1}{3}[a(0)+4a(1)+2a(2)+4a(3)+2a(4)+4a(5)+a(6)] \\ &\approx \frac{1}{3}[0+4(0.5)+2(4.1)+4(9.8)+2(12.9)+4(9.5)+0]=\frac{1}{3}(113.2)=37.7\overline{3}\,ft/s \end{aligned}$$

34. By the Net Change Theorem, the total amount of water that leaked out during the first six hours is equal to $\int_0^6 r(t)\,dt$. We use Simpson's Rule with $n=6$ and $\Delta t=\frac{6-0}{6}=1$ to estimate this integral:

$$\begin{aligned} \int_0^6 \mathrm{r(t)\,dt} \approx \mathrm{S}_6 &= \frac{1}{3}\left[r(0)+4r(1)+2r(2)+4r(3)+2r(4)+4r(5)+r(6)\right] \\ &\approx \frac{1}{3}\left[4+4(3)+2(2.4)+4(1.9)+2(1.4)+4(1.1)+1\right] \\ &= \frac{1}{3}(36.6)=12.2\text{liters} \end{aligned}$$

The function values were obtained from a high–resolution graph.

35. By the Net Change Theorem, the energy used is equal to $\int_0^6 P(t)\,dt$. We use Simpson's Rule with $n=12$ and $\Delta t=(6-0)/12=\frac{1}{2}$ to estimate this integral:

$$\int_0^6 P(t)\,dt \approx S_{12} = \frac{1/2}{3}\,[P(0)+4P(0.5)+2P(1)+4P(1.5)+2P(2)+4P(2.5)]$$
$$+2P(3)+4P(3.5)+2P(4)+4P(4.5)+2P(5)+4P(5.5)+P(6)]$$
$$=\frac{1}{6}\,[1814+4(1735)+2(1686)+4(1646)+2(1637)+4(1609)+2(1604)]$$
$$+4(1611)+2(1621)+4(1666)+2(1745)+4(1886)+2052]$$
$$=\frac{1}{6}\,(61{,}064)=10{,}177.\overline{3} \text{ megawatt–hours.}$$

36. By the Net Change Theorem, the total amount of data transmitted is equal to $\int_0^8 D(t)\,dt \times 3600$. We use Simpson's Rule with $n=8$ and $\Delta t=(8-0)/8=1$ to estimate this integral:

$$\int_0^8 D(t)\,dt \approx S_8 = \frac{1}{3}\,[D(0)+4D(1)+2D(2)+4D(3)+2D(4)+4D(5)+2D(6)+4D(7)+D(8)]$$
$$\approx \frac{1}{3}\,[0.35+4(0.32)+2(0.41)+4(0.50)+2(0.51)+4(0.56)+2(0.56)+4(0.83)+0.88]$$
$$=\frac{1}{3}\,(13.03)=4.34\overline{3}$$

Now multiply by 3600 to obtain 15 , 636 megabits.

37. Let $y=f(x)$ denote the curve. Using cylindrical shells, $V=\int_2^{10} 2\pi x f(x)\,dx=2\pi\int_2^{10} x f(x)\,dx=2\pi I$. Now use Simpson's Rule to approximate I :

$$I\approx S_8 = \frac{10-2}{3(8)}\,[2f(2)+4\cdot 3f(3)+2\cdot 4f(4)+4\cdot 5f(5)+2\cdot 6f(6)$$
$$+4\cdot 7f(7)+2\cdot 8f(8)+4\cdot 9f(9)+10f(10)]$$
$$\approx \frac{1}{3}\,[2(0)+12(1.5)+8(1.9)+20(2.2)+12(3.0)+28(3.8)+16(4.0)+36(3.1)+10(0)]$$
$$=\frac{1}{3}\,(395.2)$$

Thus, $V\approx 2\pi\cdot \frac{1}{3}\,(395.2)\approx 827.7$ or 828 cubic units.

38.

$$\text{Work } =\int_0^{18} f(x)dx\approx S_6=\frac{18-0}{6\cdot 3}\,[f(0)+4f(3)+2f(6)+4f(9)+2f(12)+4f(15)+f(18)]$$
$$=1\cdot[9.8+4(9.1)+2(8.5)+4(8.0)+2(7.7)+4(7.5)+7.4]=148 \text{ joules}$$

39. Volume $=\pi\int_0^2\left(\sqrt[3]{1+x^3}\right)^2 dx=\pi\int_0^2\left(1+x^3\right)^{2/3} dx$. $V\approx\pi\cdot S_{10}$ where $f(x)=\left(1+x^3\right)^{2/3}$ and $\Delta x=(2-0)/10=\frac{1}{5}$. Therefore,

$$V \approx \pi\cdot S_{10}=\pi\,\frac{1}{5\cdot 3}\,[f(0)+4f(0.2)+2f(0.4)+4f(0.6)+2f(0.8)+4f(1)$$
$$+2f(1.2)+4f(1.4)+2f(1.6)+4f(1.8)+f(2)] \approx 12.325078$$

40. Using Simpson's Rule with $n=10$, $\Delta x=\frac{\pi/2}{10}$, $L=1$, $\theta_0=\frac{42\pi}{180}$ radians, $g=9.8\ \mathrm{m/s}^2$, $k^2=\sin^2\left(\frac{1}{2}\theta_0\right)$, and $f(x)=1/\sqrt{1-k^2\sin^2 x}$, we get

$$T = 4\sqrt{\frac{L}{g}}\int_0^{\pi/2}\frac{dx}{\sqrt{1-k^2\sin^2 x}}\approx 4\sqrt{\frac{L}{g}}\,S_{10}$$
$$= 4\sqrt{\frac{1}{9.8}}\left(\frac{\pi/2}{10\cdot 3}\right)\left[f(0)+4f\left(\frac{\pi}{20}\right)+2f\left(\frac{2\pi}{20}\right)+\cdots+4f\left(\frac{9\pi}{20}\right)+f\left(\frac{\pi}{2}\right)\right]\approx 2.07665$$

41. $I(\theta)=\frac{N^2\sin^2 k}{k^2}$, where $k=\frac{\pi Nd\sin\theta}{\lambda}$, $N=10,000$, $d=10^{-4}$, and $\lambda=632.8\times 10^{-9}$. So $I(\theta)=\frac{\left(10^4\right)^2\sin^2 k}{k^2}$, where $k=\frac{\pi\left(10^4\right)\left(10^{-4}\right)\sin\theta}{632.8\times 10^{-9}}$. Now $n=10$ and $\Delta\theta=\frac{10^{-6}-\left(-10^{-6}\right)}{10}=2\times 10^{-7}$, so $M_{10}=2\times 10^{-7}[I(-0.0000009)+I(-0.0000007)+\cdots+I(0.0000009)]\approx 59.4$.

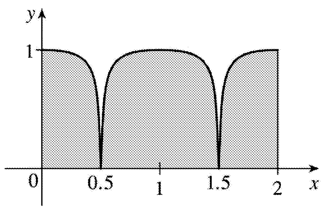
42. $f(x)=\cos(\pi x)$, $\Delta x=\frac{20-0}{10}=2\Rightarrow$

$$\begin{aligned} T_{10} &= \frac{2}{2}\,\{f(0)+2[f(2)+f(4)+\cdots+f(18)]+f(20)\}\\ &= 1[\cos 0+2(\cos 2\pi+\cos 4\pi+\cdots+\cos 18\pi)+\cos 20\pi]\\ &= 1+2(1+1+1+1+1+1+1+1+1)+1=20 \end{aligned}$$

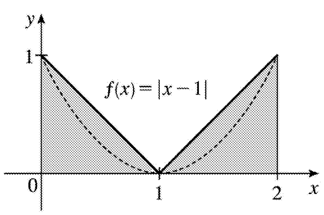
The actual value is $\int_0^{20}\cos(\pi x)\,dx=\frac{1}{\pi}\left[\sin\pi x\right]_0^{20}=\frac{1}{\pi}(\sin 20\pi-\sin 0)=0$. The discrepancy is due to the fact that the function is sampled only at points of the form $2n$, where its value is $f(2n)=\cos(2n\pi)=1$.

43. Consider the function f whose graph is shown. The area

$\int_0^2 f(x)dx$ is close to 2 . The Trapezoidal Rule gives $T_2=\frac{2-0}{2\cdot 2}\left[f(0)+2f(1)+f(2)\right]=\frac{1}{2}\left[1+2\cdot 1+1\right]=2$. The Midpoint Rule gives $M_2=\frac{2-0}{2}\left[f(0.5)+f(1.5)\right]=1[0+0]=0$,
so the Trapezoidal Rule is more accurate.



44. Consider the function $f(x)=|x-1|$, $0\leq x\leq 2$. The area $\int_0^2 f(x)dx$ is exactly 1 . So is the right endpoint approximation: $R_2=f(1)\Delta x+f(2)\Delta x=0\cdot 1+1\cdot 1=1$. But Simpson's Rule approximates f with the parabola $y=(x-1)^2$, shown dashed, and $S_2=\frac{\Delta x}{3}\left[f(0)+4f(1)+f(2)\right]=\frac{1}{3}\left[1+4\cdot 0+1\right]=\frac{2}{3}$.



45. Since the Trapezoidal and Midpoint approximations on the interval $[a,b]$ are the sums of the Trapezoidal and Midpoint approximations on the subintervals $\left[x_{i-1},x_i\right]$, $i=1,2,\ldots,n$, we can focus our attention on one such interval. The condition $f''(x)<0$ for $a\leq x\leq b$ means that the graph of f is concave down as in Figure 5. In that figure, T_n is the area of the trapezoid $AQRD$, $\int_a^b f(x)dx$ is the area of the region $AQPRD$, and M_n is the area of the trapezoid $ABCD$, so $T_n<\int_a^b f(x)dx<M_n$. In general, the condition $f''<0$ implies that the graph of f on $[a,b]$ lies above the chord joining the points $(a,f(a))$ and $(b,f(b))$. Thus, $\int_a^b f(x)dx>T_n$. Since M_n is the area under a tangent to the graph, and since $f''<0$ implies that the tangent lies above the graph, we also have $M_n>\int_a^b f(x)dx$. Thus, $T_n<\int_a^b f(x)dx<M_n$.

46. Let f be a polynomial of degree ≤ 3 ; say $f(x)=Ax^3+Bx^2+Cx+D$. It will suffice to show that Simpson's estimate is exact when there are two subintervals ($n=2$), because for a larger even number

of subintervals the sum of exact estimates is exact. As in the derivation of Simpson's Rule, we can assume that $x_0=-h$, $x_1=0$, and $x_2=h$. Then Simpson's approximation is

$$\begin{aligned}\int_{-h}^{h} f(x)dx &\approx \frac{1}{3}\, h[f(-h)+4f(0)+f(h)]\\ &= \frac{1}{3}\, h\left[\left(-Ah^3+Bh^2-Ch+D\right)+4D+\left(Ah^3+Bh^2+Ch+D\right)\right]\\ &= \frac{1}{3}\, h\left[2Bh^2+6D\right]=\frac{2}{3}\, Bh^3+2Dh\end{aligned}$$

The exact value of the integral is

$$\begin{aligned}\int_{-h}^{h}\left(Ax^3+Bx^2+Cx+D\right) dx &= 2\int_0^h\left(Bx^2+D\right)dx \quad \text{[by Theorem 5.5. (a) and (b)]}\\ &= 2\left[\frac{1}{3}\, Bx^3+Dx\right]_0^h=\frac{2}{3}\, Bh^3+2Dh\end{aligned}$$

Thus, Simpson's Rule is exact.

47. $T_n=\frac{1}{2}\,\Delta x\left[f\left(x_0\right)+2f\left(x_1\right)+\cdots+2f\left(x_{n-1}\right)+f\left(x_n\right)\right]$ and $M_n=\Delta x\left[f\left(\bar{x}_1\right)+f\left(\bar{x}_2\right)+\cdots+f\left(\bar{x}_{n-1}\right)+f\left(\bar{x}_n\right)\right]$, where $\bar{x}_i=\frac{1}{2}\left(x_{i-1}+x_i\right)$. Now

$$\begin{aligned}T_{2n} = \frac{1}{2}\left(\frac{1}{2}\,\Delta x\right)&\left[f\left(x_0\right)+2f\left(\bar{x}_1\right)+2f\left(x_1\right)+2f\left(\bar{x}_2\right)+2f\left(x_2\right)+\cdots\right.\\ &\left.+2f\left(\bar{x}_{n-1}\right)+2f\left(x_{n-1}\right)+2f\left(\bar{x}_n\right)+f\left(x_n\right)\right]\end{aligned}$$

$$\begin{aligned}\text{so } \frac{1}{2}\left(T_n+M_n\right) &= \frac{1}{2}\, T_n+\frac{1}{2}\, M_n\\ &= \frac{1}{4}\,\Delta x\left[f\left(x_0\right)+2f\left(x_1\right)+\cdots+2f\left(x_{n-1}\right)+f\left(x_n\right)\right]\\ &\quad+\frac{1}{4}\,\Delta x\left[2f\left(\bar{x}_1\right)+2f\left(\bar{x}_2\right)+\cdots+2f\left(\bar{x}_{n-1}\right)+2f\left(\bar{x}_n\right)\right]\\ &= T_{2n}\end{aligned}$$

48. $T_n=\frac{\Delta x}{2}\left[f\left(x_0\right)+2\sum_{i=1}^{n-1} f\left(x_i\right)+f\left(x_n\right)\right]$ and $M_n=\Delta x\sum_{i=1}^{n} f\left(x_i-\frac{\Delta x}{2}\right)$, so

$$\frac{1}{3}\, T_n+\frac{2}{3}\, M_n=\frac{1}{3}\left(T_n+2M_n\right)=\frac{\Delta x}{3\cdot 2}\left[f\left(x_0\right)+2\sum_{i=1}^{n-1} f\left(x_i\right)+f\left(x_n\right)+4\sum_{i=1}^{n} f\left(x_i-\frac{\Delta x}{2}\right)\right]$$

where

$\Delta x=\dfrac{b-a}{n}$. Let $\delta x=\dfrac{b-a}{2n}$. Then $\Delta x=2\delta x$, so

$$\begin{aligned}\frac{1}{3}T_n+\frac{2}{3}M_n &= \frac{\delta x}{3}\left[f\left(x_0\right)+2\sum_{i=1}^{n-1}f\left(x_i\right)+f\left(x_n\right)+4\sum_{i=1}^{n}f\left(x_i-\delta x\right)\right]\\ &= \frac{1}{3}\delta x\left[f\left(x_0\right)+4f\left(x_1-\delta x\right)+2f\left(x_1\right)+4f\left(x_2-\delta x\right)\right.\\ &\quad \left.+2f\left(x_2\right)+\cdots+2f\left(x_{n-1}\right)+4f\left(x_n-\delta x\right)+f\left(x_n\right)\right]\end{aligned}$$

Since $x_0, x_1-\delta x, x_1, x_2-\delta x, x_2, \ldots, x_{n-1}, x_n-\delta x, x_n$ are the subinterval endpoints for S_{2n} , and since $\delta x=\dfrac{b-a}{2n}$ is the width of the subintervals for S_{2n} , the last expression for $\dfrac{1}{3}T_n+\dfrac{2}{3}M_n$ is the usual expression for S_{2n} . Therefore, $\dfrac{1}{3}T_n+\dfrac{2}{3}M_n=S_{2n}$.

1. **(a)** Since $\int_1^\infty x^4 e^{-x^4}\,dx$ has an infinite interval of integration, it is an improper integral of Type I.

(b) Since $y=\sec x$ has an infinite discontinuity at $x=\frac{\pi}{2}$, $\int_0^{\pi/2} \sec x\,dx$ is a Type II improper integral.

(c) Since $y=\frac{x}{(x-2)(x-3)}$ has an infinite discontinuity at $x=2$, $\int_0^2 \frac{x}{x^2-5x+6}\,dx$ is a Type II improper integral.

(d) Since $\int_{-\infty}^0 \frac{1}{x^2+5}\,dx$ has an infinite interval of integration, it is an improper integral of Type I.

2. **(a)** Since $y=1/(2x-1)$ is defined and continuous on $[1,2]$, the integral is proper.

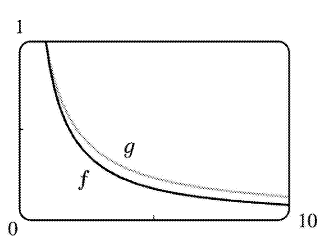
(b) Since $y=\frac{1}{2x-1}$ has an infinite discontinuity at $x=\frac{1}{2}$, $\int_0^1 \frac{1}{2x-1}\,dx$ is a Type II improper integral.

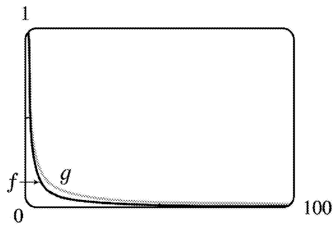
(c) Since $\int_{-\infty}^\infty \frac{\sin x}{1+x^2}\,dx$ has an infinite interval of integration, it is an improper integral of Type I.

(d) Since $y=\ln(x-1)$ has an infinite discontinuity at $x=1$, $\int_1^2 \ln(x-1)\,dx$ is a Type II improper integral.

3. The area under the graph of $y=1/x^3=x^{-3}$ between $x=1$ and $x=t$ is

$A(t)=\int_1^t x^{-3}\,dx=\left[-\frac{1}{2}x^{-2}\right]_1^t=-\frac{1}{2}t^{-2}-\left(-\frac{1}{2}\right)=\frac{1}{2}-1\Big/\left(2t^2\right)$. So the area for $1\le x\le 10$ is $A(10)=0.5-0.005=0.495$, the area for $1\le x\le 100$ is $A(100)=0.5-0.00005=0.49995$, and the area for $1\le x\le 1000$ is $A(1000)=0.5-0.0000005=0.4999995$. The total area under the curve for $x\ge 1$ is $\lim_{t\to\infty} A(t)=\lim_{t\to\infty}\left[\frac{1}{2}-1\Big/\left(2t^2\right)\right]=\frac{1}{2}$.

4. **(a)** 



(b) The area under the graph of f from $x=1$ to $x=t$ is

$$F(t) = \int_1^t f(x)\,dx = \int_1^t x^{-1.1}\,dx = \left[-\frac{1}{0.1}\, x^{-0.1} \right]_1^t$$

$$= -10\left(t^{-0.1}-1\right) = 10\left(1-t^{-0.1}\right)$$

and the area under the graph of g is

$$G(t) = \int_1^t g(x)\,dx = \int_1^t x^{-0.9}\,dx = \left[\frac{1}{0.1}\, x^{0.1} \right]_1^t$$

$$= 10\left(t^{0.1}-1\right)$$

t	$F(t)$	$G(t)$
10	2.06	2.59
100	3.69	5.85
10^4	6.02	15.12
10^6	7.49	29.81
10^{10}	9	90
10^{20}	9.9	990

(c) The total area under the graph of f is $\lim_{t\to\infty} F(t) = \lim_{t\to\infty} 10\left(1-t^{-0.1}\right) = 10$.

The total area under the graph of g does not exist, since $\lim_{t\to\infty} G(t) = \lim_{t\to\infty} 10\left(t^{0.1}-1\right) = \infty$.

5. $I = \int_1^\infty \frac{1}{(3x+1)^2}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{(3x+1)^2}\,dx$. Now

$\int \frac{1}{(3x+1)^2}\,dx = \frac{1}{3} \int \frac{1}{u^2}\,du$ [$u=3x+1$, $du=3\,dx$]

$$=-\frac{1}{3u}+C=-\frac{1}{3(3x+1)}+C\ ,$$

so $I=\lim_{t\to\infty}\left[-\frac{1}{3(3x+1)}\right]_1^t=\lim_{t\to\infty}\left[-\frac{1}{3(3t+1)}+\frac{1}{12}\right]=0+\frac{1}{12}=\frac{1}{12}$. Convergent

6. $\int_{-\infty}^{0}\frac{1}{2x-5}\,dx=\lim_{t\to-\infty}\int_t^0\frac{1}{2x-5}\,dx=\lim_{t\to-\infty}\left[\frac{1}{2}\ln|2x-5|\right]_t^0=\lim_{t\to\infty}\left[\frac{1}{2}\ln 5-\frac{1}{2}\ln|2t-5|\right]=-\infty$.
Divergent

7.
$$\int_{-\infty}^{-1}\frac{1}{\sqrt{2-w}}\,dw=\lim_{t\to-\infty}\int_t^{-1}\frac{1}{\sqrt{2-w}}\,dw=\lim_{t\to-\infty}\left[-2\sqrt{2-w}\right]_t^{-1}$$
$$=\lim_{t\to-\infty}\left[-2\sqrt{3}+2\sqrt{2-t}\right]=\infty\ \text{. Divergent}$$

8.
$$\int_0^{\infty}\frac{x}{(x^2+2)^2}\,dx=\lim_{t\to\infty}\int_0^t\frac{x}{(x^2+2)^2}\,dx=\lim_{t\to\infty}\frac{1}{2}\left[\frac{-1}{x^2+2}\right]_0^t=\frac{1}{2}\lim_{t\to\infty}\left(\frac{-1}{t^2+2}+\frac{1}{2}\right)$$
$$=\frac{1}{2}\left(0+\frac{1}{2}\right)=\frac{1}{4}\ \text{. Convergent}$$

9. $\int_4^{\infty}e^{-y/2}dy=\lim_{t\to\infty}\int_4^t e^{-y/2}dy=\lim_{t\to\infty}\left[-2e^{-y/2}\right]_4^t=\lim_{t\to\infty}(-2e^{-t/2}+2e^{-2})=0+2e^{-2}=2e^{-2}$.

Convergent

10. $\int_{-\infty}^{-1}e^{-2t}dt=\lim_{x\to-\infty}\int_x^{-1}e^{-2t}dt=\lim_{x\to-\infty}\left[-\frac{1}{2}e^{-2t}\right]_x^{-1}=\lim_{x\to-\infty}\left[-\frac{1}{2}e^2+\frac{1}{2}e^{-2x}\right]=\infty$. Divergent

11. $\int_{-\infty}^{\infty}\frac{x}{dx}1+x^2=\int_{-\infty}^{0}\frac{x}{dx}1+x^2+\int_0^{\infty}\frac{x}{dx}1+x^2$ and

$\int_{-\infty}^{0}\frac{x}{dx}1+x^2=\lim_{t\to-\infty}\left[\frac{1}{2}\ln\left(1+x^2\right)\right]_{\mathrm{t}}^0=\lim_{t\to-\infty}\left[0-\frac{1}{2}\ln\left(1+t^2\right)\right]=-\infty$. Divergent

12. $I=\int_{-\infty}^{\infty}\left(2-v^4\right)dv=I_1+I_2=\int_{-\infty}^{0}\left(2-v^4\right)dv+\int_{0}^{\infty}\left(2-v^4\right)dv$, but
$I_1=\lim_{t\to-\infty}\left[2v-\frac{1}{5}v^5\right]_t^0=\lim_{t\to-\infty}\left(-2t+\frac{1}{5}t^5\right)=-\infty$. Since I_1 is divergent, I is divergent, and there is no need to evaluate I_2 . Divergent

13. $\int_{-\infty}^{\infty}xe^{-x^2}dx=\int_{-\infty}^{0}xe^{-x^2}dx+\int_{0}^{\infty}xe^{-x^2}dx$.
$\int_{-\infty}^{0}xe^{-x^2}dx=\lim_{t\to-\infty}\left(-\frac{1}{2}\right)\left[e^{-x^2}\right]_t^0=\lim_{t\to\infty}\left(-\frac{1}{2}\right)\left(e^{-t^2}-1\right)=-\frac{1}{2}\cdot 1=-\frac{1}{2}$, and
$\int_{0}^{\infty}xe^{-x^2}dx=\lim_{t\to\infty}\left(-\frac{1}{2}\right)\left[e^{-x^2}\right]_0^t=\lim_{t\to\infty}\left(-\frac{1}{2}\right)\left(e^{-t^2}-1\right)=-\frac{1}{2}\cdot(-1)=\frac{1}{2}$.
Therefore, $\int_{-\infty}^{\infty}xe^{-x^2}dx=-\frac{1}{2}+\frac{1}{2}=0$. Convergent

14. $\int_{-\infty}^{\infty}x^2e^{-x^3}dx=\int_{-\infty}^{0}x^2e^{-x^3}dx+\int_{0}^{\infty}x^2e^{-x^3}dx$, and
$\int_{-\infty}^{0}x^2e^{-x^3}dx=\lim_{t\to-\infty}\left[-\frac{1}{3}e^{-x^3}\right]_t^0=-\frac{1}{3}+\frac{1}{3}\left(\lim_{t\to-\infty}e^{-t^3}\right)=\infty$. Divergent

15. $\int_{2\pi}^{\infty}\sin\theta\,d\theta=\lim_{t\to\infty}\int_{2\pi}^{t}\sin\theta\,d\theta=\lim_{t\to\infty}\left[-\cos\theta\right]_{2\pi}^{t}=\lim_{t\to\infty}(-\cos t+1)$. This limit does not exist, so the integral is divergent. Divergent

16. $\int_0^{\infty}\cos^2\alpha\,d\alpha=\lim_{t\to\infty}\int_0^t\frac{1}{2}(1+\cos 2\alpha)d\alpha=\lim_{t\to\infty}\left[\frac{1}{2}\alpha+\frac{1}{4}\sin 2\alpha\right]_0^t=\lim_{t\to\infty}\left[\frac{1}{2}t+\frac{1}{4}\sin 2t\right]=\infty$ since $\left|\frac{1}{4}\sin 2t\right|\le\frac{1}{4}$ for all t , but $\frac{1}{2}t\to\infty$ as $t\to\infty$. Divergent

17.
$$\int_1^{\infty}\frac{x+1}{x^2+2x}dx=\lim_{t\to\infty}\int_1^t\frac{\frac{1}{2}(2x+2)}{x^2+2x}dx=\frac{1}{2}\lim_{t\to\infty}\left[\ln(x^2+2x)\right]_1^t=\frac{1}{2}\lim_{t\to\infty}\left[\ln(t^2+2t)-\ln 3\right]$$
$=\infty$. Divergent

18.

$$\int_0^{\infty} \frac{dz}{z^2+3z+2} = \lim_{t\to\infty} \int_0^t \left[\frac{1}{z+1} - \frac{1}{z+2} \right] dz = \lim_{t\to\infty} \left[\ln \left(\frac{z+1}{z+2} \right) \right]_0^t$$

$$= \lim_{t\to\infty} \left[\ln \left(\frac{t+1}{t+2} \right) - \ln \left(\frac{1}{2} \right) \right] = \ln 1 + \ln 2 = \ln 2 \text{ . Convergent}$$

19.

$$\int_0^{\infty} se^{-5s} ds = \lim_{t\to\infty} \int_0^t se^{-5s} ds = \lim_{t\to\infty} \left[-\frac{1}{5} se^{-5s} - \frac{1}{25} e^{-5s} \right]_0^t \left[\begin{array}{c} \text{by integration by} \\ \text{parts with } u=s \end{array} \right]$$

$$= \lim_{t\to\infty} \left(-\frac{1}{5} te^{-5t} - \frac{1}{25} e^{-5t} + \frac{1}{25} \right) = 0 - 0 + \frac{1}{25} \text{ [by l'Hospital's Rule]}$$

$$= \frac{1}{25} \text{ . Convergent}$$

20.

$$\int_{-\infty}^{6} re^{r/3} dr = \lim_{t\to-\infty} \int_t^6 re^{r/3} dr = \lim_{t\to-\infty} \left[3re^{r/3} - 9e^{r/3} \right]_t^6 \left[\begin{array}{c} \text{by integration } by \\ \text{parts with } u=r \end{array} \right]$$

$$= \lim_{t\to-\infty} (18e^2 - 9e^2 - 3te^{t/3} + 9e^{t/3}) = 9e^2 - 0 + 0 \text{ [by l'Hospital's Rule]}$$

$$= 9e^2 \text{ . Convergent}$$

21. $\int_1^{\infty} \frac{\ln x}{x} dx = \lim_{t\to\infty} \left[\frac{(\ln x)^2}{2} \right]_1^t$ (by substitution with $u = \ln x$, $du = dx/x$) $= \lim_{t\to\infty} \frac{(\ln t)^2}{2} = \infty$. Divergent

22. $\int_{-\infty}^{\infty} e^{-|x|} dx = \int_{-\infty}^{0} e^{x} dx + \int_0^{\infty} e^{-x} dx$, $\int_{-\infty}^{0} e^{x} dx = \lim_{t\to-\infty} \left[e^{x} \right]_t^0 = \lim_{t\to-\infty} \left(1 - e^{t} \right) = 1$, and $\int_0^{\infty} e^{-x} dx = \lim_{t\to\infty} \left[-e^{-x} \right]_0^t = \lim_{t\to\infty} \left(1 - e^{-t} \right) = 1$. Therefore, $\int_{-\infty}^{\infty} e^{-|x|} dx = 1 + 1 = 2$. Convergent

23.

$$\int_{-\infty}^{\infty} \frac{x^2}{9+x^6}\,dx = \int_{-\infty}^{0} \frac{x^2}{9+x^6}\,dx + \int_{0}^{\infty} \frac{x^2}{9+x^6}\,dx = 2\int_{0}^{\infty} \frac{x^2}{9+x^6}\,dx \text{ [since the integrand is even].}$$

Now $\int \frac{x^2\,dx}{9+x^6} \begin{bmatrix} u=x^3 \\ du=3x^2\,dx \end{bmatrix} = \int \frac{\frac{1}{3}\,du}{9+u^2} \begin{bmatrix} u=3v \\ du=3\,dv \end{bmatrix} = \int \frac{\frac{1}{3}(3\,dv)}{9+9v^2} = \frac{1}{9}\int \frac{dv}{1+v^2}$

$$= \frac{1}{9}\tan^{-1} v + C = \frac{1}{9}\tan^{-1}\left(\frac{u}{3}\right) + C = \frac{1}{9}\tan^{-1}\left(\frac{x^3}{3}\right) + C\,,$$

so $2\int_{0}^{\infty} \frac{x^2}{9+x^6}\,dx = 2\lim_{t\to\infty} \int_{0}^{t} \frac{x^2}{9+x^6}\,dx = 2\lim_{t\to\infty} \left[\frac{1}{9}\tan^{-1}\left(\frac{x^3}{3}\right)\right]_0^t$

$$= 2\lim_{t\to\infty} \frac{1}{9}\tan^{-1}\left(\frac{t^3}{3}\right) = \frac{2}{9}\cdot\frac{\pi}{2} = \frac{\pi}{9}$$. Convergent

24. Integrate by parts with $u=\ln x$, $dv=dx/x^3 \Rightarrow du=dx/x$, $v=-1/\left(2x^2\right)$.

$$\int_{1}^{\infty} \frac{\ln x}{x^3}\,dx = \lim_{t\to\infty} \int_{1}^{t} \frac{\ln x}{x^3}\,dx = \lim_{t\to\infty} \left(\left[-\frac{1}{2x^2}\ln x\right]_1^t + \frac{1}{2}\int_{1}^{t} \frac{1}{x^3}\,dx\right)$$

$$= \lim_{t\to\infty} \left(-\frac{1}{2}\,\frac{\ln t}{t^2} + 0 - \frac{1}{4t^2} + \frac{1}{4}\right) = \frac{1}{4}$$

since $\lim_{t\to\infty} \frac{\ln t}{t^2} = \lim_{t\to\infty} \frac{1/t}{2t} = \lim_{t\to\infty} \frac{1}{2t^2} = 0$. Convergent

25. Integrate by parts with $u=\ln x$, $dv=dx/x^2 \Rightarrow du=dx/x$, $v=-1/x$.

$$\int_{1}^{\infty} \frac{\ln x}{x^2}\,dx = \lim_{t\to\infty} \int_{1}^{t} \frac{\ln x}{x^2}\,dx = \lim_{t\to\infty} \left[-\frac{\ln x}{x} - \frac{1}{x}\right]_1^t = \lim_{t\to\infty} \left(-\frac{\ln t}{t} - \frac{1}{t} + 0 + 1\right)$$

$$= -0 - 0 + 0 + 1 = 1$$

since $\lim_{t\to\infty} \frac{\ln t}{t} = \lim_{t\to\infty} \frac{1/t}{1} = 0$. Convergent

26. $\displaystyle\int_0^{\infty} \frac{x\arctan x}{(1+x^2)^2}\,dx = \lim_{t\to\infty}\int_0^{t} \frac{x\arctan x}{(1+x^2)^2}\,dx$. Let $u=\arctan x$, $dv=\dfrac{x\,dx}{(1+x^2)^2}$. Then $du=\dfrac{dx}{1+x^2}$,
$v=\dfrac{1}{2}\displaystyle\int \frac{2x\,dx}{(1+x^2)^2} = \frac{-1/2}{1+x^2}$, and

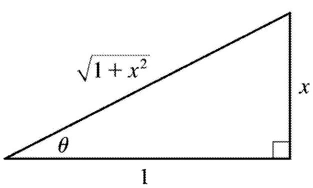
$$\int \frac{x\arctan x}{(1+x^2)^2}\,dx = -\frac{1}{2}\,\frac{\arctan x}{1+x^2} + \frac{1}{2}\int \frac{dx}{(1+x^2)^2} \quad \left[\begin{array}{l} x=\tan\theta, \\ dx=\sec^2\theta\,d\theta \end{array}\right]$$

$$= -\frac{1}{2}\,\frac{\arctan x}{1+x^2} + \frac{1}{2}\int \frac{\sec^2\theta\,d\theta}{(\sec^2\theta)^2}$$

$$= -\frac{1}{2}\,\frac{\arctan x}{1+x^2} + \frac{1}{2}\int \cos^2\theta\,d\theta$$

$$= -\frac{1}{2}\,\frac{\arctan x}{1+x^2} + \frac{\theta}{4} + \frac{\sin\theta\cos\theta}{4} + C$$

$$= -\frac{1}{2}\,\frac{\arctan x}{1+x^2} + \frac{1}{4}\arctan x + \frac{1}{4}\,\frac{x}{1+x^2} + C$$



It follows that

$$\int_0^{\infty} \frac{x\arctan x}{(1+x^2)^2}\,dx = \lim_{t\to\infty}\left[-\frac{1}{2}\,\frac{\arctan x}{1+x^2} + \frac{1}{4}\arctan x + \frac{1}{4}\,\frac{x}{1+x^2}\right]_0^t$$

$$= \lim_{t\to\infty}\left(-\frac{1}{2}\,\frac{\arctan t}{1+t^2} + \frac{1}{4}\arctan t + \frac{1}{4}\,\frac{t}{1+t^2}\right) = 0 + \frac{1}{4}\cdot\frac{\pi}{2} + 0 = \frac{\pi}{8} \ .$$

Convergent.

27. There is an infinite discontinuity at the left endpoint of $[0,3]$.

$$\int_0^3 \frac{dx}{\sqrt{x}} = \lim_{t\to 0^+} \int_t^3 \frac{dx}{\sqrt{x}} = \lim_{t\to 0^+} \left[2\sqrt{x}\right]_t^3 = \lim_{t\to 0^+} \left(2\sqrt{3} - 2\sqrt{t}\right) = 2\sqrt{3}$$. Convergent

28. There is an infinite discontinuity at the left endpoint of $[0,3]$.

$$\int_0^3 \frac{dx}{x\sqrt{x}} = \lim_{t\to 0^+} \int_t^3 \frac{dx}{\sqrt{x^{3/2}}} = \lim_{t\to 0^+} \left[\frac{-2}{\sqrt{x}} \right]_t^3 = \frac{-2}{\sqrt{3}} + \lim_{t\to 0^+} \frac{2}{\sqrt{t}} = \infty$$. Divergent

29. There is an infinite discontinuity at the right endpoint of $[-1,0]$.

$$\int_{-1}^0 \frac{dx}{x^2} = \lim_{t\to 0^-} \int_{-1}^t \frac{dx}{x^2} = \lim_{t\to 0^-} \left[\frac{-1}{x} \right]_{-1}^t = \lim_{t\to 0^-} \left[-\frac{1}{t} + \frac{1}{-1} \right] = \infty$$. Divergent

30. $$\int_1^9 \frac{dx}{\sqrt[3]{x-9}} = \lim_{t\to 9^-} \int_1^t \frac{}{dx} \sqrt[3]{x-9} = \lim_{t\to 9^-} \left[\frac{3}{2}(x-9)^{2/3} \right]_1^t = \lim_{t\to 9^-} \left[\frac{3}{2}(t-9)^{2/3} - \frac{3}{2}(4) \right] = 0-6 = -6$$. Convergent

31. $$\int_{-2}^3 \frac{}{dx} x^4 = \int_{-2}^0 \frac{dx}{x^4} + \int_0^3 \frac{dx}{x^4}$$, but $$\int_{-2}^0 \frac{dx}{x^4} = \lim_{t\to 0^-} \left[-\frac{x^{-3}}{3} \right]_{-2}^t = \lim_{t\to 0^-} \left[-\frac{1}{3t^3} - \frac{1}{24} \right] = \infty$$. Divergent

32. $$\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \lim_{t\to 1^-} \int_0^t \frac{dx}{\sqrt{1-x^2}} = \lim_{t\to 1^-} \left[\sin^{-1} x\right]_0^t = \lim_{t\to 1^-} \sin^{-1} t = \frac{\pi}{2}$$. Convergent

33. There is an infinite discontinuity at $x=1$. $\int_0^{33} (x-1)^{-1/5} dx = \int_0^1 (x-1)^{-1/5} dx + \int_1^{33} (x-1)^{-1/5} dx$. Here

$$\int_0^1 (x-1)^{-1/5} dx = \lim_{t\to 1^-} \int_0^t (x-1)^{-1/5} dx = \lim_{t\to 1^-} \left[\frac{5}{4}(x-1)^{4/5} \right]_0^t = \lim_{t\to 1^-} \left[\frac{5}{4}(t-1)^{4/5} - \frac{5}{4} \right] = -\frac{5}{4}$$ and

$\int_1^{33}(x-1)^{-1/5}dx=\lim_{t\to 1^+}\int_t^{33}(x-1)^{-1/5}dx=\lim_{t\to 1^+}\left[\frac{5}{4}(x-1)^{4/5}\right]_t^{33}=\lim_{t\to 1^+}\left[\frac{5}{4}\cdot 16-\frac{5}{4}(t-1)^{4/5}\right]=20$. Thus,

$\int_0^{33}(x-1)^{-1/5}dx=-\frac{5}{4}+20=\frac{75}{4}$. Convergent

34. $f(y)=1/(4y-1)$ has an infinite discontinuity at $y=\frac{1}{4}$.

$$\int_{1/4}^{1}\frac{1}{4y-1}\,dy=\lim_{t\to(1/4)^+}\int_t^1\frac{1}{4y-1}\,dy=\lim_{t\to(1/4)^+}\left[\frac{1}{4}\ln|4y-1|\right]_t^1$$

$$=\lim_{t\to(1/4)^+}\left[\frac{1}{4}\ln 3-\frac{1}{4}\ln(4t-1)\right]=\infty$$

so $\int_{1/4}^{1}\frac{1}{4y-1}\,dy$ diverges, and hence, $\int_0^1\frac{1}{4y-1}\,dy$ diverges. Divergent

35.

$$\int_0^{\pi}\sec x\,dx=\int_0^{\pi/2}\sec x\,dx+\int_{\pi/2}^{\pi}\sec x\,dx\ .\ \int_0^{\pi/2}\sec x\,dx=\lim_{t\to\pi/2^-}\int_0^t\sec x\,dx$$

$$=\lim_{t\to\pi/2^-}\left[\ln|\sec x+\tan x|\right]_0^t=\lim_{t\to\pi/2^-}\ln|\sec t+\tan t|=\infty$$. Divergent

36. $\int_0^4\frac{dx}{x^2+x-6}=\int_0^4\frac{dx}{(x+3)(x-2)}=\int_0^2\frac{dx}{(x-2)(x+3)}+\int_2^4\frac{dx}{(x-2)(x+3)}$, and

$$\int_0^2\frac{dx}{(x-2)(x+3)}=\lim_{t\to 2^-}\int_0^t\left[\frac{1/5}{x-2}-\frac{1/5}{x+3}\right]dx\ \text{[partial fractions]}=\lim_{t\to 2^-}\left[\frac{1}{5}\ln\left|\frac{x-2}{x+3}\right|\right]_0^t$$

$$=\lim_{t\to 2^-}\frac{1}{5}\left[\ln\left|\frac{t-2}{t+3}\right|-\ln\frac{2}{3}\right]=-\infty$$ .Divergent

37. There is an infinite discontinuity at $x=0$. $\int_{-1}^{1}\frac{e^x}{e^x-1}\,dx=\int_{-1}^{0}\frac{e^x}{e^x-1}\,dx+\int_0^1\frac{e^x}{e^x-1}\,dx$.

$$\int_{-1}^{0} \frac{e^x}{e^x-1}\,dx=\lim_{t\to 0^-}\int_{-1}^{t} \frac{e^x}{e^x-1}\,dx=\lim_{t\to 0^-}\Big[\ln\left|e^x-1\right|\Big]_{-1}^{t}=\lim_{t\to 0^-}\Big[\ln\left|e^t-1\right|-\ln\left|e^{-1}-1\right|\Big]=-\infty\ ,$$

so $\int_{-1}^{1} \frac{e^x}{e^x-1}\,dx$ is divergent. The integral $\int_{0}^{1} \frac{e^x}{e^x-1}\,dx$ also diverges since

$$\int_{0}^{1} \frac{e^x}{e^x-1}\,dx=\lim_{t\to 0^+}\int_{t}^{1} \frac{e^x}{e^x-1}\,dx=\lim_{t\to 0^+}\Big[\ln\left|e^x-1\right|\Big]_{t}^{1}=\lim_{t\to 0^+}\Big[\ln\left|e-1\right|-\ln\left|e^t-1\right|\Big]=\infty\ .$$

Divergent

38. $\int_{0}^{2} \frac{x-3}{2x-3}\,dx=\int_{0}^{3/2} \frac{x-3}{2x-3}\,dx+\int_{3/2}^{2} \frac{x-3}{2x-3}\,dx$ and

$$\int \frac{x-3}{2x-3}\,dx=\frac{1}{2}\int \frac{2x-6}{2x-3}\,dx=\frac{1}{2}\int\left[1-\frac{3}{2x-3}\right]dx=\frac{1}{2}\,x-\frac{3}{4}\ln\left|2x-3\right|+C\ ,\text{ so}$$

$$\int_{0}^{3/2} \frac{x-3}{2x-3}\,dx=\lim_{t\to 3/2^-}\frac{1}{4}\,\big[2x-3\ln\left|2x-3\right|\big]_{0}^{t}=\infty\ .\text{ Divergent}$$

39.

$$I=\int_{0}^{2} z^2\ln z\,dz=\lim_{t\to 0^+}\int_{t}^{2} z^2\ln z\,dz=\lim_{t\to 0^+}\left[\frac{z^3}{3^2}\,(3\ln z-1)\right]_{t}^{2}$$

$$=\lim_{t\to 0^+}\left[\frac{8}{9}\,(3\ln 2-1)-\frac{1}{9}\,t^3(3\ln t-1)\right]=\frac{8}{3}\ln 2-\frac{8}{9}-\frac{1}{9}\lim_{t\to 0^+}\left[t^3(3\ln t-1)\right]=\frac{8}{3}\ln 2-\frac{8}{9}-\frac{1}{9}\,L\ .$$

Now $L=\lim_{t\to 0^+}\left[t^3(3\ln t-1)\right]=\lim_{t\to 0^+}\frac{3\ln t-1}{t^{-3}}=\lim_{t\to 0^+}\frac{3/t}{-3/t^4}=\lim_{t\to 0^+}\left(-t^3\right)=0$. Thus, $L=0$ and $I=\frac{8}{3}\ln 2-\frac{8}{9}$.

Convergent

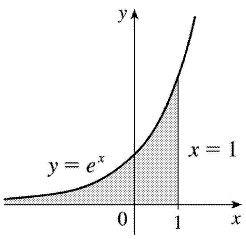
40. Integrate by parts with $u=\ln x$, $dv=dx\,/\sqrt{x}\ \Rightarrow\ du=dx/x$, $v=2\sqrt{x}$.

$$\int_{0}^{1} \frac{\ln x}{\sqrt{x}}\,dx=\lim_{t\to 0^+}\int_{t}^{1} \frac{\ln x}{\sqrt{x}}\,dx=\lim_{t\to 0^+}\left(\left[2\sqrt{x}\ln x\right]_{t}^{1}-2\int_{t}^{1}\frac{}{dx}\,\sqrt{x}\right)=\lim_{t\to 0^+}\left(-2\sqrt{t}\ln t-4\left[\sqrt{x}\,\right]_{t}^{1}\right)$$

$$=\lim_{t\to 0^+}\left(-2\sqrt{t}\ln t-4+4\sqrt{t}\,\right)=-4$$

since $\lim\limits_{t\to 0^+}\sqrt{t}\ln t=\lim\limits_{t\to 0^+}\dfrac{\ln t}{t^{-1/2}}=\lim\limits_{t\to 0^+}\dfrac{1/t}{-t^{-3/2}/2}=\lim\limits_{t\to 0^+}\left(-2\sqrt{t}\right)=0$. Convergent

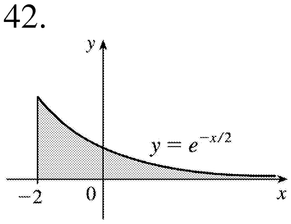
41.



$$\text{Area }=\int_{-\infty}^{1} e^{x}\,dx=\lim_{t\to-\infty}\left[e^{x}\right]_t^1$$

$$=e-\lim_{t\to-\infty}e^{t}=e$$

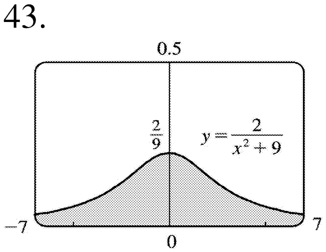
42.



$$\text{Area }=\int_{-2}^{\infty} e^{-x/2}\,dx=-2\lim_{t\to\infty}\left[e^{-x/2}\right]_{-2}^{t}$$

$$=-2\lim_{t\to\infty}e^{-t/2}+2e=2e$$

43.

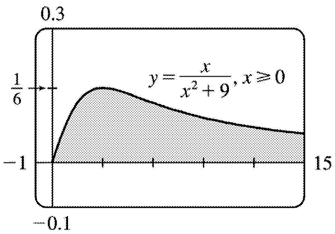


$$\text{Area }=\int_{-\infty}^{\infty}\frac{2}{x^{2}+9}\,dx=2\cdot 2\int_{0}^{\infty}\frac{1}{x^{2}+9}\,dx$$

$$=4\lim_{t\to\infty}\int_0^t \frac{1}{x^2+9}\,dx=4\lim_{t\to\infty}\left[\frac{1}{3}\tan^{-1}\frac{x}{3}\right]_0^t$$

$$=\frac{4}{3}\lim_{t\to\infty}\left[\tan^{-1}\frac{t}{3}-0\right]=\frac{4}{3}\cdot\frac{\pi}{2}=\frac{2\pi}{3}$$

44.



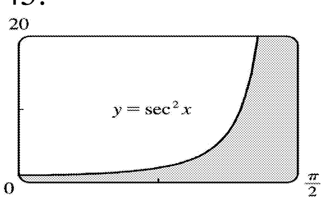
$$\text{Area }=\int_0^\infty \frac{x}{x^2+9}\,dx=\lim_{t\to\infty}\int_0^t \frac{x}{x^2+9}\,dx$$

$$=\lim_{t\to\infty}\left[\frac{1}{2}\ln\left(x^2+9\right)\right]_0^t$$

$$=\frac{1}{2}\lim_{t\to\infty}\left[\ln\left(t^2+9\right)-\ln 9\right]=\infty$$

Infinite area

45.



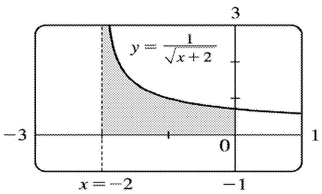
$$\text{Area }=\int_0^{\pi/2}\sec^2 x\,dx=\lim_{t\to(\pi/2)^-}\int_0^t\sec^2 x\,dx$$

$$=\lim_{t\to(\pi/2)^-}\left[\tan x\right]_0^t=\lim_{t\to(\pi/2)^-}(\tan t-0)$$

$$=\infty$$

Infinite area

46.



$$\text{Area} = \int_{-2}^{0} \frac{1}{\sqrt{x+2}}\,dx = \lim_{t\to -2^+} \int_{t}^{0} \frac{1}{\sqrt{x+2}}\,dx$$

$$= \lim_{t\to -2^+} \left[2\sqrt{x+2}\,\right]_t^0 = \lim_{t\to -2^+} \left(2\sqrt{2} - 2\sqrt{t+2}\,\right)$$

$$= 2\sqrt{2} - 0 = 2\sqrt{2}$$

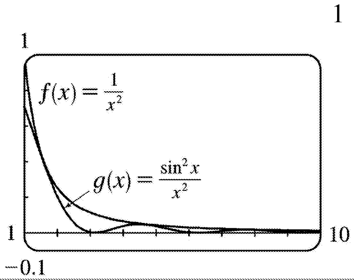
47. **(a)**

t	$\int_1^t g(x)\,dx$
2	0.447453
5	0.577101
10	0.621306
100	0.668479
1000	0.672957
10 , 000	0.673407

$g(x) = \frac{\sin^2 x}{x^2}$. It appears that the integral is convergent.

(b) $-1 \le \sin x \le 1 \Rightarrow 0 \le \sin^2 x \le 1 \Rightarrow 0 \le \frac{\sin^2 x}{x^2} \le \frac{1}{x^2}$. Since $\int_1^\infty \frac{1}{x^2}\,dx$ is convergent

(Equation 2 with $p=2>1$), $\int_1^\infty \frac{\sin^2 x}{x^2}\,dx$ is convergent by the Comparison Theorem.



(c)

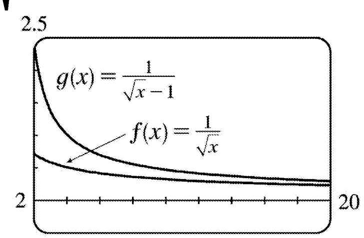
Since $\int_1^{\infty} f(x)dx$ is finite and the area under $g(x)$ is less than the area under $f(x)$ on any interval $[1,t]$, $\int_1^{\infty} g(x)dx$ must be finite; that is, the integral is convergent.

48. **(a)**

t	$\int_2^t g(x)dx$
5	3.830327
10	6.801200
100	23.328769
1000	69.023361
10 , 000	208.124560

$g(x)=\dfrac{1}{\sqrt{x}-1}$. It appears that the integral is divergent.

(b) For $x\geq 2$, $\sqrt{x}>\sqrt{x}-1\Rightarrow \dfrac{1}{\sqrt{x}}<\dfrac{1}{\sqrt{x}-1}$. Since $\int_2^{\infty} \dfrac{1}{\sqrt{x}}\,dx$ is divergent (Equation 2 with $p=\dfrac{1}{2}\leq 1$), $\int_2^{\infty} \dfrac{1}{\sqrt{x}-1}\,dx$ is divergent by the Comparison Theorem.

(c) 

Since $\int_2^{\infty} f(x)dx$ is infinite and the area under $g(x)$ is greater than the area under $f(x)$ on any interval $[2,t]$, $\int_2^{\infty} g(x)dx$ must be infinite; that is, the integral is divergent.

49. For $x\geq 1$, $\dfrac{\cos^2 x}{1+x^2}\leq \dfrac{1}{1+x^2}<\dfrac{1}{x^2}$. $\int_1^{\infty} \dfrac{1}{x^2}\,dx$ is convergent by Equation 2 with $p=2>1$, so

$\int_1^{\infty} \frac{\cos^2 x}{1+x^2}\, dx$ is convergent by the Comparison Theorem.

50. For $x \geq 1$, $\frac{2+e^{-x}}{x} > \frac{2}{x}$ [since $e^{-x} > 0$] $> \frac{1}{x}$. $\int_1^{\infty} \frac{1}{x}\, dx$ is divergent by Equation 2 with $p=1 \leq 1$, so $\int_1^{\infty} \frac{2+e^{-x}}{x}\, dx$ is divergent by the Comparison Theorem.

51. For $x \geq 1$, $x+e^{2x} > e^{2x} > 0 \Rightarrow \frac{1}{x+e^{2x}} \leq \frac{1}{e^{2x}} = e^{-2x}$ on $[1,\infty)$.

$\int_1^{\infty} e^{-2x}\, dx = \lim_{t\to\infty} \left[-\frac{1}{2} e^{-2x} \right]_1^t = \lim_{t\to\infty} \left[-\frac{1}{2} e^{-2t} + \frac{1}{2} e^{-2} \right] = \frac{1}{2} e^{-2}$. Therefore, $\int_1^{\infty} e^{-2x}\, dx$ is convergent, and by the Comparison Theorem, $\int_1^{\infty} \frac{}{dx}\, x+e^{2x}$ is also convergent.

52. For $x \geq 1$, $0 < \frac{x}{\sqrt{1+x^6}} < \frac{x}{\sqrt{x^6}} = \frac{x}{x^3} = \frac{1}{x^2}$. $\int_1^{\infty} \frac{1}{x^2}\, dx$ is convergent by Equation 2 with $p=2>1$, so

$$\int_1^{\infty} \frac{x}{\sqrt{1+x^6}}\, dx$$ is convergent by the Comparison Theorem.

53. $\frac{1}{x \sin x} \geq \frac{1}{x}$ on $\left(0, \frac{\pi}{2}\right]$ since $0 \leq \sin x \leq 1$. $\int_0^{\pi/2} \frac{dx}{x} = \lim_{t\to 0^+} \int_t^{\pi/2} \frac{dx}{x} = \lim_{t\to 0^+} [\ln x]_t^{\pi/2}$.

But $\ln t \to -\infty$ as $t \to 0^+$, so $\int_0^{\pi/2} \frac{dx}{x}$ is divergent, and by the Comparison Theorem, $\int_0^{\pi/2} \frac{dx}{x \sin x}$ is also divergent.

54. For $0 \leq x \leq 1$, $e^{-x} \leq 1 \Rightarrow \frac{e^{-x}}{\sqrt{x}} \leq \frac{1}{\sqrt{x}}$.

$\int_0^1 \frac{1}{\sqrt{x}}\, dx = \lim_{t\to 0^+} \int_t^1 \frac{1}{\sqrt{x}}\, dx = \lim_{t\to 0^+} \left[2\sqrt{x}\,\right]_t^1 = \lim_{t\to 0^+} \left(2-2\sqrt{t}\,\right) = 2$ is convergent. Therefore, $\int_0^1 \frac{e^{-x}}{\sqrt{x}}\, dx$ is convergent by the Comparison Theorem.

55. $\int_0^\infty \frac{dx}{\sqrt{x}\,(1+x)} = \int_0^1 \frac{dx}{\sqrt{x}\,(1+x)} + \int_1^\infty \frac{dx}{\sqrt{x}\,(1+x)} = \lim_{t\to 0^+} \int_t^1 \frac{dx}{\sqrt{x}\,(1+x)} + \lim_{t\to\infty} \int_1^t \frac{dx}{\sqrt{x}\,(1+x)}$. Now

$$\int \frac{dx}{\sqrt{x}\,(1+x)} = \int \frac{2u\,du}{u\left(1+u^2\right)} \quad [\, u=\sqrt{x}, x=u^2, dx=2u\,du \,]$$

$$=2\int \frac{du}{1+u^2} = 2\tan^{-1} u + C = 2\tan^{-1}\sqrt{x} + C\ ,$$

so

$$\int_0^\infty \frac{dx}{\sqrt{x}\,(1+x)} = \lim_{t\to 0^+} \left[2\tan^{-1}\sqrt{x}\,\right]_t^1 + \lim_{t\to\infty} \left[2\tan^{-1}\sqrt{x}\,\right]_1^t$$

$$= \lim_{t\to 0^+} \left[\, 2\left(\frac{\pi}{4}\right) - 2\tan^{-1}\sqrt{t}\,\right] + \lim_{t\to\infty} \left[\, 2\tan^{-1}\sqrt{t} - 2\left(\frac{\pi}{4}\right)\right] = \frac{\pi}{2} - 0 + 2\left(\frac{\pi}{2}\right) - \frac{\pi}{2} = \pi$$

.

56. $\int_2^\infty \frac{dx}{x\sqrt{x^2-4}} = \int_2^3 \frac{dx}{x\sqrt{x^2-4}} + \int_3^\infty \frac{dx}{x\sqrt{x^2-4}} = \lim_{t\to 2^+} \int_t^3 \frac{dx}{x\sqrt{x^2-4}} + \lim_{t\to\infty} \int_3^t \frac{dx}{x\sqrt{x^2-4}}$. Now

$$\int \frac{dx}{x\sqrt{x^2-4}} = \int \frac{2\sec\theta\,\tan\theta\, d\theta}{2\sec\theta\;2\tan\theta} \quad [\, x=2\sec\theta\ , \text{ where } 0\le\theta<\pi/2 \text{ or } \pi\le\theta<3\pi/2 \,]$$

$$= \frac{1}{2}\theta + C = \frac{1}{2}\sec^{-1}\left(\frac{1}{2}x\right) + C\ , \text{ so}$$

$$\int_2^\infty \frac{dx}{x\sqrt{x^2-4}} = \lim_{t\to 2^+} \left[\frac{1}{2}\sec^{-1}\left(\frac{1}{2}x\right)\right]_t^3 + \lim_{t\to\infty} \left[\frac{1}{2}\sec^{-1}\left(\frac{1}{2}x\right)\right]_3^t$$

$$=\frac{1}{2}\sec^{-1}\left(\frac{3}{2}\right)-0+\frac{1}{2}\left(\frac{\pi}{2}\right)-\frac{1}{2}\sec^{-1}\left(\frac{3}{2}\right)=\frac{\pi}{4}$$

57. If $p=1$, then $\int_0^1 \frac{}{dx}\, x^p=\lim_{t\to 0^+}\int_t^1 \frac{}{dx}\, x=\lim_{t\to 0^+}\left[\ln x\right]_t^1=\infty$. Divergent.

If $p\neq 1$, then

$\int_0^1 \frac{}{dx}\, x^p=\lim_{t\to 0^+}\int_t^1 \frac{}{dx}\, x^p$(note that the integral is not improper if $p<0$)

$$=\lim_{t\to 0^+}\left[\frac{x^{-p+1}}{-p+1}\right]_t^1=\lim_{t\to 0^+}\frac{1}{1-p}\left[1-\frac{1}{t^{p-1}}\right]$$

If $p>1$, then $p-1>0$, so $\frac{1}{t^{p-1}}\to\infty$ as $t\to 0^+$, and the integral diverges.

If $p<1$, then $p-1<0$, so $\frac{1}{t^{p-1}}\to 0$ as $t\to 0^+$ and $\int_0^1 \frac{}{dx}\, x^p=\frac{1}{1-p}\left[\lim_{t\to 0^+}\left(1-t^{1-p}\right)\right]=\frac{1}{1-p}$.

Thus, the integral converges if and only if $p<1$, and in that case its value is $\frac{1}{1-p}$.

58. Let $u=\ln x$. Then $du=dx/x\Rightarrow\int_e^\infty \frac{dx}{x(\ln x)^p}=\int_1^\infty \frac{du}{u^p}$. By Example 4, this converges to $\frac{1}{p-1}$ if $p>1$ and diverges otherwise.

59. First suppose $p=-1$. Then

$\int_0^1 x^p\ln x\,dx=\int_0^1 \frac{\ln x}{x}\,dx=\lim_{t\to 0^+}\int_t^1 \frac{\ln x}{x}\,dx=\lim_{t\to 0^+}\left[\frac{1}{2}(\ln x)^2\right]_t^1=-\frac{1}{2}\lim_{t\to 0^+}(\ln t)^2=-\infty$, so the integral diverges.

Now suppose $p\neq -1$. Then integration by parts gives

$\int x^p\ln x\,dx=\frac{x^{p+1}}{p+1}\ln x-\int\frac{x^p}{p+1}\,dx=\frac{x^{p+1}}{p+1}\ln x-\frac{x^{p+1}}{(p+1)^2}+C$. If $p<-1$, then $p+1<0$, so

$\int_0^1 x^p\ln x\,dx=\lim_{t\to 0^+}\left[\frac{x^{p+1}}{p+1}\ln x-\frac{x^{p+1}}{(p+1)^2}\right]_t^1=\frac{-1}{(p+1)^2}-\left(\frac{1}{p+1}\right)\lim_{t\to 0^+}\left[t^{p+1}\left(\ln t-\frac{1}{p+1}\right)\right]=\infty$.

If $p>-1$, then $p+1>0$ and

$$\int_0^1 x^p \ln x\,dx = \frac{-1}{(p+1)^2} - \left(\frac{1}{p+1}\right)\lim_{t\to 0^+}\frac{\ln t - 1/(p+1)}{t^{-(p+1)}} = \frac{-1}{(p+1)^2} - \left(\frac{1}{p+1}\right)\lim_{t\to 0^+}\frac{1/t}{-(p+1)t^{-(p+2)}}$$

$$= \frac{-1}{(p+1)^2} + \frac{1}{(p+1)^2}\lim_{t\to 0^+} t^{p+1} = \frac{-1}{(p+1)^2}$$

Thus, the integral converges to $-\frac{1}{(p+1)^2}$ if $p>-1$ and diverges otherwise.

60. **(a)**

$n=0$:

$$\int_0^\infty x^n e^{-x}dx = \lim_{t\to\infty}\int_0^t e^{-x}dx = \lim_{t\to\infty}\left[-e^{-x}\right]_0^t$$

$$= \lim_{t\to\infty}\left[-e^{-t}+1\right] = 0+1=1$$

$n=1$:

$\int_0^\infty x^n e^{-x}dx = \lim_{t\to\infty}\int_0^t xe^{-x}dx$. To evaluate $\int xe^{-x}dx$, we'll use integration by parts

with $u=x$, $dv=e^{-x}dx \Rightarrow du=dx$, $v=-e^{-x}$.

So $\int xe^{-x}dx = -xe^{-x} - \int -e^{-x}dx = -xe^{-x} - e^{-x} + C = (-x-1)e^{-x} + C$ and

$$\lim_{t\to\infty}\int_0^t xe^{-x}dx = \lim_{t\to\infty}\left[(-x-1)e^{-x}\right]_0^t$$

$$= \lim_{t\to\infty}\left[(-t-1)e^{-t}+1\right] = \lim_{t\to\infty}\left[-te^{-t}-e^{-t}+1\right]$$

$= 0-0+1$ [use l'Hospital's Rule] $=1$

$n=2$:

$\int_0^\infty x^n e^{-x}dx = \lim_{t\to\infty}\int_0^t x^2 e^{-x}dx$. To evaluate $\int x^2 e^{-x}dx$, we could use integration by parts

again or Formula 97. Thus,

$$\lim_{t\to\infty}\int_0^t x^2 e^{-x}dx = \lim_{t\to\infty}\left[-x^2 e^{-x}\right]_0^t + 2\lim_{t\to\infty}\int_0^t xe^{-x}dx$$

$= 0+0+2(1)$ [use l'Hospital's Rule and the result for $n=1$] $=2$

$n=3$:

$$\int_0^\infty x^n e^{-x}dx = \lim_{t\to\infty}\int_0^t x^3 e^{-x}dx = \lim_{t\to\infty}\left[-x^3e^{-x}\right]_0^t + 3\lim_{t\to\infty}\int_0^t x^2e^{-x}dx$$

$=0+0+3(2)$ [use l'Hospital's Rule and the result for $n=2$] $=6$

(b) For $n=1$, 2 , and 3 , we have $\int_0^\infty x^n e^{-x}dx=1$, 2 , and 6 . The values for the integral are equal to the factorials for n , so we guess $\int_0^\infty x^n e^{-x}dx=n!$.

(c) Suppose that $\int_0^\infty x^k e^{-x}dx=k!$ for some positive integer k . Then $\int_0^\infty x^{k+1}e^{-x}dx=\lim_{t\to\infty}\int_0^t x^{k+1}e^{-x}dx$. To evaluate $\int x^{k+1}e^{-x}dx$, we use parts with $u=x^{k+1}$, $dv=e^{-x}dx \Rightarrow du=(k+1)x^k dx$, $v=-e^{-x}$. So $\int x^{k+1}e^{-x}dx=-x^{k+1}e^{-x}-\int -(k+1)x^ke^{-x}dx=-x^{k+1}e^{-x}+(k+1)\int x^ke^{-x}dx$ and

$$\lim_{t\to\infty}\int_0^t x^{k+1}e^{-x}dx = \lim_{t\to\infty}\left[-x^{k+1}e^{-x}\right]_0^t + (k+1)\lim_{t\to\infty}\int_0^t x^ke^{-x}dx$$

$$=\lim_{t\to\infty}\left[-t^{k+1}e^{-t}+0\right]+(k+1)k!=0+0+(k+1)!=(k+1)!\ ,$$

so the formula holds for $k+1$. By induction, the formula holds for all positive integers. (Since $0!=1$, the formula holds for $n=0$, too.)

61. **(a)** $I=\int_{-\infty}^{\infty} x\,dx=\int_{-\infty}^{0} x\,dx+\int_0^\infty x\,dx$, and $\int_0^\infty x\,dx=\lim_{t\to\infty}\int_0^t x\,dx=\lim_{t\to\infty}\left[\frac{1}{2}x^2\right]_0^t=\lim_{t\to\infty}\left[\frac{1}{2}t^2-0\right]=\infty$, so I is divergent.

(b) $\int_{-t}^t x\,dx=\left[\frac{1}{2}x^2\right]_{-t}^t=\frac{1}{2}t^2-\frac{1}{2}t^2=0$, so $\lim_{t\to\infty}\int_{-t}^t x\,dx=0$. Therefore, $\int_{-\infty}^\infty x\,dx\neq\lim_{t\to\infty}\int_{-t}^t x\,dx$.

62. Let $k=\dfrac{M}{2RT}$ so that $\bar{v}=\dfrac{4}{\sqrt{\pi}}k^{3/2}\int_0^\infty v^3e^{-kv^2}dv$. Let I denote the integral and use parts to integrate I . Let $\alpha=v^2$, $d\beta=ve^{-kv^2}dv \Rightarrow d\alpha=2v\,dv$, $\beta=-\dfrac{1}{2k}e^{-kv^2}$:

$$I = \lim_{t\to\infty}\left[-\frac{1}{2k}v^2e^{-kv^2}\right]_0^t + \frac{1}{k}\int_0^\infty ve^{-kv^2}dv = -\frac{1}{2k}\lim_{t\to\infty}\left(t^2e^{-kt^2}\right) + \frac{1}{k}\lim_{t\to\infty}\left[-\frac{1}{2k}e^{-kv^2}\right]_0^t$$
$$= -\frac{1}{2k}\cdot 0 - \frac{1}{2k^2}(0-1) = \frac{1}{2k^2}$$

Thus, $\bar{v} = \frac{4}{\sqrt{\pi}}k^{3/2}\cdot\frac{1}{2k^2} = \frac{2}{(k\pi)^{1/2}} = \frac{2}{[\pi M/(2RT)]^{1/2}} = \frac{2\sqrt{2}\sqrt{RT}}{\sqrt{\pi M}} = \sqrt{\frac{8RT}{\pi M}}$.

63. Volume $= \int_1^\infty \pi\left(\frac{1}{x}\right)^2 dx = \pi\lim_{t\to\infty}\int_1^t \frac{}{dx}x^2 = \pi\lim_{t\to\infty}\left[-\frac{1}{x}\right]_1^t = \pi\lim_{t\to\infty}\left(1-\frac{1}{t}\right) = \pi < \infty$.

64. Work $= \int_R^\infty \frac{GMm}{r^2}dr = \lim_{t\to\infty}\int_R^t \frac{GMm}{r^2}dr = \lim_{t\to\infty} GMm\left[\frac{-1}{r}\right]_R^t = GMm_{t\to\infty}\left(\frac{-1}{t}+\frac{1}{R}\right) = \frac{GMm}{R}$, where M= mass of Earth $=5.98\times 10^{24}$ kg, m= mass of satellite $=10^3$ kg, R= radius of Earth $=6.37\times 10^6$ m, and G= gravitational constant $=6.67\times 10^{-11}$ N $\cdot$ m^2/ kg.

Therefore, Work $= \frac{6.67\times 10^{-11}\cdot 5.98\times 10^{24}\cdot 10^3}{6.37\times 10^6} \approx 6.26\times 10^{10}$ J.

65. Work $= \int_R^\infty F\,dr = \lim_{t\to\infty}\int_R^t \frac{GmM}{r^2}dr = \lim_{t\to\infty} GmM\left(\frac{1}{R}-\frac{1}{t}\right) = \frac{GmM}{R}$. The initial kinetic energy provides the work, so $\frac{1}{2}mv_0^2 = \frac{GmM}{R} \Rightarrow v_0 = \sqrt{\frac{2GM}{R}}$.

66. $y(s) = \int_s^R \frac{2r}{\sqrt{r^2-s^2}}x(r)dr$ and $x(r) = \frac{1}{2}(R-r)^2 \Rightarrow$

$$y(s) = \lim_{t\to s^+}\int_t^R \frac{r(R-r)^2}{\sqrt{r^2-s^2}}\,dr = \lim_{t\to s^+}\int_t^R \frac{r^3-2Rr^2+R^2r}{\sqrt{r^2-s^2}}\,dr$$

$$= \lim_{t\to s^+}\left[\int_t^R \frac{r^3}{dr}\sqrt{r^2-s^2} - 2R\int_t^R \frac{r^2}{dr}\sqrt{r^2-s^2} + R^2\int_t^R \frac{r}{dr}\sqrt{r^2-s^2}\right]$$

$$= \lim_{t\to s^+}\left(I_1 - 2RI_2 + R^2 I_3\right) = L$$

For I_1 : Let $u=\sqrt{r^2-s^2} \Rightarrow u^2=r^2-s^2$, $r^2=u^2+s^2$, $2r\,dr=2u\,du$, so, omitting limits and constant of integration,

$$I_1 = \int \frac{\left(u^2+s^2\right)u}{u}\,du = \int \left(u^2+s^2\right)du = \frac{1}{3}u^3+s^2u = \frac{1}{3}u\left(u^2+3s^2\right)$$

$$= \frac{1}{3}\sqrt{r^2-s^2}\left(r^2-s^2+3s^2\right) = \frac{1}{3}\sqrt{r^2-s^2}\left(r^2+2s^2\right)$$

For I_2 : Using Formula 44, $I_2 = \frac{r}{2}\sqrt{r^2-s^2} + \frac{s^2}{2}\ln\left|r+\sqrt{r^2-s^2}\right|$.

For I_3 : Let $u=r^2-s^2 \Rightarrow du=2r\,dr$. Then $I_3 = \frac{1}{2}\int \frac{du}{\sqrt{u}} = \frac{1}{2}\cdot 2\sqrt{u} = \sqrt{r^2-s^2}$.

Thus,

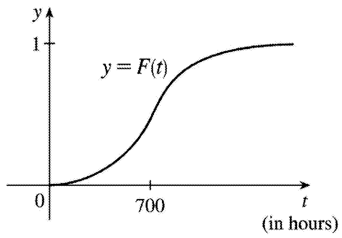
$$L = \lim_{t\to s^+}\left[\frac{1}{3}\sqrt{r^2-s^2}\left(r^2+2s^2\right) - 2R\left(\frac{r}{2}\sqrt{r^2-s^2} + \frac{s^2}{2}\ln\left|r+\sqrt{r^2-s^2}\right|\right) + R^2\sqrt{r^2-s^2}\right]_t^R$$

$$= \lim_{t\to s^+}\left[\frac{1}{3}\sqrt{R^2-s^2}\left(R^2+2s^2\right) - 2R\left(\frac{R}{2}\sqrt{R^2-s^2} + \frac{s^2}{2}\ln\left|R+\sqrt{R^2-s^2}\right|\right) + R^2\sqrt{R^2-s^2}\right]$$

$$-\lim_{t\to s^+}\left[\frac{1}{3}\sqrt{t^2-s^2}\left(t^2+2s^2\right) - 2R\left(\frac{t}{2}\sqrt{t^2-s^2} + \frac{s^2}{2}\ln\left|t+\sqrt{t^2-s^2}\right|\right) + R^2\sqrt{t^2-s^2}\right]$$

$$= \left[\frac{1}{3}\sqrt{R^2-s^2}\left(R^2+2s^2\right) - Rs^2\ln\left|R+\sqrt{R^2-s^2}\right|\right] - \left[-Rs^2\ln|s|\right]$$

$$= \frac{1}{3}\sqrt{R^2-s^2}\left(R^2+2s^2\right) - Rs^2\ln\left(\frac{R+\sqrt{R^2-s^2}}{s}\right)$$

67. **(a)** 

(b) $r(t)=F'(t)$ is the rate at which the fraction $F(t)$ of burnt–out bulbs increases as t increases. This could be interpreted as a fractional burnout rate.

(c) $\int_0^\infty r(t)\,dt=\lim_{x\to\infty} F(x)=1$, since all of the bulbs will eventually burn out.

68.

$$I=\int_0^\infty te^{kt}\,dt=\lim_{s\to\infty}\left[\frac{1}{k^2}(kt-1)e^{kt}\right]_0^s \text{ [Formula 96, or parts]}$$

$$=\lim_{s\to\infty}\left[\left(\frac{1}{k}se^{ks}-\frac{1}{k^2}e^{ks}\right)-\left(-\frac{1}{k^2}\right)\right].$$

Since $k<0$ the first two terms approach 0 (you can verify that the first term does so with l'Hospital's Rule), so the limit is equal to $1/k^2$. Thus, $M=-kI=-k\left(1/k^2\right)=-1/k=-1/(-0.000121)\approx 8264.5$ years.

69. $I=\int_a^\infty \frac{1}{x^2+1}\,dx=\lim_{t\to\infty}\int_a^t \frac{1}{x^2+1}\,dx=\lim_{t\to\infty}\left[\tan^{-1}x\right]_a^t=\lim_{t\to\infty}\left(\tan^{-1}t-\tan^{-1}a\right)=\frac{\pi}{2}-\tan^{-1}a$. $I<0.001\Rightarrow$ $\frac{\pi}{2}-\tan^{-1}a<0.001\Rightarrow\tan^{-1}a>\frac{\pi}{2}-0.001\Rightarrow a>\tan\left(\frac{\pi}{2}-0.001\right)\approx 1000$.

70. $f(x)=e^{-x^2}$ and $\Delta x=\frac{4-0}{8}=\frac{1}{2}$.

$$\int_0^4 f(x)\,dx\approx S_8=\frac{1}{2\cdot 3}[f(0)+4f(0.5)+2f(1)+\cdots+2f(3)+4f(3.5)+f(4)]$$

$$\approx\frac{1}{6}(5.31717808)\approx 0.8862$$

Now $x>4\Rightarrow -x\cdot x<-x\cdot 4\Rightarrow e^{-x^2}<e^{-4x}\Rightarrow\int_4^\infty e^{-x^2}\,dx<\int_4^\infty e^{-4x}\,dx$.

$\int_4^\infty e^{-4x}dx = \lim_{t\to\infty}\left[-\frac{1}{4}e^{-4x}\right]_4^t = -\frac{1}{4}\left(0-e^{-16}\right) = 1/\left(4e^{16}\right) \approx 0.0000000281 < 0.0000001$, as desired.

71. **(a)** $F(s) = \int_0^\infty f(t)e^{-st}dt = \int_0^\infty e^{-st}dt = \lim_{n\to\infty}\left[-\frac{e^{-st}}{s}\right]_0^n = \lim_{n\to\infty}\left(\frac{e^{-sn}}{-s}+\frac{1}{s}\right)$. This converges to $\frac{1}{s}$ only if $s>0$. Therefore $F(s)=\frac{1}{s}$ with domain $\{s\,|\,s>0\}$.

(b)

$$F(s) = \int_0^\infty f(t)e^{-st}dt = \int_0^\infty e^t e^{-st}dt = \lim_{n\to\infty}\int_0^n e^{t(1-s)}dt = \lim_{n\to\infty}\left[\frac{1}{1-s}e^{t(1-s)}\right]_0^n$$
$$= \lim_{n\to\infty}\left(\frac{e^{(1-s)n}}{1-s} - \frac{1}{1-s}\right)$$

This converges only if $1-s<0 \Rightarrow s>1$, in which case $F(s)=\frac{1}{s-1}$ with domain $\{s\,|\,s>1\}$.

(c) $F(s) = \int_0^\infty f(t)e^{-st}dt = \lim_{n\to\infty}\int_0^n te^{-st}dt$. Use integration by parts: let $u=t$, $dv=e^{-st}dt \Rightarrow du=dt$, $v=-\frac{e^{-st}}{s}$. Then $F(s) = \lim_{n\to\infty}\left[-\frac{t}{s}e^{-st}-\frac{1}{s^2}e^{-st}\right]_0^n = \lim_{n\to\infty}\left(\frac{-n}{se^{sn}} - \frac{1}{s^2e^{sn}} + 0 + \frac{1}{s^2}\right) = \frac{1}{s^2}$ only if $s>0$. Therefore, $F(s)=\frac{1}{s^2}$ and the domain of F is $\{s\,|\,s>0\}$.

72. $0 \le f(t) \le Me^{at} \Rightarrow 0 \le f(t)e^{-st} \le Me^{at}e^{-st}$ for $t \ge 0$. Now use the Comparison Theorem:

$$\int_0^\infty Me^{at}e^{-st}dt = \lim_{n\to\infty} M\int_0^n e^{t(a-s)}dt = M\cdot\lim_{n\to\infty}\left[\frac{1}{a-s}e^{t(a-s)}\right]_0^n = M\cdot\lim_{n\to\infty}\frac{1}{a-s}\left[e^{n(a-s)}-1\right]$$

This is convergent only when $a-s<0 \Rightarrow s>a$. Therefore, by the Comparison Theorem, $F(s)=\int_0^\infty f(t)e^{-st}dt$ is also convergent for $s>a$.

73. $G(s)=\int_0^\infty f'(t)e^{-st}dt$. Integrate by parts with $u=e^{-st}$, $dv=f'(t)dt \Rightarrow du=-se^{-st}$, $v=f(t)$:

$$G(s)=\lim_{n\to\infty}\left[f(t)e^{-st}\right]_0^n+s\int_0^\infty f(t)e^{-st}\,dt=\lim_{n\to\infty} f(n)e^{-sn}-f(0)+sF(s)$$

But $0\le f(t)\le Me^{at}\Rightarrow 0\le f(t)e^{-st}\le Me^{at}e^{-st}$ and $\lim_{t\to\infty} Me^{t(a-s)}=0$ for $s>a$. So by the Squeeze Theorem,

$\lim_{t\to\infty} f(t)e^{-st}=0$ for $s>a\Rightarrow G(s)=0-f(0)+sF(s)=sF(s)-f(0)$ for $s>a$ .f}(0) for $s>a$.

74. Assume without loss of generality that $a<b$. Then

$$\begin{aligned}
\int_{-\infty}^a f(x)\,dx+\int_a^\infty f(x)\,dx &= \lim_{t\to-\infty}\int_t^a f(x)\,dx+\lim_{u\to\infty}\int_a^u f(x)\,dx\\
&= \lim_{t\to-\infty}\int_t^a f(x)\,dx+\lim_{u\to\infty}\left[\int_a^b f(x)\,dx+\int_b^u f(x)\,dx\right]\\
&= \lim_{t\to-\infty}\int_t^a f(x)\,dx+\int_a^b f(x)\,dx+\lim_{u\to\infty}\int_b^u f(x)\,dx\\
&= \lim_{t\to-\infty}\left[\int_t^a f(x)\,dx+\int_a^b f(x)\,dx\right]+\int_b^\infty f(x)\,dx\\
&= \lim_{t\to-\infty}\int_t^b f(x)\,dx+\int_b^\infty f(x)\,dx\\
&= \int_{-\infty}^b f(x)\,dx+\int_b^\infty f(x)\,dx
\end{aligned}$$

75. We use integration by parts: let $u=x$, $dv=xe^{-x^2}dx\Rightarrow du=dx$, $v=-\frac{1}{2}e^{-x^2}$. So

$$\begin{aligned}
\int_0^\infty x^2e^{-x^2}\,dx &= \lim_{t\to\infty}\left[-\frac{1}{2}xe^{-x^2}\right]_0^t+\frac{1}{2}\int_0^\infty e^{-x^2}\,dx\\
&= \lim_{t\to\infty}\left[-t\Big/\left(2e^{t^2}\right)\right]+\frac{1}{2}\int_0^\infty e^{-x^2}\,dx=\frac{1}{2}\int_0^\infty e^{-x^2}\,dx
\end{aligned}$$

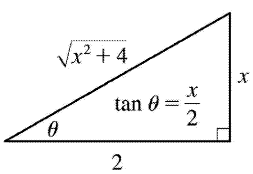
(The limit is 0 by l’Hospital’s Rule.)

76.

$\int_0^\infty e^{-x^2}\,dx$ is the area under the curve $y=e^{-x^2}$ for $0\le x<\infty$ and $0<y\le 1$. Solving $y=e^{-x^2}$ for x , we get $y=e^{-x^2}\Rightarrow \ln y=-x^2\Rightarrow -\ln y=x^2\Rightarrow x=\pm\sqrt{-\ln y}$. Since x is positive, choose $x=\sqrt{-\ln y}$, and the area is represented by $\int_0^1 \sqrt{-\ln y}\,dy$. Therefore, each integral represents the same area, so the integrals are equal.

77. For the first part of the integral, let $x=2\tan\theta\Rightarrow dx=2\sec^2\theta\,d\theta$.

$\int \frac{1}{\sqrt{x^2+4}}\,dx=\int \frac{2\sec^2\theta}{2\sec\theta}\,d\theta=\int\sec\theta\,d\theta=\ln|\sec\theta+\tan\theta|$. From the figure, $\tan\theta=\frac{x}{2}$, and $\sec\theta=\frac{\sqrt{x^2+4}}{2}$. So



$$I=\int_0^\infty\left(\frac{1}{\sqrt{x^2+4}}-\frac{C}{x+2}\right)dx=\lim_{t\to\infty}\left[\ln\left|\frac{\sqrt{x^2+4}}{2}+\frac{x}{2}\right|-C\ln|x+2|\right]_0^t$$

$$=\lim_{t\to\infty}\left[\ln\frac{\sqrt{t^2+4}+t}{2}-C\ln(t+2)-(\ln 1-C\ln 2)\right]$$

$$=\lim_{t\to\infty}\left[\ln\left(\frac{\sqrt{t^2+4}+t}{2(t+2)^C}\right)+\ln 2^C\right]$$

$$=\ln\left(\lim_{t\to\infty}\frac{t+\sqrt{t^2+4}}{(t+2)^C}\right)+\ln 2^{C-1}$$

Now $L=\lim_{t\to\infty}\frac{t+\sqrt{t^2+4}}{(t+2)^C}=\lim_{t\to\infty}\frac{1+t/\sqrt{t^2+4}}{C(t+2)^{C-1}}=\frac{2}{C\lim_{t\to\infty}(t+2)^{C-1}}$.

If $C<1$, $L=\infty$ and I diverges. If $C=1$, $L=2$ and I converges to

$\ln 2+\ln 2^0=\ln 2$. If $C>1$, $L=0$ and I diverges to $-\infty$.

78.

$$I=\int_0^\infty \left(\frac{x}{x^2+1} - \frac{C}{3x+1} \right) dx = \lim_{t\to\infty} \left[\frac{1}{2}\ln\left(x^2+1\right) - \frac{1}{3} C\ln(3x+1) \right]_0^t$$

$$= \lim_{t\to\infty} \left[\ln\left(t^2+1\right)^{1/2} - \ln(3t+1)^{C/3} \right]$$

$$= \lim_{t\to\infty} \left(\ln \frac{\left(t^2+1\right)^{1/2}}{(3t+1)^{C/3}} \right) = \ln \left(\lim_{t\to\infty} \frac{\sqrt{t^2+1}}{(3t+1)^{C/3}} \right)$$

For $C\leq 0$, the integral diverges. For $C>0$, we have

$$L=\lim_{t\to\infty} \frac{\sqrt{t^2+1}}{(3t+1)^{C/3}} = \lim_{t\to\infty} \frac{t\big/\sqrt{t^2+1}}{C(3t+1)^{(C/3)-1}} = \frac{1}{C}\lim_{t\to\infty} \frac{1}{(3t+1)^{(C/3)-1}} .$$

For $C/3<1 \Leftrightarrow C<3$, $L=\infty$ and I diverges. For $C=3$, $L=\frac{1}{3}$ and $I=\ln \frac{1}{3}$. For $C>3$, $L=0$ and I diverges to $-\infty$.

1. $y=2-3x \Rightarrow L=\int_{-2}^{1} \sqrt{1+(\{dy/dx\})^2}\,\text{italic\{dx\}}=\int_{-2}^{1} \sqrt{1+(-3)^2}\,dx=\sqrt{10}\,[1-(-2)]=3\sqrt{10}$.

The arc length can be calculated using the distance formula, since the curve is a line segment, so

$$L=[\text{ distance from } (-2,8) \text{ to } (1,-1)\ ]=\sqrt{[1-(-2)]^2+[(-1)-8]^2}=\sqrt{90}=3\sqrt{10}$$

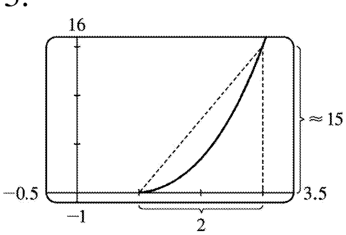
2. Using the arc length formula with $y=\sqrt{4-x^2} \Rightarrow \dfrac{dy}{dx}=-\dfrac{x}{\sqrt{4-x^2}}$, we get

$$L=\int_0^2 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx=\int_0^2 \sqrt{1+\frac{x^2}{4-x^2}}\,dx=\int_0^2 \frac{2\,dx}{\sqrt{4-x^2}}=2\lim_{t\to 2^-}\int_0^t \frac{dx}{\sqrt{2^2-x^2}}$$

$$=2\lim_{t\to 2^-}\left[\sin^{-1}(x/2)\right]_0^t=2\lim_{t\to 2^-}\left[\sin^{-1}(t/2)-\sin^{-1}0\right]=2\left(\frac{\pi}{2}-0\right)=\pi$$

The curve is a quarter of a circle with radius 2 , so the length of the arc is $\frac{1}{4}(2\pi\cdot 2)=\pi$, as above.

3.



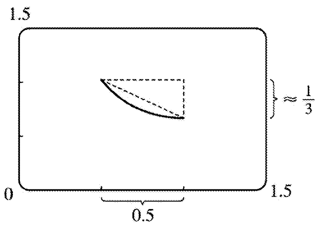
From the figure, the length of the curve is slightly larger than the hypotenuse of the triangle formed by the points $(1,0)$, $(3,0)$, and $(3,f(3))\approx(3,15)$, where $y=f(x)=\frac{2}{3}\left(x^2-1\right)^{3/2}$. This length is about $\sqrt{15^2+2^2}\approx 15$, so we might estimate the length to be 15.5 . $y=\frac{2}{3}\left(x^2-1\right)^{3/2} \Rightarrow y'=\left(x^2-1\right)^{1/2}(2x) \Rightarrow$

$1+(y')^2=1+4x^2\left(x^2-1\right)=4x^4-4x^2+1=\left(2x^2-1\right)^2$, so, using the fact that $2x^2-1>0$ for $1\le x\le 3$,

$$L=\int_1^3 \sqrt{(2x^2-1)^2}\,dx=\int_1^3 \left|2x^2-1\right|dx=\int_1^3 (2x^2-1)\,dx=\left[\frac{2}{3}x^3-x\right]_1^3$$

$$=(18-3)-\left(\frac{2}{3}-1\right)=\frac{46}{3}=15.\overline{3}$$

4.



From the figure, the length of the curve is slightly larger than the hypotenuse of the triangle formed by the points $(0.5, f(0.5) \approx 1)$, $(1, f(0.5) \approx 1)$ and $\left(1, \frac{2}{3}\right)$, where $y = f(x) = x^3/6 + 1/(2x)$. This length is about $\sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{3}\right)^2} \approx 0.6$, so we might estimate the length to be 0.65 .

$$y = \frac{x^3}{6} + \frac{1}{2x} \Rightarrow y' = \frac{x^2}{2} - \frac{x^{-2}}{2} \Rightarrow$$
$$1 + \left(y'\right)^2 = 1 + \frac{x^4}{4} - \frac{1}{2} + \frac{x^{-4}}{4} = \frac{x^4}{4} + \frac{1}{2} + \frac{x^{-4}}{4} = \left(\frac{x^2}{2} + \frac{x^{-2}}{2}\right)^2$$

so, using the fact that the parenthetical expression is positive,

$$L = \int_{1/2}^{1} \sqrt{\left(\frac{x^2}{2} + \frac{x^{-2}}{2}\right)^2}\, dx = \int_{1/2}^{1} \left(\frac{x^2}{2} + \frac{x^{-2}}{2}\right) dx = \left[\frac{x^3}{6} - \frac{1}{2x}\right]_{1/2}^{1}$$
$$= \left(\frac{1}{6} - \frac{1}{2}\right) - \left(\frac{1}{48} - 1\right) = \frac{31}{48} = 0.6458\overline{3}$$

5. $y = 1 + 6x^{3/2} \Rightarrow dy/dx = 9x^{1/2} \Rightarrow 1 + (dy/dx)^2 = 1 + 81x$. So

$$L = \int_0^1 \sqrt{1 + 81x}\, dx = \int_1^{82} u^{1/2} \left(\frac{1}{81}\right) du \text{ [where } u = 1 + 81x \text{ and } du = 81\, dx \text{]}$$
$$= \frac{1}{81} \cdot \frac{2}{3} \left[u^{3/2}\right]_1^{82} = \frac{2}{243} \left(82\sqrt{82} - 1\right)$$

6. $y^2 = 4(x+4)^3$, $y > 0 \Rightarrow y = 2(x+4)^{3/2} \Rightarrow dy/dx = 3(x+4)^{1/2} \Rightarrow$
$1 + (dy/dx)^2 = 1 + 9(x+4) = 9x + 37$. So

$$L=\int_0^2 \sqrt{9x+37}\,dx \left[\begin{array}{l} \text{u=9x+37,} \\ \text{du=9dx} \end{array}\right]=\int_{37}^{55} u^{1/2}\left(\frac{1}{9}\right)du$$

$$=\frac{1}{9}\cdot\frac{2}{3}\left[u^{3/2}\right]_{37}^{55}=\frac{2}{27}\left(55\sqrt{55}-37\sqrt{37}\right)$$

7. $y=\dfrac{x^5}{6}+\dfrac{1}{10x^3}\Rightarrow\dfrac{dy}{dx}=\dfrac{5}{6}x^4-\dfrac{3}{10}x^{-4}\Rightarrow$

$1+(dy/dx)^2=1+\dfrac{25}{36}x^8-\dfrac{1}{2}+\dfrac{9}{100}x^{-8}=\dfrac{25}{36}x^8+\dfrac{1}{2}+\dfrac{9}{100}x^{-8}=\left(\dfrac{5}{6}x^4+\dfrac{3}{10}x^{-4}\right)^2$. So

$$L=\int_1^2\sqrt{\left(\frac{5}{6}x^4+\frac{3}{10}x^{-4}\right)^2}\,dx=\int_1^2\left(\frac{5}{6}x^4+\frac{3}{10}x^{-4}\right)dx=\left[\frac{1}{6}x^5-\frac{1}{10}x^{-3}\right]_1^2$$

$$=\left(\frac{32}{6}-\frac{1}{80}\right)-\left(\frac{1}{6}-\frac{1}{10}\right)=\frac{31}{6}+\frac{7}{80}=\frac{1261}{240}$$

8. $y=\dfrac{x^2}{2}-\dfrac{\ln x}{4}\Rightarrow\dfrac{dy}{dx}=x-\dfrac{1}{4x}\Rightarrow 1+\left(\dfrac{dy}{dx}\right)^2=x^2+\dfrac{1}{2}+\dfrac{1}{16x^2}$. So

$$L=\int_2^4\left(x+\frac{1}{4x}\right)dx=\left[\frac{x^2}{2}+\frac{\ln x}{4}\right]_2^4=\left(8+\frac{2\ln 2}{4}\right)-\left(2+\frac{\ln 2}{4}\right)=6+\frac{\ln 2}{4}.$$

9. $x=\dfrac{1}{3}\sqrt{y}\,(y-3)=\dfrac{1}{3}y^{3/2}-y^{1/2}\Rightarrow dx/dy=\dfrac{1}{2}y^{1/2}-\dfrac{1}{2}y^{-1/2}\Rightarrow$

$1+(dx/dy)^2=1+\dfrac{1}{4}y-\dfrac{1}{2}+\dfrac{1}{4}y^{-1}=\dfrac{1}{4}y+\dfrac{1}{2}+\dfrac{1}{4}y^{-1}=\left(\dfrac{1}{2}y^{1/2}+\dfrac{1}{2}y^{-1/2}\right)^2$. So

$$L=\int_1^9\left(\frac{1}{2}y^{1/2}+\frac{1}{2}y^{-1/2}\right)dy=\frac{1}{2}\left[\frac{2}{3}y^{3/2}+2y^{1/2}\right]_1^9=\frac{1}{2}\left[\left(\frac{2}{3}\cdot 27+2\cdot 3\right)-\left(\frac{2}{3}\cdot 1+2\cdot 1\right)\right]$$

$$=\frac{1}{2}\left(24-\frac{8}{3}\right)=\frac{1}{2}\left(\frac{64}{3}\right)=\frac{32}{3}$$

10. $y=\ln(\cos x)\Rightarrow dy/dx=-\tan x\Rightarrow 1+(dy/dx)^2=1+\tan^2 x=\sec^2 x$. So

$$L=\int_0^{\pi/3}\sqrt{\sec^2 x}\,dx=\int_0^{\pi/3}\sec x\,dx=\left[\ln|\sec x+\tan x|\right]_0^{\pi/3}=\ln\left(2+\sqrt{3}\right)-\ln(1+0)=\ln\left(2+\sqrt{3}\right).$$

11. $y=\ln(\sec x)\Rightarrow$

$\dfrac{dy}{dx}=\dfrac{\sec x\tan x}{\sec x}=\tan x\Rightarrow 1+\left(\dfrac{dy}{dx}\right)^2=1+\tan^2 x=\sec^2 x$, so

$$L=\int_0^{\pi/4}\sqrt{\sec^2 x}\,dx=\int_0^{\pi/4}|\sec x|\,dx=\int_0^{\pi/4}\sec x\,dx=\left[\ln(\sec x+\tan x)\right]_0^{\pi/4}$$
$$=\ln\left(\sqrt{2}+1\right)-\ln(1+0)=\ln\left(\sqrt{2}+1\right)$$

12. $y=\ln x\Rightarrow \dfrac{dy}{dx}=\dfrac{1}{x}\Rightarrow\sqrt{1+\left(\dfrac{dy}{dx}\right)^2}=\sqrt{1+\left(\dfrac{1}{x}\right)^2}=\dfrac{\sqrt{1+x^2}}{x}$. So $L=\displaystyle\int_1^{\sqrt{3}}\dfrac{\sqrt{1+x^2}}{x}\,dx$. Now let $v=\sqrt{1+x^2}$, so $v^2=1+x^2$ and $v\,dv=x\,dx$. Thus

$$L=\int_{\sqrt{2}}^{2}\frac{v}{v^2-1}\,v\,dv=\int_{\sqrt{2}}^{2}\left(1+\frac{1/2}{v-1}-\frac{1/2}{v+1}\right)dv=\left[v+\frac{1}{2}\ln|v-1|-\frac{1}{2}\ln|v+1|\right]_{\sqrt{2}}^{2}$$
$$=\left[v-\frac{1}{2}\ln\left|\frac{v+1}{v-1}\right|\right]_{\sqrt{2}}^{2}=2-\frac{1}{2}\ln 3-\sqrt{2}+\frac{1}{2}\ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)=2-\sqrt{2}+\ln\left(\sqrt{2}+1\right)-\frac{1}{2}\ln 3$$

Or: Use Formula 23 in the table of integrals.

13. $y=\cosh x\Rightarrow y'=\sinh x\Rightarrow 1+\left(y'\right)^2=1+\sinh^2 x=\cosh^2 x$.

So $L=\displaystyle\int_0^1\cosh x\,dx=\left[\sinh x\right]_0^1=\sinh 1=\frac{1}{2}(e-1/e)$.

14. $y^2=4x$, $x=\dfrac{1}{4}y^2\Rightarrow\dfrac{dx}{dy}=\dfrac{1}{2}y\Rightarrow 1+\left(\dfrac{dx}{dy}\right)^2=1+\dfrac{1}{4}y^2$. So

$$L=\int_0^2\sqrt{1+\frac{1}{4}y^2}\,dy=\int_0^1\sqrt{1+u^2}\cdot 2\,du$$
$$=\left[u\sqrt{1+u^2}+\ln\left|u+\sqrt{1+u^2}\right|\right]_0^1=\sqrt{2}+\ln\left(1+\sqrt{2}\right)$$

15. $y=e^x\Rightarrow y'=e^x\Rightarrow 1+\left(y'\right)^2=1+e^{2x}$. So

$$L=\int_0^1\sqrt{1+e^{2x}}\,dx=\int_1^e\sqrt{1+u^2}\,\frac{du}{u}\quad\left[u=e^x,\text{so}\,x=\ln u,dx=du/u\right]$$

$$=\int_1^e \frac{\sqrt{1+u^2}}{u^2}\,u\,du=\int_{\sqrt{2}}^{\sqrt{1+e^2}} \frac{v}{v^2-1}\,v\,dv\left[v=\sqrt{1+u^2},\text{so}\,v^2=1+u^2,v\,dv=u\,du\right]$$

$$=\int_{\sqrt{2}}^{\sqrt{1+e^2}}\left(1+\frac{1/2}{v-1}-\frac{1/2}{v+1}\right)dv=\left[v+\frac{1}{2}\ln\frac{v-1}{v+1}\right]_{\sqrt{2}}^{\sqrt{1+e^2}}$$

$$=\sqrt{1+e^2}+\frac{1}{2}\ln\frac{\sqrt{1+e^2}-1}{\sqrt{1+e^2}+1}-\sqrt{2}-\frac{1}{2}\ln\frac{\sqrt{2}-1}{\sqrt{2}+1}$$

$$=\sqrt{1+e^2}-\sqrt{2}+\ln\left(\sqrt{1+e^2}-1\right)-1-\ln\left(\sqrt{2}-1\right)$$

Or: Use Formula 23 for $\int\left(\sqrt{1+u^2}/u\right)du$, or substitute $u=\tan\theta$.

16. $y=\ln\left(\dfrac{e^x+1}{e^x-1}\right)=\ln\left(e^x+1\right)-\ln\left(e^x-1\right)\Rightarrow y'=\dfrac{e^x}{e^x+1}-\dfrac{e^x}{e^x-1}=\dfrac{-2e^x}{e^{2x}-1}\Rightarrow$

$1+\left(y'\right)^2=1+\dfrac{4e^{2x}}{\left(e^{2x}-1\right)^2}=\dfrac{\left(e^{2x}+1\right)^2}{\left(e^{2x}-1\right)^2}\Rightarrow\sqrt{1+\left(y'\right)^2}=\dfrac{e^{2x}+1}{e^{2x}-1}=\dfrac{e^x+e^{-x}}{e^x-e^{-x}}=\dfrac{\cosh x}{\sinh x}$.

So $L=\int_a^b\dfrac{\cosh x}{\sinh x}\,dx=[\ln\sinh x]_a^b=\ln\sinh b-\ln\sinh a=\ln\left(\dfrac{\sinh b}{\sinh a}\right)=\ln\left(\dfrac{e^b-e^{-b}}{e^a-e^{-a}}\right)$.

17. $y=\cos x\Rightarrow dy/dx=-\sin x\Rightarrow 1+(dy/dx)^2=1+\sin^2x$. So $L=\int_0^{2\pi}\sqrt{1+\sin^2x}\,dx$.

18. $y=2^x\Rightarrow dy/dx=\left(2^x\right)\ln 2\Rightarrow L=\int_0^3\sqrt{1+(\ln 2)^2 2^{2x}}\,dx$

19. $x=y+y^3\Rightarrow dx/dy=1+3y^2\Rightarrow 1+(dx/dy)^2=1+(1+3y^2)^2=9y^4+6y^2+2$. So $L=\int_1^4\sqrt{9y^4+6y^2+2}\,dy$.

20. $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$, $y=\pm b\sqrt{1-\left(x^2/a^2\right)}=\pm\dfrac{b}{a}\sqrt{a^2-x^2}$.

$y=\frac{b}{a}\sqrt{a^2-x^2} \Rightarrow \frac{dy}{dx}=\frac{-bx}{a\sqrt{a^2-x^2}} \Rightarrow \left(\frac{dy}{dx}\right)^2=\frac{b^2x^2}{a^2\left(a^2-x^2\right)}$.

So $L=2\int_{-a}^{a}\left[1+\frac{b^2x^2}{a^2\left(a^2-x^2\right)}\right]^{1/2}dx=\frac{4}{a}\int_{0}^{a}\left[\frac{\left(b^2-a^2\right)x^2+a^4}{a^2-x^2}\right]^{1/2}dx$.

21. $y=xe^{-x} \Rightarrow dy/dx=e^{-x}-xe^{-x}=e^{-x}(1-x) \Rightarrow 1+(dy/dx)^2=1+e^{-2x}(1-x)^2$. Let $f(x)=\sqrt{1+(dy/dx)^2}=\sqrt{1+e^{-2x}(1-x)^2}$. Then $L=\int_0^5 f(x)\,dx$. Since $n=10$, $\Delta x=\frac{5-0}{10}=\frac{1}{2}$. Now

$$L\approx S_{10}=\frac{1/2}{3}\left[f(0)+4f\left(\frac{1}{2}\right)+2f(1)+4f\left(\frac{3}{2}\right)+2f(2)+4f\left(\frac{5}{2}\right)+2f(3)\right.$$
$$\left.+4f\left(\frac{7}{2}\right)+2f(4)+4f\left(\frac{9}{2}\right)+f(5)\right]\approx 5.115840$$

The value of the integral produced by a calculator is 5.113568 (to six decimal places).

22. $x=y+\sqrt{y} \Rightarrow dx/dy=1+\frac{1}{2\sqrt{y}} \Rightarrow 1+(dx/dy)^2=1+\left(1+\frac{1}{2\sqrt{y}}\right)^2=2+\frac{1}{\sqrt{y}}+\frac{1}{4y}$. Let $g(y)=\sqrt{1+(dx/dy)^2}$. Then $L=\int_1^2 g(y)\,dy$. Since $n=10$, $\Delta y=\frac{2-1}{10}=\frac{1}{10}$. Now

$$L\approx S_{10}=\frac{1/10}{3}\left[g(1)+4g(1.1)+2g(1.2)+4g(1.3)+2g(1.4)+4g(1.5)\right.$$
$$\left.+2g(1.6)+4g(1.7)+2g(1.8)+4g(1.9)+g(2)\right]\approx 1.732215 ,$$

which is the same value of the integral produced by a calculator to six decimal places.

23. $y=\sec x \Rightarrow dy/dx=\sec x\tan x \Rightarrow L=\int_0^{\pi/3} f(x)\,dx$, where $f(x)=\sqrt{1+\sec^2 x\tan^2 x}$. Since $n=10$, $\Delta x=\frac{\pi/3-0}{10}=\frac{\pi}{30}$. Now

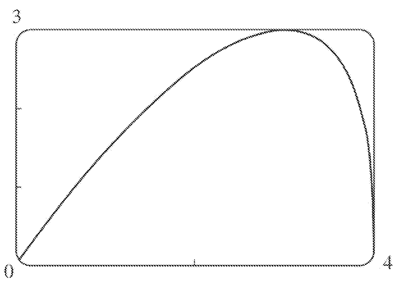
$$L\approx S_{10}=\frac{\pi/30}{3}\left[f(0)+4f\left(\frac{\pi}{30}\right)+2f\left(\frac{2\pi}{30}\right)+4f\left(\frac{3\pi}{30}\right)+2f\left(\frac{4\pi}{30}\right)+4f\left(\frac{5\pi}{30}\right)\right.$$
$$\left.+2f\left(\frac{6\pi}{30}\right)+4f\left(\frac{7\pi}{30}\right)+2f\left(\frac{8\pi}{30}\right)+4f\left(\frac{9\pi}{30}\right)+f\left(\frac{\pi}{3}\right)\right]\approx 1.569619 .$$

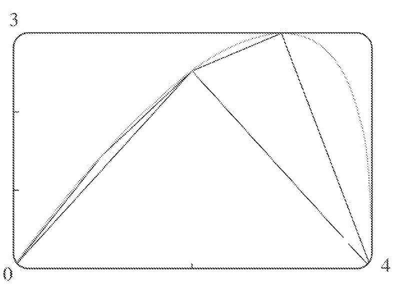
The value of the integral produced by a calculator is 1.569259 (to six decimal places).

24. $y=x\ln x \Rightarrow dy/dx=1+\ln x$. Let $f(x)=\sqrt{1+(dy/dx)^2}=\sqrt{1+(1+\ln x)^2}$.
Then $L=\int_1^3 f(x)dx$. Since $n=10$, $\Delta x=\frac{3-1}{10}=\frac{1}{5}$. Now

$$L\approx S_{10}=\frac{1/5}{3}[f(1)+4f(1.2)+2f(1.4)+4f(1.6)+2f(1.8)+4f(2)]$$
$$+2f(2.2)+4f(2.4)+2f(2.6)+4f(2.8)+f(3)]\approx 3.869618 .$$

The value of the integral produced by a calculator is 3.869617 (to six decimal places).

25. **(a)** 

(b) 

Let $f(x)=y=x\sqrt[3]{4-x}$. The polygon with one side is just the line segment joining the points $(0,f(0))=(0,0)$ and $(4,f(4))=(4,0)$, and its length is 4 . The polygon with two sides joins the points $(0,0)$, $(2,f(2))=\left(2,2\sqrt[3]{2}\right)$ and $(4,0)$.

Its length is

$$\sqrt{(2-0)^2+\left(2\sqrt[3]{2}-0\right)^2}+\sqrt{(4-2)^2+\left(0-2\sqrt[3]{2}\right)^2}=2\sqrt{4+2^{8/3}}\approx 6.43$$

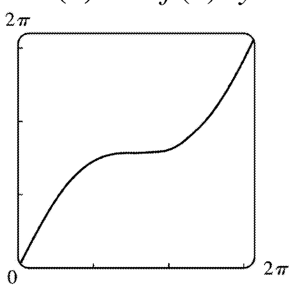
Similarly, the inscribed polygon with four sides joins the points $(0,0)$, $\left(1,\sqrt[3]{3}\right)$, $\left(2,2\sqrt[3]{2}\right)$, $(3,3)$, and $(4,0)$, so its length is

$$\sqrt{1+\left(\sqrt[3]{3}\right)^2}+\sqrt{1+\left(2\sqrt[3]{2}-\sqrt[3]{3}\right)^2}+\sqrt{1+\left(3-2\sqrt[3]{2}\right)^2}+\sqrt{1+9}\approx 7.50$$

(c) Using the arc length formula with $\frac{dy}{dx}=x\left[\frac{1}{3}(4-x)^{-2/3}(-1)\right]+\sqrt[3]{4-x}=\frac{12-4x}{3(4-x)^{2/3}}$, the length of the curve is $L=\int_0^4\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx=\int_0^4\sqrt{1+\left[\frac{12-4x}{3(4-x)^{2/3}}\right]^2}\,dx$.

(d) According to a CAS, the length of the curve is $L\approx 7.7988$. The actual value is larger than any of the approximations in part (b). This is always true, since any approximating straight line between two points on the curve is shorter than the length of the curve between the two points.

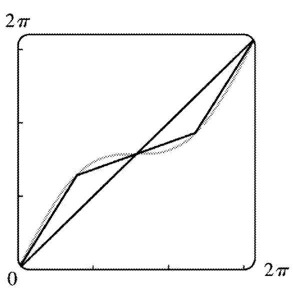
26. **(a)** Let $f(x)=y=x+\sin x$ with $0\le x\le 2\pi$.



(b) The polygon with one side is just the line segment joining the points $(0,f(0))=(0,0)$ and $(2\pi,f(2\pi))=(2\pi,2\pi)$, and its length is $\sqrt{(2\pi-0)^2+(2\pi-0)^2}=2\sqrt{2}\,\pi\approx 8.9$.
The polygon with two sides joins the points $(0,0)$, $(\pi,f(\pi))=(\pi,\pi)$, and $(2\pi,2\pi)$. Its length is

$$\sqrt{(\pi-0)^2+(\pi-0)^2}+\sqrt{(2\pi-\pi)^2+(2\pi-\pi)^2}=\sqrt{2}\,\pi+\sqrt{2}\,\pi=2\sqrt{2}\,\pi\approx 8.9$$

Note from the diagram that the two approximations are the same because the sides of the 2 –sided polygon are in fact on the same line, since $f(\pi)=\pi=\frac{1}{2}f(2\pi)$.



The four–sided polygon joins the points $(0,0)$, $\left(\frac{\pi}{2},\frac{\pi}{2}+1\right)$, (π,π) , $\left(\frac{3\pi}{2},\frac{3\pi}{2}-1\right)$, and $(2\pi,2\pi)$, so its length is

$$\sqrt{\left(\frac{\pi}{2}\right)^2+\left(\frac{\pi}{2}+1\right)^2}+\sqrt{\left(\frac{\pi}{2}\right)^2+\left(\frac{\pi}{2}-1\right)^2}+\sqrt{\left(\frac{\pi}{2}\right)^2+\left(\frac{\pi}{2}-1\right)^2}+\sqrt{\left(\frac{\pi}{2}\right)^2+\left(\frac{\pi}{2}+1\right)^2}\approx 9.4$$

(c) Using the arc length formula with $dy/dx=1+\cos x$, the length of the curve is

$$L=\int_0^{2\pi}\sqrt{1+(1+\cos x)^2}\,dx=\int_0^{2\pi}\sqrt{2+2\cos x+\cos^2 x}\,dx$$

(d) The CAS approximates the integral as 9.5076 . The actual length is larger than the approximations in part (b).

27. $x=\ln\left(1-y^2\right)\Rightarrow \frac{dx}{dy}=\frac{-2y}{1-y^2}\Rightarrow 1+\left(\frac{dx}{dy}\right)^2=1+\frac{4y^2}{\left(1-y^2\right)^2}=\frac{\left(1+y^2\right)^2}{\left(1-y^2\right)^2}$. So

$$L=\int_0^{1/2}\sqrt{\frac{\left(1+y^2\right)^2}{\left(1-y^2\right)^2}}\,dy=\int_0^{1/2}\frac{1+y^2}{1-y^2}\,dy=\ln 3-\frac{1}{2}\approx 0.599$$

28. $y=x^{4/3}\Rightarrow dy/dx=\frac{4}{3}x^{1/3}\Rightarrow 1+(dy/dx)^2=1+\frac{16}{9}x^{2/3}\Rightarrow$

$$\mathrm{L}=\int_0^1\sqrt{1+\frac{16}{9}x^{2/3}}\,\mathrm{dx}=\int_0^{4/3}\sqrt{1+u^2}\,\frac{81}{64}\mathrm{u}^2\mathrm{du}\quad\left[u=\frac{4}{3}x^{1/3},du=\frac{4}{9}x^{-2/3}dx,dx=\frac{9}{4}x^{2/3}du=\frac{9}{4}\cdot\frac{9}{16}u^2du=\frac{81}{64}u^2du\right]$$

$$=\frac{81}{64}\left[\frac{1}{8}u\left(1+2u^2\right)\sqrt{1+u^2}-\frac{1}{8}\ln\left(u+\sqrt{1+u^2}\right)\right]_0^{4/3}$$

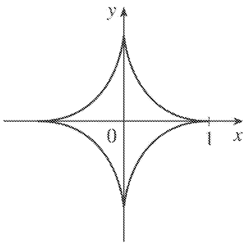
$$=\frac{81}{64}\left[\frac{1}{6}\left(1+\frac{32}{9}\right)\sqrt{\frac{25}{9}}-\frac{1}{8}\ln\left(\frac{4}{3}+\sqrt{\frac{25}{9}}\right)\right]=\frac{81}{64}\left(\frac{1}{6}\cdot\frac{41}{9}\cdot\frac{5}{3}-\frac{1}{8}\ln 3\right)$$

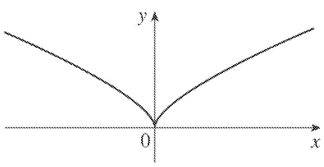
$$=\frac{205}{128}-\frac{81}{512}\ln 3\approx 1.4277586$$

29. $y^{2/3}=1-x^{2/3}\Rightarrow y=\left(1-x^{2/3}\right)^{3/2}\Rightarrow \frac{dy}{dx}=\frac{3}{2}\left(1-x^{2/3}\right)^{1/2}\left(-\frac{2}{3}x^{-1/3}\right)=-x^{-1/3}\left(1-x^{2/3}\right)^{1/2}\Rightarrow$

$\left(\frac{dy}{dx}\right)^2 = x^{-2/3}\left(1-x^{2/3}\right) = x^{-2/3}-1$. Thus

$L = 4\int_0^1 \sqrt{1+\left(x^{-2/3}-1\right)}\,dx = 4\int_0^1 x^{-1/3}\,dx = 4\lim_{t\to 0^+}\left[\frac{3}{2}x^{2/3}\right]_t^1 = 6$.



30. **(a)** 

(b) $y = x^{2/3} \Rightarrow 1+\left(\frac{dy}{dx}\right)^2 = 1+\left(\frac{2}{3}x^{-1/3}\right)^2 = 1+\frac{4}{9}x^{-2/3}$. So $L = \int_0^1 \sqrt{1+\frac{4}{9}x^{-2/3}}\,dx$ [an improper integral]. $x = y^{3/2} \Rightarrow 1+\left(\frac{dx}{dy}\right)^2 = 1+\left(\frac{3}{2}y^{1/2}\right)^2 = 1+\frac{9}{4}y$. So $L = \int_0^1 \sqrt{1+\frac{9}{4}y}\,dy$.

The second integral equals $\frac{4}{9}\cdot\frac{2}{3}\left[\left(1+\frac{9}{4}y\right)^{3/2}\right]_0^1 = \frac{8}{27}\left(\frac{13\sqrt{13}}{8}-1\right) = \frac{13\sqrt{13}-8}{27}$. The first integral can be evaluated as follows:

$$\int_0^1 \sqrt{1+\frac{4}{9}x^{-2/3}}\,dx = \lim_{t\to 0^+}\int_t^1 \frac{\sqrt{9x^{2/3}+4}}{3x^{1/3}}\,dx = \lim_{t\to 0^+}\int_{9t^{2/3}}^9 \frac{\sqrt{u+4}}{18}\,du\left[u=9x^{2/3}, du=6x^{-1/3}\,dx\right]$$

$$= \int_0^9 \frac{\sqrt{u+4}}{18}\,du = \frac{1}{18}\cdot\left[\frac{2}{3}(u+4)^{3/2}\right]_0^9 = \frac{1}{27}\left(13^{3/2}-4^{3/2}\right) = \frac{13\sqrt{13}-8}{27}$$

(c)

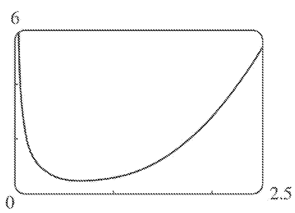
L = length of the arc of this curve from $(-1,1)$ to $(8,4)$

$$= \int_0^1 \sqrt{1+\frac{9}{4}y}\,dy + \int_0^4 \sqrt{1+\frac{9}{4}y}\,dy = \frac{13\sqrt{13}-8}{27} + \frac{8}{27}\left[\left(1+\frac{9}{4}y\right)^{3/2}\right]_0^4$$

$$=\frac{13\sqrt{13}-8}{27}+\frac{8}{27}\left(10\sqrt{10}-1\right)=\frac{13\sqrt{13}+80\sqrt{10}-16}{27}$$

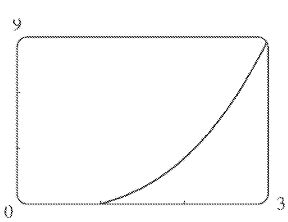
31. $y=2x^{3/2}\Rightarrow y'=3x^{1/2}\Rightarrow 1+\left(y'\right)^2=1+9x$. The arc length function with starting point $P_0(1,2)$ is

$$s(x)=\int_1^x\sqrt{1+9t}\,dt=\left[\frac{2}{27}(1+9t)^{3/2}\right]_1^x=\frac{2}{27}\left[(1+9x)^{3/2}-10\sqrt{10}\right]$$

32. **(a)** 

(b) $1+\left(\frac{dy}{dx}\right)^2=x^4+\frac{1}{2}+\frac{1}{16x^4}$,

$$\begin{aligned} s(x) &= \int_1^x\left[t^2+1/\left(4t^2\right)\right]dt \\ &= \left[\frac{1}{3}t^3-1/(4t)\right]_1^x \\ &= \frac{1}{3}x^3-1/(4x)-\left(\frac{1}{3}-\frac{1}{4}\right) \\ &= \frac{1}{3}x^3-1/(4x)-\frac{1}{12}\quad\text{for } x\geq 1 \end{aligned}$$

(c) 

33. The prey hits the ground when $y=0\Leftrightarrow 180-\frac{1}{45}x^2=0\Leftrightarrow x^2=45\cdot 180\Rightarrow x=\sqrt{8100}=90$, since x must be positive. $y'=-\frac{2}{45}x\Rightarrow 1+\left(y'\right)^2=1+\frac{4}{45^2}x^2$, so the distance traveled by the prey is

$$L=\int_0^{90}\sqrt{1+\frac{4}{45^2}x^2}\,dx=\int_0^4\sqrt{1+u^2}\left(\frac{45}{2}\,du\right)$$
$$=\frac{45}{2}\left[\frac{1}{2}u\sqrt{1+u^2}+\frac{1}{2}\ln\left(u+\sqrt{1+u^2}\right)\right]_0^4$$
$$=\frac{45}{2}\left[2\sqrt{17}+\frac{1}{2}\ln\left(4+\sqrt{17}\right)\right]=45\sqrt{17}+\frac{45}{4}\ln\left(4+\sqrt{17}\right)\approx 209.1\text{m}$$

34. $y=150-\frac{1}{40}(x-50)^2\Rightarrow y'=-\frac{1}{20}(x-50)\Rightarrow 1+\left(y'\right)^2=1+\frac{1}{20^2}(x-50)^2$, so the distance traveled by the kite is

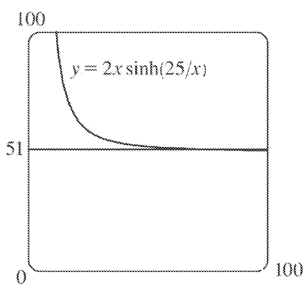
$$L=\int_0^{80}\sqrt{1+\frac{1}{20^2}(x-50)^2}\,dx=\int_{-5/2}^{3/2}\sqrt{1+u^2}\,(20\text{du})\ [\text{u}=\frac{1}{20}(\text{x}-50),\text{du}=\frac{1}{20}\text{ dx}]$$
$$=20\left[\frac{1}{2}u\sqrt{1+u^2}+\frac{1}{2}\ln\left(u+\sqrt{1+u^2}\right)\right]_{-5/2}^{3/2}$$
$$=10\left[\frac{3}{2}\sqrt{\frac{13}{4}}+\ln\left(\frac{3}{2}+\sqrt{\frac{13}{4}}\right)+\frac{5}{2}\sqrt{\frac{29}{4}}-\ln\left(-\frac{5}{2}+\sqrt{\frac{29}{4}}\right)\right]$$
$$=\frac{15}{2}\sqrt{13}+\frac{25}{2}\sqrt{29}+10\ln\left(\frac{3+\sqrt{13}}{-5+\sqrt{29}}\right)\approx 122.8\text{ft}$$

35. The sine wave has amplitude 1 and period 14 , since it goes through two periods in a distance of 28 in., so its equation is $y=1\sin\left(\frac{2\pi}{14}x\right)=\sin\left(\frac{\pi}{7}x\right)$. The width w of the flat metal sheet needed to make the panel is the arc length of the sine curve from $x=0$ to $x=28$. We set up the integral to evaluate w using the arc length formula with $\frac{dy}{dx}=\frac{\pi}{7}\cos\left(\frac{\pi}{7}x\right)$:
$L=\int_0^{28}\sqrt{1+\left[\frac{\pi}{7}\cos\left(\frac{\pi}{7}x\right)\right]^2}\,dx=2\int_0^{14}\sqrt{1+\left[\frac{\pi}{7}\cos\left(\frac{\pi}{7}x\right)\right]^2}\,dx$. This integral would be very difficult to evaluate exactly, so we use a CAS, and find that $L\approx 29.36$ inches.

36. **(a)** $y=c+a\cosh\left(\frac{x}{a}\right)\Rightarrow y'=\sinh\left(\frac{x}{a}\right)\Rightarrow 1+\left(y'\right)^2=1+\sinh^2\left(\frac{x}{a}\right)=\cosh^2\left(\frac{x}{a}\right)$. So

$$L=\int_{-b}^{b}\sqrt{\cosh^2\left(\frac{x}{a}\right)}\,dx=2\int_0^b\cosh\left(\frac{x}{a}\right)dx=2\left[a\sinh\left(\frac{x}{a}\right)\right]_0^b=2a\sinh\left(\frac{b}{a}\right)$$

(b) At $x=0$, $y=c+a$, so $c+a=20$. The poles are 50 ft apart, so $b=25$, and $L=51\Rightarrow 51=2a\sinh(b/a)$. From the figure, we see that $y=51$ intersects $y=2x\sinh(25/x)$ at $x\approx 72.3843$ for $x>0$. So $a\approx 72.3843$ and the wire should be attached at a distance of $y=c+a\cosh(25/a)=20-a+a\cosh(25/a)\approx 24.36$ ft above the ground.



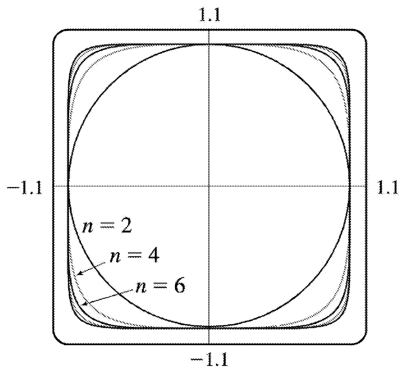
37. $y=\int_1^x\sqrt{t^3-1}\,dt\Rightarrow \frac{dy}{dx}=\sqrt{x^3-1}$ [by FTC1] $\Rightarrow 1+\left(\frac{dy}{dx}\right)^2=1+\left(\sqrt{x^3-1}\right)^2=x^3\Rightarrow$

$$L=\int_1^4\sqrt{x^3}\,dx=\int_1^4 x^{3/2}dx=\frac{2}{5}\left[x^{5/2}\right]_1^4=\frac{2}{5}(32-1)=\frac{62}{5}=12.4$$

38. By symmetry, the length of the curve in each quadrant is the same, so we'll find the length in the first quadrant and multiply by 4 . $x^{2k}+y^{2k}=1\Rightarrow y^{2k}=1-x^{2k}\Rightarrow y=\left(1-x^{2k}\right)^{1/(2k)}$ (in the first quadrant), so we use the arc length formula with

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{2k}\left(1-x^{2k}\right)^{1/(2k)-1}\left(-2kx^{2k-1}\right)\\ &= -x^{2k-1}\left(1-x^{2k}\right)^{1/(2k)-1}\end{aligned}$$

The total length is therefore



$L_{2k}=4\int_0^1 \sqrt{1+\left[-x^{2k-1}\left(1-x^{2k}\right)^{1/(2k)-1}\right]^2}\,dx=4\int_0^1 \sqrt{1+x^{2(2k-1)}\left(1-x^{2k}\right)^{1/k-2}}\,dx$ Now from the graph, we see that as k increases, the ''corners'' of these fat circles get closer to the points $(\pm 1,\pm 1)$ and $(\pm 1,\mp 1)$, and the ''edges'' of the fat circles approach the lines joining these four points. It seems plausible that as $k\to\infty$, the total length of the fat circle with $n=2k$ will approach the length of the perimeter of the square with sides of length 2 . This is supported by taking the limit as $k\to\infty$ of the equation of the fat circle in the first quadrant: $\lim_{k\to\infty}\left(1-x^{2k}\right)^{1/(2k)}=1$ for $0\le x<1$. So we guess that $\lim_{k\to\infty} L_{2k}=4\cdot 2=8$.

1. $y=\ln x \Rightarrow ds=\sqrt{1+(dy/dx)^2}\,dx=\sqrt{1+(1/x)^2}\,dx \Rightarrow S=\int_1^3 2\pi(\ln x)\sqrt{1+(1/x)^2}\,dx$

2. $y=\sin^2 x \Rightarrow ds=\sqrt{1+(dy/dx)^2}\,dx=\sqrt{1+(2\sin x\cos x)^2}\,dx \Rightarrow S=\int_0^{\pi/2} 2\pi\sin^2 x\sqrt{1+(2\sin x\cos x)^2}\,dx$ [by (7)]

3. $y=\sec x \Rightarrow ds=\sqrt{1+(dy/dx)^2}\,dx=\sqrt{1+(\sec x\tan x)^2}\,dx \Rightarrow S=\int_0^{\pi/4} 2\pi x\sqrt{1+(\sec x\tan x)^2}\,dx$ [by (8)]

4. $y=e^x \Rightarrow ds=\sqrt{1+(dy/dx)^2}\,dx=\sqrt{1+e^{2x}}\,dx \Rightarrow S=\int_0^{\ln 2} 2\pi x\sqrt{1+e^{2x}}\,dx$ [by (8)] or $\int_1^2 2\pi(\ln y)\sqrt{1+(1/y)^2}\,dy$ [by (6)]

5. $y=x^3 \Rightarrow y'=3x^2$. So

$$S=\int_0^2 2\pi y\sqrt{1+\left(y'\right)^2}\,dx=2\pi\int_0^2 x^3\sqrt{1+9x^4}\,dx\,[u=1+9x^4, du=36x^3\,dx]$$
$$=\frac{2\pi}{36}\int_1^{145}\sqrt{u}\,du=\frac{\pi}{18}\left[\frac{2}{3}u^{3/2}\right]_1^{145}=\frac{\pi}{27}\left(145\sqrt{145}-1\right)$$

6. The curve $9x=y^2+18$ is symmetric about the x –axis, so we only use its top half, given by $y=3\sqrt{x-2}$. $dy/dx=\dfrac{3}{2\sqrt{x-2}}$, so $1+(dy/dx)^2=1+\dfrac{9}{4(x-2)}$. Thus,

$$S=\int_2^6 2\pi\cdot 3\sqrt{x-2}\sqrt{1+\frac{9}{4(x-2)}}\,dx=6\pi\int_2^6\sqrt{x-2+\frac{9}{4}}\,dx=6\pi\int_2^6\left(x+\frac{1}{4}\right)^{1/2}dx$$
$$=6\pi\cdot\frac{2}{3}\left[\left(x+\frac{1}{4}\right)^{3/2}\right]_2^6=4\pi\left[\left(\frac{25}{4}\right)^{3/2}-\left(\frac{9}{4}\right)^{3/2}\right]=4\pi\left(\frac{125}{8}-\frac{27}{8}\right)=4\pi\cdot\frac{98}{8}=49\pi$$

7. $y=\sqrt{x} \Rightarrow 1+(dy/dx)^2=1+\left[1/\left(2\sqrt{x}\right)\right]^2=1+1/(4x)$. So

$$S=\int_4^9 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx=\int_4^9 2\pi\sqrt{x}\sqrt{1+\frac{1}{4x}}\,dx=2\pi\int_4^9\sqrt{x+\frac{1}{4}}\,dx$$

$$= 2\pi \left[\frac{2}{3} \left(x+\frac{1}{4} \right)^{3/2} \right]_4^9 = \frac{4\pi}{3} \left[\frac{1}{8} (4x+1)^{3/2} \right]_4^9 = \frac{\pi}{6} \left(37\sqrt{37}-17\sqrt{17} \right)$$

8. $y=\cos 2x \Rightarrow ds=\sqrt{1+(dy/dx)^2}\, dx=\sqrt{1+(-2\sin 2x)^2}\, dx \Rightarrow$

$$S=\int_0^{\pi/6} 2\pi\cos 2x\sqrt{1+4\sin^2 2x}\, dx=2\pi\int_0^{\sqrt{3}} \sqrt{1+u^2} \left(\frac{1}{4} \right) du[u=2\sin 2x, du=4\cos 2x\, dx]$$

$$=\frac{\pi}{2} \left[\frac{1}{2} u\sqrt{1+u^2}+\frac{1}{2}\ln \left(u+\sqrt{1+u^2} \right) \right]_0^{\sqrt{3}}=\frac{\pi}{2} \left[\frac{\sqrt{3}}{2}\cdot 2+\frac{1}{2}\ln \left(\sqrt{3}+2 \right) \right]=\frac{\pi\sqrt{3}}{2}+\frac{\pi}{4}\ln \left(2+\sqrt{3} \right)$$

9. $y=\cosh x \Rightarrow 1+(dy/dx)^2=1+\sinh^2 x=\cosh^2 x$. So

$$S = 2\pi\int_0^1 \cosh x\cosh x\, dx=2\pi\int_0^1 \frac{1}{2} (1+\cosh 2x)\, dx=\pi \left[x+\frac{1}{2}\sinh 2x \right]_0^1$$

$$= \pi \left(1+\frac{1}{2}\sinh 2 \right) \text{ or } \pi \left[1+\frac{1}{4} \left(e^2-e^{-2} \right) \right]$$

10. $y=\frac{x^3}{6}+\frac{1}{2x} \Rightarrow \frac{dy}{dx}=\frac{x^2}{2}-\frac{1}{2x^2} \Rightarrow$

$$\sqrt{1+(dy/dx)^2}=\sqrt{\frac{x^4}{4}+\frac{1}{2}+\frac{1}{4x^4}}=\sqrt{\left(\frac{x^2}{2}+\frac{1}{2x^2} \right)^2}=\frac{x^2}{2}+\frac{1}{2x^2} \Rightarrow$$

$$S=\int_{1/2}^1 2\pi \left(\frac{x^3}{6}+\frac{1}{2x} \right) \left(\frac{x^2}{2}+\frac{1}{2x^2} \right) dx=2\pi\int_{1/2}^1 \left(\frac{x^5}{12}+\frac{x}{12}+\frac{x}{4}+\frac{1}{4x^3} \right) dx$$

$$=2\pi\int_{1/2}^1 \left(\frac{x^5}{12}+\frac{x}{3}+\frac{x^{-3}}{4} \right) dx=2\pi \left[\frac{x^6}{72}+\frac{x^2}{6}-\frac{x^{-2}}{8} \right]_{1/2}^1$$

$$=2\pi \left[\left(\frac{1}{72}+\frac{1}{6}-\frac{1}{8} \right)-\left(\frac{1}{64\cdot 72}+\frac{1}{24}-\frac{1}{2} \right) \right]=2\pi \left(\frac{263}{512} \right)=\frac{263}{256}\pi$$

11. $x=\frac{1}{3} \left(y^2+2 \right)^{3/2} \Rightarrow dx/dy=\frac{1}{2} \left(y^2+2 \right)^{1/2}(2y)=y\sqrt{y^2+2} \Rightarrow 1+(dx/dy)^2=1+y^2 \left(y^2+2 \right)=\left(y^2+1 \right)^2$. So

$$S=2\pi\int_1^2 y\left(y^2+1\right)dy=2\pi\left[\frac{1}{4}y^4+\frac{1}{2}y^2\right]_1^2=2\pi\left(4+2-\frac{1}{4}-\frac{1}{2}\right)=\frac{21\pi}{2}$$

12. $x=1+2y^2\Rightarrow 1+(dx/dy)^2=1+(4y)^2=1+16y^2$. So

$$S=2\pi\int_1^2 y\sqrt{1+16y^2}\,dy=\frac{\pi}{16}\int_1^2\left(16y^2+1\right)^{1/2}32y\,dy=\frac{\pi}{16}\left[\frac{2}{3}\left(16y^2+1\right)^{3/2}\right]_1^2$$

$$=\frac{\pi}{24}\left(65\sqrt{65}-17\sqrt{17}\right)$$

13. $y=\sqrt[3]{x}\Rightarrow x=y^3\Rightarrow 1+(dx/dy)^2=1+9y^4$. So

$$S=2\pi\int_1^2 x\sqrt{1+(dx/dy)^2}\,dy=2\pi\int_1^2 y^3\sqrt{1+9y^4}\,dy=\frac{2\pi}{36}\int_1^2\sqrt{1+9y^4}\,36y^3\,dy$$

$$=\frac{\pi}{18}\left[\frac{2}{3}\left(1+9y^4\right)^{3/2}\right]_1^2=\frac{\pi}{27}\left(145\sqrt{145}-10\sqrt{10}\right)$$

14. $y=1-x^2\Rightarrow 1+(dy/dx)^2=1+4x^2\Rightarrow$

$$S=2\pi\int_0^1 x\sqrt{1+4x^2}\,dx=\frac{\pi}{4}\int_0^1 8x\sqrt{4x^2+1}\,dx=\frac{\pi}{4}\left[\frac{2}{3}\left(4x^2+1\right)^{3/2}\right]_0^1=\frac{\pi}{6}\left(5\sqrt{5}-1\right)$$

15. $x=\sqrt{a^2-y^2}\Rightarrow dx/dy=\frac{1}{2}(a^2-y^2)^{-1/2}(-2y)=-y/\sqrt{a^2-y^2}\Rightarrow$

$$1+(dx/dy)^2=1+\frac{y^2}{a^2-y^2}=\frac{a^2-y^2}{a^2-y^2}+\frac{y^2}{a^2-y^2}=\frac{a^2}{a^2-y^2}\Rightarrow$$

$S=\displaystyle\int_0^{a/2}2\pi\sqrt{a^2-y^2}\,\frac{a}{\sqrt{a^2-y^2}}\,dy=2\pi\int_0^{a/2}a\,dy=2\pi a\left[y\right]_0^{a/2}=2\pi a\left(\frac{a}{2}-0\right)=\pi a^2$. Note that this is $\frac{1}{4}$ the surface area of a sphere of radius a , and the length of the interval $y=0$ to $y=a/2$ is $\frac{1}{4}$ the length of the interval $y=-a$ to $y=a$.

16. $x=a\cosh(y/a)\Rightarrow 1+(dx/dy)^2=1+\sinh^2(y/a)=\cosh^2(y/a)$. So

$$S = 2\pi\int_{-a}^{a} a\cosh\left(\frac{y}{a}\right)\cosh\left(\frac{y}{a}\right)dy=4\pi a\int_0^a \cosh^2\left(\frac{y}{a}\right)dy=2\pi a\int_0^a\left[1+\cosh\left(\frac{2y}{a}\right)\right]dy$$

$$= 2\pi a\left[y+\frac{a}{2}\sinh\left(\frac{2y}{a}\right)\right]_0^a=2\pi a\left[a+\frac{a}{2}\sinh 2\right]=2\pi a^2\left[1+\frac{1}{2}\sinh 2\right] \text{ or } \frac{\pi a^2\left(e^2+4-e^{-2}\right)}{2}$$

17. $y=\ln x\Rightarrow dy/dx=1/x\Rightarrow 1+(dy/dx)^2=1+1/x^2\Rightarrow S=\int_1^3 2\pi\ln x\sqrt{1+1/x^2}\,dx$.

Let $f(x)=\ln x\sqrt{1+1/x^2}$. Since $n=10$, $\Delta x=\dfrac{3-1}{10}=\dfrac{1}{5}$. Then

$S\approx S_{10}=2\pi\cdot\dfrac{1/5}{3}[f(1)+4f(1.2)+2f(1.4)+\cdot\cdot\cdot+2f(2.6)+4f(2.8)+f(3)]\approx 9.023754$.

The value of the integral produced by a calculator is 9.024262 (to six decimal places).

18. $y=x+\sqrt{x}\Rightarrow dy/dx=1+\dfrac{1}{2}x^{-1/2}\Rightarrow 1+(dy/dx)^2=2+x^{-1/2}+\dfrac{1}{4}x^{-1}\Rightarrow$

$S=\displaystyle\int_1^2 2\pi\left(x+\sqrt{x}\right)\sqrt{2+\frac{1}{\sqrt{x}}+\frac{1}{4x}}\,dx$. Let $f(x)=(x+\sqrt{x})\sqrt{2+\dfrac{1}{\sqrt{x}}+\dfrac{1}{4x}}$.

Since $n=10$, $\Delta x=\dfrac{2-1}{10}=\dfrac{1}{10}$. Then

$S\approx S_{10}=2\pi\cdot\dfrac{1/10}{3}[f(1)+4f(1.1)+2f(1.2)+\cdot\cdot\cdot+2f(1.8)+4f(1.9)+f(2)]\approx 29.506566$.

The value of the integral produced by a calculator is 29.506568 (to six decimal places).

19. $y=\sec x\Rightarrow dy/dx=\sec x\tan x\Rightarrow 1+(dy/dx)^2=1+\sec^2 x\tan^2 x\Rightarrow$

$S=\displaystyle\int_0^{\pi/3} 2\pi\sec x\sqrt{1+\sec^2 x\tan^2 x}\,dx$. Let $f(x)=\sec x\sqrt{1+\sec^2 x\tan^2 x}$.

Since $n=10$, $\Delta x=\dfrac{\pi/3-0}{10}=\dfrac{\pi}{30}$. Then

$S\approx S_{10}=2\pi\cdot\dfrac{\pi/30}{3}\left[f(0)+4f\left(\dfrac{\pi}{30}\right)+2f\left(\dfrac{2\pi}{30}\right)+\cdot\cdot\cdot+2f\left(\dfrac{8\pi}{30}\right)+4f\left(\dfrac{9\pi}{30}\right)+f\left(\dfrac{\pi}{3}\right)\right]\approx 13.527296$

.

The value of the integral produced by a calculator is 13.516987 (to six decimal places).

20. $y=(1+e^x)^{1/2}\Rightarrow\dfrac{dy}{dx}=\dfrac{1}{2}(1+e^x)^{-1/2}\cdot e^x=\dfrac{e^x}{2(1+e^x)^{1/2}}\Rightarrow$

$$1+\left(\frac{dy}{dx}\right)^2=1+\frac{e^{2x}}{4(1+e^x)}=\frac{4+4e^x+e^{2x}}{4(1+e^x)}=\frac{(e^x+2)^2}{4(1+e^x)}\Rightarrow$$

$$S=\int_0^1 2\pi\sqrt{1+e^x}\ \frac{e^x+2}{2\sqrt{1+e^x}}\ dx=\pi\int_0^1(e^x+2)dx=\pi\Big[e^x+2x\Big]_0^1=\pi[(e+2)-(1+0)]=\pi(e+1)\ .$$

Let $f(x)=\frac{1}{2}(e^x+2)$. Since $n=10$, $\Delta x=\frac{1-0}{10}=\frac{1}{10}$. Then

$S\approx S_{10}=2\pi\cdot\frac{1/10}{3}\left[f(0)+4f(0.1)+2f(0.2)+\cdot\cdot\cdot+2f(0.8)+4f(0.9)+f(1)\right]\approx 11.681330$.

The value of the integral produced by a calculator is 11.681327 (to six decimal places).

21. $y=1/x\Rightarrow ds=\sqrt{1+(dy/dx)^2}\ dx=\sqrt{1+\left(-1/x^2\right)^2}\ dx=\sqrt{1+1/x^4}\ dx\Rightarrow$

$$S=\int_1^2 2\pi\cdot\frac{1}{x}\sqrt{1+\frac{1}{x^4}}\ dx=2\pi\int_1^2\frac{\sqrt{x^4+1}}{x^3}\ dx=2\pi\int_1^4\frac{\sqrt{u^2+1}}{u^2}\left(\frac{1}{2}\right)du[u=x^2,du=2xdx]$$

$$=\pi\int_1^4\frac{\sqrt{1+u^2}}{u^2}\ du=\pi\left[-\frac{\sqrt{1+u^2}}{u}+\ln\left(u+\sqrt{1+u^2}\right)\right]_1^4$$

$$=\pi\left[-\frac{\sqrt{17}}{4}+\ln\left(4+\sqrt{17}\right)+\frac{\sqrt{2}}{1}-\ln\left(1+\sqrt{2}\right)\right]=\pi\left[\sqrt{2}-\frac{\sqrt{17}}{4}+\ln\left(\frac{4+\sqrt{17}}{1+\sqrt{2}}\right)\right]$$

22. $y=\sqrt{x^2+1}\Rightarrow\frac{dy}{dx}=\frac{x}{\sqrt{x^2+1}}\Rightarrow ds=\sqrt{1+\left(\frac{dy}{dx}\right)^2}\ dx=\sqrt{1+\frac{x^2}{x^2+1}}\ dx\Rightarrow$

$$S=\int_0^3 2\pi\sqrt{x^2+1}\sqrt{1+\frac{x^2}{x^2+1}}\ dx=2\pi\int_0^3\sqrt{2x^2+1}\ dx=2\sqrt{2}\pi\int_0^3\sqrt{x^2+\left(\frac{1}{\sqrt{2}}\right)^2}\ dx$$

$$=2\sqrt{2}\pi\left[\frac{1}{2}x\sqrt{x^2+\frac{1}{2}}+\frac{1}{4}\ln\left(x+\sqrt{x^2+\frac{1}{2}}\right)\right]_0^3$$

$$=2\sqrt{2}\pi\left[\frac{3}{2}\sqrt{9+\frac{1}{2}}+\frac{1}{4}\ln\left(3+\sqrt{9+\frac{1}{2}}\right)-\frac{1}{4}\ln\frac{1}{\sqrt{2}}\right]=2\sqrt{2}\pi\left[\frac{3}{2}\sqrt{\frac{19}{2}}+\frac{1}{4}\ln\left(3+\sqrt{\frac{19}{2}}\right)+\right.$$

$$=2\sqrt{2}\pi\left[\frac{3}{2}\frac{\sqrt{19}}{\sqrt{2}}+\frac{1}{4}\ln\left(3\sqrt{2}+\sqrt{19}\right)\right]=3\sqrt{19}\pi+\frac{\pi}{\sqrt{2}}\ln\left(3\sqrt{2}+\sqrt{19}\right)$$

23. $y=x^3$ and $0\le y\le 1\Rightarrow y'=3x^2$ and $0\le x\le 1$.

$$S=\int_0^1 2\pi x\sqrt{1+\left(3x^2\right)^2}\,dx=2\pi\int_0^3\sqrt{1+u^2}\,\frac{1}{6}\,du[u=3x^2,du=6x\,dx]$$

$$=\frac{\pi}{3}\int_0^3\sqrt{1+u^2}\,du=\text{[or use CAS]}\,\frac{\pi}{3}\left[\frac{1}{2}u\sqrt{1+u^2}+\frac{1}{2}\ln\left(u+\sqrt{1+u^2}\right)\right]_0^3$$

$$=\frac{\pi}{3}\left[\frac{3}{2}\sqrt{10}+\frac{1}{2}\ln\left(3+\sqrt{10}\right)\right]=\frac{\pi}{6}\left[3\sqrt{10}+\ln\left(3+\sqrt{10}\right)\right]$$

24. $y=\ln(x+1)$, $0\le x\le 1$. $ds=\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx=\sqrt{1+\left(\frac{1}{x+1}\right)^2}\,dx$, so S

$$=\int_0^1 2\pi x\sqrt{1+\frac{1}{(x+1)^2}}\,dx=\int_1^2 2\pi(u-1)\sqrt{1+\frac{1}{u^2}}\,du[u=x+1,du=dx]$$

$$=2\pi\int_1^2 u\,\frac{\sqrt{1+u^2}}{u}\,du-2\pi\int_1^2\frac{\sqrt{1+u^2}}{u}\,du=2\pi\int_1^2\sqrt{1+u^2}\,du-2\pi\int_1^2\frac{\sqrt{1+u^2}}{u}\,du$$

$$=\text{[or use CAS]}\;2\pi\left[\frac{1}{2}u\sqrt{1+u^2}+\frac{1}{2}\ln\left(u+\sqrt{1+u^2}\right)\right]_1^2-2\pi\left[\sqrt{1+u^2}-\ln\left(\frac{1+\sqrt{1+u^2}}{u}\right)\right]_1^2$$

$$=2\pi\left[\sqrt{5}+\frac{1}{2}\ln\left(2+\sqrt{5}\right)-\frac{1}{2}\sqrt{2}-\frac{1}{2}\ln\left(1+\sqrt{2}\right)\right]-2\pi\left[\sqrt{5}-\ln\left(\frac{1+\sqrt{5}}{2}\right)-\sqrt{2}+\ln\left(1+\sqrt{2}\right)\right]$$

$$=2\pi\left[\frac{1}{2}\ln\left(2+\sqrt{5}\right)+\ln\left(\frac{1+\sqrt{5}}{2}\right)+\frac{\sqrt{2}}{2}-\frac{3}{2}\ln\left(1+\sqrt{2}\right)\right]$$

25. $S=2\pi\int_1^\infty y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx=2\pi\int_1^\infty\frac{1}{x}\sqrt{1+\frac{1}{x^4}}\,dx=2\pi\int_1^\infty\frac{\sqrt{x^4+1}}{x^3}\,dx$. Rather than trying to

evaluate this integral, note that $\sqrt{x^4+1} > \sqrt{x^4} = x^2$ for $x>0$. Thus, if the area is finite,

$$S=2\pi\int_1^{\infty} \frac{\sqrt{x^4+1}}{x^3}\,dx > 2\pi\int_1^{\infty} \frac{x^2}{x^3}\,dx = 2\pi\int_1^{\infty} \frac{1}{x}\,dx$$

But we know that this integral diverges, so the area S is infinite.

26. $S=\int_0^{\infty} 2\pi y\sqrt{1+(dy/dx)^2}\,dx = 2\pi\int_0^{\infty} e^{-x}\sqrt{1+(-e^{-x})^2}\,dx$.

Evaluate $I=\int e^{-x}\sqrt{1+(-e^{-x})^2}\,dx$ by using the substitution $u=-e^{-x}$, $du=e^{-x}dx$.

$$I=\int\sqrt{1+u^2}\,du = \frac{1}{2}u\sqrt{1+u^2}+\frac{1}{2}\ln\left(u+\sqrt{1+u^2}\right)+C$$

$$= \frac{1}{2}(-e^{-x})\sqrt{1+e^{-2x}}+\frac{1}{2}\ln\left(-e^{-x}+\sqrt{1+e^{-2x}}\right)+C$$

Returning to the surface area integral, we have

$$S = 2\pi\lim_{t\to\infty}\int_0^t e^{-x}\sqrt{1+(-e^{-x})^2}\,dx$$

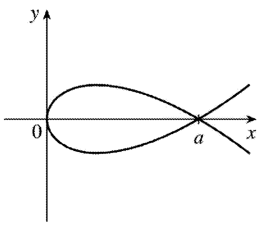
$$=2\pi\lim_{t\to\infty}\left[\frac{1}{2}(-e^{-x})\sqrt{1+e^{-2x}}+\frac{1}{2}\ln\left(-e^{-x}+\sqrt{1+e^{-2x}}\right)\right]_0^t$$

$$=2\pi\lim_{t\to\infty}\left\{\left[\frac{1}{2}(-e^{-t})\sqrt{1+e^{-2t}}+\frac{1}{2}\ln\left(-e^{-t}+\sqrt{1+e^{-2t}}\right)\right]-\left[\frac{1}{2}(-1)\sqrt{1+1}+\frac{1}{2}\ln\left(-1+\sqrt{1+1}\right)\right]\right\}$$

$$=2\pi\left\{\left[\frac{1}{2}(0)\sqrt{1}+\frac{1}{2}\ln\left(0+\sqrt{1}\right)\right]-\left[-\frac{1}{2}\sqrt{2}+\frac{1}{2}\ln\left(-1+\sqrt{2}\right)\right]\right\}$$

$$=2\pi\left\{[0]+\frac{1}{2}\left[\sqrt{2}-\ln\left(\sqrt{2}-1\right)\right]\right\} = \pi\left[\sqrt{2}-\ln\left(\sqrt{2}-1\right)\right]$$

27. Since $a>0$, the curve $3ay^2=x(a-x)^2$ only has points with $x\geq 0$. ($3ay^2\geq 0 \Rightarrow x(a-x)^2\geq 0 \Rightarrow x\geq 0$.) The curve is symmetric about the x –axis (since the equation is unchanged when y is replaced by $-y$). $y=0$ when $x=0$ or a , so the curve’s loop extends from $x=0$ to $x=a$.



$$\frac{d}{dx}(3ay^2)=\frac{d}{dx} \Rightarrow 6ay\frac{dy}{dx}=x\cdot 2(a-x)(-1)+(a-x)^2 \Rightarrow \frac{dy}{dx}=\frac{(a-x)[-2x+a-x]}{6ay} \Rightarrow$$

$$\left(\frac{dy}{dx}\right)^2=\frac{(a-x)^2(a-3x)^2}{36a^2y^2}=\frac{(a-x)^2(a-3x)^2}{36a^2}\cdot\frac{3a}{x(a-x)^2}\left[\begin{array}{c}\text{the last fraction}\\ \text{is } 1/y^2\end{array}\right]=\frac{(a-3x)^2}{12ax} \Rightarrow$$

$$1+\left(\frac{dy}{dx}\right)^2=1+\frac{a^2-6ax+9x^2}{12ax}=\frac{12ax}{12ax}+\frac{a^2-6ax+9x^2}{12ax}=\frac{a^2+6ax+9x^2}{12ax}=\frac{(a+3x)^2}{12ax} \quad \text{for } x\neq 0 .$$

(a)

$$S=\int_{x=0}^{a} 2\pi y\,ds=2\pi\int_0^a \frac{\sqrt{x}(a-x)}{\sqrt{3a}}\cdot\frac{a+3x}{\sqrt{12ax}}\,dx=2\pi\int_0^a \frac{(a-x)(a+3x)}{6a}\,dx$$

$$=\frac{\pi}{3a}\int_0^a(a^2+2ax-3x^2)dx=\frac{\pi}{3a}\left[a^2x+ax^2-x^3\right]_0^a=\frac{\pi}{3a}(a^3+a^3-a^3)=\frac{\pi}{3a}\cdot a^3=\frac{\pi a^2}{3} .$$

Note that we have rotated the top half of the loop about the x –axis. This generates the full surface.
(b) We must rotate the full loop about the y –axis, so we get double the area obtained by rotating the top half of the loop:

$$S=2\cdot 2\pi\int_{x=0}^{a} x\,ds=4\pi\int_0^a x\frac{a+3x}{\sqrt{12ax}}\,dx=\frac{4\pi}{2\sqrt{3a}}\int_0^a x^{1/2}(a+3x)\,dx$$

$$=\frac{2\pi}{\sqrt{3a}}\int_0^a(ax^{1/2}+3x^{3/2})dx=\frac{2\pi}{\sqrt{3a}}\left[\frac{2}{3}ax^{3/2}+\frac{6}{5}x^{5/2}\right]_0^a=\frac{2\pi\sqrt{3}}{3\sqrt{a}}\left(\frac{2}{3}a^{5/2}+\frac{6}{5}a^{5/2}\right)$$

$$=\frac{2\pi\sqrt{3}}{3}\left(\frac{2}{3}+\frac{6}{5}\right)a^2=\frac{2\pi\sqrt{3}}{3}\left(\frac{28}{15}\right)a^2=\frac{56\pi\sqrt{3}\,a^2}{45}$$

28. In general, if the parabola $y=ax^2$, $-c\leq x\leq c$, is rotated about the y –axis, the surface area it generates is

$$2\pi\int_0^c x\sqrt{1+(2ax)^2}\,dx = 2\pi\int_0^{2ac}\frac{u}{2a}\sqrt{1+u^2}\,\frac{1}{2a}\,du\ [u=2ax,du=2a\,dx]$$

$$= \frac{\pi}{4a^2}\int_0^{2ac}\left(1+u^2\right)^{1/2}2u\,du = \frac{\pi}{4a^2}\left[\frac{2}{3}\left(1+u^2\right)^{3/2}\right]_0^{2ac}$$
$$= \frac{\pi}{6a^2}\left[\left(1+4a^2c^2\right)^{3/2}-1\right]$$

Here $2c=10$ ft and $ac^2=2$ ft, so $c=5$ and $a=\frac{2}{25}$. Thus, the surface area is

$$S = \frac{\pi}{6}\,\frac{625}{4}\left[\left(1+4\cdot\frac{4}{625}\cdot 25\right)^{3/2}-1\right] = \frac{625\pi}{24}\left[\left(1+\frac{16}{25}\right)^{3/2}-1\right] = \frac{625\pi}{24}\left(\frac{41\sqrt{41}}{125}-1\right)$$
$$= \frac{5\pi}{24}\left(41\sqrt{41}-125\right)\approx 90.01\text{ft}^2$$

29. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \Rightarrow \frac{y(dy/dx)}{b^2}=-\frac{x}{a^2} \Rightarrow \frac{dy}{dx}=-\frac{b^2x}{a^2y} \Rightarrow$

$$1+\left(\frac{dy}{dx}\right)^2 = 1+\frac{b^4x^2}{a^4y^2} = \frac{b^4x^2+a^4y^2}{a^4y^2} = \frac{b^4x^2+a^4b^2\left(1-x^2/a^2\right)}{a^4b^2\left(1-x^2/a^2\right)} = \frac{a^4b^2+b^4x^2-a^2b^2x^2}{a^4b^2-a^2b^2x^2}$$
$$= \frac{a^4+b^2x^2-a^2x^2}{a^4-a^2x^2} = \frac{a^4-\left(a^2-b^2\right)x^2}{a^2\left(a^2-x^2\right)}$$

The ellipsoid’s surface area is twice the area generated by rotating the first quadrant portion of the ellipse about the x– axis. Thus,

$$S = 2\int_0^a 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = 4\pi\int_0^a \frac{b}{a}\sqrt{a^2-x^2}\,\frac{\sqrt{a^4-\left(a^2-b^2\right)x^2}}{a\sqrt{a^2-x^2}}\,dx$$
$$= \frac{4\pi b}{a^2}\int_0^a\sqrt{a^4-\left(a^2-b^2\right)x^2}\,dx = \frac{4\pi b}{a^2}\int_0^{a\sqrt{a^2-b^2}}\sqrt{a^4-u^2}\,\frac{du}{\sqrt{a^2-b^2}}\quad\left[u=\sqrt{a^2-b^2}\,x\right]$$
$$= \frac{4\pi b}{a^2\sqrt{a^2-b^2}}\left[\frac{u}{2}\sqrt{a^4-u^2}+\frac{a^4}{2}\sin^{-1}\frac{u}{a^2}\right]_0^{a\sqrt{a^2-b^2}}$$

$$= \frac{4\pi b}{a^2\sqrt{a^2-b^2}}\left[\frac{a\sqrt{a^2-b^2}}{2}\sqrt{a^4-a^2\left(a^2-b^2\right)}+\frac{a^4}{2}\sin^{-1}\frac{\sqrt{a^2-b^2}}{a}\right]=2\pi\left[b^2+\frac{a^2b\sin^{-1}\frac{\sqrt{a^2-b^2}}{a}}{\sqrt{a^2-b^2}}\right]$$

30. The upper half of the torus is generated by rotating the curve $(x-R)^2+y^2=r^2$, $y>0$, about the y–axis. $y\frac{dy}{dx}=-(x-R)\Rightarrow 1+\left(\frac{dy}{dx}\right)^2=1+\frac{(x-R)^2}{y^2}=\frac{y^2+(x-R)^2}{y^2}=\frac{r^2}{r^2-(x-R)^2}$. Thus,

$$S=2\int_{R-r}^{R+r}2\pi x\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx=4\pi\int_{R-r}^{R+r}\frac{rx}{\sqrt{r^2-(x-R)^2}}\,dx$$

$$=4\pi r\int_{-r}^{r}\frac{u+R}{\sqrt{r^2-u^2}}\,du\,[u=x-R]$$

$$=4\pi r\int_{-r}^{r}\frac{u\,du}{\sqrt{r^2-u^2}}+4\pi Rr\int_{-r}^{r}\frac{du}{\sqrt{r^2-u^2}}$$

$$=4\pi r\cdot 0+8\pi Rr\int_{0}^{r}\frac{du}{\sqrt{r^2-u^2}}$$ [since the first integrand is odd and the second is even]

$$=8\pi Rr\left[\sin^{-1}(u/r)\right]_0^r=8\pi Rr\left(\frac{\pi}{2}\right)=4\pi^2Rr$$

31. The analogue of $f\left(x_i^*\right)$ in the derivation of (4) is now $c-f\left(x_i^*\right)$, so

$$S=\lim_{n\to\infty}\sum_{i=1}^{n}2\pi\left[c-f\left(x_i^*\right)\right]\sqrt{1+\left[f'\left(x_i^*\right)\right]^2}\,\Delta x=\int_a^b 2\pi\,[c-f(x)]\sqrt{1+\left[f'(x)\right]^2}\,dx .$$

32. $y=x^{1/2}\Rightarrow y'=\frac{1}{2}x^{-1/2}\Rightarrow 1+\left(y'\right)^2=1+1/4x$, so by Exercise 31,

$S=\int_0^4 2\pi\left(4-\sqrt{x}\right)\sqrt{1+1/(4x)}\,dx$. Using a CAS, we get $S=2\pi\ln\left(\sqrt{17}+4\right)+\frac{\pi}{6}\left(31\sqrt{17}+1\right)\approx 80.6095$.

33. For the upper semicircle, $f(x)=\sqrt{r^2-x^2}$, $f'(x)=-x/\sqrt{r^2-x^2}$. The surface area generated is

$$S_1=\int_{-r}^{r}2\pi\left(r-\sqrt{r^2-x^2}\right)\sqrt{1+\frac{x^2}{r^2-x^2}}\,dx=4\pi\int_0^r\left(r-\sqrt{r^2-x^2}\right)\frac{r}{\sqrt{r^2-x^2}}\,dx$$

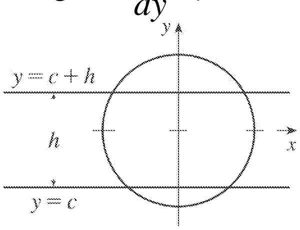
$$=4\pi\int_0^r\left(\frac{r^2}{\sqrt{r^2-x^2}}-r\right)dx$$

For the lower semicircle, $f(x)=-\sqrt{r^2-x^2}$ and $f'(x)=\frac{x}{\sqrt{r^2-x^2}}$, so $S_2=4\pi\int_0^r\left(\frac{r^2}{\sqrt{r^2-x^2}}+r\right)dx$.

Thus, the total area is

$$S=S_1+S_2=8\pi\int_0^r\left(\frac{r^2}{\sqrt{r^2-x^2}}\right)dx=8\pi\left[r^2\sin^{-1}\left(\frac{x}{r}\right)\right]_0^r=8\pi r^2\left(\frac{\pi}{2}\right)=4\pi^2r^2 .$$

34. Take the sphere $x^2+y^2+z^2=\frac{1}{4}d^2$ and let the intersecting planes be $y=c$ and $y=c+h$, where $-\frac{1}{2}d\le c\le\frac{1}{2}d-h$. The sphere intersects the $xy-$ plane in the circle $x^2+y^2=\frac{1}{4}d^2$. From this equation, we get $x\frac{dx}{dy}+y=0$, so $\frac{dx}{dy}=-\frac{y}{x}$. The desired surface area is



$$S = 2\pi \int x\,ds = 2\pi \int_c^{c+h} x\sqrt{1+(dx/dy)^2}\,dy = 2\pi \int_c^{c+h} x\sqrt{1+y^2/x^2}\,dy = 2\pi \int_c^{c+h} \sqrt{x^2+y^2}\,dy$$

$$= 2\pi \int_c^{c+h} \frac{1}{2}\,d\,dy = \pi d \int_c^{c+h} dy = \pi dh$$

35. In the derivation of (4), we computed a typical contribution to the surface area to be $2\pi \frac{y_{i-1}+y_i}{2} \left|P_{i-1}P_i\right|$, the area of a frustum of a cone. When $f(x)$ is not necessarily positive, the approximations $y_i = f\left(x_i\right) \approx f\left(x_i^*\right)$ and $y_{i-1} = f\left(x_{i-1}\right) \approx f\left(x_i^*\right)$ must be replaced by $y_i = \left|f\left(x_i\right)\right| \approx \left|f\left(x_i^*\right)\right|$ and $y_{i-1} = \left|f\left(x_{i-1}\right)\right| \approx \left|f\left(x_i^*\right)\right|$. Thus, $2\pi \frac{y_{i-1}+y_i}{2} \left|P_{i-1}P_i\right| \approx 2\pi \left|f\left(x_i^*\right)\right| \sqrt{1+\left[f'\left(x_i^*\right)\right]^2}\,\Delta x$. Continuing with the rest of the derivation as before, we obtain $S = \int_a^b 2\pi\,|f(x)|\sqrt{1+\left[f'(x)\right]^2}\,dx$.

36. Since $g(x) = f(x)+c$, we have $g'(x) = f'(x)$. Thus,

$$S_g = \int_a^b 2\pi g(x)\sqrt{1+\left[g'(x)\right]^2}\,dx = \int_a^b 2\pi\,[f(x)+c]\sqrt{1+\left[f'(x)\right]^2}\,dx$$

$$= \int_a^b 2\pi f(x)\sqrt{1+\left[f'(x)\right]^2}\,dx + 2\pi c \int_a^b \sqrt{1+\left[f'(x)\right]^2}\,dx = S_f + 2\pi cL$$

1. The weight density of water is $\delta = 62.5\ \text{lb}/\text{ft}^3$.

(a) $P = \delta d \approx \left(62.5\text{lb}/\text{ft}^3\right)(3\text{ft}) = 187.5\ \text{lb}/\text{ft}^2$

(b) $F = PA \approx \left(187.5\text{lb}/\text{ft}^2\right)(5\text{ft})(2\text{ft}) = 1875$ lb. (A is the area of the bottom of the tank.)

(c) As in Example 1, the area of the i th strip is $2(\Delta x)$ and the pressure is $\delta d = \delta x_i$. Thus,

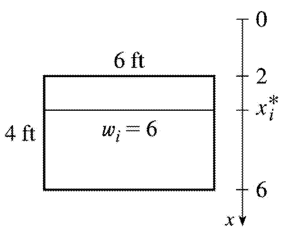
$$F = \int_0^3 \delta x \cdot 2dx \approx (62.5)(2)\int_0^3 x dx = 125\left[\frac{1}{2}x^2\right]_0^3 = 125\left(\frac{9}{2}\right) = 562.5\text{lb}$$

2. **(a)** $P = \rho g d = 1030(9.8)(2.5) = 25,235 \approx 2.52\times 10^4$ Pa $= 25.2$ kPa

(b) $F = PA \approx \left(2.52\times 10^4 \text{N/m}^2\right)\left(50\text{m}^2\right) = 1.26\times 10^6$ N

(c) $F = \int_0^{2.5} \rho g x \cdot 5dx = (1030)(9.8)(5)\int_0^{2.5} x dx \approx 2.52\times 10^4\left[x^2\right]_0^{2.5} \approx 1.58\times 10^5$ N

3. Set up a vertical x –axis as shown, with $x=0$ at the water's surface and x increasing in the downward direction. Then the area of the i th rectangular strip is $6\Delta x$ and the pressure on the strip is δx_i^* (where $\delta \approx 62.5\text{lb}/\text{ft}^3$). Thus, the hydrostatic force on the strip is $\delta x_i^* \cdot 6\Delta x$ and the total hydrostatic force $\approx \sum_{i=1}^n \delta x_i^* \cdot 6\Delta x$. The total force

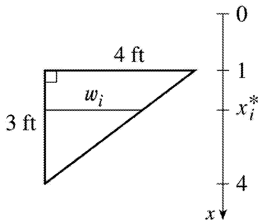


$$F = \lim_{n\to\infty} \sum_{i=1}^n \delta x_i^* \cdot 6\Delta x = \int_2^6 \delta x \cdot 6dx = 6\delta\int_2^6 x dx$$

$$= 6\delta\left[\frac{1}{2}x^2\right]_2^6 = 6\delta(18-2) = 96\delta \approx 6000\text{lb}$$

4. Set up a vertical x –axis as shown. Then the area of the i th rectangular strip is $\frac{4}{3}(4-x_i^*)\Delta x$. The pressure on the strip is δx_i^* , so the hydrostatic force on the strip is $\delta x_i^* \cdot \frac{4}{3}(4-x_i^*)\Delta x$ and the total force on the plate

$\approx \sum_{i=1}^{n} \delta x_i^* \cdot \frac{4}{3}(4-x_i^*)\Delta x$. The total force



$$F = \lim_{n\to\infty} \sum_{i=1}^{n} \delta x_i^* \cdot \frac{4}{3}(4-x_i^*)\Delta x = \int_1^4 \delta x \cdot \frac{4}{3}(4-x)dx = \frac{4}{3}\delta \int_1^4 (4x-x^2)dx$$
$$= \frac{4}{3}\delta\left[2x^2 - \frac{1}{3}x^3\right]_1^4 = \frac{4}{3}\delta\left[\left(32-\frac{64}{3}\right)-\left(2-\frac{1}{3}\right)\right] = \frac{4}{3}\delta(9) = 12\delta \approx 750 \text{ lb}$$

5. Since an equation for the shape is $x^2+y^2=10^2$ ($x \geq 0$), we have $y=\sqrt{100-x^2}$. Thus, the area of the i th strip is $2\sqrt{100-(x_i^*)^2}\,\Delta x$
and the pressure on the strip is $\rho g x_i^*$, so the hydrostatic force on the
strip is $\rho g x_i^* \cdot 2\sqrt{100-(x_i^*)^2}\,\Delta x$ and the total force on the

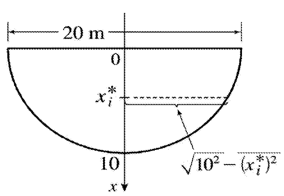
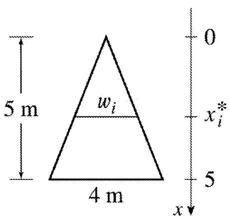


plate $\approx \sum_{i=1}^{n} \rho g x_i^* \cdot 2\sqrt{100-(x_i^*)^2}\,\Delta x$. The total force

$$F = \lim_{n\to\infty} \sum_{i=1}^{n} \rho g x_i^* \cdot 2\sqrt{100-(x_i^*)^2}\,\Delta x = \int_0^{10} 2\rho g x\sqrt{100-x^2}\,dx$$
$$= -\rho g \int_0^{10} \left(100-x^2\right)^{1/2}(-2x)dx = -\rho g\left[\frac{2}{3}(100-x^2)^{3/2}\right]_0^{10} = -\frac{2}{3}\rho g(0-1000)$$
$$= \frac{2000}{3}\rho g \approx \frac{2000}{3}\cdot 1000 \cdot 9.8 \approx 6.5\times 10^6 \text{N} \left[\rho \approx 1000 \text{kg/m}^3 \text{and} g \approx 9.8 \text{m/s}^2.\right]$$

6. By similar triangles, $w_i/4 = x_i^*/5$, so $w_i = \frac{4}{5} x_i^*$ and the area of the i th strip is $\frac{4}{5} x_i^* \Delta x$. The pressure on the strip is $\rho g x_i^*$, so the hydrostatic force on the strip is $\rho g x_i^* \cdot \frac{4}{5} x_i^* \Delta x$ and the total force on the plate $\approx \sum_{i=1}^{n} \rho g x_i^* \cdot \frac{4}{5} x_i^* \Delta x$. The total force



$$F = \lim_{n\to\infty} \sum_{i=1}^{n} \rho g x_i^* \cdot \frac{4}{5} x_i^* \Delta x = \int_0^5 \rho g x \cdot \frac{4}{5} x dx = \frac{4}{5} \rho g \left[\frac{1}{3} x^3 \right]_0^5 = \frac{4}{5} \rho g \cdot \frac{125}{3} = \frac{100}{3} \rho g$$

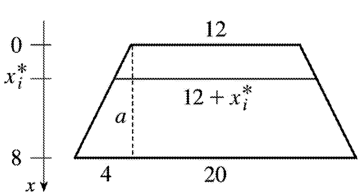
$$\approx \frac{100}{3} \cdot 1000 \cdot 9.8 \approx 3.3 \times 10^5 \text{ N.}$$

7. Using similar triangles, $\frac{4 \text{ ft wide}}{8 \text{ ft high}} = \frac{a \text{ ft wide}}{x_i^* \text{ ft high}}$, so $a = \frac{1}{2} x_i^*$ and the width of the i th rectangular strip is $12 + 2a = 12 + x_i^*$. The area of the strip is $\left(12 + x_i^*\right) \Delta x$. The pressure on the strip is δx_i^* .

$$F = \lim_{n\to\infty} \sum_{i=1}^{n} \delta x_i^* \left(12 + x_i^*\right) \Delta x = \int_0^8 \delta x \cdot (12 + x) dx$$

$$= \delta \int_0^8 \left(12x + x^2\right) dx = \delta \left[6x^2 + \frac{x^3}{3} \right]_0^8 = \delta \left(384 + \frac{512}{3} \right)$$

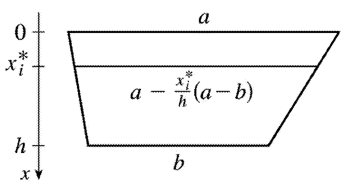
$$= (62.5) \frac{1664}{3} \approx 3.47 \times 10^4 \text{lb}$$



8. In the figure, deleting a $b \times h$ rectangle leaves a triangle with base $a-b$ and height h . By similar triangles,

$$\frac{(a-b)\ \text{ft wide}}{h\ \text{ft high}} = \frac{d\ \text{ft wide}}{\left(h-x_i^*\right)\ \text{ft high}}$$, so the width of the triangle is

$$d=\frac{h-x_i^*}{h}(a-b)=\left(1-\frac{x_i^*}{h}\right)(a-b)=a-b-\frac{x_i^*}{h}(a-b)$$



and the width of the trapezoid is $b+d=a-\frac{x_i^*}{h}(a-b)$. The area of the i th rectangular strip is $\left[a-\frac{x_i^*}{h}(a-b)\right]\Delta x$ and the pressure on it is $\rho g x_i^*$.

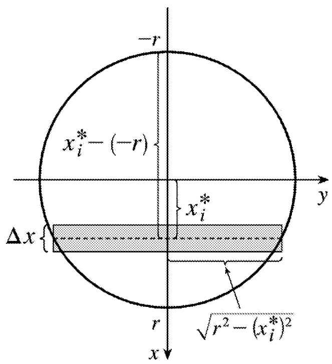
$$\begin{aligned} F &= \lim_{n\to\infty}\sum_{i=1}^{n}\rho g x_i^*\left[a-\frac{x_i^*}{h}(a-b)\right]\Delta x=\int_0^h \rho g x\left[a-\frac{x}{h}(a-b)\right]dx \\ &= \rho g a\int_0^h x\,dx+\frac{\rho g(b-a)}{h}\int_0^h x^2\,dx=\rho g a\,\frac{h^2}{2}+\rho g\,\frac{b-a}{h}\,\frac{h^3}{3} \\ &= \rho g h^2\left(\frac{a}{2}+\frac{b-a}{3}\right)=\rho g h^2\,\frac{a+2b}{6}\approx\frac{500}{3}\,gh^2(a+2b)\text{N} \end{aligned}$$

9. From the figure, the area of the i th rectangular strip is $2\sqrt{r^2-\left(x_i^*\right)^2}\,\Delta x$ and the pressure on it is $\rho g\left(x_i^*+r\right)$.

$$\begin{aligned} F &= \lim_{n\to\infty}\sum_{i=1}^{n}\rho g\left(x_i^*+r\right)2\sqrt{r^2-\left(x_i^*\right)^2}\,\Delta x \\ &= \int_{-r}^{r}\rho g(x+r)\cdot 2\sqrt{r^2-x^2}\,dx \end{aligned}$$

$$= \rho g\int_{-r}^{r}\sqrt{r^2-x^2}\,2x\,dx+2\rho gr\int_{-r}^{r}\sqrt{r^2-x^2}\,dx$$

The first integral is 0 because the integrand is an odd function. The second integral can be interpreted as the area of a semicircular disk with radius r , or we could make the trigonometric substitution $x=r\sin\theta$. Continuing: $F=\rho g\cdot 0+2\rho gr\cdot \frac{1}{2}\pi r^2=\rho g\pi r^3=1000g\pi r^3$ N (SI units assumed).



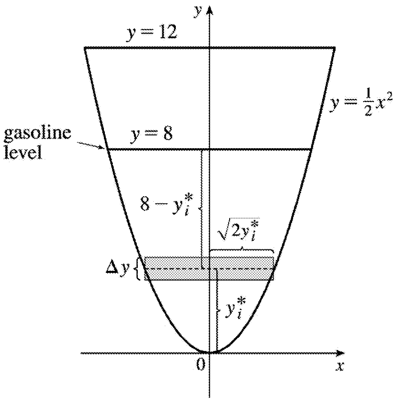
10. The area of the i th rectangular strip is $2\sqrt{2y_i^*}\,\Delta y$ and the pressure on it is $\delta d_i=\delta\left(8-y_i^*\right)$.

$$F = \int_0^8 \delta(8-y)2\sqrt{2y}\,dy=42\cdot 2\cdot\sqrt{2}\int_0^8(8-y)y^{1/2}dy$$

$$= 84\sqrt{2}\int_0^8\left(8y^{1/2}-y^{3/2}\right)dy=84\sqrt{2}\left[8\cdot\frac{2}{3}y^{3/2}-\frac{2}{5}y^{5/2}\right]_0^8$$

$$= 84\sqrt{2}\left[8\cdot\frac{2}{3}\cdot 16\sqrt{2}-\frac{2}{5}\cdot 128\sqrt{2}\right]$$

$$= 84\sqrt{2}\cdot 256\sqrt{2}\left(\frac{1}{3}-\frac{1}{5}\right)=43{,}008\cdot\frac{2}{15}=5734.4\text{lb}$$

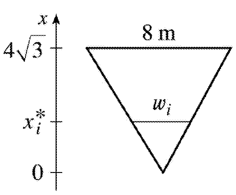


11. By similar triangles, $\dfrac{8}{4\sqrt{3}}=\dfrac{w_i}{x_i^*} \Rightarrow w_i=\dfrac{2x_i^*}{\sqrt{3}}$. The area of the i th rectangular strip is $\dfrac{2x_i^*}{\sqrt{3}}\,\Delta x$ and the pressure on it is $\rho g\left(4\sqrt{3}-x_i^*\right)$.

$$F=\int_0^{4\sqrt{3}}\rho g\left(4\sqrt{3}-x\right)\frac{2x}{\sqrt{3}}\,dx=8\rho g\int_0^{4\sqrt{3}}x\,dx-\frac{2\rho g}{\sqrt{3}}\int_0^{4\sqrt{3}}x^2\,dx$$

$$=4\rho g\left[x^2\right]_0^{4\sqrt{3}}-\frac{2\rho g}{3\sqrt{3}}\left[x^3\right]_0^{4\sqrt{3}}=192\rho g-\frac{2\rho g}{3\sqrt{3}}\,64\cdot 3\sqrt{3}$$

$$=192\rho g-128\rho g=64\rho g\approx 64(840)(9.8)\approx 5.27\times 10^5\ \mathrm{N}$$



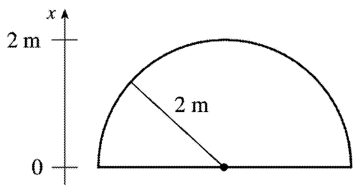
12.

$$F=\int_0^2\rho g(10-x)2\sqrt{4-x^2}\,dx$$

$$=20\rho g\int_0^2\sqrt{4-x^2}\,dx-\rho g\int_0^2\sqrt{4-x^2}\,2x\,dx$$

$$=20\rho g\,\frac{1}{4}\pi\left(2^2\right)-\rho g\int_0^4 u^{1/2}\,du\ \ [\ u=4-x^2\ ,\ du=-2x\,dx\ ]$$

$$=20\pi\rho g-\frac{2}{3}\rho g\left[u^{3/2}\right]_0^4=20\pi\rho g-\frac{16}{3}\rho g=\rho g\left(20\pi-\frac{16}{3}\right)$$
$$=(1000)(9.8)\left(20\pi-\frac{16}{3}\right)\approx 5.63\times 10^5\ \text{N}$$



13. **(a)** The top of the cube has depth $d=1$ m -20 cm $=80$ cm $=0.8$ m.

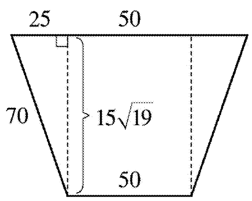
$$F=\rho gdA\approx(1000)(9.8)(0.8)(0.2)^2=313.6\approx 314\text{N}$$

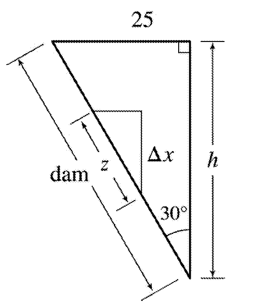
(b) The area of a strip is $0.2\Delta x$ and the pressure on it is $\rho g x_i^*$.

$$F=\int_{0.8}^{1}\rho gx(0.2)dx=0.2\rho g\left[\frac{1}{2}x^2\right]_{0.8}^{1}=(0.2\rho g)(0.18)=0.036\rho g=0.036(1000)(9.8)$$
$$=352.8\approx 353\text{N}$$

14. The height of the dam is $h=\sqrt{70^2-25^2}\cos 30^\circ=15\sqrt{19}\left(\frac{\sqrt{3}}{2}\right)$. From the solution for Exercise 8, the width of the trapezoid is $100-\frac{x}{h}(100-50)=100-\frac{50x}{h}$. From the small triangle in the second figure, $\cos 30^\circ=\frac{\Delta x}{z}\Rightarrow z=\Delta x\sec 30^\circ=2\Delta x/\sqrt{3}$.

$$F=\int_0^h\delta x\left(100-\frac{50x}{h}\right)\frac{2}{\sqrt{3}}dx=\frac{200\delta}{\sqrt{3}}\int_0^h xdx-\frac{100\delta}{h\sqrt{3}}\int_0^h x^2dx$$
$$=\frac{200\delta}{\sqrt{3}}\frac{h^2}{2}-\frac{100\delta}{h\sqrt{3}}\frac{h^3}{3}=\frac{200\delta h^2}{3\sqrt{3}}=\frac{200(62.5)}{3\sqrt{3}}\cdot\frac{12{,}825}{4}$$
$$\approx 7.71\times 10^6\text{lb}$$

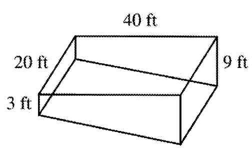




15. **(a)** The area of a strip is $20\,\Delta x$ and the pressure on it is δx_i .

$$F = \int_0^3 \delta x 20\, dx = 20\delta \left[\frac{1}{2} x^2 \right]_0^3 = 20\delta \cdot \frac{9}{2} = 90\delta$$

$$= 90(62.5) = 5625 \text{lb} \approx 5.63 \times 10^3 \text{lb}$$



(b) $F = \int_0^9 \delta x 20\, dx = 20\delta \left[\frac{1}{2} x^2 \right]_0^9 = 20\delta \cdot \frac{81}{2} = 810\delta = 810(62.5) = 50{,}625 \text{ lb} \approx 5.06 \times 10^4 \text{ lb}.$

(c) For the first 3 ft, the length of the side is constant at 40 ft. For $3 < x \leq 9$, we can use similar triangles to find the length a : $\frac{a}{40} = \frac{9-x}{6} \Rightarrow a = 40 \cdot \frac{9-x}{6}$.

$$F = \int_0^3 \delta x 40\, dx + \int_3^9 \delta x (40) \frac{9-x}{6}\, dx = 40\delta \left[\frac{1}{2} x^2 \right]_0^3 + \frac{20}{3} \delta \int_3^9 \left(9x - x^2 \right) dx$$

$$= 180\delta + \frac{20}{3} \delta \left[\frac{9}{2} x^2 - \frac{1}{3} x^3 \right]_3^9 = 180\delta + \frac{20}{3} \delta \left[\left(\frac{729}{2} - 243 \right) - \left(\frac{81}{2} - 9 \right) \right]$$

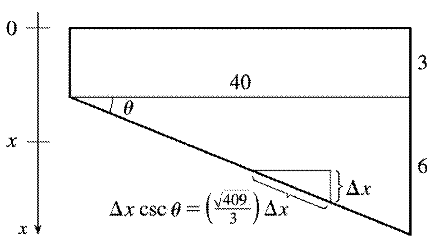
$$= 180\delta + 600\delta = 780\delta = 780(62.5) = 48{,}750 \text{lb} \approx 4.88 \times 10^4 \text{lb}$$

(d) For any right triangle with hypotenuse on the bottom, $\csc\theta = \dfrac{\Delta x}{\text{hypotenuse}} \Rightarrow$ hypotenuse $= \Delta x \csc\theta = \Delta x \dfrac{\sqrt{40^2+6^2}}{6} = \dfrac{\sqrt{409}}{3}\,\Delta x$.

$$F = \int_3^9 \delta x 20 \frac{\sqrt{409}}{3}\,dx = \frac{1}{3}\left(20\sqrt{409}\right)\delta\left[\frac{1}{2}x^2\right]_3^9$$

$$= \frac{1}{3}\cdot 10\sqrt{409}\,\delta(81-9)$$

$$\approx 303{,}356\text{lb} \approx 3.03\times 10^5\text{lb}$$



16. Partition the interval $[a,b]$ by points x_i as usual and choose $x_i^* \in [x_{i-1}, x_i]$ for each i . The i th horizontal strip of the immersed plate is approximated by a rectangle of height Δx_i and width $w\left(x_i^*\right)$, so its area is $A_i \approx w\left(x_i^*\right)\Delta x_i$. For small Δx_i , the pressure P_i on the i th strip is almost constant and $P_i \approx \rho g x_i^*$ by Equation 1. The hydrostatic force F_i acting on the i th strip is $F_i = P_i A_i \approx \rho g x_i^* w\left(x_i^*\right)\Delta x_i$. Adding these forces and taking the limit as $n\to\infty$, we obtain the hydrostatic force on the immersed plate:

$$F = \lim_{n\to\infty} \sum_{i=1}^{n} F_i = \lim_{n\to\infty} \sum_{i=1}^{n} \rho g x_i^* w\left(x_i^*\right)\Delta x_i = \int_a^b \rho g x w(x)\,dx$$

17. $F = \int_2^5 \rho g x\cdot w(x)\,dx$, where $w(x)$ is the width of the plate at depth x . Since $n=6$, $\Delta x = \dfrac{5-2}{6} = \dfrac{1}{2}$, and

$$F \approx S_6 = \rho g\cdot \frac{1/2}{3}\,[2\cdot w(2) + 4\cdot 2.5\cdot w(2.5) + 2\cdot 3\cdot w(3) + 4\cdot 3.5\cdot w(3.5)]$$

$$+2\cdot 4\cdot w(4) + 4\cdot 4.5\cdot w(4.5) + 5\cdot w(5)]$$

$$=\frac{1}{6}\rho g(2\cdot 0+10\cdot 0.8+6\cdot 1.7+14\cdot 2.4+8\cdot 2.9+18\cdot 3.3+5\cdot 3.6)$$

$$=\frac{1}{6}(1000)(9.8)(152.4)\approx 2.5\times 10^{5}\text{N}$$

18. **(a)** From Equation 8, $\bar{x}=\frac{1}{A}\int_a^b xw(x)dx \Rightarrow A\bar{x}=\int_a^b xw(x)dx \Rightarrow \rho gA\bar{x}=\rho g\int_a^b xw(x)dx \Rightarrow$

$(\rho g\bar{x})A=\int_a^b \rho gxw(x)dx=F$ by Exercise 16.

(b) The centroid of a circle is its center. In this case, the center is at a depth of r meters, so $\bar{x}=r$. Thus, $F=(\rho g\bar{x})A=(\rho gr)(\pi r^2)=\rho g\pi r^3$.

19. The moment M of the system about the origin is $M=\sum_{i=1}^{2} m_i x_i=m_1x_1+m_2x_2=40\cdot 2+30\cdot 5=230$. The mass m of the system is $m=\sum_{i=1}^{2} m_i=m_1+m_2=40+30=70$. The center of mass of the system is $M/m=\frac{230}{70}=\frac{23}{7}$.

20. $M=m_1x_1+m_2x_2+m_3x_3=25(-2)+20(3)+10(7)=80$;
$\bar{x}=M/(m_1+m_2+m_3)=\frac{80}{55}=\frac{16}{11}$.

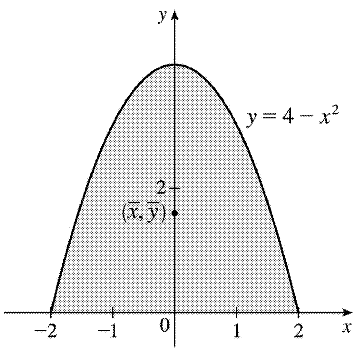
21. $m=\sum_{i=1}^{3} m_i=6+5+10=21$. $M_x=\sum_{i=1}^{3} m_iy_i=6(5)+5(-2)+10(-1)=10$;
$M_y=\sum_{i=1}^{3} m_ix_i=6(1)+5(3)+10(-2)=1$. $\bar{x}=\frac{M_y}{m}=\frac{1}{21}$ and $\bar{y}=\frac{M_x}{m}=\frac{10}{21}$, so the center of mass of the system is $\left(\frac{1}{21},\frac{10}{21}\right)$.

22. $M_x=\sum_{i=1}^{4} m_iy_i=6(-2)+5(4)+1(-7)+4(-1)=-3$, $M_y=\sum_{i=1}^{4} m_ix_i=6(1)+5(3)+1(-3)+4(6)=42$, and $m=\sum_{i=1}^{4} m_i=16$, so $\bar{x}=\frac{M_y}{m}=\frac{42}{16}=\frac{21}{8}$ and $\bar{y}=\frac{M_x}{m}=-\frac{3}{16}$; the center of mass is $(\bar{x},\bar{y})=\left(\frac{21}{8},-\frac{3}{16}\right)$.

23. Since the region in the figure is symmetric about the y–axis, we know that $\bar{x}=0$. The region is ''bottom–heavy,'' so we know that $\bar{y}<2$, and we might guess that $\bar{y}=1.5$.

$$A=\int_{-2}^{2}(4-x^2)\,dx=2\int_{0}^{2}(4-x^2)\,dx=2\left[4x-\frac{1}{3}x^3\right]_0^2$$
$$=2\left(8-\frac{8}{3}\right)=\frac{32}{3}$$

$\overline{x}=\frac{1}{A}\int_{-2}^{2}x(4-x^2)\,dx=0$ since $f(x)=x(4-x^2)$ is an odd function (or since the region is symmetric about the y-axis).



$$\overline{y}=\frac{1}{A}\int_{-2}^{2}\frac{1}{2}(4-x^2)^2\,dx=\frac{3}{32}\cdot\frac{1}{2}\cdot 2\int_{0}^{2}(16-8x^2+x^4)\,dx=\frac{3}{32}\left[16x-\frac{8}{3}x^3+\frac{1}{5}x^5\right]_0^2$$
$$=\frac{3}{32}\left(32-\frac{64}{3}+\frac{32}{5}\right)=3\left(1-\frac{2}{3}+\frac{1}{5}\right)=3\left(\frac{8}{15}\right)=\frac{8}{5}$$

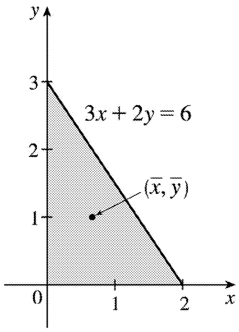
Thus, the centroid is $(\overline{x},\overline{y})=\left(0,\frac{8}{5}\right)$.

24. The region in the figure is ''left–heavy'' and ''bottom–heavy,'' so we know $\overline{x}<1$ and $\overline{y}<1.5$, and we might guess that $\overline{x}=0.7$ and $\overline{y}=1.2$.

$3x+2y=6 \Leftrightarrow 2y=6-3x \Leftrightarrow y=3-\frac{3}{2}x$.

$$A=\int_0^2\left(3-\frac{3}{2}x\right)dx=\left[3x-\frac{3}{4}x^2\right]_0^2=6-3=3\ .$$

$$\overline{x}=\frac{1}{A}\int_0^2 x\left(3-\frac{3}{2}x\right)dx=\frac{1}{3}\int_0^2\left(3x-\frac{3}{2}x^2\right)dx$$
$$=\frac{1}{3}\left[\frac{3}{2}x^2-\frac{1}{2}x^3\right]_0^2=\frac{1}{3}(6-4)=\frac{2}{3}\ ;$$



$\bar{y} = \frac{1}{A}\int_0^2 \frac{1}{2}\left(3 - \frac{3}{2}x\right)^2 dx = \frac{1}{3}\cdot\frac{1}{2}\int_0^2\left(9 - 9x + \frac{9}{4}x^2\right)dx = \frac{1}{6}\left[9x - \frac{9}{2}x^2 + \frac{3}{4}x^3\right]_0^2 = \frac{1}{6}(18 - 18 + 6) = 1$.

Thus, the centroid is $(\bar{x}, \bar{y}) = \left(\frac{2}{3}, 1\right)$.

25. The region in the figure is "right–heavy'' and "bottom–heavy,'' so we know $\bar{x} > 0.5$ and $\bar{y} < 1$, and we might guess that $\bar{x} = 0.6$ and $\bar{y} = 0.9$.

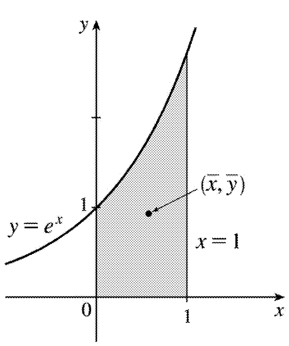
$A = \int_0^1 e^x dx = \left[e^x\right]_0^1 = e - 1$,

$\bar{x} = \frac{1}{A}\int_0^1 xe^x dx = \frac{1}{e-1}\left[xe^x - e^x\right]_0^1$ [by parts]

$= \frac{1}{e-1}\left[0 - (-1)\right] = \frac{1}{e-1}$,

$\bar{y} = \frac{1}{A}\int_0^1 \frac{1}{2}\left(e^x\right)^2 dx = \frac{1}{e-1}\cdot\frac{1}{4}\left[e^{2x}\right]_0^1 = \frac{1}{4(e-1)}\left(e^2 - 1\right) = \frac{e+1}{4}$. Thus, the centroid is

$(\bar{x}, \bar{y}) = \left(\frac{1}{e-1}, \frac{e+1}{4}\right) \approx (0.58, 0.93)$.



26. The region in the figure is "left–heavy'' and "bottom–heavy,'' so we know $\bar{x} < 1.5$ and $\bar{y} < 0.5$, and we might guess that

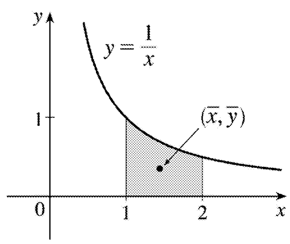
$\bar{x}=1.4$ and $\bar{y}=0.4$.

$A=\int_1^2 \frac{1}{x}\, dx=[\ln x]_1^2=\ln 2$, $\bar{x}=\frac{1}{A}\int_1^2 x\cdot \frac{1}{x}\, dx=\frac{1}{A}\,[x]_1^2=\frac{1}{A}=\frac{1}{\ln 2}$,

$$\bar{y}=\frac{1}{A}\int_1^2 \frac{1}{2}\left(\frac{1}{x}\right)^2 dx=\frac{1}{2A}\int_1^2 x^{-2}\,dx=\frac{1}{2A}\left[-\frac{1}{x}\right]_1^2$$

$$=\frac{1}{2\ln 2}\left(-\frac{1}{2}+1\right)=\frac{1}{4\ln 2} .$$

Thus, the centroid is $(\bar{x},\bar{y})=\left(\frac{1}{\ln 2},\frac{1}{4\ln 2}\right)\approx(1.44,0.36)$.



27. $A=\int_0^1 \left(\sqrt{x}-x\right)dx=\left[\frac{2}{3}x^{3/2}-\frac{1}{2}x^2\right]_0^1=\frac{2}{3}-\frac{1}{2}=\frac{1}{6}$.

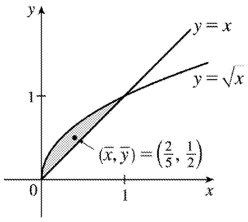
$$\bar{x}=\frac{1}{A}\int_0^1 x(\sqrt{x}-x)\,dx=6\int_0^1 (x^{3/2}-x^2)\,dx$$

$$=6\left[\frac{2}{5}x^{5/2}-\frac{1}{3}x^3\right]_0^1=6\left(\frac{2}{5}-\frac{1}{3}\right)=6\left(\frac{1}{15}\right)=\frac{2}{5} ;$$

$$\bar{y}=\frac{1}{A}\int_0^1 \frac{1}{2}\left[\left(\sqrt{x}\right)^2-x^2\right]dx=6\cdot\frac{1}{2}\int_0^1 (x-x^2)\,dx$$

$$=3\left[\frac{1}{2}x^2-\frac{1}{3}x^3\right]_0^1=3\left(\frac{1}{2}-\frac{1}{3}\right)=\frac{1}{2} .$$

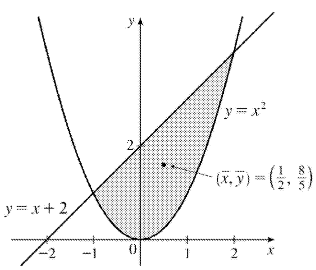
Thus, the centroid is $(\bar{x},\bar{y})=\left(\frac{2}{5},\frac{1}{2}\right)$.



28.

$$A=\int_{-1}^{2}(x+2-x^2)\,dx=\left[\frac{1}{2}x^2+2x-\frac{1}{3}x^3\right]_{-1}^{2}$$
$$=\left(2+4-\frac{8}{3}\right)-\left(\frac{1}{2}-2+\frac{1}{3}\right)=\frac{9}{2}.$$

$$\bar{x}=\frac{1}{A}\int_{-1}^{2}x(x+2-x^2)\,dx=\frac{2}{9}\int_{-1}^{2}\left(x^2+2x-x^3\right)dx$$
$$=\frac{2}{9}\left[\frac{1}{3}x^3+x^2-\frac{1}{4}x^4\right]_{-1}^{2}$$
$$=\frac{2}{9}\left[\left(\frac{8}{3}+4-4\right)-\left(-\frac{1}{3}+1-\frac{1}{4}\right)\right]=\frac{2}{9}\cdot\frac{9}{4}=\frac{1}{2}\;;$$



$$\bar{y}=\frac{1}{A}\int_{-1}^{2}\frac{1}{2}\left[(x+2)^2-(x^2)^2\right]dx=\frac{2}{9}\cdot\frac{1}{2}\int_{-1}^{2}\left(x^2+4x+4-x^4\right)dx=\frac{1}{9}\left[\frac{1}{3}x^3+2x^2+4x-\frac{1}{5}x^5\right]_{-1}^{2}$$
$$=\frac{1}{9}\left[\left(\frac{8}{3}+8+8-\frac{32}{5}\right)-\left(-\frac{1}{3}+2-4+\frac{1}{5}\right)\right]=\frac{1}{9}\left(18+\frac{9}{3}-\frac{33}{5}\right)=\frac{1}{9}\cdot\frac{72}{5}=\frac{8}{5}\;.$$

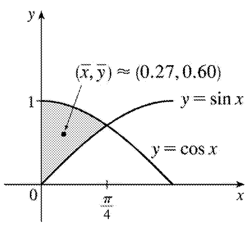
Thus, the centroid is $(\bar{x},\bar{y})=\left(\frac{1}{2},\frac{8}{5}\right)$.

29. $A=\int_{0}^{\pi/4}(\cos x-\sin x)\,dx=\left[\sin x+\cos x\right]_{0}^{\pi/4}=\sqrt{2}-1$,

$$\bar{x} = A^{-1}\int_0^{\pi/4} x(\cos x - \sin x)\,dx$$

$$= A^{-1}\left[x(\sin x + \cos x) + \cos x - \sin x\right]_0^{\pi/4} \text{ [integration by parts]}$$

$$= A^{-1}\left(\frac{\pi}{4}\sqrt{2} - 1\right) = \frac{\frac{1}{4}\pi\sqrt{2} - 1}{\sqrt{2} - 1}$$



$$\bar{y} = A^{-1}\int_0^{\pi/4} \frac{1}{2}\left(\cos^2 x - \sin^2 x\right)dx = \frac{1}{2A}\int_0^{\pi/4} \cos 2x\,dx = \frac{1}{4A}\left[\sin 2x\right]_0^{\pi/4} = \frac{1}{4A} = \frac{1}{4\left(\sqrt{2} - 1\right)}$$

Thus, the centroid is $\left(\bar{x}, \bar{y}\right) = \left(\dfrac{\pi\sqrt{2} - 4}{4\left(\sqrt{2} - 1\right)}, \dfrac{1}{4\left(\sqrt{2} - 1\right)}\right) \approx (0.27, 0.60)$.

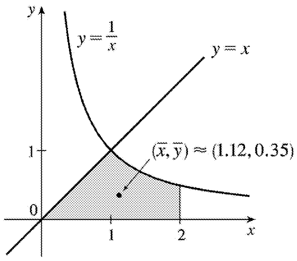
30. $A = \int_0^1 x\,dx + \int_1^2 \frac{1}{x}\,dx = \left[\frac{1}{2}x^2\right]_0^1 + \left[\ln x\right]_1^2 = \frac{1}{2} + \ln 2$,

$$\bar{x} = \frac{1}{A}\left[\int_0^1 x^2\,dx + \int_1^2 1\,dx\right] = \frac{1}{A}\left(\left[\frac{1}{3}x^3\right]_0^1 + \left[x\right]_1^2\right)$$

$$= \frac{1}{A}\left(\frac{1}{3} + 1\right) = \frac{2}{1 + 2\ln 2}\cdot\frac{4}{3} = \frac{8}{3(1 + 2\ln 2)} \ ,$$

$$\bar{y} = \frac{1}{A}\left[\int_0^1 \frac{1}{2}x^2\,dx + \int_1^2 \frac{1}{2x^2}\,dx\right] = \frac{1}{2A}\left(\left[\frac{1}{3}x^3\right]_0^1 + \left[-\frac{1}{x}\right]_1^2\right)$$

$$= \frac{1}{2A}\left(\frac{1}{3} + \frac{1}{2}\right) = \frac{5}{12A} = \frac{5}{6 + 12\ln 2} \ .$$

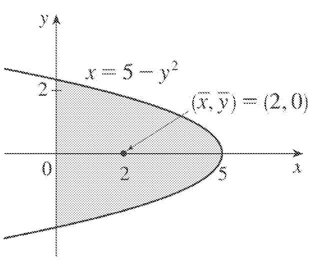


Thus, the centroid is $(\bar{x},\bar{y})=\left(\frac{8}{3(1+2\ln 2)},\frac{5}{6(1+2\ln 2)}\right)\approx(1.12,0.35)$. The principle used in this problem is stated after Example 3: the moment of the union of two nonoverlapping regions is the sum of the moments of the individual regions.

31. From the figure we see that $\bar{y}=0$. Now

$$\begin{aligned} A &= \int_0^5 2\sqrt{5-x}\,dx=2\left[-\frac{2}{3}(5-x)^{3/2}\right]_0^5 \\ &= 2\left(0+\frac{2}{3}\cdot 5^{3/2}\right)=\frac{20}{3}\sqrt{5} \end{aligned}$$

so



$$\begin{aligned} \bar{x} &= \frac{1}{A}\int_0^5 x\left[\sqrt{5-x}-\left(-\sqrt{5-x}\right)\right]dx=\frac{1}{A}\int_0^5 2x\sqrt{5-x}\,dx \\ &= \frac{1}{A}\int_{\sqrt{5}}^0 2\left(5-u^2\right)u(-2u)\,du[u=\sqrt{5-x},x=5-u^2,u^2=5-x,dx=-2u\,du] \\ &= \frac{4}{A}\int_0^{\sqrt{5}} u^2\left(5-u^2\right)du=\frac{4}{A}\left[\frac{5}{3}u^3-\frac{1}{5}u^5\right]_0^{\sqrt{5}}=\frac{3}{5\sqrt{5}}\left(\frac{25}{3}\sqrt{5}-5\sqrt{5}\right)=5-3=2 \end{aligned}$$

Thus, the centroid is $(\bar{x},\bar{y})=(2,0)$.

32. By symmetry, $M_y=0$ and $\bar{x}=0$;

$A=\frac{1}{2}\pi\cdot 1^2+4$, so $m=\rho A=5(\frac{\pi}{2}+4)=\frac{5}{2}(\pi+8)$;

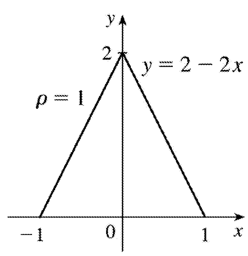
$M_x=\rho\cdot 2\int_0^1 \frac{1}{2}[(\sqrt{1-x^2})^2-(-2)^2]dx=5\int_0^1\left(-x^2-3\right)dx=-5\left[\frac{1}{3}x^3+3x\right]_0^1=-5\cdot\frac{10}{3}=-\frac{50}{3}$;

$\bar{y}=\frac{1}{m}M_x=\frac{2}{5(\pi+8)}\cdot\frac{-50}{3}=-\frac{20}{3(\pi+8)}$. Thus, the centroid is $(\bar{x},\bar{y})=\left(0,\frac{-20}{3(\pi+8)}\right)$.

33. By symmetry, $M_y=0$ and $\bar{x}=0$. $A=\frac{1}{2}bh=\frac{1}{2}\cdot 2\cdot 2=2$.

$$\begin{aligned}
M_x &= \rho\int_{-1}^1 \frac{1}{2}(2-2x)^2dx=2\rho\int_0^1\frac{1}{2}(2-2x)^2dx \\
&= \left(2\cdot 1\cdot\frac{1}{2}\cdot 2^2\right)\int_0^1(1-x)^2dx \\
&= 4\int_1^0 u^2(-du)[u=1-x,du=-dx] \\
&= -4\left[\frac{1}{3}u^3\right]_1^0=-4\left(-\frac{1}{3}\right)=\frac{4}{3}
\end{aligned}$$

$\bar{y}=\frac{1}{m}M_x=\frac{1}{\rho A}M_x=\frac{1}{1\cdot 2}\cdot\frac{4}{3}=\frac{2}{3}$. Thus, the centroid is $(\bar{x},\bar{y})=\left(0,\frac{2}{3}\right)$.



34. By symmetry about the line $y=x$, we expect that $\bar{x}=\bar{y}$. $A=\frac{1}{4}\pi r^2$, so $m=\rho A=2A=\frac{1}{2}\pi r^2$.

$M_x=\rho\int_0^r\frac{1}{2}\left(\sqrt{r^2-x^2}\right)^2dx=2\cdot\frac{1}{2}\int_0^r\left(r^2-x^2\right)dx=\left[r^2x-\frac{1}{3}x^3\right]_0^r=\frac{2}{3}r^3$.

$M_y=\rho\int_0^r x\sqrt{r^2-x^2}\,dx=\int_0^r\left(r^2-x^2\right)^{1/2}2x\,dx=\int_0^{r^2}u^{1/2}du=\left[\frac{2}{3}u^{3/2}\right]_0^{r^2}=\frac{2}{3}r^3$. $\bar{x}=\frac{1}{m}M_y=\frac{2}{\pi r^2}\left(\frac{2}{3}r^3\right)=\frac{4}{3\pi}r$

; $\bar{y}=\frac{1}{m}M_x=\frac{2}{\pi r^2}\left(\frac{2}{3}r^3\right)=\frac{4}{3\pi}r$. Thus, the centroid is $(\bar{x},\bar{y})=\left(\frac{4}{3\pi}r,\frac{4}{3\pi}r\right)$.

35.

$$A=\int_0^2\left(2^x-x^2\right)dx=\left[\frac{2^x}{\ln 2}-\frac{x^3}{3}\right]_0^2$$

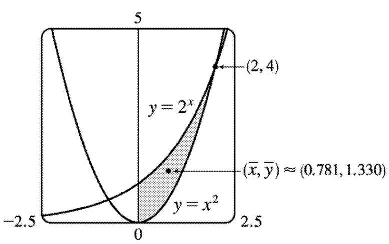
$$=\left(\frac{4}{\ln 2}-\frac{8}{3}\right)-\frac{1}{\ln 2}=\frac{3}{\ln 2}-\frac{8}{3}\approx 1.661418\ .$$

$$\bar{x}=\frac{1}{A}\int_0^2 x(2^x-x^2)dx=\frac{1}{A}\int_0^2(x2^x-x^3)dx$$

$$=\frac{1}{A}\left[\frac{x2^x}{\ln 2}-\frac{2^x}{(\ln 2)^2}-\frac{x^4}{4}\right]_0^2\text{[use parts]}$$

$$=\frac{1}{A}\left[\frac{8}{\ln 2}-\frac{4}{(\ln 2)^2}-4+\frac{1}{(\ln 2)^2}\right]$$

$$=\frac{1}{A}\left[\frac{8}{\ln 2}-\frac{3}{(\ln 2)^2}-4\right]\approx\frac{1}{A}(1.297453)\approx 0.781$$



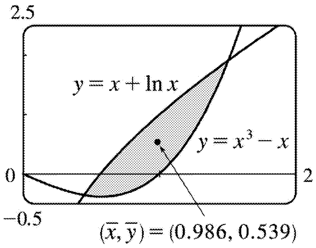
$$\bar{y}=\frac{1}{A}\int_0^2\frac{1}{2}\left[\left(2^x\right)^2-\left(x^2\right)^2\right]dx=\frac{1}{A}\int_0^2\frac{1}{2}(2^{2x}-x^4)dx=\frac{1}{A}\cdot\frac{1}{2}\left[\frac{2^{2x}}{2\ln 2}-\frac{x^5}{5}\right]_0^2$$

$$=\frac{1}{A}\cdot\frac{1}{2}\left(\frac{16}{2\ln 2}-\frac{32}{5}-\frac{1}{2\ln 2}\right)=\frac{1}{A}\left(\frac{15}{4\ln 2}-\frac{16}{5}\right)\approx\frac{1}{A}(2.210106)\approx 1.330$$

36. The curves $y=x+\ln x$ and $y=x^3-x$ intersect at $(a,c)\approx(0.447141,-0.357742)$ and $(b,d)\approx(1.507397,1.917782)$.

$$A=\int_a^b\left(x+\ln x-x^3+x\right)dx=\int_a^b\left(2x+\ln x-x^3\right)dx$$

$$=\left[x^2+x\ln x-x-\frac{1}{4}x^4\right]_a^b\approx 0.709781$$



$$\bar{x} = \frac{1}{A}\int_a^b x\left(2x+\ln x-x^3\right)dx = \frac{1}{A}\int_a^b \left(2x^2+x\ln x-x^4\right)dx$$

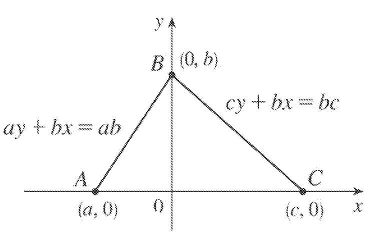
$$= \frac{1}{A}\left[\frac{2}{3}x^3+\frac{1}{4}x^2(2\ln x-1)-\frac{1}{5}x^5\right]_a^b \approx \frac{1}{A}(0.699489)\approx 0.985501$$

$$\bar{y} = \frac{1}{A}\int_a^b \frac{1}{2}\left[(x+\ln x)^2-\left(x^3-x\right)^2\right]dx = \frac{1}{2A}\int_a^b\left[2x\ln x+(\ln x)^2-x^6+2x^4\right]dx$$

$$= \frac{1}{2A}\left[x^2\ln x-\frac{1}{2}x^2+x(\ln x)^2-2x\ln x+2x-\frac{1}{7}x^7+\frac{2}{5}x^5\right]_a^b \approx \frac{1}{2A}(0.765092)\approx 0.538964$$

Thus, the centroid is $(\bar{x},\bar{y})\approx(0.986,0.539)$.

37. Choose $x-$ and $y-$ axes so that the base (one side of the triangle) lies along the $x-$ axis with the other vertex along the positive $y-$ axis as shown. From geometry, we know the medians intersect at a point $\frac{2}{3}$ of the way from each vertex (along the median) to the opposite side. The median from B goes to the midpoint $\left(\frac{1}{2}(a+c),0\right)$ of side AC , so the point of intersection of the medians is $\left(\frac{2}{3}\cdot\frac{1}{2}(a+c),\frac{1}{3}b\right)=\left(\frac{1}{3}(a+c),\frac{1}{3}b\right)$.



This can also be verified by finding the equations of two medians, and solving them simultaneously to find their point of intersection. Now let us compute the location of the centroid of the triangle. The area is $A=\frac{1}{2}(c-a)b$.

$$\bar{x} = \frac{1}{A}\left[\int_a^0 x\cdot\frac{b}{a}(a-x)\,dx+\int_0^c x\cdot\frac{b}{c}(c-x)dx\right]=\frac{1}{A}\left[\frac{b}{a}\int_a^0\left(ax-x^2\right)dx+\frac{b}{c}\int_0^c\left(cx-x^2\right)dx\right]$$

$$=\frac{b}{Aa}\left[\frac{1}{2}ax^2-\frac{1}{3}x^3\right]_a^0+\frac{b}{Ac}\left[\frac{1}{2}cx^2-\frac{1}{3}x^3\right]_0^c=\frac{b}{Aa}\left[-\frac{1}{2}a^3+\frac{1}{3}a^3\right]+\frac{b}{Ac}\left[\frac{1}{2}c^3-\frac{1}{3}c^3\right]$$

$$=\frac{2}{a(c-a)}\cdot\frac{-a^3}{6}+\frac{2}{c(c-a)}\cdot\frac{c^3}{6}=\frac{1}{3(c-a)}(c^2-a^2)=\frac{a+c}{3}$$

and

$$\bar{y} = \frac{1}{A}\left[\int_a^0\frac{1}{2}\left(\frac{b}{a}(a-x)\right)^2dx+\int_0^c\frac{1}{2}\left(\frac{b}{c}(c-x)\right)^2dx\right]$$

$$=\frac{1}{A}\left[\frac{b^2}{2a^2}\int_a^0\left(a^2-2ax+x^2\right)dx+\frac{b^2}{2c^2}\int_0^c\left(c^2-2cx+x^2\right)dx\right]$$

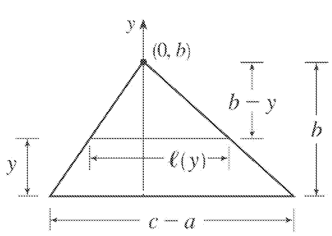
$$=\frac{1}{A}\left[\frac{b^2}{2a^2}\left[a^2x-ax^2+\frac{1}{3}x^3\right]_a^0+\frac{b^2}{2c^2}\left[c^2x-cx^2+\frac{1}{3}x^3\right]_0^c\right]$$

$$=\frac{1}{A}\left[\frac{b^2}{2a^2}\left(-a^3+a^3-\frac{1}{3}a^3\right)+\frac{b^2}{2c^2}\left(c^3-c^3+\frac{1}{3}c^3\right)\right]=\frac{1}{A}\left[\frac{b^2}{6}(-a+c)\right]=\frac{2}{(c-a)b}\cdot\frac{(c-a)b^2}{6}=\frac{b}{3}$$

Thus, the centroid is $(\bar{x},\bar{y})=\left(\frac{a+c}{3},\frac{b}{3}\right)$, as claimed.

Remarks: Actually the computation of $\bar{y}$ is all that is needed. By considering each side of the triangle in turn to be the base, we see that the centroid is $\frac{1}{3}$ of the way from each side to the opposite vertex and must therefore be the intersection of the medians.

The computation of $\bar{y}$ in this problem (and many others) can be simplified by using horizontal rather than vertical approximating rectangles. If the length of a thin rectangle at coordinate y is $\ell(y)$, then its area is $\ell(y)\Delta y$, its mass is $\rho\ell(y)\Delta y$, and its moment about the $x-$ axis is $\Delta M_x=\rho y\ell(y)\Delta y$. Thus,



In this problem, $\ell(y)=\frac{c-a}{b}(b-y)$ by similar triangles, so

$$\overline{y}=\frac{1}{A}\int_0^b \frac{c-a}{b}\,y(b-y)dy=\frac{2}{b^2}\int_0^b\left(by-y^2\right)dy=\frac{2}{b^2}\left[\frac{1}{2}by^2-\frac{1}{3}y^3\right]_0^b=\frac{2}{b^2}\cdot\frac{b^3}{6}=\frac{b}{3}$$

Notice that only one integral is needed when this method is used.

38. Divide the lamina into three rectangles with masses 2 , 2 and 6 , with centroids $\left(-\frac{3}{2},1\right)$, $\left(0,\frac{1}{2}\right)$ and $\left(2,\frac{3}{2}\right)$, respectively. The total mass of the lamina is 10 . So, using Formulas 5, 6, and 7, we have

$\overline{x}=\frac{M_y}{m}=\frac{1}{m}\Sigma_{i=1}^3 m_i x_i=\frac{1}{10}\left[2\left(-\frac{3}{2}\right)+2(0)+6(2)\right]=\frac{1}{10}(9)$, and

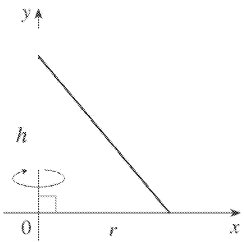
$\overline{y}=\frac{M_x}{m}=\frac{1}{m}\Sigma_{i=1}^3 m_i y_i=\frac{1}{10}\left[2(1)+2\left(\frac{1}{2}\right)+6\left(\frac{3}{2}\right)\right]=\frac{1}{10}(12)$.

Thus, the centroid is $(\overline{x},\overline{y})=\left(\frac{9}{10},\frac{6}{5}\right)$.

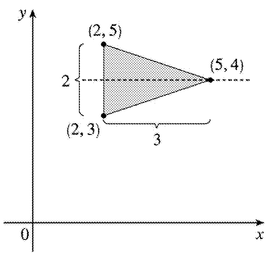
39. Divide the lamina into two triangles and one rectangle with respective masses of 2 , 2 and 4 , so that the total mass is 8 . Using the result of Exercise 37, the triangles have centroids $\left(-1,\frac{2}{3}\right)$ and $\left(1,\frac{2}{3}\right)$. The centroid of the rectangle (its center) is $\left(0,-\frac{1}{2}\right)$. So, using Formulas 5 and 7, we have $\overline{y}=\frac{M_x}{m}=\frac{1}{m}\Sigma_{i=1}^3 m_i y_i=\frac{1}{8}\left[2\left(\frac{2}{3}\right)+2\left(\frac{2}{3}\right)+4\left(-\frac{1}{2}\right)\right]=\frac{1}{8}\left(\frac{2}{3}\right)=\frac{1}{12}$, and $\overline{x}=0$, since the lamina is symmetric about the line $x=0$. Thus, the centroid is $(\overline{x},\overline{y})=\left(0,\frac{1}{12}\right)$.

40. A sphere can be generated by rotating a semicircle about its diameter. By Example 4, the center of mass travels a distance $2\pi\overline{y}=2\pi\left(\frac{4r}{3\pi}\right)=\frac{8r}{3}$, so by the Theorem of Pappus, the volume of the sphere is $V=Ad=\frac{\pi r^2}{2}\cdot\frac{8r}{3}=\frac{4}{3}\pi r^3$.

41. A cone of height h and radius r can be generated by rotating a right triangle about one of its legs as shown. By Exercise 37, $\overline{x}=\frac{1}{3}r$, so by the Theorem of Pappus, the volume of the cone is $V=Ad=\left(\frac{1}{2}\cdot\text{base}\cdot\text{height}\right)\cdot(2\pi\overline{x})=\frac{1}{2}rh\cdot 2\pi\left(\frac{1}{3}r\right)=\frac{1}{3}\pi r^2 h$.



42. From the symmetry in the figure, $\overline{y}=4$. So the distance traveled by the centroid when rotating the triangle about the x –axis is $d=2\pi\cdot 4=8\pi$. The area of the triangle is $A=\frac{1}{2}bh=\frac{1}{2}(2)(3)=3$. By the Theorem of Pappus, the volume of the resulting solid is $Ad=3(8\pi)=24\pi$.

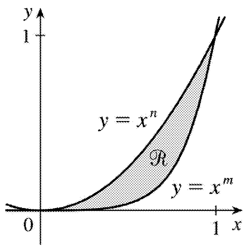


43. Suppose the region lies between two curves $y=f(x)$ and $y=g(x)$ where $f(x)\geq g(x)$, as illustrated in Figure 13. Choose points x_i with $a=x_0<x_1<\cdots<x_n=b$ and choose x_i^* to be the midpoint of the i th subinterval; that is, $x_i^*=\overline{x}_i=\frac{1}{2}\left(x_{i-1}+x_i\right)$. Then the centroid of the i th approximating rectangle R_i is its center $C_i=\left(\overline{x}_i,\frac{1}{2}\left[f\left(\overline{x}_i\right)+g\left(\overline{x}_i\right)\right]\right)$. Its area is $\left[f\left(\overline{x}_i\right)-g\left(\overline{x}_i\right)\right]\Delta x$, so its mass is $\rho\left[f\left(\overline{x}_i\right)-g\left(\overline{x}_i\right)\right]\Delta x$. Thus, $M_y\left(R_i\right)=\rho\left[f\left(\overline{x}_i\right)-g\left(\overline{x}_i\right)\right]\Delta x\cdot\overline{x}_i=\rho\overline{x}_i\left[f\left(\overline{x}_i\right)-g\left(\overline{x}_i\right)\right]\Delta x$ and $M_x\left(R_i\right)=\rho\left[f\left(\overline{x}_i\right)-g\left(\overline{x}_i\right)\right]\Delta x\cdot\frac{1}{2}\left[f\left(\overline{x}_i\right)+g\left(\overline{x}_i\right)\right]=\rho\cdot\frac{1}{2}\left[f\left(\overline{x}_i\right)^2-g\left(\overline{x}_i\right)^2\right]\Delta x$. Summing over i and taking the limit as $n\to\infty$, we get $M_y=\lim_{n\to\infty}\sum_i\rho\overline{x}_i\left[f\left(\overline{x}_i\right)-g\left(\overline{x}_i\right)\right]\Delta x=\rho\int_a^b x[f(x)-g(x)]\,dx$ and $M_x=\lim_{n\to\infty}\sum_i\rho\cdot\frac{1}{2}\left[f\left(\overline{x}_i\right)^2-g\left(\overline{x}_i\right)^2\right]\Delta x=\rho\int_a^b\frac{1}{2}\left[f(x)^2-g(x)^2\right]dx$. Thus,

$$\overline{x}=\frac{M_y}{m}=\frac{M_y}{\rho A}=\frac{1}{A}\int_a^b x[f(x)-g(x)]\,dx \text{ and } \overline{y}=\frac{M_x}{m}=\frac{M_x}{\rho A}=\frac{1}{A}\int_a^b\frac{1}{2}\left[f(x)^2-g(x)^2\right]dx$$

44. **(a)** Let $0\leq x\leq 1$. If $n<m$, then $x^n>x^m$; that is, raising x to a larger power produces a smaller

number.



(b) Using Formulas 9 and the fact that the area of is

$$A=\int_0^1 \left(x^n - x^m\right)dx = \frac{1}{n+1} - \frac{1}{m+1} = \frac{m-n}{(n+1)(m+1)}$$, we get

$$\bar{x} = \frac{(n+1)(m+1)}{m-n} \int_0^1 x\left[x^n - x^m\right]dx = \frac{(n+1)(m+1)}{m-n} \int_0^1 \left(x^{n+1} - x^{m+1}\right)dx$$

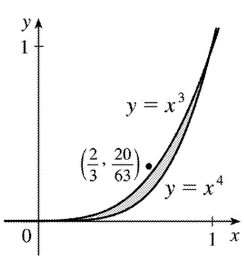
$$= \frac{(n+1)(m+1)}{m-n} \left[\frac{1}{n+2} - \frac{1}{m+2} \right] = \frac{(n+1)(m+1)}{(n+2)(m+2)} \quad \textit{and} \quad \bar{y} = \frac{(n+1)(m+1)}{m-n} \int_0^1 \frac{1}{2}\left[\left(x^n\right)^2 - \left(x^m\right)^2\right]dx = \frac{(n+1}{2(}$$

$$= \frac{(n+1)(m+1)}{2(m-n)} \left[\frac{1}{2n+1} - \frac{1}{2m+1} \right] = \frac{(n+1)(m+1)}{(2n+1)(2m+1)}$$

(c) If we take $n=3$ and $m=4$, then

$$\left(\bar{x},\bar{y}\right) = \left(\frac{4\cdot 5}{5\cdot 6}, \frac{4\cdot 5}{7\cdot 9} \right) = \left(\frac{2}{3}, \frac{20}{63} \right)$$

which lies outside R since $\left(\frac{2}{3} \right)^3 = \frac{8}{27} < \frac{20}{63}$. This is the simplest of many possibilities.



1. By the Net Change Theorem, $C(2000)-C(0)=\int_0^{2000} C'(x)\,dx \Rightarrow$

$$C(2000)=20,000+\int_0^{2000}(5-0.008x+0.000009x^2)\,dx=20,000+\left[5x-0.004x^2+0.000003x^3\right]_0^{2000}$$

$$=20,000+10,000-0.004(4,000,000)+0.000003(8,000,000,000)=30,000-16,000+24,000$$

$$=\$38,000$$

2. By the Net Change Theorem, $R(5000)-R(1000)=\int_{1000}^{5000} R'(x)\,dx \Rightarrow$

$$\mathrm{R}(5000)=12{,}400+\int_{1000}^{5000}(12-0.0004\mathrm{x})\,\mathrm{dx}=12{,}400+\left[12x-0.0002x^2\right]_{1000}^{5000}$$

$$=12{,}400+(60{,}000-5{,}000)-(12{,}000-200)=\$55{,}600$$

3. If the production level is raised from 1200 units to 1600 units, then the increase in cost is

$$C(1600)-C(1200)=\int_{1200}^{1600} C'(x)\,dx=\int_{1200}^{1600}\left(74+1.1x-0.002x^2+0.00004x^3\right)dx$$

$$=\left[74x+0.55x^2-\frac{0.002}{3}x^3+0.00001x^4\right]_{1200}^{1600}$$

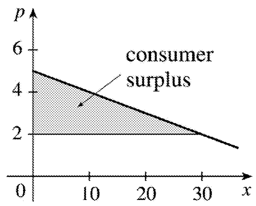
$$=64{,}331{,}733.33-20{,}464{,}800=\$43{,}866{,}933.33$$

4.

$$\text{Consumer surplus}=\int_0^{30}[p(x)-p(30)]\,dx$$

$$=\int_0^{30}\left[5-\frac{1}{10}x-\left(5-\frac{30}{10}\right)\right]dx$$

$$=\left[3x-\frac{1}{20}x^2\right]_0^{30}=90-45=\$45$$



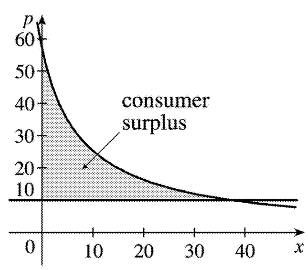
5. $p(x)=10 \Rightarrow \dfrac{450}{x+8}=10 \Rightarrow x+8=45 \Rightarrow x=37$.

$$\text{Consumersurplus} = \int_0^{37}[p(x)-10]\,dx = \int_0^{37}\left(\frac{450}{x+8}-10\right)dx$$

$$= \left[450\ln(x+8)-10x\right]_0^{37}$$

$$= (450\ln 45-370)-450\ln 8$$

$$= 450\ln\left(\frac{45}{8}\right)-370 \approx \$407.25$$

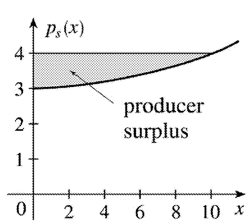


6. $p_S(x)=3+0.01x^2$. $P=p_S(10)=3+1=4$.

$$\text{Producersurplus} = \int_0^{10}[P-p_S(x)]\,dx$$

$$= \int_0^{10}\left[4-3-0.01x^2\right]dx = \left[x-\frac{0.01}{3}x^3\right]_0^{10}$$

$$\approx 10-3.33=\$6.67$$



7. $P=p_S(x) \Rightarrow 400=200+0.2x^{3/2} \Rightarrow 200=0.2x^{3/2} \Rightarrow 1000=x^{3/2} \Rightarrow x=1000^{2/3}=100$.

$$\text{Producer surplus} =\int_0^{100}\left[P-p_S(x)\right]dx=\int_0^{100}\left[400-\left(200+0.2x^{3/2}\right)\right]dx=\int_0^{100}\left(200-\frac{1}{5}x^{3/2}\right)dx$$

$$=\left[200x-\frac{2}{25}x^{5/2}\right]_0^{100}=20,000-8,000=\$12,000$$

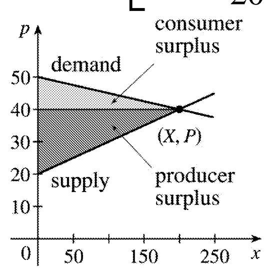
8. $p=50-\frac{1}{20}x$ and $p=20+\frac{1}{10}x$ intersect at $p=40$ and $x=200$.

$$\text{Consumer surplus}=\int_0^{200}\left(50-\frac{1}{20}x-40\right)dx$$

$$=\left[10x-\frac{1}{40}x^2\right]_0^{200}=\$1000$$

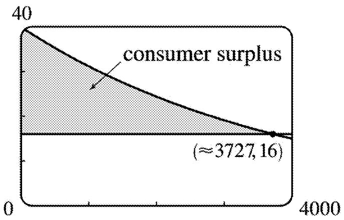
$$\text{Producer surplus}=\int_0^{200}\left(40-20-\frac{1}{10}x\right)\text{dx}$$

$$=\left[20x-\frac{1}{20}x^2\right]_0^{200}=\$2000$$



9. $p(x)=\dfrac{800,000e^{-x/5000}}{x+20,000}=16 \Rightarrow x=x_1 \approx 3727.04$.

$$\text{Consumersurplus}=\int_0^{x_1}[p(x)-16]\,dx \approx \$37,753$$



10. The demand function is linear with slope $\frac{-0.5}{35}=-\frac{1}{70}$ and $p(400)=7.5$, so an equation is $p-7.5=-\frac{1}{70}(x-400)$ or $p=-\frac{1}{70}x+\frac{185}{14}$. A selling price of \$6 implies that $6=-\frac{1}{70}x+\frac{185}{14}\Rightarrow$ $\frac{1}{70}x=\frac{185}{14}-\frac{84}{14}=\frac{101}{14}\Rightarrow x=505$.

$$\text{Consumer surplus}=\int_0^{505}\left(-\frac{1}{70}x+\frac{185}{14}-6\right)dx=\left[-\frac{1}{140}x^2+\frac{101}{14}x\right]_0^{505}\approx\$1821.61$$

11. $f(8)-f(4)=\int_4^8 f'(t)\,dt=\int_4^8\sqrt{t}\,dt=\left[\frac{2}{3}t^{3/2}\right]_4^8=\frac{2}{3}\left(16\sqrt{2}-8\right)\approx\$9.75$ million

12.

$$n(9)-n(5)=\int_5^9\left(2200+10e^{0.8t}\right)dt=\left[2200t+\frac{10e^{0.8t}}{0.8}\right]_5^9=\left[2200t\right]_5^9+\frac{25}{2}\left[e^{0.8t}\right]_5^9$$
$$=2200(9-5)+12.5\left(e^{7.2}-e^4\right)\approx 24,860$$

13. $F=\frac{\pi PR^4}{8\eta l}=\frac{\pi(4000)(0.008)^4}{8(0.027)(2)}\approx 1.19\times 10^{-4}\ \text{cm}^3/\text{s}$

14. If the flux remains constant, then $\frac{\pi P_0R_0^4}{8\eta l}=\frac{\pi PR^4}{8\eta l}\Rightarrow P_0R_0^4=PR^4\Rightarrow\frac{P}{P_0}=\left(\frac{R_0}{R}\right)^4$.

$R=\frac{3}{4}R_0\Rightarrow\frac{P}{P_0}=\left(\frac{R_0}{\frac{3}{4}R_0}\right)^4\Rightarrow P=P_0\left(\frac{4}{3}\right)^4\approx 3.1605P_0>3P_0$; that is, the blood pressure is more than tripled.

15.

$$\int_0^{12} c(t)dt = \int_0^{12} \frac{1}{4} t(12-t)dt = \int_0^{12} \left(3t - \frac{1}{4} t^2\right) dt = \left[\frac{3}{2} t^2 - \frac{1}{12} t^3 \right]_0^{12} = (216-144) = 72 \text{ mg} \cdot \text{ s / L}.$$

Thus, the cardiac output is $F = \dfrac{A}{\int_0^{12} c(t)\,dt} = \dfrac{8\text{mg}}{72\text{mg}\cdot \text{s/L}} = \dfrac{1}{9}$ L / s = $\dfrac{60}{9}$ L / min.

16. As in Example 2, we will estimate the cardiac output using Simpson's Rule with $\Delta t = 2$.

$$\begin{aligned}\int_0^{20} c(t)\,dt &\approx \frac{2}{3}\,[1(0)+4(2.4)+2(5.1)+4(7.8)+2(7.6)\\ &\quad +4(5.4)+2(3.9)+4(2.3)+2(1.6)+4(0.7)+1(0)]\\ &= \frac{2}{3}(110.8) \approx 73.87\text{mg}\cdot \text{s/L}\end{aligned}$$

Therefore, $F \approx \dfrac{A}{73.87} = \dfrac{8}{73.87} \approx 0.1083$ L / s or 6.498 L / min.

1. **(a)** $\int_{30,000}^{40,000} f(x)dx$ is the probability that a randomly chosen tire will have a lifetime between 30 , 000 and 40 , 000 miles.

(b) $\int_{25,000}^{\infty} f(x)dx$ is the probability that a randomly chosen tire will have a lifetime of at least 25 , 000 miles.

2. **(a)** The probability that you drive to school in less than 15 minutes is $\int_0^{15} f(t)dt$.

(b) The probability that it takes you more than half an hour to get to school is $\int_{30}^{\infty} f(t)dt$.

3. **(a)** In general, we must satisfy the two conditions that are mentioned before Example 1 — namely, (1) $f(x) \geq 0$ for all x , and (2) $\int_{-\infty}^{\infty} f(x)dx = 1$. For $0 \leq x \leq 4$, we have $f(x) = \frac{3}{64} x\sqrt{16-x^2} \geq 0$, so $f(x) \geq 0$ for all x . Also,

$$\int_{-\infty}^{\infty} f(x)dx = \int_0^4 \frac{3}{64} x\sqrt{16-x^2}\,dx = -\frac{3}{128}\int_0^4 (16-x^2)^{1/2}(-2x)dx = -\frac{3}{128}\left[\frac{2}{3}(16-x^2)^{3/2}\right]_0^4$$

$$= -\frac{1}{64}\left[\left(16-x^2\right)^{3/2}\right]_0^4 = -\frac{1}{64}(0-64) = 1 .$$

Therefore, f is a probability density function.

(b)

$$P(X<2) = \int_{-\infty}^{2} f(x)dx = \int_0^2 \frac{3}{64} x\sqrt{16-x^2}\,dx = -\frac{3}{128}\int_0^2 (16-x^2)^{1/2}(-2x)dx$$

$$= -\frac{3}{128}\left[\frac{2}{3}(16-x^2)^{3/2}\right]_0^2 = -\frac{1}{64}\left[(16-x^2)^{3/2}\right]_0^2 = -\frac{1}{64}(12^{3/2}-16^{3/2})$$

$$= \frac{1}{64}\left(64-12\sqrt{12}\right) = \frac{1}{64}\left(64-24\sqrt{3}\right) = 1-\frac{3}{8}\sqrt{3} \approx 0.350481$$

4. **(a)** For $0 \leq x \leq 1$, we have $f(x) = kx^2(1-x)$, which is nonnegative if and only if $k \geq 0$. Also, $\int_{-\infty}^{\infty} f(x)dx = \int_0^1 kx^2(1-x)dx = k\int_0^1 (x^2-x^3)dx = k\left[\frac{1}{3}x^3 - \frac{1}{4}x^4\right]_0^1 = k/12$. Now $k/12 = 1 \Leftrightarrow k = 12$. Therefore, f is

a probability density function if and only if $k=12$.
(b) Let $k=12$.

$$P\left(X\geq\frac{1}{2}\right)=\int_{1/2}^{\infty}f(x)dx=\int_{1/2}^{1}12x^{2}(1-x)dx=\int_{1/2}^{1}(12x^{2}-12x^{3})dx=\left[4x^{3}-3x^{4}\right]_{1/2}^{1}$$

$$=(4-3)-\left(\frac{1}{2}-\frac{3}{16}\right)=1-\frac{5}{16}=\frac{11}{16}$$

(c) The mean

$$\mu=\int_{-\infty}^{\infty}xf(x)dx=\int_{0}^{1}x\cdot 12x^{2}(1-x)dx=12\int_{0}^{1}(x^{3}-x^{4})dx=12\left[\frac{1}{4}x^{4}-\frac{1}{5}x^{5}\right]_{0}^{1}$$

$$=12\left(\frac{1}{4}-\frac{1}{5}\right)=\frac{12}{20}=\frac{3}{5}$$

5. **(a)** In general, we must satisfy the two conditions that are mentioned before Example 1 — namely, (1) $f(x)\geq 0$ for all x , and (2) $\int_{-\infty}^{\infty}f(x)dx=1$. Since $f(x)=0$ or $f(x)=0.1$, condition (1) is satisfied. For condition (2), we see that $\int_{-\infty}^{\infty}f(x)dx=\int_{0}^{10}0.1\,dx=\left[\frac{1}{10}x\right]_{0}^{10}=1$. Thus, $f(x)$ is a probability density function for the spinner's values.
(b) Since all the numbers between 0 and 10 are equally likely to be selected, we expect the mean to be halfway between the endpoints of the interval; that is, $x=5$.

$$\mu=\int_{-\infty}^{\infty}xf(x)dx=\int_{0}^{10}x(0.1)dx=\left[\frac{1}{20}x^{2}\right]_{0}^{10}=\frac{100}{20}=5\text{ , as expected.}$$

6. **(a)** As in the preceding exercise, (1) $f(x)\geq 0$ and (2) $\int_{-\infty}^{\infty}f(x)dx=\int_{0}^{10}f(x)dx=\frac{1}{2}(10)(0.2)=1$. So $f(x)$ is a probability density function.
(b)

(a)
$$P(X<3)=\int_{0}^{3}f(x)dx=\frac{1}{2}(3)(0.1)=\frac{3}{20}=0.15$$

(b)
$$P(X<3)=\int_{0}^{3}f(x)dx=\frac{1}{2}(3)(0.1)=\frac{3}{20}=0.15$$

(c) We first compute $P(X>8)$ and then subtract that value and our answer in (i) from 1 (the total probability). $P(X>8)=\int_8^{10} f(x)dx=\frac{1}{2}(2)(0.1)=\frac{2}{20}=0.10$. So $P(3\leq X\leq 8)=1-0.15-0.10=0.75$.

(d) We first compute $P(X>8)$ and then subtract that value and our answer in (i) from 1 (the total

probability). $P(X>8)=\int_8^{10} f(x)dx=\frac{1}{2}(2)(0.1)=\frac{2}{20}=0.10$. So $P(3\leq X\leq 8)=1-0.15-0.10=0.75$.

(c) We find equations of the lines from $(0,0)$ to $(6,0.2)$ and from $(6,0.2)$ to $(10,0)$, and find that

$$f(x)=\begin{cases} \frac{1}{30}x & \text{if } 0\leq x<6 \\ -\frac{1}{20}x+\frac{1}{2} & \text{if } 6\leq x<10 \\ 0 & \text{otherwise} \end{cases}$$

$$\mu = \int_{-\infty}^{\infty} xf(x)dx=\int_0^6 x\left(\frac{1}{30}x\right)dx+\int_6^{10} x\left(-\frac{1}{20}x+\frac{1}{2}\right)dx=\left[\frac{1}{90}x^3\right]_0^6+\left[-\frac{1}{60}x^3+\frac{1}{4}x^2\right]_6^{10}$$

$$= \frac{216}{90}+\left(-\frac{1000}{60}+\frac{100}{4}\right)-\left(-\frac{216}{60}+\frac{36}{4}\right)=\frac{16}{3}=5.\overline{3}$$

7. We need to find m so that $\int_m^{\infty} f(t)dt=\frac{1}{2}\Rightarrow \lim_{x\to\infty}\int_m^x \frac{1}{5}e^{-t/5}dt=\frac{1}{2}\Rightarrow \lim_{x\to\infty}\left[\frac{1}{5}(-5)e^{-t/5}\right]_m^x=\frac{1}{2}\Rightarrow$ $(-1)(0-e^{-m/5})=\frac{1}{2}\Rightarrow e^{-m/5}=\frac{1}{2}\Rightarrow -m/5=\ln\frac{1}{2}\Rightarrow m=-5\ln\frac{1}{2}=5\ln 2\approx 3.47$ min.

8. **(a)**

(a) $P(0\leq X\leq 200)=\int_0^{200}\frac{1}{1000}e^{-t/1000}dt=\left[-e^{-t/1000}\right]_0^{200}=-e^{-1/5}+1\approx 0.181$

(b) $P(0\leq X\leq 200)=\int_0^{200}\frac{1}{1000}e^{-t/1000}dt=\left[-e^{-t/1000}\right]_0^{200}=-e^{-1/5}+1\approx 0.181$

(c) $P(X>800)=\int_{800}^{\infty}\frac{1}{1000}e^{-t/1000}dt=\lim_{x\to\infty}\left[-e^{-t/1000}\right]_{800}^{x}=0+e^{-4/5}\approx 0.449$

(d) $P(X>800)=\int_{800}^{\infty}\frac{1}{1000}e^{-t/1000}dt=\lim_{x\to\infty}\left[-e^{-t/1000}\right]_{800}^{x}=0+e^{-4/5}\approx 0.449$

(b) We need to find m so that $\int_m^\infty f(t)dt = \frac{1}{2} \Rightarrow \lim_{x\to\infty}\int_m^x \frac{1}{1000} e^{-t/1000} dt = \frac{1}{2} \Rightarrow \lim_{x\to\infty}\left[-e^{-t/1000}\right]_m^x = \frac{1}{2} \Rightarrow$ $0+e^{-m/1000} = \frac{1}{2} \Rightarrow -m/1000 = \ln\frac{1}{2} \Rightarrow m = -1000\ln\frac{1}{2} = 1000\ln 2 \approx 693.1$ h.

9. We use an exponential density function with $\mu=2.5$ min.

(a) $P(X>4) = \int_4^\infty f(t)dt = \lim_{x\to\infty}\int_4^x \frac{1}{2.5} e^{-t/2.5} dt = \lim_{x\to\infty}\left[-e^{-t/2.5}\right]_4^x = 0+e^{-4/2.5} \approx 0.202$

(b) $P(0 \le X \le 2) = \int_0^2 f(t)dt = \left[-e^{-t/2.5}\right]_0^2 = -e^{-2/2.5}+1 \approx 0.551$

(c) We need to find a value a so that $P(X \ge a)=0.02$, or, equivalently, $P(0 \le X \le a)=0.98 \Leftrightarrow$ $\int_0^a f(t)dt = 0.98 \Leftrightarrow \left[-e^{-t/2.5}\right]_0^a = 0.98 \Leftrightarrow -e^{-a/2.5}+1 = 0.98 \Leftrightarrow e^{-a/2.5} = 0.02 \Leftrightarrow -a/2.5 = \ln 0.02 \Leftrightarrow$ $a = -2.5\ln\frac{1}{50} = 2.5\ln 50 \approx 9.78 \text{ min} \approx 10$ min. The ad should say that if you aren’t served within 10 minutes, you get a free hamburger.

10. **(a)** With $\mu=69$ and $\sigma=2.8$, we have $P(65 \le X \le 73) = \int_{65}^{73} \frac{1}{2.8\sqrt{2\pi}} \exp\left(-\frac{(x-69)^2}{2\cdot 2.8^2}\right)dx \approx 0.847$ (using a calculator or computer to estimate the integral).

(b) $P(X>6 \text{ feet }) = P(X>72 \text{ inches }) = 1-P(0 \le X \le 72) \approx 1-0.858 = 0.142$, so 14.2% of the adult male population is more than 6 feet tall.

11. $P(X \ge 10) = \int_{10}^\infty \frac{1}{4.2\sqrt{2\pi}} \exp\left(-\frac{(x-9.4)^2}{2\cdot 4.2^2}\right)dx$. To avoid the improper integral we approximate it by the integral from 10 to 100 . Thus, $P(X \ge 10) \approx \int_{10}^{100} \frac{1}{4.2\sqrt{2\pi}} \exp\left(-\frac{(x-9.4)^2}{2\cdot 4.2^2}\right)dx \approx 0.443$ (using a calculator or computer to estimate the integral), so about 44 percent of the households throw out at least 10 lb of paper a week.

Note : We can’t evaluate $1-P(0 \le X \le 10)$ for this problem since a significant amount of area lies to the left of $X=0$.

12. **(a)**

$P(0\le X\le 480)=\int_0^{480}\frac{1}{12\sqrt{2\pi}}\exp\left(-\frac{(x-500)^2}{2\cdot 12^2}\right)dx\approx 0.0478$ (using a calculator or computer to estimate the integral), so there is about a 4.78% chance that a particular box contains less than 480 g of cereal.

(b) We need to find μ so that $P(0\le X<500)=0.05$. Using our calculator or computer to find $P(0\le X\le 500)$ for various values of μ , we find that if $\mu=519.73$, $P=0.05007$; and if $\mu=519.74$, $P=0.04998$. So a good target weight is at least 519.74 g.

13. $P(\mu-2\sigma\le X\le \mu+2\sigma)=\int_{\mu-2\sigma}^{\mu+2\sigma}\frac{1}{\sigma\sqrt{2\pi}}\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)dx$. Substituting $t=\frac{x-\mu}{\sigma}$ and $dt=\frac{1}{\sigma}\,dx$ gives us

$$\int_{-2}^{2}\frac{1}{\sigma\sqrt{2\pi}}\,e^{-t^2/2}(\sigma\,dt)=\frac{1}{\sqrt{2\pi}}\int_{-2}^{2}e^{-t^2/2}dt\approx 0.9545$$

14. Let $f(x)=\begin{cases}0 & \text{if } x<0\\ ce^{-cx} & \text{if } x\ge 0\end{cases}$ where $c=1/\mu$. By using parts, tables, or a CAS, we find that

(1): $\int xe^{bx}dx=\left(e^{bx}/b^2\right)(bx-1)$

(2): $\int x^2e^{bx}dx=\left(e^{bx}/b^3\right)\left(b^2x^2-2bx+2\right)$

Now

$$\begin{aligned}\sigma^2 &= \int_{-\infty}^{\infty}(x-\mu)^2f(x)dx=\int_{-\infty}^{0}(x-\mu)^2f(x)dx+\int_0^{\infty}(x-\mu)^2f(x)dx\\ &= 0+\lim_{t\to\infty}c\int_0^t(x-\mu)^2e^{-cx}dx=c\cdot\lim_{t\to\infty}\int_0^t\left(x^2e^{-cx}-2x\mu e^{-cx}+\mu^2e^{-cx}\right)dx\end{aligned}$$

Next we use (2) and (1) with $b=-c$ to get

$$\sigma^2=c\lim_{t\to\infty}\left[-\frac{e^{-cx}}{c^3}\left(c^2x^2+2cx+2\right)-2\mu\frac{e^{-cx}}{c^2}(-cx-1)+\mu^2\frac{e^{-cx}}{-c}\right]_0^t$$

Using l'Hospital's Rule several times, along with the fact that $\mu=1/c$, we get

$$\sigma^2=c\left[0-\left(-\frac{2}{c^3}+\frac{2}{c}\cdot\frac{1}{c^2}+\frac{1}{c^2}\cdot\frac{1}{-c}\right)\right]=c\left(\frac{1}{c^3}\right)=\frac{1}{c^2}\Rightarrow\sigma=\frac{1}{c}=\mu$$

15. **(a)** First $p(r)=\frac{4}{a_0^3}r^2e^{-2r/a_0}\geq 0$ for $r\geq 0$. Next,

$$\int_{-\infty}^{\infty}p(r)\,dr=\int_0^{\infty}\frac{4}{a_0^3}r^2e^{-2r/a_0}dr=\frac{4}{a_0^3}\lim_{t\to\infty}\int_0^t r^2e^{-2r/a_0}dr$$

By using parts, tables, or a CAS , we find that $\int x^2e^{bx}dx=(e^{bx}/b^3)(b^2x^2-2bx+2)$. (*)

Next, we use (*) (with $b=-2/a_0$) and l'Hospital's Rule to get $\frac{4}{a_0^3}\left[\frac{a_0^3}{-8}(-2)\right]=1$. This satisfies the second condition for a function to be a probability density function.

(b) Using l'Hospital's Rule, $\frac{4}{a_0^3}\lim_{r\to\infty}\frac{r^2}{e^{2r/a_0}}=\frac{4}{a_0^3}\lim_{r\to\infty}\frac{2r}{\left(2/a_0\right)e^{2r/a_0}}=\frac{2}{a_0^2}\lim_{r\to\infty}\frac{2}{\left(2/a_0\right)e^{2r/a_0}}=0$.

To find the maximum of p , we differentiate:

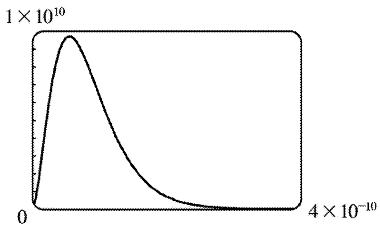
$$p'(r)=\frac{4}{a_0^3}\left[r^2e^{-2r/a_0}\left(-\frac{2}{a_0}\right)+e^{-2r/a_0}(2r)\right]=\frac{4}{a_0^3}e^{-2r/a_0}(2r)\left(-\frac{r}{a_0}+1\right)$$

$p'(r)=0\Leftrightarrow r=0$ or $1=\frac{r}{a_0}\Leftrightarrow r=a_0$. $p'(r)$ changes from positive to negative at $r=a_0$, so $p(r)$ has its maximum value at $r=a_0$.

(c) It is fairly difficult to find a viewing rectangle, but knowing the maximum value from part (b) helps.

$$p\left(a_0\right)=\frac{4}{a_0^3}a_0^2e^{-2a_0/a_0}=\frac{4}{a_0}e^{-2}\approx 9{,}684{,}098{,}979$$

With a maximum of nearly 10 billion and a total area under the curve of 1 , we know that the ''hump'' in the graph must be extremely narrow.



(d) $P(r) = \displaystyle\int_0^r \frac{4}{a_0^3} s^2 e^{-2s/a_0} ds \Rightarrow P\left(4a_0\right) = \int_0^{4a_0} \frac{4}{a_0^3} s^2 e^{-2s/a_0} ds$. Using (*) from part (a)

(with $b = -2/a_0$),

$$P\left(4a_0\right) = \frac{4}{a_0^3}\left[\frac{e^{-2s/a_0}}{-8/a_0^3}\left(\frac{4}{a_0^2}s^2 + \frac{4}{a_0}s + 2\right)\right]_0^{4a_0} = \frac{4}{a_0^3}\left(\frac{a_0^3}{-8}\right)\left[e^{-8}(64+16+2) - 1(2)\right]$$

$$= -\frac{1}{2}\left(82e^{-8} - 2\right) = 1 - 41e^{-8} \approx 0.986$$

(e) $\mu = \displaystyle\int_{-\infty}^{\infty} rp(r)dr = \frac{4}{a_0^3}\lim_{t\to\infty}\int_0^t r^3 e^{-2r/a_0} dr$. Integrating by parts three times or using a CAS, we find that

$\displaystyle\int x^3 e^{bx} dx = \frac{e^{bx}}{b^4}\left(b^3x^3 - 3b^2x^2 + 6bx - 6\right)$. So with $b = -\dfrac{2}{a_0}$, we use l'Hospital's Rule, and get

$$\mu = \frac{4}{a_0^3}\left[-\frac{a_0^4}{16}(-6)\right] = \frac{3}{2}a_0 .$$

1. $y=x-x^{-1} \Rightarrow y'=1+x^{-2}$. To show that y is a solution of the differential equation, we will substitute the expressions for y and y' in the left–hand side of the equation and show that the left–hand side is equal to the right–hand side.

$$\text{LHS}=xy'+y=x\left(1+x^{-2}\right)+\left(x-x^{-1}\right)=x+x^{-1}+x-x^{-1}=2x=\text{RHS}$$

2. $y=\sin x\cos x-\cos x \Rightarrow y'=\sin x(-\sin x)+\cos x(\cos x)-(-\sin x)=\cos^2 x-\sin^2 x+\sin x$.

$$\begin{aligned}\text{LHS} &= y'+(\tan x)y=\cos^2 x-\sin^2 x+\sin x+(\tan x)(\sin x\cos x-\cos x)\\ &= \cos^2 x-\sin^2 x+\sin x+\sin^2 x-\sin x=\cos^2 x=\text{RHS},\end{aligned}$$

so y is a solution of the differential equation. Also, $y(0)=\sin 0\cos 0-\cos 0=0\cdot 1-1=-1$, so the initial condition is satisfied.

3. **(a)** $y=\sin kt \Rightarrow y'=k\cos kt \Rightarrow y''=-k^2\sin kt$. $y''+9y=0 \Rightarrow$
$-k^2\sin kt+9\sin kt=0$ for all $t \Leftrightarrow \left(9-k^2\right)\sin kt=0$ for all $t \Leftrightarrow 9-k^2=0 \Leftrightarrow k=\pm 3$
(b) $y=A\sin kt+B\cos kt \Rightarrow y'=Ak\cos kt-Bk\sin kt \Rightarrow y''=-Ak^2\sin kt-Bk^2\cos kt$.
Thus, $y''+9y=0 \Rightarrow -Ak^2\sin kt-Bk^2\cos kt+9(A\sin kt+B\cos kt)=0 \Rightarrow \left(9-k^2\right)A\sin kt+\left(9-k^2\right)B\cos kt=0$.
The last equation is true for all values of A and B if $k=\pm 3$.

4. $y=e^{rt} \Rightarrow y'=re^{rt} \Rightarrow y''=r^2e^{rt}$. $y''+y'-6y=0 \Rightarrow r^2e^{rt}+re^{rt}-6e^{rt}=0 \Rightarrow \left(r^2+r-6\right)e^{rt}=0 \Rightarrow (r+3)(r-2)=0$
$\Rightarrow r=-3$ or 2

5. **(a)** $y=e^t \Rightarrow y'=e^t \Rightarrow y''=e^t$. LHS $=y''+2y'+y=e^t+2e^t+e^t=4e^t\neq 0$, so $y=e^t$ is not a solution of the differential equation.
(b) $y=e^{-t} \Rightarrow y'=-e^{-t} \Rightarrow y''=e^{-t}$. LHS $=y''+2y'+y=e^{-t}-2e^{-t}+e^{-t}=0=$ RHS, so $y=e^{-t}$ is a solution.
(c) $y=te^{-t} \Rightarrow y'=t\left(-e^{-t}\right)+e^{-t}(1)=e^{-t}(1-t) \Rightarrow y''=e^{-t}(t-2)$.

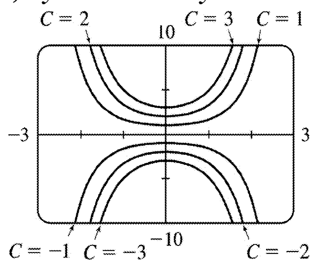
$$\begin{aligned}\text{LHS} &= y''+2y'+y=e^{-t}(t-2)+2e^{-t}(1-t)+te^{-t}\\ &= e^{-t}\left[(t-2)+2(1-t)+t\right]=e^{-t}(0)=0=\text{RHS},\end{aligned}$$

so $y=te^{-t}$ is a solution.
(d) $y=t^2e^{-t} \Rightarrow y'=te^{-t}(2-t) \Rightarrow y''=e^{-t}\left(t^2-4t+2\right)$.

$$\begin{aligned}\text{LHS} &= y''+2y'+y=e^{-t}\left(t^2-4t+2\right)+2te^{-t}(2-t)+t^2e^{-t}\\ &= e^{-t}\left[\left(t^2-4t+2\right)+2t(2-t)+t^2\right]=e^{-t}(2)\neq 0,\end{aligned}$$

so $y=t^2e^{-t}$ is not a solution.

6. **(a)** $y=Ce^{x^2/2} \Rightarrow y'=Ce^{x^2/2}(2x/2)=xCe^{x^2/2}=xy$.



(b)

(c) $y(0)=5 \Rightarrow Ce^0=5 \Rightarrow C=5$, so the solution is $y=5e^{x^2/2}$.

(d) $y(1)=2 \Rightarrow Ce^{1/2}=2 \Rightarrow C=2e^{-1/2}$, so the solution is $y=2e^{-1/2}e^{x^2/2}=2e^{\left(x^2-1\right)/2}$.

7. **(a)** Since the derivative $y'=-y^2$ is always negative (or 0 if $y=0$), the function y must be decreasing (or equal to 0) on any interval on which it is defined.

(b) $y=\dfrac{1}{x+C} \Rightarrow y'=-\dfrac{1}{(x+C)^2}$. LHS $=y'=-\dfrac{1}{(x+C)^2}=-\left(\dfrac{1}{x+C}\right)^2=-y^2=$ RHS

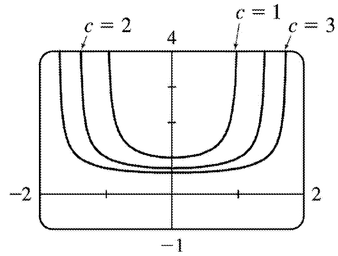
(c) $y=0$ is a solution of $y'=-y^2$ that is not a member of the family in part (b).

(d) If $y(x)=\dfrac{1}{x+C}$, then $y(0)=\dfrac{1}{0+C}=\dfrac{1}{C}$. Since $y(0)=0.5$, $\dfrac{1}{C}=\dfrac{1}{2} \Rightarrow C=2$, so $y=\dfrac{1}{x+2}$.

8. **(a)** If x is close to 0 , then xy^3 is close to 0 , and hence, y' is close to 0 . Thus, the graph of y must have a tangent line that is nearly horizontal. If x is large, then xy^3 is large, and the graph of y must have a tangent line that is nearly vertical. (In both cases, we assume reasonable values for y .)

(b) $y=\left(c-x^2\right)^{-1/2} \Rightarrow y'=x\left(c-x^2\right)^{-3/2}$.

RHS $=xy^3=x\left[\left(c-x^2\right)^{-1/2}\right]^3=x\left(c-x^2\right)^{-3/2}=y'=$ LHS



(c)

When x is close to 0 , y' is also close to 0 . As x gets larger, so does $\left|y'\right|$.

(d) $y(0)=(c-0)^{-1/2}=1/\sqrt{c}$ and $y(0)=2\Rightarrow\sqrt{c}=\frac{1}{2}\Rightarrow c=\frac{1}{4}$, so $y=\left(\frac{1}{4}-x^2\right)^{-1/2}$.

9. **(a)** $\frac{dP}{dt}=1.2P\left(1-\frac{P}{4200}\right)$. Now $\frac{dP}{dt}>0\Rightarrow 1-\frac{P}{4200}>0\Rightarrow\frac{P}{4200}<1\Rightarrow P<4200\Rightarrow$ the population is increasing for $0<P<4200$.

(b) $\frac{dP}{dt}<0\Rightarrow P>4200$

(c) $\frac{dP}{dt}=0\Rightarrow P=4200$ or $P=0$

10. **(a)** $y=k\Rightarrow y'=0$, so $\frac{dy}{dt}=y^4-6y^3+5y^2\Leftrightarrow 0=k^4-6k^3+5k^2\Leftrightarrow k^2\left(k^2-6k+5\right)=0\Leftrightarrow k^2(k-1)(k-5)=0\Leftrightarrow k=0$, 1 , or 5

(b) y is increasing $\Leftrightarrow\frac{dy}{dt}>0\Leftrightarrow y^2(y-1)(y-5)>0\Leftrightarrow y\in(-\infty,0)\cup(0,1)\cup(5,\infty)$

(c) y is decreasing $\Leftrightarrow\frac{dy}{dt}<0\Leftrightarrow y\in(1,5)$

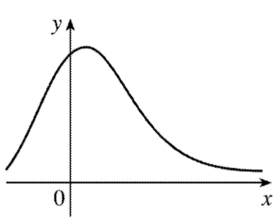
11. **(a)** This function is increasing *and* also decreasing. But $dy/dt=e^t(y-1)^2\geq 0$ for all t , implying that the graph of the solution of the differential equation cannot be decreasing on any interval.
(b) When $y=1$, $dy/dt=0$, but the graph does not have a horizontal tangent line.

12. The graph for this exercise is shown in the figure at the right.

A. $y'=1+xy>1$ for points in the first quadrant, but we can see that $y'<0$ for some points in the first quadrant. So equation A is incorrect.

B. $y'=-2xy=0$ when $x=0$, but we can see that $y'>0$ for $x=0$. So equation B is incorrect.

C. $y'=1-2xy$ seems reasonable since:



(1) When $x=0$, y' could be 1 .

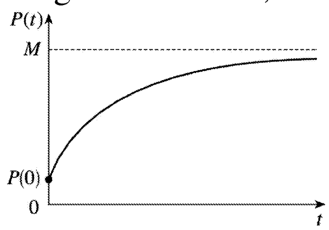
(2) When $x<0$, y' could be greater than 1 .

(3)

Solving $y^{/}=1-2xy$ for y gives us $y=\dfrac{1-y^{/}}{2x}$. If $y^{/}$ takes on small negative values, then as $x\rightarrow\infty$, $y\rightarrow 0^{+}$, as shown in the figure. Thus, the correct equation is C.

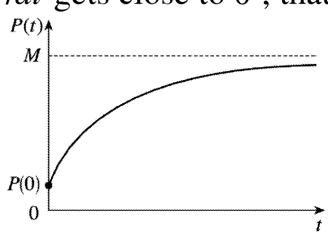
13. **(a)** *P* increases most rapidly at the beginning, since there are usually many simple, easily–learned sub–skills associated with learning a skill. As t increases, we would expect dP/dt to remain positive, but decrease. This is because as time progresses, the only points left to learn are the more difficult ones.

(b) $\dfrac{dP}{dt}=k(M-P)$ is always positive, so the level of performance P is increasing. As P gets close to M , dP/dt gets close to 0 ; that is, the performance levels off, as explained in part (a).

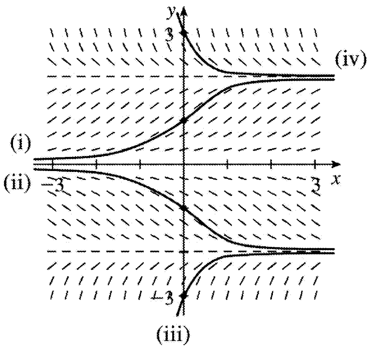
(c) 

14. **(a)** *P* increases most rapidly at the beginning, since there are usually many simple, easily–learned sub–skills associated with learning a skill. As t increases, we would expect dP/dt to remain positive, but decrease. This is because as time progresses, the only points left to learn are the more difficult ones.

(b) $\dfrac{dP}{dt}=k(M-P)$ is always positive, so the level of performance P is increasing. As P gets close to M , dP/dt gets close to 0 ; that is, the performance levels off, as explained in part (a).

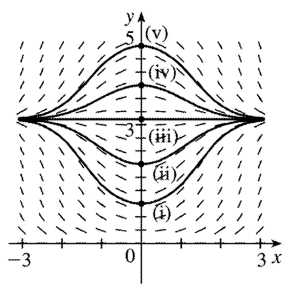
(c) 

1. **(a)**



(b) It appears that the constant functions $y=0$, $y=-2$, and $y=2$ are equilibrium solutions. Note that these three values of y satisfy the given differential equation $y'=y\left(1-\frac{1}{4}y^2\right)$.

2. **(a)**



(b) From the figure, it appears that $y=\pi$ is an equilibrium solution. From the equation $y'=x\sin y$, we see that $y=n\pi$ (n an integer) describes all the equilibrium solutions.

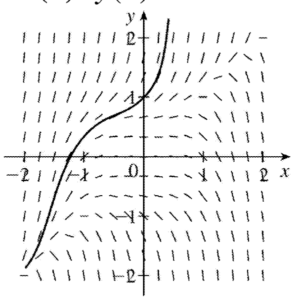
3. $y'=y-1$. The slopes at each point are independent of x , so the slopes are the same along each line parallel to the $x-$ axis. Thus, IV is the direction field for this equation. Note that for $y=1$, $y'=0$.

4. $y'=y-x=0$ on the line $y=x$, when $x=0$ the slope is y , and when $y=0$ the slope is $-x$. Direction field II satisfies these conditions.

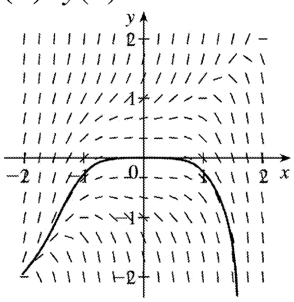
5. $y'=y^2-x^2=0\Rightarrow y=\pm x$. There are horizontal tangents on these lines only in graph III, so this equation corresponds to direction field III.

6. $y'=y^3-x^3=0$ on the line $y=x$, when $x=0$ the slope is y^3 , and when $y=0$ the slope is $-x^3$. The graph is similar to the graph for Exercise 4, but the segments must get steeper very rapidly as they move away from the origin, because x and y are raised to the third power. This is the case in direction field I.

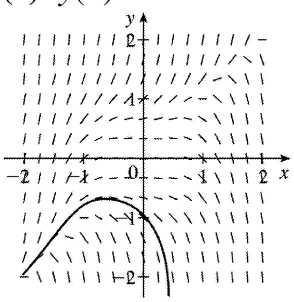
7. **(a)** $y(0)=1$



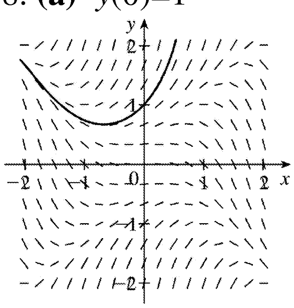
(b) $y(0)=0$



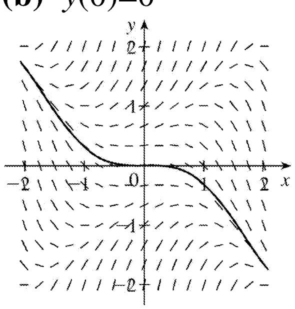
(c) $y(0)=-1$



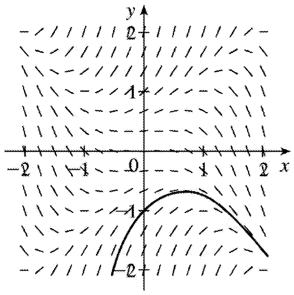
8. **(a)** $y(0)=1$



(b) $y(0)=0$



(c) $y(0)=-1$

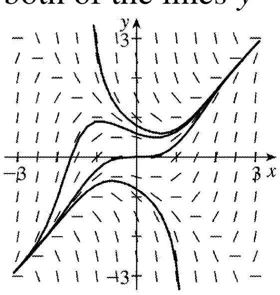


9.

10.

x	y	$y'=x^2-y^2$
± 1	± 3	-8
± 3	± 1	8
± 1	± 0.5	0.75
± 0.5	± 1	-0.75

Note that $y'=0$ for $y=\pm x$. If $|x|<|y|$, then $y'<0$; that is, the slopes are negative for all points in quadrants I and II above both of the lines $y=x$ and $y=-x$, and all points in quadrants III and IV below both of the lines $y=-x$ and $y=x$. A similar statement holds for positive slopes.

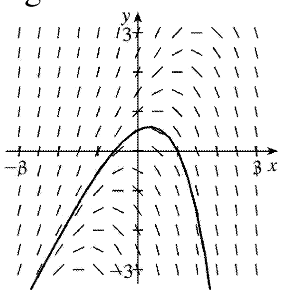


11.

x	y	$y'=y-2x$
-2	-2	2
-2	2	6
2	2	-2
2	-2	-6

Note that

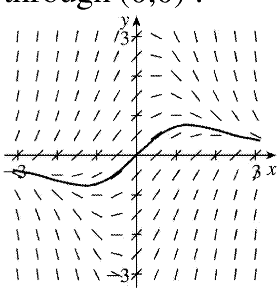
$y' = 0$ for any point on the line $y = 2x$. The slopes are positive to the left of the line and negative to the right of the line. The solution curve in the graph passes through $(1,0)$.



12.

x	y	$y' = 1 - xy$
± 1	± 1	0
± 2	± 2	-3
± 2	∓ 2	5

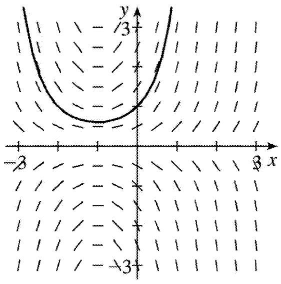
Note that $y' = 0$ for any point on the hyperbola $xy = 1$ (or $y = 1/x$). The slopes are negative at points ''inside'' the branches and positive at points everywhere else. The solution curve in the graph passes through $(0,0)$.



13.

x	y	$y' = y + xy$
0	± 2	± 2
1	± 2	± 4
-3	± 2	∓ 4

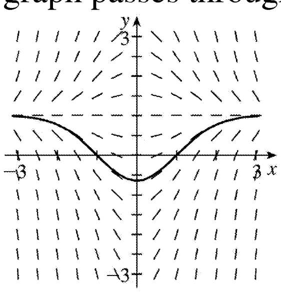
Note that $y' = y(x+1) = 0$ for any point on $y = 0$ or on $x = -1$. The slopes are positive when the factors y and $x+1$ have the same sign and negative when they have opposite signs. The solution curve in the graph passes through $(0,1)$.



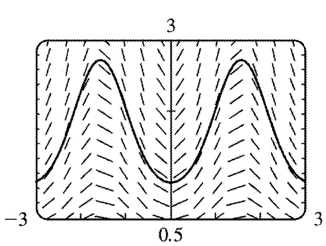
14.

x	y	$y'=x-xy$
± 2	0	± 2
± 2	3	∓ 4
± 2	−1	± 4

Note that $y'=x(1-y)=0$ for any point on $x=0$ or on $y=1$. The slopes are positive when the factors x and $1-y$ have the same sign and negative when they have opposite signs. The solution curve in the graph passes through $(1,0)$.

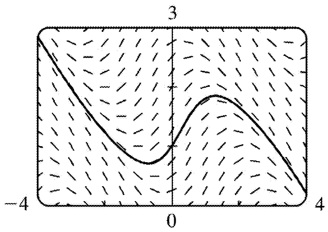


15. In Maple, we can use either directionfield (in Maple's share library) or plots [fieldplot] to plot the direction field. To plot the solution, we can either use the initial–value option in directionfield, or actually solve the equation. In Mathematica, we use PlotVectorField for the direction field, and the Plot [Evaluate[...]] construction to plot the solution, which

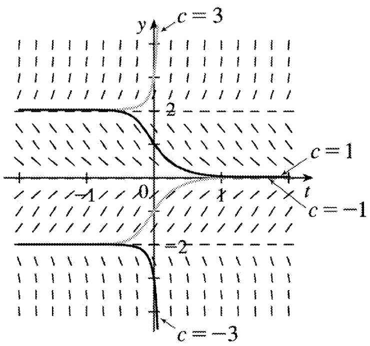


is $y=e^{(1-\cos 2x)/2}$. In Derive, use Direction_Field (in utility file ODE_APPR) to plot the direction field. Then use DSOLVE1(−y*SIN(2*x),1,x,y,0,1) (in utility file ODE1) to solve the equation. Simplify each result.

16. See Exercise 15 for specific CAS directions. The exact solution is

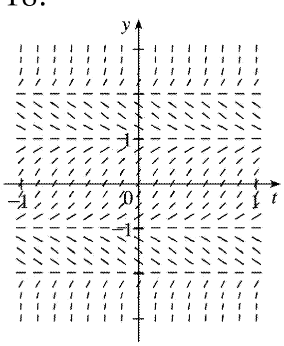


17.



$L = \lim_{t \to \infty} y(t)$ exists for $-2 \le c \le 2$; $L = \pm 2$ for $c = \pm 2$ and $L = 0$ for $-2 < c < 2$. For other values of c , L does not exist.

18.



Note that when $f(y) = 0$ on the graph in the text, we have $y' = f(y) = 0$; so we get horizontal segments at $y = \pm 1$, ± 2 . We get segments with negative slopes only for $1 < |y| < 2$. All other segments have positive slope. For the limiting behavior of solutions:

- If $y(0) > 2$, then $\lim_{t \to \infty} y = \infty$ and $\lim_{t \to -\infty} y = 2$.
- If $1 < y(0) < 2$, then $\lim_{t \to \infty} y = 1$ and $\lim_{t \to -\infty} y = 2$.
- If $-1 < y(0) < 1$, then $\lim_{t \to \infty} y = 1$ and $\lim_{t \to -\infty} y = -1$.
- If $-2 < y(0) < -1$, then $\lim_{t \to \infty} y = -2$ and $\lim_{t \to -\infty} y = -1$.

- If $y<-2$, then $\lim_{t\to\infty} y=-2$ and $\lim_{t\to-\infty} y=-\infty$.

19. **(a)**

(a) $h=0.4$ and $y_1=y_0+hF\left(x_0,y_0\right)\Rightarrow y_1=1+0.4\cdot 1=1.4$. $x_1=x_0+h=0+0.4=0.4$, so $y_1=y(0.4)=1.4$.

(b) $h=0.4$ and $y_1=y_0+hF\left(x_0,y_0\right)\Rightarrow y_1=1+0.4\cdot 1=1.4$. $x_1=x_0+h=0+0.4=0.4$, so $y_1=y(0.4)=1.4$.

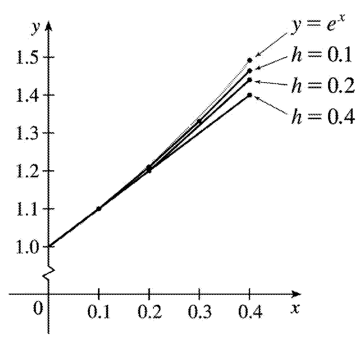
(c) $h=0.2\Rightarrow x_1=0.2$ and $x_2=0.4$, so we need to find y_2 . $y_1=y_0+hF\left(x_0,y_0\right)=1+0.2y_0=1+0.2\cdot 1=1.2$,
$y_2=y_1+hF\left(x_1,y_1\right)=1.2+0.2y_1=1.2+0.2\cdot 1.2=1.44$.

(d) $h=0.2\Rightarrow x_1=0.2$ and $x_2=0.4$, so we need to find y_2 . $y_1=y_0+hF\left(x_0,y_0\right)=1+0.2y_0=1+0.2\cdot 1=1.2$,
$y_2=y_1+hF\left(x_1,y_1\right)=1.2+0.2y_1=1.2+0.2\cdot 1.2=1.44$.

(e) $h=0.1\Rightarrow x_4=0.4$, so we need to find y_4 . $y_1=y_0+hF\left(x_0,y_0\right)=1+0.1y_0=1+0.1\cdot 1=1.1$,
$y_2=y_1+hF\left(x_1,y_1\right)=1.1+0.1y_1=1.1+0.1\cdot 1.1=1.21$,
$y_3=y_2+hF\left(x_2,y_2\right)=1.21+0.1y_2=1.21+0.1\cdot 1.21=1.331$,
$y_4=y_3+hF\left(x_3,y_3\right)=1.331+0.1y_3=1.331+0.1\cdot 1.331=1.4641$.

(f) $h=0.1\Rightarrow x_4=0.4$, so we need to find y_4 . $y_1=y_0+hF\left(x_0,y_0\right)=1+0.1y_0=1+0.1\cdot 1=1.1$,
$y_2=y_1+hF\left(x_1,y_1\right)=1.1+0.1y_1=1.1+0.1\cdot 1.1=1.21$,
$y_3=y_2+hF\left(x_2,y_2\right)=1.21+0.1y_2=1.21+0.1\cdot 1.21=1.331$,
$y_4=y_3+hF\left(x_3,y_3\right)=1.331+0.1y_3=1.331+0.1\cdot 1.331=1.4641$.

(b)



We see that the estimates are underestimates since they are all below the graph of $y=e^x$.

(c)

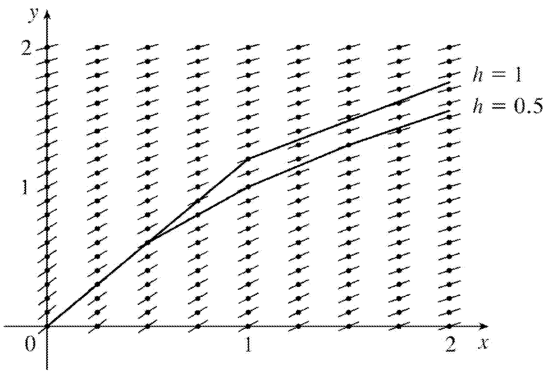
(a) For $h=0.4$: $(\text{exactvalue})-(\text{approximatevalue})=e^{0.4}-1.4\approx 0.0918$

(b) For $h=0.2$: $(\text{exactvalue})-(\text{approximatevalue})=e^{0.4}-1.44\approx 0.0518$

(c) For $h=0.1$: $(\text{exactvalue})-(\text{approximatevalue})=e^{0.4}-1.4641\approx 0.0277$

Each time the step size is halved, the error estimate also appears to be halved (approximately).

20.



As x increases, the slopes decrease and all of the estimates are above the true values. Thus, all of the estimates are overestimates.

21. $h=0.5$, $x_0=1$, $y_0=0$, and $F(x,y)=y-2x$.

Note that $x_1=x_0+h=1+0.5=1.5$, $x_2=2$, and $x_3=2.5$.

$y_1=y_0+hF\left(x_0,y_0\right)=0+0.5F(1,0)=0.5[0-2(1)]=-1$.

$y_2=y_1+hF\left(x_1,y_1\right)=-1+0.5F(1.5,-1)=-1+0.5[-1-2(1.5)]=-3$.

$y_3=y_2+hF\left(x_2,y_2\right)=-3+0.5F(2,-3)=-3+0.5[-3-2(2)]=-6.5$.

$y_4=y_3+hF\left(x_3,y_3\right)=-6.5+0.5F(2.5,-6.5)=-6.5+0.5[-6.5-2(2.5)]=-12.25$.

22. $h=0.2$, $x_0=0$, $y_0=0$, and $F(x,y)=1-xy$.

Note that $x_1=x_0+h=0+0.2=0.2$, $x_2=0.4$, $x_3=0.6$, and $x_4=0.8$.

$y_1=y_0+hF\left(x_0,y_0\right)=0+0.2F(0,0)=0.2[1-(0)(0)]=0.2$.

$y_2=y_1+hF\left(x_1,y_1\right)=0.2+0.2F(0.2,0.2)=0.2+0.2[1-(0.2)(0.2)]=0.392$.

$y_3=y_2+hF\left(x_2,y_2\right)=0.392+0.2F(0.4,0.392)=0.392+0.2[1-(0.4)(0.392)]=0.56064$.

$y_4=y_3+hF\left(x_3,y_3\right)=0.56064+0.2[1-(0.6)(0.56064)]=0.6933632$.

$y_5=y_4+hF\left(x_4,y_4\right)=0.6933632+0.2[1-(0.8)(0.6933632)]=0.782425088$.

Thus, $y(1)\approx 0.7824$.

23. $h=0.1$, $x_0=0$, $y_0=1$, and $F(\text{x,y})=y+xy$.

Note that $x_1=x_0+h=0+0.1=0.1$, $x_2=0.2$, $x_3=0.3$, and $x_4=0.4$.

$y_1=y_0+hF\left(x_0,\text{y}_0\right)=1+0.1F(0,1)=1+0.1[1+(0)(1)]=1.1$.

$y_2=y_1+hF\left(x_1,\text{y}_1\right)=1.1+0.1F(0.1,1.1)=1.1+0.1[1.1+(0.1)(1.1)]=1.221$.

$y_3=y_2+hF\left(x_2,\text{y}_2\right)=1.221+0.1F(0.2,1.221)=1.221+0.1[1.221+(0.2)(1.221)]=1.36752$.

$$y_4=y_3+hF\left(x_3,\text{y}_3\right) =1.36752+0.1F(0.3,1.36752)=1.36752+0.1[1.36752+(0.3)(1.36752)]$$
$$=1.5452976 .$$

$$y_5=y_4+hF\left(x_4,\text{y}_4\right) =1.5452976+0.1F(0.4,1.5452976)$$
$$=1.5452976+0.1[1.5452976+(0.4)(1.5452976)]=1.761639264 .$$

Thus, $y(0.5)\approx 1.7616$.

24. **(a)** $h=0.2$, $x_0=1$, $y_0=0$, and $F(\text{x,y})=x-xy$.

We need to find y_2 , because $x_1=1.2$ and $x_2=1.4$.

$y_1=y_0+hF\left(x_0,\text{y}_0\right)=0+0.2F(1,0)=0.2[1-(1)(0)]=0.2$.

$y_2=y_1+hF\left(x_1,\text{y}_1\right)=0.2+0.2F(1.2,0.2)=0.2+0.2[1.2-(1.2)(0.2)]=0.392\approx y(1.4)$.

(b) Now $h=0.1$, so we need to find y_4 .

$y_1=0+0.1[1-(1)(0)]=0.1$,

$y_2=0.1+0.1[1.1-(1.1)(0.1)]=0.199$,

$y_3=0.199+0.1[1.2-(1.2)(0.199)]=0.29512$, and

$y_4=0.29512+0.1[1.3-(1.3)(0.29512)]=0.3867544\approx y(1.4)$.

25. **(a)**

(a) H $=1$, N $=1\Rightarrow y(1)=3$

(b) H $=1$, N $=1\Rightarrow y(1)=3$

(c) H $=0.1$, N $=10\Rightarrow y(1)\approx 2.3928$

(d) H $=0.1$, N $=10\Rightarrow y(1)\approx 2.3928$

(e) H $=0.01$, N $=100\Rightarrow y(1)\approx 2.3701$

(f) H $=0.01$, N $=100\Rightarrow y(1)\approx 2.3701$

(g) H =0.001, N =1000 $\Rightarrow y(1)\approx 2.3681$

(h) H =0.001, N =1000 $\Rightarrow y(1)\approx 2.3681$

(b) $y=2+e^{-x^3} \Rightarrow y'=-3x^2e^{-x^3}$

$$\text{LHS}=y'+3x^2y=-3x^2e^{-x^3}+3x^2\left(2+e^{-x^3}\right)=-3x^2e^{-x^3}+6x^2+3x^2e^{-x^3}=6x^2=\text{RHS}$$

$y(0)=2+e^{-0}=2+1=3$

(c)

(a) For $h=1$: $(\text{exactvalue})-(\text{approximatevalue})=2+e^{-1}-3\approx -0.6321$

(b) For $h=1$: $(\text{exactvalue})-(\text{approximatevalue})=2+e^{-1}-3\approx -0.6321$

(c) For $h=0.1$: $(\text{exactvalue})-(\text{approximatevalue})=2+e^{-1}-2.3928\approx -0.0249$

(d) For $h=0.1$: $(\text{exactvalue})-(\text{approximatevalue})=2+e^{-1}-2.3928\approx -0.0249$

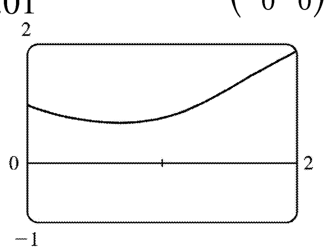
(e) For $h=0.01$: $(\text{exactvalue})-(\text{approximatevalue})=2+e^{-1}-2.3701\approx -0.0022$

(f) For $h=0.01$: $(\text{exactvalue})-(\text{approximatevalue})=2+e^{-1}-2.3701\approx -0.0022$

(g) For $h=0.001$: $(\text{exactvalue})-(\text{approximatevalue})=2+e^{-1}-2.3681\approx -0.0002$

(h) For $h=0.001$: $(\text{exactvalue})-(\text{approximatevalue})=2+e^{-1}-2.3681\approx -0.0002$

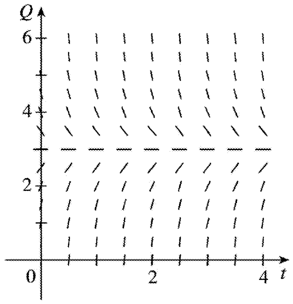
26. **(a)** We use the program from the solution to Exercise 25 with $Y_1=x^3-y^3$, H =0.01 , and N $=\dfrac{2-0}{0.01}=200$. With $(x_0,y_0)=(0,1)$, we get $y(2)\approx 1.9000$.



(b)

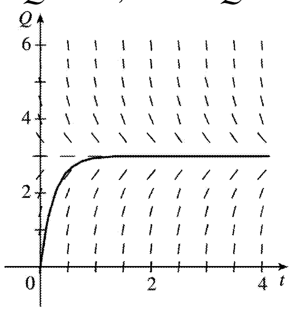
Notice from the graph that $y(2)\approx 1.9$, which serves as a check on our calculation in part (a).

27. **(a)** $R\dfrac{dQ}{dt}+\dfrac{1}{C}Q=E(t)$ becomes $5Q'+\dfrac{1}{0.05}Q=60$ or $Q'+4Q=12$.



(b) From the graph, it appears that the limiting value of the charge Q is about 3 .

(c) If $Q'=0$, then $4Q=12\Rightarrow Q=3$ is an equilibrium solution.



(d)

(e) $Q'+4Q=12\Rightarrow Q'=12-4Q$. Now $Q(0)=0$, so $t_0=0$ and $Q_0=0$.

$$Q_1 = Q_0+hF\left(t_0,Q_0\right)=0+0.1(12-4\cdot 0)=1.2$$

$$Q_2 = Q_1+hF\left(t_1,Q_1\right)=1.2+0.1(12-4\cdot 1.2)=1.92$$

$$Q_3 = Q_2+hF\left(t_2,Q_2\right)=1.92+0.1(12-4\cdot 1.92)=2.352$$

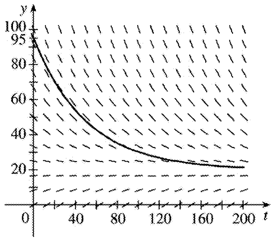
$$Q_4 = Q_3+hF\left(t_3,Q_3\right)=2.352+0.1(12-4\cdot 2.352)=2.6112$$

$$Q_5 = Q_4+hF\left(t_4,Q_4\right)=2.6112+0.1(12-4\cdot 2.6112)=2.76672$$

Thus, $Q_5=Q(0.5)\approx 2.77$ C.

28. **(a)** From Exercise .1.14, we have $dy/dt=k(y-R)$. We are given that $R=20^\circ$ C and $dy/dt=-1^\circ$ C / min when $y=70^\circ$ C. Thus, $-1=k(70-20)\Rightarrow k=-\frac{1}{50}$ and the differential equation becomes $dy/dt=-\frac{1}{50}(y-20)$.

(b)



The limiting value of the temperature is 20° C;
that is, the temperature of the room.

(c) From part (a), $dy/dt=-\frac{1}{50}(y-20)$. With $t_0=0$, $y_0=95$, and $h=2$ min, we get

$$y_1=y_0+hF\left(t_0,y_0\right)=95+2\left[-\frac{1}{50}(95-20)\right]=92$$

$$y_2=y_1+hF\left(t_1,y_1\right)=92+2\left[-\frac{1}{50}(92-20)\right]=89.12$$

$$y_3=y_2+hF\left(t_2,y_2\right)=89.12+2\left[-\frac{1}{50}(89.12-20)\right]=86.3552$$

$$y_4=y_3+hF\left(t_3,y_3\right)=86.3552+2\left[-\frac{1}{50}(86.3552-20)\right]=83.700992$$

$$y_5=y_4+hF\left(t_4,y_4\right)=83.700992+2\left[-\frac{1}{50}(83.700992-20)\right]=81.15295232$$

Thus, $y(10)\approx 81.15^\circ$ C.

1. $\frac{dy}{dx}=\frac{y}{x}\Rightarrow\frac{dy}{y}=\frac{dx}{x}\Rightarrow\int\frac{dy}{y}=\int\frac{dx}{x}\Rightarrow\ln|y|=\ln|x|+C\Rightarrow|y|=e^{\ln|x|+C}=e^{\ln|x|}e^{C}=e^{C}|x|\Rightarrow y=Kx$, where $K=\pm e^{C}$ is a constant. (In our derivation, K was nonzero, but we can restore the excluded case $y=0$ by allowing K to be zero.)

2. $\frac{dy}{dx}=\frac{e^{2x}}{4y^{3}}\Rightarrow 4y^{3}dy=e^{2x}dx\Rightarrow\int 4y^{3}dy=\int e^{2x}dx\Rightarrow y^{4}=\frac{1}{2}e^{2x}+C\Rightarrow\ y=\pm\sqrt[4]{\frac{1}{2}e^{2x}+C}$

3. $(x^{2}+1)y'=xy\Rightarrow\frac{dy}{dx}=\frac{xy}{x^{2}+1}\Rightarrow\frac{dy}{y}=\frac{x\,dx}{x^{2}+1}$ $[\,y\neq 0\,]\Rightarrow\int\frac{dy}{y}=\int\frac{x\,dx}{x^{2}+1}\Rightarrow\ln|y|=\frac{1}{2}\ln(x^{2}+1)+C$ $[$ $u=x^{2}+1$, $du=2x\,dx$ $]$ $=\ln(x^{2}+1)^{1/2}+\ln e^{C}=\ln\left(e^{C}\sqrt{x^{2}+1}\right)\Rightarrow|y|=e^{C}\sqrt{x^{2}+1}\Rightarrow y=K\sqrt{x^{2}+1}$, where $K=\pm e^{C}$ is a constant. (In our derivation, K was nonzero, but we can restore the excluded case $y=0$ by allowing K to be zero.)

4. $y'=y^{2}\sin x\Rightarrow\frac{dy}{dx}=y^{2}\sin x\Rightarrow\frac{dy}{y^{2}}=\sin x\,dx\Rightarrow\int\frac{dy}{y^{2}}=\int\sin x\,dx\Rightarrow-\frac{1}{y}=-\cos x+C\Rightarrow\frac{1}{y}=\cos x-C\Rightarrow$ $y=\frac{1}{\cos x+K}$, where $K=-C$. $y=0$ is also a solution.

5. $(1+\tan y)y'=x^{2}+1\Rightarrow(1+\tan y)\frac{dy}{dx}=x^{2}+1\Rightarrow\left(1+\frac{\sin y}{\cos y}\right)dy=(x^{2}+1)dx\Rightarrow$ $\int\left(1-\frac{-\sin y}{\cos y}\right)dy=\int(x^{2}+1)dx\Rightarrow y-\ln|\cos y|=\frac{1}{3}x^{3}+x+C$. Note: The left side is equivalent to $y+\ln|\sec y|$.

6. $\frac{du}{dr}=\frac{1+\sqrt{r}}{1+\sqrt{u}}\Rightarrow(1+\sqrt{u})du=(1+\sqrt{r})dr\Rightarrow\int(1+u^{1/2})du=\int(1+r^{1/2})dr\Rightarrow\ u+\frac{2}{3}u^{3/2}=r+\frac{2}{3}r^{3/2}+C$

7. $\frac{dy}{dt}=\frac{te^{t}}{y\sqrt{1+y^{2}}}\Rightarrow y\sqrt{1+y^{2}}\,dy=te^{t}dt\Rightarrow\int y\sqrt{1+y^{2}}\,dy=\int te^{t}dt\Rightarrow\frac{1}{3}\left(1+y^{2}\right)^{3/2}=te^{t}-e^{t}+C\Rightarrow$ $1+y^{2}=[3(te^{t}-e^{t}+C)]^{2/3}\Rightarrow\ y=\pm\sqrt{[3(te^{t}-e^{t}+C)]^{2/3}-1}$

8. $y'=\frac{xy}{2\ln y}\Rightarrow\frac{2\ln y}{y}dy=x\,dx\Rightarrow\int\frac{2\ln y}{y}dy=\int x\,dx\Rightarrow(\ln y)^{2}=\frac{x^{2}}{2}+C\Rightarrow\ln y=\pm\sqrt{x^{2}/2+C}\Rightarrow$

$y=e^{\pm\sqrt{x^2/2+C}}$

9. $\frac{du}{dt}=2+2u+t+tu\Rightarrow\frac{du}{dt}=(1+u)(2+t)\Rightarrow\int\frac{du}{1+u}=\int(2+t)dt\Rightarrow\ln|1+u|=\frac{1}{2}t^2+2t+C\Rightarrow$ $|1+u|=e^{t^2/2+2t+C}=Ke^{t^2/2+2t}$, where $K=e^C\Rightarrow 1+u=\pm Ke^{t^2/2+2t}\Rightarrow u=-1\pm Ke^{t^2/2+2t}$ where $K>0$. $u=-1$ is also a solution, so $u=-1+Ae^{t^2/2+2t}$, where A is an arbitrary constant.

10. $\frac{dz}{dt}+e^{t+z}=0\Rightarrow\frac{dz}{dt}=-e^te^z\Rightarrow\int e^{-z}dz=-\int e^tdt\Rightarrow-e^{-z}=-e^t+C\Rightarrow e^{-z}=e^t-C\Rightarrow\frac{1}{e^z}=e^t-C\Rightarrow e^z=\frac{1}{e^t-C}\Rightarrow$ $z=\ln\left(\frac{1}{e^t-C}\right)\Rightarrow z=-\ln\left(e^t-C\right)$

11. $\frac{dy}{dx}=y^2+1$, $y(1)=0$. $\int\frac{dy}{y^2+1}=\int dx\Rightarrow\tan^{-1}y=x+C$. $y=0$ when $x=1$, so $1+C=\tan^{-1}0=0\Rightarrow C=-1$. Thus, $\tan^{-1}y=x-1$ and $y=\tan(x-1)$.

12. $\frac{dy}{dx}=\frac{y\cos x}{1+y^2}$, $y(0)=1$. $\left(1+y^2\right)dy=y\cos x\,dx\Rightarrow\frac{1+y^2}{y}dy=\cos x\,dx\Rightarrow\int\left(\frac{1}{y}+y\right)dy=\int\cos x\,dx\Rightarrow$ $\ln|y|+\frac{1}{2}y^2=\sin x+C$. $y(0)=1\Rightarrow\ln 1+\frac{1}{2}=\sin 0+C\Rightarrow C=\frac{1}{2}$, so $\ln|y|+\frac{1}{2}y^2=\sin x+\frac{1}{2}$. We cannot solve explicitly for y.

13. $x\cos x=(2y+e^{3y})y'\Rightarrow x\cos x\,dx=(2y+e^{3y})dy\Rightarrow\int(2y+e^{3y})dy=\int x\cos x\,dx\Rightarrow$ $y^2+\frac{1}{3}e^{3y}=x\sin x+\cos x+C$ [where the second integral is evaluated using integration by parts]. Now $y(0)=0\Rightarrow 0+\frac{1}{3}=0+1+C\Rightarrow C=-\frac{2}{3}$. Thus, a solution is $y^2+\frac{1}{3}e^{3y}=x\sin x+\cos x-\frac{2}{3}$. We cannot solve explicitly for y.

14. $\frac{dP}{dt}=\sqrt{Pt}\Rightarrow dP/\sqrt{P}=\sqrt{t}\,dt\Rightarrow\int P^{-1/2}dP=\int t^{1/2}dt\Rightarrow 2P^{1/2}=\frac{2}{3}t^{3/2}+C$. $P(1)=2\Rightarrow 2\sqrt{2}=\frac{2}{3}+C\Rightarrow C=2\sqrt{2}-\frac{2}{3}$, so $2P^{1/2}=\frac{2}{3}t^{3/2}+2\sqrt{2}-\frac{2}{3}\Rightarrow\sqrt{P}=\frac{1}{3}t^{3/2}+\sqrt{2}-\frac{1}{3}\Rightarrow$ $P=\left(\frac{1}{3}t^{3/2}+\sqrt{2}-\frac{1}{3}\right)^2$.

15. $\frac{du}{dt}=\frac{2t+\sec^2 t}{2u}$, $u(0)=-5$. $\int 2u\,du=\int\left(2t+\sec^2 t\right)dt \Rightarrow u^2=t^2+\tan t+C$, where $[u(0)]^2=0^2+\tan 0+C$ $\Rightarrow C=(-5)^2=25$. Therefore, $u^2=t^2+\tan t+25$, so $u=\pm\sqrt{t^2+\tan t+25}$. Since $u(0)=-5$, we must have $u=-\sqrt{t^2+\tan t+25}$.

16. $\frac{dy}{dt}=te^y$, $y(1)=0$. $\int e^{-y}dy=\int t\,dt \Rightarrow -e^{-y}=\frac{1}{2}t^2+C$. Since $y(1)=0$, $-e^0=\frac{1}{2}\cdot 1^2+C$. Therefore, $C=-1-\frac{1}{2}=-\frac{3}{2}$ and $-e^{-y}=\frac{1}{2}t^2-\frac{3}{2}$. So $e^{-y}=\frac{3}{2}-\frac{1}{2}t^2=\frac{3-t^2}{2} \Rightarrow e^y=\frac{2}{3-t^2} \Rightarrow y=\ln 2-\ln\left(3-t^2\right)$ for $|t|<\sqrt{3}$.

17. $y'\tan x=a+y$, $0<x<\pi/2 \Rightarrow \frac{dy}{dx}=\frac{a+y}{\tan x} \Rightarrow \frac{dy}{a+y}=\cot x\,dx$ [$a+y\neq 0$] $\Rightarrow$
$\int\frac{dy}{a+y}=\int\frac{\cos x}{\sin x}dx \Rightarrow \ln|a+y|=\ln|\sin x|+C \Rightarrow |a+y|=e^{\ln|\sin x|+C}=e^{\ln|\sin x|}\cdot e^C=e^C|\sin x| \Rightarrow$
$a+y=K\sin x$, where $K=\pm e^C$. (In our derivation, K was nonzero, but we can restore the excluded case $y=-a$ by allowing K to be zero.) $y(\pi/3)=a \Rightarrow$
$a+a=K\sin\left(\frac{\pi}{3}\right) \Rightarrow 2a=K\frac{\sqrt{3}}{2} \Rightarrow K=\frac{4a}{\sqrt{3}}$. Thus, $a+y=\frac{4a}{\sqrt{3}}\sin x$ and so $y=\frac{4a}{\sqrt{3}}\sin x-a$.

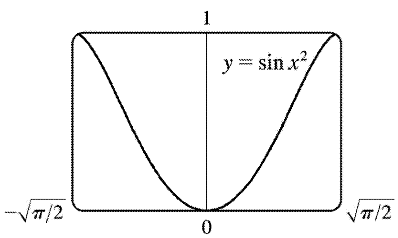
18. $xy'+y=y^2 \Rightarrow x\frac{dy}{dx}=y^2-y \Rightarrow x\,dy=(y^2-y)\,dx \Rightarrow \frac{dy}{y^2-y}=\frac{dx}{x} \Rightarrow$
$\int\frac{dy}{y(y-1)}=\int\frac{dx}{x}$ [$y\neq 0$, 1] $\Rightarrow \int\left(\frac{1}{y-1}-\frac{1}{y}\right)dy=\int\frac{dx}{x} \Rightarrow \ln|y-1|-\ln|y|=\ln|x|+C \Rightarrow$
$\ln\left|\frac{y-1}{y}\right|=\ln\left(e^C|x|\right) \Rightarrow \left|\frac{y-1}{y}\right|=e^C|x| \Rightarrow \frac{y-1}{y}=Kx$, where $K=\pm e^C \Rightarrow 1-\frac{1}{y}=Kx \Rightarrow \frac{1}{y}=1-Kx \Rightarrow$
$y=\frac{1}{1-Kx}$. Now $y(1)=-1 \Rightarrow -1=\frac{1}{1-K} \Rightarrow 1-K=-1 \Rightarrow K=2$,
so $y=\frac{1}{1-2x}$.

19. $\frac{dy}{dx}=4x^3y$, $y(0)=7$. $\frac{dy}{y}=4x^3dx \Rightarrow \int\frac{dy}{y}=\int 4x^3dx \Rightarrow \ln|y|=x^4+C \Rightarrow e^{\ln|y|}=e^{x^4+C} \Rightarrow |y|=e^{x^4}e^C \Rightarrow$
$y=Ae^{x^4}$; $y(0)=7 \Rightarrow A=7 \Rightarrow y=7e^{x^4}$.

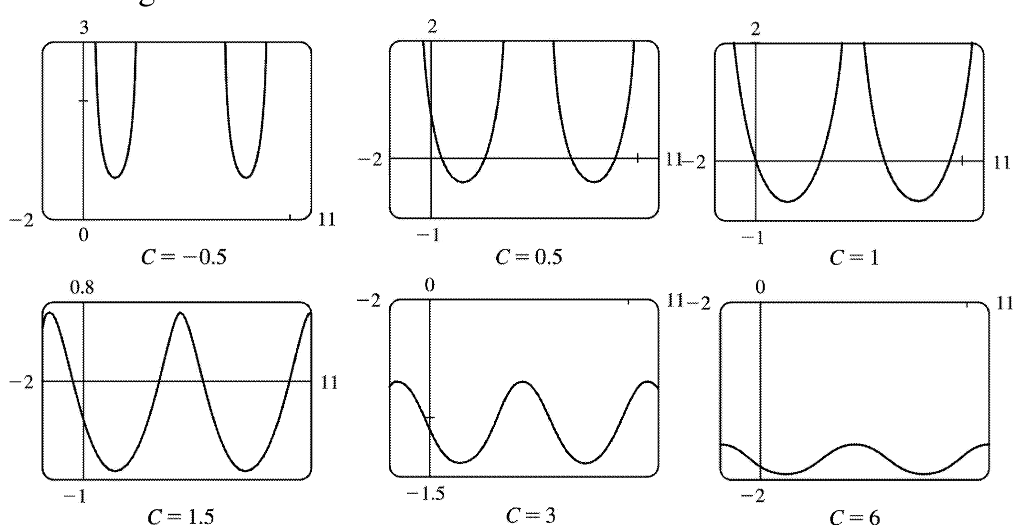
20.

$\dfrac{dy}{dx}=\dfrac{y^2}{x^3}$, $y(1)=1$. $\int \dfrac{dy}{y^2}=\int \dfrac{dx}{x^3} \Rightarrow -\dfrac{1}{y}=-\dfrac{1}{2x^2}+C$. $y(1)=1 \Rightarrow -1=-\dfrac{1}{2}+C \Rightarrow C=-\dfrac{1}{2}$. So
$\dfrac{1}{y}=\dfrac{1}{2x^2}+\dfrac{1}{2}=\dfrac{2+2x^2}{2\cdot 2x^2} \Rightarrow y=\dfrac{2x^2}{x^2+1}$.

21. **(a)** $y'=2x\sqrt{1-y^2} \Rightarrow \dfrac{dy}{dx}=2x\sqrt{1-y^2} \Rightarrow \dfrac{dy}{\sqrt{1-y^2}}=2x\,dx \Rightarrow \int \dfrac{dy}{\sqrt{1-y^2}}=\int 2x\,dx \Rightarrow \sin^{-1} y=x^2+C$ for
$-\dfrac{\pi}{2} \le x^2+C \le \dfrac{\pi}{2}$.
(b) $y(0)=0 \Rightarrow \sin^{-1} 0=0^2+C \Rightarrow C=0$, so $\sin^{-1} y=x^2$ and $y=\sin\left(x^2\right)$ for $-\sqrt{\pi/2} \le x \le \sqrt{\pi/2}$.

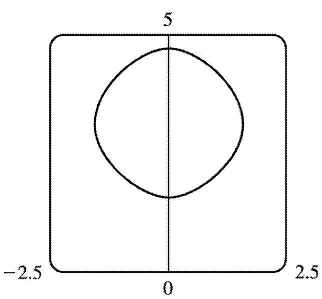


(c) For $\sqrt{1-y^2}$ to be a real number, we must have $-1 \le y \le 1$; that is, $-1 \le y(0) \le 1$. Thus, the initial–value problem $y'=2x\sqrt{1-y^2}$, $y(0)=2$ does *not* have a solution.
22. $e^{-y}y'+\cos x=0 \Leftrightarrow \int e^{-y}dy=-\int \cos x\,dx \Leftrightarrow -e^{-y}=-\sin x+C_1 \Leftrightarrow y=-\ln(\sin x+C)$. The solution is periodic, with period 2π . Note that for $C>1$, the domain of the solution is R , but for $-1<C\le 1$ it is only defined on the intervals where $\sin x+C>0$, and it is meaningless for $C\le -1$, since then $\sin x+C\le 0$, and the logarithm is undefined.

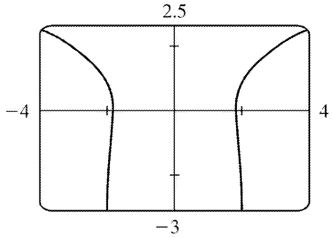


For $-1<C<1$, the solution curve consists of concave–up pieces separated by intervals on which the solution is not defined (where $\sin x+C\leq 0$). For $C=1$, the solution curve consists of concave–up pieces separated by vertical asymptotes at the points where $\sin x+C=0\Leftrightarrow\sin x=-1$. For $C>1$, the curve is continuous, and as C increases, the graph moves downward, and the amplitude of the oscillations decreases.

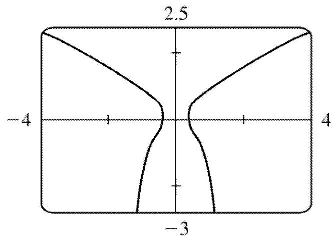
23. $\frac{dy}{dx}=\frac{\sin x}{\sin y}$, $y(0)=\frac{\pi}{2}$. So $\int\sin y\,dy=\int\sin x\,dx\Leftrightarrow-\cos y=-\cos x+C\Leftrightarrow\cos y=\cos x-C$. From the initial condition, we need $\cos\frac{\pi}{2}=\cos 0-C\Rightarrow 0=1-C\Rightarrow C=1$, so the solution is $\cos y=\cos x-1$. Note that we cannot take $\cos^{-1}$ of both sides, since that would unnecessarily restrict the solution to the case where $-1\leq\cos x-1\Leftrightarrow 0\leq\cos x$, as $\cos^{-1}$ is defined only on $[-1,1]$. Instead we plot the graph using Maple’s plots [implicitplot] or Mathematica’s Plot [Evaluate[..]].



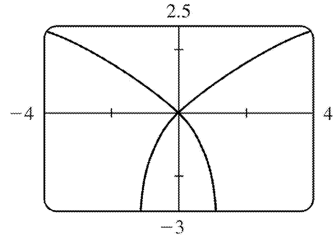
24. $\frac{dy}{dx}=\frac{x\sqrt{x^2+1}}{ye^y}\Leftrightarrow\int ye^y\,dy=\int x\sqrt{x^2+1}\,dx$. We use parts on the LHS with $u=y$, $dv=e^y\,dy$, and on the RHS we use the substitution $z=x^2+1$, so $dz=2x\,dx$. The equation becomes $ye^y-\int e^y\,dy=\frac{1}{2}\int\sqrt{z}\,dz\Leftrightarrow$ $e^y(y-1)=\frac{1}{3}\left(x^2+1\right)^{3/2}+C$, so we see that the curves are symmetric about the $y-$ axis. Every point (x,y) in the plane lies on one of the curves, namely the one for which $C=(y-1)e^y-\frac{1}{3}\left(x^2+1\right)^{3/2}$. For example,
along the $y-$ axis, $C=(y-1)e^y-\frac{1}{3}$, so the origin lies on the curve with $C=-\frac{4}{3}$. We use Maple’s plots command or Plot] in Mathematica to plot the solution curves for various values of C .



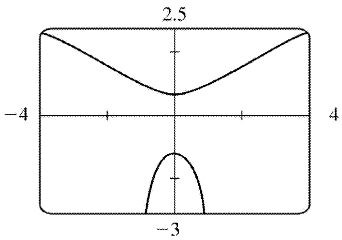
$C = -4$



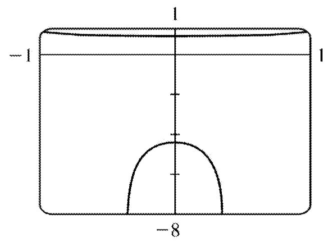
$C = -1.4$



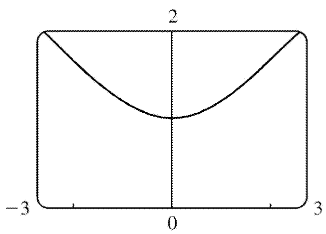
$C = -4/3$



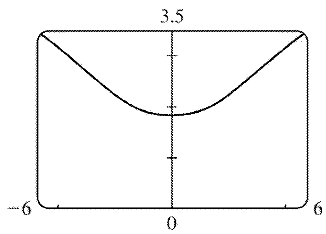
$C = -1$



$C = -0.4$



$C = -1/3$



$C = 5$

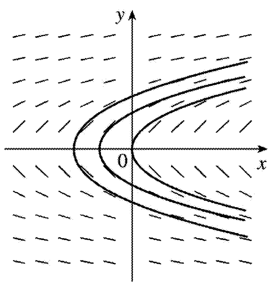
It seems that the transitional values of C are $-\frac{4}{3}$ and $-\frac{1}{3}$. For $C<-\frac{4}{3}$, the graph consists of left and right branches. At $C=-\frac{4}{3}$, the two branches become connected at the origin, and as C increases, the graph splits into top and bottom branches. At $C=-\frac{1}{3}$, the bottom half disappears. As C increases further, the graph moves upward, but doesn't change shape much.

25. **(a)**

x	y	$y'=1/y$
0	0.5	2
0	–0.5	–2
0	1	1
0	–1	–1
0	2	0.5

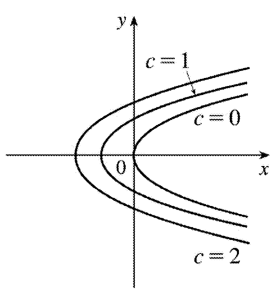
x	y	$y'=1/y$
0	–2	–0.5
0	4	0.25

0	3	$0.\overline{3}$
0	0.25	4
0	$0.\overline{3}$	3



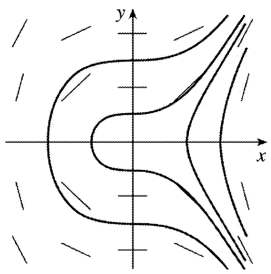
(b) $y'=1/y \Rightarrow dy/dx=1/y \Rightarrow$
$y\,dy=dx \Rightarrow \int y\,dy=\int dx \Rightarrow \frac{1}{2}y^2=x+c \Rightarrow y^2=2(x+c)$ or $y=\pm\sqrt{2(x+c)}$.

(c)

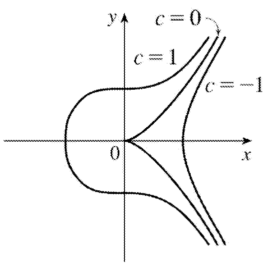


26. **(a)**

x	y	$y'=x^2/y$
1	1	1
–1	1	1
–1	–1	–1
1	–1	–1
1	2	0.5
2	1	4
2	2	2
1	0.5	2
0.5	1	0.25
2	0.5	8

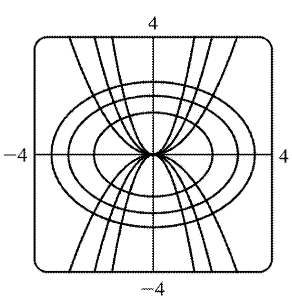


(b) $y' = x^2/y \Rightarrow y\,dy = x^2\,dx$, so $\frac{1}{2}y^2 = \frac{1}{3}x^3 + c_1$, or $y = \pm\left(\frac{2}{3}x^3 + c\right)^{1/2}$.



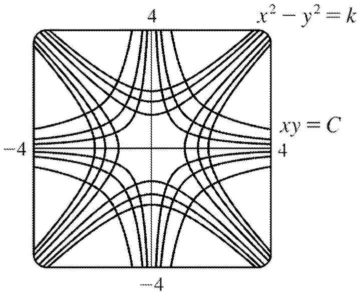
(c)

27. The curves $y = kx^2$ form a family of parabolas with axis the y– axis. Differentiating gives $y' = 2kx$, but $k = y/x^2$, so $y' = 2y/x$. Thus, the slope of the tangent line at any point (x,y) on one of the parabolas is $y' = 2y/x$, so the orthogonal trajectories must satisfy $y' = -x/(2y) \Leftrightarrow 2y\,dy = -x\,dx \Leftrightarrow y^2 = \left(-x^2\right)/2 + C_1 \Leftrightarrow$ $x^2 + 2y^2 = C$. This is a family of ellipses.

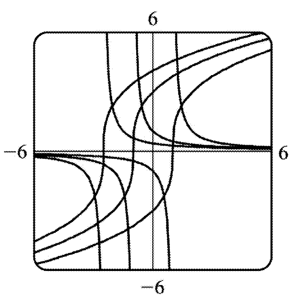


28. The curves $x^2 - y^2 = k$ form a family of hyperbolas. Differentiating gives $2x - 2y(dy/dx) = 0$ or $y' = x/y$, the slope of the tangent line at (x,y) on one of the hyperbolas. Thus, the orthogonal trajectories must satisfy $y' = -y/x \Leftrightarrow dy/y = -dx/x \Leftrightarrow \ln|y| = -\ln|x| + C_1 \Leftrightarrow \ln|x| + \ln|y| = C_1 \Leftrightarrow \ln|xy| = C_1 \Leftrightarrow$

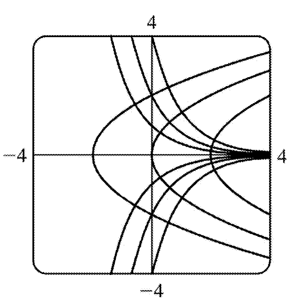
$|xy| = e^{C_1} \Leftrightarrow xy = C$. This is a family of hyperbolas.



29. Differentiating $y=(x+k)^{-1}$ gives $y'=-\dfrac{1}{(x+k)^2}$, but $k=\dfrac{1}{y}-x$, so $y'=-\dfrac{1}{(1/y)^2}=-y^2$. Thus, the orthogonal trajectories must satisfy $y'=-\dfrac{1}{-y^2}=\dfrac{1}{y^2}\Leftrightarrow y^2dy=dx\Leftrightarrow\dfrac{y^3}{3}=x+C$ or $y=[3(x+C)]^{1/3}$



30. Differentiating $y=ke^{-x}$ gives $y'=-ke^{-x}$, but $k=ye^{x}$, so $y'=-y$. Thus, the orthogonal trajectories must satisfy $y'=-1/(-y)=1/y\Leftrightarrow ydy=dx\Leftrightarrow\dfrac{1}{2}y^2=x+C\Leftrightarrow y=\pm[2(C+x)]^{1/2}$. This is a family of parabolas with axis the x –axis.



31. From Exercise .2.27, $\dfrac{dQ}{dt}=12-4Q\Leftrightarrow\int\dfrac{dQ}{12-4Q}=\int dt\Leftrightarrow-\dfrac{1}{4}\ln|12-4Q|=t+C\Leftrightarrow\ln|12-4Q|=-4t-4C$ $\Leftrightarrow|12-4Q|=e^{-4t-4C}\Leftrightarrow 12-4Q=Ke^{-4t}\Leftrightarrow 4Q=12-Ke^{-4t}\Leftrightarrow Q=3-Ae^{-4t}$. $Q(0)=0\Leftrightarrow 0=3-A\Leftrightarrow A=3\Leftrightarrow$ $Q(t)=3-3e^{-4t}$. As $t\to\infty$, $Q(t)\to 3-0=3$ (the limiting value).

32. From Exercise .2.28, $\frac{dy}{dt}=-\frac{1}{50}(y-20)\Leftrightarrow\int\frac{dy}{y-20}=\int\left(-\frac{1}{50}\right)dt\Leftrightarrow\ln|y-20|=-\frac{1}{50}t+C\Leftrightarrow$ $y-20=Ke^{-t/50}\Leftrightarrow y(t)=Ke^{-t/50}+20$. $y(0)=95\Leftrightarrow 95=K+20\Leftrightarrow K=75\Leftrightarrow y(t)=75e^{-t/50}+20$.

33. $\frac{dP}{dt}=k(M-P)\Leftrightarrow\int\frac{dP}{P-M}=\int(-k)\,dt\Leftrightarrow\ln|P-M|=-kt+C\Leftrightarrow|P-M|=e^{-kt+C}\Leftrightarrow P-M=Ae^{-kt}\Leftrightarrow P=M+Ae^{-kt}$. If we assume that performance is at level 0 when $t=0$, then $P(0)=0\Leftrightarrow 0=M+A\Leftrightarrow A=-M\Leftrightarrow$ $P(t)=M-Me^{-kt}$. $\lim_{t\to\infty}P(t)=M-M\cdot 0=M$.

34. **(a)** $\frac{dx}{dt}=k(a-x)(b-x)$, $a\neq b$. Using partial fractions, $\frac{1}{(a-x)(b-x)}=\frac{1/(b-a)}{a-x}-\frac{1/(b-a)}{b-x}$, so $\int\frac{dx}{(a-x)(b-x)}=\int k\,dt\Rightarrow\frac{1}{b-a}(-\ln|a-x|+\ln|b-x|)=kt+C\Rightarrow\ln\left|\frac{b-x}{a-x}\right|=(b-a)(kt+C)$. The concentrations $[\mathrm{A}]=a-x$ and $[\mathrm{B}]=b-x$ cannot be negative, so $\frac{b-x}{a-x}\geq 0$ and $\left|\frac{b-x}{a-x}\right|=\frac{b-x}{a-x}$. We now have $\ln\left(\frac{b-x}{a-x}\right)=(b-a)(kt+C)$. Since $x(0)=0$, we get $\ln\left(\frac{b}{a}\right)=(b-a)C$. Hence, $\ln\left(\frac{b-x}{a-x}\right)=(b-a)kt+\ln\left(\frac{b}{a}\right)\Rightarrow\frac{b-x}{a-x}=\frac{b}{a}e^{(b-a)kt}\Rightarrow x=\frac{b\left[e^{(b-a)kt}-1\right]}{\left|be^{(b-a)kt}\right/a-1}=\frac{ab\left[e^{(b-a)kt}-1\right]}{be^{(b-a)kt}-a}$ moles / L.

(b) If $b=a$, then $\frac{dx}{dt}=k(a-x)^2$, so $\int\frac{dx}{(a-x)^2}=\int k\,dt$ and $\frac{1}{a-x}=kt+C$. Since $x(0)=0$, we get $C=\frac{1}{a}$.

Thus, $a-x=\frac{1}{kt+1/a}$ and $x=a-\frac{a}{akt+1}=\frac{a^2kt}{akt+1}\ \frac{\text{moles}}{\text{L}}$.

Suppose $x=[\mathrm{C}]=a/2$ when $t=20$. Then $x(20)=a/2\Rightarrow\frac{a}{2}=\frac{20a^2k}{20ak+1}\Rightarrow 40a^2k=20a^2k+a\Rightarrow 20a^2k=a\Rightarrow$ $k=\frac{1}{20a}$, so $x=\frac{a^2t/(20a)}{1+at/(20a)}=\frac{at/20}{1+t/20}=\frac{at}{t+20}\ \frac{\text{moles}}{\text{L}}$.

35. **(a)** If $a=b$, then $\frac{dx}{dt}=k(a-x)(b-x)^{1/2}$ becomes $\frac{dx}{dt}=k(a-x)^{3/2}\Rightarrow(a-x)^{-3/2}dx=k\,dt\Rightarrow$ $\int(a-x)^{-3/2}dx=\int k\,dt\Rightarrow 2(a-x)^{-1/2}=kt+C\Rightarrow\frac{2}{kt+C}=\sqrt{a-x}\Rightarrow\left(\frac{2}{kt+C}\right)^2=a-x\Rightarrow x(t)=a-\frac{4}{(kt+C)^2}$. The initial concentration of HBr is 0 , so $x(0)=0\Rightarrow 0=a-\frac{4}{C^2}\Rightarrow\frac{4}{C^2}=a\Rightarrow C^2=\frac{4}{a}\Rightarrow C=2/\sqrt{a}$ (C is positive since

$kt+C=2(a-x)^{-1/2}>0$). Thus, $x(t)=a-\dfrac{4}{\left(kt+2/\sqrt{a}\,\right)^2}$.

(b) $\dfrac{dx}{dt}=k(a-x)(b-x)^{1/2}\Rightarrow \dfrac{dx}{(a-x)\sqrt{b-x}}=k\,dt\Rightarrow \int \dfrac{dx}{(a-x)\sqrt{b-x}}=\int k\,dt$ (*) . From the hint, $u=\sqrt{b-x}\Rightarrow$ $u^2=b-x\Rightarrow 2u\,du=-dx$, so

$\displaystyle\int \frac{dx}{(a-x)\sqrt{b-x}}=\int \frac{-2u\,du}{\left[a-\left(b-u^2\right)\right]u}=-2\int \frac{du}{a-b+u^2}=-2\int \frac{du}{\left(\sqrt{a-b}\,\right)^2+u^2}=-2\left(\frac{1}{\sqrt{a-b}}\tan^{-1}\frac{u}{\sqrt{a-b}}\right)$. So (*) becomes $\dfrac{-2}{\sqrt{a-b}}\tan^{-1}\dfrac{\sqrt{b-x}}{\sqrt{a-b}}=kt+C$. Now $x(0)=0\Rightarrow C=\dfrac{-2}{\sqrt{a-b}}\tan^{-1}\dfrac{\sqrt{b}}{\sqrt{a-b}}$ and we have

$\dfrac{-2}{\sqrt{a-b}}\tan^{-1}\dfrac{\sqrt{b-x}}{\sqrt{a-b}}=kt-\dfrac{2}{\sqrt{a-b}}\tan^{-1}\dfrac{\sqrt{b}}{\sqrt{a-b}}\Rightarrow \dfrac{2}{\sqrt{a-b}}\left(\tan^{-1}\sqrt{\dfrac{b}{a-b}}-\tan^{-1}\sqrt{\dfrac{b-x}{a-b}}\right)=kt\Rightarrow$ $t(x)=\dfrac{2}{k\sqrt{a-b}}\left(\tan^{-1}\sqrt{\dfrac{b}{a-b}}-\tan^{-1}\sqrt{\dfrac{b-x}{a-b}}\right)$.

36. If $S=\dfrac{dT}{dr}$, then $\dfrac{dS}{dr}=\dfrac{d^2T}{dr^2}$. The differential equation $\dfrac{d^2T}{dr^2}+\dfrac{2}{r}\dfrac{dT}{dr}=0$ can be written as $\dfrac{dS}{dr}+\dfrac{2}{r}S=0$. Thus, $\dfrac{dS}{dr}=\dfrac{-2S}{r}\Rightarrow \dfrac{dS}{S}=-\dfrac{2}{r}dr\Rightarrow \int \dfrac{1}{S}dS=\int -\dfrac{2}{r}dr\Rightarrow \ln|S|=-2\ln|r|+C$. Assuming $S=dT/dr>0$ and $r>0$, we have $S=e^{-2\ln r+C}=e^{\ln r^{-2}}e^{C}=r^{-2}k\Rightarrow S=\dfrac{1}{r^2}k\Rightarrow$

$\dfrac{dT}{dr}=\dfrac{1}{r^2}k\Rightarrow dT=\dfrac{1}{r^2}k\,dr\Rightarrow \int dT=\int \dfrac{1}{r^2}k\,dr\Rightarrow T(r)=-\dfrac{k}{r}+A$.

$T(1)=15\Rightarrow 15=-k+A$ **(1)** and $T(2)=25\Rightarrow 25=-\dfrac{1}{2}k+A$ **(2)** .

Now solve for k and A : -2 **(2)** + **(1)** $\Rightarrow -35=-A$, so $A=35$ and $k=20$, and $T(r)=-20/r+35$.

37. **(a)** $\dfrac{dC}{dt}=r-kC\Rightarrow \dfrac{dC}{dt}=-(kC-r)\Rightarrow \int \dfrac{dC}{kC-r}=\int -dt\Rightarrow (1/k)\ln|kC-r|=-t+M_1\Rightarrow \ln|kC-r|=-kt+M_2$ $\Rightarrow |kC-r|=e^{-kt+M_2}\Rightarrow kC-r=M_3e^{-kt}\Rightarrow kC=M_3e^{-kt}+r\Rightarrow C(t)=M_4e^{-kt}+r/k$. $C(0)=C_0\Rightarrow C_0=M_4+r/k\Rightarrow$ $M_4=C_0-r/k\Rightarrow C(t)=\left(C_0-r/k\right)e^{-kt}+r/k$.

(b) If $C_0<r/k$, then $C_0-r/k<0$ and the formula for $C(t)$ shows that $C(t)$ increases and $\lim_{t\to\infty} C(t)=r/k$. As t increases, the formula for $C(t)$ shows how the role of C_0 steadily diminishes as that of r/k increases.

38. **(a)** Use 1 billion dollars as the $x-$ unit and 1 day as the $t-$ unit. Initially, there is \$10 billion of old currency in circulation, so all of the \$50 million returned to the banks is old. At time t , the amount of new currency is $x(t)$ billion dollars, so $10-x(t)$ billion dollars of currency is old. The fraction of circulating money that is old is $[10-x(t)]/10$, and the amount of old currency being returned to the banks each day is $\frac{10-x(t)}{10}$ 0.05 billion dollars. This amount of new currency per day is introduced into circulation, so $\frac{dx}{dt}=\frac{10-x}{10}\cdot 0.05=0.005(10-x)$ billion dollars per day.

(b) $\frac{dx}{10-x}=0.005\,dt \Rightarrow \frac{-dx}{10-x}=-0.005\,dt \Rightarrow \ln(10-x)=-0.005t+c \Rightarrow 10-x=Ce^{-0.005t}$, where $C=e^{c} \Rightarrow$ $x(t)=10-Ce^{-0.005t}$. From $x(0)=0$, we get $C=10$, so $x(t)=10\left(1-e^{-0.005t}\right)$.

(c) The new bills make up 90% of the circulating currency when $x(t)=0.9\cdot 10=9$ billion dollars. $9=10\left(1-e^{-0.005t}\right) \Rightarrow 0.9=1-e^{-0.005t} \Rightarrow e^{-0.005t}=0.1 \Rightarrow -0.005t=-\ln 10 \Rightarrow t=200\ln 10\approx 460.517$ days ≈ 1.26 years.

39. **(a)** Let $y(t)$ be the amount of salt (in kg) after t minutes. Then $y(0)=15$. The amount of liquid in the tank is 1000 L at all times, so the concentration at time t (in minutes) is $y(t)/1000$ kg/L and $\frac{dy}{dt}=-\left[\frac{y(t)}{1000}\frac{\text{kg}}{\text{L}}\right]\left(10\frac{\text{L}}{\text{min}}\right)=-\frac{y(t)}{100}\frac{\text{kg}}{\text{min}}$. $\int\frac{dy}{y}=-\frac{1}{100}\int dt \Rightarrow \ln y=-\frac{t}{100}+C$, and $y(0)=15 \Rightarrow$ $\ln 15=C$, so $\ln y=\ln 15-\frac{t}{100}$. It follows that $\ln\left(\frac{y}{15}\right)=-\frac{t}{100}$ and $\frac{y}{15}=e^{-t/100}$, so $y=15e^{-t/100}$ kg.

(b) After 20 minutes, $y=15e^{-20/100}=15e^{-0.2}\approx 12.3$ kg.

40. **(a)** If $y(t)$ is the amount of salt (in kg) after t minutes, then $y(0)=0$ and the total amount of liquid in the tank remains constant at 1000 L.

$$\begin{aligned}\frac{dy}{dt} &= \left(0.05\frac{\text{kg}}{\text{L}}\right)\left(5\frac{\text{L}}{\text{min}}\right)+\left(0.04\frac{\text{kg}}{\text{L}}\right)\left(10\frac{\text{L}}{\text{min}}\right)-\left(\frac{y(t)}{1000}\frac{\text{kg}}{\text{L}}\right)\left(15\frac{\text{L}}{\text{min}}\right)\\ &= 0.25+0.40-0.015y=0.65-0.015y=\frac{130-3y}{200}\frac{\text{kg}}{\text{min}}\end{aligned}$$

so $\int\frac{dy}{130-3y}=\int\frac{dt}{200}$ and $-\frac{1}{3}\ln|130-3y|=\frac{1}{200}t+C$; since $y(0)=0$, we have $-\frac{1}{3}\ln 130=C$, so $-\frac{1}{3}\ln|130-3y|=\frac{1}{200}t-\frac{1}{3}\ln 130 \Rightarrow \ln|130-3y|=-\frac{3}{200}t+\ln 130=\ln\left(130e^{-3t/200}\right)$, and $|130-3y|=130e^{-3t/200}$. Since y is continuous, $y(0)=0$, and the right–hand side is never zero, we deduce that $130-3y$ is always positive. Thus, $130-3y=130e^{-3t/200}$ and $y=\frac{130}{3}\left(1-e^{-3t/200}\right)$ kg.

(b) After one hour, $y=\frac{130}{3}\left(1-e^{-3\cdot 60/200}\right)=\frac{130}{3}\left(1-e^{-0.9}\right)\approx 25.7$ kg.

Note: As $t\to\infty$,

$y(t) \to \dfrac{130}{3} = 43\dfrac{1}{3}$ kg.

41. Assume that the raindrop begins at rest, so that $v(0)=0$. $dm/dt=km$ and $(mv)'=gm \Rightarrow$ $mv'+vm'=gm \Rightarrow mv'+v(km)=gm \Rightarrow v'+vk=g \Rightarrow dv/dt=g-kv \Rightarrow \int \dfrac{dv}{g-kv} = \int dt \Rightarrow -(1/k)\ln|g-kv| = t+C \Rightarrow$ $\ln|g-kv| = -kt-kC \Rightarrow g-kv=Ae^{-kt}$. $v(0)=0 \Rightarrow A=g$. So $kv=g-ge^{-kt} \Rightarrow v=(g/k)\left(1-e^{-kt}\right)$. Since $k>0$, as $t\to\infty$, $e^{-kt}\to 0$ and therefore, $\lim\limits_{t\to\infty} v(t)=g/k$.

42. **(a)** $m\dfrac{dv}{dt}=-kv \Rightarrow \dfrac{dv}{v}=-\dfrac{k}{m}\,dt \Rightarrow \ln|v| = -\dfrac{k}{m}t+C$. Since $v(0)=v_0$, $\ln\left|v_0\right|=C$. Therefore, $\ln\left|\dfrac{v}{v_0}\right| = -\dfrac{k}{m}t \Rightarrow \left|\dfrac{v}{v_0}\right| = e^{-kt/m} \Rightarrow v(t)=\pm v_0 e^{-kt/m}$. The sign is + when $t=0$, and we assume v is continuous, so that the sign is + for all t . Thus, $v(t)=v_0e^{-kt/m}$. $ds/dt=v_0e^{-kt/m} \Rightarrow s(t)=-\dfrac{mv_0}{k}e^{-kt/m}+C'$. From $s(0)=s_0$, we get $s_0=-\dfrac{mv_0}{k}+C'$, so $C'=s_0+\dfrac{mv_0}{k}$ and $s(t)=s_0+\dfrac{mv_0}{k}\left(1-e^{-kt/m}\right)$. The distance traveled from time 0 to time t is $s(t)-s_0$, so the total distance traveled is $\lim\limits_{t\to\infty}\left[s(t)-s_0\right]=\dfrac{mv_0}{k}$.

Note: In finding the limit, we use the fact that $k>0$ to conclude that $\lim\limits_{t\to\infty} e^{-kt/m}=0$.

(b) $m\dfrac{dv}{dt}=-kv^2 \Rightarrow \dfrac{dv}{v^2}=-\dfrac{k}{m}\,dt \Rightarrow \dfrac{-1}{v}=-\dfrac{kt}{m}+C \Rightarrow \dfrac{1}{v}=\dfrac{kt}{m}-C$. Since $v(0)=v_0$, $C=-\dfrac{1}{v_0}$ and $\dfrac{1}{v}=\dfrac{kt}{m}+\dfrac{1}{v_0}$. Therefore, $v(t)=\dfrac{1}{kt/m+1/v_0}=\dfrac{mv_0}{kv_0t+m}$. $\dfrac{ds}{dt}=\dfrac{mv_0}{kv_0t+m} \Rightarrow s(t)=\dfrac{m}{k}\int\dfrac{kv_0\,dt}{kv_0t+m}=\dfrac{m}{k}\ln\left|kv_0t+m\right|+C'$.

Since $s(0)=s_0$, we get $s_0=\dfrac{m}{k}\ln m+C' \Rightarrow C'=s_0-\dfrac{m}{k}\ln m \Rightarrow$

$s(t)=s_0+\dfrac{m}{k}\left(\ln\left|kv_0t+m\right|-\ln m\right)=s_0+\dfrac{m}{k}\ln\left|\dfrac{kv_0t+m}{m}\right|$. We can rewrite the formulas for $v(t)$ and $s(t)$ as $v(t)=\dfrac{v_0}{1+\left(kv_0/m\right)t}$ and $s(t)=s_0+\dfrac{m}{k}\ln\left|1+\dfrac{kv_0}{m}t\right|$.

Remarks: This model of horizontal motion through a resistive medium was designed to handle the case in which

$v_0>0$. Then the term $-kv^2$ representing the resisting force causes the object to decelerate. The absolute value in the expression for $s(t)$ is unnecessary (since k , v_0 , and m are all positive), and $\lim_{t\to\infty} s(t)=\infty$. In other words, the object travels infinitely far. However, $\lim_{t\to\infty} v(t)=0$. When $v_0<0$, the term $-kv^2$ increases the magnitude of the object's negative velocity. According to the formula for $s(t)$, the position of the object approaches $-\infty$ as t approaches $m/k\left(-v_0\right)$: $\lim_{t\to -m/\left(kv_0\right)} s(t)=-\infty$. Again the object travels infinitely far, but this time the feat is accomplished in a finite amount of time. Notice also that $\lim_{t\to -m/\left(kv_0\right)} v(t)=-\infty$ when $v_0<0$, showing that the speed of the object increases without limit.

43. **(a)** The rate of growth of the area is jointly proportional to $\sqrt{A(t)}$ and $M-A(t)$; that is, the rate is proportional to the product of those two quantities. So for some constant k , $dA/dt=k\sqrt{A}\,(M-A)$. We are interested in the maximum of the function dA/dt (when the tissue grows the fastest), so we differentiate, using the Chain Rule and then substituting for dA/dt from the differential equation:

$$\frac{d}{dt}\left(\frac{dA}{dt}\right) = k\left[\sqrt{A}\,(-1)\frac{dA}{dt}+(M-A)\cdot\frac{1}{2}A^{-1/2}\frac{dA}{dt}\right]=\frac{1}{2}kA^{-1/2}\frac{dA}{dt}\left[-2A+(M-A)\right]$$
$$= \frac{1}{2}kA^{-1/2}\left[k\sqrt{A}(M-A)\right]\left[M-3A\right]=\frac{1}{2}k^2(M-A)(M-3A)$$

This is 0 when $M-A=0$ and when $M-3A=0\Leftrightarrow A(t)=M/3$. This represents a maximum by the First Derivative Test, since $\frac{d}{dt}\left(\frac{dA}{dt}\right)$ goes from positive to negative when $A(t)=M/3$.

(b) From the CAS, we get $A(t)=M\left(\dfrac{Ce^{\sqrt{M}kt}-1}{Ce^{\sqrt{M}kt}+1}\right)^2$. To get C in terms of the initial area A_0 and the maximum area M , we substitute $t=0$ and $A=A_0=A(0)$: $A_0=M\left(\dfrac{C-1}{C+1}\right)^2\Leftrightarrow(C+1)\sqrt{A_0}=(C-1)\sqrt{M}\Leftrightarrow$
$C\sqrt{A_0}+\sqrt{A_0}=C\sqrt{M}-\sqrt{M}\Leftrightarrow\sqrt{M}+\sqrt{A_0}=C\sqrt{M}-C\sqrt{A_0}\Leftrightarrow$
$\sqrt{M}+\sqrt{A_0}=C\left(\sqrt{M}-\sqrt{A_0}\right)\Leftrightarrow C=\dfrac{\sqrt{M}+\sqrt{A_0}}{\sqrt{M}-\sqrt{A_0}}$. (Notice that if $A_0=0$, then $C=1$.)

44. **(a)** According to the hint we use the Chain Rule: $m\dfrac{dv}{dt}=m\dfrac{dv}{dx}\cdot\dfrac{dx}{dt}=mv\dfrac{dv}{dx}=-\dfrac{mgR^2}{(x+R)^2}\Rightarrow$

$\int v\,dv = \int \dfrac{-gR^2\,dx}{(x+R)^2} \Rightarrow \dfrac{v^2}{2} = \dfrac{gR^2}{x+R} + C$. When $x=0$, $v=v_0$, so $\dfrac{v_0^2}{2} = \dfrac{gR^2}{0+R} + C \Rightarrow C = \dfrac{1}{2}v_0^2 - gR \Rightarrow$ $\dfrac{1}{2}v^2 - \dfrac{1}{2}v_0^2 = \dfrac{gR^2}{x+R} - gR$. Now at the top of its flight, the rocket's velocity will be 0 , and its height will be $x=h$. Solving for v_0 : $-\dfrac{1}{2}v_0^2 = \dfrac{gR^2}{h+R} - gR \Rightarrow \dfrac{v_0^2}{2} = g\left[-\dfrac{R^2}{R+h} + \dfrac{R(R+h)}{R+h}\right] = \dfrac{gRh}{R+h} \Rightarrow v_0 = \sqrt{\dfrac{2gRh}{R+h}}$.

(b) $v_e = \lim_{h\to\infty} v_0 = \lim_{h\to\infty} \sqrt{\dfrac{2gRh}{R+h}} = \lim_{h\to\infty} \sqrt{\dfrac{2gR}{(R/h)+1}} = \sqrt{2gR}$

(c) $v_e = \sqrt{2 \cdot 32\text{ft/s}^2 \cdot 3960\text{mi} \cdot 5280\text{ft/mi}} \approx 36,581 \text{ ft / s} \approx 6.93 \text{ mi / s}$

1. The relative growth rate is $\frac{1}{P}\frac{dP}{dt}=0.7944$, so $\frac{dP}{dt}=0.7944P$ and, by Theorem 2, $P(t)=P(0)e^{0.7944t}=2e^{0.7944t}$. Thus, $P(6)=2e^{0.7944(6)}\approx 234.99$ or about 235 members.

2. **(a)** By Theorem 2, $P(t)=P(0)e^{kt}=60e^{kt}$. In 20 minutes ($\frac{1}{3}$ hour), there are 120 cells, so $P\left(\frac{1}{3}\right)=60e^{k/3}=120\Rightarrow e^{k/3}=2\Rightarrow k/3=\ln 2\Rightarrow k=3\ln 2=\ln\left(2^3\right)=\ln 8$.

(b) $P(t)=60e^{(\ln 8)t}=60\cdot 8^t$

(c) $P(8)=60\cdot 8^8=60\cdot 2^{24}=1,006,632,960$

(d) $dP/dt=kP\Rightarrow P'(8)=kP(8)=(\ln 8)P(8)\approx 2.093$ billion cells / h

(e) $P(t)=20,000\Rightarrow 60\cdot 8^t=20,000\Rightarrow 8^t=1000/3\Rightarrow t\ln 8=\ln(1000/3)\Rightarrow t=\frac{\ln(1000/3)}{\ln 8}\approx 2.79$ h

3. **(a)** By Theorem 2, $y(t)=y(0)e^{kt}=500e^{kt}$. Now $y(3)=500e^{k(3)}=8000\Rightarrow e^{3k}=\frac{8000}{500}\Rightarrow 3k=\ln 16\Rightarrow$ $k=(\ln 16)/3$. So $y(t)=500e^{(\ln 16)t/3}=500\cdot 16^{t/3}$

(b) $y(4)=500\cdot 16^{4/3}\approx 20,159$

(c) $dy/dt=ky\Rightarrow y'(4)=ky(4)=\frac{1}{3}\ln 16\left(500\cdot 16^{4/3}\right)\approx 18,631$ cells / h

(d) $y(t)=500\cdot 16^{t/3}=30,000\Rightarrow 16^{t/3}=60\Rightarrow \frac{1}{3}t\ln 16=\ln 60\Rightarrow t=3(\ln 60)/(\ln 16)\approx 4.4$h

4. **(a)** $y(t)=y(0)e^{kt}\Rightarrow y(2)=y(0)e^{2k}=600$, $y(8)=y(0)e^{8k}=75,000$. Dividing these equations, we get $e^{8k}/e^{2k}=75,000/600\Rightarrow e^{6k}=125\Rightarrow 6k=\ln 125=\ln 5^3=3\ln 5\Rightarrow k=\frac{3}{6}\ln 5=\frac{1}{2}\ln 5$. Thus, $y(0)=600/e^{2k}=600/e^{\ln 5}=\frac{600}{5}=120$.

(b) $y(t)=y(0)e^{kt}=120e^{(\ln 5)t/2}$ or $y=120\cdot 5^{t/2}$

(c) $y(5)=120\cdot 5^{5/2}=120\cdot 25\sqrt{5}=3000\sqrt{5}\approx 6708$ bacteria.

(d) $y(t)=120\cdot 5^{t/2}\Rightarrow y'(t)=120\cdot 5^{t/2}\cdot\ln 5\cdot\frac{1}{2}=60\cdot\ln 5\cdot 5^{t/2}$.
$y'(5)=60\cdot\ln 5\cdot 5^{5/2}=60\cdot\ln 5\cdot 25\sqrt{5}\approx 5398$ bacteria / hour.

(e) $y(t)=200,000\Leftrightarrow 120e^{(\ln 5)t/2}=200,000\Leftrightarrow e^{(\ln 5)t/2}=\frac{5000}{3}\Leftrightarrow(\ln 5)t/2=\ln\frac{5000}{3}\Leftrightarrow$ $t=\left(2\ln\frac{5000}{3}\right)/\ln 5\approx 9.2$ h.

5. **(a)** Let the population (in millions) in the year t be $P(t)$. Since the initial time is the year 1750, we substitute $t-1750$ for t in Theorem 2, so the exponential model gives $P(t)=P(1750)e^{k(t-1750)}$. Then $P(1800)=980=790e^{k(1800-1750)} \Rightarrow \frac{980}{790}=e^{k(50)} \Rightarrow \ln \frac{980}{790}=50k \Rightarrow k=\frac{1}{50}\ln\frac{980}{790}\approx 0.0043104$. So with this model, we have $P(1900)=790e^{k(1900-1750)}\approx 1508$ million, and $P(1950)=790e^{k(1950-1750)}\approx 1871$ million. Both of these estimates are much too low.

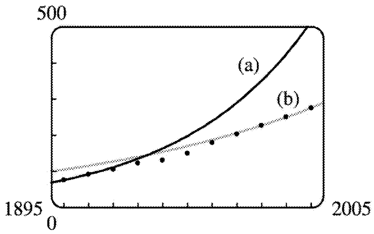
(b) In this case, the exponential model gives $P(t)=P(1850)e^{k(t-1850)} \Rightarrow$ $P(1900)=1650=1260e^{k(1900-1850)} \Rightarrow \ln\frac{1650}{1260}=k(50) \Rightarrow k=\frac{1}{50}\ln\frac{1650}{1260}\approx 0.005393$. So with this model, we estimate $P(1950)=1260e^{k(1950-1850)}\approx 2161$ million. This is still too low, but closer than the estimate of $P(1950)$ in part (a).

(c) The exponential model gives $P(t)=P(1900)e^{k(t-1900)} \Rightarrow P(1950)=2560=1650e^{k(1950-1900)} \Rightarrow$ $\ln\frac{2560}{1650}=k(50) \Rightarrow k=\frac{1}{50}\ln\frac{2560}{1650}\approx 0.008785$. With this model, we estimate $P(2000)=1650e^{k(2000-1900)}\approx 3972$ million. This is much too low. The discrepancy is explained by the fact that the world birth rate (average yearly number of births per person) is about the same as always, whereas the mortality rate (especially the infant mortality rate) is much lower, owing mostly to advances in medical science and to the wars in the first part of the twentieth century. The exponential model assumes, among other things, that the birth and mortality rates will remain constant.

6. **(a)** Let $P(t)$ be the population (in millions) in the year t . Since the initial time is the year 1900, we substitute $t-1900$ for t in Theorem 2, and find that the exponential model gives $P(t)=P(1900)e^{k(t-1900)}$ $\Rightarrow P(1910)=92=76e^{k(1910-1900)} \Rightarrow k=\frac{1}{10}\ln\frac{92}{76}\approx 0.0191$. With this model, we estimate $P(2000)=76e^{k(2000-1900)}\approx 514$ million. This estimate is much too high. The discrepancy is explained by the fact that, between the years 1900 and 1910, an enormous number of immigrants (compared to the total population) came to the United States. Since that time, immigration (as a proportion of total population) has been much lower. Also, the birth rate in the United States has declined since the turn of the century. So our calculation of the constant k was based partly on factors which no longer exist.

(b) Substituting $t-1980$ for t in Theorem 2, we find that the exponential model gives $P(t)=P(1980)e^{k(t-1980)} \Rightarrow P(1990)=250=227e^{k(1990-1980)} \Rightarrow k=\frac{1}{10}\ln\frac{250}{227}\approx 0.00965$. With this model, we estimate $P(2000)=227e^{k(2000-1980)}\approx 275.3$ million. This is quite accurate. The further estimates are $P(2010)=227e^{30k}\approx 303$ million and $P(2020)=227e^{40k}\approx 334$ million.

(c)



The model in part (a) is quite inaccurate after 1910 (off by 5 million in 1920 and 12 million in 1930). The model in part (b) is more accurate (which is not surprising, since it is based on more recent information).

7. **(a)** If $y=\left[N_2O_5\right]$ then by Theorem 2, $\dfrac{dy}{dt}=-0.0005y \Rightarrow y(t)=y(0)e^{-0.0005t}=Ce^{-0.0005t}$.

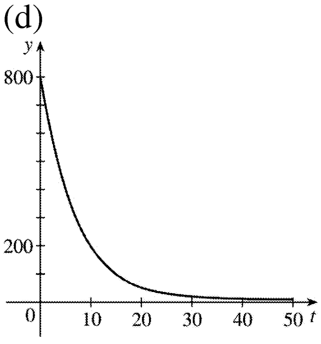
(b) $y(t)=Ce^{-0.0005t}=0.9C \Rightarrow e^{-0.0005t}=0.9 \Rightarrow -0.0005t=\ln 0.9 \Rightarrow t=-2000\ln 0.9 \approx 211$ s

8. **(a)** The mass remaining after t days is $y(t)=y(0)e^{kt}=800e^{kt}$. Since the half–life is 5.0 days,
$y(5)=800e^{5k}=400 \Rightarrow e^{5k}=\dfrac{1}{2} \Rightarrow$
$5k=\ln \dfrac{1}{2} \Rightarrow k=-(\ln 2)/5$, so $y(t)=800e^{-(\ln 2)t/5}=800\cdot 2^{-t/5}$.

(b) $y(30)=800\cdot 2^{-30/5}=12.5$ mg

(c) $800e^{-(\ln 2)t/5}=1 \Leftrightarrow -(\ln 2)\dfrac{t}{5}=\ln \dfrac{1}{800}=-\ln 800 \Leftrightarrow t=5\dfrac{\ln 800}{\ln 2} \approx 48$ days

(d)



9. **(a)** If $y(t)$ is the mass (in mg) remaining after t years, then $y(t)=y(0)e^{kt}=100e^{kt}$.
$y(30)=100e^{30k}=\dfrac{1}{2}(100) \Rightarrow e^{30k}=\dfrac{1}{2} \Rightarrow k=-(\ln 2)/30 \Rightarrow y(t)=100e^{-(\ln 2)t/30}=100\cdot 2^{-t/30}$

(b) $y(100)=100\cdot 2^{-100/30} \approx 9.92$ mg

(c) $100e^{-(\ln 2)t/30}=1 \Rightarrow$

$-(\ln 2)t/30=\ln \frac{1}{100} \Rightarrow t=-30 \frac{\ln 0.01}{\ln 2} \approx 199.3$ years

10. **(a)** If $y(t)$ is the mass after t days and $y(0)=A$, then $y(t)=Ae^{kt}$. $y(3)=Ae^{3k}=0.58A\Rightarrow$ $e^{3k}=0.58\Rightarrow 3k=\ln 0.58\Rightarrow k=\frac{1}{3}\ln 0.58$. Then $Ae^{(\ln 0.58)t/3}=\frac{1}{2}A\Leftrightarrow$ $\ln e^{(\ln 0.58)t/3}=\ln \frac{1}{2} \Leftrightarrow \frac{(\ln 0.58)t}{3}=\ln \frac{1}{2}$, so the half–life is $t=-\frac{3\ln 2}{\ln 0.58}\approx 3.82$ days.
(b) $Ae^{(\ln 0.58)t/3}=0.10A\Leftrightarrow \frac{(\ln 0.58)t}{3}=\ln \frac{1}{10} \Leftrightarrow t=-\frac{3\ln 10}{\ln 0.58}\approx 12.68$ days

11. Let $y(t)$ be the level of radioactivity. Thus, $y(t)=y(0)e^{-kt}$ and k is determined by using the half–life: $y(5730)=\frac{1}{2}y(0)\Rightarrow y(0)e^{-k(5730)}=\frac{1}{2}y(0)\Rightarrow e^{-5730k}=\frac{1}{2}\Rightarrow -5730k=\ln \frac{1}{2} \Rightarrow k=-\frac{\ln \frac{1}{2}}{5730}=\frac{\ln 2}{5730}$. If 74% of the ^{14}C remains, then we know that $y(t)=0.74y(0)\Rightarrow 0.74=e^{-t(\ln 2)/5730}\Rightarrow \ln 0.74=-\frac{t\ln 2}{5730}\Rightarrow$ $t=-\frac{5730(\ln 0.74)}{\ln 2}\approx 2489\approx 2500$ years.

12. From the information given, we know that $\frac{dy}{dx}=2y\Rightarrow y=Ce^{2x}$ by Theorem 2. To calculate C we use the point $(0,5)$: $5=Ce^{2(0)}\Rightarrow C=5$. Thus, the equation of the curve is $y=5e^{2x}$.

13. **(a)** Using Newton's Law of Cooling, $\frac{dT}{dt}=k(T-T_s)$, we have $\frac{dT}{dt}=k(T-75)$.
Now let $y=T-75$, so $y(0)=T(0)-75=185-75=110$, so y is a solution of the initial–value problem $dy/dt=ky$ with $y(0)=110$ and by Theorem 2 we have $y(t)=y(0)e^{kt}=110e^{kt}$.
$y(30)=110e^{30k}=150-75\Rightarrow e^{30k}=\frac{75}{110}=\frac{15}{22}\Rightarrow k=\frac{1}{30}\ln \frac{15}{22}$,
so $y(t)=110e^{\frac{1}{30}t\ln\left(\frac{15}{22}\right)}$ and $y(45)=110e^{\frac{45}{30}\ln\left(\frac{15}{22}\right)}\approx 62^\circ$ F. Thus, $T(45)\approx 62+75=137^\circ$ F.
(b) $T(t)=100\Rightarrow y(t)=25$. $y(t)=110e^{\frac{1}{30}t\ln\left(\frac{15}{22}\right)}=25\Rightarrow e^{\frac{1}{30}t\ln\left(\frac{15}{22}\right)}=\frac{25}{110}\Rightarrow \frac{1}{30}t\ln \frac{15}{22}=\ln \frac{25}{110}\Rightarrow$
$t=\frac{30\ln \frac{25}{110}}{\ln \frac{15}{22}}\approx 116$ min.

14. **(a)** Let $T(t)$= temperature after t minutes. Newton's Law of Cooling implies that $\dfrac{dT}{dt}=k(T-5)$. Let $y(t)=T(t)-5$. Then $\dfrac{dy}{dt}=ky$, so $y(t)=y(0)e^{kt}=15e^{kt}\Rightarrow T(t)=5+15e^{kt}\Rightarrow$ $T(1)=5+15e^{k}=12\Rightarrow e^{k}=\dfrac{7}{15}\Rightarrow k=\ln\dfrac{7}{15}$, so $T(t)=5+15e^{\ln(7/15)t}$ and $T(2)=5+15e^{2\ln(7/15)}\approx 8.3^\circ$ C.

(b) $5+15e^{\ln(7/15)t}=6$ when $e^{\ln(7/15)t}=\dfrac{1}{15}\Rightarrow \ln\left(\dfrac{7}{15}\right)t=\ln\dfrac{1}{15}\Rightarrow t=\dfrac{\ln\frac{1}{15}}{\ln\frac{7}{15}}\approx 3.6$ min.

15. $\dfrac{dT}{dt}=k(T-20)$. Letting $y=T-20$, we get $\dfrac{dy}{dt}=ky$, so $y(t)=y(0)e^{kt}$. $y(0)=T(0)-20=5-20=-15$, so $y(25)=y(0)e^{25k}=-15e^{25k}$, and $y(25)=T(25)-20=10-20=-10$, so $-15e^{25k}=-10\Rightarrow e^{25k}=\dfrac{2}{3}$. Thus, $25k=\ln\left(\dfrac{2}{3}\right)$ and $k=\dfrac{1}{25}\ln\left(\dfrac{2}{3}\right)$, so $y(t)=y(0)e^{kt}=-15e^{(1/25)\ln(2/3)t}$. More simply, $e^{25k}=\dfrac{2}{3}\Rightarrow$ $e^{k}=\left(\dfrac{2}{3}\right)^{1/25}\Rightarrow e^{kt}=\left(\dfrac{2}{3}\right)^{t/25}\Rightarrow y(t)=-15\cdot\left(\dfrac{2}{3}\right)^{t/25}$.

(a) $T(50)=20+y(50)=20-15\cdot\left(\dfrac{2}{3}\right)^{50/25}=20-15\cdot\left(\dfrac{2}{3}\right)^{2}=20-\dfrac{20}{3}=13.\bar{3}^\circ$ C

(b) $15=T(t)=20+y(t)=20-15\cdot\left(\dfrac{2}{3}\right)^{t/25}\Rightarrow 15\cdot\left(\dfrac{2}{3}\right)^{t/25}=5\Rightarrow\left(\dfrac{2}{3}\right)^{t/25}=\dfrac{1}{3}\Rightarrow$ $(t/25)\ln\left(\dfrac{2}{3}\right)=\ln\left(\dfrac{1}{3}\right)\Rightarrow t=25\ln\left(\dfrac{1}{3}\right)\Big/\ln\left(\dfrac{2}{3}\right)\approx 67.74$ min.

16. $\dfrac{dT}{dt}=k(T-20)$. Let $y=T-20$. Then $\dfrac{dy}{dt}=ky$, so $y(t)=y(0)e^{kt}$. $y(0)=T(0)-20=95-20=75$, so $y(t)=75e^{kt}$. When $T(t)=70$, $\dfrac{dT}{dt}=-1^\circ$ C / min. Equivalently, $\dfrac{dy}{dt}=-1$ when $y(t)=50$. Thus, $-1=\dfrac{dy}{dt}=ky(t)=50k$ and $50=y(t)=75e^{kt}$. The first relation implies $k=-1/50$, so the second relation says $50=75e^{-t/50}$. Thus, $e^{-t/50}=\dfrac{2}{3}\Rightarrow -t/50=\ln\left(\dfrac{2}{3}\right)\Rightarrow t=-50\ln\left(\dfrac{2}{3}\right)\approx 20.27$ min.

17. **(a)** Let $P(h)$ be the pressure at altitude h . Then $dP/dh=kP\Rightarrow P(h)=P(0)e^{kh}=101.3e^{kh}$. $P(1000)=101.3e^{1000k}=87.14\Rightarrow 1000k=\ln\left(\dfrac{87.14}{101.3}\right)\Rightarrow k=\dfrac{1}{1000}\ln\left(\dfrac{87.14}{101.3}\right)\Rightarrow P(h)=101.3$ $e^{\frac{1}{1000}h\ln\left(\frac{87.14}{101.3}\right)}$, so $P(3000)=101.3e^{3\ln\left(\frac{87.14}{101.3}\right)}\approx 64.5$ kPa.

(b) $P(6187)=101.3e^{\frac{6187}{1000}\ln\left(\frac{87.14}{101.3}\right)}\approx 39.9$ kPa

18. **(a)** Using $A=A_0\left(1+\frac{r}{n}\right)^{nt}$ with $A_0=500$, $r=0.14$, and $t=2$, we have:

(i) Annually: $n=1$; $A=500\left(1+\frac{0.14}{1}\right)^{1\cdot 2}=\$649.80$

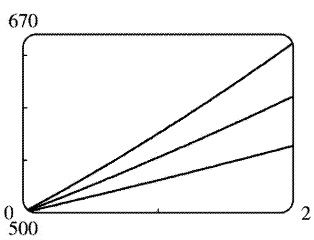
(ii) Quarterly: $n=4$; $A=500\left(1+\frac{0.14}{4}\right)^{4\cdot 2}=\$658.40$

(iii) Monthly: $n=12$; $A=500\left(1+\frac{0.14}{12}\right)^{12\cdot 2}=\$660.49$

(iv) Daily: $n=365$; $A=500\left(1+\frac{0.14}{365}\right)^{365\cdot 2}=\$661.53$

(v) Hourly: $n=365\cdot 24$; $A=500\left(1+\frac{0.14}{365\cdot 24}\right)^{365\cdot 24\cdot 2}=\$661.56$

(vi) Continuously: $A=500e^{(0.14)2}=\$661.56$

(b) 

$A_{0.14}(2)=\$661.56$, $A_{0.10}(2)=\$610.70$, and $A_{0.06}(2)=\$563.75$.

19. **(a)** Using $A=A_0\left(1+\frac{r}{n}\right)^{nt}$ with $A_0=3000$, $r=0.05$, and $t=5$, we have:

(i) Annually: $n=1$; $A=3000\left(1+\frac{0.05}{1}\right)^{1\cdot 5}=\$3828.84$

(ii) Semiannually: $n=2$; $A=3000\left(1+\frac{0.05}{2}\right)^{2\cdot 5}=\$3840.25$

(iii) Monthly: $n=12$; $A=3000\left(1+\frac{0.05}{12}\right)^{12\cdot 5}=\$3850.08$

(iv) Weekly: $n=52$; $A=3000\left(1+\frac{0.05}{52}\right)^{52\cdot 5}=\$3851.61$

(v) Daily: $n=365$; $A=3000\left(1+\frac{0.05}{365}\right)^{365\cdot 5}=\$3852.01$

(vi) Continuously: $A=3000e^{(0.05)5}=\$3852.08$

(b) $dA/dt=0.05A$ and $A(0)=3000$.

20. **(a)** $A_0e^{0.06t}=2A_0\Leftrightarrow e^{0.06t}=2\Leftrightarrow 0.06t=\ln 2\Leftrightarrow t=\frac{50}{3}\ln 2\approx 11.55$, so the investment will double in about 11.55 years.

(b) The annual interest rate in $A=A_0(1+r)^t$ is r . From part (a), we have $A=A_0e^{0.06t}$. These amounts must be equal, so $(1+r)^t=e^{0.06t}\Rightarrow 1+r=e^{0.06}\Rightarrow r=e^{0.06}-1\approx 0.0618=6.18\%$, which is the equivalent annual interest rate.

21. **(a)** $\frac{dP}{dt}=kP-m=k\left(P-\frac{m}{k}\right)$. Let $y=P-\frac{m}{k}$, so $\frac{dy}{dt}=\frac{dP}{dt}$ and the differential equation becomes $\frac{dy}{dt}=ky$. The solution is $y=y_0e^{kt}\Rightarrow P-\frac{m}{k}=\left(P_0-\frac{m}{k}\right)e^{kt}\Rightarrow P(t)=\frac{m}{k}+\left(P_0-\frac{m}{k}\right)e^{kt}$.

(b) Since $k>0$, there will be an exponential expansion $\Leftrightarrow P_0-\frac{m}{k}>0\Leftrightarrow m<kP_0$.

(c) The population will be constant if $P_0-\frac{m}{k}=0\Leftrightarrow m=kP_0$. It will decline if $P_0-\frac{m}{k}<0\Leftrightarrow m>kP_0$.

(d) $P_0=8,000,000$, $k=\alpha-\beta=0.016$, $m=210,000\Rightarrow m>kP_0$ ($=128,000$), so by part (c), the population was declining.

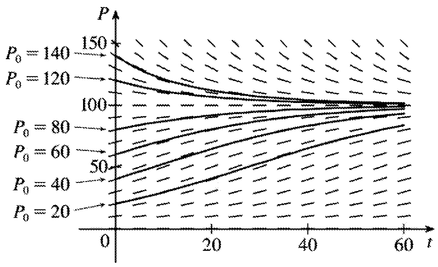
22. **(a)** $\frac{dy}{dt}=ky^{1+c}\Rightarrow y^{-1-c}dy=k\,dt\Rightarrow \frac{y^{-c}}{-c}=kt+C$. Since $y(0)=y_0$, we have $C=\frac{y_0^{-c}}{-c}$. Thus, $\frac{y^{-c}}{-c}=kt+\frac{y_0^{-c}}{-c}$, or $y^{-c}=y_0^{-c}-ckt$. So $y^c=\frac{1}{y_0^{-c}-ckt}=\frac{y_0^c}{1-cy_0^ckt}$ and $y(t)=\frac{y_0}{\left(1-cy_0^ckt\right)^{1/c}}$.

(b) $y(t)\to\infty$ as $1-cy_0^ckt\to 0$, that is, as $t\to\frac{1}{cy_0^ck}$. Define $T=\frac{1}{cy_0^ck}$. Then $\lim_{t\to T^-}y(t)=\infty$.

(c) According to the data given, we have $c=0.01$, $y(0)=2$, and $y(3)=16$, where the time t is given in months. Thus, $y_0=2$ and $16=y(3)=\frac{y_0}{\left(1-cy_0^ck\cdot 3\right)^{1/c}}$. Since $T=\frac{1}{cy_0^ck}$, we will solve for cy_0^ck .

$16=\dfrac{2}{\left(1-3cy_0^c k\right)^{100}} \Rightarrow 1-3cy_0^c k=\left(\dfrac{1}{8}\right)^{0.01}=8^{-0.01} \Rightarrow cy_0^c k=\dfrac{1}{3}\left(1-8^{-0.01}\right)$. Thus, doomsday occurs when $t=T=\dfrac{1}{cy_0^c k}=\dfrac{3}{1-8^{-0.01}} \approx 145.77$ months or 12.15 years.

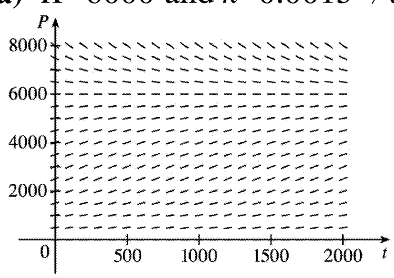
1. **(a)** $dP/dt = 0.05P - 0.0005P^2 = 0.05P(1-0.01P) = 0.05P\left(1-P/100\right)$. Comparing to Equation 1, $dP/dt = kP\left(1-P/K\right)$, we see that the carrying capacity is $K=100$ and the value of k is 0.05 .
(b) The slopes close to 0 occur where P is near 0 or 100 . The largest slopes appear to be on the line $P=50$. The solutions are increasing for $0<P_0<100$ and decreasing for $P_0>100$.



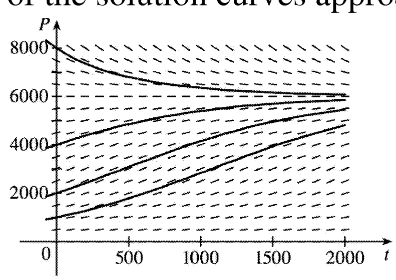
(c)
All of the solutions approach $P=100$ as t increases. As in part (b), the solutions differ since for $0<P_0<100$ they are increasing, and for $P_0>100$ they are decreasing. Also, some have an IP and some don't. It appears that the solutions which have $P_0=20$ and $P_0=40$ have inflection points at $P=50$.

(d) The equilibrium solutions are $P=0$ (trivial solution) and $P=100$. The increasing solutions move away from $P=0$ and all nonzero solutions approach $P=100$ as $t\to\infty$.

2. **(a)** $K=6000$ and $k=0.0015 \Rightarrow dP/dt = 0.0015P\left(1-P/6000\right)$.



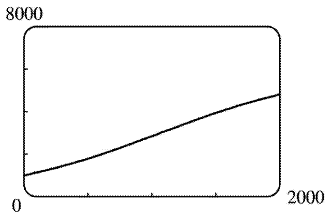
(b)
All of the solution curves approach 6000 as $t\to\infty$.



(c)
The curves with $P_0=1000$ and $P_0=2000$ appear to be concave upward at first and then concave downward. The curve with $P_0=4000$ appears to be concave downward everywhere. The curve with $P_0=8000$ appears to be concave upward everywhere. The inflection points are where the population grows the fastest.

(d) See the solution to Exercise .2.25 for a possible program to calculate $P(50)$. We find that $P(50)\approx 1064$.

(e) Using Equation 4 with $K=6000$, $k=0.0015$, and $P_0=1000$, we have $P(t)=\dfrac{K}{1+Ae^{-kt}}=\dfrac{6000}{1+Ae^{-0.0015t}}$, where $A=\dfrac{K-P_0}{P_0}=\dfrac{6000-1000}{1000}=5$. Thus, $P(50)=\dfrac{6000}{1+5e^{-0.0015(50)}}\approx 1064.1$, which is extremely close to the estimate obtained in part (d).

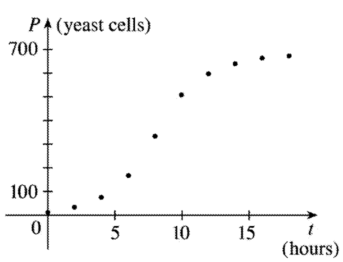


(f)

The curves are very similar.

3. **(a)** $\dfrac{dy}{dt}=ky\left(1-\dfrac{y}{K}\right)\Rightarrow y(t)=\dfrac{K}{1+Ae^{-kt}}$ with $A=\dfrac{K-y(0)}{y(0)}$. With $K=8\times 10^7$, $k=0.71$, and $y(0)=2\times 10^7$, we get the model $y(t)=\dfrac{8\times 10^7}{1+3e^{-0.71t}}$, so $y(1)=\dfrac{8\times 10^7}{1+3e^{-0.71}}\approx 3.23\times 10^7$ kg.

(b) $y(t)=4\times 10^7\Rightarrow \dfrac{8\times 10^7}{1+3e^{-0.71t}}=4\times 10^7\Rightarrow 2=1+3e^{-0.71t}\Rightarrow e^{-0.71t}=\dfrac{1}{3}\Rightarrow -0.71t=\ln\dfrac{1}{3}\Rightarrow t=\dfrac{\ln 3}{0.71}\approx 1.55$ years



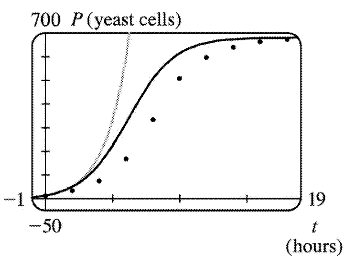
4. **(a)**
From the graph, we estimate the carrying capacity K for the yeast population to be 680 .

(b) An estimate of the initial relative growth rate is $\dfrac{1}{P_0}\dfrac{dP}{dt}=\dfrac{1}{18}\cdot\dfrac{39-18}{2-0}=\dfrac{7}{12}=0.58\overline{3}$.

(c) An exponential model is $P(t)=18e^{7t/12}$. A logistic model is $P(t)=\dfrac{680}{1+Ae^{-7t/12}}$, where $A=\dfrac{680-18}{18}=\dfrac{331}{9}$.

(d)

Time in Hours	Observed Values	Exponential Model	Logistic Model
0	18	18	18
2	39	58	55
4	80	186	149
6	171	596	322
8	336	1914	505
10	509	6147	614
12	597	19 , 739	658
14	640	63 , 389	673
16	664	203 , 558	678
18	672	653 , 679	679



The exponential model is a poor fit for anything beyond the first two observed values. The logistic model varies more for the middle values than it does for the values at either end, but provides a good general fit, as shown in the figure.

(e) $P(7)=\dfrac{680}{1+\dfrac{331}{9}e^{-7(7/12)}}\approx 420$ yeast cells

5. **(a)** We will assume that the difference in the birth and death rates is 20 million / year. Let $t=0$ correspond to the year 1990 and use a unit of 1 billion for all calculations.

$k\approx\dfrac{1}{P}\dfrac{dP}{dt}=\dfrac{1}{5.3}(0.02)=\dfrac{1}{265}$, so

$$\frac{dP}{dt}=kP\left(1-\frac{P}{K}\right)=\frac{1}{265}P\left(1-\frac{P}{100}\right), P \text{ in billions}$$

(b) $A=\dfrac{K-P_0}{P_0}=\dfrac{100-5.3}{5.3}=\dfrac{947}{53}\approx 17.8679$. $P(t)=\dfrac{K}{1+Ae^{-kt}}=\dfrac{100}{1+\dfrac{947}{53}e^{-(1/265)t}}$, so $P(10)\approx 5.49$

billion.
(c) $P(110)\approx 7.81$, and $P(510)\approx 27.72$. The predictions are 7.81 billion in the year 2100 and 27.72 billion in 2500.

(d) If $K=50$, then $P(t)=\dfrac{50}{1+\dfrac{447}{53}e^{-(1/265)t}}$. So $P(10)\approx 5.48$, $P(110)\approx 7.61$, and $P(510)\approx 22.41$. The predictions become 5.48 billion in the year 2000, 7.61 billion in 2100, and 22.41 billion in the year 2500.

6. **(a)** If we assume that the carrying capacity for the world population is 100 billion, it would seem reasonable that the carrying capacity for the U.S. is 3 — 5 billion by using current populations and simple proportions. We will use $K=4$ billion or 4000 million. With $t=0$ corresponding to 1980, we have

$$P(t)=\frac{4000}{1+\left(\dfrac{4000-250}{250}\right)e^{-kt}}=\frac{4000}{1+15e^{-kt}}$$

(b) $P(10)=275\Rightarrow \dfrac{4000}{1+15e^{-10k}}=275\Rightarrow 1+15e^{-10k}=\dfrac{4000}{275}\Rightarrow e^{-10k}=\dfrac{\frac{160}{11}-1}{15}\Rightarrow -10k=\ln\dfrac{149}{165}\Rightarrow$
$k=-\dfrac{1}{10}\ln\dfrac{149}{165}\approx 0.01019992$.
(c) $2100-1990=110$ and $P(110)\approx 680$ million.
$2200-1990=210$ and $P(210)\approx 1449$ million, or about 1.4 billion.
(d) $P(t)=300\Rightarrow \dfrac{4000}{1+15e^{-kt}}=300\Rightarrow 1+15e^{-kt}=\dfrac{40}{3}\Rightarrow e^{-kt}=\dfrac{37}{3}\cdot\dfrac{1}{15}\Rightarrow -kt=\ln\dfrac{37}{45}\Rightarrow$
$t=10\dfrac{\ln\frac{37}{45}}{\ln\frac{149}{165}}\approx 19.19\approx 19$. So we predict that the U.S. population will exceed 300 million in the year $1990+19=2009$.

7. **(a)** Our assumption is that $\dfrac{dy}{dt}=ky(1-y)$, where y is the fraction of the population that has heard the rumor.
(b) Using the logistic equation (1), $\dfrac{dP}{dt}=kP\left(1-\dfrac{P}{K}\right)$, we substitute $y=\dfrac{P}{K}$, $P=Ky$, and $\dfrac{dP}{dt}=K\dfrac{dy}{dt}$, to obtain

$K\dfrac{dy}{dt}=k(Ky)(1-y)\Leftrightarrow\dfrac{dy}{dt}=ky(1-y)$, our equation in part (a). Now the solution to (1) is $P(t)=\dfrac{K}{1+Ae^{-kt}}$,

where $A=\dfrac{K-P_0}{P_0}$. We use the same substitution to obtain $Ky=\dfrac{K}{1+\dfrac{K-Ky_0}{Ky_0}e^{-kt}}$

$\Rightarrow y=\dfrac{y_0}{y_0+\left(1-y_0\right)e^{-kt}}$.

Alternatively, we could use the same steps as outlined in ''The Analytic Solution,'' following Example 2.

(c) Let t be the number of hours since 8 A.M. Then $y_0=y(0)=\dfrac{80}{1000}=0.08$ and $y(4)=\dfrac{1}{2}$, so $\dfrac{1}{2}=y(4)=\dfrac{0.08}{0.08+0.92e^{-4k}}$. Thus, $0.08+0.92e^{-4k}=0.16$, $e^{-4k}=\dfrac{0.08}{0.92}=\dfrac{2}{23}$, and $e^{-k}=\left(\dfrac{2}{23}\right)^{1/4}$, so $y=\dfrac{0.08}{0.08+0.92\,(2/23)^{t/4}}=\dfrac{2}{2+23\,(2/23)^{t/4}}$. Solving this equation for t , we get $2y+23y\left(\dfrac{2}{23}\right)^{t/4}=2\Rightarrow\left(\dfrac{2}{23}\right)^{t/4}=\dfrac{2-2y}{23y}\Rightarrow\left(\dfrac{2}{23}\right)^{t/4}=\dfrac{2}{23}\cdot\dfrac{1-y}{y}\Rightarrow\left(\dfrac{2}{23}\right)^{t/4-1}=\dfrac{1-y}{y}$. It follows that $\dfrac{t}{4}-1=\dfrac{\ln\left[(1-y)/y\right]}{\ln\dfrac{2}{23}}$, so $t=4\left[1+\dfrac{\ln\left((1-y)/y\right)}{\ln\dfrac{2}{23}}\right]$.

When $y=0.9$, $\dfrac{1-y}{y}=\dfrac{1}{9}$, so $t=4\left(1-\dfrac{\ln 9}{\ln\dfrac{2}{23}}\right)\approx7.6$ h or 7 h 36 min. Thus, 90% of the population will have heard the rumor by 3:36 P.M.

8. **(a)** $P(0)=P_0=400$, $P(1)=1200$ and $K=10\,,000$. From the solution to the logistic differential equation $P(t)=\dfrac{P_0K}{P_0+\left(K-P_0\right)e^{-kt}}$, we get $P=\dfrac{400(10{,}000)}{400+(9600)e^{-kt}}=\dfrac{10{,}000}{1+24e^{-kt}}$. $P(1)=1200\Rightarrow$ $1+24e^{-k}=\dfrac{100}{12}\Rightarrow e^{k}=\dfrac{288}{88}\Rightarrow k=\ln\dfrac{36}{11}$. So $P=\dfrac{10{,}000}{1+24e^{-t\ln(36/11)}}=\dfrac{10{,}000}{1+24\cdot(11/36)^{t}}$.

(b) $5000=\dfrac{10{,}000}{1+24\,(11/36)^{t}}\Rightarrow24\left(\dfrac{11}{36}\right)^{t}=1\Rightarrow t\ln\dfrac{11}{36}=\ln\dfrac{1}{24}\Rightarrow t\approx2.68$ years.

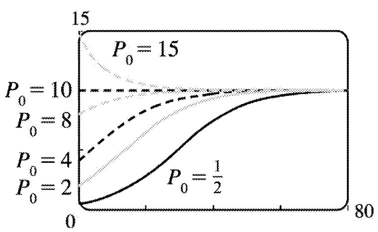
9. **(a)** $\frac{dP}{dt}=kP\left(1-\frac{P}{K}\right)\Rightarrow$

$$\frac{d^2P}{dt^2}=k\left[P\left(-\frac{1}{K}\frac{dP}{dt}\right)+\left(1-\frac{P}{K}\right)\frac{dP}{dt}\right]=k\frac{dP}{dt}\left(-\frac{P}{K}+1-\frac{P}{K}\right)$$

$$=k\left[kP\left(1-\frac{P}{K}\right)\right]\left(1-\frac{2P}{K}\right)=k^2P\left(1-\frac{P}{K}\right)\left(1-\frac{2P}{K}\right)$$

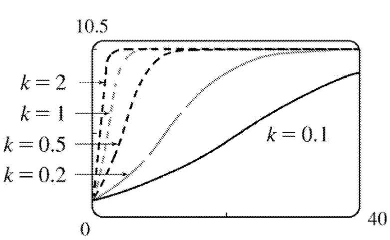
(b) P grows fastest when P' has a maximum, that is, when $P''=0$. From part (a), $P''=0\Leftrightarrow P=0$, $P=K$, or $P=K/2$. Since $0<P<K$, we see that $P''=0\Leftrightarrow P=K/2$.

10.



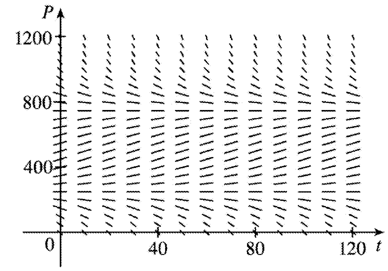
First we keep k constant (at 0.1 , say) and change P_0 in the function $P=\frac{10P_0}{P_0+\left(10-P_0\right)e^{-0.1t}}$. (Notice that P_0 is the $P-$ intercept.) If $P_0=0$, the function is 0 everywhere. For $0<P_0<5$, the curve has an inflection point, which moves to the right as P_0 decreases. If $5<P_0<10$, the graph is concave down everywhere. (We are

considering only $t\geq 0$.) If $P_0=10$, the function is the constant function $P=10$, and if $P_0>10$, the function decreases. For all $P_0\neq 0$, $\lim_{t\to\infty} P=10$.

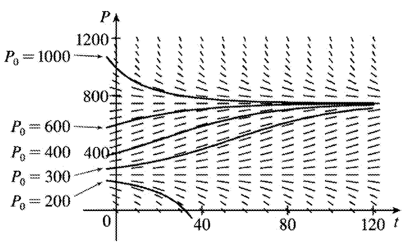


Now we instead keep P_0 constant (at $P_0=1$) and change k in the function $P=\frac{10}{1+9e^{-kt}}$. It seems that as k increases, the graph approaches the line $P=10$ more and more quickly. (Note that the only difference in the shape of the curves is in the horizontal scaling; if we choose suitable x –scales, the graphs all look the same.)

11. **(a)** The term -15 represents a harvesting of fish at a constant rate — in this case, 15 fish / week. This is the rate at which fish are caught.

(b) 

(c) From the graph in part (b), it appears that $P(t)=250$ and $P(t)=750$ are the equilibrium solutions. We confirm this analytically by solving the equation $dP/dt=0$ as follows: $0.08P\,(1-P/1000)-15=0 \Rightarrow$ $0.08P-0.00008P^2-15=0 \Rightarrow -0.00008\left(P^2-1000P+187{,}500\right)=0 \Rightarrow (P-250)(P-750)=0 \Rightarrow P=250$ or 750 .

(d) 

For $0<P_0<250$, $P(t)$ decreases to 0 . For $P_0=250$, $P(t)$ remains constant. For $250<P_0<750$, $P(t)$ increases and approaches 750 . For $P_0=750$, $P(t)$ remains constant. For $P_0>750$, $P(t)$ decreases and approaches 750 .

(e) $\dfrac{dP}{dt}=0.08P\left(1-\dfrac{P}{1000}\right)-15 \Leftrightarrow -\dfrac{100{,}000}{8}\cdot\dfrac{dP}{dt}=\left(0.08P-0.00008P^2-15\right)\cdot\left(-\dfrac{100{,}000}{8}\right)\Leftrightarrow -12$,

$500\dfrac{dP}{dt}=P^2-1000P+187$, $500 \Leftrightarrow \dfrac{dP}{(P-250)(P-750)}=-\dfrac{1}{12{,}500}\,dt \Leftrightarrow$

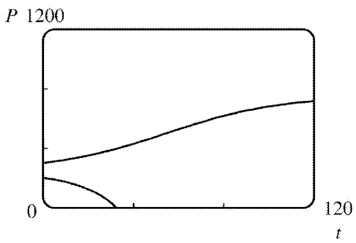
$\int\left(\dfrac{-1/500}{P-250}+\dfrac{1/500}{P-750}\right)dP=-\dfrac{1}{12{,}500}\,dt \Leftrightarrow \int\left(\dfrac{1}{P-250}-\dfrac{1}{P-750}\right)dP=\dfrac{1}{25}\,dt \Leftrightarrow$

$\ln|P-250|-\ln|P-750|=\dfrac{1}{25}t+C \Leftrightarrow \ln\left|\dfrac{P-250}{P-750}\right|=\dfrac{1}{25}t+C \Leftrightarrow \left|\dfrac{P-250}{P-750}\right|=e^{t/25+C}=ke^{t/25} \Leftrightarrow$

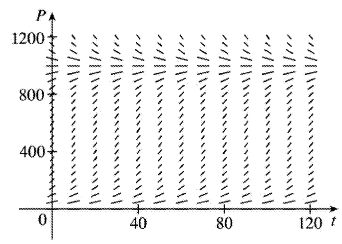
$\dfrac{P-250}{P-750}=ke^{t/25} \Leftrightarrow P-250=Pke^{t/25}-750ke^{t/25} \Leftrightarrow P-Pke^{t/25}=250-750ke^{t/25} \Leftrightarrow P(t)=\dfrac{250-750ke^{t/25}}{1-ke^{t/25}}$. If $t=0$ and $P=200$, then

$200=\dfrac{250-750k}{1-k} \Leftrightarrow 200-200k=250-750k \Leftrightarrow 550k=50 \Leftrightarrow k=\dfrac{1}{11}$. Similarly, if $t=0$ and $P=300$, then

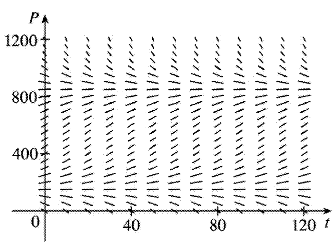
$k=-\dfrac{1}{9}$. Simplifying P with these two values of k gives us $P(t)=\dfrac{250\left(3e^{t/25}-11\right)}{e^{t/25}-11}$ and $P(t)=\dfrac{750\left(e^{t/25}+3\right)}{e^{t/25}+9}$.



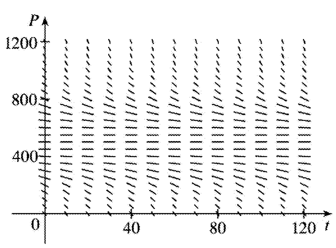
12. **(a)**



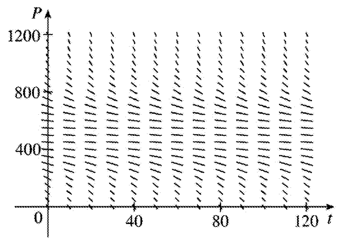
$c=0$



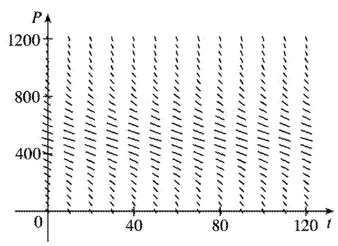
c=10



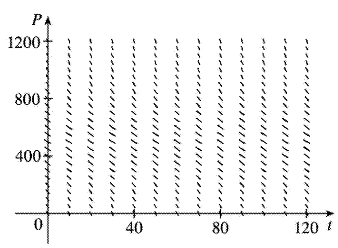
$c=20$



c=21



c=25



c=30

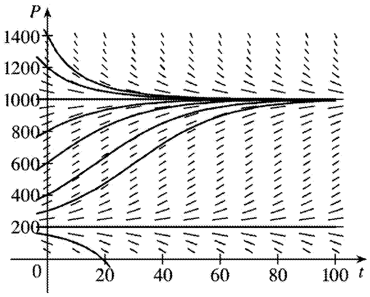
(b) For $0\leq c\leq 20$, there is at least one equilibrium solution. For $c>20$, the population always dies out.

(c) $\dfrac{dP}{dt}=0.08P-0.00008P^2-c$. $\dfrac{dP}{dt}=0\Leftrightarrow P=\dfrac{-0.08\pm\sqrt{(0.08)^2-4(-0.00008)(-c)}}{2(-0.00008)}$, which has at least one solution when the discriminant is nonnegative $\Rightarrow 0.0064-0.00032c\geq 0\Leftrightarrow c\leq 20$. For $0\leq c\leq 20$, there is at least one value of P such that $dP/dt=0$ and hence, at least one equilibrium solution. For $c>20$, $dP/dt<0$ and the population always dies out.

(d) The weekly catch should be less than 20 fish per week.

13. **(a)** $\dfrac{dP}{dt}=(kP)\left(1-\dfrac{P}{K}\right)\left(1-\dfrac{m}{P}\right)$. If $m<P<K$, then $dP/dt=(+)(+)(+)=+\Rightarrow P$ is increasing. If $0<P<m$, then $dP/dt=(+)(+)(-)=-\Rightarrow P$ is decreasing.

(b)



$k=0.08$, $K=1000$, and $m=200 \Rightarrow$

$$\frac{dP}{dt}=0.08P\left(1-\frac{P}{1000}\right)\left(1-\frac{200}{P}\right)$$

For $0<P_0<200$, the population dies out. For $P_0=200$, the population is steady. For $200<P_0<1000$, the population increases and approaches 1000 . For $P_0>1000$, the population decreases and approaches 1000 .

The equilibrium solutions are $P(t)=200$ and $P(t)=1000$.

(c) $\frac{dP}{dt}=kP\left(1-\frac{P}{K}\right)\left(1-\frac{m}{P}\right)=kP\left(\frac{K-P}{K}\right)\left(\frac{P-m}{P}\right)=\frac{k}{K}(K-P)(P-m) \Leftrightarrow \int \frac{dP}{(K-P)(P-m)}=\int \frac{k}{K}\,dt$.

By partial fractions, $\frac{1}{(K-P)(P-m)}=\frac{A}{K-P}+\frac{B}{P-m}$, so $A(P-m)+B(K-P)=1$.

If $P=m$, $B=\frac{1}{K-m}$; if $P=K$, $A=\frac{1}{K-m}$, so $\frac{1}{K-m}\int\left(\frac{1}{K-P}+\frac{1}{P-m}\right)dP=\int \frac{k}{K}\,dt \Rightarrow$

$\frac{1}{K-m}\left(-\ln|K-P|+\ln|P-m|\right)=\frac{k}{K}t+M \Rightarrow \frac{1}{K-m}\ln\left|\frac{P-m}{K-P}\right|=\frac{k}{K}t+M \Rightarrow \ln\left|\frac{P-m}{K-P}\right|=(K-m)\frac{k}{K}t+M_1$

$\Leftrightarrow \frac{P-m}{K-P}=De^{(K-m)(k/K)t}$.

Let $t=0$: $\frac{P_0-m}{K-P_0}=D$. So $\frac{P-m}{K-P}=\frac{P_0-m}{K-P_0}e^{(K-m)(k/K)t}$. Solving for P , we get

$$P(t)=\frac{m\left(K-P_0\right)+K\left(P_0-m\right)e^{(K-m)(k/K)t}}{K-P_0+\left(P_0-m\right)e^{(K-m)(k/K)t}}\ .$$

(d) If $P_0<m$, then $P_0-m<0$. Let $N(t)$ be the numerator of the expression for $P(t)$ in part (c). Then $N(0)=P_0(K-m)>0$, and $P_0-m<0 \Leftrightarrow \lim_{t\to\infty} K\left(P_0-m\right)e^{(K-m)(k/K)t}=-\infty \Rightarrow \lim_{t\to\infty} N(t)=-\infty$. Since N is continuous, there is a number t such that $N(t)=0$ and thus $P(t)=0$. So the species will become extinct.

14. **(a)**

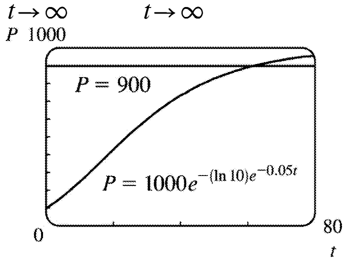
$\frac{dP}{dt}=c\ln\left(\frac{K}{P}\right)P \Rightarrow \int \frac{dP}{P\ln(K/P)}=\int c\,dt$. Let $u=\ln\left(\frac{K}{P}\right)=\ln K-\ln P \Rightarrow du=-\frac{dP}{P}$

$\Rightarrow \int -\frac{du}{u}=ct+D \Rightarrow \ln|u|=-ct-D \Rightarrow |u|=e^{-(ct+D)} \Rightarrow |\ln(K/P)|=e^{-(ct+D)} \Rightarrow$

$\ln(K/P)=\pm e^{-(ct+D)}$. Letting $t=0$, we get $\ln\left(K/P_0\right)=\pm e^{-D}$, so

$\ln(K/P)=\pm e^{-ct-D}=\pm e^{-ct}e^{-D}=\ln\left(K/P_0\right)e^{-ct} \Rightarrow K/P=e^{\ln\left(K/P_0\right)e^{-ct}} \Rightarrow P(t)=Ke^{-\ln\left(K/P_0\right)e^{-ct}}$, $c\neq 0$.

(b) $\lim_{t\to\infty} P(t)=\lim_{t\to\infty} Ke^{-\ln\left(K/P_0\right)e^{-ct}}=Ke^{-\ln\left(K/P_0\right)\cdot 0}=Ke^{0}=K$



(c)

The graphs look very similar. For the Gompertz function, $P(40)\approx 732$, nearly the same as the logistic function. The Gompertz function reaches $P=900$ at $t\approx 61.7$ and its value at $t=80$ is about 959 , so it doesn't increase quite as fast as the logistic curve.

(d) $\frac{dP}{dt}=c\ln\left(\frac{K}{P}\right)P=cP(\ln K-\ln P) \Rightarrow$

$$\frac{d^2P}{dt^2} = c\left[P\left(-\frac{1}{P}\frac{dP}{dt}\right)+(\ln K-\ln P)\frac{dP}{dt}\right]=c\frac{dP}{dt}\left[-1+\ln\left(\frac{K}{P}\right)\right]$$

$$= c[c\ln(K/P)P][\ln(K/P)-1]=c^2P\ln(K/P)[\ln(K/P)-1]$$

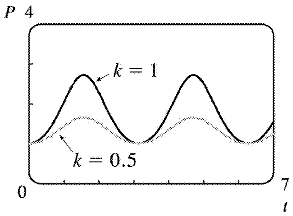
Since $0<P<K$, $P''=0 \Leftrightarrow \ln(K/P)=1 \Leftrightarrow K/P=e \Leftrightarrow P=K/e$. $P''>0$ for $0<P<K/e$ and $P''<0$ for $K/e<P<K$, so P' is a maximum (and P grows fastest) when $P=K/e$.

Note: If $P>K$, then $\ln(K/P)<0$, so $P''(t)>0$.

15. **(a)** $dP/dt=kP\cos(rt-\phi) \Rightarrow (dP)/P=k\cos(rt-\phi)\,dt \Rightarrow \int (dP)/P=k\square(rt-\phi)\,dt \Rightarrow$
$\ln P=(k/r)\sin(rt-\phi)+C$. (Since this is a growth model, $P>0$ and we can write $\ln P$ instead of $\ln|P|$.) Since $P(0)=P_0$, we obtain $\ln P_0=(k/r)\sin(-\phi)+C=-(k/r)\sin\phi+C \Rightarrow C=\ln P_0+(k/r)\sin\phi$. Thus, $\ln P=(k/r)\sin(rt-\phi)+\ln P_0+(k/r)\sin\phi$, which we can rewrite as $\ln\left(P/P_0\right)=(k/r)[\sin(rt-\phi)+\sin\phi]$ or, after exponentiation, $P(t)=P_0e^{(k/r)[\sin(rt-\phi)+\sin\phi]}$.

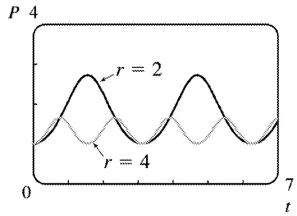
(b)

As k increases, the amplitude increases, but the minimum value stays the same.



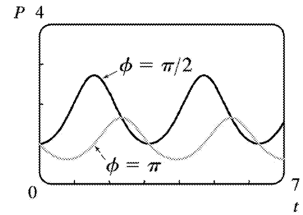
Comparing values of k with $P_0 = 1$, $r = 2$, and $\phi = \pi/2$

As r increases, the amplitude and the period decrease.



Comparing values of r with $P_0 = 1$, $k = 1$, and $\phi = \pi/2$

A change in ϕ produces slight adjustments in the phase shift and amplitude.



Comparing values of ϕ with $P_0 = 1$, $k = 1$, and $r = 2$

$P(t)$ oscillates between $P_0 e^{(k/r)\,(1+\sin\phi)}$ and $P_0 e^{(k/r)\,(-1+\sin\phi)}$ (the extreme values are attained when $rt-\phi$ is an odd multiple of $\frac{\pi}{2}$), so $\lim\limits_{t\to\infty} P(t)$ does not exist.

16. **(a)** $dP/dt=kP\cos^2(rt-\phi) \Rightarrow (dP)/P=k\cos^2(rt-\phi)\,dt \Rightarrow \int (dP)/P=k\square^2(rt-\phi)\,dt \Rightarrow$

$\ln P=k\int \frac{1+\cos(2(rt-\phi))}{2}\,dt=\frac{k}{2}t+\frac{k}{4r}(2(rt-\phi))+C$. From $P(0)=P_0$, we get

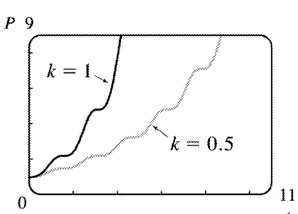
$\ln P_0=\frac{k}{4r}(-2\phi)+C=C-\frac{k}{4r}\sin 2\phi$, so $C=\ln P_0+\frac{k}{4r}\sin 2\phi$ and

$\ln P=\frac{k}{2}t+\frac{k}{4r}\sin(2(rt-\phi))+\ln P_0+\frac{k}{4r}\sin 2\phi$. Simplifying, we get

$\ln\frac{P}{P_0}=\frac{k}{2}t+\frac{k}{4r}\left[\sin(2(rt-\phi))+\sin 2\phi\right]=f(t)$, or $P(t)=P_0 e^{f(t)}$.

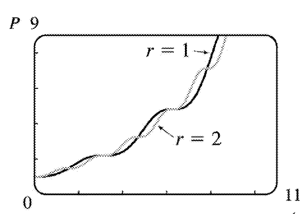
(b)

An increase in k stretches the graph of P vertically while maintaining $P(0)=P_0$.



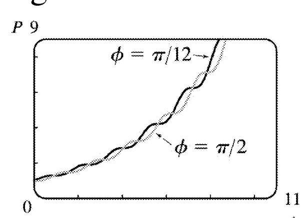
Comparing values of k with $P_0 = 1$, $r = 2$, and $\phi = \pi/2$

An increase in r compresses the graph of P horizontally — similar to changing the period in Exercise 15.



Comparing values of r with $P_0 = 1$, $k = 0.5$, and $\phi = \pi/2$

As in Exercise 15, a change in ϕ only makes slight adjustments in the growth of P , as shown in the figure.



Comparing values of ϕ with $P_0 = 1$, $k = 0.5$, and $r = 2$

$f'(t)=k/2+[k/(4r)]\,[2r\cos(2(rt-\phi))]=(k/2)\,[1+\cos(2(rt-\phi))]\geq 0$. Since $P(t)=P_0 e^{f(t)}$, we have $P'(t)=P_0 f'(t)e^{f(t)}\geq 0$, with equality only when $\cos(2(rt-\phi))=-1$; that is, when $rt-\phi$ is an odd multiple of $\frac{\pi}{2}$. Therefore, $P(t)$ is an increasing function on $(0,\infty)$. P can also be written as $P(t)=P_0 e^{kt/2} e^{(k/4r)\left[\sin\left(2(rt-\phi)\right)+\sin 2\phi\right]}$. The second exponential oscillates between $e^{(k/4r)(1+\sin 2\phi)}$ and $e^{(k/4r)(-1+\sin 2\phi)}$, while the first one, $e^{kt/2}$, grows without bound. So $\lim_{t\to\infty} P(t)=\infty$.

17. By Equation (4), $P(t)=\dfrac{K}{1+Ae^{-kt}}$. By comparison, if $c=(\ln A)/k$ and $u=\frac{1}{2}k(t-c)$, then

$1+\tanh u=1+\dfrac{e^{u}-e^{-u}}{e^{u}+e^{-u}}=\dfrac{e^{u}+e^{-u}}{e^{u}+e^{-u}}+\dfrac{e^{u}-e^{-u}}{e^{u}+e^{-u}}=\dfrac{2e^{u}}{e^{u}+e^{-u}}\cdot\dfrac{e^{-u}}{e^{-u}}=\dfrac{2}{1+e^{-2u}}$ and

$e^{-2u}=e^{-k(t-c)}=e^{kc}e^{-kt}=e^{\ln A}e^{-kt}=Ae^{-kt}$, so

$\frac{1}{2}K\left[1+\tanh\left(\frac{1}{2}k(t-c)\right)\right]=\frac{K}{2}\left[1+\tanh u\right]=\frac{K}{2}\cdot\dfrac{2}{1+e^{-2u}}=\dfrac{K}{1+e^{-2u}}=\dfrac{K}{1+Ae^{-kt}}=P(t)$.

1. $y'+e^x y=x^2 y^2$ is not linear since it cannot be put into the standard linear form (1), $y'+P(x)y=Q(x)$.

2. $y+\sin x=x^3 y' \Rightarrow x^3 y'-y=\sin x \Rightarrow y'+\left(-\dfrac{1}{x^3}\right)y=\dfrac{\sin x}{x^3}$. This equation is in the standard linear form (1), so it is linear.

3. $xy'+\ln x-x^2 y=0 \Rightarrow xy'-x^2 y=-\ln x \Rightarrow y'+(-x)y=-\dfrac{\ln x}{x}$, which is in the standard linear form (1), so this equation is linear.

4. $y'+\cos y=\tan x$ is not linear since it cannot be put into the standard linear form (1), $y'+P(x)y=Q(x)$.

5. Comparing the given equation, $y'+2y=2e^x$, with the general form, $y'+P(x)y=Q(x)$, we see that $P(x)=2$ and the integrating factor is $I(x)=e^{\int P(x)\,dx}=e^{\int 2\,dx}=e^{2x}$. Multiplying the differential equation by $I(x)$ gives $e^{2x}y'+2e^{2x}y=2e^{3x} \Rightarrow \left(e^{2x}y\right)'=2e^{3x} \Rightarrow e^{2x}y=\int 2e^{3x}dx \Rightarrow e^{2x}y=\frac{2}{3}e^{3x}+C \Rightarrow y=\frac{2}{3}e^{x}+Ce^{-2x}$.

6. $y'=x+5y \Rightarrow y'-5y=x$. $I(x)=e^{\int P(x)\,dx}=e^{\int(-5)\,dx}=e^{-5x}$. Multiplying the differential equation by $I(x)$ gives $e^{-5x}y'-5e^{-5x}y=xe^{-5x} \Rightarrow \left(e^{-5x}y\right)'=xe^{-5x} \Rightarrow e^{-5x}y=\int xe^{-5x}dx=-\frac{1}{5}xe^{-5x}-\frac{1}{25}e^{-5x}+C$ [by parts] $\Rightarrow$ $y=-\frac{1}{5}x-\frac{1}{25}+Ce^{5x}$

7. $xy'-2y=x^2$ [divide by x] $\Rightarrow y'+\left(-\dfrac{2}{x}\right)y=x$ (*).
$I(x)=e^{\int P(x)dx}=e^{\int(-2/x)dx}=e^{-2\ln|x|}=e^{\ln|x|^{-2}}=e^{\ln(1/x^2)}=1/x^2$. Multiplying the differential equation (*) by $I(x)$ gives $\dfrac{1}{x^2}y'-\dfrac{2}{x^3}y=\dfrac{1}{x} \Rightarrow \left(\dfrac{1}{x^2}y\right)'=\dfrac{1}{x} \Rightarrow \dfrac{1}{x^2}y=\ln|x|+C \Rightarrow y=x^2(\ln|x|+C)=x^2\ln|x|+Cx^2$.

8. $x^2y'+2xy=\cos^2 x \Rightarrow y'+\dfrac{2}{x}y=\dfrac{\cos^2 x}{x^2}$. $I(x)=e^{\int P(x)dx}=e^{\int 2/x\,dx}=e^{2\ln|x|}=e^{\ln(x^2)}=x^2$. Multiplying by $I(x)$ gives us our original equation back. You may have noticed this immediately, since $P(x)$ is the derivative of the coefficient of y' . We rewrite it as $(x^2y)'=\cos^2 x$. Thus,
$x^2y=\int\cos^2 x\,dx=\int\frac{1}{2}(1+\cos 2x)\,dx=\frac{1}{2}x+\frac{1}{4}\sin 2x+C \Rightarrow$

$y=\frac{1}{2x}+\frac{1}{4x^2}\sin 2x+\frac{C}{x^2}$ or $y=\frac{1}{2x}+\frac{1}{2x^2}\sin x\cos x+\frac{C}{x^2}$.

9. Since $P(x)$ is the derivative of the coefficient of y' , we can write the differential equation $xy'+y=\sqrt{x}$ in the easily integrable form $(xy)'=\sqrt{x}\Rightarrow xy=\frac{2}{3}x^{3/2}+C\Rightarrow y=\frac{2}{3}\sqrt{x}+C/x$.

10. $y'-y=1/x$, so $I(x)=e^{\int(-1)\,dx}=e^{-x}$. Multiplying the differential equation by $I(x)$ gives $e^{-x}y'-e^{-x}y=e^{-x}/x\Rightarrow\left(e^{-x}y\right)'=e^{-x}/x\Rightarrow y=e^{x}\left[\int\left(e^{-x}/x\right)dx+C\right]$.

11. $I(x)=e^{\int 2x dx}=e^{x^2}$. Multiplying the differential equation $y'+2xy=x^2$ by $I(x)$ gives $e^{x^2}y'+2xe^{x^2}y=x^2e^{x^2}\Rightarrow\left(e^{x^2}y\right)'=x^2e^{x^2}$. Thus
$y=e^{-x^2}\left[\int x^2e^{x^2}dx+C\right]=e^{-x^2}\left[\frac{1}{2}xe^{x^2}-\int\frac{1}{2}e^{x^2}dx+C\right]=\frac{1}{2}x+Ce^{-x^2}-e^{-x^2}\int\frac{1}{2}e^{x^2}dx$.

12. $I(x)=e^{\int -\tan x dx}=e^{\ln|\cos x|}=\cos x$ (since $-\frac{\pi}{2}<x<\frac{\pi}{2}$). Multiplying the differential equation by $I(x)$ gives $y'\cos x-y\tan x\cos x=x\cos x\sin 2x\Rightarrow(y\cos x)'=x\cos x\sin 2x$. So

$$y=\frac{1}{\cos x}\left[\int x\cos x\sin 2x\,dx+C\right]=\frac{1}{\cos x}\left[\int 2x\cos^2 x\sin x\,dx+C\right]$$

$$=\frac{1}{\cos x}\left[\frac{-2x\cos^3 x}{3}+\frac{2}{3}\left(\sin x-\frac{\sin^3 x}{3}\right)+C\right]=\frac{-2x\cos^2 x}{3}+\frac{C}{\cos x}+2\tan x\,\frac{3-\sin^2 x}{9}$$

13. $(1+t)\frac{du}{dt}+u=1+t$, $t>0$ [divide by $1+t$] $\Rightarrow\frac{du}{dt}+\frac{1}{1+t}u=1$ (*), which has the form $u'+P(t)u=Q(t)$. The integrating factor is $I(t)=e^{\int P(t)dt}=e^{\int[1/(1+t)]\,dt}=e^{\ln(1+t)}=1+t$.
Multiplying (*) by $I(t)$ gives us our original equation back. We rewrite it as $[(1+t)u]'=1+t$. Thus,
$(1+t)u=\int(1+t)dt=t+\frac{1}{2}t^2+C\Rightarrow u=\dfrac{t+\frac{1}{2}t^2+C}{1+t}$ or $u=\dfrac{t^2+2t+2C}{2(t+1)}$.

14. $t\ln t\,\frac{dr}{dt}+r=te^t\Rightarrow\frac{dr}{dt}+\frac{1}{t\ln t}r=\frac{e^t}{\ln t}$. $I(t)=e^{\int dt/(t\ln t)}=e^{\ln(\ln t)}=\ln t$. Multiplying by $\ln t$ gives

$\ln t\,\frac{dr}{dt}+\frac{1}{t}\,r=e^t \Rightarrow [(\ln t)r]'=e^t \Rightarrow (\ln t)r=e^t+C \Rightarrow r=\frac{e^t+C}{\ln t}$.

15. $y'=x+y \Rightarrow y'+(-1)y=x$. $I(x)=e^{\int(-1)dx}=e^{-x}$. Multiplying by e^{-x} gives $e^{-x}y'-e^{-x}y=xe^{-x} \Rightarrow$ $(e^{-x}y)'=xe^{-x} \Rightarrow e^{-x}y=\int xe^{-x}dx=-xe^{-x}-e^{-x}+C$ [integration by parts with $u=x$, $dv=e^{-x}dx$] $\Rightarrow y=-x-1+Ce^{x}$. $y(0)=2 \Rightarrow -1+C=2 \Rightarrow C=3$, so $y=-x-1+3e^{x}$.

16. $t\,\frac{dy}{dt}+2y=t^3$, $t>0$, $y(1)=0$. Divide by t to get $\frac{dy}{dt}+\frac{2}{t}\,y=t^2$, which is linear. $I(t)=e^{\int(2/t)\,dt}=e^{2\ln t}=t^2$. Multiplying by t^2 gives $t^2\,\frac{dy}{dt}+2ty=t^4 \Rightarrow \left(t^2y\right)'=t^4 \Rightarrow t^2y=\frac{1}{5}\,t^5+C \Rightarrow y=\frac{t^3}{5}+\frac{C}{t^2}$. Thus, $0=y(1)=\frac{1}{5}+C$ $\Rightarrow C=-\frac{1}{5}$, so $y=\frac{t^3}{5}-\frac{1}{5t^2}$.

17. $\frac{dv}{dt}-2tv=3t^2e^{t^2}$, $v(0)=5$. $I(t)=e^{\int(-2t)\,dt}=e^{-t^2}$. Multiply the differential equation by $I(t)$ to get $e^{-t^2}\frac{dv}{dt}-2te^{-t^2}v=3t^2 \Rightarrow \left(e^{-t^2}v\right)'=3t^2 \Rightarrow e^{-t^2}v=\int 3t^2\,dt=t^3+C \Rightarrow v=t^3e^{t^2}+Ce^{t^2}$. $5=v(0)=0\cdot 1+C\cdot 1=C$, so $v=t^3e^{t^2}+5e^{t^2}$.

18. $2xy'+y=6x$, $x>0 \Rightarrow y'+\frac{1}{2x}\,y=3$. $I(x)=e^{\int 1/(2x)dx}=e^{(1/2)\ln x}=e^{\ln x^{1/2}}=\sqrt{x}$. Multiplying by $\sqrt{x}$ gives $\sqrt{x}\,y'+\frac{1}{2\sqrt{x}}\,y=3\sqrt{x} \Rightarrow \left(\sqrt{x}\,y\right)'=3\sqrt{x} \Rightarrow \sqrt{x}\,y=\int 3\sqrt{x}\,dx=2x^{3/2}+C \Rightarrow y=2x+\frac{C}{\sqrt{x}}$. $y(4)=20 \Rightarrow$ $8+\frac{C}{2}=20 \Rightarrow C=24$, so $y=2x+\frac{24}{\sqrt{x}}$.

19. $xy'=y+x^2\sin x \Rightarrow y'-\frac{1}{x}\,y=x\sin x$. $I(x)=e^{\int(-1/x)dx}=e^{-\ln x}=e^{\ln x^{-1}}=\frac{1}{x}$.
Multiplying by $\frac{1}{x}$ gives $\frac{1}{x}\,y'-\frac{1}{x^2}\,y=\sin x \Rightarrow \left(\frac{1}{x}\,y\right)'=\sin x \Rightarrow \frac{1}{x}\,y=-\cos x+C \Rightarrow y=-x\cos x+Cx$.
$y(\pi)=0 \Rightarrow -\pi\cdot(-1)+C\pi=0 \Rightarrow C=-1$, so $y=-x\cos x-x$.

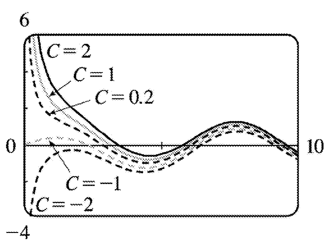
20. $x\,\frac{dy}{dx}-\frac{y}{x+1}=x \Rightarrow y'-\frac{y}{x(x+1)}=1\,(x>0)$, so $I(x)=e^{-\int 1/[x(x+1)]dx}=e^{-(\ln|x|-\ln|x+1|)}=\frac{x+1}{x}$.
Multiplying the differential equation by $I(x)$ gives

$\frac{x+1}{x} y' - \frac{y}{x(x+1)} \frac{x+1}{x} = \frac{x+1}{x} \Rightarrow \left(\frac{x+1}{x} y \right)' = \frac{x+1}{x}$. Then

$y=\frac{x}{x+1} \left[\int \left(1+\frac{1}{x} \right) dx+C \right] = \frac{x}{x+1} (x+\ln x+C)$. But $0=y(1)=\frac{1}{2} [1+C]$ so $C=-1$ and the solution to the initial–value problem is $y=\frac{x}{x+1} (x-1+\ln x)$.

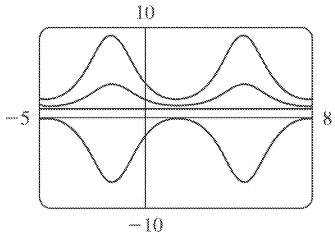
21. $y'+\frac{1}{x} y=\cos x$ ($x\neq 0$), so $I(x)=e^{\int (1/x)\,dx}=e^{\ln |x|}=x$ (for $x>0$). Multiplying the differential equation by $I(x)$ gives $xy'+y=x\cos x \Rightarrow (xy)'=x\cos x$. Thus,

$$y = \frac{1}{x} \left[\int x\cos x\,dx+C \right] = \frac{1}{x} [x\sin x+\cos x+C]$$
$$= \sin x+\frac{\cos x}{x}+\frac{C}{x}$$



The solutions are asymptotic to the y– axis (except for $C=-1$). In fact, for $C>-1$, $y\to\infty$ as $x\to 0^+$, whereas for $C<-1$, $y\to -\infty$ as $x\to 0^+$. As x gets larger, the solutions approximate $y=\sin x$ more closely. The graphs for larger C lie above those for smaller C . The distance between the graphs lessens as x increases.

22. $I(x)=e^{\int \cos x\,dx}=e^{\sin x}$. Multiplying the differential equation by $I(x)$ gives $e^{\sin x} y'+\cos x\cdot e^{\sin x} y=\cos x\cdot e^{\sin x} \Rightarrow \left(e^{\sin x} y\right)'=\cos x\cdot e^{\sin x} \Rightarrow y=e^{-\sin x}\left[\int \cos x\cdot e^{\sin x} dx+C \right]=1+Ce^{-\sin x}$. The graphs for $C=-3$, 0 , 1 , and 3 are shown. As the values of C get further from zero the graph is stretched away from the line $y=1$, which is the value for $C=0$. The graphs are all periodic in x , with a period of 2π .



23. Setting $u=y^{1-n}$, $\frac{du}{dx}=(1-n)\,y^{-n}\frac{dy}{dx}$ or $\frac{dy}{dx}=\frac{y^{n}}{1-n}\frac{du}{dx}=\frac{u^{n/(1-n)}}{1-n}\frac{du}{dx}$. Then the Bernoulli differential equation becomes $\frac{u^{n/(1-n)}}{1-n}\frac{du}{dx}+P(x)u^{1/(1-n)}=Q(x)u^{n/(1-n)}$ or $\frac{du}{dx}+(1-n)P(x)u=Q(x)(1-n)$.

24. Here $y'+\frac{y}{x}=-y^{2}$, so $n=2$, $P(x)=\frac{1}{x}$ and $Q(x)=-1$. Setting $u=y^{-1}$, u satisfies $u'-\frac{1}{x}u=1$. Then $I(x)=e^{\int(-1/x)\,dx}=\frac{1}{x}$ (for $x>0$) and $u=x\left(\int\frac{1}{x}\,dx+C\right)=x\,(\ln|x|+C)$. Thus, $y=\frac{1}{x\,(C+\ln|x|)}$.

25. $y'+\frac{2}{x}\,y=\frac{y^{3}}{x^{2}}$. Here $n=3$, $P(x)=\frac{2}{x}$, $Q(x)=\frac{1}{x^{2}}$ and setting $u=y^{-2}$, u satisfies $u'-\frac{4u}{x}=-\frac{2}{x^{2}}$.
Then $I(x)=e^{\int(-4/x)\,dx}=x^{-4}$ and $u=x^{4}\left(\int-\frac{2}{x^{6}}\,dx+C\right)=x^{4}\left(\frac{2}{5x^{5}}+C\right)=Cx^{4}+\frac{2}{5x}$.
Thus, $y=\pm\left(Cx^{4}+\frac{2}{5x}\right)^{-1/2}$.

26. Here $n=3$, $P(x)=1$, $Q(x)=x$ and setting $u=y^{-2}$, u satisfies $u'-2u=-2x$. Then $I(x)=e^{\int(-2)\,dx}=e^{-2x}$ and $u=e^{2x}\left[\int-2xe^{-2x}dx+C\right]=e^{2x}\left(xe^{-2x}+\frac{1}{2}e^{-2x}+C\right)=x+\frac{1}{2}+Ce^{2x}$.
So $y^{-2}=x+\frac{1}{2}+Ce^{2x}\Rightarrow y=\pm\left[x+\frac{1}{2}+Ce^{2x}\right]^{-1/2}$.

27. **(a)** $2\frac{dI}{dt}+10I=40$ or $\frac{dI}{dt}+5I=20$. Then the integrating factor is $e^{\int 5\,dt}=e^{5t}$. Multiplying the differential equation by the integrating factor gives $e^{5t}\frac{dI}{dt}+5Ie^{5t}=20e^{5t}\Rightarrow\left(e^{5t}I\right)'=20e^{5t}\Rightarrow$ $I(t)=e^{-5t}\left[\int 20e^{5t}\,dt+C\right]=4+Ce^{-5t}$. But $0=I(0)=4+C$, so $I(t)=4-4e^{-5t}$.
(b) $I(0.1)=4-4e^{-0.5}\approx 1.57$ A

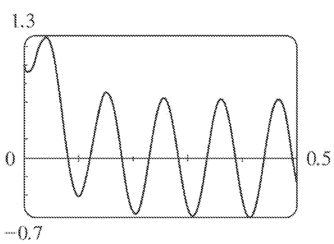
28. **(a)** $\frac{dI}{dt}+20I=40\sin 60t$, so the integrating factor is e^{20t} . Multiplying the differential equation by the integrating factor gives $e^{20t}\frac{dI}{dt}+20Ie^{20t}=40e^{20t}\sin 60t\Rightarrow\left(e^{20t}I\right)'=40e^{20t}\sin 60t\Rightarrow$

$$
\begin{aligned}
I(t) &= e^{-20t}\left[\int 40e^{20t}\sin 60t\,dt+C\right]\\
&= e^{-20t}\left[40e^{20t}\left(\frac{1}{4000}\right)(20\sin 60t-60\cos 60t)\right]+Ce^{-20t}\\
&= \frac{\sin 60t-3\cos 60t}{5}+Ce^{-20t}
\end{aligned}
$$

But $1=I(0)=-\frac{3}{5}+C$, so $I(t)=\frac{\sin 60t-3\cos 60t+8e^{-20t}}{5}$.

(b)

$$
\begin{aligned}
I(0.1) &= \frac{\sin 6-3\cos 6+8e^{-2}}{5}\\
&\approx -0.42\text{ A}
\end{aligned}
$$



(c)

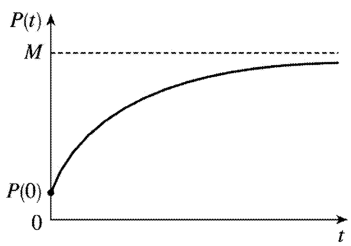
29. $5\frac{dQ}{dt}+20Q=60$ with $Q(0)=0$ C. Then the integrating factor is $e^{\int 4\,dt}=e^{4t}$, and multiplying the differential equation by the integrating factor gives $e^{4t}\frac{dQ}{dt}+4e^{4t}Q=12e^{4t}\Rightarrow\left(e^{4t}Q\right)'=12e^{4t}\Rightarrow$ $Q(t)=e^{-4t}\left[\int 12e^{4t}\,dt+C\right]=3+Ce^{-4t}$. But $0=Q(0)=3+C$ so $Q(t)=3\left(1-e^{-4t}\right)$ is the charge at time t and $I=dQ/dt=12e^{-4t}$ is the current at time t .

30. $2\frac{dQ}{dt}+100Q=10\sin 60t$ or $\frac{dQ}{dt}+50Q=5\sin 60t$. Then the integrating factor is $e^{\int 50\,dt}=e^{50t}$, and multiplying the differential equation by the integrating factor gives $e^{50t}\frac{dQ}{dt}+50e^{50t}Q=5e^{50t}\sin 60t\Rightarrow$ $\left(e^{50t}Q\right)'=5e^{50t}\sin 60t\Rightarrow$

$$Q(t) = e^{-50t}\left[\int 5e^{50t}\sin 60t\,dt+C\right]=e^{-50t}\left[5e^{50t}\left(\frac{1}{6100}\right)(50\sin 60t-60\cos 60t)\right]+Ce^{-50t}$$
$$= \frac{1}{122}(5\sin 60t-6\cos 60t)+Ce^{-50t}$$

But $0=Q(0)=-\frac{6}{122}+C$ so $C=\frac{3}{61}$ and $Q(t)=\frac{5\sin 60t-6\cos 60t}{122}+\frac{3e^{-50t}}{61}$ is the charge at time t , while the current is $I(t)=\frac{dQ}{dt}=\frac{150\cos 60t+180\sin 60t-150e^{-50t}}{61}$.

31. $\frac{dP}{dt}+kP=kM$, so $I(t)=e^{\int k\,dt}=e^{kt}$. Multiplying the differential equation by $I(t)$ gives $e^{kt}\frac{dP}{dt}+kPe^{kt}=kMe^{kt}\Rightarrow\left(e^{kt}P\right)'=kMe^{kt}\Rightarrow P(t)=e^{-kt}\left(\int kMe^{kt}dt+C\right)=M+Ce^{-kt}$, $k>0$. Furthermore, it is reasonable to assume that $0\le P(0)\le M$, so $-M\le C\le 0$.



32. Since $P(0)=0$, we have $P(t)=M(1-e^{-kt})$. If $P_1(t)$ is Jim's learning curve, then $P_1(1)=25$ and $P_1(2)=45$. Hence, $25=M_1(1-e^{-k})$ and $45=M_1(1-e^{-2k})$, so $1-25/M_1=e^{-k}$ or $k=-\ln\left(1-\frac{25}{M_1}\right)=\ln\left(\frac{M_1}{M_1-25}\right)$. But $45=M_1(1-e^{-2k})$ so $45=M_1\left[1-\left(\frac{M_1-25}{M_1}\right)^2\right]$ or $45=\frac{50M_1-625}{M_1}$. Thus, $M_1=125$ is the maximum number of units per hour Jim is capable of processing. Similarly, if $P_2(t)$ is Mark's learning curve, then $P_2(1)=35$ and $P_2(2)=50$. So $k=\ln\left(\frac{M_2}{M_2-35}\right)$ and $50=M_2\left[1-\left(\frac{M_2-35}{M_2}\right)^2\right]$ or $M_2=61.25$. Hence the maximum number of units per hour for Mark is approximately 61 . Another approach would be to use the midpoints of the intervals so that $P_1(0.5)=25$ and $P_1(1.5)=45$. Doing so gives us $M_1\approx 52.6$ and $M_2\approx 51.8$.

33. $y(0)=0$ kg. Salt is added at a rate of $\left(0.4\,\frac{\text{kg}}{\text{L}}\right)\left(5\,\frac{\text{L}}{\text{min}}\right)=2\,\frac{\text{kg}}{\text{min}}$. Since solution is drained from the tank at a rate of 3 L / min, but salt solution is added at a rate of 5 L / min, the tank, which starts out with 100 L of water, contains $(100+2t)$ L of liquid after t min. Thus, the salt concentration at time t is $\frac{y(t)}{100+2t}\,\frac{\text{kg}}{\text{L}}$. Salt therefore leaves the tank at a rate of $\left(\frac{y(t)}{100+2t}\,\frac{\text{kg}}{\text{L}}\right)\left(3\,\frac{\text{L}}{\text{min}}\right)=\frac{3y}{100+2t}\,\frac{\text{kg}}{\text{min}}$. Combining the rates at which salt enters and leaves the tank, we get $\frac{dy}{dt}=2-\frac{3y}{100+2t}$. Rewriting this equation as $\frac{dy}{dt}+\left(\frac{3}{100+2t}\right)y=2$, we see that it is linear. $I(t)=\exp\left(\int\frac{3\,dt}{100+2t}\right)=\exp\left(\frac{3}{2}\ln(100+2t)\right)=(100+2t)^{3/2}$. Multiplying the differential equation by $I(t)$ gives $(100+2t)^{3/2}\frac{dy}{dt}+3(100+2t)^{1/2}y=2(100+2t)^{3/2}\Rightarrow\left[(100+2t)^{3/2}y\right]'=2(100+2t)^{3/2}\Rightarrow$ $(100+2t)^{3/2}y=\frac{2}{5}(100+2t)^{5/2}+C\Rightarrow y=\frac{2}{5}(100+2t)+C(100+2t)^{-3/2}$. Now $0=y(0)=\frac{2}{5}(100)+C\cdot 100^{-3/2}=40+\frac{1}{1000}C\Rightarrow C=-40{,}000$, so $y=\left[\frac{2}{5}(100+2t)-40{,}000(100+2t)^{-3/2}\right]$ kg. From this solution (no pun intended), we calculate the salt concentration at time t to be $C(t)=\frac{y(t)}{100+2t}=\left[\frac{-40{,}000}{(100+2t)^{5/2}}+\frac{2}{5}\right]\frac{\text{kg}}{\text{L}}$. In particular, $C(20)=\frac{-40{,}000}{140^{5/2}}+\frac{2}{5}\approx 0.2275\,\frac{\text{kg}}{\text{L}}$ and $y(20)=\frac{2}{5}(140)-40{,}000(140)^{-3/2}\approx 31.85$ kg.

34. Let $y(t)$ denote the amount of chlorine in the tank at time t (in seconds). $y(0)=(0.05\text{g/L})(400\text{L})=20$ g. The amount of liquid in the tank at time t is $(400-6t)$ L since 4 L of water enters the tank each second and 10 L of liquid leaves the tank each second. Thus, the concentration of chlorine at time t is $\frac{y(t)}{400-6t}\,\frac{\text{g}}{\text{L}}$. Chlorine doesn't enter the tank, but it leaves at a rate of $\left[\frac{y(t)}{400-6t}\,\frac{\text{g}}{\text{L}}\right]\left[10\,\frac{\text{L}}{\text{s}}\right]=\frac{10\,y(t)}{400-6t}\,\frac{\text{g}}{\text{s}}=\frac{5\,y(t)}{200-3t}\,\frac{\text{g}}{\text{s}}$. Therefore, $\frac{dy}{dt}=-\frac{5y}{200-3t}\Rightarrow\int\frac{dy}{y}=\int\frac{-5\,dt}{200-3t}\Rightarrow\ln y=\frac{5}{3}\ln(200-3t)+C\Rightarrow$ $y=\exp\left(\frac{5}{3}\ln(200-3t)+C\right)=e^{C}(200-3t)^{5/3}$. Now $20=y(0)=e^{C}\cdot 200^{5/3}\Rightarrow e^{C}=\frac{20}{200^{5/3}}$, so $y(t)=20\,\frac{(200-3t)^{5/3}}{200^{5/3}}=20(1-0.015t)^{5/3}$ g for $0\le t\le 66\frac{2}{3}$ s, at which time the tank is empty.

35. **(a)**

$\frac{dv}{dt}+\frac{c}{m}v=g$ and $I(t)=e^{\int (c/m)\,dt}=e^{(c/m)t}$, and multiplying the differential equation by $I(t)$ gives $e^{(c/m)t}\frac{dv}{dt}+\frac{vce^{(c/m)t}}{m}=ge^{(c/m)t}\Rightarrow\left[e^{(c/m)t}v\right]'=ge^{(c/m)t}$. Hence, $v(t)=e^{-(c/m)t}\left[\int ge^{(c/m)t}dt+K\right]=mg/c+Ke^{-(c/m)t}$. But the object is dropped from rest, so $v(0)=0$ and $K=-mg/c$. Thus, the velocity at time t is $v(t)=(mg/c)\left[1-e^{-(c/m)t}\right]$.

(b) $\lim_{t\to\infty} v(t)=mg/c$

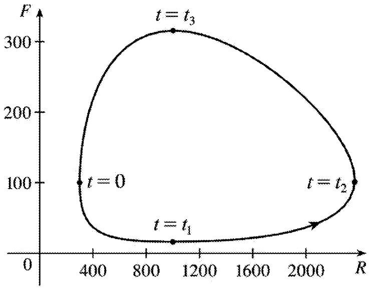
(c) $s(t)=\int v(t)\,dt=(mg/c)\left[t+(m/c)e^{-(c/m)t}\right]+c_1$ where $c_1=s(0)-m^2g/c^2$. $s(0)$ is the initial position, so $s(0)=0$ and $s(t)=(mg/c)\left[t+(m/c)e^{-(c/m)t}\right]-m^2g/c^2$.

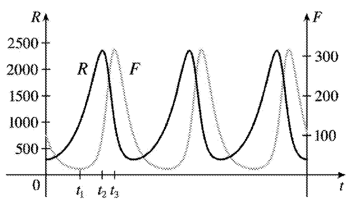
36. $v=(mg/c)(1-e^{-ct/m})\Rightarrow$
$\frac{dv}{dm}=\frac{mg}{c}\left(0-e^{-ct/m}\cdot\frac{ct}{m^2}\right)+\frac{g}{c}\left(1-e^{-ct/m}\right)\cdot 1=-\frac{gt}{m}e^{-ct/m}+\frac{g}{c}-\frac{g}{c}e^{-ct/m}=\frac{g}{c}\left(1-e^{-ct/m}-\frac{ct}{m}e^{-ct/m}\right)\Rightarrow$
$\frac{c}{g}\frac{dv}{dm}=1-\left(1+\frac{ct}{m}\right)e^{-ct/m}=1-\frac{1+ct/m}{e^{ct/m}}=1-\frac{1+Q}{e^{Q}}$, where $Q=\frac{ct}{m}\geq 0$. Since $e^{Q}>1+Q$ for all $Q>0$, it follows that $dv/dm>0$ for $t>0$. In other words, for all $t>0$, v increases as m increases.

1. **(a)** $dx/dt=-0.05x+0.0001xy$. If $y=0$, we have $dx/dt=-0.05x$, which indicates that in the absence of y , x declines at a rate proportional to itself. So x represents the predator population and y represents the prey population. The growth of the prey population, $0.1y$ (from $dy/dt=0.1y-0.005xy$), is restricted only by encounters with predators (the term $-0.005xy$). The predator population increases only through the term $0.0001xy$; that is, by encounters with the prey and not through additional food sources.
(b) $dy/dt=-0.015y+0.00008xy$. If $x=0$, we have $dy/dt=-0.015y$, which indicates that in the absence of x , y would decline at a rate proportional to itself. So y represents the predator population and x represents the prey population. The growth of the prey population, $0.2x$ (from $dx/dt=0.2x-0.0002x^2-0.006xy=0.2x(1-0.001x)-0.006xy$), is restricted by a carrying capacity of 1000 and by encounters with predators (the term $-0.006xy$). The predator population increases only through the term $0.00008xy$; that is, by encounters with the prey and not through additional food sources.

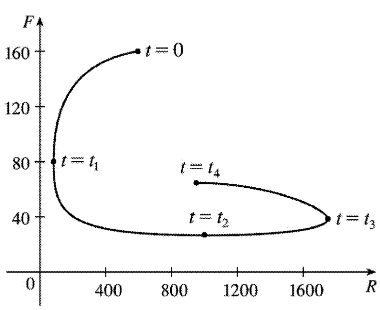
2. **(a)** $dx/dt=0.12x-0.0006x^2+0.00001xy$. $dy/dt=0.08y+0.00004xy$.
The xy terms represent encounters between the two species x and y . An increase in y makes dx/dt (the growth rate of x) larger due to the positive term $0.00001xy$. An increase in x makes dy/dt (the growth rate of y) larger due to the positive term $0.00004xy$. Hence, the system describes a cooperation model.
(b) $dx/dt=0.15x-0.0002x^2-0.0006xy=0.15x\left(1-x/750\right)-0.0006xy$.
$dy/dt=0.2y-0.00008y^2-0.0002xy=0.2y\left(1-y/2500\right)-0.0002xy$.
The system shows that x and y have carrying capacities of 750 and 2500 . An increase in x reduces the growth rate of y due to the negative term $-0.0002xy$. An increase in y reduces the growth rate of x due to the negative term $-0.0006xy$. Hence, the system describes a competition model.

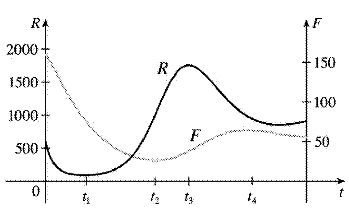
3. **(a)** At $t=0$, there are about 300 rabbits and 100 foxes. At $t=t_1$, the number of foxes reaches a minimum of about 20 while the number of rabbits is about 1000 . At $t=t_2$, the number of rabbits reaches a maximum of about 2400 , while the number of foxes rebounds to 100 . At $t=t_3$, the number of rabbits decreases to about 1000 and the number of foxes reaches a maximum of about 315 . As t increases, the number of foxes decreases greatly to 100 , and the number of rabbits decreases to 300 (the initial populations), and the cycle starts again.



(b) 

4. **(a)** At t=0 , there are about 600 rabbits and 160 foxes. At t=t_1 , the number of rabbits reaches a minimum of about 80 and the number of foxes is also 80 . At t=t_2 , the number of foxes reaches a minimum of about 25 while the number of rabbits rebounds to 1000 . At t=t_3 , the number of foxes has increased to 40 and the rabbit population has reached a maximum of about 1750 . The curve ends at t=t_4 , where the number of foxes has increased to 65 and the number of rabbits has decreased to about 950 .



(b) 

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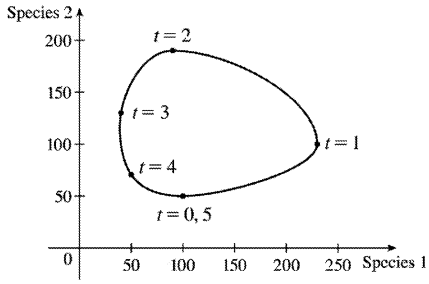

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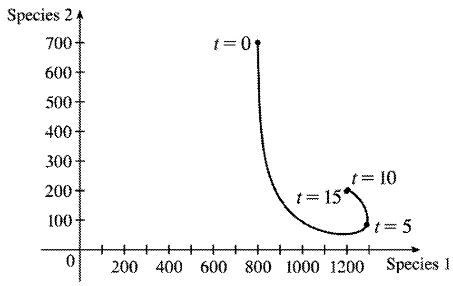


7. $\dfrac{dW}{dR}=\dfrac{-0.02W+0.00002RW}{0.08R-0.001RW}\Leftrightarrow(0.08-0.001W)R\,dW=(-0.02+0.00002R)W\,dR\Leftrightarrow$
$\dfrac{0.08-0.001W}{W}\,dW=\dfrac{-0.02+0.00002R}{R}\,dR\Leftrightarrow\int\left(\dfrac{0.08}{W}-0.001\right)dW=\int\left(-\dfrac{0.02}{R}+0.00002\right)dR\Leftrightarrow$
$0.08\ln|W|-0.001W=-0.02\ln|R|+0.00002R+K\Leftrightarrow0.08\ln W+0.02\ln R=0.001W+0.00002R+K\Leftrightarrow$
$\ln\left(W^{0.08}R^{0.02}\right)=0.00002R+0.001W+K\Leftrightarrow W^{0.08}R^{0.02}=e^{0.00002R+0.001W+K}\Leftrightarrow$
$R^{0.02}W^{0.08}=Ce^{0.00002R}e^{0.001W}\Leftrightarrow\dfrac{R^{0.02}W^{0.08}}{e^{0.00002R}e^{0.001W}}=C$.

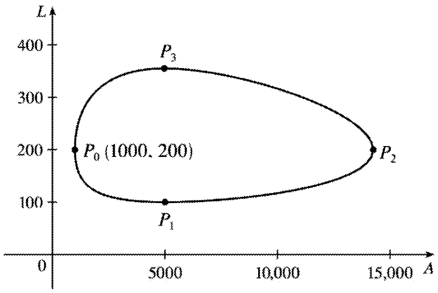
In general, if $\dfrac{dy}{dx}=\dfrac{-ry+bxy}{kx-axy}$, then $C=\dfrac{x^ry^k}{e^{bx}e^{ay}}$.

8. **(a)** A and L are constant $\Rightarrow A'=0$ and $L'=0\Rightarrow\left\{\begin{array}{l}0=2A-0.01AL\\0=-0.5L+0.0001AL\end{array}\right\}\Rightarrow$
$\left\{\begin{array}{l}0=A(2-0.01L)\\0=L(-0.5+0.0001A)\end{array}\right.$

So either $A=L=0$ or $L=\dfrac{2}{0.01}=200$ and $A=\dfrac{0.5}{0.0001}=5000$. The trivial solution $A=L=0$ just says that if there aren't any aphids or ladybugs, then the populations will not change. The non–trivial solution, $L=200$ and $A=5000$, indicates the population sizes needed so that there are no changes in either the number of aphids or the number of ladybugs.

(b) $\dfrac{dL}{dA}=\dfrac{dL/dt}{dA/dt}=\dfrac{-0.5L+0.0001AL}{2A-0.01AL}$

(c) C100708cC100708c.texC100708c.tex

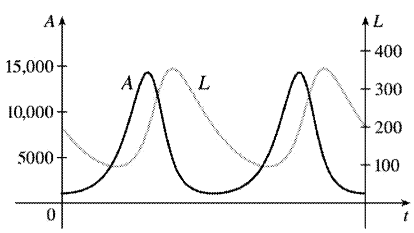


(d)

At $P_0(1000,200)$, $dA/dt=0$ and $dL/dt=-80<0$, so the number of ladybugs is decreasing and hence, we are proceeding in a counterclockwise direction. At P_0 , there aren't enough aphids to support the ladybug population, so the number of ladybugs decreases and the number of aphids begins to increase. The ladybug population reaches a minimum at $P_1(5000,100)$ while the aphid population increases in a dramatic way, reaching its maximum at $P_2(14,250,200)$.

Meanwhile, the ladybug population is increasing from P_1 to $P_3(5000,355)$, and as we pass through P_2 , the increasing number of ladybugs starts to deplete the aphid population. At P_3 the ladybugs reach a maximum population, and start to decrease due to the reduced aphid population. Both populations then decrease until P_0 , where the cycle starts over again.

(e) Both graphs have the same period and the graph of L peaks about a quarter of a cycle after the graph of A .



9. **(a)** Letting $W=0$ gives us $dR/dt=0.08R(1-0.0002R)$. $dR/dt=0 \Leftrightarrow R=0$ or 5000 . Since $dR/dt>0$ for $0<R<5000$, we would expect the rabbit population to *increase* to 5000 for these values of R . Since $dR/dt<0$ for $R>5000$, we would expect the rabbit population to *decrease* to 5000 for these values of R . Hence, in the absence of wolves, we would expect the rabbit population to stabilize at 5000 .

(b) R and W are constant $\Rightarrow R'=0$ and $W'=0 \Rightarrow$

$$\left\{ \begin{array}{l} 0=0.08R(1-0.0002R)-0.001RW \\ 0=-0.02W+0.00002RW \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} 0=R\left[0.08(1-0.0002R)-0.001W\right] \\ 0=W(-0.02+0.00002R) \end{array} \right.$$

The second equation is true if $W=0$ or

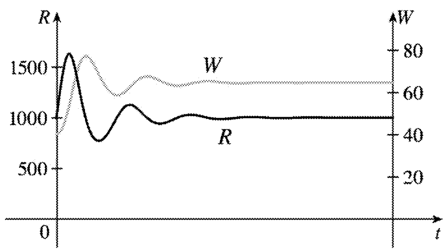
$R=\dfrac{0.02}{0.00002}=1000$. If $W=0$ in the first equation, then either $R=0$ or $R=\dfrac{1}{0.0002}=5000$. If $R=1000$, then $0=1000\left[0.08(1-0.0002\cdot 1000)-0.001W\right]\Leftrightarrow 0=80(1-0.2)-W\Leftrightarrow W=64$.

Case (i): $W=0$, $R=0$: both populations are zero

Case (ii): $W=0$, $R=5000$: see part (a)

Case (iii): $R=1000$, $W=64$: the predator/prey interaction balances and the populations are stable.

(c) The populations of wolves and rabbits fluctuate around 64 and 1000 , respectively, and eventually stabilize at those values.

(d) 

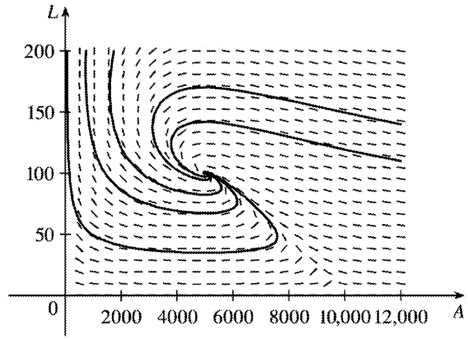
10. **(a)** If $L=0$, $dA/dt=2A(1-0.0001A)$, so $dA/dt=0\Leftrightarrow A=0$ or $A=\dfrac{1}{0.0001}=10,000$. Since $dA/dt>0$ for $0<A<10,000$, we expect the aphid population to *increase* to 10 , 000 for these values of A . Since $dA/dt<0$ for $A>10,000$, we expect the aphid population to *decrease* to 10 , 000 for these values of A . Hence, in the absence of ladybugs we expect the aphid population to stabilize at 10 , 000 .

(b) A and L are constant $\Rightarrow A'=0$ and $L'=0\Rightarrow$

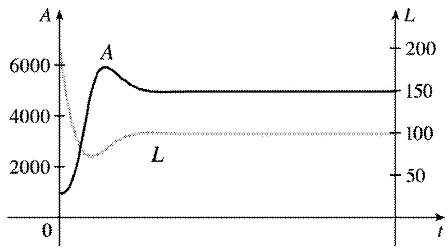
$$\left\{\begin{array}{l}0=2A(1-0.0001A)-0.01AL\\0=-0.5L+0.0001AL\end{array}\right\}\Rightarrow\left\{\begin{array}{l}0=A\left[2(1-0.0001A)-0.01L\right]\\0=L(-0.5+0.0001A)\end{array}\right.$$

The second equation is true if $L=0$ or $A=\dfrac{0.5}{0.0001}=5000$. If $L=0$ in the first equation, then either $A=0$ or $A=\dfrac{1}{0.0001}=10,000$. If $A=5000$, then $0=5000\left[2(1-0.0001\cdot 5000)-0.01L\right]\Leftrightarrow 0=10,000(1-0.5)-50L\Leftrightarrow 50L=5000\Leftrightarrow L=100$. The equilibrium solutions are: (i) $L=0, A=0$ (ii) $L=0, A=10,000$ (iii) $A=5000, L=100$

(c) $\dfrac{dL}{dA}=\dfrac{dL/dt}{dA/dt}=\dfrac{-0.5L+0.0001AL}{2A(1-0.0001A)-0.01AL}$

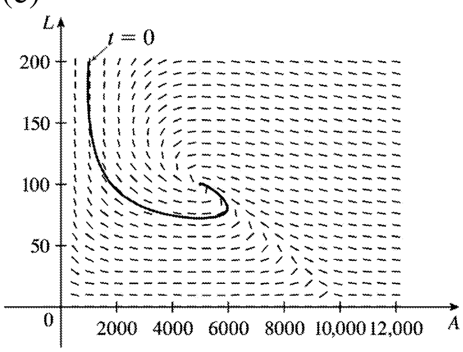


(d) All of the phase trajectories spiral tightly around the equilibrium solution $(5000,100)$.



The graph of A peaks just after the graph of L has a minimum.

(e)



At t=0 , the ladybug population decreases rapidly and the aphid population decreases slightly before beginning to increase. As the aphid population continues to increase, the ladybug population reaches a minimum at about $(5000,75)$. The ladybug population starts to increase and quickly stabilizes at 100 , while the aphid population stabilizes at 5000 .