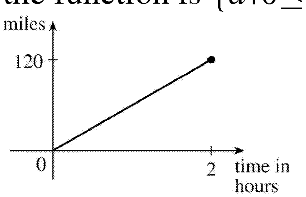
1. **(a)** The point $(-1,-2)$ is on the graph of f, so $f(-1)=-2$.
(b) When $x=2$, y is about 2.8, so $f(2)\approx 2.8$.
(c) $f(x)=2$ is equivalent to $y=2$. When $y=2$, we have $x=-3$ and $x=1$.
(d) Reasonable estimates for x when $y=0$ are $x=-2.5$ and $x=0.3$.
(e) The domain of f consists of all x-values on the graph of f. For this function, the domain is $-3\le x\le 3$, or $[-3,3]$. The range of f consists of all y-values on the graph of f. For this function, the range is $-2\le y\le 3$, or $[-2,3]$.
(f) As x increases from -1 to 3, y increases from -2 to 3. Thus, f is increasing on the interval $[-1,3]$.

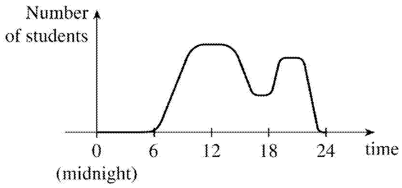
2. **(a)** The point $(-4,-2)$ is on the graph of f, so $f(-4)=-2$. The point $(3,4)$ is on the graph of g, so $g(3)=4$.
(b) We are looking for the values of x for which the y-values are equal. The y-values for f and g are equal at the points $(-2,1)$ and $(2,2)$, so the desired values of x are -2 and 2.
(c) $f(x)=-1$ is equivalent to $y=-1$. When $y=-1$, we have $x=-3$ and $x=4$.
(d) As x increases from 0 to 4, y decreases from 3 to -1. Thus, f is decreasing on the interval $[0,4]$.
(e) The domain of f consists of all x-values on the graph of f. For this function, the domain is $-4\le x\le 4$, or $[-4,4]$. The range of f consists of all y-values on the graph of f. For this function, the range is $-2\le y\le 3$, or $[-2,3]$.
(f) The domain of g is $[-4,3]$ and the range is $[0.5,4]$.

3. From Figure 1 in the text, the lowest point occurs at about $(t,a)=(12,-85)$. The highest point occurs at about $(17,115)$. Thus, the range of the vertical ground acceleration is $-85\le a\le 115$. In Figure 11, the range of the north–south acceleration is approximately $-325\le a\le 485$. In Figure 12, the range of the east–west acceleration is approximately $-210\le a\le 200$.

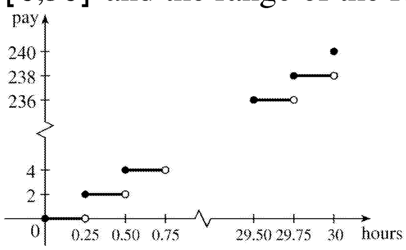
4. *Example 1:* A car is driven at 60 mi / h for 2 hours. The distance d traveled by the car is a function of the time t. The domain of the function is $\{t \mid 0\le t\le 2\}$, where t is measured in hours. The range of the function is $\{d \mid 0\le d\le 120\}$, where d is measured in miles.



Example 2: At a certain university, the number of students N on campus at any time on a particular day is a function of the time t after midnight. The domain of the function is $\{t \mid 0\le t\le 24\}$, where t is measured in hours. The range of the function is $\{N \mid 0\le N\le k\}$, where N is an integer and k is the largest number of students on campus at once.



Example 3: A certain employee is paid \$8.00 per hour and works a maximum of 30 hours per week. The number of hours worked is rounded down to the nearest quarter of an hour. This employee's gross weekly pay P is a function of the number of hours worked h . The domain of the function is $[0,30]$ and the range of the function is $\{0,2.00,4.00,\ldots,238.00,240.00\}$.



5. No, the curve is not the graph of a function because a vertical line intersects the curve more than once. Hence, the curve fails the Vertical Line Test.

6. Yes, the curve is the graph of a function because it passes the Vertical Line Test. The domain is $[-2,2]$ and the range is $[-1,2]$.

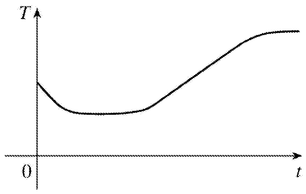
7. Yes, the curve is the graph of a function because it passes the Vertical Line Test. The domain is $[-3,2]$ and the range is $[-3,-2)\cup[-1,3]$.

8. No, the curve is not the graph of a function since for $x=0$, ± 1 , and ± 2 , there are infinitely many points on the curve.

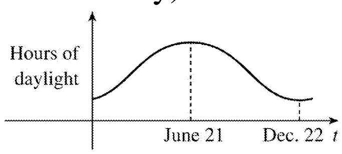
9. The person's weight increased to about 160 pounds at age 20 and stayed fairly steady for 10 years. The person's weight dropped to about 120 pounds for the next 5 years, then increased rapidly to about 170 pounds. The next 30 years saw a gradual increase to 190 pounds. Possible reasons for the drop in weight at 30 years of age: diet, exercise, health problems.

10. The salesman travels away from home from 8 to 9 A.M. and is then stationary until 10 : 00 . The salesman travels farther away from 10 until noon. There is no change in his distance from home until 1 : 00 , at which time the distance from home decreases until 3 : 00 . Then the distance starts increasing again, reaching the maximum distance away from home at 5 : 00 . There is no change from 5 until 6 , and then the distance decreases rapidly until 7 : 00 P.M., at which time the salesman reaches home.

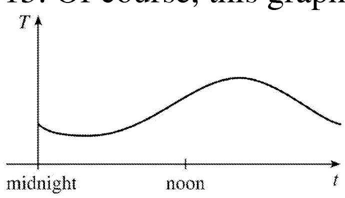
11. The water will cool down almost to freezing as the ice melts. Then, when the ice has melted, the water will slowly warm up to room temperature.



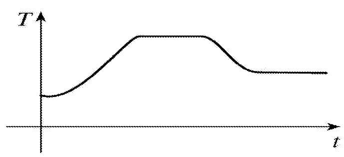
12. The summer solstice (the longest day of the year) is around June 21, and the winter solstice (the shortest day) is around December 22.

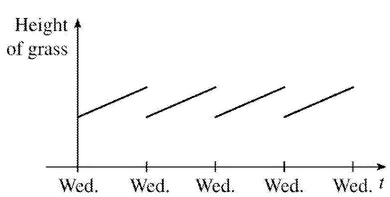


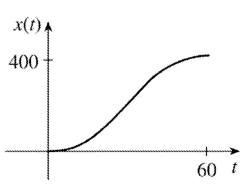
13. Of course, this graph depends strongly on the geographical location!

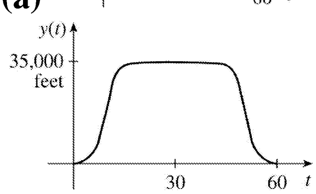


14. The temperature of the pie would increase rapidly, level off to oven temperature, decrease rapidly, and then level off to room temperature.

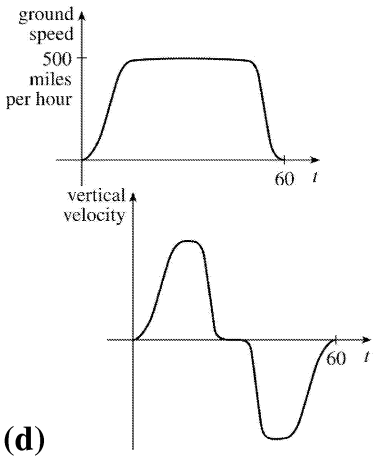


15. 

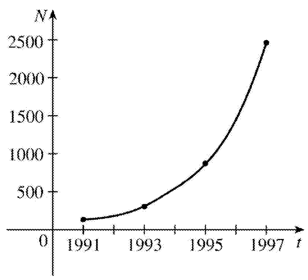
16. **(a)** 

(b) 

(c)

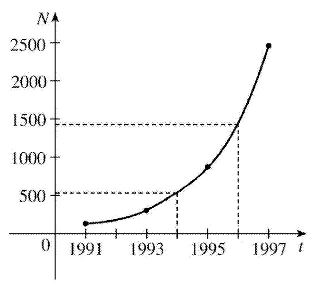


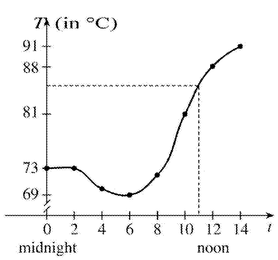
(d)



17. **(a)**

(b) From the graph, we estimate the number of cell–phone subscribers in Malaysia to be about 540 in 1994 and 1450 in 1996.





18. **(a)**

(b) From the graph in part (a), we estimate the temperature at 11:00 A.M. to be about 84.5 $^\circ$ C.

19. $f(x)=3x^2-x+2$.

$f(2)=3(2)^2-2+2=12-2+2=12$.

$f(-2)=3(-2)^2-(-2)+2=12+2+2=16$.

$f(a)=3a^2-a+2.$

$f(-a)=3(-a)^2-(-a)+2=3a^2+a+2.$

$f(a+1)=3(a+1)^2-(a+1)+2=3(a^2+2a+1)-a-1+2=3a^2+6a+3-a+1=3a^2+5a+4.$

$2f(a)=2\cdot f(a)=2(3a^2-a+2)=6a^2-2a+4.$

$f(2a)=3(2a)^2-(2a)+2=3(4a^2)-2a+2=12a^2-2a+2.$

$f(a^2)=3(a^2)^2-(a^2)+2=3(a^4)-a^2+2=3a^4-a^2+2.$

$$\begin{aligned}[f(a)]^2 &=\left[3a^2-a+2\right]^2=\left(3a^2-a+2\right)\left(3a^2-a+2\right)\\ &=9a^4-3a^3+6a^2-3a^3+a^2-2a+6a^2-2a+4=9a^4-6a^3+13a^2-4a+4.\end{aligned}$$

$f(a+h)=3(a+h)^2-(a+h)+2=3(a^2+2ah+h^2)-a-h+2=3a^2+6ah+3h^2-a-h+2.$

20. A spherical balloon with radius $r+1$ has volume $V(r+1)=\frac{4}{3}\pi(r+1)^3=\frac{4}{3}\pi\left(r^3+3r^2+3r+1\right)$. We wish to find the amount of air needed to inflate the balloon from a radius of r to $r+1$. Hence, we need to find the difference $V(r+1)-V(r)=\frac{4}{3}\pi\left(r^3+3r^2+3r+1\right)-\frac{4}{3}\pi r^3=\frac{4}{3}\pi\left(3r^2+3r+1\right)$.

21. $f(x)=x-x^2$, so $f(2+h)=2+h-(2+h)^2=2+h-(4+4h+h^2)=2+h-4-4h-h^2=-\left(h^2+3h+2\right)$, $f(x+h)=x+h-(x+h)^2=x+h-(x^2+2xh+h^2)=x+h-x^2-2xh-h^2$, and

$$\frac{f(x+h)-f(x)}{h}=\frac{x+h-x^2-2xh-h^2-x+x^2}{h}=\frac{h-2xh-h^2}{h}=\frac{h(1-2x-h)}{h}=1-2x-h.$$

22. $f(x)=\frac{x}{x+1}$, so $f(2+h)=\frac{2+h}{2+h+1}=\frac{2+h}{3+h}$, $f(x+h)=\frac{x+h}{x+h+1}$, and

$$\frac{f(x+h)-f(x)}{h}=\frac{\frac{x+h}{x+h+1}-\frac{x}{x+1}}{h}=\frac{(x+h)(x+1)-x(x+h+1)}{h(x+h+1)(x+1)}=\frac{1}{(x+h+1)(x+1)}.$$

23. $f(x)=x/(3x-1)$ is defined for all x except when $0=3x-1 \Leftrightarrow x=\frac{1}{3}$, so the domain is $\left\{x\in R \mid x\neq\frac{1}{3}\right\}=\left(-\infty,\frac{1}{3}\right)\cup\left(\frac{1}{3},\infty\right)$.

24. $f(x)=(5x+4)/\left(x^2+3x+2\right)$ is defined for all x except when $0=x^2+3x+2\Leftrightarrow 0=(x+2)(x+1)\Leftrightarrow x=-2$ or -1, so the domain is $\{x\in R \mid x\neq -2,-1\}=(-\infty,-2)\cup(-2,-1)\cup(-1,\infty)$.

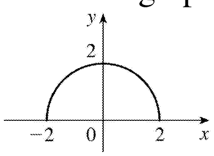
25.

$f(t)=\sqrt{t}+\sqrt[3]{t}$ is defined when $t\geq 0$. These values of t give real number results for $\sqrt{t}$, whereas any value of t gives a real number result for $\sqrt[3]{t}$. The domain is $[0,\infty)$.

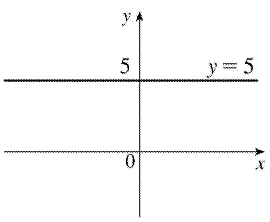
26. $g(u)=\sqrt{u}+\sqrt{4-u}$ is defined when $u\geq 0$ and $4-u\geq 0 \Leftrightarrow u\leq 4$. Thus, the domain is $0\leq u\leq 4=[0,4]$.

27. $h(x)=1/\sqrt[4]{x^2-5x}$ is defined when $x^2-5x>0 \Leftrightarrow x(x-5)>0$. Note that $x^2-5x\neq 0$ since that would result in division by zero. The expression $x(x-5)$ is positive if $x<0$ or $x>5$. (See Appendix A for methods for solving inequalities.) Thus, the domain is $(-\infty,0)\cup(5,\infty)$.

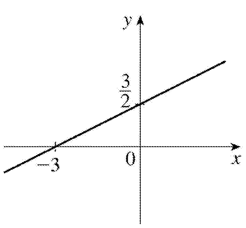
28. $h(x)=\sqrt{4-x^2}$. Now $y=\sqrt{4-x^2} \Rightarrow y^2=4-x^2 \Leftrightarrow x^2+y^2=4$, so the graph is the top half of a circle of radius 2 with center at the origin. The domain is $\left\{ x \mid 4-x^2\geq 0\right\} = \left\{ x \mid 4\geq x^2\right\} =\{x \mid 2\geq |x|\}=[-2,2]$. From the graph, the range is $0\leq y\leq 2$, or $[0,2]$.



29. $f(x)=5$ is defined for all real numbers, so the domain is R , or $(-\infty,\infty)$. The graph of f is a horizontal line with y –intercept 5 .

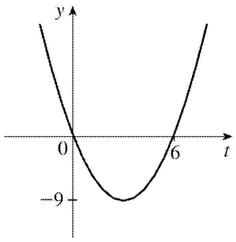


30. $F(x)=\frac{1}{2}(x+3)$ is defined for all real numbers, so the domain is R , or $(-\infty,\infty)$. The graph of F is a line with x –intercept -3 and y –intercept $\frac{3}{2}$.

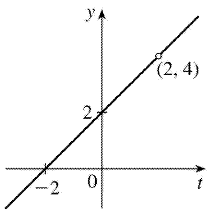


31. $f(t)=t^2-6t$ is defined for all real numbers, so the domain is R , or $(-\infty,\infty)$. The graph of f is a parabola opening upward since the coefficient of t^2 is positive. To find the t –intercepts, let $y=0$ and solve for t . $0=t^2-6t=t(t-6) \Rightarrow t=0$ and $t=6$. The t –coordinate of the vertex is halfway between the t –

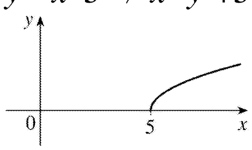
intercepts, that is, at $t=3$. Since $f(3)=3^2-6\cdot 3=-9$, the vertex is $(3,-9)$.



32. $H(t)=\dfrac{4-t^2}{2-t}=\dfrac{(2+t)(2-t)}{2-t}$, so for $t\neq 2$, $H(t)=2+t$. The domain is $\{t\,|\,t\neq 2\}$. So the graph of H is the same as the graph of the function $f(t)=t+2$ (a line) except for the hole at $(2,4)$.



33. $g(x)=\sqrt{x-5}$ is defined when $x-5\geq 0$ or $x\geq 5$, so the domain is $[5,\infty)$. Since $y=\sqrt{x-5}\Rightarrow$ $y^2=x-5\Rightarrow x=y^2+5$, we see that g is the top half of a parabola.

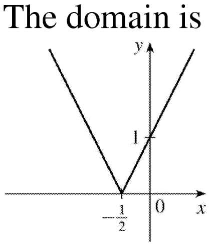


34.

$$F(x)=|2x+1|=\begin{cases} 2x+1 & \text{if } 2x+1\geq 0 \\ -(2x+1) & \text{if } 2x+1<1 \end{cases}$$

$$=\begin{cases} 2x+1 & \text{if } x\geq -\frac{1}{2} \\ -2x-1 & \text{if } x<-\frac{1}{2} \end{cases}$$

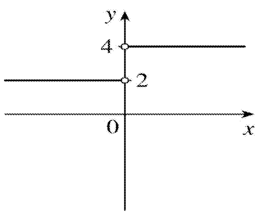
The domain is R , or $(-\infty,\infty)$.



35.

$G(x)=\dfrac{3x+|x|}{x}$. Since $|x|=\begin{cases} x & \text{if } x\geq 0 \\ -x & \text{if } x<0 \end{cases}$, we have

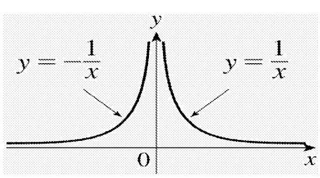
$$G(x)=\begin{cases} \dfrac{3x+x}{x} & \text{if } x>0 \\ \dfrac{3x-x}{x} & \text{if } x<0 \end{cases} = \begin{cases} \dfrac{4x}{x} & \text{if } x>0 \\ \dfrac{2x}{x} & \text{if } x<0 \end{cases} = \begin{cases} 4 & \text{if } x>0 \\ 2 & \text{if } x<0 \end{cases}$$



Note that G is not defined for $x=0$. The domain is $(-\infty,0)\cup(0,\infty)$.

36. $g(x)=\dfrac{|x|}{x^2}$. Since $|x|=\begin{cases} x & \text{if } x\geq 0 \\ -x & \text{if } x<0 \end{cases}$, we have

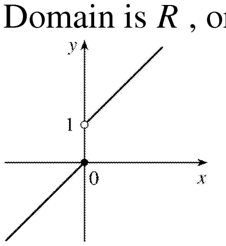
$$g(x)=\begin{cases} \dfrac{x}{x^2} & \text{if } x>0 \\ \dfrac{-x}{x^2} & \text{if } x<0 \end{cases} = \begin{cases} \dfrac{1}{x} & \text{if } x>0 \\ \dfrac{1}{-x} & \text{if } x<0 \end{cases}$$



Note that g is not defined for $x=0$. The domain is $(-\infty,0)\cup(0,\infty)$.

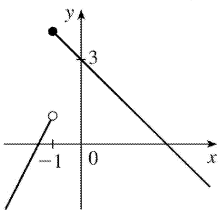
37. $f(x)=\begin{cases} x & \text{if } x\leq 0 \\ x+1 & \text{if } x>0 \end{cases}$

Domain is R , or $(-\infty,\infty)$.



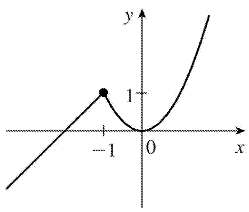
38. $f(x)=\begin{cases} 2x+3 & \text{if } x<-1 \\ 3-x & \text{if } x\geq -1 \end{cases}$

Domain is R , or $(-\infty,\infty)$.



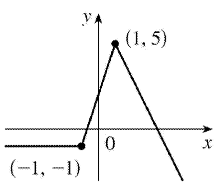
39. $f(x)=\begin{cases} x+2 & \text{if } x\leq -1 \\ x^2 & \text{if } x>-1 \end{cases}$

Note that for $x=-1$, both $x+2$ and x^2 are equal to 1. Domain is R .



40. $f(x)=\begin{cases} -1 & \text{if } x\leq -1 \\ 3x+2 & \text{if } -1<x<1 \\ 7-2x & \text{if } x\geq 1 \end{cases}$

Domain is R .



41. Recall that the slope m of a line between the two points (x_1,y_1) and (x_2,y_2) is $m=\frac{y_2-y_1}{x_2-x_1}$ and an equation of the line connecting those two points is $y-y_1=m(x-x_1)$. The slope of this line segment is $\frac{-6-1}{4-(-2)}=-\frac{7}{6}$, so an equation is $y-1=-\frac{7}{6}(x+2)$. The function is $f(x)=-\frac{7}{6}x-\frac{4}{3}$, $-2\leq x\leq 4$.

42. The slope of this line segment is $\frac{3-(-2)}{6-(-3)}=\frac{5}{9}$, so an equation is $y+2=\frac{5}{9}(x+3)$. The function is $f(x)=\frac{5}{9}x-\frac{1}{3}$, $-3\leq x\leq 6$.

43. We need to solve the given equation for y .

$x+(y-1)^2=0 \Leftrightarrow (y-1)^2=-x \Leftrightarrow y-1=\pm\sqrt{-x} \Leftrightarrow y=1\pm\sqrt{-x}$. The expression with the positive radical represents the top half of the parabola, and the one with the negative radical represents the bottom half. Hence, we want $f(x)=1-\sqrt{-x}$. Note that the domain is $x\leq 0$.

44. $(x-1)^2+y^2=1 \Leftrightarrow y=\pm\sqrt{1-(x-1)^2}=\pm\sqrt{2x-x^2}$. The top half is given by the function $f(x)=\sqrt{2x-x^2}$, $0\leq x\leq 2$.

45. For $-1\leq x\leq 2$, the graph is the line with slope 1 and $y-$ intercept 1 , that is, the line $y=x+1$. For $2<x\leq 4$, the graph is the line with slope $-\frac{3}{2}$ and $x-$ intercept 4 , so $y-0=-\frac{3}{2}(x-4)=-\frac{3}{2}x+6$. So the function is $f(x)=\begin{cases} x+1 & \text{if } -1\leq x\leq 2 \\ -\frac{3}{2}x+6 & \text{if } 2<x\leq 4 \end{cases}$

46. For $x\leq 0$, the graph is the line $y=2$. For $0<x\leq 1$, the graph is the line with slope -2 and $y-$ intercept 2 , that is, the line $y=-2x+2$. For $x>1$, the graph is the line with slope 1 and $x-$ intercept 1 , that is, the line $y=1(x-1)=x-1$. So the function is $f(x)=\begin{cases} 2 & \text{if } x\leq 0 \\ -2x+2 & \text{if } 0<x\leq 1 \\ x-1 & \text{if } 1<x \end{cases}$.

47. Let the length and width of the rectangle be L and W . Then the perimeter is $2L+2W=20$ and the area is $A=LW$. Solving the first equation for W in terms of L gives $W=\frac{20-2L}{2}=10-L$. Thus, $A(L)=L(10-L)=10L-L^2$. Since lengths are positive, the domain of A is $0<L<10$. If we further restrict L to be larger than W , then $5<L<10$ would be the domain.

48. Let the length and width of the rectangle be L and W . Then the area is $LW=16$, so that $W=16/L$. The perimeter is $P=2L+2W$, so $P(L)=2L+2(16/L)=2L+32/L$, and the domain of P is $L>0$, since lengths must be positive quantities. If we further restrict L to be larger than W , then $L>4$ would be the domain.

49. Let the length of a side of the equilateral triangle be x . Then by the Pythagorean Theorem, the height y of the triangle satisfies $y^2+\left(\frac{1}{2}x\right)^2=x^2$, so that $y^2=x^2-\frac{1}{4}x^2=\frac{3}{4}x^2$ and $y=\frac{\sqrt{3}}{2}x$. Using the formula for the area A of a triangle, $A=\frac{1}{2}(\text{base})(\text{height})$, we obtain $A(x)=\frac{1}{2}(x)\left(\frac{\sqrt{3}}{2}x\right)=\frac{\sqrt{3}}{4}x^2$, with domain $x>0$.

50. Let the volume of the cube be V and the length of an edge be L . Then $V=L^3$ so $L=\sqrt[3]{V}$, and the

surface area is $S(V)=6\left(\sqrt[3]{V}\right)^2=6V^{2/3}$, with domain $V>0$.

51. Let each side of the base of the box have length x , and let the height of the box be h . Since the volume is 2 , we know that $2=hx^2$, so that $h=2/x^2$, and the surface area is $S=x^2+4xh$. Thus, $S(x)=x^2+4x(2/x^2)=x^2+(8/x)$, with domain $x>0$.

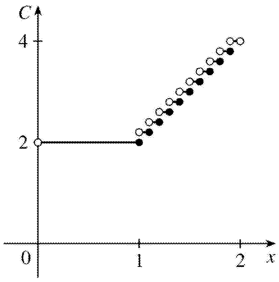
52. The area of the window is $A=xh+\frac{1}{2}\pi\left(\frac{1}{2}x\right)^2=xh+\frac{\pi x^2}{8}$, where h is the height of the rectangular portion of the window. The perimeter is $P=2h+x+\frac{1}{2}\pi x=30\Leftrightarrow 2h=30-x-\frac{1}{2}\pi x\Leftrightarrow$ $h=\frac{1}{4}(60-2x-\pi x)$. Thus,

$$A(x)=x\frac{60-2x-\pi x}{4}+\frac{\pi x^2}{8}=15x-\frac{1}{2}x^2-\frac{\pi}{4}x^2+\frac{\pi}{8}x^2=15x-\frac{4}{8}x^2-\frac{\pi}{8}x^2=15x-x^2\left(\frac{\pi+4}{8}\right)$$

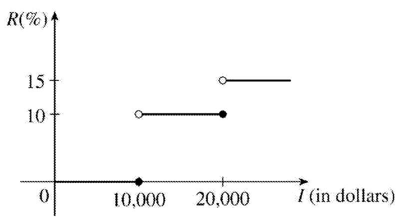
Since the lengths x and h must be positive quantities, we have $x>0$ and $h>0$. For $h>0$, we have $2h>0$ $\Leftrightarrow 30-x-\frac{1}{2}\pi x>0\Leftrightarrow 60>2x+\pi x\Leftrightarrow x<\frac{60}{2+\pi}$. Hence, the domain of A is $0<x<\frac{60}{2+\pi}$.

53. The height of the box is x and the length and width are $L=20-2x$, $W=12-2x$. Then $V=LWx$ and so
$V(x)=(20-2x)(12-2x)(x)=4(10-x)(6-x)(x)=4x\left(60-16x+x^2\right)=4x^3-64x^2+240x$.
The sides L , W , and x must be positive. Thus, $L>0\Leftrightarrow 20-2x>0\Leftrightarrow x<10$; $W>0\Leftrightarrow 12-2x>0\Leftrightarrow x<6$; and $x>0$. Combining these restrictions gives us the domain $0<x<6$.
54.

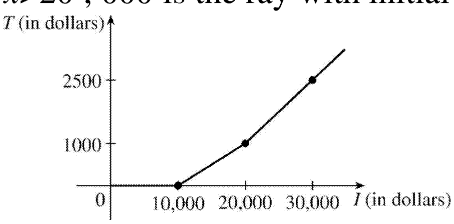
$$C(x)=\begin{cases} \$2.00 & \text{if } 0.0<x\le 1.0 \\ 2.20 & \text{if } 1.0<x\le 1.1 \\ 2.40 & \text{if } 1.1<x\le 1.2 \\ 2.60 & \text{if } 1.2<x\le 1.3 \\ 2.80 & \text{if } 1.3<x\le 1.4 \\ 3.00 & \text{if } 1.4<x\le 1.5 \\ 3.20 & \text{if } 1.5<x\le 1.6 \\ 3.40 & \text{if } 1.6<x\le 1.7 \\ 3.60 & \text{if } 1.7<x\le 1.8 \\ 3.80 & \text{if } 1.8<x\le 1.9 \\ 4.00 & \text{if } 1.9<x<2.0 \end{cases}$$



55. **(a)**



(b) On \$14 , 000 , tax is assessed on \$4000 , and 10%(\$4000)=\$400 .
On \$26 , 000 , tax is assessed on \$16 , 000 , and 10%(\$10 , 000)+15%(\$6000)=\$1000+\$900=\$1900 .
(c) As in part (b), there is \$1000 tax assessed on \$20,000 of income, so the graph of T is a line segment from (10 , 000,0) to (20 , 000,1000) . The tax on \$30,000 is \$2500, so the graph of T for x>20 , 000 is the ray with initial point (20 , 000,1000) that passes through (30 , 000,2500) .



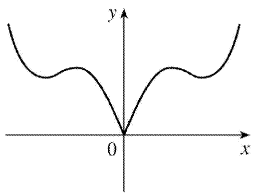
56. One example is the amount paid for cable or telephone system repair in the home, usually measured to the nearest quarter hour. Another example is the amount paid by a student in tuition fees, if the fees vary according to the number of credits for which the student has registered.

57. f is an odd function because its graph is symmetric about the origin. g is an even function because its graph is symmetric with respect to the y –axis.

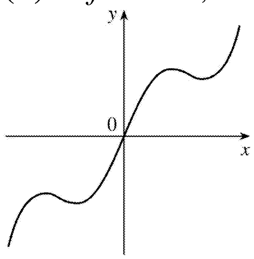
58. f is not an even function since it is not symmetric with respect to the y –axis. f is not an odd function since it is not symmetric about the origin. Hence, f is *neither* even nor odd. g is an even function because its graph is symmetric with respect to the y –axis.

59. **(a)** Because an even function is symmetric with respect to the y –axis, and the point $(5,3)$ is on the graph of this even function, the point $(-5,3)$ must also be on its graph.
(b) Because an odd function is symmetric with respect to the origin, and the point $(5,3)$ is on the graph of this odd function, the point $(-5,-3)$ must also be on its graph.

60. **(a)** If f is even, we get the rest of the graph by reflecting about the y –axis.



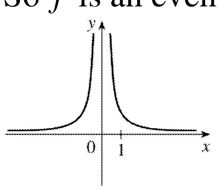
(b) If f is odd, we get the rest of the graph by rotating 180° about the origin.



61. $f(x)=x^{-2}$.

$$f(-x)=(-x)^{-2}=\frac{1}{(-x)^{2}}=\frac{1}{x^{2}}$$
$$=x^{-2}=f(x)$$

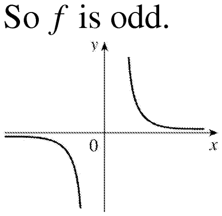
So f is an even function.



62. $f(x)=x^{-3}$.

$$f(-x)=(-x)^{-3}=\frac{1}{(-x)^{3}}=\frac{1}{-x^{3}}$$
$$=-\frac{1}{x^{3}}=-\left(x^{-3}\right)=-f(x)$$

So f is odd.



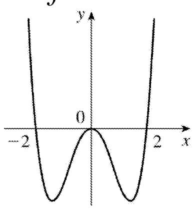
63. $f(x)=x^{2}+x$, so $f(-x)=(-x)^{2}+(-x)=x^{2}-x$. Since this is neither $f(x)$ nor $-f(x)$, the function f is

neither even nor odd.

64. $f(x)=x^4-4x^2$.

$$f(-x) =(-x)^4-4(-x)^2 =x^4-4x^2=f(x)$$

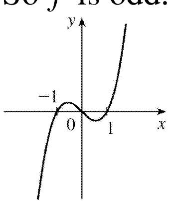
So f is even.



65. $f(x)=x^3-x$.

$$f(-x) =(-x)^3-(-x)=-x^3+x =-\left(x^3-x\right)=-f(x)$$

So f is odd.



66. $f(x)=3x^3+2x^2+1$, so $f(-x)=3(-x)^3+2(-x)^2+1=-3x^3+2x^2+1$. Since this is neither $f(x)$ nor $-f(x)$, the function f is neither even nor odd.

1. **(a)** $f(x)=\sqrt[5]{x}$ is a root function with $n=5$.

(b) $g(x)=\sqrt{1-x^2}$ is an algebraic function because it is a root of a polynomial.

(c) $h(x)=x^9+x^4$ is a polynomial of degree 9 .

(d) $r(x)=\dfrac{x^2+1}{x^3+x}$ is a rational function because it is a ratio of polynomials.

(e) $s(x)=\tan\ 2x$ is a trigonometric function.

(f) $t(x)=\log_{10}\ x$ is a logarithmic function.

2. **(a)** $y=(x-6)\,/\,(x+6)$ is a rational function because it is a ratio of polynomials.

(b) $y=x+x^2\,/\sqrt{x-1}$ is an algebraic function because it involves polynomials and roots of polynomials.

(c) $y=10^x$ is an exponential function (notice that x is the *exponent*).

(d) $y=x^{10}$ is a power function (notice that x is the *base*).

(e) $y=2t^6+t^4-\pi$ is a polynomial of degree 6 .

(f) $y=\cos\ \theta+\sin\ \theta$ is a trigonometric function.

3. We notice from the figure that g and h are even functions (symmetric with respect to the y –axis) and that f is an odd function (symmetric with respect to the origin). So (b) $\left[y=x^5\right]$ must be f . Since g is flatter than h near the origin, we must have (c) $\left[y=x^8\right]$ matched with g and (a) $\left[y=x^2\right]$ matched with h .

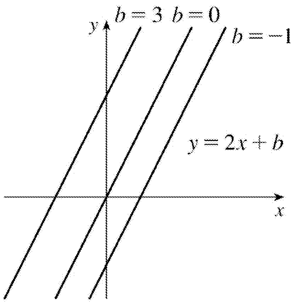
4. **(a)** The graph of $y=3x$ is a line (choice G).

(b) $y=3^x$ is an exponential function (choice f).

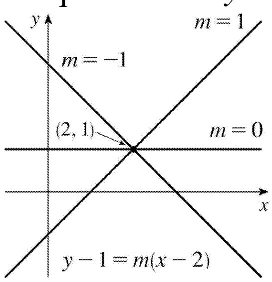
(c) $y=x^3$ is an odd polynomial function or power function (choice F).

(d) $y=\sqrt[3]{x}=x^{1/3}$ is a root function (choice g).

5. **(a)** An equation for the family of linear functions with slope 2 is $y=f(x)=2x+b$, where b is the y –intercept.

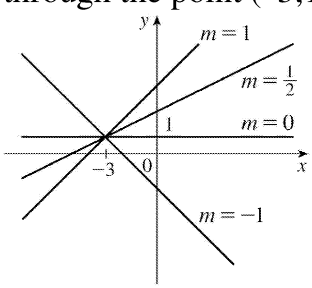


(b) $f(2)=1$ means that the point $(2,1)$ is on the graph of f . We can use the point–slope form of a line to obtain an equation for the family of linear functions through the point $(2,1)$. $y-1=m(x-2)$, which is equivalent to $y=mx+(1-2m)$ in slope–intercept form.

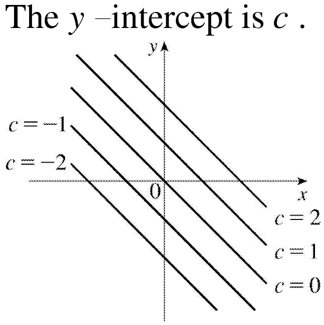


(c) To belong to both families, an equation must have slope $m=2$, so the equation in part (b), $y=mx+(1-2m)$, becomes $y=2x-3$. It is the *only* function that belongs to both families.

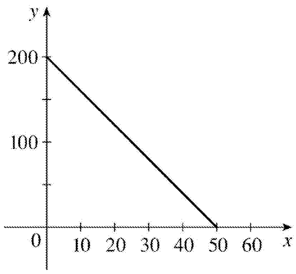
6. All members of the family of linear functions $f(x)=1+m(x+3)$ have graphs that are lines passing through the point $(-3,1)$.



7. All members of the family of linear functions $f(x)=c-x$ have graphs that are lines with slope -1 . The y –intercept is c .

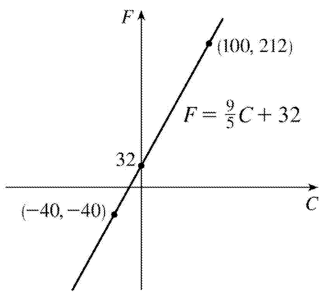


8. **(a)**



(b) The slope of -4 means that for each increase of 1 dollar for a rental space, the number of spaces rented *decreases* by 4 . The y –intercept of 200 is the number of spaces that would be occupied if there were no charge for each space. The x –intercept of 50 is the smallest rental fee that results in no spaces rented.

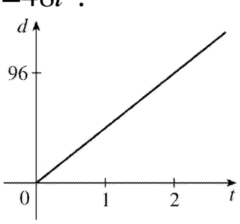
9. **(a)**



(b) The slope of $\frac{9}{5}$ means that F increases $\frac{9}{5}$ degrees for each increase of 1° C. (Equivalently, F increases by 9 when C increases by 5 and F decreases by 9 when C decreases by 5 .) The F –intercept of 32 is the Fahrenheit temperature corresponding to a Celsius temperature of 0 .

10. **(a)** Let $d=$ distance traveled (in miles) and $t=$ time elapsed (in hours). At $t=0$, $d=0$ and at $t=50$ minutes $=50\cdot\frac{1}{60}=\frac{5}{6}$ h, $d=40$. Thus we have two points: $(0,0)$ and $\left(\frac{5}{6},40\right)$, so $m=\frac{40-0}{\frac{5}{6}-0}=48$ and so $d=48t$.

(b)



(c) The slope is 48 and represents the car’s speed in mi / h.

11. **(a)** Using N in place of x and T in place of y , we find the slope to be $\frac{T_2-T_1}{N_2-N_1}=\frac{80-70}{173-113}=\frac{10}{60}=\frac{1}{6}$. So a linear equation is $T-80=\frac{1}{6}(N-173)\Leftrightarrow T-80=\frac{1}{6}N-\frac{173}{6}\Leftrightarrow$

$T=\frac{1}{6}N+\frac{307}{6}\left[\frac{307}{6}=51.1\overline{6}\right]$.

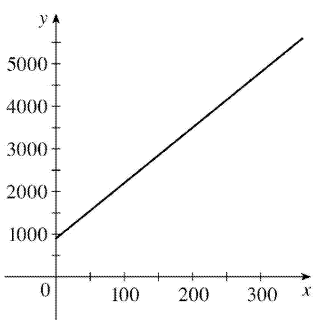
(b) The slope of $\frac{1}{6}$ means that the temperature in Fahrenheit degrees increases one–sixth as rapidly as the number of cricket chirps per minute. Said differently, each increase of 6 cricket chirps per minute corresponds to an increase of 1° F.

(c) When $N=150$, the temperature is given approximately by $T=\frac{1}{6}(150)+\frac{307}{6}=76.1\overline{6}^\circ$ F $\approx 76^\circ$ F.

12. **(a)** Let x denote the number of chairs produced in one day and y the associated cost. Using the points $(100,2200)$ and $(300,4800)$ we get the slope $\frac{4800-2200}{300-100}=\frac{2600}{200}=13$. So $y-2200=13(x-100)$ $\Leftrightarrow y=13x+900$.

(b) The slope of the line in part (a) is 13 and it represents the cost (in dollars) of producing each additional chair.

(c) The y –intercept is 900 and it represents the fixed daily costs of operating the factory.



13. **(a)** We are given $\frac{\text{change in pressure}}{10 \text{ feet change in depth}}=\frac{4.34}{10}=0.434$. Using P for pressure and d for depth with the point $(d,P)=(0,15)$, we have the slope–intercept form of the line, $P=0.434d+15$.

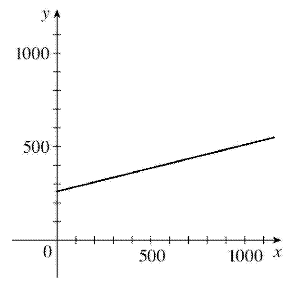
(b) When $P=100$, then $100=0.434d+15 \Leftrightarrow 0.434d=85 \Leftrightarrow d=\frac{85}{0.434}\approx 195.85$ feet. Thus, the pressure is 100 lb/in^2 at a depth of approximately 196 feet.

14. **(a)** Using d in place of x and C in place of y , we find the slope to be

$$\frac{C_2-C_1}{d_2-d_1}=\frac{460-380}{800-480}=\frac{80}{320}=\frac{1}{4}$$

So a linear equation is $C-460=\frac{1}{4}(d-800) \Leftrightarrow C-460=\frac{1}{4}d-200 \Leftrightarrow C=\frac{1}{4}d+260$.

(b) Letting $d=1500$ we get $C=\frac{1}{4}(1500)+260=635$. The cost of driving 1500 miles is \$635.

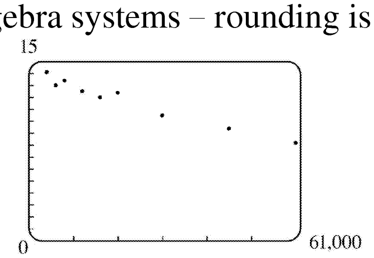


(c)
The slope of the line represents the cost per mile, \$0.25 .
(d) The y –intercept represents the fixed cost, \$260 .
(e) A linear function gives a suitable model in this situation because you have fixed monthly costs such as insurance and car payments, as well as costs that increase as you drive, such as gasoline, oil, and tires, and the cost of these for each additional mile driven is a constant.

15. **(a)** The data appear to be periodic and a sine or cosine function would make the best model. A model of the form $f(x)=a\cos(bx)+c$ seems appropriate.
(b) The data appear to be decreasing in a linear fashion. A model of the form $f(x)=mx+b$ seems appropriate.

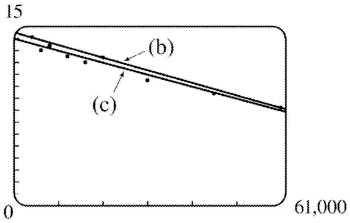
16. **(a)** The data appear to be increasing exponentially. A model of the form $f(x)=a\cdot b^x$ or $f(x)=a\cdot b^x+c$ seems appropriate.
(b) The data appear to be decreasing similarly to the values of the reciprocal function. A model of the form $f(x)=a/x$ seems appropriate.

17. Some values are given to many decimal places. These are the results given by several computer algebra systems – rounding is left to the reader.



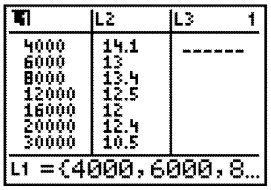
(a)
A linear model does seem appropriate.

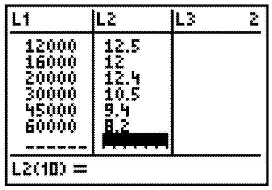
(b) Using the points $(4000,14.1)$ and $(60{,}000,8.2)$, we obtain $y-14.1=\frac{8.2-14.1}{60{,}000-4000}(x-4000)$ or, equivalently, $y\approx -0.000105357x+14.521429$.



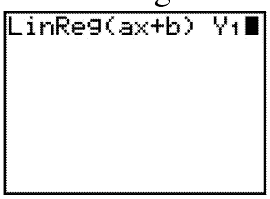
(c) Using a computing device, we obtain the least squares regression line $y = -0.0000997855x + 13.950764$.

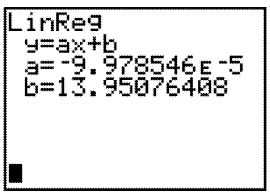
The following commands and screens illustrate how to find the least squares regression line on a TI–83 Plus. Enter the data into list one (L1) and list two (L2). Press STAT 1 to enter the editor.

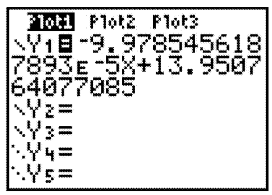




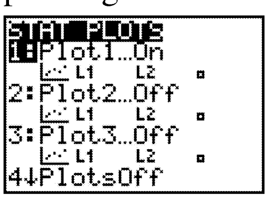
Find the regession line and store it in Y_1 . Press 2nd QUIT STAT ▸ 4 VARS ▸ 1 1 ENTER .

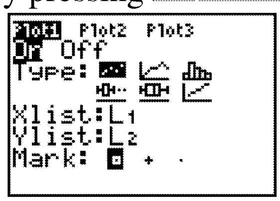




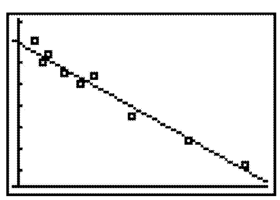


Note from the last figure that the regression line has been stored in Y_1 and that Plot1 has been turned on (Plot1 is highlighted). You can turn on Plot1 from the Y= menu by placing the cursor on Plot1 and pressing ENTER or by pressing 2nd STAT PLOT 1 ENTER .





Now press ZOOM 9 to produce a graph of the data and the regression line. Note that choice 9 of the ZOOM menu automatically selects a window that displays all of the data.

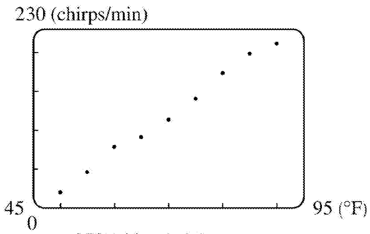


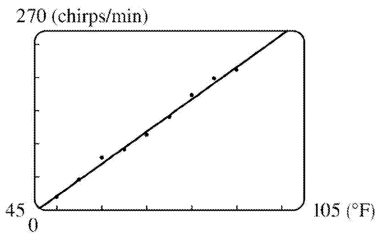
(d) When $x = 25,000$, $y \approx 11.456$; or about 11.5 per 100 population.

(e) When $x = 80,000$, $y \approx 5.968$; or about a 6% chance.

(f) When $x = 200,000$, y is negative, so the model does not apply.

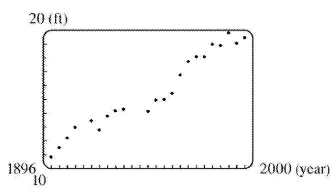
18. **(a)**



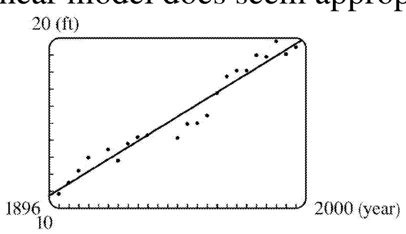


(b)
Using a computing device, we obtain the least squares regression line $y=4.85\overline{6}x-220.9\overline{6}$.

(c) When $x=100^{\circ}$ F, $y=264.7\approx 265$ chirps / min.

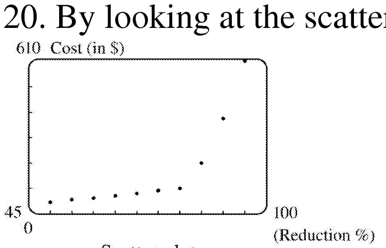


19. **(a)**
A linear model does seem appropriate.



(b)
Using a computing device, we obtain the least squares regression line $y=0.089119747x-158.2403249$, where x is the year and y is the height in feet.
(c) When $x=2000$, the model gives $y\approx 20.00$ ft. Note that the actual winning height for the 2000 Olympics is *less than* the winning height for 1996 – so much for that prediction.
(d) When $x=2100$, $y\approx 28.91$ ft. This would be an increase of 9.49 ft from 1996 to 2100. Even though there was an increase of 8.59 ft from 1900 to 1996, it is unlikely that a similar increase will occur over the next 100 years.

20. By looking at the scatter plot of the data, we rule out the linear and logarithmic models.



Scatter plot

We try various models:

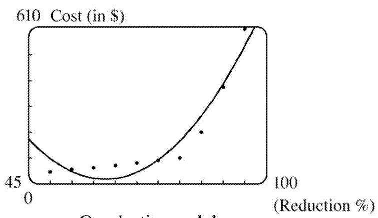
Quadratic: $y=0.49\overline{6}x^2-62.28\overline{93}x+1970.6\overline{39}$

Cubic: $y=0.0201243201x^3-3.88037296x^2+247.6754468x-5163.935198$

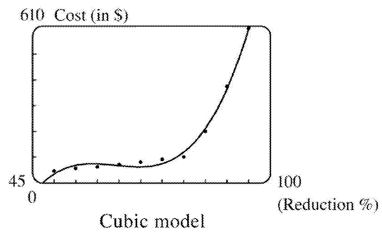
Quartic: $y=0.0002951049x^4-0.0654560995x^3+5.27525641x^2-180.2266511x+2203.210956$

Exponential: $y=2.41422994(1.054516914)^x$

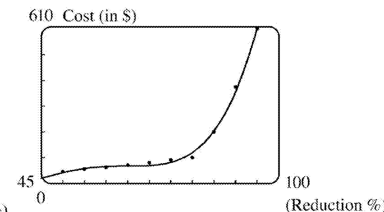
Power: $y=0.000022854971x^{3.616078251}$



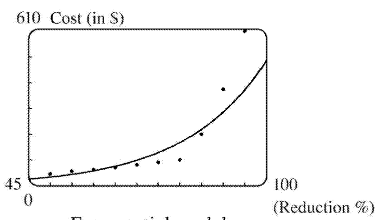
Quadratic model



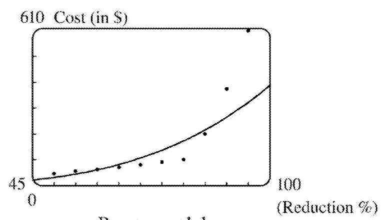
Cubic model



Quartic model

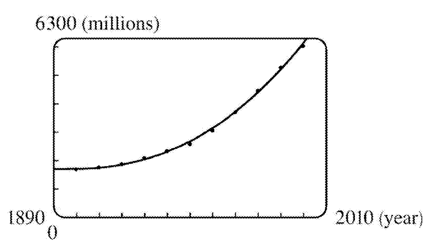


Exponential model



Power model

After examining the graphs of these models, we see that the cubic and quartic models are clearly the best.

21. 

Using a computing device, we obtain the cubic function $y=ax^3+bx^2+cx+d$ with $a=0.0012937$, $b=-7.06142$, $c=12$, 823 , and $d=-7$, 743 , 770 . When $x=1925$, $y\approx 1914$ (million).

22. **(a)** $T=1.000396048d^{1.499661718}$

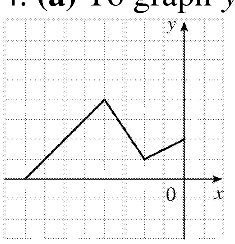
(b) The power model in part (a) is approximately $T=d^{1.5}$. Squaring both sides gives us $T^2=d^3$, so the model matches Kepler's Third Law, $T^2=kd^3$.

1. **(a)** If the graph of f is shifted 3 units upward, its equation becomes $y=f(x)+3$.
(b) If the graph of f is shifted 3 units downward, its equation becomes $y=f(x)-3$.
(c) If the graph of f is shifted 3 units to the right, its equation becomes $y=f(x-3)$.
(d) If the graph of f is shifted 3 units to the left, its equation becomes $y=f(x+3)$.
(e) If the graph of f is reflected about the x –axis, its equation becomes $y=-f(x)$.
(f) If the graph of f is reflected about the y –axis, its equation becomes $y=f(-x)$.
(g) If the graph of f is stretched vertically by a factor of 3 , its equation becomes $y=3f(x)$.
(h) If the graph of f is shrunk vertically by a factor of 3 , its equation becomes $y=\frac{1}{3}f(x)$.

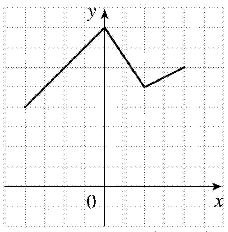
2. **(a)** To obtain the graph of $y=5f(x)$ from the graph of $y=f(x)$, stretch the graph vertically by a factor of 5 .
(b) To obtain the graph of $y=f(x-5)$ from the graph of $y=f(x)$, shift the graph 5 units to the right.
(c) To obtain the graph of $y=-f(x)$ from the graph of $y=f(x)$, reflect the graph about the x –axis.
(d) To obtain the graph of $y=-5f(x)$ from the graph of $y=f(x)$, stretch the graph vertically by a factor of 5 and reflect it about the x –axis.
(e) To obtain the graph of $y=f(5x)$ from the graph of $y=f(x)$, shrink the graph horizontally by a factor of 5 .
(f) To obtain the graph of $y=5f(x)-3$ from the graph of $y=f(x)$, stretch the graph vertically by a factor of 5 and shift it 3 units downward.

3. **(a)** (graph 3) The graph of f is shifted 4 units to the right and has equation $y=f(x-4)$.
(b) (graph 1) The graph of f is shifted 3 units upward and has equation $y=f(x)+3$.
(c) (graph 4) The graph of f is shrunk vertically by a factor of 3 and has equation $y=\frac{1}{3}f(x)$.
(d) (graph 5) The graph of f is shifted 4 units to the left and reflected about the x –axis. Its equation is $y=-f(x+4)$.
(e) (graph 2) The graph of f is shifted 6 units to the left and stretched vertically by a factor of 2 . Its equation is $y=2f(x+6)$.

4. **(a)** To graph $y=f(x+4)$ we shift the graph of f , 4 units to the left.

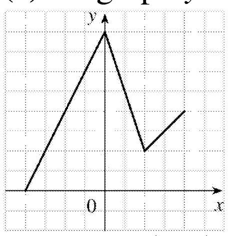


The point $(2,1)$ on the graph of f corresponds to the point $(2-4,1)=(-2,1)$.
(b) To graph $y=f(x)+4$ we shift the graph of f , 4 units upward.



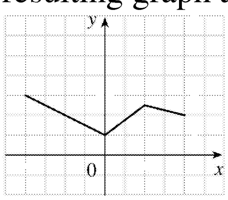
The point $(2,1)$ on the graph of f corresponds to the point $(2,1+4)=(2,5)$.

(c) To graph $y=2f(x)$ we stretch the graph of f vertically by a factor of 2 .



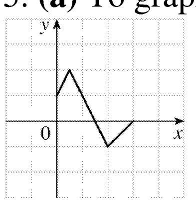
The point $(2,1)$ on the graph of f corresponds to the point $(2,2\cdot 1)=(2,2)$.

(d) To graph $y=-\frac{1}{2}f(x)+3$, we shrink the graph of f vertically by a factor of 2 , then reflect the resulting graph about the x –axis, then shift the resulting graph 3 units upward.



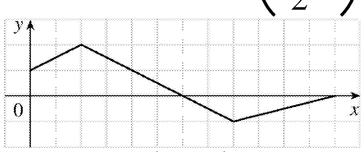
The point $(2,1)$ on the graph of f corresponds to the point $\left(2,-\frac{1}{2}\cdot 1+3\right)=(2,2.5)$.

5. **(a)** To graph $y=f(2x)$ we shrink the graph of f horizontally by a factor of 2 .



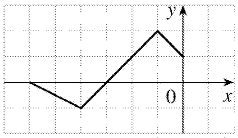
The point $(4,-1)$ on the graph of f corresponds to the point $\left(\frac{1}{2}\cdot 4,-1\right)=(2,-1)$.

(b) To graph $y=f\left(\frac{1}{2}x\right)$ we stretch the graph of f horizontally by a factor of 2 .

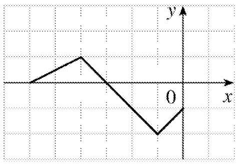


The point $(4,-1)$ on the graph of f corresponds to the point $(2\cdot 4,-1)=(8,-1)$.

(c) To graph $y=f(-x)$ we reflect the graph of f about the y –axis.



The point $(4,-1)$ on the graph of f corresponds to the point $(-1\cdot 4,-1)=(-4,-1)$.
(d) To graph $y=-f(-x)$ we reflect the graph of f about the y –axis, then about the x –axis.



The point $(4,-1)$ on the graph of f corresponds to the point $(-1\cdot 4,-1\cdot -1)=(-4,1)$.

6. The graph of $y=f(x)=\sqrt{3x-x^2}$ has been shifted 2 units to the right and stretched vertically by a factor of 2 . Thus, a function describing the graph is

$$\begin{aligned} y &= 2f(x-2)=2\sqrt{3(x-2)-(x-2)^2} \\ &= 2\sqrt{3x-6-(x2-4x+4)}=2\sqrt{-x2+7x-10} \end{aligned}$$

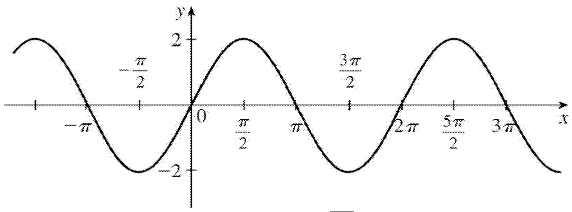
7. The graph of $y=f(x)=\sqrt{3x-x^2}$ has been shifted 4 units to the left, reflected about the x –axis, and shifted downward 1 unit. Thus, a function describing the graph is

$$y=\underbrace{-1\ \cdot}_{\substack{\text{reflect}\\\text{about}\\\text{x–axis}}}\quad f\underbrace{\ x+4\ }_{\substack{\text{shift}\\\text{4units}\\\text{left}}}\quad\underbrace{-1}_{\substack{\text{shift}\\\text{1unit}\\\text{down}}}$$

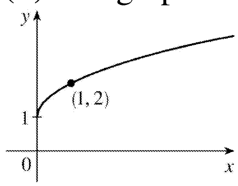
This function can be written as

$$\begin{aligned} y &=-f(x+4)-1=-\sqrt{3(x+4)-(x+4)^2}-1=-\sqrt{3x+12-\left(x^2+8x+16\right)}-1 \\ &=-\sqrt{-x^2-5x-4}-1 \end{aligned}$$

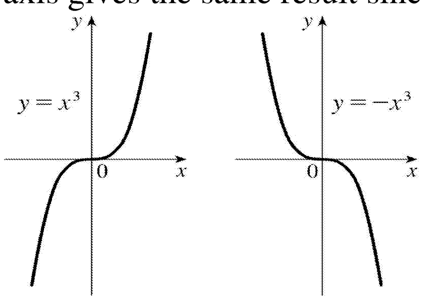
8. **(a)** The graph of $y=2\sin\ x$ can be obtained from the graph of $y=\sin\ x$ by stretching it vertically by a factor of 2 .



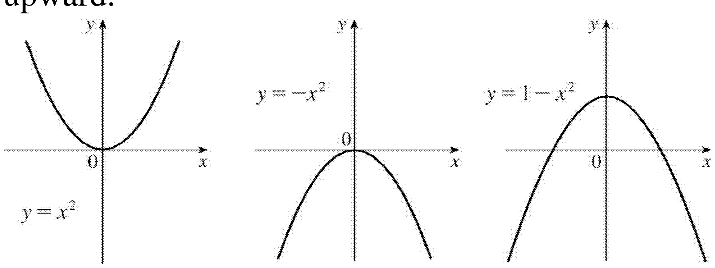
(b) The graph of $y=1+\sqrt{x}$ can be obtained from the graph of $y=\sqrt{x}$ by shifting it upward 1 unit.



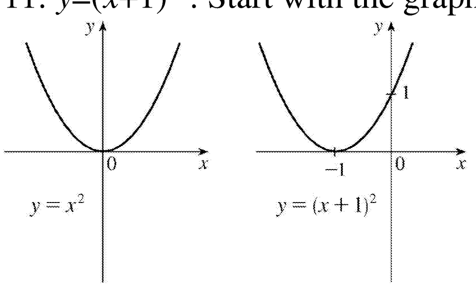
9. $y=-x^3$: Start with the graph of $y=x^3$ and reflect about the x –axis. Note: Reflecting about the y –axis gives the same result since substituting $-x$ for x gives us $y=(-x)^3=-x^3$.



10. $y=1-x^2=-x^2+1$: Start with the graph of $y=x^2$, reflect about the x –axis, and then shift 1 unit upward.

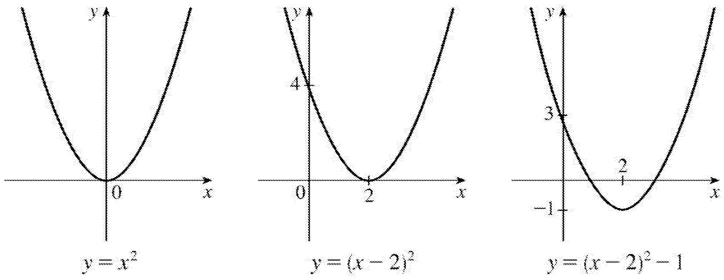


11. $y=(x+1)^2$: Start with the graph of $y=x^2$ and shift 1 unit to the left.

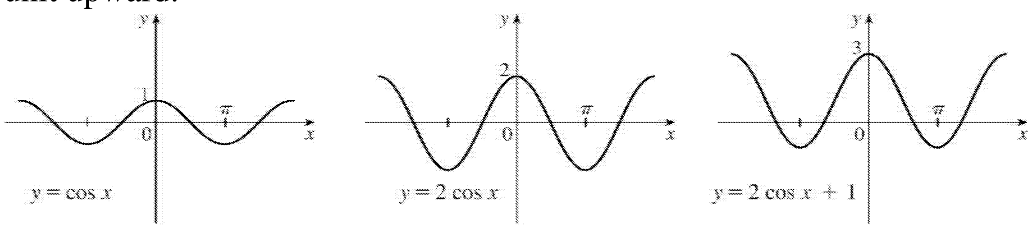


12.

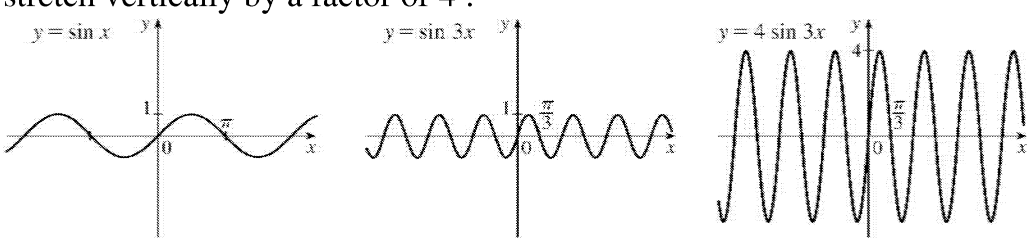
$y=x^2-4x+3=(x^2-4x+4)-1=(x-2)^2-1$: Start with the graph of $y=x^2$, shift 2 units to the right, and then shift 1 unit downward.



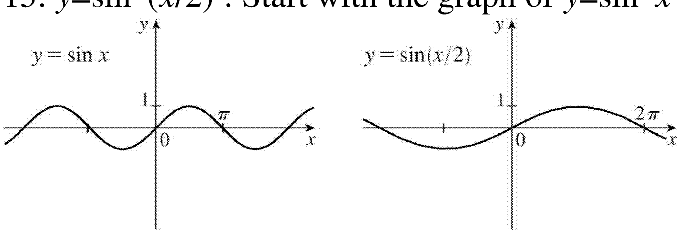
13. $y=1+2\cos\ x$: Start with the graph of $y=\cos\ x$, stretch vertically by a factor of 2 , and then shift 1 unit upward.



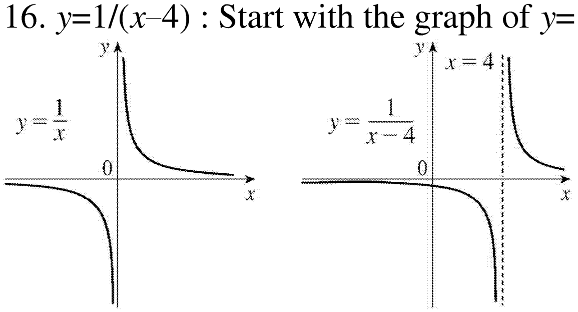
14. $y=4\sin\ 3x$: Start with the graph of $y=\sin\ x$, compress horizontally by a factor of 3 , and then stretch vertically by a factor of 4 .



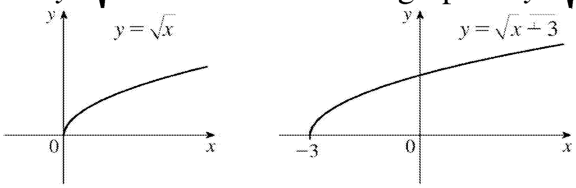
15. $y=\sin\ (x/2)$: Start with the graph of $y=\sin\ x$ and stretch horizontally by a factor of 2 .



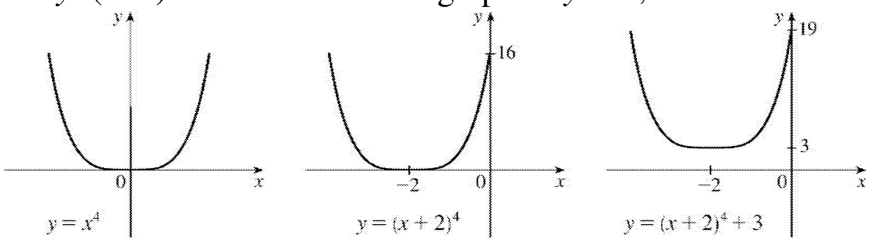
16. $y=1/(x-4)$: Start with the graph of $y=1/x$ and shift 4 units to the right.



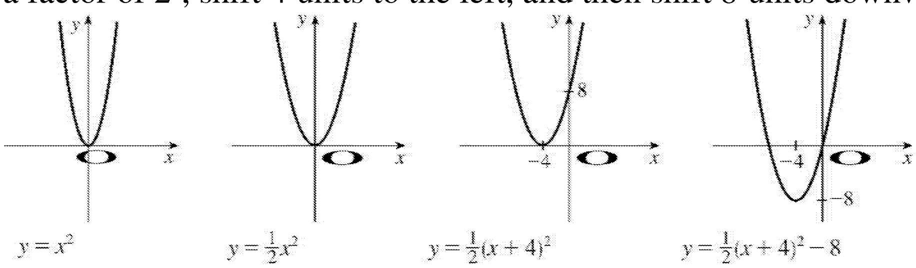
17. $y=\sqrt{x+3}$: Start with the graph of $y=\sqrt{x}$ and shift 3 units to the left.



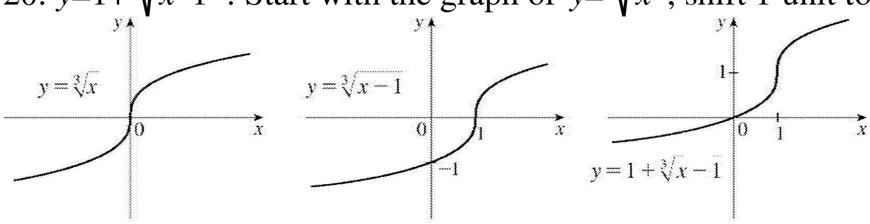
18. $y=(x+2)^4+3$: Start with the graph of $y=x^4$, shift 2 units to the left, and then shift 3 units upward.



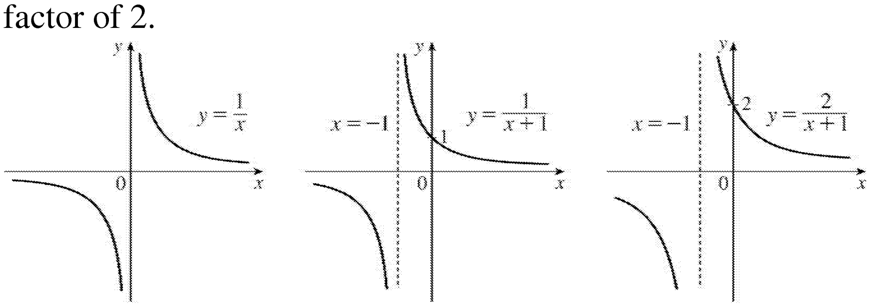
19. $y=\frac{1}{2}(x^2+8x)=\frac{1}{2}(x^2+8x+16)-8=\frac{1}{2}(x+4)^2-8$: Start with the graph of $y=x^2$, compress vertically by a factor of 2 , shift 4 units to the left, and then shift 8 units downward.



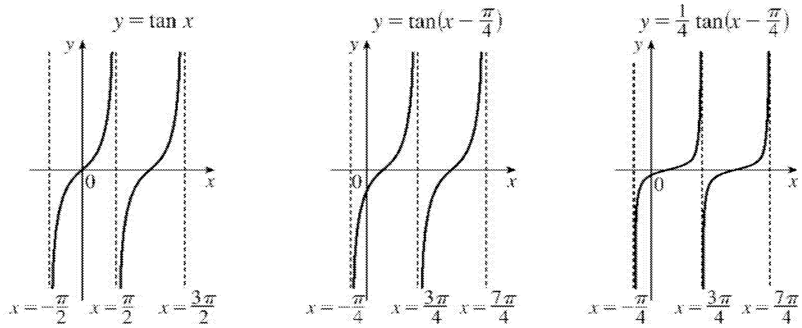
20. $y=1+\sqrt[3]{x-1}$: Start with the graph of $y=\sqrt[3]{x}$, shift 1 unit to the right, and then shift 1 unit upward.



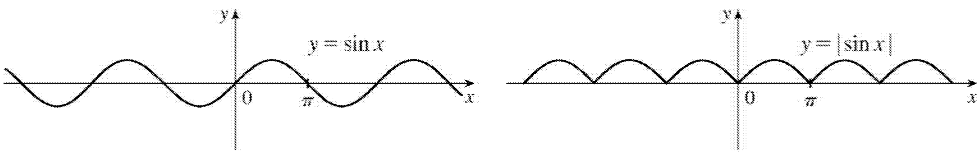
21. $y=2/(x+1)$: Start with the graph of $y=1/x$, shift 1 unit to the left, and then stretch vertically by a factor of 2.



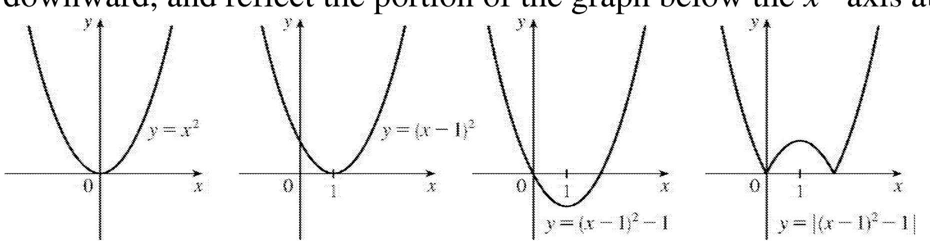
22. $y=\frac{1}{4}\tan\left(x-\frac{\pi}{4}\right)$: Start with the graph of $y=\tan x$, shift $\frac{\pi}{4}$ units to the right, and then compress vertically by a factor of 4 .



23. $y=|\sin x|$: Start with the graph of $y=\sin x$ and reflect all the parts of the graph below the x –axis about the x –axis.



24. $y=\left|x^2-2x\right|=\left|x^2-2x+1-1\right|=\left|(x-1)^2-1\right|$: Start with the graph of $y=x^2$, shift 1 unit right, shift 1 unit downward, and reflect the portion of the graph below the x –axis about the x –axis.

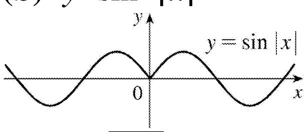


25. This is just like the solution to Example 4 except the amplitude of the curve (the 30 $^\circ$ N curve in Figure 9 on June 21) is $14-12=2$. So the function is $L(t)=12+2\sin\left[\frac{2\pi}{365}(t-80)\right]$. March 31 is the 90 th day of the year, so the model gives $L(90)\approx 12.34$ h. The daylight time (5:51 A.M. to 6:18 P.M.) is 12 hours and 27 minutes, or 12.45 h. The model value differs from the actual value by $\frac{12.45-12.34}{12.45}\approx 0.009$, less than 1% .

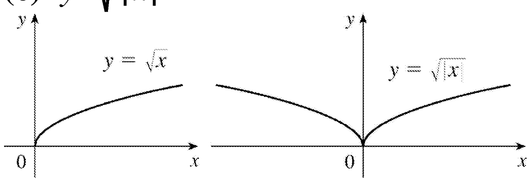
26. Using a sine function to model the brightness of Delta Cephei as a function of time, we take its period to be 5.4 days, its amplitude to be 0.35 (on the scale of magnitude), and its average magnitude to be 4.0 . If we take $t=0$ at a time of average brightness, then the magnitude (brightness) as a function of time t in days can be modeled by the formula $M(t)=4.0+0.35\sin\left(\frac{2\pi}{5.4}t\right)$.

27. **(a)** To obtain $y=f(|x|)$, the portion of the graph of $y=f(x)$ to the right of the y –axis is reflected about the y –axis.

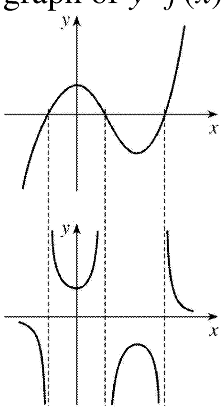
(b) $y=\sin\ |x|$



(c) $y=\sqrt{|x|}$

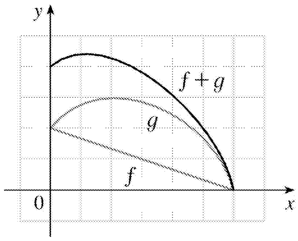


28. The most important features of the given graph are the x –intercepts and the maximum and minimum points. The graph of $y=1/f(x)$ has vertical asymptotes at the x –values where there are x –intercepts on the graph of $y=f(x)$. The maximum of 1 on the graph of $y=f(x)$ corresponds to a minimum of 1/1=1 on $y=1/f(x)$. Similarly, the minimum on the graph of $y=f(x)$ corresponds to a maximum on the graph of $y=1/f(x)$. As the values of y get large (positively or negatively) on the graph of $y=f(x)$, the values of y get close to zero on the graph of $y=1/f(x)$.



29. Assuming that successive horizontal and vertical gridlines are a unit apart, we can make a table of approximate values as follows.

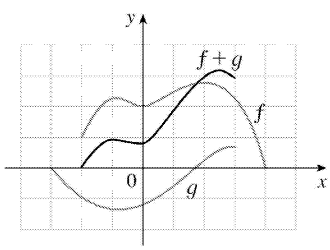
x	0	1	2	3	4	5	6
$f(x)$	2	1.7	1.3	1.0	0.7	0.3	0
$g(x)$	2	2.7	3	2.8	2.4	1.7	0
$g(x)+g(x)$	4	4.4	4.3	3.8	3.1	2.0	0



Connecting the points $(x,f(x)+g(x))$ with a smooth curve gives an approximation to the graph of $f+g$. Extra points can be plotted between those listed above if necessary.

30. First note that the domain of $f+g$ is the intersection of the domains of f and g ; that is, $f+g$ is only defined where both f and g are defined. Taking the horizontal and vertical units of length to be the distances between successive vertical and horizontal gridlines, we can make a table of approximate values as follows:

x	-2	-1	0	1	2	2.5	3
$f(x)$	-1	2.2	2.0	2.4	2.7	2.7	2.3
$g(x)$	1	-1.3	-1.2	-0.6	0.3	0.5	0.7
$f(x)+g(x)$	0	0.9	0.8	1.8	3.0	3.2	3.0



Extra values of x (like the value 2.5 in the table above) can be added as needed.

31. $f(x)=x^3+2x^2$; $g(x)=3x^2-1$. $D=R$ for both f and g .

$(f+g)(x)=(x^3+2x^2)+(3x^2-1)=x^3+5x^2-1$, $D=R$.

$(f-g)(x)=(x^3+2x^2)-(3x^2-1)=x^3-x^2+1$, $D=R$.

$(fg)(x)=(x^3+2x^2)(3x^2-1)=3x^5+6x^4-x^3-2x^2$, $D=R$.

$$\left(\frac{f}{g}\right)(x)=\frac{x^3+2x^2}{3x^2-1} \text{ , } D=\left\{ x \mid x\neq\pm\frac{1}{\sqrt{3}} \right\} \text{ since } 3x^2-1\neq 0 \text{ .}$$

32. $f(x)=\sqrt{1+x}$, $D=[-1,\infty)$; $g(x)=\sqrt{1-x}$, $D=(-\infty,1]$.

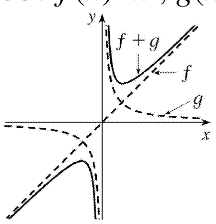
$(f+g)(x)=\sqrt{1+x}+\sqrt{1-x}$, $D=(-\infty,1]\cap[-1,\infty)=[-1,1]$.

$(f-g)(x)=\sqrt{1+x}-\sqrt{1-x}$, $D=[-1,1]$.

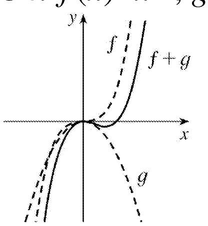
$(fg)(x)=\sqrt{1+x}\cdot\sqrt{1-x}=\sqrt{1-x^2}$, $D=[-1,1]$.

$\left(\frac{f}{g}\right)(x)=\frac{\sqrt{1+x}}{\sqrt{1-x}}$, $D=[-1,1)$. We must exclude $x=1$ since it would make $\frac{f}{g}$ undefined.

33. $f(x)=x$, $g(x)=1/x$



34. $f(x)=x^3$, $g(x)=-x^2$



35. $f(x)=2x^2-x$; $g(x)=3x+2$. $D=R$ for both f and g , and hence for their composites.
$(f\circ g)(x)=f(g(x))=f(3x+2)=2(3x+2)^2-(3x+2)=2(9x^2+12x+4)-3x-2=18x^2+21x+6$.
$(g\circ f)(x)=g(f(x))=g(2x^2-x)=3(2x^2-x)+2=6x^2-3x+2$.
$(f\circ f)(x)=f(f(x))=f(2x^2-x)=2(2x^2-x)^2-(2x^2-x)=2(4x^4-4x^3+x^2)-2x^2+x=8x^4-8x^3+x$.
$(g\circ g)(x)=g(g(x))=g(3x+2)=3(3x+2)+2=9x+6+2=9x+8$.

36. $f(x)=1-x^3$, $D=R$; $g(x)=1/x$, $D=\{x|x\neq 0\}$.

$(f\circ g)(x)=f(g(x))=f(1/x)=1-(1/x)^3=1-1/x^3$, $D=\{x|x\neq 0\}$.

$(g\circ f)(x)=g(f(x))=g\left(1-x^3\right)=1/(1-x^3)$, $D=\{x|1-x^3\neq 0\}=\{x|x\neq 1\}$.
$(f\circ f)(x)=f(f(x))=f(1-x^3)=1-(1-x^3)^3$ $[=x^9-3x^6+3x^3]$, $D=R$.
$(g\circ g)(x)=g(g(x))=g(1/x)=1/(1/x)=x$, $D=\{x|x\neq 0\}$ because 0 is not in the domain of g .

37. $f(x)=\sin x$, $D=R$; $g(x)=1-\sqrt{x}$, $D=[0,\infty)$.
$(f\circ g)(x)=f(g(x))=f(1-\sqrt{x})=\sin(1-\sqrt{x})$, $D=[0,\infty]$.
$(g\circ f)(x)=g(f(x))=g(\sin x)=1-\sqrt{\sin x}$. For $\sqrt{\sin x}$ to be defined, we must have $\sin x\geq 0\Leftrightarrow$
$x\in[0,\pi]\cup[2\pi,3\pi]\cup[-2\pi,-\pi]\cup[4\pi,5\pi]\cup[-4\pi,-3\pi]\cup\ldots$, so
$D=\{x|x\in[2n\pi,\pi+2n\pi],\text{where } n \text{ is an integer}\}$.

$(f\circ f)(x)=f(f(x))=f(\sin x)=\sin(\sin x)$, $D=R$.
$(g\circ g)(x)=g(g(x))=g\left(1-\sqrt{x}\right)=1-\sqrt{1-\sqrt{x}}$,
$D=\left\{x\geq 0|1-\sqrt{x}\geq 0\right\}=\left\{x\geq 0|1\geq\sqrt{x}\right\}=\left\{x\geq 0|\sqrt{x}\leq 1\right\}=[0,1]$.

38. $f(x)=1-3x$, $D=R$; $g(x)=5x^2+3x+2$, $D=R$.

$$(f\circ g)(x)=f(g(x))=f(5x^2+3x+2)=1-3(5x^2+3x+2)$$
$$=1-15x^2-9x-6=-15x^2-9x-5\text{ , }D=R\text{ .}$$
$$(g\circ f)(x)=g(f(x))=g(1-3x)=5(1-3x)^2+3(1-3x)+2=5(1-6x+9x^2)+3-9x+2$$
$$=5-30x+45x^2-9x+5=45x^2-39x+10\text{ , }D=R\text{ .}$$
$$(f\circ f)(x)=f(f(x))=f(1-3x)=1-3(1-3x)=1-3+9x=9x-2,\ D=R\text{ .}$$
$$(g\circ g)(x)=g(g(x))=g(5x^2+3x+2)=5(5x^2+3x+2)^2+3(5x^2+3x+2)+2$$
$$=5(25x^4+30x^3+29x^2+12x+4)+15x^2+9x+6+2$$
$$=125x^4+150x^3+145x^2+60x+20+15x^2+9x+8$$
$$=125x^4+150x^3+160x^2+69x+28,\ D=R\text{ .}$$

39. $f(x)=x+\frac{1}{x}$, $D=\{x|x\neq 0\}$; $g(x)=\frac{x+1}{x+2}$, $D=\{x|x\neq -2\}$.

$$(f\circ g)(x)=f(g(x))=f\left(\frac{x+1}{x+2}\right)=\frac{x+1}{x+2}+\frac{1}{\frac{x+1}{x+2}}=\frac{x+1}{x+2}+\frac{x+2}{x+1}$$
$$=\frac{(x+1)(x+1)+(x+2)(x+2)}{(x+2)(x+1)}=\frac{\left(x^2+2x+1\right)+\left(x^2+4x+4\right)}{(x+2)(x+1)}=\frac{2x^2+6x+5}{(x+2)(x+1)}$$

Since $g(x)$ is not defined for $x=-2$ and $f(g(x))$ is not defined for $x=-2$ and $x=-1$, the domain of $(f\circ g)(x)$ is $D=\{x|x\neq -2,-1\}$.

$$(g\circ f)(x)=g(f(x))=g\left(x+\frac{1}{x}\right)=\frac{\left(x+\frac{1}{x}\right)+1}{\left(x+\frac{1}{x}\right)+2}=\frac{\frac{x^2+1+x}{x}}{\frac{x^2+1+2x}{x}}=\frac{x^2+x+1}{x^2+2x+1}=\frac{x^2+x+1}{(x+1)^2}\text{ .}$$

Since $f(x)$ is not defined for $x=0$ and $g(f(x))$ is not defined for $x=-1$, the domain of $(g\circ f)(x)$ is $D=\{x|x\neq -1,0\}$.

$$(f\circ f)(x)=f(f(x))=f\left(x+\frac{1}{x}\right)=\left(x+\frac{1}{x}\right)+\frac{1}{x+\frac{1}{x}}=x+\frac{1}{x}+\frac{1}{\frac{x^2+1}{x}}=x+\frac{1}{x}+\frac{x}{x^2+1}$$

$$=\frac{x(x)\left(x^2+1\right)+1\left(x^2+1\right)+x(x)}{x(x^2+1)}=\frac{x^4+x^2+x^2+1+x^2}{x(x^2+1)}$$

$$=\frac{x^4+3x^2+1}{x(x^2+1)},\ D=\{x\mid x\neq 0\}.$$

$$(g\circ g)(x)=g(g(x))=g\left(\frac{x+1}{x+2}\right)=\frac{\frac{x+1}{x+2}+1}{\frac{x+1}{x+2}+2}=\frac{\frac{x+1+1(x+2)}{x+2}}{\frac{x+1+2(x+2)}{x+2}}=\frac{x+1+x+2}{x+1+2x+4}=\frac{2x+3}{3x+5}$$. Since $g(x)$ is not defined for $x=-2$ and $g(g(x))$ is not defined for $x=-\frac{5}{3}$, the domain of $(g\circ g)(x)$ is $D=\left\{x\mid x\neq -2,-\frac{5}{3}\right\}$.

40. $f(x)=\sqrt{2x+3}$, $D=\left\{x\mid x\geq -\frac{3}{2}\right\}$; $g(x)=x^2+1$, $D=R$.

$(f\circ g)(x)=f(x^2+1)=\sqrt{2(x^2+1)+3}=\sqrt{2x^2+5}$, $D=R$.

$(g\circ f)(x)=g\left(\sqrt{2x+3}\right)=\left(\sqrt{2x+3}\right)^2+1=(2x+3)+1=2x+4$, $D=\left\{x\mid x\geq -\frac{3}{2}\right\}$.

$(f\circ f)(x)=f\left(\sqrt{2x+3}\right)=\sqrt{2\left(\sqrt{2x+3}\right)+3}=\sqrt{2\sqrt{2x+3}+3}$, $D=\left\{x\mid x\geq -\frac{3}{2}\right\}$.

$(g\circ g)(x)=g(x^2+1)=(x^2+1)^2+1=(x^4+2x^2+1)+1=x^4+2x^2+2$, $D=R$.

41.

$$(f\circ g\circ h)(x)=f(g(h(x)))=f(g(x-1))=f(2(x-1))$$
$$=2(x-1)+1=2x-1$$

42.

$$(f\circ g\circ h)(x)=f(g(h(x)))=f(g(1-x))=f((1-x)^2)$$
$$=2(1-x)^2-1=2x^2-4x+1$$

43.

$(f\circ g\circ h)(x)=f(g(h(x)))=f(g(x+3))=f((x+3)^2+2)$

$=f(x^2+6x+11)=\sqrt{(x^2+6x+11)-1}=\sqrt{x^2+6x+10}$

44. $(f\circ g\circ h)(x)=f(g(h(x)))=f\left(g\left(\sqrt{x+3}\right)\right)=f\left(\cos\sqrt{x+3}\right)=\dfrac{2}{\cos\sqrt{x+3}+1}$

45. Let $g(x)=x^2+1$ and $f(x)=x^{10}$. Then $(f\circ g)(x)=f(g(x))=(x^2+1)^{10}=F(x)$.

46. Let $g(x)=\sqrt{x}$ and $f(x)=\sin x$. Then $(f\circ g)(x)=f(g(x))=\sin\left(\sqrt{x}\right)=F(x)$.

47. Let $g(x)=x^2$ and $f(x)=\dfrac{x}{x+4}$. Then $(f\circ g)(x)=f(g(x))=\dfrac{x^2}{x^2+4}=G(x)$.

48. Let $g(x)=x+3$ and $f(x)=1/x$. Then $(f\circ g)(x)=f(g(x))=1/(x+3)=G(x)$.

49. Let $g(t)=\cos t$ and $f(t)=\sqrt{t}$. Then $(f\circ g)(t)=f(g(t))=\sqrt{\cos t}=u(t)$.

50. Let $g(t)=\tan t$ and $f(t)=\dfrac{t}{1+t}$. Then $(f\circ g)(t)=f(g(t))=\dfrac{\tan t}{1+\tan t}=u(t)$.

51. Let $h(x)=x^2$, $g(x)=3^x$, and $f(x)=1-x$. Then $(f\circ g\circ h)(x)=1-3^{x^2}=H(x)$.

52. Let $h(x)=\sqrt{x}$, $g(x)=x-1$, and $f(x)=\sqrt[3]{x}$. Then $(f\circ g\circ h)(x)=\sqrt[3]{\sqrt{x}-1}=H(x)$.

53. Let $h(x)=\sqrt{x}$, $g(x)=\sec x$, and $f(x)=x^4$. Then $(f\circ g\circ h)(x)=\left(\sec\sqrt{x}\right)^4=\sec^4\left(\sqrt{x}\right)=H(x)$.

54. **(a)** $f(g(1))=f(6)=5$
(b) $g(f(1))=g(3)=2$
(c) $f(f(1))=f(3)=4$
(d) $g(g(1))=g(6)=3$
(e) $(g\circ f)(3)=g(f(3))=g(4)=1$
(f) $(f\circ g)(6)=f(g(6))=f(3)=4$

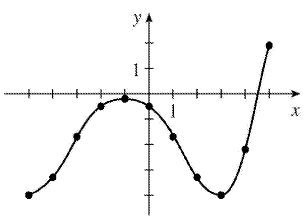
55. **(a)** $g(2)=5$, because the point $(2,5)$ is on the graph of g. Thus, $f(g(2))=f(5)=4$, because the point $(5,4)$ is on the graph of f.
(b) $g(f(0))=g(0)=3$

(c) $(f\circ g)(0)=f(g(0))=f(3)=0$
(d) $(g\circ f)(6)=g(f(6))=g(6)$. This value is not defined, because there is no point on the graph of g that has x –coordinate 6 .
(e) $(g\circ g)(-2)=g(g(-2))=g(1)=4$
(f) $(f\circ f)(4)=f(f(4))=f(2)=-2$

56. To find a particular value of $f(g(x))$, say for $x=0$, we note from the graph that $g(0)\approx 2.8$ and $f(2.8)\approx -0.5$. Thus, $f(g(0))\approx f(2.8)\approx -0.5$. The other values listed in the table were obtained in a similar fashion.

x	$g(x)$	$f(g(x))$
–5	–0.2	–4
–4	1.2	–3.3
–3	2.2	–1.7
–2	2.8	–0.5
–1	3	–0.2

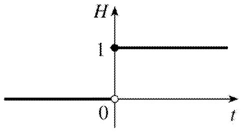
x	$g(x)$	$f(g(x))$
0	2.8	–0.5
1	2.2	–1.7
2	1.2	–3.3
3	–0.2	–4
4	–1.9	–2.2
5	–4.1	1.9



57. **(a)** Using the relationship *distance* = *rate* · *time* with the radius r as the distance, we have $r(t)=60t$.
(b) $A=\pi r^2 \Rightarrow (A\circ r)(t)=A(r(t))=\pi(60t)^2=3600\pi t^2$. This formula gives us the extent of the rippled area (in cm^2) at any time t .

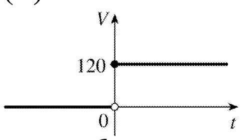
58. **(a)** $d=rt\Rightarrow d(t)=350t$
(b) There is a Pythagorean relationship involving the legs with lengths d and 1 and the hypotenuse with length s : $d^2+1^2=s^2$. Thus, $s(d)=\sqrt{d^2+1}$.
(c) $(s\circ d)(t)=s(d(t))=s(350t)=\sqrt{(350t)^2+1}$

59. **(a)**

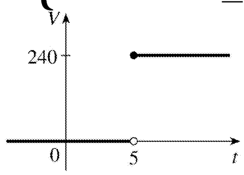


$H(t)=\begin{cases} 0 & \text{if } t<0 \\ 1 & \text{if } t\geq 0 \end{cases}$

(b)



$V(t)=\begin{cases} 0 & \text{if } t<0 \\ 120 & \text{if } t\geq 0 \end{cases}$ so $V(t)=120H(t)$.



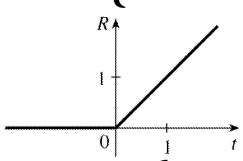
(c)

Starting with the formula in part (b), we replace 120 with 240 to reflect the different voltage. Also, because we are starting 5 units to the right of $t=0$, we replace t with $t-5$ Thus, the formula is $V(t)=240H(t-5)$.

60. **(a)**

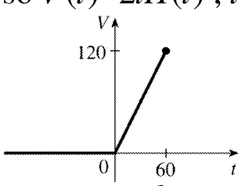
$R(t)=tH(t)$

$=\begin{cases} 0 & \text{if } t<0 \\ t & \text{if } t\geq 0 \end{cases}$

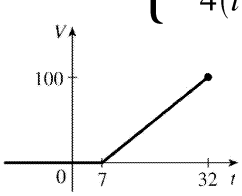


(b) $V(t)=\begin{cases} 0 & \text{if } t<0 \\ 2t & \text{if } 0\leq t\leq 60 \end{cases}$

so $V(t)=2tH(t)$, $t\leq 60$.



(c) $V(t)=\begin{cases} 0 & \text{if } t<7 \\ 4(t-7) & \text{if } 7\leq t\leq 32 \end{cases}$ so $V(t)=4(t-7)H(t-7)$, $t\leq 32$.



61. **(a)** By examining the variable terms in g and h , we deduce that we must square g to get the terms $4x^2$ and $4x$ in h . If we let $f(x)=x^2+c$, then $(f\circ g)(x)=f(g(x))=f(2x+1)=(2x+1)^2+c=4x^2+4x+(1+c)$.

Since $h(x)=4x^2+4x+7$, we must have $1+c=7$. So $c=6$ and $f(x)=x^2+6$.
(b) We need a function g so that $f(g(x))=3(g(x))+5=h(x)$. But
$h(x)=3x^2+3x+2=3(x^2+x)+2=3(x^2+x-1)+5$, so we see that $g(x)=x^2+x-1$.

62. We need a function g so that $g(f(x))=g(x+4)=h(x)=4x-1=4(x+4)-17$. So we see that the function g must be $g(x)=4x-17$.

63. We need to examine $h(-x)$.

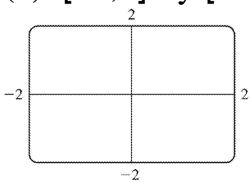
$$h(-x)=(f\circ g)(-x)=f(g(-x))=f(g(x)) \quad \text{[because } g \text{ is even]} \quad =h(x)$$

Because $h(-x)=h(x)$, h is an even function.

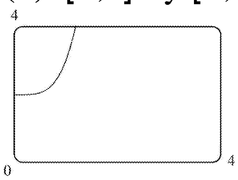
64. $h(-x)=f(g(-x))=f(-g(x))$. At this point, we can't simplify the expression, so we might try to find a counterexample to show that h is not an odd function. Let $g(x)=x$, an odd function, and $f(x)=x^2+x$. Then $h(x)=x^2$+x, which is neither even nor odd.
Now suppose f is an odd function. Then $f(-g(x))=-f(g(x))=-h(x)$. Hence, $h(-x)=-h(x)$, and so h is odd if both f and g are odd.
Now suppose f is an even function. Then $f(-g(x))=f(g(x))=h(x)$. Hence, $h(-x)=h(x)$, and so h is even if g is odd and f is even.

1. $f(x) = x^4 + 2$

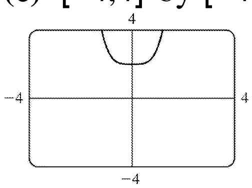
(a) $[-2,2]$ by $[-2,2]$



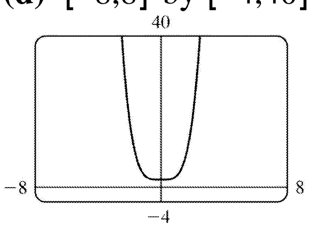
(b) $[0,4]$ by $[0,4]$



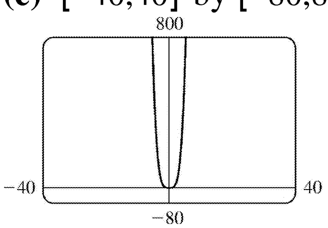
(c) $[-4,4]$ by $[-4,4]$



(d) $[-8,8]$ by $[-4,40]$



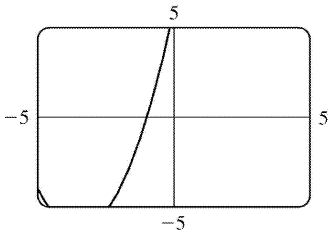
(e) $[-40,40]$ by $[-80,800]$



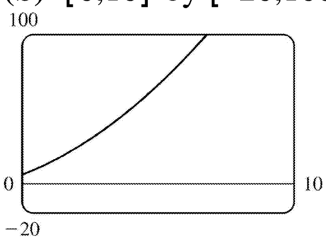
The most appropriate graph is produced in viewing rectangle (d).

2. $f(x) = x^2 + 7x + 6$

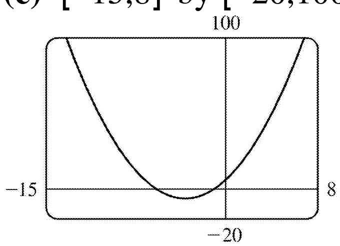
(a) $[-5,5]$ by $[-5,5]$



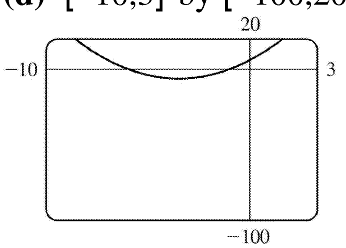
(b) $[0,10]$ by $[-20,100]$



(c) $[-15,8]$ by $[-20,100]$



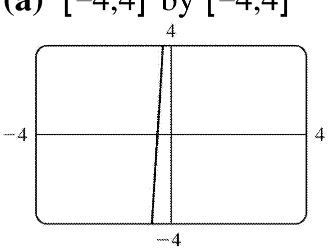
(d) $[-10,3]$ by $[-100,20]$



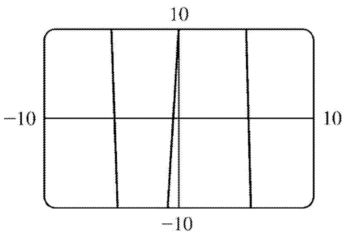
The most appropriate graph is produced in viewing rectangle (c).

3. $f(x)=10+25x-x^3$

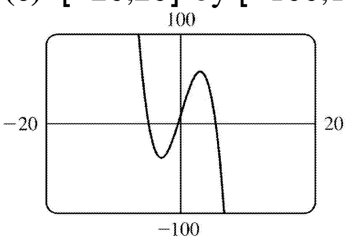
(a) $[-4,4]$ by $[-4,4]$



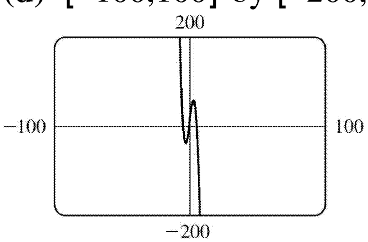
(b) $[-10,10]$ by $[-10,10]$



(c) $[-20,20]$ by $[-100,100]$



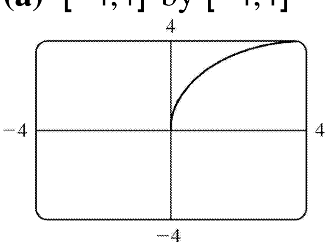
(d) $[-100,100]$ by $[-200,200]$



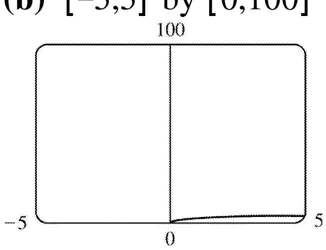
The most appropriate graph is produced in viewing rectangle (c) because the maximum and minimum points are fairly easy to see and estimate.

4. $f(x)=\sqrt{8x-x^2}$

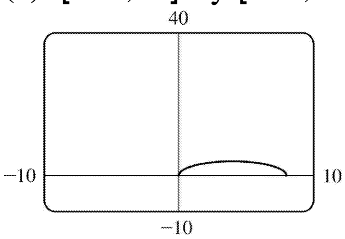
(a) $[-4,4]$ by $[-4,4]$



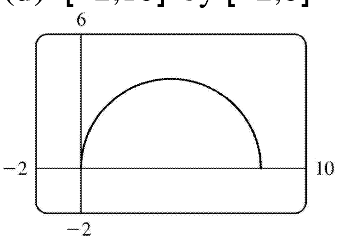
(b) $[-5,5]$ by $[0,100]$



(c) $[-10,10]$ by $[-10,40]$

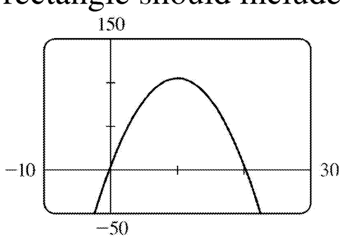


(d) $[-2,10]$ by $[-2,6]$

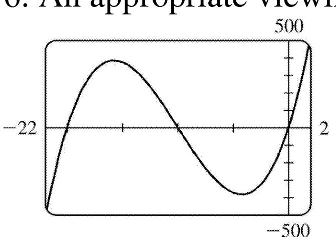


The most appropriate graph is produced in viewing rectangle (d).

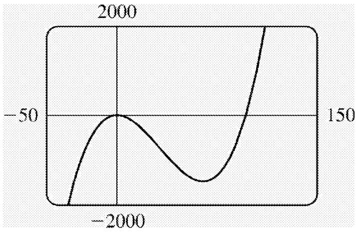
5. Since the graph of $f(x)=5+20x-x^2$ is a parabola opening downward, an appropriate viewing rectangle should include the maximum point.



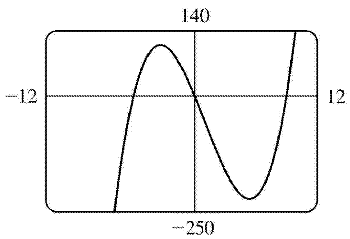
6. An appropriate viewing rectangle for $f(x)=x^3+30x^2+200x$ should include the high and low points.



7. $f(x)=0.01x^3-x^2+5$. Graphing f in a standard viewing rectangle, $[-10,10]$ by $[-10,10]$, shows us what appears to be a parabola. But since this is a cubic polynomial, we know that a larger viewing rectangle will reveal a minimum point as well as the maximum point. After some trial and error, we choose the viewing rectangle $[-50,150]$ by $[-2000,2000]$.

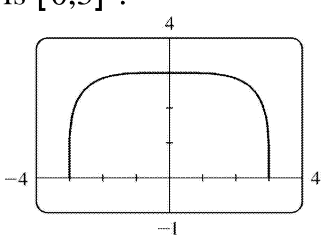


8. $f(x)=x(x+6)(x-9)$

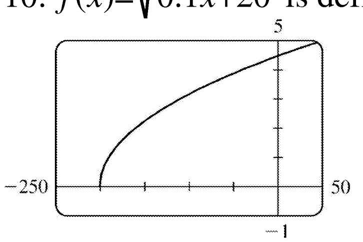


9. $f(x)=\sqrt[4]{81-x^4}$ is defined when

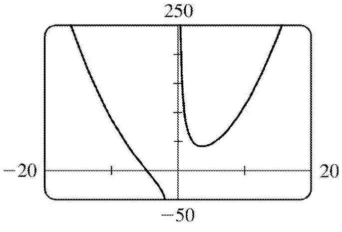
$81-x^4\geq 0\Leftrightarrow x^4\leq 81\Leftrightarrow |x|\leq 3$, so the domain of f is $[-3,3]$. Also $0\leq\sqrt[4]{81-x^4}\leq\sqrt[4]{81}=3$, so the range is $[0,3]$.



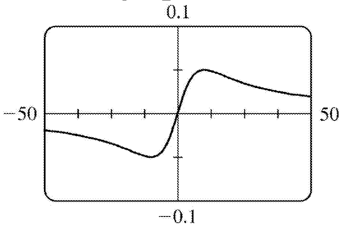
10. $f(x)=\sqrt{0.1x+20}$ is defined when $0.1x+20\geq 0\Leftrightarrow x\geq -200$, so the domain of f is $[-200,\infty)$.



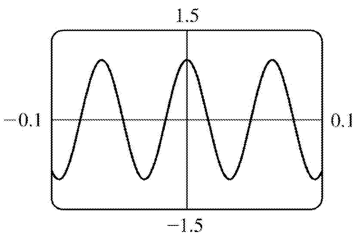
11. The graph of $f(x)=x^2+(100/x)$ has a vertical asymptote of $x=0$. As you zoom out, the graph of f looks more and more like that of $y=x^2$.



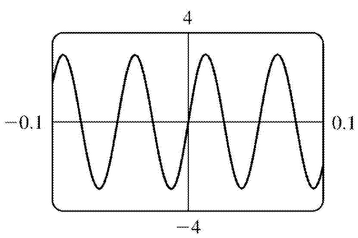
12. The graph of $f(x)=x/(x^2+100)$ is symmetric with respect to the origin.



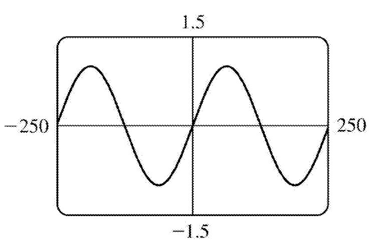
13. $f(x)=\cos\ (100x)$



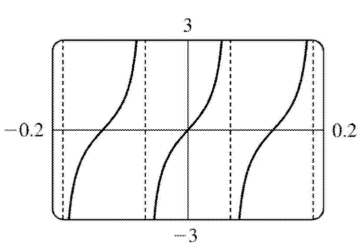
14. $f(x)=3\sin\ (120x)$



15. $f(x)=\sin\ (x/40)$

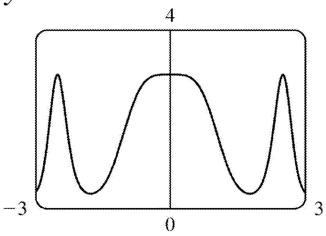


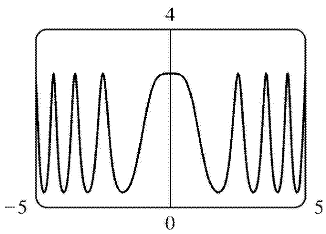
16. $f(x)=\tan\ (25x)$



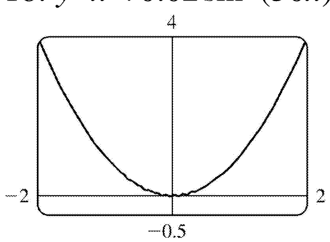
17.

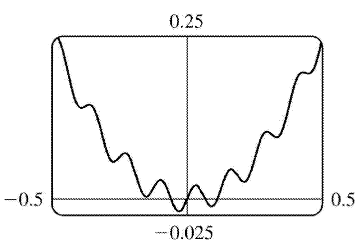
$y=3^{\cos(x^2)}$





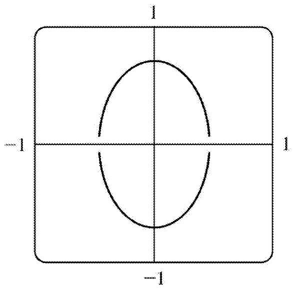
18. $y=x^2+0.02\sin(50x)$



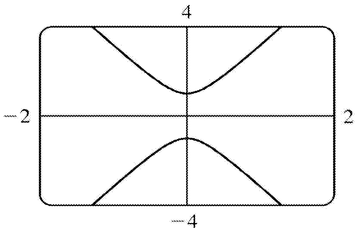


19. We must solve the given equation for y to obtain equations for the upper and lower halves of the ellipse.

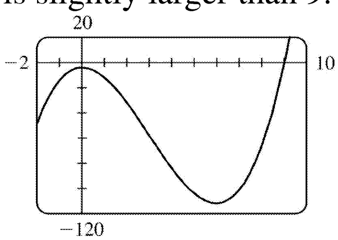
$$4x^2+2y^2=1 \Leftrightarrow 2y^2=1-4x^2 \Leftrightarrow y^2=\frac{1-4x^2}{2} \Leftrightarrow y=\pm\sqrt{\frac{1-4x^2}{2}}$$



20. $y^2-9x^2=1 \Leftrightarrow y^2=1+9x^2 \Leftrightarrow y=\pm\sqrt{1+9x^2}$

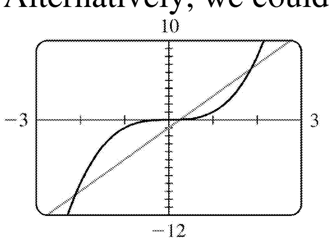


21. From the graph of $f(x)=x^3-9x^2-4$, we see that there is one solution of the equation $f(x)=0$ and it is slightly larger than 9. By zooming in or using a root or zero feature, we obtain $x\approx 9.05$.

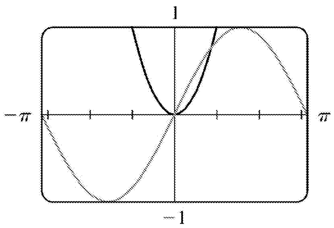


22. We see that the graphs of $f(x)=x^3$ and $g(x)=4x-1$ intersect three times. The x –coordinates of these points (which are the solutions of the equation) are approximately $-2.11, 0.25$, and 1.86 .

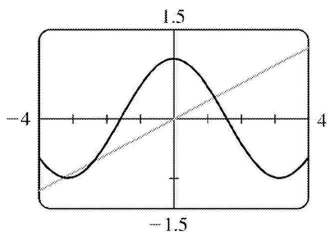
Alternatively, we could find these values by finding the zeros of $h(x)=x^3-4x+1$.



23. We see that the graphs of $f(x)=x^2$ and $g(x)=\sin\ x$ intersect twice. One solution is $x=0$. The other solution of $f=g$ is the x –coordinate of the point of intersection in the first quadrant. Using an intersect feature or zooming in, we find this value to be approximately 0.88. Alternatively, we could find that value by finding the positive zero of $h(x)=x^2-\sin\ x$.

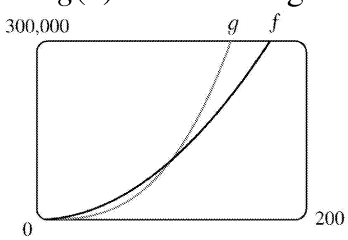


Note : After producing the graph on a TI–83 Plus, we can find the approximate value 0.88 by using the following keystrokes:
2nd CALC 5 ENTER ENTER 1 ENTER . The ‘‘1’’ is just a guess for 0.88.

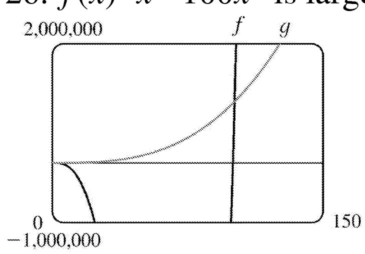


24. **(a)**
The x –coordinates of the three points of intersection are $x\approx-3.29$, -2.36 and 1.20 .
(b) Using trial and error, we find that $m\approx0.3365$. Note that m could also be negative.

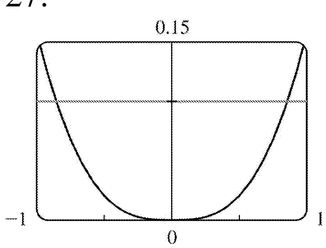
25. $g(x)=x^3/10$ is larger than $f(x)=10x^2$ whenever $x>100$.



26. $f(x)=x^4-100x^3$ is larger than $g(x)=x^3$ whenever $x>101$.

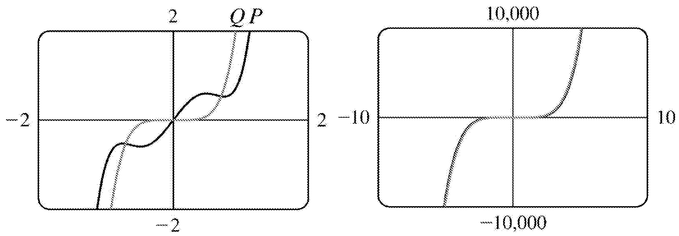


27.



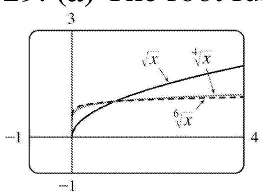
We see from the graphs of $y=|\sin\ x-x|$ and $y=0.1$ that there are two solutions to the equation

$|\sin\ x - x| = 0.1$: $x \approx -0.85$ and $x \approx 0.85$. The condition $|\sin\ x - x| < 0.1$ holds for any x lying between these two values.

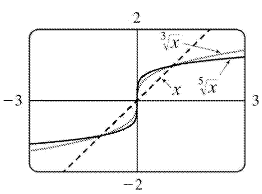
28. 

$P(x) = 3x^5 - 5x^3 + 2x$, $Q(x) = 3x^5$. These graphs are significantly different only in the region close to the origin. The larger a viewing rectangle one chooses, the more similar the two graphs look.

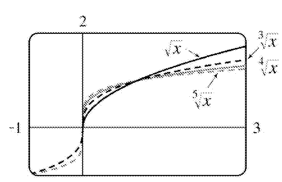
29. **(a)** The root functions $y = \sqrt{x}$, $y = \sqrt[4]{x}$ and $y = \sqrt[6]{x}$



(b) The root functions $y = x$, $y = \sqrt[3]{x}$ and $y = \sqrt[5]{x}$



(c) The root functions $y = \sqrt{x}$, $y = \sqrt[3]{x}$, $y = \sqrt[4]{x}$ and $y = \sqrt[5]{x}$

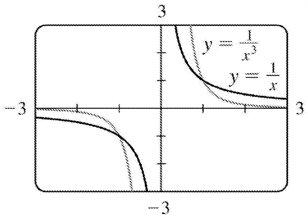


(d)

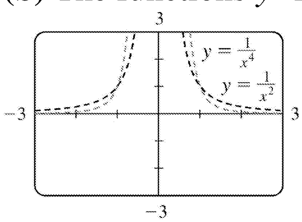
- For any n , the n th root of 0 is 0 and the n th root of 1 is 1 ; that is, all n th root functions pass through the points $(0,0)$ and $(1,1)$.
- For odd n , the domain of the n th root function is R , while for even n , it is $\{x \in R | x \geq 0\}$.
- Graphs of even root functions look similar to that of $\sqrt{x}$, while those of odd root functions resemble that of $\sqrt[3]{x}$.

- As n increases, the graph of $\sqrt[n]{x}$ becomes steeper near 0 and flatter for $x>1$.

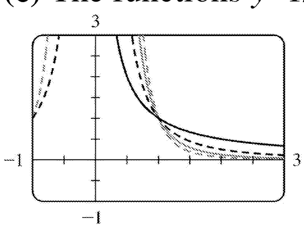
30. **(a)** The functions $y=1/x$ and $y=1/x^3$



(b) The functions $y=1/x^2$ and $y=1/x^4$



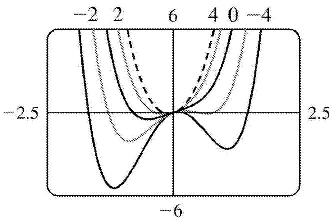
(c) The functions $y=1/x$, $y=1/x^2$, $y=1/x^3$ and $y=1/x^4$



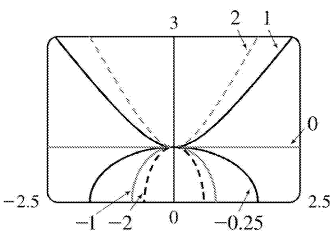
(d)

- The graphs of all functions of the form $y=1/x^n$ pass through the point $(1,1)$.
- If n is even, the graph of the function is entirely above the x– axis. The graphs of $1/x^n$ for n even are similar to one another.
- If n is odd, the function is positive for positive x and negative for negative x . The graphs of $1/x^n$ for n odd are similar to one another.
- As n increases, the graphs of $1/x^n$ approach 0 faster as $x\to\infty$.

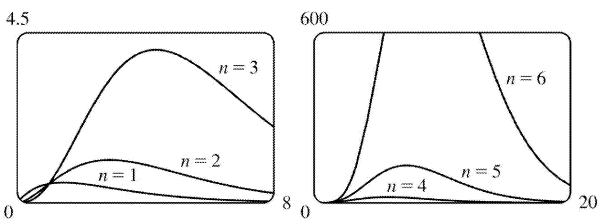
31. $f(x)=x^4+cx^2+x$. If $c<0$, there are three humps: two minimum points and a maximum point. These humps get flatter as c increases, until at $c=0$ two of the humps disappear and there is only one minimum point. This single hump then moves to the right and approaches the origin as c increases.



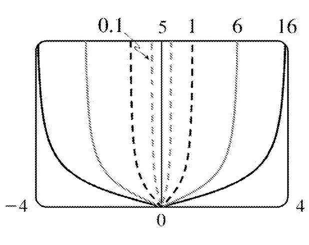
32. $f(x)=\sqrt{1+cx^2}$. If $c<0$, the function is only defined on $\left[-1/\sqrt{-c}\ ,1/\sqrt{-c}\ \right]$, and its graph is the top half of an ellipse. If $c=0$, the graph is the line $y=1$. If $c>0$, the graph is the top half of a hyperbola. As c approaches 0 , these curves become flatter and approach the line $y=1$.



33. $y=x^n 2^{-x}$. As n increases, the maximum of the function moves further from the origin, and gets larger. Note, however, that regardless of n , the function approaches 0 as $x\to\infty$.



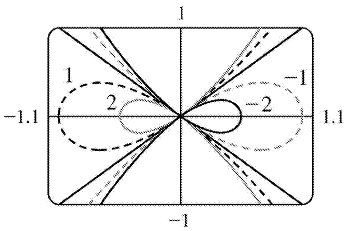
34. $y=\dfrac{|x|}{\sqrt{c-x^2}}$. The ''bullet'' becomes broader as c increases.



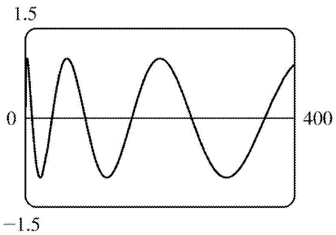
35. $y^2=cx^3+x^2$

If $c<0$, the loop is to the right of the origin, and if c is positive, it is to the left. In both cases, the closer c is to 0 , the larger the loop is.

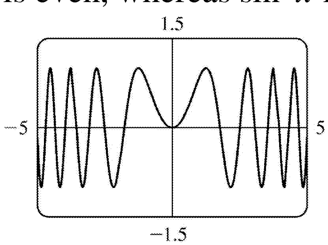
(In the limiting case, $c=0$, the loop is ''infinite'', that is, it doesn't close.) Also, the larger $|c|$ is, the steeper the slope is on the loopless side of the origin.



36. **(a)** $y=\sin\left(\sqrt{x}\right)$
This function is not periodic; it oscillates less frequently as x increases.



(b) $y=\sin\left(x^2\right)$
This function oscillates more frequently as $|x|$ increases. Note also that this function is even, whereas $\sin x$ is odd.



37. The graphing window is 95 pixels wide and we want to start with $x=0$ and end with $x=2\pi$. Since there are 94 ''gaps'' between pixels, the distance between pixels is $\frac{2\pi-0}{94}$. Thus, the x –values that the calculator actually plots are $x=0+\frac{2\pi}{94}\cdot n$, where $n=0, 1, 2, \ldots, 93, 94$. For $y=\sin 2x$, the actual points plotted by the calculator are $\left(\frac{2\pi}{94}\cdot\sin\left(2\cdot\frac{2\pi}{94}\cdot n\right)\right)$ for $n=0, 1, \ldots, 94$. For $y=\sin 96x$, the points plotted are $\left(\frac{2\pi}{94}\cdot\sin\left(96\cdot\frac{2\pi}{94}\cdot n\right)\right)$ for $n=0, 1, \ldots, 94$. But

$$\sin\left(96\cdot\frac{2\pi}{94}\cdot n\right)=\sin\left(94\cdot\frac{2\pi}{94}\cdot n+2\cdot\frac{2\pi}{94}\cdot n\right)=\sin\left(2\pi n+2\cdot\frac{2\pi}{94}\cdot n\right)$$

$$=\sin\left(2\cdot\frac{2\pi}{94}\cdot n\right) \text{ [by periodicity of sine]}, n=0\ ,\ 1\ ,\ \dots\ ,\ 94$$

So the y –values, and hence the points, plotted for $y=\sin\ 96x$ are identical to those plotted for $y=\sin\ 2x$.

Note: Try graphing $y=\sin\ 94x$. Can you see why all the y –values are zero?

38. As in Exercise 37, we know that the points being plotted for $y=\sin\ 45x$ are $\left(\frac{2\pi}{94}\cdot\sin\left(45\cdot\frac{2\pi}{94}\cdot n\right)\right)$ for $n=0\ ,\ 1\ ,\ \dots\ ,\ 94$. But

$$\sin\left(45\cdot\frac{2\pi}{94}\cdot n\right)=\sin\left(47\cdot\frac{2\pi}{94}\cdot n-2\cdot\frac{2\pi}{94}\cdot n\right)=\sin\left(n\pi-2\cdot\frac{2\pi}{94}\cdot n\right)$$

$$=\sin\ (n\pi)\cos\left(2\cdot\frac{2\pi}{94}\cdot n\right)-\cos\ (n\pi)\sin\left(2\cdot\frac{2\pi}{94}\cdot n\right) \text{ [Subtraction formula for the sine]}$$

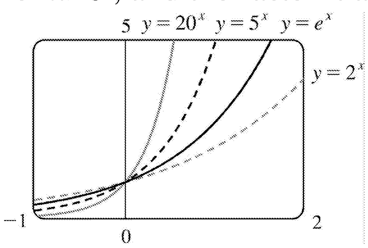
$$=0\cdot\cos\left(2\cdot\frac{2\pi}{94}\cdot n\right)-(\pm 1)\sin\left(2\cdot\frac{2\pi}{94}\cdot n\right)=\pm\sin\left(2\cdot\frac{2\pi}{94}\cdot n\right), n=0\ ,\ 1\ ,\ \dots\ ,\ 94$$

So the y –values, and hence the points, plotted for $y=\sin\ 45x$ lie on either $y=\sin\ 2x$ or $y=-\sin\ 2x$.

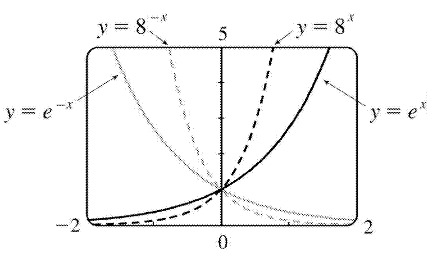
1. **(a)** $f(x)=a^x$, $a>0$
(b) R
(c) $(0,\infty)$
(d) See Figures (c), (b), and (a), respectively.

2. **(a)** The number e is the value of a such that the slope of the tangent line at $x=0$ on the graph of $y=a^x$ is exactly 1 .
(b) $e\approx 2.71828$
(c) $f(x)=e^x$

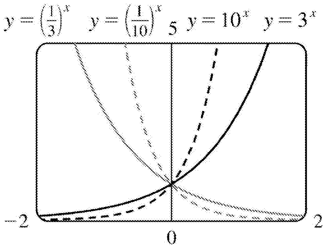
3. All of these graphs approach 0 as $x\to-\infty$, all of them pass through the point $(0,1)$, and all of them are increasing and approach ∞ as $x\to\infty$. The larger the base, the faster the function increases for $x>0$, and the faster it approaches 0 as $x\to-\infty$.



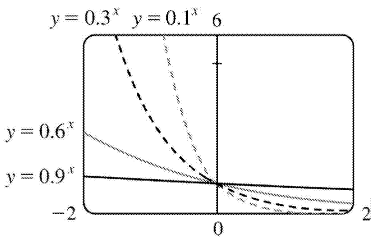
4. The graph of e^{-x} is the reflection of the graph of e^x about the $y-$ axis, and the graph of 8^{-x} is the reflection of that of 8^x about the y –axis. The graph of 8^x increases more quickly than that of e^x for $x>0$, and approaches 0 faster as $x\to-\infty$.



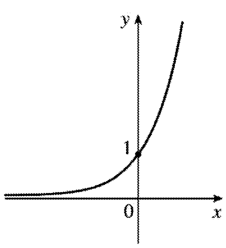
5. The functions with bases greater than 1 (3^x and 10^x) are increasing, while those with bases less than 1 $\left[\left(\frac{1}{3}\right)^x \text{and} \left(\frac{1}{10}\right)^x\right]$ are decreasing. The graph of $\left(\frac{1}{3}\right)^x$ is the reflection of that of 3^x about the $y-$ axis, and the graph of $\left(\frac{1}{10}\right)^x$ is the reflection of that of 10^x about the $y-$ axis. The graph of 10^x increases more quickly than that of 3^x for $x>0$, and approaches 0 faster as $x\to-\infty$.

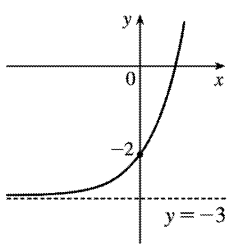


6. Each of the graphs approaches ∞ as $x\to-\infty$, and each approaches 0 as $x\to\infty$. The smaller the base, the faster the function grows as $x\to-\infty$, and the faster it approaches 0 as $x\to\infty$.



7. We start with the graph of $y=4^x$ (Figure 3) and then shift 3 units downward. This shift doesn't affect the domain, but the range of $y=4^x-3$ is $(-3,\infty)$. There is a horizontal asymptote of $y=-3$.

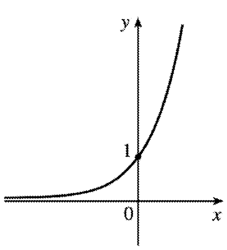


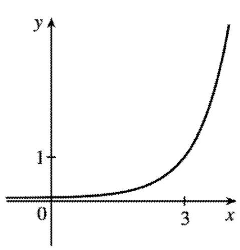


$y=4^x$

$y=4^x-3$

8. We start with the graph of $y=4^x$ (Figure 3) and then shift 3 units to the right. There is a horizontal asymptote of $y=0$.

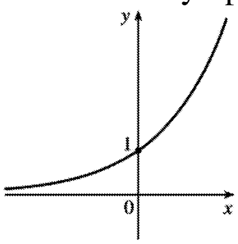


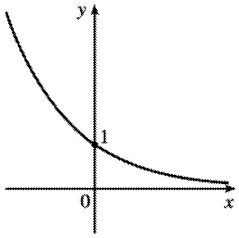


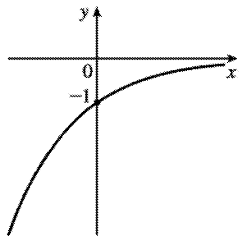
$y=4^x$

$y=4^{x-3}$

9. We start with the graph of $y=2^{x}$ (Figure 2), reflect it about the y –axis, and then about the x –axis (or just rotate 180° to handle both reflections) to obtain the graph of $y=-2^{-x}$. In each graph, $y=0$ is the horizontal asymptote.

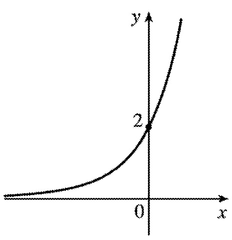


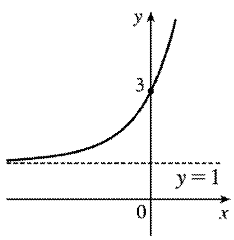




$y=2^{x}$ $y=2^{-x}$ $y=-2^{-x}$

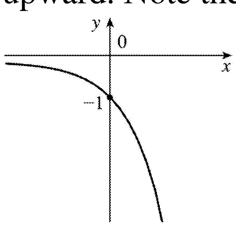
10. We start with the graph of $y=e^{x}$ (Figure 13), vertically stretch by a factor of 2, and then shift 1 unit upward. There is a horizontal asymptote of $y=1$.



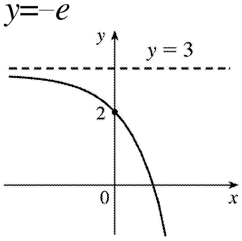


$y=2e^{x}$ $y=1+2e^{x}$

11. We start with the graph of $y=e^{x}$ (Figure 13), reflect it about the x –axis, and then shift 3 units upward. Note the horizontal asymptote of $y=3$.



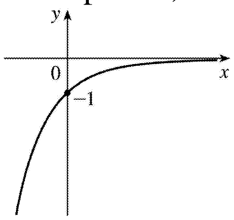
$y=-e^{x}$



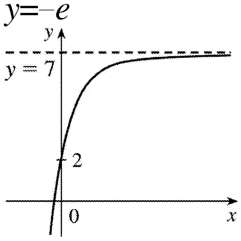
$y=3-e^{x}$

12. We start with the graph of $y=e^{x}$ (Figure 13), reflect it about the y –axis, and then about the x –axis

(or just rotate 180° to handle both reflections) to obtain the graph of $y=-e^{-x}$. Now shift this graph 1 unit upward, vertically stretch by a factor of 5 , and then shift 2 units upward.



$y=-e^{-x}$



$y=2+5(1-e^{-x})$

13. **(a)** To find the equation of the graph that results from shifting the graph of $y=e^{x}$ 2 units downward, we subtract 2 from the original function to get $y=e^{x}-2$.
(b) To find the equation of the graph that results from shifting the graph of $y=e^{x}$ 2 units to the right, we replace x with $x-2$ in the original function to get $y=e^{(x-2)}$.
(c) To find the equation of the graph that results from reflecting the graph of $y=e^{x}$ about the x –axis, we multiply the original function by -1 to get $y=-e^{x}$.
(d) To find the equation of the graph that results from reflecting the graph of $y=e^{x}$ about the y –axis, we replace x with $-x$ in the original function to get $y=e^{-x}$.
(e) To find the equation of the graph that results from reflecting the graph of $y=e^{x}$ about the x –axis and then about the y –axis, we first multiply the original function by -1 (to get $y=-e^{x}$) and then replace x with $-x$ in this equation to get $y=-e^{-x}$.

14. **(a)** This reflection consists of first reflecting the graph about the x –axis (giving the graph with equation $y=-e^{x}$) and then shifting this graph $2\cdot 4=8$ units upward. So the equation is $y=-e^{x}+8$.
(b) This reflection consists of first reflecting the graph about the y –axis (giving the graph with equation $y=e^{-x}$) and then shifting this graph $2\cdot 2=4$ units to the right. So the equation is $y=e^{-(x-4)}$.

15. **(a)** The denominator $1+e^{x}$ is never equal to zero because $e^{x}>0$, so the domain of $f(x)=1/(1+e^{x})$ is R .

(b) $1-e^x=0 \Leftrightarrow e^x=1 \Leftrightarrow x=0$, so the domain of $f(x)=1/(1-e^x)$ is $(-\infty,0)\cup(0,\infty)$.

16. **(a)** The sine and exponential functions have domain R , so $g(t)=\sin(e^{-t})$ also has domain R .
(b) The function $g(t)=\sqrt{1-2^t}$ has domain $\{t \mid 1-2^t \geq 0\}=\{t \mid 2^t \leq 1\}=\{t \mid t \leq 0\}=(-\infty,0]$.

17. Use $y=Ca^x$ with the points $(1,6)$ and $(3,24)$. $6=Ca^1$ $\left[C=\frac{6}{a}\right]$ and $24=Ca^3 \Rightarrow 24=\left(\frac{6}{a}\right)a^3 \Rightarrow$ $4=a^2 \Rightarrow a=2$ [since $a>0$] and $C=\frac{6}{2}=3$. The function is $f(x)=3\cdot 2^x$.

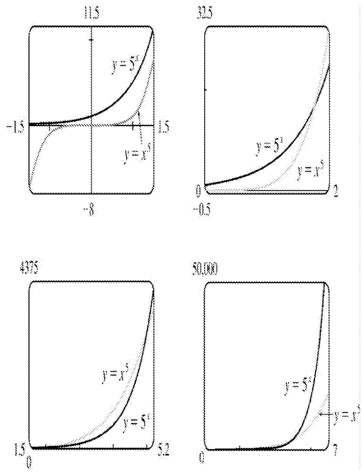
18. Given the y –intercept $(0,2)$, we have $y=Ca^x=2a^x$. Using the point $\left(2,\frac{2}{9}\right)$ gives us $\frac{2}{9}=2a^2 \Rightarrow$ $\frac{1}{9}=a^2 \Rightarrow a=\frac{1}{3}$ [since $a>0$]. The function is $f(x)=2\left(\frac{1}{3}\right)^x$ or $f(x)=2(3)^{-x}$.

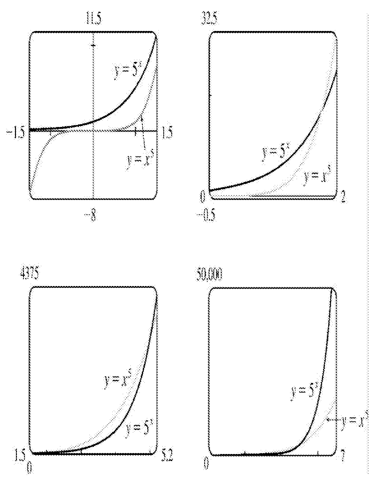
19. If $f(x)=5^x$, then $\frac{f(x+h)-f(x)}{h}=\frac{5^{x+h}-5^x}{h}=\frac{5^x5^h-5^x}{h}=\frac{5^x\left(5^h-1\right)}{h}=5^x\left(\frac{5^h-1}{h}\right)$.

20. Suppose the month is February. Your payment on the 28th day would be $2^{28-1}=2^{27}=134,217,728$ cents, or \$1 , 342 , 177 . 28 . Clearly, the second method of payment results in a larger amount for any month.

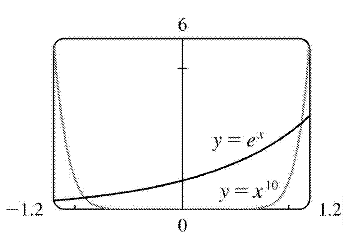
21. 2 ft =24 in, $f(24)=24^2$ in =576 in =48 ft. $g(24)=2^{24}$ in $=2^{24}/(12\cdot 5280)$ mi ≈ 265 mi

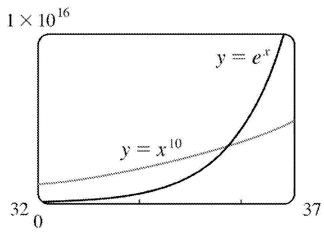
22. We see from the graphs that for x less than about 1.8 , $g(x)=5^x>f(x)=x^5$, and then near the point $(1.8,17.1)$ the curves intersect. Then $f(x)>g(x)$ from $x\approx 1.8$ until $x=5$. At $(5,3125)$ there is another point of intersection, and for $x>5$ we see that $g(x)>f(x)$. In fact, g increases much more rapidly than f beyond that point.





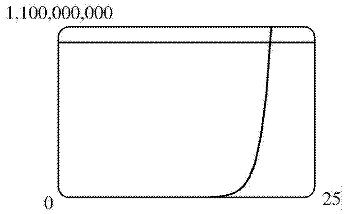
23. The graph of g finally surpasses that of f at $x \approx 35.8$.



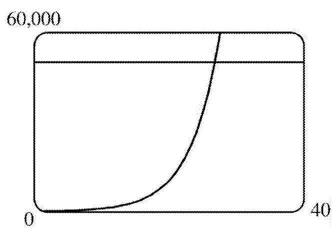


24. We graph $y = e^x$ and $y = 1,000,000,000$ and determine where $e^x = 1 \times 10^9$. This seems to be true at $x \approx 20.723$, so

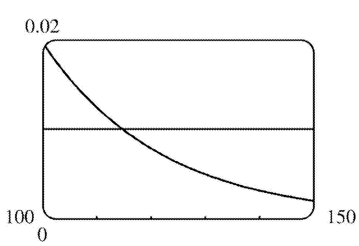
$e^x > 1\times 10^9$ for $x > 20.723$.



25. **(a)** Fifteen hours represents 5 doubling periods (one doubling period is three hours).
$100\cdot 2^5 = 3200$
(b) In t hours, there will be $t/3$ doubling periods. The initial population is 100 , so the population y at time t is $y = 100\cdot 2^{t/3}$.
(c) $t = 20 \Rightarrow y = 100\cdot 2^{20/3} \approx 10 , 159$
(d) We graph $y_1 = 100\cdot 2^{x/3}$ and $y_2 = 50 , 000$. The two curves intersect at $x \approx 26.9$, so the population reaches 50 , 000 in about 26.9 hours.

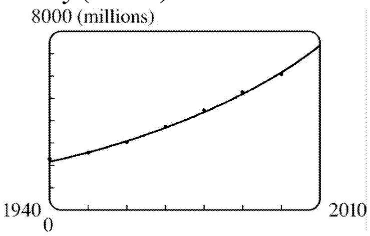


26. **(a)** Sixty hours represents 4 half–life periods. $2\cdot \left(\frac{1}{2}\right)^4 = \frac{1}{8}$ g
(b) In t hours, there will be $t/15$ half–life periods. The initial mass is 2 g, so the mass y at time t is $y = 2\cdot \left(\frac{1}{2}\right)^{t/15}$.
(c) 4 days $= 4\cdot 24 = 96$ hours. $t = 96 \Rightarrow y = 2\cdot \left(\frac{1}{2}\right)^{96/15} \approx 0.024$ g
(d) $y = 0.01 \Rightarrow t \approx 114.7$ hours

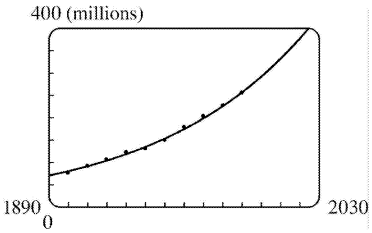


27. An exponential model is

$y = ab^t$, where $a = 3.154832569 \times 10^{-12}$ and $b = 1.017764706$. This model gives $y(1993) \approx 5498$ million and $y(2010) \approx 7417$ million.



28. An exponential model is $y = ab^t$, where $a = 1.9976760197589 \times 10^{-9}$ and $b = 1.0129334321697$. This model gives $y(1925) \approx 111$ million, $y(2010) \approx 330$ million, and $y(2020) \approx 375$ million.



1. **(a)** See Definition 1.
(b) It must pass the Horizontal Line Test.

2. **(a)** $f^{-1}(y)=x \Leftrightarrow f(x)=y$ for any y in B . The domain of f^{-1} is B and the range of f^{-1} is A .
(b) See the steps in (5).
(c) Reflect the graph of f about the line $y=x$.

3. f is not one–to–one because $2\neq 6$, but $f(2)=2.0=f(6)$.

4. f is one–to–one since for any two different domain values, there are different range values.

5. No horizontal line intersects the graph of f more than once. Thus, by the Horizontal Line Test, f is one–to–one.

6. The horizontal line $y=0$ (the x –axis) intersects the graph of f in more than one point. Thus, by the Horizontal Line Test, f is not one–to–one.

7. The horizontal line $y=0$ (the x –axis) intersects the graph of f in more than one point. Thus, by the Horizontal Line Test, f is not one–to–one.

8. No horizontal line intersects the graph of f more than once. Thus, by the Horizontal Line Test, f is one–to–one.

9. The graph of $f(x)=\frac{1}{2}(x+5)$ is a line with slope $\frac{1}{2}$. It passes the Horizontal Line Test, so f is one–to–one.
Algebraic solution : If $x_1\neq x_2$, then $x_1+5\neq x_2+5 \Rightarrow \frac{1}{2}\left(x_1+5\right)\neq \frac{1}{2}\left(x_2+5\right) \Rightarrow f\left(x_1\right)\neq f\left(x_2\right)$, so f is one–to–one.

10. The graph of $f(x)=1+4x-x^2$ is a parabola with axis of symmetry $x=-\frac{b}{2a}=-\frac{4}{2(-1)}=2$. Pick any x –values equidistant from 2 to find two equal function values. For example, $f(1)=4$ and $f(3)=4$, so f is not $1-1$.

11. $g(x)=|x| \Rightarrow g(-1)=1=g(1)$, so g is not one–to–one.

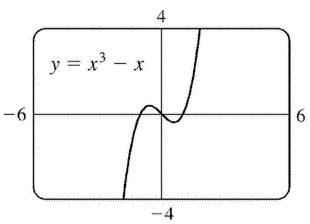
12. $x_1\neq x_2 \Rightarrow \sqrt{x_1}\neq\sqrt{x_2} \Rightarrow g\left(x_1\right)\neq g\left(x_2\right)$, so g is $1-1$.

13. A football will attain every height h up to its maximum height twice: once on the way up, and

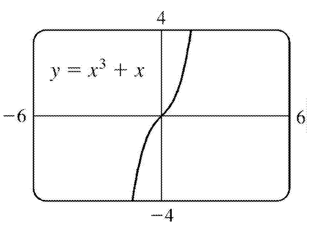
again on the way down. Thus, even if t_1 does not equal t_2 , $f(t_1)$ may equal $f(t_2)$, so f is not $1-1$.

14. f is not $1-1$ because eventually we all stop growing and therefore, there are two times at which we have the same height.

15. f does not pass the Horizontal Line Test, so f is not $1-1$.



16. f passes the Horizontal Line Test,
so f is $1-1$.



17. Since $f(2)=9$ and f is $1-1$, we know that $f^{-1}(9)=2$. Remember, if the point $(2,9)$ is on the graph of f , then the point $(9,2)$ is on the graph of f^{-1} .

18. **(a)** First, we must determine x such that $f(x)=3$. By inspection, we see that if $x=0$, then $f(x)=3$. Since f is $1-1$ (f is an increasing function), it has an inverse, and $f^{-1}(3)=0$.
(b) By the second cancellation equation in (4), we have $f\left(f^{-1}(5)\right)=5$.

19. First, we must determine x such that $g(x)=4$. By inspection, we see that if $x=0$, then $g(x)=4$. Since g is $1-1$ (g is an increasing function), it has an inverse, and $g^{-1}(4)=0$.

20. **(a)** f is $1-1$ because it passes the Horizontal Line Test.
(b) Domain of $f=[-3,3]=$ Range of f^{-1} . Range of $f=[-2,2]=$ Domain of f^{-1} .
(c) Since $f(-2)=1$, $f^{-1}(1)=-2$.

21. We solve

$C=\frac{5}{9}(F-32)$ for F : $\frac{9}{5}C=F-32 \Rightarrow F=\frac{9}{5}C+32$. This gives us a formula for the inverse function, that is, the Fahrenheit temperature F as a function of the Celsius temperature C . $F \geq -459.67 \Rightarrow$ $\frac{9}{5}C+32 \geq -459.67 \Rightarrow \frac{9}{5}C \geq -491.67 \Rightarrow C \geq -273.15$, the domain of the inverse function.

22. $m=\frac{m_0}{\sqrt{1-v^2/c^2}} \Rightarrow 1-\frac{v^2}{c^2}=\frac{m_0^2}{m^2} \Rightarrow \frac{v^2}{c^2}=1-\frac{m_0^2}{m^2} \Rightarrow v^2=c^2\left(1-\frac{m_0^2}{m^2}\right) \Rightarrow v=c\sqrt{1-\frac{m_0^2}{m^2}}$. This formula gives us the speed v of the particle in terms of its mass m , that is, $v=f^{-1}(m)$.

23. $f(x)=\sqrt{10-3x} \Rightarrow y=\sqrt{10-3x}$ $(y \geq 0) \Rightarrow y^2=10-3x \Rightarrow 3x=10-y^2 \Rightarrow x=-\frac{1}{3}y^2+\frac{10}{3}$. Interchange x and y : $y=-\frac{1}{3}x^2+\frac{10}{3}$. So $f^{-1}(x)=-\frac{1}{3}x^2+\frac{10}{3}$. Note that the domain of f^{-1} is $x \geq 0$.

24. $f(x)=\frac{4x-1}{2x+3} \Rightarrow y=\frac{4x-1}{2x+3} \Rightarrow y(2x+3)=4x-1 \Rightarrow 2xy+3y=4x-1 \Rightarrow 3y+1=4x-2xy \Rightarrow 3y+1=(4-2y)x \Rightarrow$ $x=\frac{3y+1}{4-2y}$. Interchange x and y : $y=\frac{3x+1}{4-2x}$.
So $f^{-1}(x)=\frac{3x+1}{4-2x}$.

25. $f(x)=e^{x^3} \Rightarrow y=e^{x^3} \Rightarrow \ln y=x^3 \Rightarrow x=\sqrt[3]{\ln y}$. Interchange x and y : $y=\sqrt[3]{\ln x}$.
So $f^{-1}(x)=\sqrt[3]{\ln x}$.

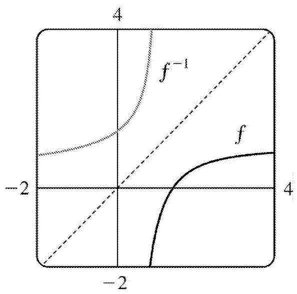
26. $y=f(x)=2x^3+3 \Rightarrow y-3=2x^3 \Rightarrow \frac{y-3}{2}=x^3 \Rightarrow x=\sqrt[3]{\frac{y-3}{2}}$.
Interchange x and y : $y=\sqrt[3]{\frac{x-3}{2}}$. So $f^{-1}(x)=\sqrt[3]{\frac{x-3}{2}}$.

27. $y=\ln(x+3) \Rightarrow x+3=e^y \Rightarrow x=e^y-3$. Interchange x and y : $y=e^x-3$. So $f^{-1}(x)=e^x-3$.

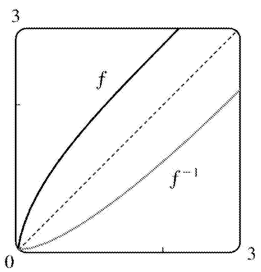
28. $y=\frac{1+e^x}{1-e^x} \Rightarrow y-ye^x=1+e^x \Rightarrow y-1=ye^x+e^x \Rightarrow y-1=e^x(y+1) \Rightarrow$

$e^x=\frac{y-1}{y+1} \Rightarrow x=\ln\left(\frac{y-1}{y+1}\right)$. Interchange x and y : $y=\ln\left(\frac{x-1}{x+1}\right)$. So $f^{-1}(x)=\ln\left(\frac{x-1}{x+1}\right)$. Note that the domain of f^{-1} is $|x|>1$.

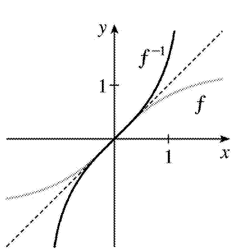
29. $y=f(x)=1-\frac{2}{x^2} \Rightarrow 1-y=\frac{2}{x^2} \Rightarrow x^2=\frac{2}{1-y} \Rightarrow x=\sqrt{\frac{2}{1-y}}$, since $x>0$. Interchange x and y : $y=\sqrt{\frac{2}{1-x}}$. So $f^{-1}(x)=\sqrt{\frac{2}{1-x}}$.



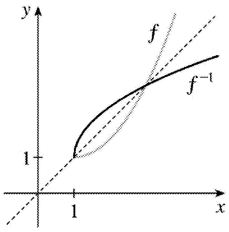
30. $y=f(x)=\sqrt{x^2+2x}$, $x>0 \Rightarrow y>0$ and $y^2=x^2+2x \Rightarrow x^2+2x-y^2=0$. Now we use the quadratic formula: $x=\frac{-2\pm\sqrt{2^2-4\cdot 1\cdot\left(-y^2\right)}}{2\cdot 1}=-1\pm\sqrt{1+y^2}$. But $x>0$, so the negative root is inadmissible. Interchange x and y : $y=-1+\sqrt{1+x^2}$. So $f^{-1}(x)=-1+\sqrt{1+x^2}$, $x>0$.

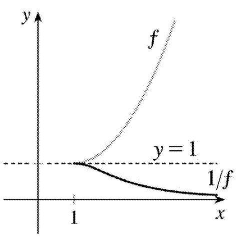


31. The function f is one–to–one, so its inverse exists and the graph of its inverse can be obtained by reflecting the graph of f about the line $y=x$.



32. The function f is one–to–one, so its inverse exists and the graph of its inverse can be obtained by reflecting the graph of f about the line $y=x$. For the graph of $1/f$, the y –coordinates are simply the reciprocals of f . For example, if $f(5)=9$, then $1/f(5)=\frac{1}{9}$. If we draw the horizontal line $y=1$, we see that the only place where the graphs intersect is on that line.





33. **(a)** It is defined as the inverse of the exponential function with base a , that is, $\log_a x=y \Leftrightarrow a^y=x$.
(b) $(0,\infty)$
(c) $\mathbb{R}$
(d) See Figure .

34. **(a)** The natural logarithm is the logarithm with base e , denoted $\ln x$.
(b) The common logarithm is the logarithm with base 10 , denoted $\log x$.
(c) See Figure .

35. **(a)** $\log_2 64=6$ since $2^6=64$.

(b) $\log_6 \frac{1}{36}=-2$ since $6^{-2}=\frac{1}{36}$.

36. **(a)** $\log_8 2=\frac{1}{3}$ since $8^{1/3}=2$.

(b) $\ln e^{\sqrt{2}}=\sqrt{2}$

37. **(a)** $\log_{10} 1.25+\log_{10} 80=\log_{10}(1.25\cdot 80)=\log_{10} 100=\log_{10} 10^2=2$

(b) $\log_5 10+\log_5 20-3\log_5 2=\log_5(10\cdot 20)-\log_5 2^3=\log_5 \frac{200}{8}=\log_5 25=\log_5 5^2=2$

38. **(a)** $2^{(\log_2 3+\log_2 5)}=2^{\log_2 15}=15$ [*Or:* $2^{(\log_2 3+\log_2 5)}=2^{\log_2 3}\cdot 2^{\log_2 5}=3\cdot 5=15$]

(b) $e^{3\ln 2}=e^{\ln (2^3)}=e^{\ln 8}=8$ [*Or:* $e^{3\ln 2}=\left(e^{\ln 2}\right)^3=2^3=8$]

39. $2\ln 4-\ln 2=\ln 4^2-\ln 2=\ln 16-\ln 2=\ln \dfrac{16}{2}=\ln 8$

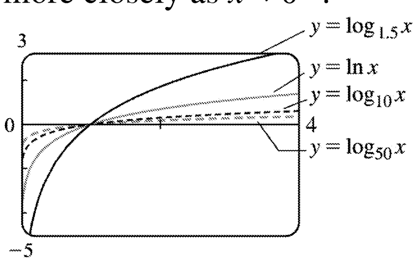
40. $\ln x+a\ln y-b\ln z=\ln x+\ln y^a-\ln z^b=\ln (x\cdot y^a)-\ln z^b=\ln (xy^a/z^b)$

41. $\ln (1+x^2)+\dfrac{1}{2}\ln x-\ln \sin x=\ln (1+x^2)+\ln x^{1/2}-\ln \sin x=\ln [(1+x^2)\sqrt{x}]-\ln \sin x=\ln \dfrac{(1+x^2)\sqrt{x}}{\sin x}$

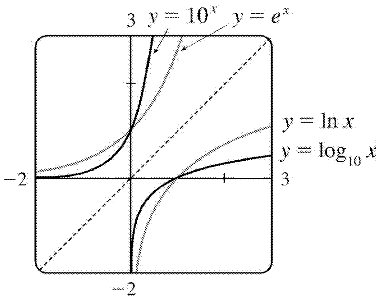
42. **(a)** $\log_{12}10=\dfrac{\ln 10}{\ln 12}\approx 0.926628$

(b) $\log_2 8.4=\dfrac{\ln 8.4}{\ln 2}\approx 3.070389$

43. To graph these functions, we use $\log_{1.5}x=\dfrac{\ln x}{\ln 1.5}$ and $\log_{50}x=\dfrac{\ln x}{\ln 50}$. These graphs all approach $-\infty$ as $x\to 0^+$, and they all pass through the point $(1,0)$. Also, they are all increasing, and all approach ∞ as $x\to\infty$. The functions with larger bases increase extremely slowly, and the ones with smaller bases do so somewhat more quickly. The functions with large bases approach the y– axis more closely as $x\to 0^+$.



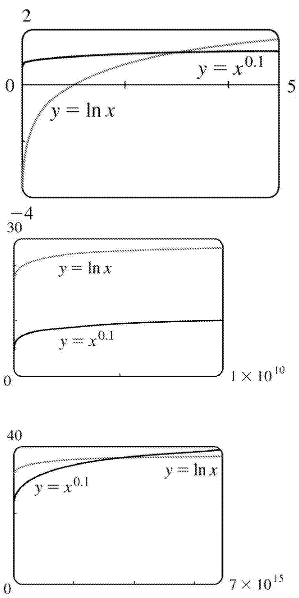
44. We see that the graph of $\ln x$ is the reflection of the graph of e^x about the line $y=x$, and that the graph of $\log_{10}x$ is the reflection of the graph of 10^x about the same line. The graph of 10^x increases more quickly than that of e^x. Also note that $\log_{10}x\to\infty$ as $x\to\infty$ more slowly than $\ln x$.



45. 3 ft $=36$ in, so we need x such that $\log_2 x=36 \Leftrightarrow x=2^{36}=68,719,476,736$. In miles, this is

$68,719,476,736 \text{ in} \cdot \dfrac{1\text{ft}}{12\text{in}} \cdot \dfrac{1\text{mi}}{5280\text{ft}} \approx 1,084,587.7$ mi.

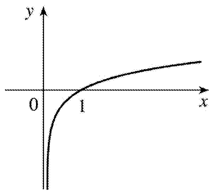
46.



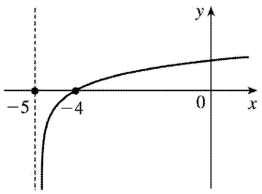
From the graphs, we see that $f(x)=x^{0.1}>g(x)=\ln x$ for approximately $0<x<3.06$, and then $g(x)>f(x)$ for $3.06<x<3.43\times 10^{15}$ (approximately). At that point, the graph of f finally surpasses the graph of g for good.

47. **(a)** Shift the graph of $y=\log_{10} x$ five units to the left to obtain the graph of $y=\log_{10}(x+5)$. Note the vertical asymptote of $x=-5$.

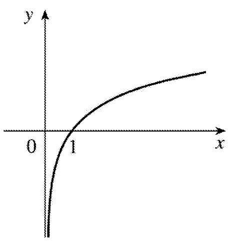
$y=\log_{10} x$



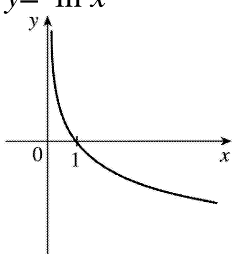
$y=\log_{10}(x+5)$



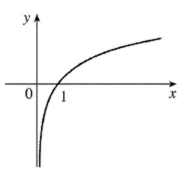
(b) Reflect the graph of $y=\ln x$ about the x –axis to obtain the graph of $y=-\ln x$.
$y=\ln x$



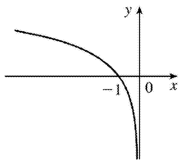
$y=-\ln x$



48. **(a)** Reflect the graph of $y=\ln x$ about the y –axis to obtain the graph of $y=\ln(-x)$.
$y=\ln x$

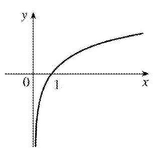


$y=\ln(-x)$

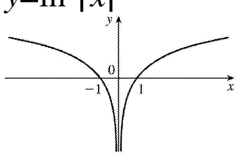


(b) Reflect the portion of the graph of $y=\ln x$ to the right of the y –axis about the y –axis. The graph of $y=\ln |x|$ is that reflection in addition to the original portion.

$y=\ln x$



$y=\ln |x|$



49. **(a)** $2\ln x=1 \Rightarrow \ln x=\frac{1}{2} \Rightarrow x=e^{1/2}=\sqrt{e}$

(b) $e^{-x}=5 \Rightarrow -x=\ln 5 \Rightarrow x=-\ln 5$

50. **(a)** $e^{2x+3}-7=0 \Rightarrow e^{2x+3}=7 \Rightarrow 2x+3=\ln 7 \Rightarrow 2x=\ln 7-3 \Rightarrow x=\frac{1}{2}(\ln 7-3)$

(b) $\ln (5-2x)=-3 \Rightarrow 5-2x=e^{-3} \Rightarrow 2x=5-e^{-3} \Rightarrow x=\frac{1}{2}(5-e^{-3})$

51. **(a)** $2^{x-5}=3 \Leftrightarrow \log_2 3=x-5 \Leftrightarrow x=5+\log_2 3$.

Or: $2^{x-5}=3 \Leftrightarrow \ln \left(2^{x-5}\right)=\ln 3 \Leftrightarrow (x-5)\ln 2=\ln 3 \Leftrightarrow x-5=\frac{\ln 3}{\ln 2} \Leftrightarrow x=5+\frac{\ln 3}{\ln 2}$

(b) $\ln x+\ln (x-1)=\ln (x(x-1))=1 \Leftrightarrow x(x-1)=e^1 \Leftrightarrow x^2-x-e=0$. The quadratic formula (with $a=1$, $b=-1$, and $c=-e$) gives $x=\frac{1}{2}\left(1\pm\sqrt{1+4e}\right)$, but we reject the negative root since the natural logarithm is not defined for $x<0$. So $x=\frac{1}{2}\left(1+\sqrt{1+4e}\right)$.

52. **(a)** $\ln(\ln x)=1 \Leftrightarrow e^{\ln(\ln x)}=e^{1} \Leftrightarrow \ln x=e^{1}=e \Leftrightarrow e^{\ln x}=e^{e} \Leftrightarrow x=e^{e}$

(b) $e^{ax}=Ce^{bx} \Leftrightarrow \ln e^{ax}=\ln[C(e^{bx})] \Leftrightarrow ax=\ln C+bx+\ln e^{bx} \Leftrightarrow ax=\ln C+bx \Leftrightarrow ax-bx=\ln C \Leftrightarrow (a-b)x=\ln C \Leftrightarrow$
$x=\dfrac{\ln C}{a-b}$

53. **(a)** $e^{x}<10 \Rightarrow \ln e^{x}<\ln 10 \Rightarrow x<\ln 10 \Rightarrow x\in(-\infty,\ln 10)$

(b) $\ln x>-1 \Rightarrow e^{\ln x}>e^{-1} \Rightarrow x>e^{-1} \Rightarrow x\in(1/\mathrm{e},\infty)$

54. **(a)** $2<\ln x<9 \Rightarrow e^{2}<e^{\ln x}<e^{9} \Rightarrow e^{2}<x<e^{9} \Rightarrow x\in\left(e^{2},\mathrm{e}^{9}\right)$

(b) $e^{2-3x}>4 \Rightarrow \ln e^{2-3x}>\ln 4 \Rightarrow 2-3x>\ln 4 \Rightarrow -3x>\ln 4-2 \Rightarrow x<-\frac{1}{3}(\ln 4-2) \Rightarrow x\in\left(-\infty,\frac{1}{3}(2-\ln 4)\right)$

55. **(a)** For $f(x)=\sqrt{3-e^{2x}}$, we must have $3-e^{2x}\geq 0 \Rightarrow e^{2x}\leq 3 \Rightarrow 2x\leq \ln 3 \Rightarrow x\leq \frac{1}{2}\ln 3$.
Thus, the domain of f is $(-\infty,\frac{1}{2}\ln 3]$.

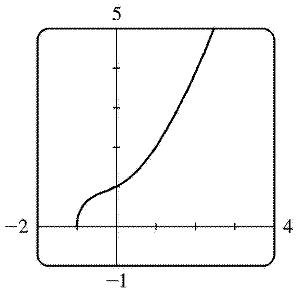
(b) $y=f(x)=\sqrt{3-e^{2x}}$ [note that $y\geq 0$] $\Rightarrow y^{2}=3-e^{2x} \Rightarrow e^{2x}=3-y^{2} \Rightarrow 2x=\ln(3-y^{2}) \Rightarrow x=\frac{1}{2}\ln(3-y^{2})$.
Interchange x and y : $y=\frac{1}{2}\ln(3-x^{2})$. So $f^{-1}(x)=\frac{1}{2}\ln(3-x^{2})$. For the domain of f^{-1} , we must have $3-x^{2}>0 \Rightarrow x^{2}<3 \Rightarrow |x|<\sqrt{3} \Rightarrow -\sqrt{3}<x<\sqrt{3} \Rightarrow 0\leq x<\sqrt{3}$ since $x\geq 0$. Note that the domain of f^{-1} , $[0,\sqrt{3})$, equals the range of f .

56. **(a)** For $f(x)=\ln(2+\ln x)$, we must have $2+\ln x>0 \Rightarrow \ln x>-2 \Rightarrow x>e^{-2}$. Thus, the domain of f is (e^{-2},∞) .

(b) $y=f(x)=\ln(2+\ln x) \Rightarrow e^{y}=2+\ln x \Rightarrow \ln x=e^{y}-2 \Rightarrow x=e^{e^{y}-2}$. Interchange x and y : $y=e^{e^{x}-2}$. So $f^{-1}(x)=e^{e^{x}-2}$. The domain of f^{-1} , as well as the range of f , is $\mathbb{R}$.

57. We see that the graph of $y=f(x)=\sqrt{x^{3}+x^{2}+x+1}$ is increasing, so f is 1 – 1 . Enter $x=\sqrt{y^{3}+y^{2}+y+1}$ and use your CAS to solve the equation for y . Using Derive, we get two (irrelevant) solutions involving imaginary expressions, as well as one which can be simplified to the following:

$$y=f^{-1}(x)=-\frac{\sqrt[3]{4}}{6}\left(\sqrt[3]{D-27x^{2}+20}-\sqrt[3]{D+27x^{2}-20}+\sqrt[3]{2}\right)$$



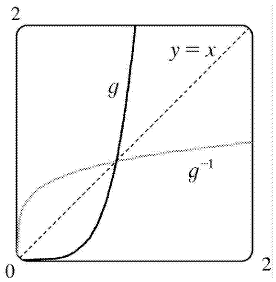
where $D=3\sqrt{3}\sqrt{27x^4-40x^2+16}$. Maple and Mathematica each give two complex expressions and one real expression, and the real expression is equivalent to that given by Derive. For example, Maple's expression simplifies to $\frac{1}{6}\ \frac{M^{2/3}-8-2M^{1/3}}{2M^{1/3}}$, where $M=108x^2+12\sqrt{48-120x^2+81x^4}-80$.

58. **(a)** If we use Derive, then solving $x=y^6+y^4$ for y gives us six solutions of the form $y=\pm\frac{\sqrt{3}}{3}\sqrt{B-1}$, where $B\in\left\{-2\sin\frac{A}{3},2\sin\left(\frac{A}{3}+\frac{\pi}{3}\right),-2\cos\left(\frac{A}{3}+\frac{\pi}{6}\right)\right\}$ and $A=\sin^{-1}\left(\frac{27x-2}{2}\right)$. The inverse for $y=x^6+x^4$ ($x\geq 0$) is $y=\frac{\sqrt{3}}{3}\sqrt{B-1}$ with $B=2\sin\left(\frac{A}{3}+\frac{\pi}{3}\right)$, but because the domain of A is $\left[0,\frac{4}{27}\right]$, this expression is only valid for $x\in\left[0,\frac{4}{27}\right]$.

Happily, Maple gives us the rest of the solution! We solve $x=y^6+y^4$ for y to get the two real solutions $\pm\frac{\sqrt{6}}{6}\ \frac{\sqrt{C^{1/3}\left(C^{2/3}-2C^{1/3}+4\right)}}{C^{1/3}}$, where $C=108x+12\sqrt{3}\sqrt{x(27x-4)}$, and the inverse for $y=x^6+x^4$ ($x\geq 0$) is the positive solution, whose domain is $\left[\frac{4}{27},\infty\right)$.

Mathematica also gives two real solutions, equivalent to those of Maple. The positive one is $\frac{\sqrt{6}}{6}\left(\sqrt[3]{4}\,D^{1/3}+2\sqrt[3]{2}\,D^{-1/3}-2\right)$, where $D=-2+27x+3\sqrt{3}\sqrt{x}\sqrt{27x-4}$. Although this expression also has domain $\left[\frac{4}{27},\infty\right)$, Mathematica is mysteriously able to plot the solution for all $x\geq 0$.

(b)



59. **(a)** $n=100\cdot 2^{t/3}\Rightarrow \frac{n}{100}=2^{t/3}\Rightarrow \log_2\left(\frac{n}{100}\right)=\frac{t}{3}\Rightarrow t=3\log_2\left(\frac{n}{100}\right)$. Using formula ( ), we can write this as $t=3\cdot\frac{\ln(n/100)}{\ln 2}$. This function tells us how long it will take to obtain n bacteria (given the number n).
(b) $n=50,000\Rightarrow t=3\log_2\frac{50,000}{100}=3\log_2 500=3\left(\frac{\ln 500}{\ln 2}\right)\approx 26.9$ hours

60. **(a)** $Q=Q_0\left(1-e^{-t/a}\right)\Rightarrow \frac{Q}{Q_0}=1-e^{-t/a}\Rightarrow e^{-t/a}=1-\frac{Q}{Q_0}\Rightarrow -\frac{t}{a}=\ln\left(1-\frac{Q}{Q_0}\right)\Rightarrow t=-a\ln(1-Q/Q_0)$. This gives us the time t necessary to obtain a given charge Q .
(b) $Q=0.9Q_0$ and $a=2\Rightarrow t=-2\ln\left(1-0.9\left(Q_0/Q_0\right)\right)=-2\ln 0.1\approx 4.6$ seconds.

61. **(a)** To find the equation of the graph that results from shifting the graph of $y=\ln x$3 units upward, we add 3 to the original function to get $y=\ln x+3$.
(b) To find the equation of the graph that results from shifting the graph of $y=\ln x$3 units to the left, we replace x with $x+3$ in the original function to get $y=\ln(x+3)$.
(c) To find the equation of the graph that results from reflecting the graph of $y=\ln x$ about the x –axis, we multiply the original equation by -1 to get $y=-\ln x$.
(d) To find the equation of the graph that results from reflecting the graph of $y=\ln x$ about the y –axis, we replace x with $-x$ in the original equation to get $y=\ln(-x)$.
(e) To find the equation of the graph that results from reflecting the graph of $y=\ln x$ about the line $y=x$, we interchange x and y in the original equation to get $x=\ln y\Leftrightarrow y=e^x$.
(f) To find the equation of the graph that results from reflecting the graph of $y=\ln x$ about the x –axis and then about the line $y=x$, we first multiply the original equation by -1 and then interchange x and y in this equation to get $x=-\ln y\Leftrightarrow \ln y=-x\Leftrightarrow y=e^{-x}$.
(g) To find the equation of the graph that results from reflecting the graph of $y=\ln x$ about the y –axis and then about the line $y=x$, we first replace x with $-x$ in the original equation and then interchange x and y to get $x=\ln(-y)\Leftrightarrow -y=e^x\Leftrightarrow y=-e^x$.
(h) To find the equation of the graph that results from shifting the graph of $y=\ln x$3 units to the left and then reflecting it about the line $y=x$, we first replace x with $x+3$ in the original equation and then

interchange x and y in this equation to get $x=\ln(y+3) \Leftrightarrow y+3=e^x \Leftrightarrow y=e^x-3$.

62. **(a)** If the point (x,y) is on the graph of $y=f(x)$, then the point $(x-c,y)$ is that point shifted c units to the left. Since f is $1-1$, the point (y,x) is on the graph of $y=f^{-1}(x)$ and the point corresponding to $(x-c,y)$ on the graph of f is $(y,x-c)$ on the graph of f^{-1} . Thus, the curve's reflection is shifted *down* the same number of units as the curve itself is shifted to the left. So an expression for the inverse function is $g^{-1}(x)=f^{-1}(x)-c$.
(b) If we compress (or stretch) a curve horizontally, the curve's reflection in the line $y=x$ is compressed (or stretched) *vertically* by the same factor. Using this geometric principle, we see that the inverse of $h(x)=f(cx)$ can be expressed as $h^{-1}(x)=(1/c)f^{-1}(x)$.

63. **(a)** $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)=\frac{\pi}{3}$ since $\sin\frac{\pi}{3}=\frac{\sqrt{3}}{2}$ and $\frac{\pi}{3}$ is in $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$.
(b) $\cos^{-1}(-1)=\pi$ since $\cos\pi=-1$ and π is in $[0,\pi]$.

64. **(a)** $\arctan(-1)=-\frac{\pi}{4}$ since $\tan\left(-\frac{\pi}{4}\right)=-1$ and $-\frac{\pi}{4}$ is in $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$.
(b) $\csc^{-1}2=\frac{\pi}{6}$ since $\csc\frac{\pi}{6}=2$ and $\frac{\pi}{6}$ is in $\left(0,\frac{\pi}{2}\right]\cup\left(\pi,\frac{3\pi}{2}\right]$.

65. **(a)** $\tan^{-1}\sqrt{3}=\frac{\pi}{3}$ since $\tan\frac{\pi}{3}=\sqrt{3}$ and $\frac{\pi}{3}$ is in $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$.
(b) $\arcsin\left(-\frac{1}{\sqrt{2}}\right)=-\frac{\pi}{4}$ since $\sin\left(-\frac{\pi}{4}\right)=-\frac{1}{\sqrt{2}}$ and $-\frac{\pi}{4}$ is in $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$.

66. **(a)** $\sec^{-1}\sqrt{2}=\frac{\pi}{4}$ since $\sec\frac{\pi}{4}=\sqrt{2}$ and $\frac{\pi}{4}$ is in $\left[0,\frac{\pi}{2}\right)\cup\left[\pi,\frac{3\pi}{2}\right)$.
(b) $\arcsin 1=\frac{\pi}{2}$ since $\sin\frac{\pi}{2}=1$ and $\frac{\pi}{2}$ is in $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$.

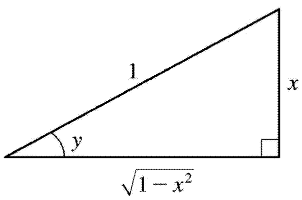
67. **(a)** $\sin(\sin^{-1}0.7)=0.7$ since 0.7 is in $[-1,1]$.
(b) $\tan^{-1}\left(\tan\frac{4\pi}{3}\right)=\tan^{-1}\sqrt{3}=\frac{\pi}{3}$ since $\frac{\pi}{3}$ is in $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$.

68. **(a)** Let $\theta=\arctan 2$, so $\tan\theta=2 \Rightarrow \sec^2\theta=1+\tan^2\theta=1+4=5 \Rightarrow \sec\theta=\sqrt{5} \Rightarrow \sec(\arctan 2)=\sec\theta=\sqrt{5}$.
(b) Let

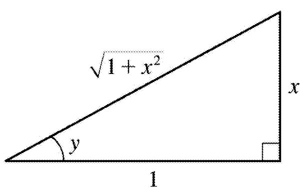
$\theta = \sin^{-1}\frac{5}{13}$. Then $\sin\theta = \frac{5}{13}$, so $\cos\left(2\sin^{-1}\frac{5}{13}\right) = \cos 2\theta = 1-2\sin^2\theta = 1-2\left(\frac{5}{13}\right)^2 = \frac{119}{169}$.

69. Let $y=\sin^{-1}x$. Then $-\frac{\pi}{2} \le y \le \frac{\pi}{2} \Rightarrow \cos y \ge 0$, so $\cos(\sin^{-1}x) = \cos y = \sqrt{1-\sin^2 y} = \sqrt{1-x^2}$

70. Let $y=\sin^{-1}x$. Then $\sin y = x$, so from the triangle we see that $\tan(\sin^{-1}x) = \tan y = \frac{x}{\sqrt{1-x^2}}$.



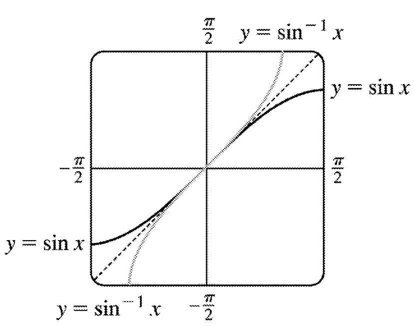
71. Let $y=\tan^{-1}x$. Then $\tan y = x$, so from the triangle we see that $\sin(\tan^{-1}x) = \sin y = \frac{x}{\sqrt{1+x^2}}$.



72.

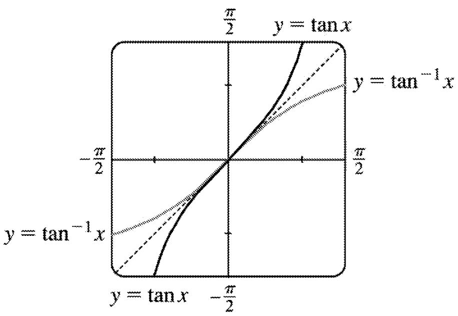
Let $y=\cos^{-1}x$. Then $\cos y = x \Rightarrow \sin y = \sqrt{1-x^2}$ since $0 \le y \le \pi$. So
$\sin(2\cos^{-1}x) = \sin 2y = 2\sin y\cos y = 2x\sqrt{1-x^2}$.

73.



The graph of $\sin^{-1}x$ is the reflection of the graph of $\sin x$ about the line $y=x$.

74.



The graph of $\tan^{-1} x$ is the reflection of the graph of $\tan x$ about the line $y=x$.

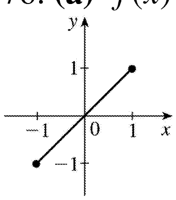
75.

$g(x)=\sin^{-1}(3x+1)$.

Domain $(g)=\{x \mid -1\leq 3x+1\leq 1\}=\{x \mid -2\leq 3x\leq 0\}=\left\{ x \mid -\frac{2}{3}\leq x\leq 0\right\}=\left[-\frac{2}{3},0\right]$.

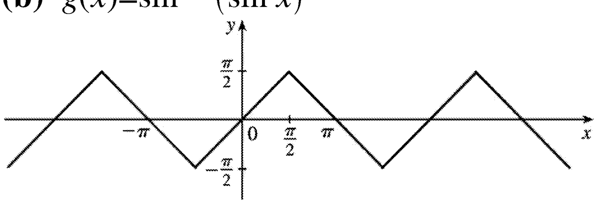
Range $(g)=\left\{ y \mid -\frac{\pi}{2}\leq y\leq \frac{\pi}{2}\right\}=\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$.

76. **(a)** $f(x)=\sin\left(\sin^{-1} x\right)$



Since one function undoes what the other one does, we get the identity function, $y=x$, on the restricted domain $-1\leq x\leq 1$.

(b) $g(x)=\sin^{-1}(\sin x)$



This is similar to part (a), but with domain R . Equations for g on intervals of the form $\left(-\frac{\pi}{2}+\pi n,\frac{\pi}{2}+\pi n\right)$, for any integer n , can be found using $g(x)=(-1)^{n}x+(-1)^{n+1}n\pi$. The sine function is monotonic on each of these intervals, and hence, so is g (but in a linear fashion).

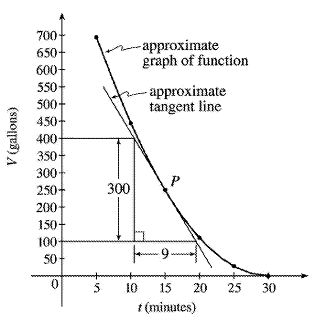
1. **(a)** Using $P(15{,}250)$, we construct the following table:

t	Q	slope=m_{PQ}
5	$(5{,}694)$	$\frac{694-250}{5-15}=-\frac{444}{10}=-44.4$
10	$(10{,}444)$	$\frac{444-250}{10-15}=-\frac{194}{5}=-38.8$
20	$(20{,}111)$	$\frac{111-250}{20-15}=-\frac{139}{5}=-27.8$
25	$(25{,}28)$	$\frac{28-250}{25-15}=-\frac{222}{10}=-22.2$
30	$(30{,}0)$	$\frac{0-250}{30-15}=-\frac{250}{15}=-16.\overline{6}$

(b) Using the values of t that correspond to the points closest to P ($t=10$ and $t=20$), we have

$$\frac{-38.8+(-27.8)}{2}=-33.3$$

(c) From the graph, we can estimate the slope of the tangent line at P to be $\frac{-300}{9}=-33.\overline{3}$.



2.

(a) $\text{Slope}=\frac{2948-2530}{42-36}=\frac{418}{6}\approx 69.67$ (b) $\text{Slope}=\frac{2948-2661}{42-38}=\frac{287}{4}=71.75$

(c) $\text{Slope}=\frac{2948-2806}{42-40}=\frac{142}{2}=71$ (d) $\text{Slope}=\frac{3080-2948}{44-42}=\frac{132}{2}=66$

From the data, we see that the patient's heart rate is decreasing from 71 to 66 heartbeats / minute after 42 minutes. After being stable for a while, the patient's heart rate is dropping.

3. **(a)** For the curve $y=x/(1+x)$ and the point $P\left(1,\frac{1}{2}\right)$

	x	Q	m_{PQ}
(i)	0.5	(0.5,0.333333)	0.333333
(ii)	0.9	(0.9,0.473684)	0.263158
(iii)	0.99	(0.99,0.497487)	0.251256
(iv)	0.999	(0.999,0.499750)	0.250125
(v)	1.1	(1.5,06)	0.2
(vi)	1.5	(1.1,0.523810)	0.238095
(vii)	1.01	(1.01,0.502488)	0.248756
(viii)	1.001	(1.001,0.500250)	0.249875

(b) The slope appears to be $\frac{1}{4}$.

(c) $y-\frac{1}{2}=\frac{1}{4}(x-1)$ or $y=\frac{1}{4}x+\frac{1}{4}$.

4. For the curve $y=\ln x$ and the point $P(2,\ln 2)$:

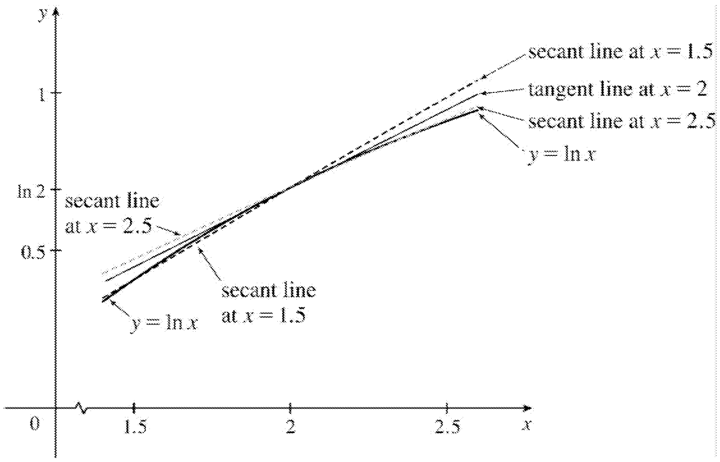
(a)

	x	Q	m_{PQ}
(i)	1.5	(1.5,0.405465)	0.575364
(ii)	1.9	(1.9,0.641854)	0.512933
(iii)	1.99	(1.99,0.688135)	0.501254
(iv)	1.999	(1.999,0.692647)	0.500125
(v)	2.5	(2.5,0.916291)	0.446287
(vi)	2.1	(2.1,0.741937)	0.487902
(vii)	2.01	(2.01,0.698135)	0.498754
(viii)	2.001	(2.001,0.693647)	0.499875

(b) The slope appears to be $\frac{1}{2}$.

(c) $y-\ln 2=\frac{1}{2}(x-2)$ or $y=\frac{1}{2}x-1+\ln 2$

(d)



5. **(a)** $y=y(t)=40t-16t^2$. At $t=2$, $y=40(2)-16(2)^2=16$. The average velocity between times 2 and $2+h$ is $v_{ave}=\dfrac{y(2+h)-y(2)}{(2+h)-2}=\dfrac{\left[40(2+h)-16(2+h)^2\right]-16}{h}=\dfrac{-24h-16h^2}{h}=-24-16h$, if $h\neq 0$.

(i) $[2,2.5]$: $h=0.5$, $v_{ave}=-32$ ft / s (ii) $[2,2.1]$: $h=0.1$, $v_{ave}=-25.6$ ft / s

(iii) $[2,2.05]$: $h=0.05$, $v_{ave}=-24.8$ ft / s (iv) $[2,2.01]$: $h=0.01$, $v_{ave}=-24.16$ ft / s

(b) The instantaneous velocity when $t=2$ (h approaches 0) is -24 ft / s.

6. The average velocity between t and $t+h$ seconds is

$$\frac{58(t+h)-0.83(t+h)^2-\left(58t-0.83t^2\right)}{h}=\frac{58h-1.66th-0.83h^2}{h}=58-1.66t-0.83h \text{ if } h\neq 0 .$$

(a) Here $t=1$, so the average velocity is $58-1.66-0.83h=56.34-0.83h$.

(i) $[1,2]$: $h=1, 55.51$ m / s (ii) $[1,1.5]$: $h=0.5$, 55.925 m / s

(iii) $[1,1.1]$: $h=0.1$, 56.257 m / s (iv) $[1,1.01]$: $h=0.01$, 56.3317 m / s

(v) $[1,1.001]$: $h=0.001$, 56.33917 m / s

(b) The instantaneous velocity after 1 second is 56.34 m / s.

7. **(a)**

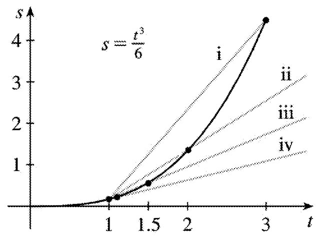
(i) $[1,3]$: $h=2$, $v_{ave}=\dfrac{13}{6}$ ft / s (ii) $[1,2]$: $h=1$, $v_{ave}=\dfrac{7}{6}$ ft / s

(iii) $[1,1.5]$: $h=0.5$, $v_{ave}=\dfrac{19}{24}$ ft / s (iv) $[1,1.1]$: $h=0.1$, $v_{ave}=\dfrac{331}{600}$ ft / s

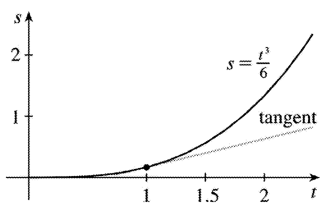
(b) As h approaches 0 , the velocity approaches

$\frac{3}{6}=\frac{1}{2}$ ft / s.

(c)



(d)



8. Average velocity between times $t=2$ and $t=2+h$ is given by $\frac{s(2+h)-s(2)}{h}$.

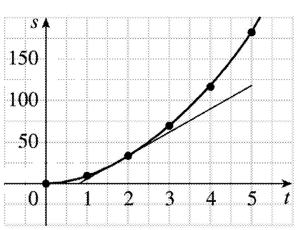
(a)

(i) $h=3 \Rightarrow v_{\text{av}}=\frac{s(5)-s(2)}{5-2}=\frac{178-32}{3}=\frac{146}{3}\approx 48.7$ ft / s

(ii) $h=2 \Rightarrow v_{\text{av}}=\frac{s(4)-s(2)}{4-2}=\frac{119-32}{2}=\frac{87}{2}=43.5$ ft / s

(iii) $h=1 \Rightarrow v_{\text{av}}=\frac{s(3)-s(2)}{3-2}=\frac{70-32}{1}=38$ ft / s

(b) Using the points $(0.8,0)$ and $(5,118)$ from the approximate tangent line, the instantaneous velocity at $t=2$ is about $\frac{118-0}{5-0.8}\approx 28$ ft / s.



9. For the curve $y=\sin\ (10\pi/x)$ and the point $P\,(1,0)$:

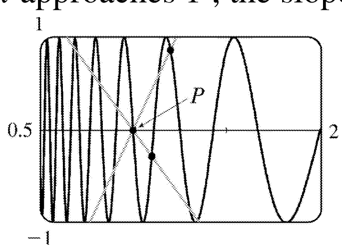
(a)

x	Q	m_{PQ}
2	$(2,0)$	0

1.5	$(1.5,0.8660)$	1.7321
1.4	$(1.4,-0.4339)$	-1.0847
1.3	$(1.3,-0.8230)$	-2.7433
1.2	$(1.2,0.8660)$	4.3301
1.1	$(1.1,-0.2817)$	-2.8173

x	Q	m_{PQ}
0.5	$(0.5,0)$	0
0.6	$(0.6,0.8660)$	-2.1651
0.7	$(0.7,0.7818)$	-2.6061
0.8	$(0.8,1)$	-5
0.9	$(0.9,-0.3420)$	3.4202

As x approaches 1 , the slopes do not appear to be approaching any particular value.



(b)
We see that problems with estimation are caused by the frequent oscillations of the graph. The tangent is so steep at *P* that we need to take x –values much closer to 1 in order to get accurate estimates of its slope.

(c) If we choose $x=1.001$, then the point Q is $(1.001,-0.0314)$ and $m_{PQ}\approx-31.3794$. If $x=0.999$, then Q is $(0.999,0.0314)$ and $m_{PQ}=-31.4422$. The average of these slopes is -31.4108 . So we estimate that the slope of the tangent line at *P* is about -31.4 .

1. As x approaches 2, $f(x)$ approaches 5. [Or, the values of $f(x)$ can be made as close to 5 as we like by taking x sufficiently close to 2 (but $x \neq 2$).] Yes, the graph could have a hole at $(2,5)$ and be defined such that $f(2)=3$.

2. As x approaches 1 from the left, $f(x)$ approaches 3; and as x approaches 1 from the right, $f(x)$ approaches 7. No, the limit does not exist because the left– and right–hand limits are different.

3. **(a)** $\lim_{x \to -3} f(x)=\infty$ means that the values of $f(x)$ can be made arbitrarily large (as large as we please) by taking x sufficiently close to -3 (but not equal to -3).
(b) $\lim_{x \to 4^+} f(x)=-\infty$ means that the values of $f(x)$ can be made arbitrarily large negative by taking x sufficiently close to 4 through values larger than 4.

4. **(a)** $\lim_{x \to 0} f(x)=3$
(b) $\lim_{x \to 3^-} f(x)=4$
(c) $\lim_{x \to 3^+} f(x)=2$
(d) $\lim_{x \to 3} f(x)$ does not exist because the limits in part (b) and part (c) are not equal.
(e) $f(3)=3$

5. **(a)** $f(x)$ approaches 2 as x approaches 1 from the left, so $\lim_{x \to 1^-} f(x)=2$.
(b) $f(x)$ approaches 3 as x approaches 1 from the right, so $\lim_{x \to 1^+} f(x)=3$.
(c) $\lim_{x \to 1} f(x)$ does not exist because the limits in part (a) and part (b) are not equal.
(d) $f(x)$ approaches 4 as x approaches 5 from the left and from the right, so $\lim_{x \to 5} f(x)=4$.
(e) $f(5)$ is not defined, so it doesn't exist.

6. **(a)** $\lim_{x \to -2^-} g(x)=-1$
(b) $\lim_{x \to -2^+} g(x)=1$
(c) $\lim_{x \to -2} g(x)$ doesn't exist
(d) $g(-2)=1$
(e) $\lim_{x \to 2^-} g(x)=1$

(f) $\lim_{x \to 2^+} g(x) = 2$

(g) $\lim_{x \to 2} g(x)$ doesn't exist

(h) $g(2) = 2$

(i) $\lim_{x \to 4^+} g(x)$ doesn't exist

(j) $\lim_{x \to 4^-} g(x) = 2$

(k) $g(0)$ doesn't exist

(l) $\lim_{x \to 0} g(x) = 0$

7. **(a)** $\lim_{t \to 0^-} g(t) = -1$

(b) $\lim_{t \to 0^+} g(t) = -2$

(c) $\lim_{t \to 0} g(t)$ does not exist because the limits in part (a) and part (b) are not equal.

(d) $\lim_{t \to 2^-} g(t) = 2$

(e) $\lim_{t \to 2^+} g(t) = 0$

(f) $\lim_{t \to 2} g(t)$ does not exist because the limits in part (d) and part (e) are not equal.

(g) $g(2) = 1$

(h) $\lim_{t \to 4} g(t) = 3$

8. **(a)** $\lim_{x \to 2} R(x) = -\infty$

(b) $\lim_{x \to 5} R(x) = \infty$

(c) $\lim_{x \to -3^-} R(x) = -\infty$

(d) $\lim_{x \to -3^+} R(x) = \infty$

(e) The equations of the vertical asymptotes are $x = -3$, $x = 2$, and $x = 5$.

9. **(a)** $\lim_{x \to -7} f(x) = -\infty$

(b) $\lim_{x \to -3} f(x) = \infty$

(c)

$\lim_{x\to 0} f(x)=\infty$

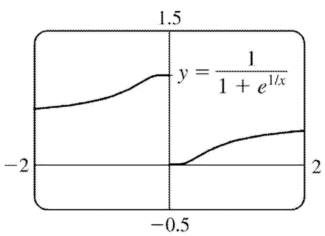
(d) $\lim_{x\to 6^-} f(x)=-\infty$

(e) $\lim_{x\to 6^+} f(x)=\infty$

(f) The equations of the vertical asymptotes are $x=-7$, $x=-3$, $x=0$, and $x=6$.

10. $\lim_{t\to 12^-} f(t)=150$ mg and $\lim_{t\to 12^+} f(t)=300$ mg. These limits show that there is an abrupt change in the amount of drug in the patient's bloodstream at $t=12$ h. The left–hand limit represents the amount of the drug just before the fourth injection. The right–hand limit represents the amount of the drug just after the fourth injection.

11.

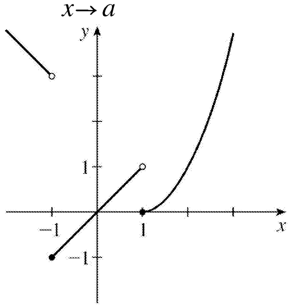


(a) $\lim_{x\to 0^-} f(x)=1$

(b) $\lim_{x\to 0^+} f(x)=0$

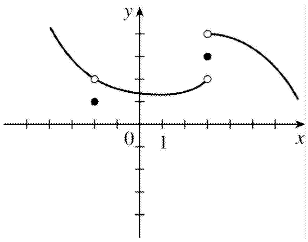
(c) $\lim_{x\to 0} f(x)$ does not exist because the limits in part (a) and part (b) are not equal.

12. $\lim_{x\to a} f(x)$ exists for all a except $a=\pm 1$.

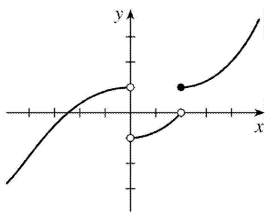


13. $\lim_{x\to 3^+} f(x)=4$, $\lim_{x\to 3^-} f(x)=2$,

$\lim_{x\to -2} f(x)=2$, $f(3)=3$, $f(-2)=1$



14. $\lim_{x\to 0^-} f(x)=1$, $\lim_{x\to 0^+} f(x)=-1$, $\lim_{x\to 2^-} f(x)=0$, $\lim_{x\to 2^+} f(x)=1$, $f(2)=1$, $f(0)$ is undefined



15. For $f(x)=\dfrac{x^2-2x}{x^2-x-2}$:

x	$f(x)$
2.5	0.714286
2.1	0.677419
2.05	0.672131
2.01	0.667774
2.005	0.667221
2.001	0.666778

x	$f(x)$
1.9	0.655172
1.95	0.661017
1.99	0.665552
1.995	0.666110
1.999	0.666556

It appears that $\lim_{x\to 2} \dfrac{x^2-2x}{x^2-x-2}=0.\overline{6}=\dfrac{2}{3}$.

16. For $f(x)=\dfrac{x^2-2x}{x^2-x-2}$:

x	$f(x)$
0	0
−0.5	−1
−0.9	−9
−0.95	−19
−0.99	−99
−0.999	−999

x	$f(x)$
−2	2
−1.5	3
−1.1	11
−1.01	101
−1.001	1001

It appears that $\lim_{x\to -1} \dfrac{x^2-2x}{x^2-x-2}$ *does not exist* since $f(x)\to -\infty$ as $x\to -1^-$ and $f(x)\to \infty$ as $x\to -1^+$.

17. For $f(x)=\dfrac{e^x-1-x}{x^2}$:

x	$f(x)$
1	0.718282
0.5	0.594885
0.1	0.517092
0.05	0.508439
0.01	0.501671

x	$f(x)$
−1	0.367879

–0.5	0.426123
–0.1	0.483742
–0.05	0.491770
–0.01	0.498337

It appears that $\lim_{x\to 0} \frac{e^x - 1 - x}{x^2} = 0.5 = \frac{1}{2}$.

18. For $f(x) = x\ln\left(x+x^2\right)$:

x	$f(x)$
1	0.693147
0.5	–0.143841
0.1	–0.220727
0.05	–0.147347
0.01	–0.045952
0.005	–0.026467
0.001	–0.006907

It appears that $\lim_{x\to 0^+} x\ln\left(x+x^2\right) = 0$.

19. For $f(x) = \frac{\sqrt{x+4} - 2}{x}$:

x	$f(x)$
1	0.236068
0.5	0.242641
0.1	0.248457
0.05	0.249224
0.01	0.249844

x	$f(x)$
–1	0.267949

–0.5	0.258343
–0.1	0.251582
–0.05	0.250786
–0.01	0.250156

It appears that $\lim_{x\to 0} \frac{\sqrt{x+4}-2}{x} = 0.25 = \frac{1}{4}$.

20. For $f(x)=\frac{\tan 3x}{\tan 5x}$:

x	$f(x)$
± 0.2	0.439279
± 0.1	0.566236
± 0.05	0.591893
± 0.01	0.599680
± 0.001	0.599997

It appears that $\lim_{x\to 0} \frac{\tan 3x}{\tan 5x} = 0.6 = \frac{3}{5}$.

21. For $f(x)=\frac{x^6-1}{x^{10}-1}$:

x	$f(x)$
0.5	0.985337
0.9	0.719397
0.95	0.660186
0.99	0.612018
0.999	0.601200

x	$f(x)$
1.5	0.183369
1.1	0.484119
1.05	0.540783
1.01	0.588022

1.001	0.598800

It appears that $\lim_{x\to 1} \frac{x^6-1}{x^{10}-1} = 0.6 = \frac{3}{5}$.

22. For $f(x) = \frac{9^x - 5^x}{x}$:

x	$f(x)$
0.5	1.527864
0.1	0.711120
0.05	0.646496
0.01	0.599082
0.001	0.588906

x	$f(x)$
–0.5	0.227761
–0.1	0.485984
–0.05	0.534447
–0.01	0.576706
–0.001	0.586669

It appears that $\lim_{x\to 0} \frac{9^x - 5^x}{x} = 0.59$. Later we will be able to show that the exact value is ln (9/5) .

23. $\lim_{x\to 5^+} \frac{6}{x-5} = \infty$ since $(x-5) \to 0$ as $x \to 5^+$ and $\frac{6}{x-5} > 0$ for $x > 5$.

24. $\lim_{x\to 5^-} \frac{6}{x-5} = -\infty$ since $(x-5) \to 0$ as $x \to 5^-$ and $\frac{6}{x-5} < 0$ for $x < 5$.

25. $\lim_{x\to 1} \frac{2-x}{(x-1)^2} = \infty$ since the numerator is positive and the denominator approaches 0 through positive values as $x \to 1$.

26.

$\lim_{x\to 0} \frac{x-1}{x^2(x+2)} = -\infty$ since $x^2 \to 0$ as $x \to 0$ and $\frac{x-1}{x^2(x+2)} < 0$ for $0<x<1$ and for $-2<x<0$.

27. $\lim_{x\to -2^+} \frac{x-1}{x^2(x+2)} = -\infty$ since $(x+2) \to 0$ as $x \to -2^+$ and $\frac{x-1}{x^2(x+2)} < 0$ for $-2<x<0$.

28. $\lim_{x\to \pi^-} \csc x = \lim_{x\to \pi^-} (1/\sin x) = \infty$ since $\sin x \to 0$ as $x \to \pi^-$ and $\sin x > 0$ for $0<x<\pi$.

29. $\lim_{x\to (-\pi/2)^-} \sec x = \lim_{x\to (-\pi/2)^-} (1/\cos x) = -\infty$ since $\cos x \to 0$ as $x \to (-\pi/2)^-$ and $\cos x < 0$ for $-\pi < x < -\pi/2$.

30. $\lim_{x\to 5^+} \ln(x-5) = -\infty$ since $x-5 \to 0^+$ as $x \to 5^+$.

31. **(a)** $f(x) = 1/(x^3-1)$

x	$f(x)$
0.5	−1.14
0.9	−3.69
0.99	−33.7
0.999	−333.7
0.9999	−3333.7
0.99999	−33 , 333.7

x	$f(x)$
1.5	0.42
1.1	3.02
1.01	33.0
1.001	333.0
1.0001	3333.0
1.00001	33 , 333.3

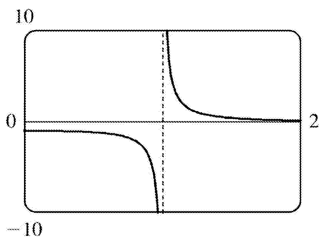
From these calculations, it seems that $\lim_{x\to 1^-} f(x) = -\infty$ and $\lim_{x\to 1^+} f(x) = \infty$.

(b) If x is slightly smaller than 1 , then

x^3-1 will be a negative number close to 0 , and the reciprocal of x^3-1 , that is, $f(x)$, will be a negative number with large absolute value. So $\lim_{x\to 1^-} f(x)=-\infty$.

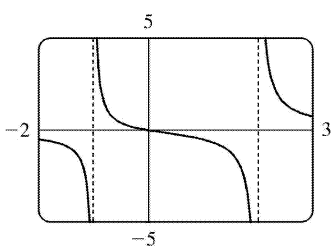
If x is slightly larger than 1 , then x^3-1 will be a small positive number, and its reciprocal, $f(x)$, will be a large positive number. So $\lim_{x\to 1^+} f(x)=\infty$.

(c) It appears from the graph of f that $\lim_{x\to 1^-} f(x)=-\infty$ and $\lim_{x\to 1^+} f(x)=\infty$.



32. **(a)** $y=\dfrac{x}{x^2-x-2}=\dfrac{x}{(x-2)(x+1)}$. Therefore, as $x\to -1^+$ or $x\to 2^+$, the denominator approaches 0 , and $y>0$ for $x<-1$ and for $x>2$, so $\lim_{x\to -1^+} y=\lim_{x\to 2^+} y=\infty$. Also, as $x\to -1^-$ or $x\to 2^-$, the denominator approaches 0 and $y<0$ for $-1<x<2$, so $\lim_{x\to -1^-} y=\lim_{x\to 2^-} y=-\infty$.

(b)



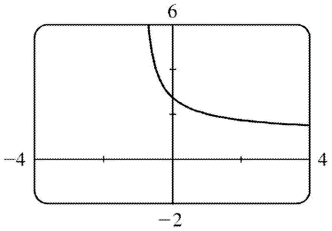
33. **(a)** Let $h(x)=(1+x)^{1/x}$.

x	$h(x)$
−0.001	2.71964
−0.0001	2.71842
−0.00001	2.71830
−0.000001	2.71828
0.000001	2.71828
0.00001	2.71827

0.0001	2.71815
0.001	2.71692

It appears that $\lim_{x\to 0}(1+x)^{1/x}\approx 2.71828$, which is approximately e . In Section 7.4 we will see that the value of the limit is exactly e .

(b)



34. For the curve $y=2^x$ and the points $P(0,1)$ and $Q\left(x,2^x\right)$:

x	Q	m_{PQ}
0.1	(0.1,1.0717735)	0.71773
0.01	(0.01,1.0069556)	0.69556
0.001	(0.001,1.0006934)	0.69339
0.0001	(0.0001,1.0000693)	0.69317

The slope appears to be about 0.693 .

35. **(a)**

x	$f(x)$
1	0.998000
0.8	0.638259
0.6	0.358484
0.4	0.158680
0.2	0.038851
0.1	0.008928
0.05	0.001465

It appears that $\lim_{x\to 0} f(x)=0$.

(b)

x	$f(x)$
0.04	0.000572

0.02	–0.000614
0.01	–0.000907
0.005	–0.000978
0.003	–0.000993
0.001	–0.001000

It appears that $\lim_{x \to 0} f(x) = -0.001$.

36. $h(x) = \dfrac{\tan x - x}{x^3}$

(a)

x	$h(x)$
1.0	0.55740773
0.5	0.37041992
0.1	0.33467209
0.05	0.33366700
0.01	0.33334667
0.005	0.33333667

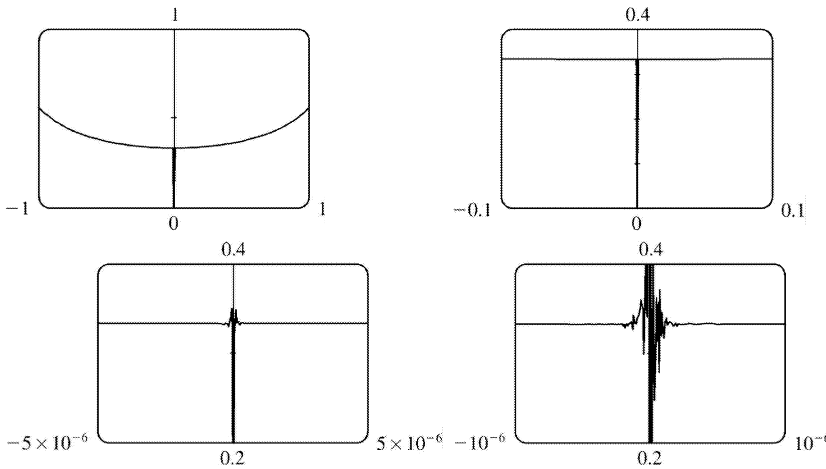
(b) It seems that $\lim_{x \to 0} h(x) = \dfrac{1}{3}$.

(c)

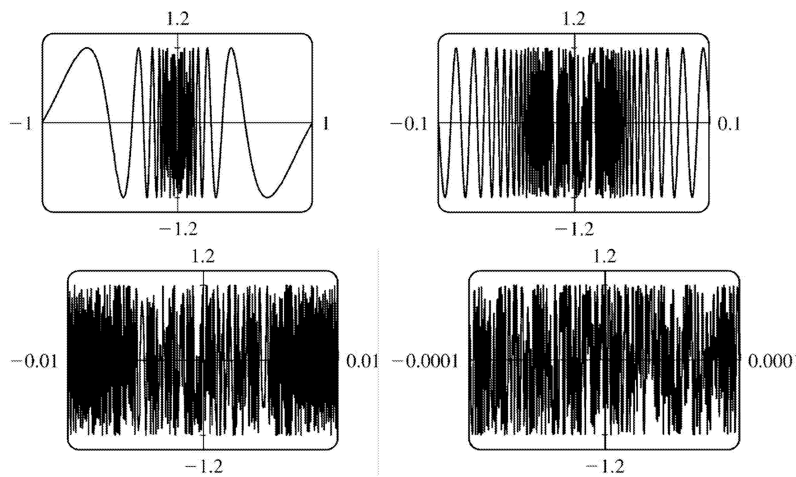
x	$h(x)$
0.001	0.33333350
0.0005	0.33333344
0.0001	0.33333000
0.00005	0.33333600
0.00001	0.33300000
0.000001	0.00000000

Here the values will vary from one calculator to another. Every calculator will eventually give *false values* .

(d) As in part (c), when we take a small enough viewing rectangle we get incorrect output.

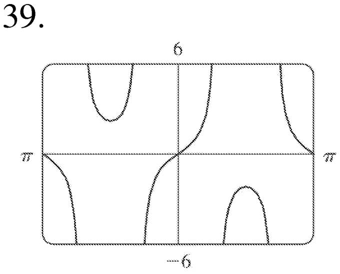


37. No matter how many times we zoom in toward the origin, the graphs of $f(x)=\sin(\pi/x)$ appear to consist of almost–vertical lines. This indicates more and more frequent oscillations as $x\rightarrow 0$.



38. $\lim_{v\rightarrow c^-} m=\lim_{v\rightarrow c^-} \dfrac{m_0}{\sqrt{1-v^2/c^2}}$. As $v\rightarrow c^-$, $\sqrt{1-v^2/c^2}\rightarrow 0^+$, and $m\rightarrow\infty$.

39.

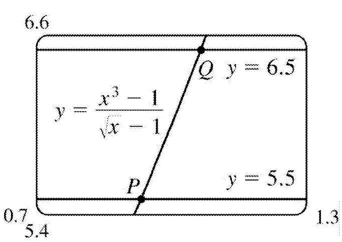


There appear to be vertical asymptotes of the curve $y=\tan(2\sin x)$ at $x\approx\pm 0.90$ and $x\approx\pm 2.24$. To find the exact equations of these asymptotes, we note that the graph of the tangent function has vertical asymptotes at $x=\frac{\pi}{2}+\pi n$. Thus, we must have $2\sin x=\frac{\pi}{2}+\pi n$, or equivalently, $\sin x=\frac{\pi}{4}+\frac{\pi}{2}n$. Since $-1\leq \sin x\leq 1$, we must have $\sin x=\pm\frac{\pi}{4}$ and so $x=\pm\sin^{-1}\frac{\pi}{4}$ (corresponding to $x\approx\pm 0.90$).

Just as 150° is the reference angle for 30° , $\pi-\sin^{-1}\frac{\pi}{4}$ is the reference angle for $\sin^{-1}\frac{\pi}{4}$. So $x=\pm\left(\pi-\sin^{-1}\frac{\pi}{4}\right)$ are also equations of the vertical asymptotes (corresponding to $x\approx\pm 2.24$).

40. **(a)** Let $y=\left(x^3-1\right)/\left(\sqrt{x}-1\right)$.

x	y
0.99	5.92531
0.999	5.99250
0.9999	5.99925
1.01	6.07531
1.001	6.00750
1.0001	6.00075



From the table and the graph, we guess that the limit of y as x approaches 1 is 6 .

(b) We need to have $5.5<\frac{x^3-1}{\sqrt{x}-1}<6.5$. From the graph we obtain the approximate points of intersection $P(0.9313853,5.5)$ and $Q(1.0649004,6.5)$. Now $1-0.9313853\approx 0.0686$ and $1.0649004-1\approx 0.0649$, so by requiring that x be within 0.0649 of 1 , we ensure that y is within 0.5 of 6 .

1. **(a)**

$$\lim_{x\to a}\left[f(x)+h(x)\right]=\lim_{x\to a}f(x)+\lim_{x\to a}h(x)$$
$$=-3+8=5$$

(b) $\lim_{x\to a}\left[f(x)\right]^2=\left[\lim_{x\to a}f(x)\right]^2=(-3)^2=9$

(c) $\lim_{x\to a}\sqrt[3]{h(x)}=\sqrt[3]{\lim_{x\to a}h(x)}=\sqrt[3]{8}=2$

(d) $\lim_{x\to a}\dfrac{1}{f(x)}=\dfrac{1}{\lim_{x\to a}f(x)}=\dfrac{1}{-3}=-\dfrac{1}{3}$

(e) $\lim_{x\to a}\dfrac{f(x)}{h(x)}=\dfrac{\lim_{x\to a}f(x)}{\lim_{x\to a}h(x)}=\dfrac{-3}{8}=-\dfrac{3}{8}$

(f) $\lim_{x\to a}\dfrac{g(x)}{f(x)}=\dfrac{\lim_{x\to a}g(x)}{\lim_{x\to a}f(x)}=\dfrac{0}{-3}=0$

(g) The limit does not exist, since $\lim_{x\to a}g(x)=0$ but $\lim_{x\to a}f(x)\neq 0$.

(h) $\lim_{x\to a}\dfrac{2f(x)}{h(x)-f(x)}=\dfrac{2\lim_{x\to a}f(x)}{\lim_{x\to a}h(x)-\lim_{x\to a}f(x)}=\dfrac{2(-3)}{8-(-3)}=-\dfrac{6}{11}$

2. **(a)** $\lim_{x\to 2}\left[f(x)+g(x)\right]=\lim_{x\to 2}f(x)+\lim_{x\to 2}g(x)=2+0=2$

(b) $\lim_{x\to 1}g(x)$ does not exist since its left– and right–hand limits are not equal, so the given limit does not exist.

(c) $\lim_{x\to 0}\left[f(x)g(x)\right]=\lim_{x\to 0}f(x)\cdot\lim_{x\to 0}g(x)=0\cdot 1.3=0$

(d) Since $\lim_{x\to -1}g(x)=0$ and g is in the denominator, but $\lim_{x\to -1}f(x)=-1\neq 0$, the given limit does not exist.

(e) $\lim_{x\to 2}x^3f(x)=\left[\lim_{x\to 2}x^3\right]\left[\lim_{x\to 2}f(x)\right]=2^3\cdot 2=16$

(f) $\lim_{x\to 1}\sqrt{3+f(x)}=\sqrt{3+\lim_{x\to 1}f(x)}=\sqrt{3+1}=2$

3.

$$\lim_{x\to -2}(3x^4+2x^2-x+1)=\lim_{x\to -2}3x^4+\lim_{x\to -2}2x^2-\lim_{x\to -2}x+\lim_{x\to -2}1 \quad \text{[Limit Laws 1 and 2]}$$

$$=3\lim_{x\to -2}x^4+2\lim_{x\to -2}x^2-\lim_{x\to -2}x+\lim_{x\to -2}1 \quad [3]$$

$$=3(-2)^4+2(-2)^2-(-2)+(1) \quad \text{[9, 8, and 7]}$$

$$=48+8+2+1=59$$

4.

$$\lim_{x\to 2}\frac{2x^2+1}{x^2+6x-4}=\frac{\lim_{x\to 2}\left(2x^2+1\right)}{\lim_{x\to 2}\left(x^2+6x-4\right)} \quad \text{[Limit Law 5]}$$

$$=\frac{2\lim_{x\to 2}x^2+\lim_{x\to 2}1}{\lim_{x\to 2}x^2+6\lim_{x\to 2}x-\lim_{x\to 2}4} \quad \text{[2, 1, and 3]}$$

$$=\frac{2(2)^2+1}{(2)^2+6(2)-4}=\frac{9}{12}=\frac{3}{4} \quad \text{[9, 7, and 8]}$$

5.

$$\lim_{x\to 3}(x^2-4)(x^3+5x-1)=\lim_{x\to 3}(x^2-4)\cdot\lim_{x\to 3}(x^3+5x-1) \quad \text{[Limit Law 4]}$$

$$=\left(\lim_{x\to 3}x^2-\lim_{x\to 3}4\right)\cdot\left(\lim_{x\to 3}x^3+5\lim_{x\to 3}x-\lim_{x\to 3}1\right) \quad \text{[2, 1, and 3]}$$

$$=(3^2-4)\cdot(3^3+5\cdot 3-1) \quad \text{[7, 8, and 9]}$$

$$=5\cdot 41=205$$

6.

$$\lim_{t\to -1}(t^2+1)^3(t+3)^5=\lim_{t\to -1}(t^2+1)^3\cdot\lim_{t\to -1}(t+3)^5 \quad \text{[Limit Law 4]}$$

$$=\left[\lim_{t\to -1}(t^2+1)\right]^3\cdot\left[\lim_{t\to -1}(t+3)\right]^5 \quad [6]$$

$$=\left[\lim_{t\to -1} t^2+\lim_{t\to -1} 1\right]^3\cdot\left[\lim_{t\to -1} t+\lim_{t\to -1} 3\right]^5 \quad [1]$$

$$=\left[(-1)^2+1\right]^3\cdot\left[-1+3\right]^5=8\cdot 32=256 \qquad [9, 7, \text{and } 8]$$

7.

$$\lim_{x\to 1}\left(\frac{1+3x}{1+4x^2+3x^4}\right)^3=\left(\lim_{x\to 1}\frac{1+3x}{1+4x^2+3x^4}\right)^3 \qquad [6]$$

$$=\left[\frac{\lim_{x\to 1}(1+3x)}{\lim_{x\to 1}(1+4x^2+3x^4)}\right]^3 \qquad [5]$$

$$=\left[\frac{\lim_{x\to 1} 1+3\lim_{x\to 1} x}{\lim_{x\to 1} 1+4\lim_{x\to 1} x^2+3\lim_{x\to 1} x^4}\right]^3 \qquad [2, 1, \text{and } 3]$$

$$=\left[\frac{1+3(1)}{1+4(1)^2+3(1)^4}\right]^3=\left[\frac{4}{8}\right]^3=\left(\frac{1}{2}\right)^3=\frac{1}{8} \qquad [7, 8, \text{and } 9]$$

8.

$$\lim_{u\to -2}\sqrt{u^4+3u+6}=\sqrt{\lim_{u\to -2}\left(u^4+3u+6\right)} \qquad [11]$$

$$=\sqrt{\lim_{u\to -2} u^4+3\lim_{u\to -2} u+\lim_{u\to -2} 6} \qquad [1, 2, \text{and } 3]$$

$$=\sqrt{(-2)^4+3(-2)+6} \qquad [9, 8, \text{and } 7]$$

$$=\sqrt{16-6+6}=\sqrt{16}=4$$

9.

$$\lim_{x\to 4^-}\sqrt{16-x^2}=\sqrt{\lim_{x\to 4^-}\left(16-x^2\right)} \qquad [11]$$

$$=\sqrt{\lim_{x\to 4^-} 16-\lim_{x\to 4^-} x^2} \qquad [2]$$

$=\sqrt{16-(4)^2}=0$ [7 and 9]

10. **(a)** The left–hand side of the equation is not defined for $x=2$, but the right–hand side is.
(b) Since the equation holds for all $x\neq 2$, it follows that both sides of the equation approach the same limit as $x\to 2$, just as in Example 3. Remember that in finding $\lim\limits_{x\to a} f(x)$, we never consider $x=a$.

11. $\lim\limits_{x\to 2}\dfrac{x^2+x-6}{x-2}=\lim\limits_{x\to 2}\dfrac{(x+3)(x-2)}{x-2}=\lim\limits_{x\to 2}(x+3)=2+3=5$

12. $\lim\limits_{x\to -4}\dfrac{x^2+5x+4}{x^2+3x-4}=\lim\limits_{x\to -4}\dfrac{(x+4)(x+1)}{(x+4)(x-1)}=\lim\limits_{x\to -4}\dfrac{x+1}{x-1}=\dfrac{-4+1}{-4-1}=\dfrac{-3}{-5}=\dfrac{3}{5}$

13. $\lim\limits_{x\to 2}\dfrac{x^2-x+6}{x-2}$ does not exist since $x-2\to 0$ but $x^2-x+6\to 8$ as $x\to 2$.

14. $\lim\limits_{x\to 4}\dfrac{x^2-4x}{x^2-3x-4}=\lim\limits_{x\to 4}\dfrac{x(x-4)}{(x-4)(x+1)}=\lim\limits_{x\to 4}\dfrac{x}{x+1}=\dfrac{4}{4+1}=\dfrac{4}{5}$

15. $\lim\limits_{t\to -3}\dfrac{t^2-9}{2t^2+7t+3}=\lim\limits_{t\to -3}\dfrac{(t+3)(t-3)}{(2t+1)(t+3)}=\lim\limits_{t\to -3}\dfrac{t-3}{2t+1}=\dfrac{-3-3}{2(-3)+1}=\dfrac{-6}{-5}=\dfrac{6}{5}$

16. $\lim\limits_{x\to -1}\dfrac{x^2-4x}{x^2-3x-4}$ does not exist since $x^2-3x-4\to 0$ but $x^2-4x\to 5$ as $x\to -1$.

17. $\lim\limits_{h\to 0}\dfrac{(4+h)^2-16}{h}=\lim\limits_{h\to 0}\dfrac{(16+8h+h^2)-16}{h}=\lim\limits_{h\to 0}\dfrac{8h+h^2}{h}=\lim\limits_{h\to 0}\dfrac{h(8+h)}{h}=\lim\limits_{h\to 0}(8+h)=8+0=8$

18. $\lim\limits_{x\to 1}\dfrac{x^3-1}{x^2-1}=\lim\limits_{x\to 1}\dfrac{(x-1)\left(x^2+x+1\right)}{(x-1)(x+1)}=\lim\limits_{x\to 1}\dfrac{x^2+x+1}{x+1}=\dfrac{1^2+1+1}{1+1}=\dfrac{3}{2}$

19.

$$\lim_{h\to 0}\frac{(1+h)^4-1}{h}=\lim_{h\to 0}\frac{\left(1+4h+6h^2+4h^3+h^4\right)-1}{h}=\lim_{h\to 0}\frac{4h+6h^2+4h^3+h^4}{h}$$

$$=\lim_{h\to 0}\frac{h(4+6h+4h^2+h^3)}{h}=\lim_{h\to 0}\left(4+6h+4h^2+h^3\right)=4+0+0+0=4$$

20.

$$\lim_{h\to 0}\frac{(2+h)^3-8}{h}=\lim_{h\to 0}\frac{\left(8+12h+6h^2+h^3\right)-8}{h}=\lim_{h\to 0}\frac{12h+6h^2+h^3}{h}$$

$$=\lim_{h\to 0}\left(12+6h+h^2\right)=12+0+0=12$$

21. $\lim_{t\to 9}\frac{9-t}{3-\sqrt{t}}=\lim_{t\to 9}\frac{\left(3+\sqrt{t}\right)\left(3-\sqrt{t}\right)}{3-\sqrt{t}}=\lim_{t\to 9}\left(3+\sqrt{t}\right)=3+\sqrt{9}=6$

22.

$$\lim_{h\to 0}\frac{\sqrt{1+h}-1}{h}=\lim_{h\to 0}\frac{\sqrt{1+h}-1}{h}\cdot\frac{\sqrt{1+h}+1}{\sqrt{1+h}+1}=\lim_{h\to 0}\frac{(1+h)-1}{h\left(\sqrt{1+h}+1\right)}=\lim_{h\to 0}\frac{h}{h\left(\sqrt{1+h}+1\right)}$$

$$=\lim_{h\to 0}\frac{1}{\sqrt{1+h}+1}=\frac{1}{\sqrt{1}+1}=\frac{1}{2}$$

23.

$$\lim_{x\to 7}\frac{\sqrt{x+2}-3}{x-7}=\lim_{x\to 7}\frac{\sqrt{x+2}-3}{x-7}\cdot\frac{\sqrt{x+2}+3}{\sqrt{x+2}+3}=\lim_{x\to 7}\frac{(x+2)-9}{(x-7)\left(\sqrt{x+2}+3\right)}$$

$$=\lim_{x\to 7}\frac{x-7}{(x-7)\left(\sqrt{x+2}+3\right)}=\lim_{x\to 7}\frac{1}{\sqrt{x+2}+3}=\frac{1}{\sqrt{9}+3}=\frac{1}{6}$$

24.

$$\lim_{x\to 2}\frac{x^4-16}{x-2}=\lim_{x\to 2}\frac{(x+2)(x-2)\left(x^2+4\right)}{x-2}=\lim_{x\to 2}(x+2)\left(x^2+4\right)=\lim_{x\to 2}(x+2)\lim_{x\to 2}\left(x^2+4\right)$$

$$=(2+2)\left(2^2+4\right)=32$$

25.

$$\lim_{x\to -4}\frac{\frac{1}{4}+\frac{1}{x}}{4+x}=\lim_{x\to -4}\frac{\frac{x+4}{4x}}{4+x}=\lim_{x\to -4}\frac{x+4}{4x(4+x)}=\lim_{x\to -4}\frac{1}{4x}=\frac{1}{4(-4)}=-\frac{1}{16}$$

26. $$\lim_{t\to 0}\left(\frac{1}{t}-\frac{1}{t^2+t}\right)=\lim_{t\to 0}\frac{\left(t^2+t\right)-t}{t\left(t^2+t\right)}=\lim_{t\to 0}\frac{t^2}{t\cdot t(t+1)}=\lim_{t\to 0}\frac{1}{t+1}=\frac{1}{0+1}=1$$

27.

$$\lim_{x\to 9}\frac{x^2-81}{\sqrt{x}-3}=\lim_{x\to 9}\frac{(x-9)(x+9)}{\sqrt{x}-3}=\lim_{x\to 9}\frac{(\sqrt{x}-3)(\sqrt{x}+3)(x+9)}{\sqrt{x}-3}\quad\left[\begin{array}{c}\text{factor } x-9 \text{ as a}\\ \text{difference of squares}\end{array}\right]$$

$$=\lim_{x\to 9}\left[(\sqrt{x}+3)(x+9)\right]=\left(\sqrt{9}+3\right)(9+9)=6\cdot 18=108$$

28.

$$\lim_{h\to 0}\frac{(3+h)^{-1}-3^{-1}}{h}=\lim_{h\to 0}\frac{\frac{1}{3+h}-\frac{1}{3}}{h}=\lim_{h\to 0}\frac{3-(3+h)}{h(3+h)3}=\lim_{h\to 0}\frac{-h}{h(3+h)3}$$

$$=\lim_{h\to 0}\left[-\frac{1}{3(3+h)}\right]=-\frac{1}{\lim_{h\to 0}\left[3(3+h)\right]}=-\frac{1}{3(3+0)}=-\frac{1}{9}$$

29.

$$\lim_{t\to 0}\left(\frac{1}{t\sqrt{1+t}}-\frac{1}{t}\right)=\lim_{t\to 0}\frac{1-\sqrt{1+t}}{t\sqrt{1+t}}=\lim_{t\to 0}\frac{\left(1-\sqrt{1+t}\right)\left(1+\sqrt{1+t}\right)}{t\sqrt{t+1}\left(1+\sqrt{1+t}\right)}=\lim_{t\to 0}\frac{-t}{t\sqrt{1+t}\left(1+\sqrt{1+t}\right)}$$

$$=\lim_{t\to 0}\frac{-1}{\sqrt{1+t}\left(1+\sqrt{1+t}\right)}=\frac{-1}{\sqrt{1+0}\left(1+\sqrt{1+0}\right)}=-\frac{1}{2}$$

30.

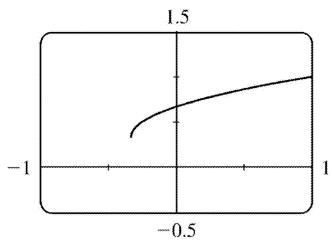
$$\lim_{x\to 1}\frac{\sqrt{x}-x^2}{1-\sqrt{x}}=\lim_{x\to 1}\frac{\sqrt{x}\left(1-x^{3/2}\right)}{1-\sqrt{x}}=\lim_{x\to 1}\frac{\sqrt{x}\left(1-\sqrt{x}\right)\left(1+\sqrt{x}+x\right)}{1-\sqrt{x}}\quad\text{[difference of cubes]}$$

$$=\lim_{x\to 1}\left[\sqrt{x}\left(1+\sqrt{x}+x\right)\right]=\lim_{x\to 1}\left[1(1+1+1)\right]=3$$

Another method: We ''add and subtract'' 1 in the numerator, and then split up the fraction:

$$\lim_{x\to 1}\frac{\sqrt{x}-x^2}{1-\sqrt{x}}=\lim_{x\to 1}\frac{\left(\sqrt{x}-1\right)+\left(1-x^2\right)}{1-\sqrt{x}}=\lim_{x\to 1}\left[-1+\frac{(1-x)(1+x)}{1-\sqrt{x}}\right]$$

$$=\lim_{x\to 1}\left[-1+\frac{\left(1-\sqrt{x}\right)\left(1+\sqrt{x}\right)(1+x)}{1-\sqrt{x}}\right]=-1+\left(1+\sqrt{1}\right)(1+1)=3$$



31. **(a)**

$$\lim_{x\to 0}\frac{x}{\sqrt{1+3x}-1}\approx\frac{2}{3}$$

(b)

x	$f(x)$
−0.001	0.6661663
−0.0001	0.6666167
−0.00001	0.6666617
−0.000001	0.6666662
0.000001	0.6666672
0.00001	0.6666717
0.0001	0.6667167
0.001	0.6671663

The limit appears to be $\frac{2}{3}$.

(c)

$$\lim_{x\to 0}\left(\frac{x}{\sqrt{1+3x}-1}\cdot\frac{\sqrt{1+3x}+1}{\sqrt{1+3x}+1}\right)=\lim_{x\to 0}\frac{x\left(\sqrt{1+3x}+1\right)}{(1+3x)-1}=\lim_{x\to 0}\frac{x\left(\sqrt{1+3x}+1\right)}{3x}$$

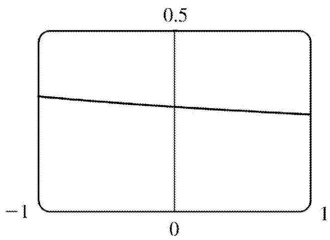
$$=\frac{1}{3}\lim_{x\to 0}\left(\sqrt{1+3x}+1\right) \qquad \text{[Limit Law 3]}$$

$$=\frac{1}{3}\left[\sqrt{\lim_{x\to 0}(1+3x)+\lim_{x\to 0}1}\right] \qquad \text{[1 and 11]}$$

$$=\frac{1}{3}\left(\sqrt{\lim_{x\to 0}1+3\lim_{x\to 0}x}+1\right) \qquad \text{[1, 3, and 7]}$$

$$= \frac{1}{3}\left(\sqrt{1+3\cdot 0}+1\right) \qquad \text{[7 and 8]}$$

$$= \frac{1}{3}(1+1) = \frac{2}{3}$$



32. **(a)**

$$\lim_{x\to 0} \frac{\sqrt{3+x}-\sqrt{3}}{x} \approx 0.29$$

(b)

x	$f(x)$
–0.001	0.2886992
–0.0001	0.2886775
–0.00001	0.2886754
–0.000001	0.2886752
0.000001	0.2886751
0.00001	0.2886749
0.0001	0.2886727
0.001	0.2886511

The limit appears to be approximately 0.2887 .

(c)

$$\lim_{x\to 0}\left(\frac{\sqrt{3+x}-\sqrt{3}}{x}\cdot\frac{\sqrt{3+x}+\sqrt{3}}{\sqrt{3+x}+\sqrt{3}}\right) = \lim_{x\to 0}\frac{(3+x)-3}{x\left(\sqrt{3+x}+\sqrt{3}\right)} = \lim_{x\to 0}\frac{1}{\sqrt{3+x}+\sqrt{3}}$$

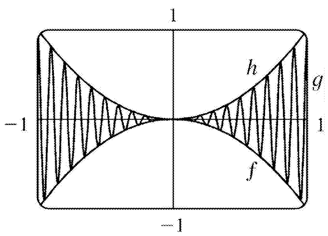
$$= \frac{\lim_{x\to 0} 1}{\lim_{x\to 0}\sqrt{3+x}+\lim_{x\to 0}\sqrt{3}} \qquad \text{[Limit Laws 5 and 1]}$$

$$= \frac{1}{\sqrt{\lim_{x\to 0}(3+x)}+\sqrt{3}} \qquad \text{[7 and 11]}$$

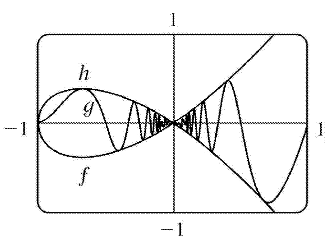
$$= \frac{1}{\sqrt{3+0}+\sqrt{3}} \qquad \text{[1, 7, and 8]}$$

$$=\frac{1}{2\sqrt{3}}$$

33. Let $f(x)=-x^2$, $g(x)=x^2\cos 20\pi x$ and $h(x)=x^2$. Then $-1\le \cos 20\pi x\le 1\Rightarrow -x^2\le x^2\cos 20\pi x\le x^2\Rightarrow$ $f(x)\le g(x)\le h(x)$. So since $\lim_{x\to 0} f(x)=\lim_{x\to 0} h(x)=0$, by the Squeeze Theorem we have $\lim_{x\to 0} g(x)=0$.



34. Let $f(x)=-\sqrt{x^3+x^2}$, $g(x)=\sqrt{x^3+x^2}\sin(\pi/x)$, and $h(x)=\sqrt{x^3+x^2}$. Then $-1\le \sin(\pi/x)\le 1\ \Rightarrow$ $-\sqrt{x^3+x^2}\le\sqrt{x^3+x^2}\sin(\pi/x)\le\sqrt{x^3+x^2}\ \Rightarrow f(x)\le g(x)\le h(x)$. So since $\lim_{x\to 0} f(x)=\lim_{x\to 0} h(x)=0$, by the Squeeze Theorem we have $\lim_{x\to 0} g(x)=0$.



35. $1\le f(x)\le x^2+2x+2$ for all x. Now $\lim_{x\to -1} 1=1$ and $\lim_{x\to -1}\left(x^2+2x+2\right)=\lim_{x\to -1} x^2+2\lim_{x\to -1} x+\lim_{x\to -1} 2=(-1)^2+2(-1)+2=1$. Therefore, by the Squeeze Theorem, $\lim_{x\to -1} f(x)=1$.

36. $3x\le f(x)\le x^3+2$ for $0\le x\le 2$. Now $\lim_{x\to 1} 3x=3$ and $\lim_{x\to 1}\left(x^3+2\right)=\lim_{x\to 1} x^3+\lim_{x\to 1} 2=1^3+2=3$. Therefore, by the Squeeze Theorem, $\lim_{x\to 1} f(x)=3$.

37. $-1\le \cos(2/x)\le 1\Rightarrow -x^4\le x^4\cos(2/x)\le x^4$. Since $\lim_{x\to 0}\left(-x^4\right)=0$ and $\lim_{x\to 0} x^4=0$, we have

$\lim_{x\to 0}\left[x^4\cos(2/x)\right]=0$ by the Squeeze Theorem.

38. $-1\le \sin(\pi/x)\le 1 \Rightarrow e^{-1}\le e^{\sin(\pi/x)}\le e^{1}\Rightarrow \sqrt{x}/e\le \sqrt{x}\,e^{\sin(\pi/x)}\le \sqrt{x}\,e$. Since $\lim_{x\to 0^+}\left(\sqrt{x}/e\right)=0$ and $\lim_{x\to 0^+}\left(\sqrt{x}\,e\right)=0$, we have $\lim_{x\to 0^+}\left[\sqrt{x}\,e^{\sin(\pi/x)}\right]=0$ by the Squeeze Theorem.

39. If $x>-4$, then $|x+4|=x+4$, so $\lim_{x\to -4^+}|x+4|=\lim_{x\to -4^+}(x+4)=-4+4=0$.
If $x<-4$, then $|x+4|=-(x+4)$, so $\lim_{x\to -4^-}|x+4|=\lim_{x\to -4^-}-(x+4)=-(-4+4)=0$.
Since the right and left limits are equal, $\lim_{x\to -4}|x+4|=0$.

40. If $x<-4$, then $|x+4|=-(x+4)$, so $\lim_{x\to -4^-}\frac{|x+4|}{x+4}=\lim_{x\to -4^-}\frac{-(x+4)}{x+4}=\lim_{x\to -4^-}(-1)=-1$.

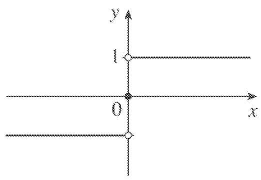
41. If $x>2$, then $|x-2|=x-2$, so $\lim_{x\to 2^+}\frac{|x-2|}{x-2}=\lim_{x\to 2^+}\frac{x-2}{x-2}=\lim_{x\to 2^+}1=1$. If $x<2$, then $|x-2|=-(x-2)$, so $\lim_{x\to 2^-}\frac{|x-2|}{x-2}=\lim_{x\to 2^-}\frac{-(x-2)}{x-2}=\lim_{x\to 2^-}-1=-1$. The right and left limits are different, so $\lim_{x\to 2}\frac{|x-2|}{x-2}$ does not exist.

42. If $x>\frac{3}{2}$, then $|2x-3|=2x-3$, so $\lim_{x\to 1.5^+}\frac{2x^2-3x}{|2x-3|}=\lim_{x\to 1.5^+}\frac{2x^2-3x}{2x-3}=\lim_{x\to 1.5^+}\frac{x(2x-3)}{2x-3}=\lim_{x\to 1.5^+}x=1.5$. If $x<\frac{3}{2}$, then $|2x-3|=3-2x$, so $\lim_{x\to 1.5^-}\frac{2x^2-3x}{|2x-3|}=\lim_{x\to 1.5^-}\frac{2x^2-3x}{-(2x-3)}=\lim_{x\to 1.5^-}\frac{x(2x-3)}{-(2x-3)}=\lim_{x\to 1.5^-}-x=-1.5$. The right and left limits are different, so $\lim_{x\to 1.5}\frac{2x^2-3x}{|2x-3|}$ does not exist.

43. Since $|x|=-x$ for $x<0$, we have $\lim_{x\to 0^-}\left(\frac{1}{x}-\frac{1}{|x|}\right)=\lim_{x\to 0^-}\left(\frac{1}{x}-\frac{1}{-x}\right)=\lim_{x\to 0^-}\frac{2}{x}$, which does not exist since the denominator approaches 0 and the numerator does not.

44. Since $|x|=x$ for $x>0$, we have $\lim_{x\to 0^+}\left(\frac{1}{x}-\frac{1}{|x|}\right)=\lim_{x\to 0^+}\left(\frac{1}{x}-\frac{1}{x}\right)=\lim_{x\to 0^+} 0=0$.

45. **(a)**



(b)

(i) Since $\operatorname{sgn}x=1$ for $x>0$, $\lim_{x\to 0^+}\operatorname{sgn}x=\lim_{x\to 0^+}1=1$.

(ii) Since $\operatorname{sgn}x=-1$ for $x<0$, $\lim_{x\to 0^-}\operatorname{sgn}x=\lim_{x\to 0^-}-1=-1$.

(iii) Since $\lim_{x\to 0^-}\operatorname{sgn}x\neq\lim_{x\to 0^+}\operatorname{sgn}x$, $\lim_{x\to 0}\operatorname{sgn}x$ does not exist.

(iv) Since $|\operatorname{sgn}x|=1$ for $x\neq 0$, $\lim_{x\to 0}|\operatorname{sgn}x|=\lim_{x\to 0}1=1$.

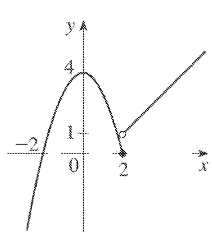
46. **(a)**

$$\lim_{x\to 2^-}f(x)=\lim_{x\to 2^-}\left(4-x^2\right)=\lim_{x\to 2^-}4-\lim_{x\to 2^-}x^2$$
$$=4-4=0$$

$$\lim_{x\to 2^+}f(x)=\lim_{x\to 2^+}(x-1)=\lim_{x\to 2^+}x-\lim_{x\to 2^+}1$$
$$=2-1=1$$

(b) No, $\lim_{x\to 2}f(x)$ does not exist since $\lim_{x\to 2^-}f(x)\neq\lim_{x\to 2^+}f(x)$.

(c)



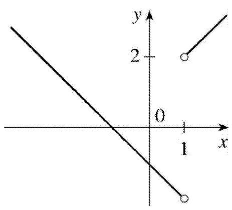
47. **(a)**

(i) $\lim_{x\to 1^+} \frac{x^2-1}{|x-1|} = \lim_{x\to 1^+} \frac{x^2-1}{x-1} = \lim_{x\to 1^+} (x+1) = 2$

(ii) $\lim_{x\to 1^-} \frac{x^2-1}{|x-1|} = \lim_{x\to 1^-} \frac{x^2-1}{-(x-1)} = \lim_{x\to 1^-} -(x+1) = -2$

(b) No, $\lim_{x\to 1} F(x)$ does not exist since $\lim_{x\to 1^+} F(x) \neq \lim_{x\to 1^-} F(x)$.

(c)



48. **(a)**

(i) $\lim_{x\to 0^+} h(x) = \lim_{x\to 0^+} x^2 = 0^2 = 0$

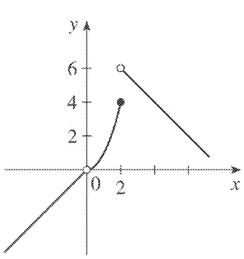
(ii) $\lim_{x\to 0^-} h(x) = \lim_{x\to 0^-} x = 0$, so $\lim_{x\to 0} h(x) = 0$.

(iii) $\lim_{x\to 1} h(x) = \lim_{x\to 1} x^2 = 1^2 = 1$

(iv) $\lim_{x\to 2^-} h(x) = \lim_{x\to 2^-} x^2 = 2^2 = 4$

(v) $\lim_{x\to 2^+} h(x) = \lim_{x\to 2^+} (8-x) = 8-2 = 6$

(vi) Since $\lim_{x\to 2^-} h(x) \neq \lim_{x\to 2^+} h(x)$, $\lim_{x\to 2} h(x)$ does not exist.



(b)

49. **(a)**

(i) $[x]=-2$ for $-2\leq x<-1$, so $\lim_{x\to -2^+}[x]=\lim_{x\to -2^+}(-2)=-2$

(ii) $[x]=-3$ for $-3\leq x<-2$, so $\lim_{x\to -2^-}[x]=\lim_{x\to -2^-}(-3)=-3$.

The right and left limits are different, so $\lim_{x\to -2}[x]$ does not exist.

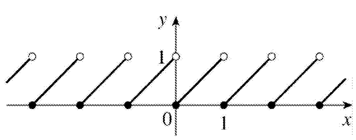
(iii) $[x]=-3$ for $-3\leq x<-2$, so $\lim_{x\to -2.4}[x]=\lim_{x\to -2.4}(-3)=-3$.

(b)

(i) $[x]=n-1$ for $n-1\leq x<n$, so $\lim_{x\to n^-}[x]=\lim_{x\to n^-}(n-1)=n-1$.

(ii) $[x]=n$ for $n\leq x<n+1$, so $\lim_{x\to n^+}[x]=\lim_{x\to n^+}n=n$.

(c) $\lim_{x\to a}[x]$ exists $\Leftrightarrow a$ is not an integer.

50. **(a)** 

(b)

(i) $\lim_{x\to n^-}f(x)=\lim_{x\to n^-}(x-[x])=\lim_{x\to n^-}[x-(n-1)]=n-(n-1)=1$

(ii) $\lim_{x\to n^+}f(x)=\lim_{x\to n^+}(x-[x])=\lim_{x\to n^+}(x-n)=n-n=0$

(c) $\lim_{x\to a}f(x)$ exists $\Leftrightarrow a$ is not an integer.

51. The graph of $f(x)=[x]+[-x]$ is the same as the graph of $g(x)=-1$ with holes at each integer, since $f(a)=0$ for any integer a . Thus, $\lim_{x\to 2^-}f(x)=-1$ and $\lim_{x\to 2^+}f(x)=-1$, so $\lim_{x\to 2}f(x)=-1$. However, $f(2)=[2]+[-2]=2+(-2)=0$, so $\lim_{x\to 2}f(x)\neq f(2)$.

52. $\lim_{v\to c^-}\left(L_0\sqrt{1-\frac{v^2}{c^2}}\right)=L_0\sqrt{1-1}=0$. As the velocity approaches the speed of light, the length

approaches 0 .
A left–hand limit is necessary since L is not defined for $v>c$.

53. Since $p(x)$ is a polynomial, $p(x)=a_0+a_1x+a_2x^2+\cdots+a_nx^n$. Thus, by the Limit Laws,

$$\begin{aligned}\lim_{x\to a} p(x) &= \lim_{x\to a}\left(a_0+a_1x+a_2x^2+\cdots+a_nx^n\right)\\ &= a_0+a_1\lim_{x\to a} x+a_2\lim_{x\to a} x^2+\cdots+a_n\lim_{x\to a} x^n\\ &= a_0+a_1a+a_2a^2+\cdots+a_na^n=p(a)\end{aligned}$$

Thus, for any polynomial p , $\lim_{x\to a} p(x)=p(a)$.

54. Let $r(x)=\dfrac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are any polynomials, and suppose that $q(a)\neq 0$. Thus,

$$\lim_{x\to a} r(x)=\lim_{x\to a}\frac{p(x)}{q(x)}=\frac{\lim_{x\to a} p(x)}{\lim_{x\to a} q(x)}\ \text{[Limit Law 5]}=\frac{p(a)}{q(a)}\ \text{[Exercise 53]}=r(a)\ .$$

55. Observe that $0\le f(x)\le x^2$ for all x , and $\lim_{x\to 0} 0=0=\lim_{x\to 0} x^2$. So, by the Squeeze Theorem, $\lim_{x\to 0} f(x)=0$.

56. Let $f(x)=[x]$ and $g(x)=-[x]$. Then $\lim_{x\to 3} f(x)$ and $\lim_{x\to 3} g(x)$ do not exist (Example 10) but $\lim_{x\to 3}\left[f(x)+g(x)\right]=\lim_{x\to 3}\left([x]-[x]\right)=\lim_{x\to 3} 0=0$.

57. Let $f(x)=H(x)$ and $g(x)=1-H(x)$, where H is the Heaviside function defined in Exercise 1.3.59. Thus, either f or g is 0 for any value of x . Then $\lim_{x\to 0} f(x)$ and $\lim_{x\to 0} g(x)$ do not exist, but $\lim_{x\to 0}\left[f(x)g(x)\right]=\lim_{x\to 0} 0=0$.

58.

$$\lim_{x\to 2}\frac{\sqrt{6-x}-2}{\sqrt{3-x}-1}=\lim_{x\to 2}\left(\frac{\sqrt{6-x}-2}{\sqrt{3-x}-1}\cdot\frac{\sqrt{6-x}+2}{\sqrt{6-x}+2}\cdot\frac{\sqrt{3-x}+1}{\sqrt{3-x}+1}\right)$$

$$=\lim_{x\to 2}\left[\frac{\left(\sqrt{6-x}\right)^2-2^2}{\left(\sqrt{3-x}\right)^2-1^2}\cdot\frac{\sqrt{3-x}+1}{\sqrt{6-x}+2}\right]=\lim_{x\to 2}\left(\frac{6-x-4}{3-x-1}\cdot\frac{\sqrt{3-x}+1}{\sqrt{6-x}+2}\right)$$

$$=\lim_{x\to 2}\frac{(2-x)\left(\sqrt{3-x}+1\right)}{(2-x)\left(\sqrt{6-x}+2\right)}=\lim_{x\to 2}\frac{\sqrt{3-x}+1}{\sqrt{6-x}+2}=\frac{1}{2}$$

59. Since the denominator approaches 0 as $x\to -2$, the limit will exist only if the numerator also approaches 0 as $x\to -2$. In order for this to happen, we need $\lim_{x\to -2}\left(3x^2+ax+a+3\right)=0\Leftrightarrow$ $3(-2)^2+a(-2)+a+3=0\Leftrightarrow 12-2a+a+3=0\Leftrightarrow a=15$. With $a=15$, the limit becomes

$$\lim_{x\to -2}\frac{3x^2+15x+18}{x^2+x-2}=\lim_{x\to -2}\frac{3(x+2)(x+3)}{(x-1)(x+2)}=\lim_{x\to -2}\frac{3(x+3)}{x-1}=\frac{3(-2+3)}{-2-1}=\frac{3}{-3}=-1 .$$

60. *Solution 1:* First, we find the coordinates of P and Q as functions of r . Then we can find the equation of the line determined by these two points, and thus find the $x-$ intercept (the point R), and take the limit as $r\to 0$.

The coordinates of P are $(0,r)$. The point Q is the point of intersection of the two circles $x^2+y^2=r^2$ and $(x-1)^2+y^2=1$. Eliminating y from these equations, we get $r^2-x^2=1-(x-1)^2\Leftrightarrow r^2=1+2x-1\Leftrightarrow x=\frac{1}{2}r^2$. Substituting back into the equation of the shrinking circle to find the $y-$ coordinate, we get $\left(\frac{1}{2}r^2\right)^2+y^2=r^2\Leftrightarrow y^2=r^2\left(1-\frac{1}{4}r^2\right)\Leftrightarrow y=r\sqrt{1-\frac{1}{4}r^2}$ (the positive $y-$ value). So the coordinates of Q are $\left(\frac{1}{2}r^2, r\sqrt{1-\frac{1}{4}r^2}\right)$. The equation of the line joining P and Q is thus

$$y-r=\frac{r\sqrt{1-\frac{1}{4}r^2}-r}{\frac{1}{2}r^2-0}(x-0)$$

. We set $y=0$ in order to find the $x-$ intercept, and get

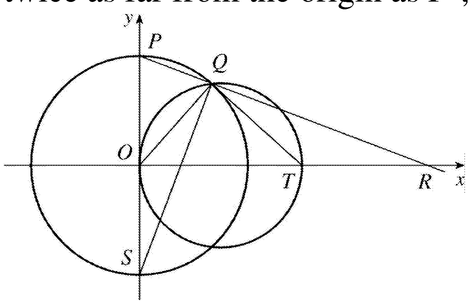
$$x=-r\frac{\frac{1}{2}r^2}{r\left(\sqrt{1-\frac{1}{4}r^2}-1\right)}=\frac{-\frac{1}{2}r^2\left(\sqrt{1-\frac{1}{4}r^2}+1\right)}{1-\frac{1}{4}r^2-1}=2\left(\sqrt{1-\frac{1}{4}r^2}+1\right) .$$

Now we take the limit as $r\to 0^+$: $\lim_{r\to 0^+}x=\lim_{r\to 0^+}2\left(\sqrt{1-\frac{1}{4}r^2}+1\right)=\lim_{r\to 0^+}2\left(\sqrt{1}+1\right)=4$.

So the limiting position of R is the point $(4,0)$.

Solution 2: We add a few lines to the diagram, as shown. Note that $\angle PQS = 90^\circ$ (subtended by diameter PS).

So $\angle SQR = 90^\circ = \angle OQT$ (subtended by diameter OT). It follows that $\angle OQS = \angle TQR$. Also $\angle PSQ = 90^\circ - \angle SPQ = \angle ORP$. Since $\triangle QOS$ is isosceles, so is $\triangle QTR$, implying that $QT = TR$. As the circle C_2 shrinks, the point Q plainly approaches the origin, so the point R must approach a point twice as far from the origin as T , that is, the point $(4,0)$, as above.



1. **(a)** To have $5x+3$ within a distance of 0.1 of 13, we must have $12.9 \le 5x+3 \le 13.1 \Rightarrow 9.9 \le 5x \le 10.1 \Rightarrow 1.98 \le x \le 2.02$. Thus, x must be within 0.02 units of 2 so that $5x+3$ is within 0.1 of 13.
(b) Use 0.01 in place of 0.1 in part (a) to obtain 0.002.

2. **(a)** To have $6x-1$ within a distance of 0.01 of 29, we must have $28.99 \le 6x-1 \le 29.01 \Rightarrow 29.99 \le 6x \le 30.01 \Rightarrow 4.998\overline{3} \le x \le 5.001\overline{6}$. Thus, x must be within $0.001\overline{6}$ units of 5 so that $6x-1$ is within 0.01 of 29.
(b) As in part (a) with 0.001 in place of 0.01, we obtain $0.0001\overline{6}$.
(c) As in part (a) with 0.0001 in place of 0.01, we obtain $0.00001\overline{6}$.

3. On the left side of $x=2$, we need $|x-2| < \left|\frac{10}{7}-2\right| = \frac{4}{7}$. On the right side, we need $|x-2| < \left|\frac{10}{3}-2\right| = \frac{4}{3}$. For both of these conditions to be satisfied at once, we need the more restrictive of the two to hold, that is, $|x-2| < \frac{4}{7}$. So we can choose $\delta = \frac{4}{7}$, or any smaller positive number.

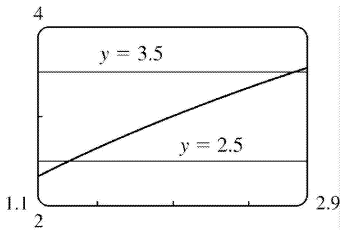
4. On the left side, we need $|x-5| < |4-5| = 1$. On the right side, we need $|x-5| < |5.7-5| = 0.7$. For both conditions to be satisfied at once, we need the more restrictive condition to hold; that is, $|x-5| < 0.7$. So we can choose $\delta = 0.7$, or any smaller positive number.

5. The leftmost question mark is the solution of $\sqrt{x} = 1.6$ and the rightmost, $\sqrt{x} = 2.4$. So the values are $1.6^2 = 2.56$ and $2.4^2 = 5.76$. On the left side, we need $|x-4| < |2.56-4| = 1.44$. On the right side, we need $|x-4| < |5.76-4| = 1.76$. To satisfy both conditions, we need the more restrictive condition to hold — namely, $|x-4| < 1.44$. Thus, we can choose $\delta = 1.44$, or any smaller positive number.

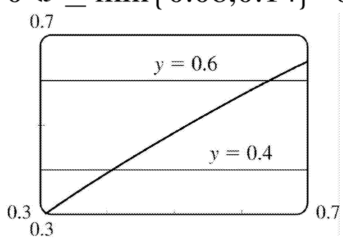
6. The left–hand question mark is the positive solution of $x^2 = \frac{1}{2}$, that is, $x = \frac{1}{\sqrt{2}}$, and the right–hand question mark is the positive solution of $x^2 = \frac{3}{2}$, that is, $x = \sqrt{\frac{3}{2}}$. On the left side, we need $|x-1| < \left|\frac{1}{\sqrt{2}}-1\right| \approx 0.292$ (rounding down to be safe). On the right side, we need $|x-1| < \left|\sqrt{\frac{3}{2}}-1\right| \approx 0.224$. The more restrictive of these two conditions must apply, so we choose $\delta = 0.224$ (or any smaller positive number).

7. $\left|\sqrt{4x+1}-3\right| < 0.5 \Leftrightarrow 2.5 < \sqrt{4x+1} < 3.5$. We plot the three parts of this inequality on the same screen

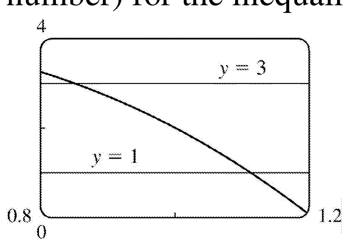
and identify the x –coordinates of the points of intersection using the cursor. It appears that the inequality holds for $1.3125\le x\le 2.8125$. Since $|2-1.3125|=0.6875$ and $|2-2.8125|=0.8125$, we choose $0<\delta<\min\{0.6875,0.8125\}=0.6875$.

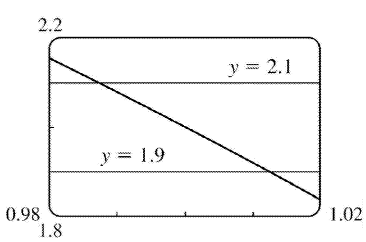


8. $\left|\sin x-\dfrac{1}{2}\right|<0.1\Leftrightarrow 0.4<\sin x<0.6$. From the graph, we see that for this inequality to hold, we need $0.42\le x\le 0.64$. So since $|0.5-0.42|=0.08$ and $|0.5-0.64|=0.14$, we choose $0<\delta\le\min\{0.08,0.14\}=0.08$.

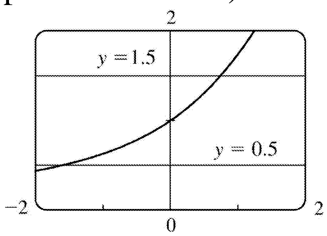


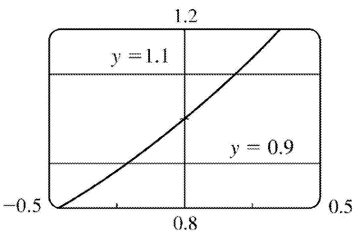
9. For $\varepsilon=1$, the definition of a limit requires that we find δ such that $\left|\left(4+x-3x^3\right)-2\right|<1\Leftrightarrow$ $1<4+x-3x^3<3$ whenever $0<|x-1|<\delta$. If we plot the graphs of $y=1$, $y=4+x-3x^3$ and $y=3$ on the same screen, we see that we need $0.86\le x\le 1.11$. So since $|1-0.86|=0.14$ and $|1-1.11|=0.11$, we choose $\delta=0.11$ (or any smaller positive number). For $\varepsilon=0.1$, we must find δ such that $\left|\left(4+x-3x^3\right)-2\right|<0.1$ $\Leftrightarrow 1.9<4+x-3x^3<2.1$ whenever $0<|x-1|<\delta$. From the graph, we see that we need $0.988\le x\le 1.012$. So since $|1-0.988|=0.012$ and $|1-1.012|=0.012$, we choose $\delta=0.012$ (or any smaller positive number) for the inequality to hold.



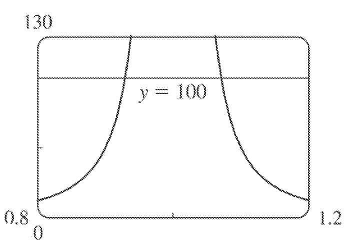


10. For $\varepsilon=0.5$, the definition of a limit requires that we find δ such that $\left|\frac{e^x-1}{x}-1\right|<0.5\Leftrightarrow$ $0.5<\frac{e^x-1}{x}<1.5$ whenever $0<|x-0|<\delta$. If we plot the graphs of $y=0.5$, $y=\frac{e^x-1}{x}$, and $y=1.5$ on the same screen, we see that we need $-1.59\leq x\leq 0.76$. So since $|0-(-1.59)|=1.59$ and $|0-0.76|=0.76$, we choose $\delta=0.76$ (or any smaller positive number). For $\varepsilon=0.1$, we must find δ such that $\left|\frac{e^x-1}{x}-1\right|<0.1\ \Leftrightarrow 0.9<\frac{e^x-1}{x}<1.1$ whenever $0<|x-0|<\delta$. From the graph, we see that we need $-0.21\leq x\leq 0.18$. So since $|0-(-0.21)|=0.21$ and $|0-0.18|=0.18$, we choose $\delta=0.18$ (or any smaller positive number) for the inequality to hold.

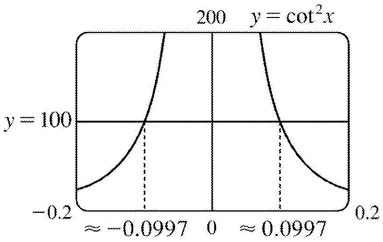




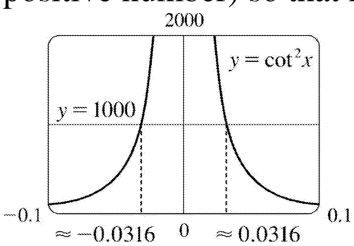
11. From the graph, we see that $\frac{x}{\left(x^2+1\right)(x-1)^2}>100$ whenever $0.93\leq x\leq 1.07$. So since $|1-0.93|=0.07$ and $|1-1.07|=0.07$, we can take $\delta=0.07$ (or any smaller positive number).



12. For $M=100$, we need $-0.0997<x<0$ or $0<x<0.0997$. Thus, we choose $\delta=0.0997$ (or any smaller positive number) so that if $0<|x|<\delta$, then $\cot^2 x>100$.



For $M=1000$, we need $-0.0316<x<0$ or $0<x<0.0316$. Thus, we choose $\delta=0.0316$ (or any smaller positive number) so that if $0<|x|<\delta$, then $\cot^2 x>1000$.



13. **(a)** $A=\pi r^2$ and $A=1000 \text{ cm}^2 \Rightarrow \pi r^2=1000 \Rightarrow r^2=\dfrac{1000}{\pi} \Rightarrow$

$r=\sqrt{\dfrac{1000}{\pi}}\ [r>0] \approx 17.8412$ cm.

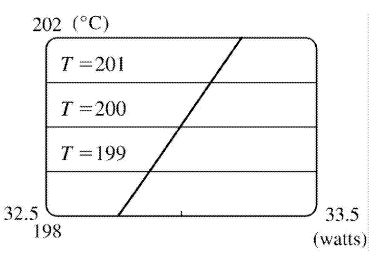
(b) $|A-1000| \leq 5 \Rightarrow -5 \leq \pi r^2-1000 \leq 5 \Rightarrow 1000-5 \leq \pi r^2 \leq 1000+5 \Rightarrow$

$\sqrt{\dfrac{995}{\pi}} \leq r \leq \sqrt{\dfrac{1005}{\pi}} \Rightarrow 17.7966 \leq r \leq 17.8858$. $\sqrt{\dfrac{1000}{\pi}}-\sqrt{\dfrac{995}{\pi}} \approx 0.04466$ and

$\sqrt{\dfrac{1005}{\pi}}-\sqrt{\dfrac{1000}{\pi}} \approx 0.04455$. So if the machinist gets the radius within 0.0445 cm of 17.8412 , the area will be within 5 cm^2 of 1000 .

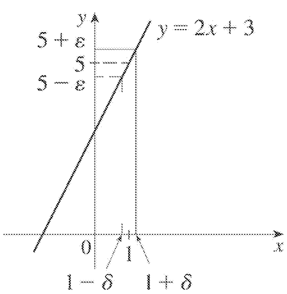
(c) x is the radius, $f(x)$ is the area, a is the target radius given in part (a), L is the target area (1000) , ε is the tolerance in the area (5), and δ is the tolerance in the radius given in part (b).

14. **(a)** $T=0.1w^2+2.155w+20$ and $T=200 \Rightarrow 0.1w^2+2.155w+20=200 \Rightarrow$ [by the quadratic formula or from the graph] $w \approx 33.0$ watts ($w>0$)

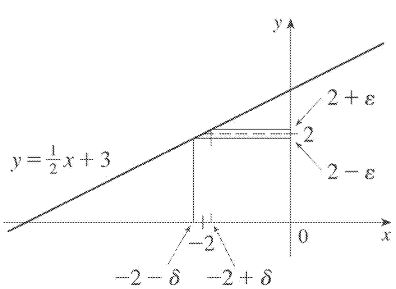


(b) From the graph, $199 \leq T \leq 201 \Rightarrow 32.89 < w < 33.11$.
(c) x is the input power, $f(x)$ is the temperature, a is the target input power given in part (a), L is the target temperature (200), ε is the tolerance in the temperature (1), and δ is the tolerance in the power input in watts indicated in part (b) (0.11 watts).

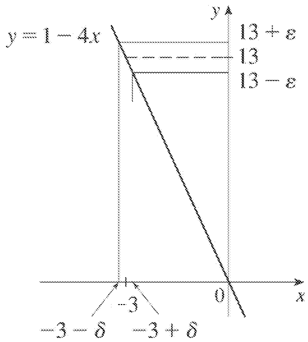
15. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x-1| < \delta$, then $|(2x+3)-5| < \varepsilon$. But $|(2x+3)-5| < \varepsilon \Leftrightarrow |2x-2| < \varepsilon \Leftrightarrow 2|x-1| < \varepsilon \Leftrightarrow |x-1| < \varepsilon/2$. So if we choose $\delta = \varepsilon/2$, then $0 < |x-1| < \delta \Rightarrow |(2x+3)-5| < \varepsilon$. Thus, $\lim_{x \to 1} (2x+3) = 5$ by the definition of a limit.



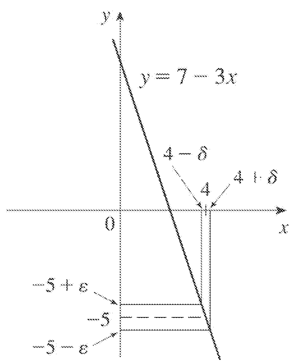
16. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x-(-2)| < \delta$, then $\left|\left(\frac{1}{2}x+3\right)-2\right| < \varepsilon$. But $\left|\left(\frac{1}{2}x+3\right)-2\right| < \varepsilon \Leftrightarrow \left|\frac{1}{2}x+1\right| < \varepsilon \Leftrightarrow \frac{1}{2}|x+2| < \varepsilon \Leftrightarrow |x-(-2)| < 2\varepsilon$. So if we choose $\delta = 2\varepsilon$, then $0 < |x-(-2)| < \delta \Rightarrow \left|\left(\frac{1}{2}x+3\right)-2\right| < \varepsilon$. Thus, $\lim_{x \to -2} \left(\frac{1}{2}x+3\right) = 2$ by the definition of a limit.



17. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x-(-3)| < \delta$, then $|(1-4x)-13| < \varepsilon$. But $|(1-4x)-13| < \varepsilon \Leftrightarrow |-4x-12| < \varepsilon \Leftrightarrow |-4|\ |x+3| < \varepsilon \Leftrightarrow |x-(-3)| < \varepsilon/4$. So if we choose $\delta = \varepsilon/4$, then $0 < |x-(-3)| < \delta \Rightarrow |(1-4x)-13| < \varepsilon$. Thus, $\lim_{x \to -3} (1-4x) = 13$ by the definition of a limit.



18. Given $\varepsilon>0$, we need $\delta>0$ such that if $0<|x-4|<\delta$, then $|(7-3x)-(-5)|<\varepsilon$. But $|(7-3x)-(-5)|<\varepsilon \Leftrightarrow |-3x+12|<\varepsilon \Leftrightarrow |-3|\ |x-4|<\varepsilon \Leftrightarrow |x-4|<\varepsilon/3$. So if we choose $\delta=\varepsilon/3$, then $0<|x-4|<\delta \Rightarrow |(7-3x)-(-5)|<\varepsilon$. Thus, $\lim_{x\to 4}(7-3x)=-5$ by the definition of a limit.



19. Given $\varepsilon>0$, we need $\delta>0$ such that if $0<|x-3|<\delta$, then $\left|\frac{x}{5}-\frac{3}{5}\right|<\varepsilon \Leftrightarrow \frac{1}{5}|x-3|<\varepsilon \Leftrightarrow |x-3|<5\varepsilon$. So choose $\delta=5\varepsilon$. Then $0<|x-3|<\delta \Rightarrow |x-3|<5\varepsilon \Rightarrow \frac{|x-3|}{5}<\varepsilon \Rightarrow \left|\frac{x}{5}-\frac{3}{5}\right|<\varepsilon$. By the definition of a limit, $\lim_{x\to 3}\frac{x}{5}=\frac{3}{5}$.

20. Given $\varepsilon>0$, we need $\delta>0$ such that if $0<|x-6|<\delta$, then $\left|\left(\frac{x}{4}+3\right)-\frac{9}{2}\right|<\varepsilon \Leftrightarrow \left|\frac{x}{4}-\frac{3}{2}\right|<\varepsilon \Leftrightarrow \frac{1}{4}|x-6|<\varepsilon \Leftrightarrow |x-6|<4\varepsilon$. So choose $\delta=4\varepsilon$. Then $0<|x-6|<\delta \Rightarrow |x-6|<4\varepsilon \Rightarrow \frac{|x-6|}{4}<\varepsilon \Rightarrow \left|\frac{x}{4}-\frac{6}{4}\right|<\varepsilon \Rightarrow \left|\left(\frac{x}{4}+3\right)-\frac{9}{2}\right|<\varepsilon$. By the definition of a limit, $\lim_{x\to 6}\left(\frac{x}{4}+3\right)=\frac{9}{2}$.

21. Given $\varepsilon>0$, we need $\delta>0$ such that if $0<|x-(-5)|<\delta$, then $\left|\left(4-\frac{3}{5}x\right)-7\right|<\varepsilon \Leftrightarrow$

$\left|-\frac{3}{5}x-3\right|<\varepsilon \Leftrightarrow \frac{3}{5}|x+5|<\varepsilon \Leftrightarrow |x-(-5)|<\frac{5}{3}\varepsilon$. So choose $\delta=\frac{5}{3}\varepsilon$. Then $|x-(-5)|<\delta \Rightarrow$ $\left|\left(4-\frac{3}{5}x\right)-7\right|<\varepsilon$. Thus, $\lim\limits_{x\to -5}\left(4-\frac{3}{5}x\right)=7$ by the definition of a limit.

22. Given $\varepsilon>0$, we need $\delta>0$ such that if $0<|x-3|<\delta$, then $\left|\frac{x^2+x-12}{x-3}-7\right|<\varepsilon$. Notice that if $0<|x-3|$, then $x\neq 3$, so $\frac{x^2+x-12}{x-3}=\frac{(x+4)(x-3)}{x-3}=x+4$. Thus, when $0<|x-3|$, we have $\left|\frac{x^2+x-12}{x-3}-7\right|<\varepsilon \Leftrightarrow |(x+4)-7|<\varepsilon \Leftrightarrow |x-3|<\varepsilon$. We take $\delta=\varepsilon$ and see that $0<|x-3|<\delta \Rightarrow$ $\left|\frac{x^2+x-12}{x-3}-7\right|<\varepsilon$. By the definition of a limit, $\lim\limits_{x\to 3}\frac{x^2+x-12}{x-3}=7$.

23. Given $\varepsilon>0$, we need $\delta>0$ such that if $0<|x-a|<\delta$, then $|x-a|<\varepsilon$. So $\delta=\varepsilon$ will work.

24. Given $\varepsilon>0$, we need $\delta>0$ such that if $0<|x-a|<\delta$, then $|c-c|<\varepsilon$. But $|c-c|=0$, so this will be true no matter what δ we pick.

25. Given $\varepsilon>0$, we need $\delta>0$ such that if $0<|x-0|<\delta$, then $\left|x^2-0\right|<\varepsilon \Leftrightarrow x^2<\varepsilon \Leftrightarrow |x|<\sqrt{\varepsilon}$. Take $\delta=\sqrt{\varepsilon}$. Then $0<|x-0|<\delta \Rightarrow \left|x^2-0\right|<\varepsilon$. Thus, $\lim\limits_{x\to 0}x^2=0$ by the definition of a limit.

26. Given $\varepsilon>0$, we need $\delta>0$ such that if $0<|x-0|<\delta$, then $\left|x^3-0\right|<\varepsilon \Leftrightarrow |x|^3<\varepsilon \Leftrightarrow |x|<\sqrt[3]{\varepsilon}$. Take $\delta=\sqrt[3]{\varepsilon}$. Then $0<|x-0|<\delta \Rightarrow \left|x^3-0\right|<\delta^3=\varepsilon$. Thus, $\lim\limits_{x\to 0}x^3=0$ by the definition of a limit.

27. Given $\varepsilon>0$, we need $\delta>0$ such that if $0<|x-0|<\delta$, then $\left||x|-0\right|<\varepsilon$. But $\left||x|\right|=|x|$. So this is true if we pick $\delta=\varepsilon$. Thus, $\lim\limits_{x\to 0}|x|=0$ by the definition of a limit.

28. Given $\varepsilon>0$, we need $\delta>0$ such that if $9-\delta<x<9$, then $\left|\sqrt[4]{9-x}-0\right|<\varepsilon \Leftrightarrow \sqrt[4]{9-x}<\varepsilon \Leftrightarrow 9-x<\varepsilon^4 \Leftrightarrow$ $9-\varepsilon^4<x<9$. So take $\delta=\varepsilon^4$. Then $9-\delta<x<9 \Rightarrow \left|\sqrt[4]{9-x}-0\right|<\varepsilon$. Thus, $\lim\limits_{x\to 9^-}\sqrt[4]{9-x}=0$ by the definition of a limit.

29. Given $\varepsilon>0$, we need $\delta>0$ such that if $0<|x-2|<\delta$, then $\left|\left(x^2-4x+5\right)-1\right|<\varepsilon \Leftrightarrow \left|x^2-4x+4\right|<\varepsilon \Leftrightarrow$

$\left|(x-2)^2\right|<\varepsilon$. So take $\delta=\sqrt{\varepsilon}$. Then $0<|x-2|<\delta \Leftrightarrow |x-2|<\sqrt{\varepsilon} \Leftrightarrow \left|(x-2)^2\right|<\varepsilon$. Thus, $\lim_{x\to 2}\left(x^2-4x+5\right)=1$ by the definition of a limit.

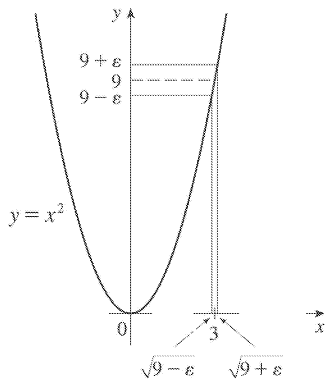
30. Given $\varepsilon>0$, we need $\delta>0$ such that if $0<|x-3|<\delta$, then $\left|\left(x^2+x-4\right)-8\right|<\varepsilon \Leftrightarrow \left|x^2+x-12\right|<\varepsilon \Leftrightarrow$ $|(x-3)(x+4)|<\varepsilon$. Notice that if $|x-3|<1$, then $-1<x-3<1 \Rightarrow 6<x+4<8 \Rightarrow |x+4|<8$. So take $\delta=\min\{1,\varepsilon/8\}$. Then $0<|x-3|<\delta \Leftrightarrow |(x-3)(x+4)| \le |8(x-3)|=8\cdot|x-3|<8\delta \le \varepsilon$. Thus, $\lim_{x\to 3}\left(x^2+x-4\right)=8$ by the definition of a limit.

31. Given $\varepsilon>0$, we need $\delta>0$ such that if $0<|x-(-2)|<\delta$, then $\left|\left(x^2-1\right)-3\right|<\varepsilon$ or upon simplifying we need $\left|x^2-4\right|<\varepsilon$ whenever $0<|x+2|<\delta$. Notice that if $|x+2|<1$, then $-1<x+2<1 \Rightarrow -5<x-2<-3 \Rightarrow$ $|x-2|<5$. So take $\delta=\min\{\varepsilon/5,1\}$. Then $0<|x+2|<\delta \Rightarrow |x-2|<5$ and $|x+2|<\varepsilon/5$, so $\left|\left(x^2-1\right)-3\right|=|(x+2)(x-2)|=|x+2|\,|x-2|<(\varepsilon/5)(5)=\varepsilon$. Thus, by the definition of a limit, $\lim_{x\to -2}\left(x^2-1\right)=3$.

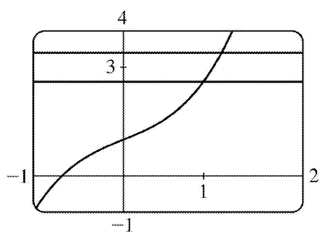
32. Given $\varepsilon>0$, we need $\delta>0$ such that if $0<|x-2|<\delta$, then $\left|x^3-8\right|<\varepsilon$. Now $\left|x^3-8\right|=\left|(x-2)\left(x^2+2x+4\right)\right|$. If $|x-2|<1$, that is, $1<x<3$, then $x^2+2x+4<3^2+2(3)+4=19$ and so $\left|x^3-8\right|=|x-2|\left(x^2+2x+4\right)<19|x-2|$. So if we take $\delta=\min\left\{1,\frac{\varepsilon}{19}\right\}$, then $0<|x-2|<\delta \Rightarrow$ $\left|x^3-8\right|=|x-2|\left(x^2+2x+4\right)<\frac{\varepsilon}{19}\cdot 19=\varepsilon$. Thus, by the definition of a limit, $\lim_{x\to 2} x^3=8$.

33. Given $\varepsilon>0$, we let $\delta=\min\left\{2,\frac{\varepsilon}{8}\right\}$. If $0<|x-3|<\delta$, then $|x-3|<2 \Rightarrow -2<x-3<2 \Rightarrow 4<x+3<8 \Rightarrow$ $|x+3|<8$. Also $|x-3|<\frac{\varepsilon}{8}$, so $\left|x^2-9\right|=|x+3|\,|x-3|<8\cdot\frac{\varepsilon}{8}=\varepsilon$. Thus, $\lim_{x\to 3} x^2=9$.

34. From the figure, our choices for δ are $\delta_1=3-\sqrt{9-\varepsilon}$ and $\delta_2=\sqrt{9+\varepsilon}-3$. The *largest* possible choice for δ is the minimum value of $\{\delta_1,\delta_2\}$; that is, $\delta=\min\{\delta_1,\delta_2\}=\delta_2=\sqrt{9+\varepsilon}-3$.



35. **(a)** The points of intersection in the graph are $(x_1,2.6)$ and $(x_2,3.4)$ with $x_1\approx 0.891$ and $x_2\approx 1.093$. Thus, we can take δ to be the smaller of $1-x_1$ and x_2-1 . So $\delta=x_2-1\approx 0.093$.



(b) Solving $x^3+x+1=3+\varepsilon$ gives us two nonreal complex roots and one real root, which is

$$x(\varepsilon)=\frac{\left(216+108\varepsilon+12\sqrt{336+324\varepsilon+81\varepsilon^2}\right)^{2/3}-12}{6\left(216+108\varepsilon+12\sqrt{336+324\varepsilon+81\varepsilon^2}\right)^{1/3}}$$

. Thus, $\delta=x(\varepsilon)-1$.

(c) If $\varepsilon=0.4$, then $x(\varepsilon)\approx 1.093272342$ and $\delta=x(\varepsilon)-1\approx 0.093$, which agrees with our answer in part (a).

36. *1. Guessing a value for* δ Let $\varepsilon>0$ be given. We have to find a number $\delta>0$ such that $\left|\frac{1}{x}-\frac{1}{2}\right|<\varepsilon$ whenever $0<|x-2|<\delta$. But $\left|\frac{1}{x}-\frac{1}{2}\right|=\left|\frac{2-x}{2x}\right|=\frac{|x-2|}{|2x|}<\varepsilon$. We find a positive constant C such that $\frac{1}{|2x|}<C\Rightarrow\frac{|x-2|}{|2x|}<C|x-2|$ and we can make $C|x-2|<\varepsilon$ by taking $|x-2|<\frac{\varepsilon}{C}=\delta$. We restrict x to lie in the interval $|x-2|<1\Rightarrow 1<x<3$ so $1>\frac{1}{x}>\frac{1}{3}\Rightarrow\frac{1}{6}<\frac{1}{2x}<\frac{1}{2}\Rightarrow\frac{1}{|2x|}<\frac{1}{2}$. So $C=\frac{1}{2}$ is suitable. Thus, we should choose $\delta=\min\{1,2\varepsilon\}$.

2. Showing that δ *works* Given $\varepsilon>0$ we let $\delta=\min\{1,2\varepsilon\}$. If $0<|x-2|<\delta$, then $|x-2|<1\Rightarrow 1<x<3\Rightarrow\frac{1}{|2x|}<\frac{1}{2}$ (as in part 1). Also $|x-2|<2\varepsilon$, so $\left|\frac{1}{x}-\frac{1}{2}\right|=\frac{|x-2|}{|2x|}<\frac{1}{2}\cdot 2\varepsilon=\varepsilon$. This shows that

$\lim_{x\to 2}(1/x)=\frac{1}{2}$.

37. *1. Guessing a value for* δ Given $\varepsilon>0$, we must find $\delta>0$ such that $|\sqrt{x}-\sqrt{a}|<\varepsilon$ whenever $0<|x-a|<\delta$. But $|\sqrt{x}-\sqrt{a}|=\frac{|x-a|}{\sqrt{x}+\sqrt{a}}<\varepsilon$ (from the hint). Now if we can find a positive constant C such that $\sqrt{x}+\sqrt{a}>C$ then $\frac{|x-a|}{\sqrt{x}+\sqrt{a}}<\frac{|x-a|}{C}<\varepsilon$, and we take $|x-a|<C\varepsilon$. We can find this number by restricting x to lie in some interval centered at a . If $|x-a|<\frac{1}{2}a$, then $-\frac{1}{2}a<x-a<\frac{1}{2}a\Rightarrow\frac{1}{2}a<x<\frac{3}{2}a$ $\Rightarrow\sqrt{x}+\sqrt{a}>\sqrt{\frac{1}{2}a}+\sqrt{a}$, and so $C=\sqrt{\frac{1}{2}a}+\sqrt{a}$ is a suitable choice for the constant. So $|x-a|<\left(\sqrt{\frac{1}{2}a}+\sqrt{a}\right)\varepsilon$. This suggests that we let $\delta=\min\left\{\frac{1}{2}a,\left(\sqrt{\frac{1}{2}a}+\sqrt{a}\right)\varepsilon\right\}$.

2. Showing that δ *works* Given $\varepsilon>0$, we let $\delta=\min\left\{\frac{1}{2}a,\left(\sqrt{\frac{1}{2}a}+\sqrt{a}\right)\varepsilon\right\}$. If $0<|x-a|<\delta$, then $|x-a|<\frac{1}{2}a\Rightarrow\sqrt{x}+\sqrt{a}>\sqrt{\frac{1}{2}a}+\sqrt{a}$ (as in part 1). Also $|x-a|<\left(\sqrt{\frac{1}{2}a}+\sqrt{a}\right)\varepsilon$, so $|\sqrt{x}-\sqrt{a}|=\frac{|x-a|}{\sqrt{x}+\sqrt{a}}<\frac{\left(\sqrt{a/2}+\sqrt{a}\right)\varepsilon}{\left(\sqrt{a/2}+\sqrt{a}\right)}=\varepsilon$. Therefore, $\lim_{x\to a}\sqrt{x}=\sqrt{a}$ by the definition of a limit.

38. Suppose that $\lim_{t\to 0}H(t)=L$. Given $\varepsilon=\frac{1}{2}$, there exists $\delta>0$ such that $0<|t|<\delta\Rightarrow|H(t)-L|<\frac{1}{2}\Leftrightarrow$ $L-\frac{1}{2}<H(t)<L+\frac{1}{2}$. For $0<t<\delta$, $H(t)=1$, so $1<L+\frac{1}{2}\Rightarrow L>\frac{1}{2}$. For $-\delta<t<0$, $H(t)=0$, so $L-\frac{1}{2}<0\Rightarrow$ $L<\frac{1}{2}$. This contradicts $L>\frac{1}{2}$. Therefore, $\lim_{t\to 0}H(t)$ does not exist.

39. Suppose that $\lim_{x\to 0}f(x)=L$. Given $\varepsilon=\frac{1}{2}$, there exists $\delta>0$ such that $0<|x|<\delta\Rightarrow|f(x)-L|<\frac{1}{2}$. Take any rational number r with $0<|r|<\delta$. Then $f(r)=0$, so $|0-L|<\frac{1}{2}$, so $L\leq|L|<\frac{1}{2}$. Now take any irrational number s with $0<|s|<\delta$. Then $f(s)=1$, so $|1-L|<\frac{1}{2}$. Hence, $1-L<\frac{1}{2}$, so $L>\frac{1}{2}$. This contradicts $L<\frac{1}{2}$, so $\lim_{x\to 0}f(x)$ does not exist.

40. First suppose that $\lim_{x\to a}f(x)=L$. Then, given $\varepsilon>0$ there exists $\delta>0$ so that $0<|x-a|<\delta\Rightarrow$ $|f(x)-L|<\varepsilon$. Then $a-\delta<x<a\Rightarrow 0<|x-a|<\delta$ so $|f(x)-L|<\varepsilon$. Thus,

$\lim_{x\to a^-} f(x)=L$. Also $a<x<a+\delta \Rightarrow 0<|x-a|<\delta$ so $|f(x)-L|<\varepsilon$. Hence, $\lim_{x\to a^+} f(x)=L$.

Now suppose $\lim_{x\to a^-} f(x)=L=\lim_{x\to a^+} f(x)$. Let $\varepsilon>0$ be given. Since $\lim_{x\to a^-} f(x)=L$, there exists $\delta_1>0$ so that $a-\delta_1<x<a \Rightarrow |f(x)-L|<\varepsilon$. Since $\lim_{x\to a^+} f(x)=L$, there exists $\delta_2>0$ so that $a<x<a+\delta_2 \Rightarrow |f(x)-L|<\varepsilon$. Let δ be the smaller of δ_1 and δ_2 . Then $0<|x-a|<\delta \Rightarrow a-\delta_1<x<a$ or $a<x<a+\delta_2$ so $|f(x)-L|<\varepsilon$. Hence, $\lim_{x\to a} f(x)=L$. So we have proved that $\lim_{x\to a} f(x)=L \Leftrightarrow \lim_{x\to a^-} f(x)=L=\lim_{x\to a^+} f(x)$.

41. $\dfrac{1}{(x+3)^4}>10,000 \Leftrightarrow (x+3)^4<\dfrac{1}{10,000} \Leftrightarrow |x+3|<\dfrac{1}{\sqrt[4]{10,000}} \Leftrightarrow |x-(-3)|<\dfrac{1}{10}$

42. Given $M>0$, we need $\delta>0$ such that $0<|x+3|<\delta \Rightarrow 1/(x+3)^4>M$. Now $\dfrac{1}{(x+3)^4}>M \Leftrightarrow (x+3)^4<\dfrac{1}{M} \Leftrightarrow$ $|x+3|<\dfrac{1}{\sqrt[4]{M}}$. So take $\delta=\dfrac{1}{\sqrt[4]{M}}$. Then $0<|x+3|<\delta=\dfrac{1}{\sqrt[4]{M}} \Rightarrow \dfrac{1}{(x+3)^4}>M$, so $\lim_{x\to -3} \dfrac{1}{(x+3)^4}=\infty$.

43. Given $M<0$ we need $\delta>0$ so that $\ln x<M$ whenever $0<x<\delta$; that is, $x=e^{\ln x}<e^M$ whenever $0<x<\delta$. This suggests that we take $\delta=e^M$. If $0<x<e^M$, then $\ln x<\ln e^M=M$. By the definition of a limit, $\lim_{x\to 0^+} \ln x=-\infty$.

44. **(a)** Let M be given. Since $\lim_{x\to a} f(x)=\infty$, there exists $\delta_1>0$ such that $0<|x-a|<\delta_1 \Rightarrow f(x)>M+1-c$. Since $\lim_{x\to a} g(x)=c$, there exists $\delta_2>0$ such that $0<|x-a|<\delta_2 \Rightarrow |g(x)-c|<1 \Rightarrow g(x)>c-1$. Let δ be the smaller of δ_1 and δ_2 . Then $0<|x-a|<\delta \Rightarrow f(x)+g(x)>(M+1-c)+(c-1)=M$. Thus, $\lim_{x\to a} [f(x)+g(x)]=\infty$.

(b) Let $M>0$ be given. Since $\lim_{x\to a} g(x)=c>0$, there exists $\delta_1>0$ such that $0<|x-a|<\delta_1 \Rightarrow |g(x)-c|<c/2$ $\Rightarrow g(x)>c/2$. Since $\lim_{x\to a} f(x)=\infty$, there exists $\delta_2>0$ such that $0<|x-a|<\delta_2 \Rightarrow f(x)>2M/c$. Let $\delta=\min\left\{\delta_1,\delta_2\right\}$. Then $0<|x-a|<\delta \Rightarrow f(x)\,g(x)>\dfrac{2M}{c}\;\dfrac{c}{2}=M$, so $\lim_{x\to a} f(x)\,g(x)=\infty$.

(c) Let $N<0$ be given. Since $\lim_{x\to a} g(x)=c<0$, there exists $\delta_1>0$ such that $0<|x-a|<\delta_1 \Rightarrow |g(x)-c|<-c/2$ $\Rightarrow g(x)<c/2$. Since $\lim_{x\to a} f(x)=\infty$, there exists $\delta_2>0$ such that $0<|x-a|<\delta_2 \Rightarrow f(x)>2N/c$. (Note that

$c<0$ and $N<0 \Rightarrow 2N/c>0$.) Let $\delta = \min\left\{\delta_1, \delta_2\right\}$. Then $0<|x-a|<\delta \Rightarrow f(x)>2N/c \Rightarrow$
$f(x)\,g(x)<\dfrac{2N}{c}\cdot\dfrac{c}{2}=N$, so $\lim_{x\to a} f(x)\,g(x)=-\infty$.

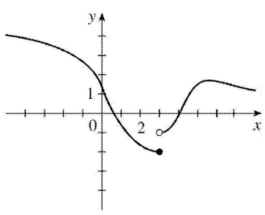
1. From Definition 1, $\lim_{x \to 4} f(x) = f(4)$.

2. The graph of f has no hole, jump, or vertical asymptote.

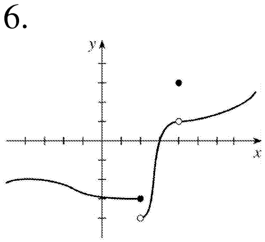
3. **(a)** The following are the numbers at which f is discontinuous and the type of discontinuity at that number: -4 (removable), -2 (jump), 2 (jump), 4 (infinite).
(b) f is continuous from the left at -2 since $\lim_{x \to -2^-} f(x) = f(-2)$. f is continuous from the right at 2 and 4 since $\lim_{x \to 2^+} f(x) = f(2)$ and $\lim_{x \to 4^+} f(x) = f(4)$. It is continuous from neither side at -4 since $f(-4)$ is undefined.

4. g is continuous on $[-4,-2)$, $(-2,2)$, $[2,4)$, $(4,6)$, and $(6,8)$.

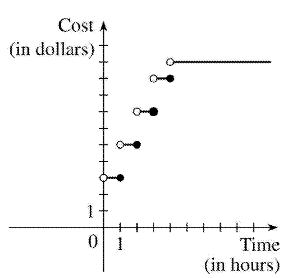
5. The graph of $y=f(x)$ must have a discontinuity at $x=3$ and must show that $\lim_{x \to 3^-} f(x) = f(3)$.



6.



7. **(a)**



(b) There are discontinuities at times $t=1$, 2 , 3 , and 4 . A person parking in the lot would want to keep in mind that the charge will jump at the beginning of each hour.

8. **(a)** Continuous; at the location in question, the temperature changes smoothly as time passes, without any instantaneous jumps from one temperature to another.

(b) Continuous; the temperature at a specific time changes smoothly as the distance due west from New York City increases, without any instantaneous jumps.
(c) Discontinuous; as the distance due west from New York City increases, the altitude above sea level may jump from one height to another without going through all of the intermediate values — at a cliff, for example.
(d) Discontinuous; as the distance traveled increases, the cost of the ride jumps in small increments.
(e) Discontinuous; when the lights are switched on (or off), the current suddenly changes between 0 and some nonzero value, without passing through all of the intermediate values. This is debatable, though, depending on your definition of current.

9. Since f and g are continuous functions,

$$\begin{aligned}\lim_{x\to 3}\left[2f(x)-g(x)\right] &= 2\lim_{x\to 3} f(x)-\lim_{x\to 3} g(x) && \text{[by Limit Laws 2 and 3]}\\ &= 2f(3)-g(3) && \text{[by continuity of } f \text{ and } g \text{ at } x=3\text{]}\\ &= 2\cdot 5-g(3)=10-g(3)\end{aligned}$$

Since it is given that $\lim_{x\to 3}\left[2f(x)-g(x)\right]=4$, we have $10-g(3)=4$, so $g(3)=6$.

10. $\lim_{x\to 4} f(x)=\lim_{x\to 4}\left(x^2+\sqrt{7-x}\right)=\lim_{x\to 4} x^2+\sqrt{\lim_{x\to 4} 7-\lim_{x\to 4} x}=4^2+\sqrt{7-4}=16+\sqrt{3}=f(4)$.
By the definition of continuity, f is continuous at $a=4$.

11. $\lim_{x\to -1} f(x)=\lim_{x\to -1}\left(x+2x^3\right)^4=\left(\lim_{x\to -1} x+2\lim_{x\to -1} x^3\right)^4=\left[-1+2(-1)^3\right]^4=(-3)^4=81=f(-1)$.
By the definition of continuity, f is continuous at $a=-1$.

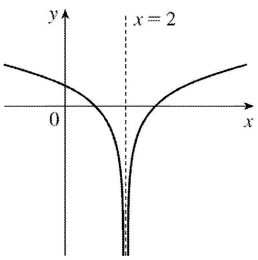
12. $\lim_{x\to 4} g(x)=\lim_{x\to 4}\dfrac{x+1}{2x^2-1}=\dfrac{\lim_{x\to 4} x+\lim_{x\to 4} 1}{2\lim_{x\to 4} x^2-\lim_{x\to 4} 1}=\dfrac{4+1}{2(4)^2-1}=\dfrac{5}{31}=g(4)$. So g is continuous at 4 .

13. For $a>2$, we have $\lim_{x\to a} f(x)=\lim_{x\to a}\dfrac{2x+3}{x-2}=\dfrac{\lim_{x\to a}(2x+3)}{\lim_{x\to a}(x-2)}$ [Limit Law 5] $=\dfrac{2\lim_{x\to a} x+\lim_{x\to a} 3}{\lim_{x\to a} x-\lim_{x\to a} 2}$ [1, 2, and 3] $=\dfrac{2a+3}{a-2}$ [7 and 8] $=f(a)$. Thus, f is continuous at $x=a$ for every a in $(2,\infty)$; that is, f is continuous on $(2,\infty)$.

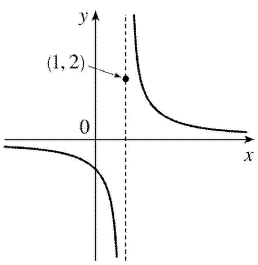
14. For $a<3$, we have $\lim_{x\to a} g(x)=\lim_{x\to a} 2\sqrt{3-x}=2\lim_{x\to a}\sqrt{3-x}$ [Limit Law 3] $=2\sqrt{\lim_{x\to a}(3-x)}$ [11]

$=2\sqrt{\lim_{x\to a} 3-\lim_{x\to a} x}$ [2] $=2\sqrt{3-a}$ [7 and 8] $=g(a)$, so g is continuous at $x=a$ for every a in $(-\infty ,3)$. Also, $\lim_{x\to 3^-} g(x)=0=g(3)$, so g is continuous from the left at 3 . Thus, g is continuous on $(-\infty ,3]$.

15. $f(x)=\ln |x-2|$ is discontinuous at 2 since $f(2)=\ln 0$ is not defined.

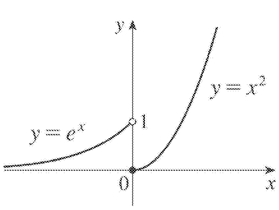


16. $f(x)=\begin{cases} 1/(x-1) & \text{if } x\neq 1 \\ 2 & \text{if } x=1 \end{cases}$ is discontinuous at 1 because $\lim_{x\to 1} f(x)$ does not exist.



17. $f(x)=\begin{cases} e^x & \text{if} x<0 \\ x^2 & \text{if} x\geq 0 \end{cases}$

The left–hand limit of f at $a=0$ is $\lim_{x\to 0^-} f(x)=\lim_{x\to 0^-} e^x=1$. The right–hand limit of f at $a=0$ is $\lim_{x\to 0^+} f(x)=\lim_{x\to 0^+} x^2=0$. Since these limits are not equal, $\lim_{x\to 0} f(x)$ does not exist and f is discontinuous at 0.



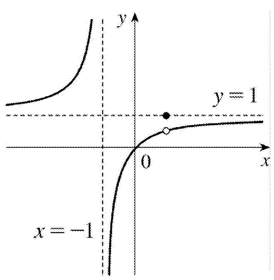
18.

$$f(x)=\begin{cases}\dfrac{x^2-x}{x^2-1} & \text{if } x\neq 1\\ 1 & \text{if } x=1\end{cases}$$

$$\lim_{x\to 1} f(x)=\lim_{x\to 1}\frac{x^2-x}{x^2-1}=\lim_{x\to 1}\frac{x(x-1)}{(x+1)(x-1)}$$

$$=\lim_{x\to 1}\frac{x}{x+1}=\frac{1}{2}\ ,$$

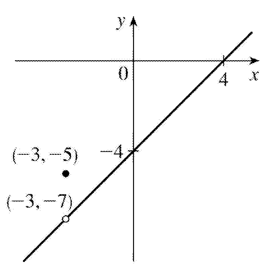
but $f(1)=1$, so f is discontinous at 1.



19. $f(x)=\begin{cases}\dfrac{x^2-x-12}{x+3} & \text{if } x\neq -3\\ -5 & \text{if } x=-3\end{cases}=\begin{cases}x-4 & \text{if } x\neq -3\\ -5 & \text{if } x=-3\end{cases}$

So $\lim_{x\to -3} f(x)=\lim_{x\to -3}(x-4)=-7$ and $f(-3)=-5$.

Since $\lim_{x\to -3} f(x)\neq f(-3)$, f is discontinuous at -3 .

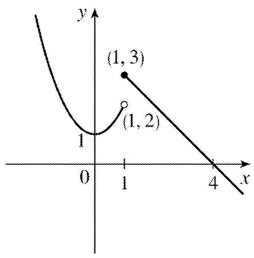


20. $f(x)=\begin{cases}1+x^2 & \text{if } x<1\\ 4-x & \text{if } x\geq 1\end{cases}$

$\lim_{x\to 1^-} f(x)=\lim_{x\to 1^-}(1+x^2)=1+1^2=2$ and

$\lim_{x\to 1^+} f(x)=\lim_{x\to 1^+} (4-x)=4-1=3$.

Thus, f is discontinuous at 1 because $\lim_{x\to 1} f(x)$ does not exist.



21. $F(x)=\dfrac{x}{x^2+5x+6}$ is a rational function. So by Theorem 5 (or Theorem 7), F is continuous at every number in its domain, $\left\{x\,|\,x^2+5x+6\neq 0\right\}=\{x\,|\,(x+3)(x+2)\neq 0\}=\{x\,|\,x\neq -3,-2\}$ or $(-\infty,-3)\cup(-3,-2)\cup(-2,\infty)$.

22. By Theorem 7, the root function $\sqrt[3]{x}$ and the polynomial function $1+x^3$ are continuous on R . By part 4 of Theorem 4, the product $G(x)=\sqrt[3]{x}\left(1+x^3\right)$ is continuous on its domain, R .

23. By Theorem 5, the polynomials x^2 and $2x-1$ are continuous on $(-\infty,\infty)$. By Theorem 7, the root function $\sqrt{x}$ is continuous on $[0,\infty)]$. By Theorem 9, the composite function $\sqrt{2x-1}$ is continuous on its domain, $[\frac{1}{2},\infty)]$. By part 1 of Theorem 4, the sum $R(x)=x^2+\sqrt{2x-1}$ is continuous on $[\frac{1}{2},\infty)]$.

24. By Theorem 7, the trigonometric function $\sin x$ and the polynomial function $x+1$ are continuous on R . By part 5 of Theorem 4, $h(x)=\dfrac{\sin x}{x+1}$ is continuous on its domain, $\{x\,|\,x\neq -1\}$.

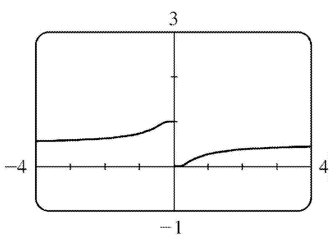
25. By Theorem 5, the polynomial $5x$ is continuous on $(-\infty,\infty)$. By Theorems 9 and 7, $\sin 5x$ is continuous on $(-\infty,\infty)$. By Theorem 7, e^x is continuous on $(-\infty,\infty)$. By part 4 of Theorem 4, the product of e^x and $\sin 5x$ is continuous at all numbers which are in both of their domains, that is, on $(-\infty,\infty)$.

26. By Theorem 5, the polynomial x^2-1 is continuous on $(-\infty,\infty)$. By Theorem 7, $\sin^{-1}$ is continuous on its domain, $[-1,1]$. By Theorem 9, $\sin^{-1}\left(x^2-1\right)$ is continuous on its domain, which is $\left\{x\,|\,-1\leq x^2-1\leq 1\right\}=\left\{x\,|\,0\leq x^2\leq 2\right\}=\left\{x\,|\,|x|\leq\sqrt{2}\right\}=[-\sqrt{2},\sqrt{2}]$.

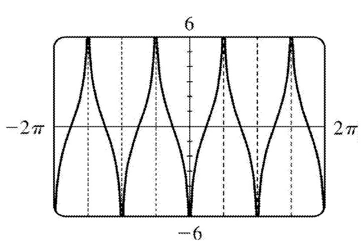
27. By Theorem 5, the polynomial t^4-1 is continuous on $(-\infty,\infty)$. By Theorem 7, ln x is continuous on its domain, $(0,\infty)$. By Theorem 9, $\ln\left(t^4-1\right)$ is continuous on its domain, which is $\left\{t\,|\,t^4-1>0\right\}=\left\{t\,|\,t^4>1\right\}=\{t\,|\,|t|>1\}=(-\infty,-1)\cup(1,\infty)$.

28. By Theorem 7, $\sqrt{x}$ is continuous on $[0,\infty)$. By Theorems 7 and 9, $e^{\sqrt{x}}$ is continuous on $[0,\infty)$. Also by Theorems 7 and 9, $\cos\left(e^{\sqrt{x}}\right)$ is continuous on $[0,\infty)$.

29. The function $y=\dfrac{1}{1+e^{1/x}}$ is discontinuous at $x=0$ because the left– and right–hand limits at $x=0$ are different.



30. The function $y=\tan^2 x$ is discontinuous at $x=\dfrac{\pi}{2}+\pi k$, where k is any integer. The function $y=\ln\left(\tan^2 x\right)$ is also discontinuous where $\tan^2 x$ is 0 , that is, at $x=\pi k$. So $y=\ln\left(\tan^2 x\right)$ is discontinuous at $x=\dfrac{\pi}{2}n$, n any integer.



31. Because we are dealing with root functions, $5+\sqrt{x}$ is continuous on $[0,\infty)$, $\sqrt{x+5}$ is continuous on $[-5,\infty)$, so the quotient $f(x)=\dfrac{5+\sqrt{x}}{\sqrt{5+x}}$ is continuous on $[0,\infty)$. Since f is continuous at $x=4$, $\lim_{x\to 4} f(x)=f(4)=\dfrac{7}{3}$.

32. Because x is continuous on R , $\sin x$ is continuous on R , and $x+\sin x$ is continuous on R , the

composite function $f(x)=\sin(x+\sin x)$ is continuous on R, so $\lim_{x\to\pi} f(x)=f(\pi)=\sin(\pi+\sin\pi)=\sin\pi=0$.

33. Because x^2-x is continuous on R, the composite function $f(x)=e^{x^2-x}$ is continuous on R, so $\lim_{x\to 1} f(x)=f(1)=e^{1-1}=e^0=1$.

34. Because arctan is a continuous function, we can apply Theorem 8.

$$\lim_{x\to 2} \arctan\left(\frac{x^2-4}{3x^2-6x}\right)=\arctan\left(\lim_{x\to 2}\frac{(x+2)(x-2)}{3x(x-2)}\right)=\arctan\left(\lim_{x\to 2}\frac{x+2}{3x}\right)=\arctan\frac{2}{3}\approx 0.588$$

35. $f(x)=\begin{cases} x^2 & \text{if } x<1 \\ \sqrt{x} & \text{if } x\geq 1 \end{cases}$

By Theorem 5, since $f(x)$ equals the polynomial x^2 on $(-\infty,1)$, f is continuous on $(-\infty,1)$. By Theorem 7, since $f(x)$ equals the root function $\sqrt{x}$ on $(1,\infty)$, f is continuous on $(1,\infty)$. At $x=1$, $\lim_{x\to 1^-} f(x)=\lim_{x\to 1^-} x^2=1$ and $\lim_{x\to 1^+} f(x)=\lim_{x\to 1^+}\sqrt{x}=1$. Thus, $\lim_{x\to 1} f(x)$ exists and equals 1. Also, $f(1)=\sqrt{1}=1$. Thus, f is continuous at $x=1$. We conclude that f is continuous on $(-\infty,\infty)$.

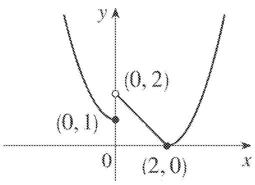
36. $f(x)=\begin{cases} \sin x & \text{if } x<\pi/4 \\ \cos x & \text{if } x\geq \pi/4 \end{cases}$

By Theorem 7, the trigonometric functions are continuous. Since $f(x)=\sin x$ on $(-\infty,\pi/4)$ and $f(x)=\cos x$ on $(\pi/4,\infty)$, f is continuous on $(-\infty,\pi/4)\cup(\pi/4,\infty)$.
$\lim_{x\to(\pi/4)^-} f(x)=\lim_{x\to(\pi/4)^-}\sin x=\sin\frac{\pi}{4}=1/\sqrt{2}$ since the sine function is continuous at $\pi/4$. Similarly, $\lim_{x\to(\pi/4)^+} f(x)=\lim_{x\to(\pi/4)^+}\cos x=1/\sqrt{2}$ by continuity of the cosine function at $\pi/4$. Thus, $\lim_{x\to(\pi/4)} f(x)$ exists and equals $1/\sqrt{2}$, which agrees with the value $f(\pi/4)$. Therefore, f is continuous at $\pi/4$, so f is continuous on $(-\infty,\infty)$.

37. $f(x)=\begin{cases} 1+x^2 & \text{if } x\leq 0 \\ 2-x & \text{if } 0<x\leq 2 \\ (x-2)^2 & \text{if } x>2 \end{cases}$

f is continuous on $(-\infty,0)$, $(0,2)$, and $(2,\infty)$ since it is a polynomial on each of these intervals. Now

$\lim_{x\to 0^-} f(x)=\lim_{x\to 0^-} (1+x^2)=1$ and

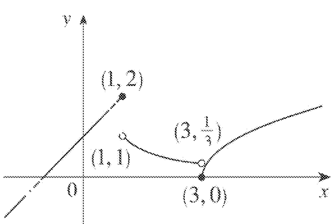


$\lim_{x\to 0^+} f(x)=\lim_{x\to 0^+} (2-x)=2$, so f is discontinuous at 0 . Since $f(0)=1$, f is continuous from the left at 0.
Also, $\lim_{x\to 2^-} f(x)=\lim_{x\to 2^-} (2-x)=0, \lim_{x\to 2^+} f(x)=\lim_{x\to 2^+} (x-2)^2=0$, and $f(2)=0$, so f is continuous at 2. The only number at which f is discontinuous is 0 .

38. $f(x)=\begin{cases} x+1 & \text{if } x\le 1 \\ 1/x & \text{if } 1<x<3 \\ \sqrt{x-3} & \text{if } x\ge 3 \end{cases}$

f is continuous on $(-\infty ,1)$, $(1,3)$, and $(3,\infty)$, where it is a polynomial, a rational function, and a composite of a root function with a polynomial, respectively. Now $\lim_{x\to 1^-} f(x)=\lim_{x\to 1^-} (x+1)=2$ and

$\lim_{x\to 1^+} f(x)=\lim_{x\to 1^+} (1/x)=1$, so f is discontinuous at 1 .



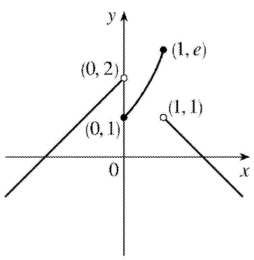
Since $f(1)=2$, f is continuous from the left at 1 . Also, $\lim_{x\to 3^-} f(x)=\lim_{x\to 3^-} (1/x)=1/3$, and

$\lim_{x\to 3^+} f(x)=\lim_{x\to 3^+} \sqrt{x-3}=0=f(3)$, so f is discontinuous at 3 , but it is continuous from the right at 3.

39. $f(x)=\begin{cases} x+2 & \text{if } x<0 \\ e^x & \text{if } 0\le x\le 1 \\ 2-x & \text{if } x>1 \end{cases}$

f is continuous on $(-\infty ,0)$ and $(1,\infty)$ since on each of these intervals it is a polynomial; it is continuous on $(0,1)$ since it is an exponential. Now

$\lim_{x\to 0^-} f(x)=\lim_{x\to 0^-}(x+2)=2$ and $\lim_{x\to 0^+} f(x)=\lim_{x\to 0^+} e^x=1$, so f is discontinuous at 0 . Since $f(0)=1$, f is continuous from the right at 0 . Also $\lim_{x\to 1^-} f(x)=\lim_{x\to 1^-} e^x=e$ and $\lim_{x\to 1^+} f(x)=\lim_{x\to 1^+}(2-x)=1$, so f is discontinuous at 1 . Since $f(1)=e$, f is continuous from the left at 1 .



40. By Theorem 5, each piece of F is continuous on its domain. We need to check for continuity at $r=R$.
$\lim_{r\to R^-} F(r)=\lim_{r\to R^-}\frac{GMr}{R^3}=\frac{GM}{R^2}$ and $\lim_{r\to R^+} F(r)=\lim_{r\to R^+}\frac{GM}{r^2}=\frac{GM}{R^2}$, so $\lim_{r\to R} F(r)=\frac{GM}{R^2}$. Since $F(R)=\frac{GM}{R^2}$, F is continuous at R . Therefore, F is a continuous function of r .

41. f is continuous on $(-\infty,3)$ and $(3,\infty)$. Now $\lim_{x\to 3^-} f(x)=\lim_{x\to 3^-}(cx+1)=3c+1$ and $\lim_{x\to 3^+} f(x)=\lim_{x\to 3^+}\left(cx^2-1\right)=9c-1$. So f is continuous $\Leftrightarrow 3c+1=9c-1 \Leftrightarrow 6c=2 \Leftrightarrow c=\frac{1}{3}$. Thus, for f to be continuous on $(-\infty,\infty)$, $c=\frac{1}{3}$.

42. The functions x^2-c^2 and $cx+20$, considered on the intervals $(-\infty,4)$ and $[4,\infty)$ respectively, are continuous for any value of c . So the only possible discontinuity is at $x=4$. For the function to be continuous at $x=4$, the left–hand and right–hand limits must be the same. Now
$\lim_{x\to 4^-} g(x)=\lim_{x\to 4^-}\left(x^2-c^2\right)=16-c^2$ and $\lim_{x\to 4^+} g(x)=\lim_{x\to 4^+}(cx+20)=4c+20=g(4)$. Thus, $16-c^2=4c+20 \Leftrightarrow$ $c^2+4c+4=0 \Leftrightarrow c=-2$.

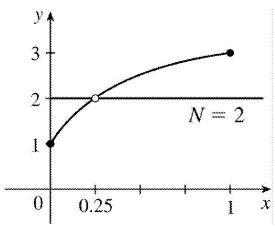
43. **(a)** $f(x)=\frac{x^2-2x-8}{x+2}=\frac{(x-4)(x+2)}{x+2}$ has a removable discontinuity at -2 because $g(x)=x-4$ is continuous on R and $f(x)=g(x)$ for $x\neq -2$. [The discontinuity is removed by defining $f(-2)=-6$.]
(b)

$f(x)=\frac{x-7}{|x-7|} \Rightarrow \lim_{x\to 7^-} f(x)=-1$ and $\lim_{x\to 7^+} f(x)=1$. Thus, $\lim_{x\to 7} f(x)$ does not exist, so the discontinuity is not removable. (It is a jump discontinuity.)

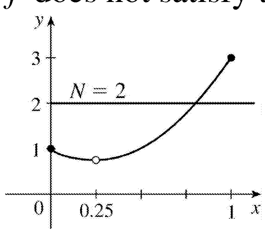
(c) $f(x)=\frac{x^3+64}{x+4}=\frac{(x+4)(x^2-4x+16)}{x+4}$ has a removable discontinuity at -4 because $g(x)=x^2-4x+16$ is continuous on R and $f(x)=g(x)$ for $x\neq-4$.[The discontinuity is removed by defining $f(-4)=48$.]

(d) $f(x)=\frac{3-\sqrt{x}}{9-x}=\frac{3-\sqrt{x}}{(3-\sqrt{x})(3+\sqrt{x})}$ has a removable discontinuity at 9 because $g(x)=\frac{1}{3+\sqrt{x}}$ is continuous on R and $f(x)=g(x)$ for $x\neq 9$. [The discontinuity is removed by defining $f(9)=\frac{1}{6}$.]

44.



f does not satisfy the conclusion of the Intermediate Value Theorem.



f does satisfy the conclusion of the Intermediate Value Theorem.

45. $f(x)=x^3-x^2+x$ is continuous on the interval $[2,3]$, $f(2)=6$, and $f(3)=21$. Since $6<10<21$, there is a number c in $(2,3)$ such that $f(c)=10$ by the Intermediate Value Theorem.

46. $f(x)=x^2$ is continuous on the interval $[1,2]$, $f(1)=1$, and $f(2)=4$. Since $1<2<4$, there is a number c in $(1,2)$ such that $f(c)=c^2=2$ by the Intermediate Value Theorem.

47. $f(x)=x^4+x-3$ is continuous on the interval $[1,2]$, $f(1)=-1$, and $f(2)=15$. Since $-1<0<15$, there is a number c in $(1,2)$ such that $f(c)=0$ by the Intermediate Value Theorem. Thus, there is a root of the equation $x^4+x-3=0$ in the interval $(1,2)$.

48. $f(x)=\sqrt[3]{x}+x-1$ is continuous on the interval $[0,1]$, $f(0)=-1$, and $f(1)=1$. Since $-1<0<1$, there is a number c in $(0,1)$ such that $f(c)=0$ by the Intermediate Value Theorem. Thus, there is a root of the

equation $\sqrt[3]{x}+x-1=0$, or $\sqrt[3]{x}=1-x$, in the interval (0,1).

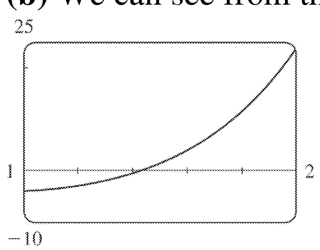
49. $f(x)=\cos x-x$ is continuous on the interval $[0,1]$, $f(0)=1$, and $f(1)=\cos 1-1\approx-0.46$. Since $-0.46<0<1$, there is a number c in $(0,1)$ such that $f(c)=0$ by the Intermediate Value Theorem. Thus, there is a root of the equation $\cos x-x=0$, or $\cos x=x$, in the interval $(0,1)$.

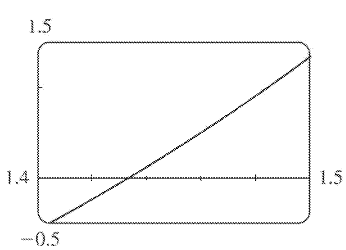
50. $f(x)=\ln x-e^{-x}$ is continuous on the interval $[1,2]$, $f(1)=-e^{-1}\approx-0.37$, and $f(2)=\ln 2-e^{-2}\approx0.56$. Since $-0.37<0<0.56$, there is a number c in $(1,2)$ such that $f(c)=0$ by the Intermediate Value Theorem. Thus, there is a root of the equation $\ln x-e^{-x}=0$, or $\ln x=e^{-x}$, in the interval $(1,2)$.

51. **(a)** $f(x)=e^{x}+x-2$ is continuous on the interval $[0,1]$, $f(0)=-1<0$, and $f(1)=e-1\approx1.72>0$. Since $-1<0<1.72$, there is a number c in $(0,1)$ such that $f(c)=0$ by the Intermediate Value Theorem. Thus, there is a root of the equation $e^{x}+x-2=0$, or $e^{x}=2-x$, in the interval $(0,1)$.
(b) $f(0.44)\approx-0.007<0$ and $f(0.45)\approx0.018>0$, so there is a root between 0.44 and 0.45 .

52. **(a)** $f(x)=\sin x-2+x$ is continuous on $[0,2]$, $f(0)=-2$, and $f(2)=\sin 2\approx0.91$. Since $-2<0<0.91$, there is a number c in $(0,2)$ such that $f(c)=0$ by the Intermediate Value Theorem. Thus, there is a root of the equation $\sin x-2+x=0$, or $\sin x=2-x$, in the interval $(0,2)$.
(b) $f(1.10)\approx-0.009<0$ and $f(1.11)\approx0.006>0$, so there is a root between 1.10 and 1.11 .

53. **(a)** Let $f(x)=x^{5}-x^{2}-4$. Then $f(1)=1^{5}-1^{2}-4=-4<0$ and $f(2)=2^{5}-2^{2}-4=24>0$. So by the Intermediate Value Theorem, there is a number c in $(1,2)$ such that $f(c)=c^{5}-c^{2}-4=0$.
(b) We can see from the graphs that, correct to three decimal places, the root is $x\approx1.434$.

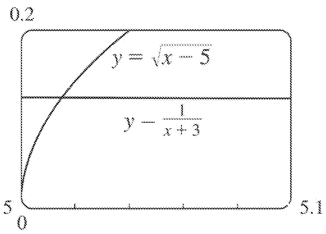




54. **(a)** Let $f(x)=\sqrt{x-5}-\frac{1}{x+3}$. Then $f(5)=-\frac{1}{8}<0$ and $f(6)=\frac{8}{9}>0$, and f is continuous on $[5,\infty)$. So

by the Intermediate Value Theorem, there is a number c in $(5,6)$ such that $f(c)=0$. This implies that $\frac{1}{c+3}=\sqrt{c-5}$.

(b) Using the intersect feature of the graphing device, we find that the root of the equation is $x=5.016$, correct to three decimal places.



55. ($\Rightarrow$) If f is continuous at a , then by Theorem 8 with $g(h)=a+h$, we have $\lim_{h\to 0} f(a+h)=f\left(\lim_{h\to 0}(a+h)\right)=f(a)$.

($\Leftarrow$) Let $\varepsilon>0$. Since $\lim_{h\to 0} f(a+h)=f(a)$, there exists $\delta>0$ such that $0<|h|<\delta \Rightarrow |f(a+h)-f(a)|<\varepsilon$. So if $0<|x-a|<\delta$, then $|f(x)-f(a)|=|f(a+(x-a))-f(a)|<\varepsilon$. Thus, $\lim_{x\to a} f(x)=f(a)$ and so f is continuous at a .

56.

$$\begin{aligned}\lim_{h\to 0}\sin(a+h) &=\lim_{h\to 0}(\sin a\cos h+\cos a\sin h)=\lim_{h\to 0}(\sin a\cos h)+\lim_{h\to 0}(\cos a\sin h)\\ &=\left(\lim_{h\to 0}\sin a\right)\left(\lim_{h\to 0}\cos h\right)+\left(\lim_{h\to 0}\cos a\right)\left(\lim_{h\to 0}\sin h\right)\\ &=(\sin a)(1)+(\cos a)(0)=\sin a\end{aligned}$$

57. As in the previous exercise, we must show that $\lim_{h\to 0}\cos(a+h)=\cos a$ to prove that the cosine function is continuous.

$$\begin{aligned}\lim_{h\to 0}\cos(a+h) &=\lim_{h\to 0}(\cos a\cos h-\sin a\sin h)\\ &=\lim_{h\to 0}(\cos a\cos h)-\lim_{h\to 0}(\sin a\sin h)\\ &=\left(\lim_{h\to 0}\cos a\right)\left(\lim_{h\to 0}\cos h\right)-\left(\lim_{h\to 0}\sin a\right)\left(\lim_{h\to 0}\sin h\right)\\ &=(\cos a)(1)-(\sin a)(0)=\cos a\end{aligned}$$

58. **(a)** Since f is continuous at a , $\lim_{x\to a} f(x)=f(a)$ .Thus, using the Constant Multiple Law of Limits,

we have $\lim_{x\to a}(cf)(x)=\lim_{x\to a}cf(x)=c\lim_{x\to a}f(x)=cf(a)=(cf)(a)$. Therefore, cf is continuous at a .
(b) Since f and g are continuous at a , $\lim_{x\to a}f(x)=f(a)$ and $\lim_{x\to a}g(x)=g(a)$. Since $g(a)\neq 0$, we can use the Quotient Law of Limits: $\lim_{x\to a}\left(\frac{f}{g}\right)(x)=\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{\lim_{x\to a}f(x)}{\lim_{x\to a}g(x)}=\frac{f(a)}{g(a)}=\left(\frac{f}{g}\right)(a)$. Thus, $\frac{f}{g}$ is continuous at a .

59. $f(x)=\begin{cases}0 & \text{if } x \text{ is rational}\\ 1 & \text{if } x \text{ is irrational}\end{cases}$ is continuous nowhere. For, given any number a and any $\delta>0$, the interval $(a-\delta,a+\delta)$ contains both infinitely many rational and infinitely many irrational numbers. Since $f(a)=0$ or 1 , there are infinitely many numbers x with $0<|x-a|<\delta$ and $|f(x)-f(a)|=1$. Thus, $\lim_{x\to a}f(x)\neq f(a)$. [In fact $\lim_{x\to a}f(x)$ does not even exist.]

60. $g(x)=\begin{cases}0 & \text{if } x \text{ is rational}\\ x & \text{if } x \text{ is irrational}\end{cases}$ is continuous at 0 . To see why, note that $-|x|\leq g(x)\leq |x|$, so by the Squeeze Theorem $\lim_{x\to 0}g(x)=0=g(0)$. But g is continuous nowhere else. For if $a\neq 0$ and $\delta>0$, the interval $(a-\delta,a+\delta)$ contains both infinitely many rational and infinitely many irrational numbers. Since $g(a)=0$ or a , there are infinitely many numbers x with $0<|x-a|<\delta$ and $|g(x)-g(a)|>|a|/2$. Thus, $\lim_{x\to a}g(x)\neq g(a)$.

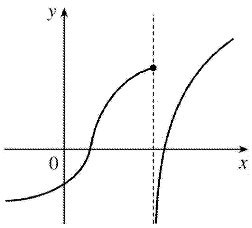
61. If there is such a number, it satisfies the equation $x^3+1=x\Leftrightarrow x^3-x+1=0$. Let the left–hand side of this equation be called $f(x)$. Now $f(-2)=-5<0$, and $f(-1)=1>0$. Note also that $f(x)$ is a polynomial, and thus continuous. So by the Intermediate Value Theorem, there is a number c between -2 and -1 such that $f(c)=0$, so that $c=c^3+1$.

62. **(a)** $\lim_{x\to 0^+}F(x)=0$ and $\lim_{x\to 0^-}F(x)=0$, so $\lim_{x\to 0}F(x)=0$, which is $F(0)$, and hence F is continuous at $x=a$ if $a=0$. For $a>0$, $\lim_{x\to a}F(x)=\lim_{x\to a}x=a=F(a)$. For $a<0$, $\lim_{x\to a}F(x)=\lim_{x\to a}(-x)=-a=F(a)$. Thus, F is continuous at $x=a$; that is, continuous everywhere.
(b) Assume that f is continuous on the interval I . Then for $a\in I$, $\lim_{x\to a}|f(x)|=\left|\lim_{x\to a}f(x)\right|=|f(a)|$ by Theorem 8. (If a is an endpoint of I , use the appropriate one–sided limit.) So $|f|$ is continuous on I .
(c) No, the converse is false. For example, the function $f(x)=\begin{cases}1 & \text{if } x\geq 0\\ -1 & \text{if } x<0\end{cases}$ is not continuous at $x=0$, but $|f(x)|=1$ is continuous on R .

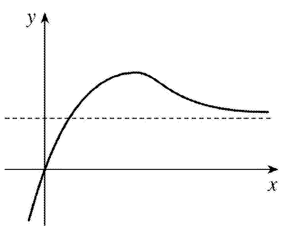
63. Define $u(t)$ to be the monk's distance from the monastery, as a function of time, on the first day, and define $d(t)$ to be his distance from the monastery, as a function of time, on the second day. Let D be the distance from the monastery to the top of the mountain. From the given information we know that $u(0)=0$, $u(12)=D$, $d(0)=D$ and $d(12)=0$. Now consider the function $u-d$, which is clearly continuous. We calculate that $(u-d)(0)=-D$ and $(u-d)(12)=D$. So by the Intermediate Value Theorem, there must be some time t_0 between 0 and 12 such that $(u-d)(t_0)=0\Leftrightarrow u(t_0)=d(t_0)$. So at time t_0 after 7:00 A.M., the monk will be at the same place on both days.

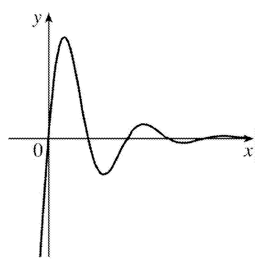
1. **(a)** As x becomes large, the values of $f(x)$ approach 5 .
(b) As x becomes large negative, the values of $f(x)$ approach 3 .

2. **(a)** The graph of a function can intersect a vertical asymptote in the sense that it can meet but not cross it.

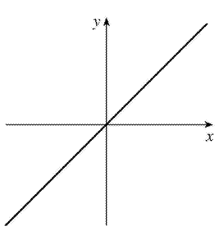


The graph of a function can intersect a horizontal asymptote. It can even intersect its horizontal asymptote an infinite number of times.

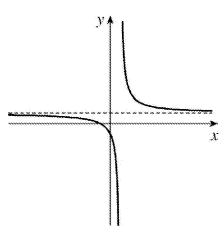




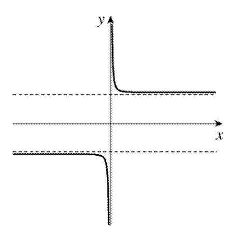
(b) The graph of a function can have 0 , 1 , or 2 horizontal asymptotes. Representative examples are shown.



No horizontal asymptote



One horizontal asymptote



Two horizontal asymptotes

3. **(a)** $\lim_{x \to 2} f(x) = \infty$

(b) $\lim_{x \to -1^-} f(x) = \infty$

(c) $\lim_{x \to -1^+} f(x) = -\infty$

(d) $\lim_{x \to \infty} f(x) = 1$

(e) $\lim_{x \to -\infty} f(x) = 2$

(f) Vertical: $x = -1$, $x = 2$; Horizontal: $y = 1$, $y = 2$

4. **(a)** $\lim_{x \to \infty} g(x) = 2$

(b) $\lim_{x \to -\infty} g(x) = -2$

(c) $\lim_{x \to 3} g(x) = \infty$

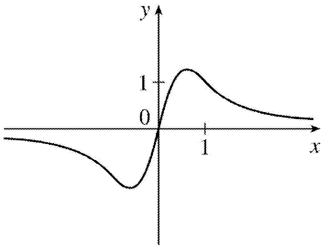
(d) $\lim_{x \to 0} g(x) = -\infty$

(e) $\lim_{x \to -2^+} g(x) = -\infty$

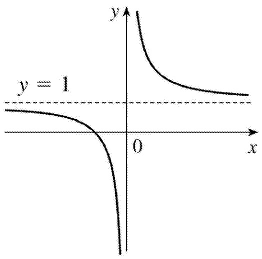
(f) Vertical: $x = -2$, $x = 0$, $x = 3$; Horizontal: $y = -2$, $y = 2$

5. $f(0) = 0$, $f(1) = 1$, $\lim_{x \to \infty} f(x) = 0$,

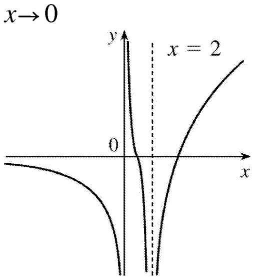
f is odd



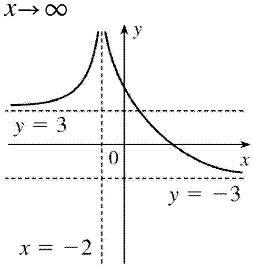
6. $\lim_{x\to 0^+} f(x)=\infty$, $\lim_{x\to 0^-} f(x)=-\infty$,
$\lim_{x\to \infty} f(x)=1$, $\lim_{x\to -\infty} f(x)=1$



7. $\lim_{x\to 2} f(x)=-\infty$, $\lim_{x\to \infty} f(x)=\infty$,
$\lim_{x\to -\infty} f(x)=0$, $\lim_{x\to 0^+} f(x)=\infty$,
$\lim_{x\to 0^-} f(x)=-\infty$



8. $\lim_{x\to -2} f(x)=\infty$, $\lim_{x\to -\infty} f(x)=3$,
$\lim_{x\to \infty} f(x)=-3$



9. If $f(x)=x^2/2^x$, then a calculator gives $f(0)=0$, $f(1)=0.5$, $f(2)=1$, $f(3)=1.125$, $f(4)=1$, $f(5)=0.78125$, $f(6)=0.5625$, $f(7)=0.3828125$, $f(8)=0.25$, $f(9)=0.158203125$, $f(10)=0.09765625$, $f(20)\approx 0.00038147$, $f(50)\approx 2.2204\times 10^{-12}$, $f(100)\approx 7.8886\times 10^{-27}$.
It appears that $\lim_{x\to \infty} \left(x^2/2^x\right)=0$.

10. **(a)** From a graph of $f(x)=(1-2/x)^x$ in a window of $[0,10{,}000]$ by $[0,0.2]$, we estimate that $\lim_{x\to\infty} f(x)=0.14$ (to two decimal places.)

(b)

x	$f(x)$
10,000	0.135308
100,000	0.135333
1,000,000	0.135335

From the table, we estimate that $\lim_{x\to\infty} f(x)=0.1353$ (to four decimal places.)

11.

$$\lim_{x\to\infty}\frac{3x^2-x+4}{2x^2+5x-8}=\lim_{x\to\infty}\frac{(3x^2-x+4)/x^2}{(2x^2+5x-8)/x^2}$$

[divide both the numerator and denominator by x^2 (the highest power of x that appears in the denominator)]

$$=\frac{\lim_{x\to\infty}(3-1/x+4/x^2)}{\lim_{x\to\infty}(2+5/x-8/x^2)}$$ [Limit Law 5]

$$=\frac{\lim_{x\to\infty}3-\lim_{x\to\infty}(1/x)+\lim_{x\to\infty}(4/x^2)}{\lim_{x\to\infty}2+\lim_{x\to\infty}(5/x)-\lim_{x\to\infty}(8/x^2)}$$ [Limit Laws 1 and 2]

$$=\frac{3-\lim_{x\to\infty}(1/x)+4\lim_{x\to\infty}(1/x^2)}{2+5\lim_{x\to\infty}(1/x)-8\lim_{x\to\infty}(1/x^2)}$$ [Limit Laws 7 and 3]

$$=\frac{3-0+4(0)}{2+5(0)-8(0)}$$ [Theorem 5 of Section 2.5]

$$=\frac{3}{2}$$

12.

$$\lim_{x\to\infty}\sqrt{\frac{12x^3-5x+2}{1+4x^2+3x^3}}=\sqrt{\lim_{x\to\infty}\frac{12x^3-5x+2}{1+4x^2+3x^3}}$$ [Limit Law 11]

$$=\sqrt{\lim_{x\to\infty}\frac{12-5/x^2+2/x^3}{1/x^3+4/x+3}}$$ [divide by x^3]

$$=\sqrt{\frac{\lim_{x\to\infty}(12-5/x^2+2/x^3)}{\lim_{x\to\infty}(1/x^3+4/x+3)}}$$ [Limit Law 5]

$$=\sqrt{\frac{\lim_{x\to\infty}12-\lim_{x\to\infty}(5/x^2)+\lim_{x\to\infty}(2/x^3)}{\lim_{x\to\infty}(1/x^3)+\lim_{x\to\infty}(4/x)+\lim_{x\to\infty}3}}$$ [Limit Laws 1 and 2]

$$=\sqrt{\frac{12-5\lim_{x\to\infty}(1/x^2)+2\lim_{x\to\infty}(1/x^3)}{\lim_{x\to\infty}(1/x^3)+4\lim_{x\to\infty}(1/x)+3}}$$ [Limit Laws 7 and 3]

$$=\sqrt{\frac{12-5(0)+2(0)}{0+4(0)+3}}$$ [Theorem 5 of Section 2.5]

$$=\sqrt{\frac{12}{3}}=\sqrt{4}=2$$

13. $$\lim_{x\to\infty}\frac{1}{2x+3}=\lim_{x\to\infty}\frac{1/x}{(2x+3)/x}=\frac{\lim_{x\to\infty}(1/x)}{\lim_{x\to\infty}(2+3/x)}=\frac{\lim_{x\to\infty}(1/x)}{\lim_{x\to\infty}2+3\lim_{x\to\infty}(1/x)}=\frac{0}{2+3(0)}=\frac{0}{2}=0$$

14. $$\lim_{x\to\infty}\frac{3x+5}{x-4}=\lim_{x\to\infty}\frac{(3x+5)/x}{(x-4)/x}=\lim_{x\to\infty}\frac{3+5/x}{1-4/x}=\frac{\lim_{x\to\infty}3+5\lim_{x\to\infty}\frac{1}{x}}{\lim_{x\to\infty}1-4\lim_{x\to\infty}\frac{1}{x}}=\frac{3+5(0)}{1-4(0)}=3$$

15.

$$\lim_{x\to-\infty}\frac{1-x-x^2}{2x^2-7}=\lim_{x\to-\infty}\frac{(1-x-x^2)/x^2}{(2x^2-7)/x^2}=\frac{\lim_{x\to-\infty}(1/x^2-1/x-1)}{\lim_{x\to-\infty}(2-7/x^2)}$$

$$=\frac{\lim_{x\to-\infty}(1/x^2)-\lim_{x\to-\infty}(1/x)-\lim_{x\to-\infty}1}{\lim_{x\to-\infty}2-7\lim_{x\to-\infty}(1/x^2)}=\frac{0-0-1}{2-7(0)}=-\frac{1}{2}$$

16. $$\lim_{y\to\infty}\frac{2-3y^2}{5y^2+4y}=\lim_{y\to\infty}\frac{(2-3y^2)/y^2}{(5y^2+4y)/y^2}=\frac{\lim_{y\to\infty}(2/y^2-3)}{\lim_{y\to\infty}(5+4/y)}=\frac{2\lim_{y\to\infty}(1/y^2)-\lim_{y\to\infty}3}{\lim_{y\to\infty}5+4\lim_{y\to\infty}(1/y)}=\frac{2(0)-3}{5+4(0)}=-\frac{3}{5}$$

17. Divide both the numerator and denominator by x^3 (the highest power of x that occurs in the denominator).

$$\lim_{x\to\infty}\frac{x^3+5x}{2x^3-x^2+4}=\lim_{x\to\infty}\frac{\frac{x^3+5x}{x^3}}{\frac{2x^3-x^2+4}{x^3}}=\lim_{x\to\infty}\frac{1+\frac{5}{x^2}}{2-\frac{1}{x}+\frac{4}{x^3}}=\frac{\lim_{x\to\infty}\left(1+\frac{5}{x^2}\right)}{\lim_{x\to\infty}\left(2-\frac{1}{x}+\frac{4}{x^3}\right)}$$

$$=\frac{\lim_{x\to\infty}1+5\lim_{x\to\infty}\frac{1}{x^2}}{\lim_{x\to\infty}2-\lim_{x\to\infty}\frac{1}{x}+4\lim_{x\to\infty}\frac{1}{x^3}}=\frac{1+5(0)}{2-0+4(0)}=\frac{1}{2}$$

18. $$\lim_{t\to-\infty}\frac{t^2+2}{t^3+t^2-1}=\lim_{t\to-\infty}\frac{\left(t^2+2\right)/t^3}{\left(t^3+t^2-1\right)/t^3}=\lim_{t\to-\infty}\frac{1/t+2/t^3}{1+1/t-1/t^3}=\frac{0+0}{1+0-0}=0$$

19. First, multiply the factors in the denominator. Then divide both the numerator and denominator by u^4.

$$\lim_{u\to\infty}\frac{4u^4+5}{(u^2-2)(2u^2-1)}=\lim_{u\to\infty}\frac{4u^4+5}{2u^4-5u^2+2}=\lim_{u\to\infty}\frac{\frac{4u^4+5}{u^4}}{\frac{2u^4-5u^2+2}{u^4}}=\lim_{u\to\infty}\frac{4+\frac{5}{u^4}}{2-\frac{5}{u^2}+\frac{2}{u^4}}$$

$$=\frac{\lim_{u\to\infty}\left(4+\frac{5}{u^4}\right)}{\lim_{u\to\infty}\left(2-\frac{5}{u^2}+\frac{2}{u^4}\right)}=\frac{\lim_{u\to\infty}4+5\lim_{u\to\infty}\frac{1}{u^4}}{\lim_{u\to\infty}2-5\lim_{u\to\infty}\frac{1}{u^2}+2\lim_{u\to\infty}\frac{1}{u^4}}=\frac{4+5(0)}{2-5(0)+2(0)}$$

$$=\frac{4}{2}=2$$

20. $$\lim_{x\to\infty}\frac{x+2}{\sqrt{9x^2+1}}=\lim_{x\to\infty}\frac{(x+2)/x}{\sqrt{9x^2+1}/\sqrt{x^2}}=\lim_{x\to\infty}\frac{1+2/x}{\sqrt{9+1/x^2}}=\frac{1+0}{\sqrt{9+0}}=\frac{1}{3}$$

21.

$$\lim_{x\to\infty}\frac{\sqrt{9x^6-x}}{x^3+1}=\lim_{x\to\infty}\frac{\sqrt{9x^6-x}/x^3}{(x^3+1)/x^3}=\frac{\lim_{x\to\infty}\sqrt{(9x^6-x)/x^6}}{\lim_{x\to\infty}(1+1/x^3)}\qquad\left[\text{ since } x^3=\sqrt{x^6}\text{ for } x>0\ \right]$$

$$=\frac{\lim_{x\to\infty}\sqrt{9-1/x^5}}{\lim_{x\to\infty}1+\lim_{x\to\infty}(1/x^3)}=\frac{\sqrt{\lim_{x\to\infty}9-\lim_{x\to\infty}(1/x^5)}}{1+0}$$

$$=\sqrt{9-0}=3$$

22.

$$\lim_{x\to-\infty}\frac{\sqrt{9x^6-x}}{x^3+1}=\lim_{x\to-\infty}\frac{\sqrt{9x^6-x}/x^3}{(x^3+1)/x^3}=\frac{\lim_{x\to-\infty}-\sqrt{(9x^6-x)/x^6}}{\lim_{x\to-\infty}(1+1/x^3)}\qquad\left[\text{ since } x^3=-\sqrt{x^6}\text{ for } x<0\ \right]$$

$$=\frac{\lim\limits_{x\to-\infty} -\sqrt{9-1/x^5}}{\lim\limits_{x\to-\infty} 1+\lim\limits_{x\to-\infty} (1/x^3)}=\frac{-\sqrt{\lim\limits_{x\to-\infty} 9-\lim\limits_{x\to-\infty} (1/x^5)}}{1+0}$$

$$=-\sqrt{9-0}=-3$$

23.

$$\lim_{x\to\infty}\left(\sqrt{9x^2+x}-3x\right)=\lim_{x\to\infty}\frac{\left(\sqrt{9x^2+x}-3x\right)\left(\sqrt{9x^2+x}+3x\right)}{\sqrt{9x^2+x}+3x}=\lim_{x\to\infty}\frac{\left(\sqrt{9x^2+x}\right)^2-(3x)^2}{\sqrt{9x^2+x}+3x}$$

$$=\lim_{x\to\infty}\frac{\left(9x^2+x\right)-9x^2}{\sqrt{9x^2+x}+3x}=\lim_{x\to\infty}\frac{x}{\sqrt{9x^2+x}+3x}\cdot\frac{1/x}{1/x}$$

$$=\lim_{x\to\infty}\frac{x/x}{\sqrt{9x^2/x^2+x/x^2}+3x/x}=\lim_{x\to\infty}\frac{1}{\sqrt{9+1/x}+3}=\frac{1}{\sqrt{9}+3}=\frac{1}{3+3}=\frac{1}{6}$$

24.

$$\lim_{x\to-\infty}\left(x+\sqrt{x^2+2x}\right)=\lim_{x\to-\infty}\left(x+\sqrt{x^2+2x}\right)\left[\frac{x-\sqrt{x^2+2x}}{x-\sqrt{x^2+2x}}\right]=\lim_{x\to-\infty}\frac{x^2-\left(x^2+2x\right)}{x-\sqrt{x^2+2x}}$$

$$=\lim_{x\to-\infty}\frac{-2x}{x-\sqrt{x^2+2x}}=\lim_{x\to-\infty}\frac{-2}{1+\sqrt{1+2/x}}=\frac{-2}{1+\sqrt{1+2(0)}}=-1$$

Note: In dividing numerator and denominator by x, we used the fact that for $x<0$, $x=-\sqrt{x^2}$.

25.

$$\lim_{x\to\infty}\left(\sqrt{x^2+ax}-\sqrt{x^2+bx}\right)=\lim_{x\to\infty}\frac{\left(\sqrt{x^2+ax}-\sqrt{x^2+bx}\right)\left(\sqrt{x^2+ax}+\sqrt{x^2+bx}\right)}{\sqrt{x^2+ax}+\sqrt{x^2+bx}}$$

$$=\lim_{x\to\infty}\frac{(x^2+ax)-(x^2+bx)}{\sqrt{x^2+ax}+\sqrt{x^2+bx}}=\lim_{x\to\infty}\frac{[(a-b)x]/x}{\left(\sqrt{x^2+ax}+\sqrt{x^2+bx}\right)/\sqrt{x^2}}$$

$$=\lim_{x\to\infty}\frac{a-b}{\sqrt{1+a/x}+\sqrt{1+b/x}}=\frac{a-b}{\sqrt{1+0}+\sqrt{1+0}}=\frac{a-b}{2}$$

26. $\lim_{x\to\infty}\cos x$ does not exist because as x increases $\cos x$ does not approach any one value, but oscillates between 1 and -1 .

27. $\sqrt{x}$ is large when x is large, so $\lim_{x\to\infty}\sqrt{x}=\infty$.

28. $\sqrt[3]{x}$ is large negative when x is large negative, so $\lim_{x\to-\infty}\sqrt[3]{x}=-\infty$.

29. $\lim_{x\to\infty}\left(x-\sqrt{x}\right)=\lim_{x\to\infty}\sqrt{x}\left(\sqrt{x}-1\right)=\infty$ since $\sqrt{x}\to\infty$ and $\sqrt{x}-1\to\infty$ as $x\to\infty$.

30. $\lim_{x\to\infty}\frac{x^3-2x+3}{5-2x^2}=\lim_{x\to\infty}\frac{(x^3-2x+3)/x^2}{(5-2x^2)/x^2}$ [divide by the highest power of x in the denominator]

$=\lim_{x\to\infty}\frac{x-2/x+3/x^2}{5/x^2-2}=-\infty$ because $x-2/x+3/x^2\to\infty$ and $5/x^2-2\to-2$ as $x\to\infty$.

31. $\lim_{x\to-\infty}(x^4+x^5)=\lim_{x\to-\infty}x^5(\frac{1}{x}+1)=-\infty$ because $x^5\to-\infty$ and $1/x+1\to1$ as $x\to-\infty$.

32. $\lim_{x\to\infty}\tan^{-1}\left(x^2-x^4\right)=\lim_{x\to\infty}\tan^{-1}\left(x^2\left(1-x^2\right)\right)$. If we let $t=x^2\left(1-x^2\right)$, we know that $t\to-\infty$ as $x\to\infty$, since $x^2\to\infty$ and $1-x^2\to-\infty$. So $\lim_{x\to\infty}\tan^{-1}\left(x^2\left(1-x^2\right)\right)=\lim_{t\to-\infty}\tan^{-1}t=-\frac{\pi}{2}$.

33.

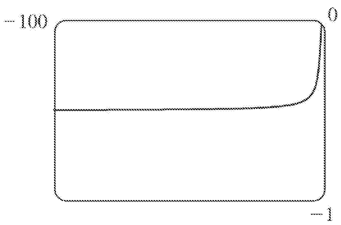
$\lim_{x\to\infty}\frac{x+x^3+x^5}{1-x^2+x^4}=\lim_{x\to\infty}\frac{(x+x^3+x^5)/x^4}{(1-x^2+x^4)/x^4}$ [divide by the highest power of x in the denominator]

$$=\lim_{x\to\infty}\frac{1/x^3+1/x+x}{1/x^4-1/x^2+1}=\infty$$

because $(1/x^3+1/x+x)\to\infty$ and $(1/x^4-1/x^2+1)\to1$ as $x\to\infty$.

34. If we let $t=\tan x$, then as $x\to(\pi/2)^+$, $t\to-\infty$. Thus,

$$\lim_{x\to(\pi/2)^+} e^{\tan x} = \lim_{t\to-\infty} e^t = 0 \ .$$

35. **(a)** 

From the graph of $f(x)=\sqrt{x^2+x+1}+x$, we estimate the value of $\lim_{x\to-\infty} f(x)$ to be -0.5 .

(b)

x	$f(x)$
−10,000	−0.4999625
−100,000	−0.4999962
−1,000,000	−0.4999996

From the table, we estimate the limit to be -0.5 .

(c)

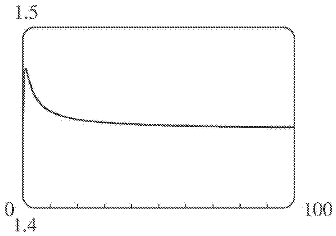
$$\lim_{x\to-\infty}\left(\sqrt{x^2+x+1}+x\right) = \lim_{x\to-\infty}\left(\sqrt{x^2+x+1}+x\right)\left[\frac{\sqrt{x^2+x+1}-x}{\sqrt{x^2+x+1}-x}\right] = \lim_{x\to-\infty}\frac{\left(x^2+x+1\right)-x^2}{\sqrt{x^2+x+1}-x}$$

$$= \lim_{x\to-\infty}\frac{(x+1)(1/x)}{\left(\sqrt{x^2+x+1}-x\right)(1/x)} = \lim_{x\to-\infty}\frac{1+(1/x)}{-\sqrt{1+(1/x)+\left(1/x^2\right)}-1}$$

$$= \frac{1+0}{-\sqrt{1+0+0}-1} = -\frac{1}{2}$$

Note that for $x<0$, we have $\sqrt{x^2}=|x|=-x$, so when we divide the radical by x , with $x<0$, we get $\frac{1}{x}\sqrt{x^2+x+1} = -\frac{1}{\sqrt{x^2}}\sqrt{x^2+x+1} = -\sqrt{1+(1/x)+\left(1/x^2\right)}$.

36. **(a)**



From the graph of $f(x)=\sqrt{3x^2+8x+6}-\sqrt{3x^2+3x+1}$, we estimate (to one decimal place) the value of $\lim_{x\to\infty} f(x)$ to be 1.4 .

(b)

x	$f(x)$
10,000	1.44339
100,000	1.44338
1,000,000	1.44338

From the table, we estimate (to four decimal places) the limit to be 1.4434 .

(c)

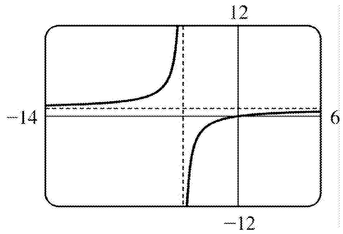
$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{\left(\sqrt{3x^2+8x+6}-\sqrt{3x^2+3x+1}\right)\left(\sqrt{3x^2+8x+6}+\sqrt{3x^2+3x+1}\right)}{\sqrt{3x^2+8x+6}+\sqrt{3x^2+3x+1}}$$

$$= \lim_{x\to\infty} \frac{\left(3x^2+8x+6\right)-\left(3x^2+3x+1\right)}{\sqrt{3x^2+8x+6}+\sqrt{3x^2+3x+1}}$$

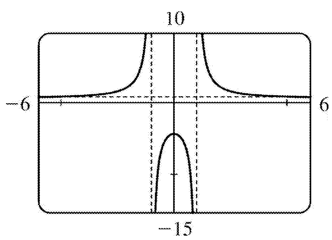
$$= \lim_{x\to\infty} \frac{(5x+5)(1/x)}{\left(\sqrt{3x^2+8x+6}+\sqrt{3x^2+3x+1}\right)(1/x)}$$

$$= \lim_{x\to\infty} \frac{5+5/x}{\sqrt{3+8/x+6/x^2}+\sqrt{3+3/x+1/x^2}} = \frac{5}{\sqrt{3}+\sqrt{3}} = \frac{5}{2\sqrt{3}} = \frac{5\sqrt{3}}{6} \approx 1.443376$$

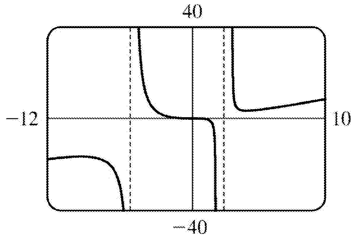
37. $\lim_{x\to\pm\infty} \frac{x}{x+4} = \lim_{x\to\pm\infty} \frac{1}{1+4/x} = \frac{1}{1+0} = 1$, so $y=1$ is a horizontal asymptote. $\lim_{x\to -4^-} \frac{x}{x+4} = \infty$ and $\lim_{x\to -4^+} \frac{x}{x+4} = -\infty$, so $x=-4$ is a vertical asymptote. The graph confirms these calculations.



38. Since $x^2-1\to 0$ as $x\to\pm 1$ and $y<0$ for $-1<x<1$ and $y>0$ for $x<-1$ and $x>1$, we have $\lim_{x\to 1^-}\frac{x^2+4}{x^2-1}=-\infty$, $\lim_{x\to 1^+}\frac{x^2+4}{x^2-1}=\infty$, $\lim_{x\to -1^-}\frac{x^2+4}{x^2-1}=\infty$, and $\lim_{x\to -1^+}\frac{x^2+4}{x^2-1}=-\infty$, so $x=1$ and $x=-1$ are vertical asymptotes. Also $\lim_{x\to\pm\infty}\frac{x^2+4}{x^2-1}=\lim_{x\to\pm\infty}\frac{1+4/x^2}{1-1/x^2}=\frac{1+0}{1-0}=1$, so $y=1$ is a horizontal asymptote. The graph confirms these calculations.

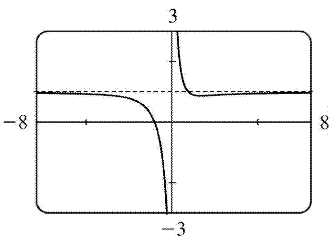


39. $\lim_{x\to\pm\infty}\frac{x^3}{x^2+3x-10}=\lim_{x\to\pm\infty}\frac{x}{1+(3/x)-\left(10/x^2\right)}=\pm\infty$, so there is no horizontal asymptote.
$\lim_{x\to 2^+}\frac{x^3}{x^2+3x-10}=\lim_{x\to 2^+}\frac{x^3}{(x+5)(x-2)}=\infty$, since $\frac{x^3}{(x+5)(x-2)}>0$ for $x>2$. Similarly, $\lim_{x\to 2^-}\frac{x^3}{x^2+3x-10}=-\infty$, $\lim_{x\to -5^-}\frac{x^3}{x^2+3x-10}=-\infty$, and $\lim_{x\to -5^+}\frac{x^3}{x^2+3x-10}=\infty$, so $x=2$ and $x=-5$ are vertical asymptotes. The graph confirms these calculations.



40.

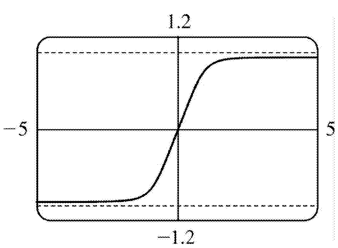
$\lim_{x\to\pm\infty} \frac{x^3+1}{x^3+x} = \lim_{x\to\pm\infty} \frac{1+1/x^3}{1+1/x^2} = 1$, so $y=1$ is a horizontal asymptote. Since $y=\frac{x^3+1}{x^3+x}=\frac{x^3+1}{x\left(x^2+1\right)}>0$ for $x>0$ and $y<0$ for $-1<x<0$, $\lim_{x\to 0^+} \frac{x^3+1}{x^3+x}=\infty$ and $\lim_{x\to 0^-} \frac{x^3+1}{x^3+x}=-\infty$, so $x=0$ is a vertical asymptote.



41. $\lim_{x\to\infty} \frac{x}{\sqrt[4]{x^4+1}} \cdot \frac{1/x}{1/\sqrt[4]{x^4}} = \lim_{x\to\infty} \frac{1}{\sqrt[4]{1+\frac{1}{x^4}}} = \frac{1}{\sqrt[4]{1+0}} = 1$ and

$\lim_{x\to-\infty} \frac{x}{\sqrt[4]{x^4+1}} \cdot \frac{1/x}{-1/\sqrt[4]{x^4}} = \lim_{x\to-\infty} \frac{1}{-\sqrt[4]{1+\frac{1}{x^4}}} = \frac{1}{-\sqrt[4]{1+0}} = -1$, so $y=\pm 1$ are horizontal asymptotes.

There is no vertical asymptote.

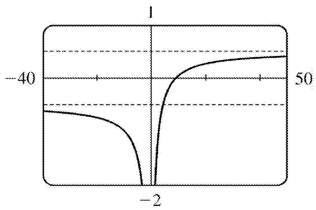


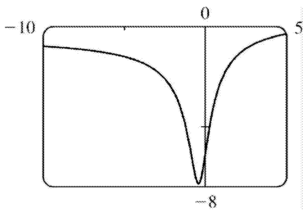
42. $\lim_{x\to\infty} \frac{x-9}{\sqrt{4x^2+3x+2}} = \lim_{x\to\infty} \frac{1-9/x}{\sqrt{4+(3/x)+\left(2/x^2\right)}} = \frac{1-0}{\sqrt{4+0+0}} = \frac{1}{2}$.

Using the fact that $\sqrt{x^2}=|x|=-x$ for $x<0$, we divide the numerator by $-x$ and the denominator by $\sqrt{x^2}$.

Thus, $\lim_{x\to-\infty} \frac{x-9}{\sqrt{4x^2+3x+2}} = \lim_{x\to-\infty} \frac{-1+9/x}{\sqrt{4+(3/x)+\left(2/x^2\right)}} = \frac{-1+0}{\sqrt{4+0+0}} = -\frac{1}{2}$.

The horizontal asymptotes are $y=\pm\frac{1}{2}$. The polynomial $4x^2+3x+2$ is positive for all x , so the denominator never approaches zero, and thus there is no vertical asymptote.





43. Let's look for a rational function.

(1) $\lim_{x\to\pm\infty} f(x)=0 \Rightarrow$ degree of numerator < degree of denominator

(2) $\lim_{x\to 0} f(x)=-\infty \Rightarrow$ there is a factor of x^2 in the denominator (not just x , since that would produce a sign change at $x=0$), and the function is negative near $x=0$.

(3) $\lim_{x\to 3^-} f(x)=\infty$ and $\lim_{x\to 3^+} f(x)=-\infty \Rightarrow$ vertical asymptote at $x=3$; there is a factor of $(x-3)$ in the denominator.

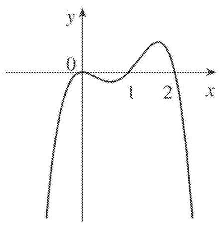
(4) $f(2)=0 \Rightarrow$ 2 is an x –intercept; there is at least one factor of $(x-2)$ in the numerator.

Combining all of this information and putting in a negative sign to give us the desired left– and right–hand limits gives us $f(x)=\dfrac{2-x}{x^2(x-3)}$ as one possibility.

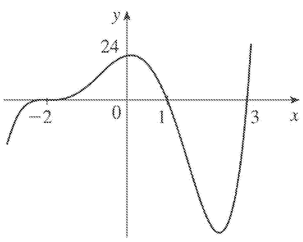
44. Since the function has vertical asymptotes $x=1$ and $x=3$, the denominator of the rational function we are looking for must have factors $(x-1)$ and $(x-3)$. Because the horizontal asymptote is $y=1$, the degree of the numerator must equal the degree of the denominator, and the ratio of the leading coefficients must be 1 . One possibility is $f(x)=\dfrac{x^2}{(x-1)(x-3)}$.

45. $y=f(x)=x^2(x-2)(1-x)$. The $y-$ intercept is $f(0)=0$, and the $x-$ intercepts occur when $y=0\Rightarrow x=0$, 1 , and 2 . Notice that, since x^2 is always positive, the graph does not cross the $x-$ axis at 0 , but does cross the $x-$ axis at 1 and 2 . $\lim_{x\to\infty} x^2(x-2)(1-x)=-\infty$, since the first two factors are large positive and the third large negative when x is large positive.

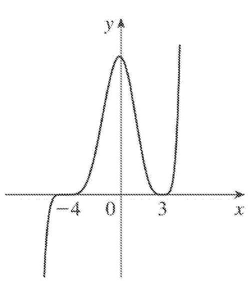
$\lim_{x\to -\infty} x^2(x-2)(1-x)=-\infty$ because the first and third factors are large positive and the second large negative as $x\to -\infty$.



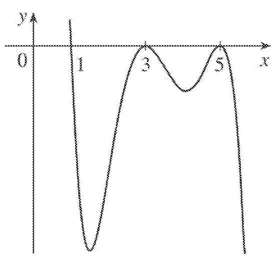
46. $y=(2+x)^3(1-x)(3-x)$. As $x\to \infty$, the first factor is large positive, and the second and third factors are large negative. Therefore, $\lim_{x\to \infty} f(x)=\infty$. As $x\to -\infty$, the first factor is large negative, and the second and third factors are large positive. Therefore, $\lim_{x\to -\infty} f(x)=-\infty$. Now the $y-$ intercept is $f(0)=(2)^3(1)(3)=24$ and the $x-$ intercepts are the solutions to $f(x)=0\Rightarrow x=-2$, 1 and 3 , and the graph crosses the $x-$ axis at all of these points.



47. $y=f(x)=(x+4)^5(x-3)^4$. The $y-$ intercept is $f(0)=4^5(-3)^4=82$, 944 . The $x-$ intercepts occur when $y=0\Rightarrow x=-4$, 3 . Notice that the graph does not cross the $x-$ axis at 3 because $(x-3)^4$ is always positive, but does cross the $x-$ axis at -4 . $\lim_{x\to \infty} (x+4)^5(x-3)^4=\infty$ since both factors are large positive when x is large positive. $\lim_{x\to -\infty} (x+4)^5(x-3)^4=-\infty$ since the first factor is large negative and the second factor is large positive when x is large negative.

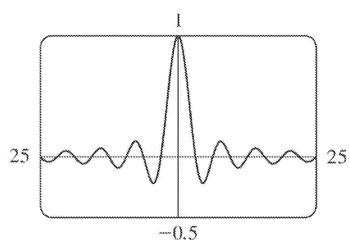


48. $y=(1-x)(x-3)^2(x-5)^2$. As $x\to\infty$, the first factor approaches $-\infty$ while the second and third factors approach ∞ . Therefore, $\lim_{x\to\infty}(x)=-\infty$. As $x\to-\infty$, the factors all approach ∞ . Therefore, $\lim_{x\to-\infty}(x)=\infty$. Now the $y-$ intercept is $f(0)=(1)(-3)^2(-5)^2=225$ and the $x-$ intercepts are the solutions to $f(x)=0\Rightarrow x=1$, 3 , and 5 . Notice that $f(x)$ does not change sign at $x=3$ or $x=5$ because the factors $(x-3)^2$ and $(x-5)^2$ are always positive, so the graph does not cross the $x-$ axis at $x=3$ or $x=5$, but does cross the $x-$ axis at $x=1$.

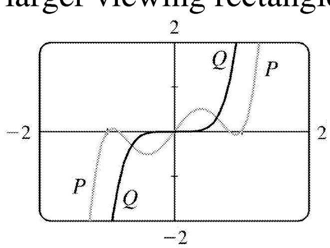


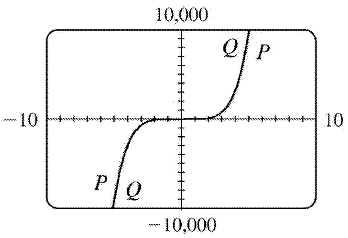
49. **(a)** Since $-1\le\sin x\le 1$ for all x,$-\frac{1}{x}\le\frac{\sin x}{x}\le\frac{1}{x}$ for $x>0$. As $x\to\infty$, $-1/x\to 0$ and $1/x\to 0$, so by the Squeeze Theorem, $(\sin x)/x\to 0$. Thus, $\lim_{x\to\infty}\frac{\sin x}{x}=0$.

(b) From part (a), the horizontal asymptote is $y=0$. The function $y=(\sin x)/x$ crosses the horizontal asymptote whenever $\sin x=0$; that is, at $x=\pi n$ for every integer n . Thus, the graph crosses the asymptote *an infinite number of times.*



50. **(a)** In both viewing rectangles, $\lim_{x\to\infty}P(x)=\lim_{x\to\infty}Q(x)=\infty$ and $\lim_{x\to-\infty}P(x)=\lim_{x\to-\infty}Q(x)=-\infty$. In the larger viewing rectangle, P and Q become less distinguishable.

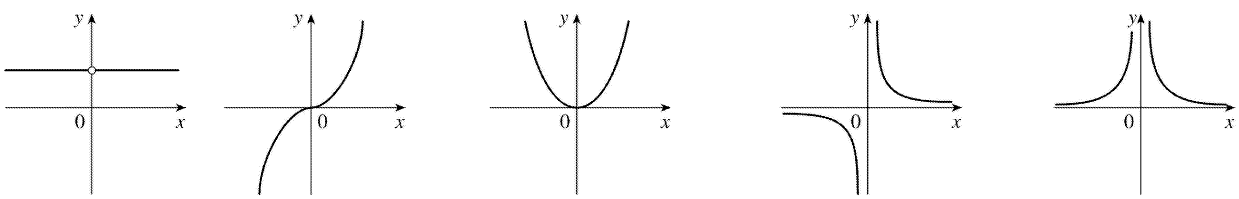




(b) $\lim\limits_{x\to\infty}\dfrac{P(x)}{Q(x)}=\lim\limits_{x\to\infty}\dfrac{3x^5-5x^3+2x}{3x^5}=\lim\limits_{x\to\infty}\left(1-\dfrac{5}{3}\cdot\dfrac{1}{x^2}+\dfrac{2}{3}\cdot\dfrac{1}{x^4}\right)=1-\dfrac{5}{3}(0)+\dfrac{2}{3}(0)=1\Rightarrow$
P and Q have the same end behavior.

51. Divide the numerator and the denominator by the highest power of x in $Q(x)$.
(a) If $\deg P<\deg Q$, then the numerator $\to 0$ but the denominator doesn't. So $\lim\limits_{x\to\infty}[P(x)/Q(x)]=0$.
(b) If $\deg P>\deg Q$, then the numerator $\to\pm\infty$ but the denominator doesn't, so $\lim\limits_{x\to\infty}[P(x)/Q(x)]=\pm\infty$ (depending on the ratio of the leading coefficients of P and Q).

52.



(i) $n=0$ (ii) $n>0$ (n odd) (iii) $n>0$ (n even) (iv) $n<0$ (n odd) (v) $n<0$ (n even)

(a) $\lim\limits_{x\to 0^+}x^n=\begin{cases}1 & \text{if } n=0\\ 0 & \text{if } n>0\\ \infty & \text{if } n<0\end{cases}$

(b) $\lim\limits_{x\to 0^-}x^n=\begin{cases}1 & \text{if } n=0\\ 0 & \text{if } n>0\\ -\infty & \text{if } n<0, \quad n \text{ odd}\\ \infty & \text{if } n<0, \quad n \text{ even}\end{cases}$

(c) $\lim\limits_{x\to\infty}x^n=\begin{cases}1 & \text{if } n=0\\ \infty & \text{if } n>0\\ 0 & \text{if } n<0\end{cases}$

(d) $\lim\limits_{x\to-\infty}x^n=\begin{cases}1 & \text{if } n=0\\ -\infty & \text{if } n>0, \quad n \text{ odd}\\ \infty & \text{if } n>0, \quad n \text{ even}\\ 0 & \text{if } n<0\end{cases}$

53.

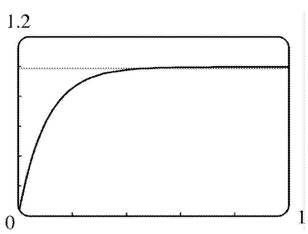
$\lim_{x\to\infty} \frac{4x-1}{x} = \lim_{x\to\infty} \left(4-\frac{1}{x}\right) = 4$, and $\lim_{x\to\infty} \frac{4x^2+3x}{x^2} = \lim_{x\to\infty} \left(4+\frac{3}{x}\right) = 4$. Therefore, by the Squeeze Theorem, $\lim_{x\to\infty} f(x) = 4$.

54. **(a)** After t minutes, $25t$ liters of brine with 30 g of salt per liter has been pumped into the tank, so it contains $(5000+25t)$ liters of water and $25t\cdot 30=750t$ grams of salt. Therefore, the salt concentration at time t will be $C(t) = \frac{750t}{5000+25t} = \frac{30t}{200+t}\ \frac{\text{g}}{\text{L}}$.

(b) $\lim_{t\to\infty} C(t) = \lim_{t\to\infty} \frac{30t}{200+t} = \lim_{t\to\infty} \frac{30t/t}{200/t+t/t} = \frac{30}{0+1} = 30$. So the salt concentration approaches that of the brine being pumped into the tank.

55. **(a)** $\lim_{t\to\infty} v(t) = \lim_{t\to\infty} v^*\left(1-e^{-gt/v^*}\right) = v^*(1-0) = v^*$

(b) We graph $v(t)=1-e^{-9.8t}$ and $v(t)=0.99v^*$, or in this case, $v(t)=0.99$. Using an intersect feature or zooming in on the point of intersection, we find that $t\approx 0.47$ s.

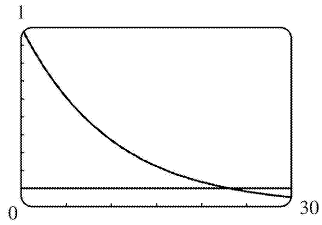


56. **(a)** $y=e^{-x/10}$ and $y=0.1$ intersect at $x_1\approx 23.03$.

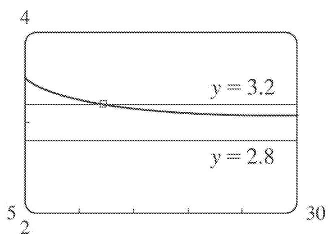
If $x>x_1$, then $e^{-x/10}<0.1$.

(b) $e^{-x/10}<0.1 \Rightarrow -x/10<\ln 0.1 \Rightarrow$

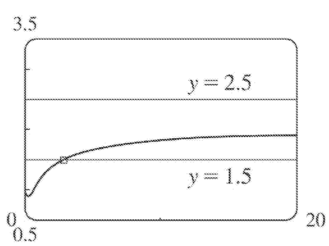
$x>-10\ln \frac{1}{10} = -10\ln 10^{-1} = 10\ln 10$

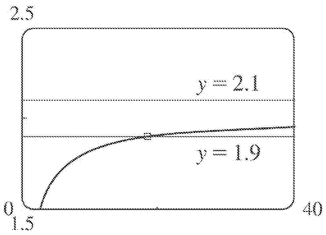


57. $\left|\dfrac{6x^2+5x-3}{2x^2-1}-3\right|<0.2 \Leftrightarrow 2.8<\dfrac{6x^2+5x-3}{2x^2-1}<3.2$. So we graph the three parts of this inequality on the same screen, and find that the curve $y=\dfrac{6x^2+5x-3}{2x^2-1}$ seems to lie between the lines $y=2.8$ and $y=3.2$ whenever $x>12.8$. So we can choose $N=13$ (or any larger number) so that the inequality holds whenever $x\geq N$.



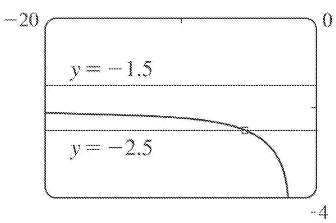
58. For $\varepsilon=0.5$, we must find N such that whenever $x\geq N$, we have $\left|\dfrac{\sqrt{4x^2+1}}{x+1}-2\right|<0.5 \Leftrightarrow$ $1.5<\dfrac{\sqrt{4x^2+1}}{x+1}<2.5$. We graph the three parts of this inequality on the same screen, and find that it holds whenever $x\geq 3$. So we choose $N=3$ (or any larger number). For $\varepsilon=0.1$, we must have $1.9<\dfrac{\sqrt{4x^2+1}}{x+1}<2.1$, and the graphs show that this holds whenever $x\geq 19$. So we choose $N=19$ (or any larger number).

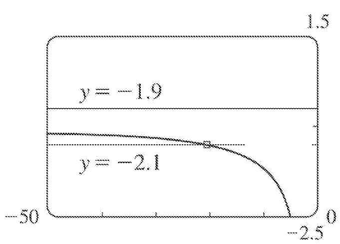




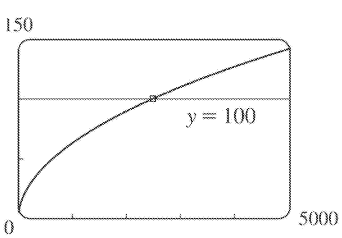
59. For $\varepsilon = 0.5$, we need to find N such that $\left| \frac{\sqrt{4x^2+1}}{x+1} - (-2) \right| < 0.5 \Leftrightarrow -2.5 < \frac{\sqrt{4x^2+1}}{x+1} < -1.5$ whenever $x \le N$. We graph the three parts of this inequality on the same screen, and see that the inequality holds for $x \le -6$. So we choose $N = -6$ (or any smaller number).

For $\varepsilon = 0.1$, we need $-2.1 < \frac{\sqrt{4x^2+1}}{x+1} < -1.9$ whenever $x \le N$. From the graph, it seems that this inequality holds for $x \le -22$. So we choose $N = -22$ (or any smaller number).





60. We need N such that $\frac{2x+1}{\sqrt{x+1}} > 100$ whenever $x \ge N$. From the graph, we see that this inequality holds for $x \ge 2500$. So we choose $N = 2500$ (or any larger number).



61. **(a)** $1/x^2 < 0.0001 \Leftrightarrow x^2 > 1/0.0001 = 10,000 \Leftrightarrow x > 100$ ($x > 0$)

(b) If $\varepsilon > 0$ is given, then $1/x^2 < \varepsilon \Leftrightarrow x^2 > 1/\varepsilon \Leftrightarrow x > 1/\sqrt{\varepsilon}$. Let $N = 1/\sqrt{\varepsilon}$.
Then $x > N \Rightarrow x > \frac{1}{\sqrt{\varepsilon}} \Rightarrow \left| \frac{1}{x^2} - 0 \right| = \frac{1}{x^2} < \varepsilon$, so $\lim_{x\to\infty} \frac{1}{x^2} = 0$.

62. **(a)** $1/\sqrt{x} < 0.0001 \Leftrightarrow \sqrt{x} > 1/0.0001 = 10^4 \Leftrightarrow x > 10^8$
(b) If $\varepsilon > 0$ is given, then $1/\sqrt{x} < \varepsilon \Leftrightarrow \sqrt{x} > 1/\varepsilon \Leftrightarrow x > 1/\varepsilon^2$. Let $N = 1/\varepsilon^2$.
Then $x > N \Rightarrow x > \frac{1}{\varepsilon^2} \Rightarrow \left| \frac{1}{\sqrt{x}} - 0 \right| = \frac{1}{\sqrt{x}} < \varepsilon$, so $\lim_{x\to\infty} \frac{1}{\sqrt{x}} = 0$.

63. For $x < 0$, $|1/x - 0| = -1/x$. If $\varepsilon > 0$ is given, then $-1/x < \varepsilon \Leftrightarrow x < -1/\varepsilon$.
Take $N = -1/\varepsilon$. Then $x < N \Rightarrow x < -1/\varepsilon \Rightarrow |(1/x) - 0| = -1/x < \varepsilon$, so $\lim_{x\to-\infty} (1/x) = 0$.

64. Given $M > 0$, we need $N > 0$ such that $x > N \Rightarrow x^3 > M$. Now $x^3 > M \Leftrightarrow x > \sqrt[3]{M}$, so take $N = \sqrt[3]{M}$. Then $x > N = \sqrt[3]{M} \Rightarrow x^3 > M$, so $\lim_{x\to\infty} x^3 = \infty$.

65. Given $M > 0$, we need $N > 0$ such that $x > N \Rightarrow e^x > M$. Now $e^x > M \Leftrightarrow x > \ln M$, so take $N = \max(1, \ln M)$. (This ensures that $N > 0$.) Then $x > N = \max(1, \ln M) \Rightarrow e^x > \max(e, M) \geq M$, so $\lim_{x\to\infty} e^x = \infty$.

66. **Definition** Let f be a function defined on some interval $(-\infty, a)$. Then $\lim_{x\to-\infty} f(x) = -\infty$ means that for every negative number M there is a corresponding negative number N such that $f(x) < M$ whenever $x < N$. Now we use the definition to prove that $\lim_{x\to-\infty} \left(1 + x^3\right) = -\infty$. Given a negative number M , we need a negative number N such that $x < N \Rightarrow 1 + x^3 < M$. Now $1 + x^3 < M \Leftrightarrow x^3 < M - 1 \Leftrightarrow x < \sqrt[3]{M-1}$. Thus, we take $N = \sqrt[3]{M-1}$ and find that $x < N \Rightarrow 1 + x^3 < M$. This proves that $\lim_{x\to-\infty} \left(1 + x^3\right) = -\infty$.

67. Suppose that $\lim_{x\to\infty} f(x) = L$. Then for every $\varepsilon > 0$ there is a corresponding positive number N such that $|f(x) - L| < \varepsilon$ whenever $x > N$. If $t = 1/x$, then $x > N \Leftrightarrow 0 < 1/x < 1/N \Leftrightarrow 0 < t < 1/N$. Thus, for every $\varepsilon > 0$ there is a corresponding $\delta > 0$ (namely $1/N$) such that $|f(1/t) - L| < \varepsilon$ whenever $0 < t < \delta$. This proves that

$\lim_{t\to 0^+} f(1/t)=L=\lim_{x\to\infty} f(x)$.

Now suppose that $\lim_{x\to -\infty} f(x)=L$. Then for every $\varepsilon>0$ there is a corresponding negative number N such that $|f(x)-L|<\varepsilon$ whenever $x<N$. If $t=1/x$, then $x<N \Leftrightarrow 1/N<1/x<0 \Leftrightarrow 1/N<t<0$. Thus, for every $\varepsilon>0$ there is a corresponding $\delta>0$ (namely $-1/N$) such that $|f(1/t)-L|<\varepsilon$ whenever $-\delta<t<0$. This proves that $\lim_{t\to 0^-} f(1/t)=L=\lim_{x\to -\infty} f(x)$.

1. **(a)** This is just the slope of the line through two points: $m_{PQ}=\dfrac{\Delta y}{\Delta x}=\dfrac{f(x)-f(3)}{x-3}$.

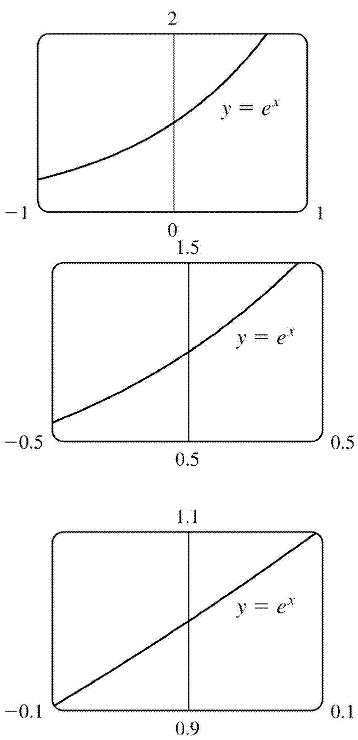
(b) This is the limit of the slope of the secant line PQ as Q approaches P : $m=\lim\limits_{x\to 3}\dfrac{f(x)-f(3)}{x-3}$.

2. **(a)** Average velocity = $\dfrac{\Delta s}{\Delta t}=\dfrac{f(a+h)-f(a)}{(a+h)-a}=\dfrac{f(a+h)-f(a)}{h}$

(b) Instantaneous velocity $=\lim\limits_{h\to 0}\dfrac{f(a+h)-f(a)}{h}$

3. The slope at D is the largest positive slope, followed by the positive slope at E . The slope at C is zero. The slope at B is steeper than at A (both are negative). In decreasing order, we have the slopes at: D , E , C , A , and B .

4. The curve looks more like a line as the viewing rectangle gets smaller.



5. **(a)**

(i) Using Definition 1,

$$m=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}\lim_{x\to -3}\frac{f(x)-f(-3)}{x-(-3)}=\lim_{x\to -3}\frac{\left(x^2+2x\right)-(3)}{x-(-3)}=\lim_{x\to -3}\frac{(x+3)(x-1)}{x+3}$$

$$=\lim_{x\to -3}(x-1)=-4$$

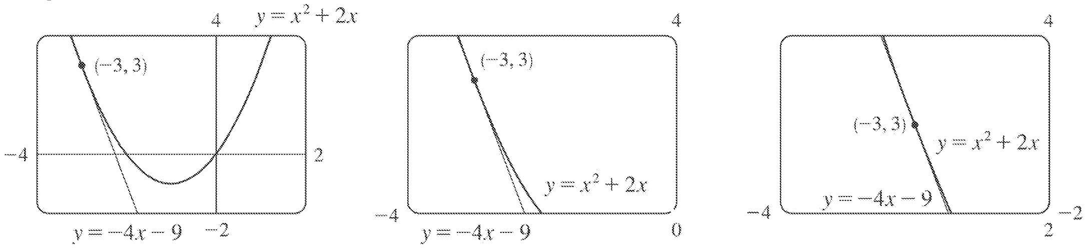
(ii) Using Equation 2,

$$m=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}=\lim_{h\to 0}\frac{f(-3+h)-f(-3)}{h}=\lim_{h\to 0}\frac{\left[(-3+h)^2+2(-3+h)\right]-(3)}{h}$$

$$=\lim_{h\to 0}\frac{9-6h+h^2-6+2h-3}{h}=\lim_{h\to 0}\frac{h(h-4)}{h}=\lim_{h\to 0}(h-4)=-4$$

(b) Using the point–slope form of the equation of a line, an equation of the tangent line is $y-3=-4(x+3)$. Solving for y gives us $y=-4x-9$, which is the slope–intercept form of the equation of the tangent line.

(c)



6. **(a)**

(i)

$$m=\lim_{x\to -1}\frac{f(x)-f(-1)}{x-(-1)}=\lim_{x\to -1}\frac{x^3-(-1)}{x+1}=\lim_{x\to -1}\frac{(x+1)\left(x^2-x+1\right)}{x+1}$$

$$=\lim_{x\to -1}\left(x^2-x+1\right)=3$$

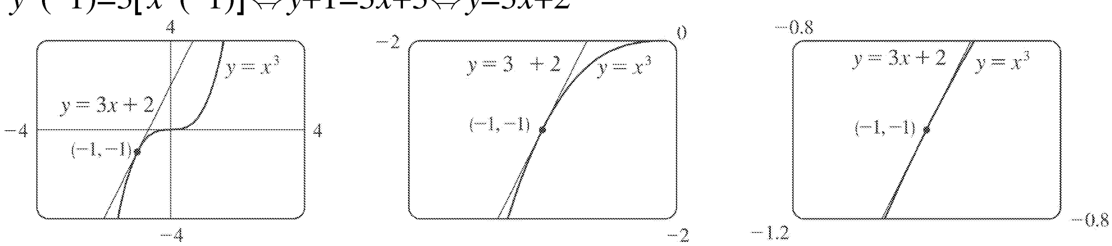
(ii)

$$m=\lim_{h\to 0}\frac{f(-1+h)-f(-1)}{h}=\lim_{h\to 0}\frac{(-1+h)^3-(-1)}{h}=\lim_{h\to 0}\frac{h^3-3h^2+3h-1+1}{h}$$

$$=\lim_{h\to 0}\left(h^2-3h+3\right)=3$$

(b) $y-(-1)=3[x-(-1)]\Leftrightarrow y+1=3x+3\Leftrightarrow y=3x+2$

(c)



7. Using (2) with $f(x)=1+2x-x^3$ and $P(1,2)$,

$$m=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}=\lim_{h\to 0}\frac{f(1+h)-f(1)}{h}=\lim_{h\to 0}\frac{\left[1+2(1+h)-(1+h)^3\right]-2}{h}$$

$$=\lim_{h\to 0}\frac{1+2+2h-(1+3h+3h^2+h^3)-2}{h}=\lim_{h\to 0}\frac{-h^3-3h^2-h}{h}$$

$$=\lim_{h\to 0}\frac{h(-h^2-3h-1)}{h}=\lim_{h\to 0}(-h^2-3h-1)=-1$$

Tangent line: $y-2=-1(x-1)\Leftrightarrow y-2=-x+1\Leftrightarrow y=-x+3$

8. Using (1),

$$m=\lim_{x\to 4}\frac{\sqrt{2x+1}-\sqrt{2(4)+1}}{x-4}=\lim_{x\to 4}\frac{\sqrt{2x+1}-3}{x-4}\cdot\frac{\sqrt{2x+1}+3}{\sqrt{2x+1}+3}$$

$$=\lim_{x\to 4}\frac{(2x+1)-3^2}{(x-4)(\sqrt{2x+1}+3)}=\lim_{x\to 4}\frac{2(x-4)}{(x-4)(\sqrt{2x+1}+3)}$$

$$=\lim_{x\to 4}\frac{2}{(\sqrt{2x+1}+3)}=\frac{2}{3+3}=\frac{1}{3}\ .$$

Tangent line: $y-3=\frac{1}{3}(x-4)\Leftrightarrow y-3=\frac{1}{3}x-\frac{4}{3}\Leftrightarrow y=\frac{1}{3}x+\frac{5}{3}$

9. Using (1) with $f(x)=\frac{x-1}{x-2}$ and $P(3,2)$,

$$m=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}=\lim_{x\to 3}\frac{\frac{x-1}{x-2}-2}{x-3}=\lim_{x\to 3}\frac{\frac{x-1-2(x-2)}{x-2}}{x-3}=\lim_{x\to 3}\frac{3-x}{(x-2)(x-3)}$$

$$=\lim_{x\to 3}\frac{-1}{x-2}=\frac{-1}{1}=-1\ .$$

Tangent line: $y-2=-1(x-3)\Leftrightarrow y-2=-x+3\Leftrightarrow y=-x+5$

10. Using (1), $m=\lim_{x\to 0}\frac{\frac{2x}{(x+1)^2}-0}{x-0}=\lim_{x\to 0}\frac{2x}{x(x+1)^2}=\lim_{x\to 0}\frac{2}{(x+1)^2}=\frac{2}{1^2}=2$.

Tangent line: $y-0=2(x-0)\Leftrightarrow y=2x$

11. **(a)**

$$m = \lim_{x\to a} \frac{f(x)-f(a)}{x-a} = \lim_{x\to a} \frac{2/(x+3)-2/(a+3)}{x-a} = \lim_{x\to a} \frac{2(a+3)-2(x+3)}{(x-a)(x+3)(a+3)}$$

$$= \lim_{x\to a} \frac{2(a-x)}{(x-a)(x+3)(a+3)} = \lim_{x\to a} \frac{-2}{(x+3)(a+3)} = \frac{-2}{(a+3)^2}$$

(b)

(i) $a=-1 \Rightarrow m=\frac{-2}{(-1+3)^2}=-\frac{1}{2}$

(ii) $a=0 \Rightarrow m=\frac{-2}{(0+3)^2}=-\frac{2}{9}$

(iii) $a=1 \Rightarrow m=\frac{-2}{(1+3)^2}=-\frac{1}{8}$

12. **(a)** Using (1),

$$m = \lim_{x\to a} \frac{\left(1+x+x^2\right)-\left(1+a+a^2\right)}{x-a} = \lim_{x\to a} \frac{x+x^2-a-a^2}{x-a} = \lim_{x\to a} \frac{x-a+(x-a)(x+a)}{x-a}$$

$$= \lim_{x\to a} \frac{(x-a)(1+x+a)}{x-a} = \lim_{x\to a} (1+x+a) = 1+2a$$

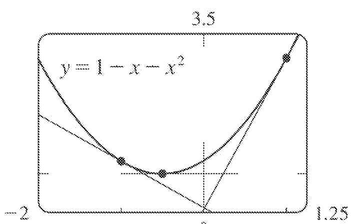
(b)

(i) $x=-1 \Rightarrow m=1+2(-1)=-1$

(ii) $x=-\frac{1}{2} \Rightarrow m=1+2\left(-\frac{1}{2}\right)=0$

(iii) $x=1 \Rightarrow m=1+2(1)=3$

(c)

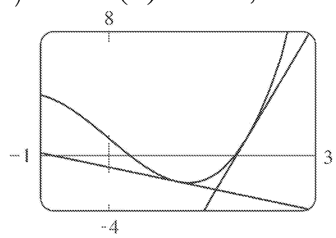


13. **(a)** Using (1),

$$m = \lim_{x\to a} \frac{\left(x^3-4x+1\right)-\left(a^3-4a+1\right)}{x-a} = \lim_{x\to a} \frac{\left(x^3-a^3\right)-4(x-a)}{x-a}$$

$$= \lim_{x\to a} \frac{(x-a)\left(x^2+ax+a^2\right)-4(x-a)}{x-a} = \lim_{x\to a} \left(x^2+ax+a^2-4\right)=3a^2-4$$

(b) At $(1,-2)$: $m=3(1)^2-4=-1$, so an equation of the tangent line is $y-(-2)=-1(x-1)\Leftrightarrow y=-x-1$. At $(2,1)$: $m=3(2)^2-4=8$, so an equation of the tangent line is $y-1=8(x-2)\Leftrightarrow y=8x-15$.

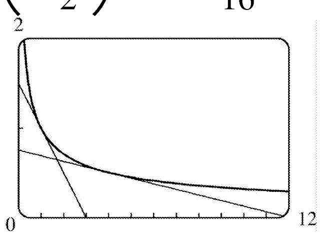
(c) 

14. **(a)** Using (1),

$$m=\lim_{x\to a} \frac{\frac{1}{\sqrt{x}}-\frac{1}{\sqrt{a}}}{x-a} = \lim_{x\to a} \frac{\frac{\sqrt{a}-\sqrt{x}}{\sqrt{ax}}}{x-a} = \lim_{x\to a} \frac{\left(\sqrt{a}-\sqrt{x}\right)\left(\sqrt{a}+\sqrt{x}\right)}{\sqrt{ax}\,(x-a)\left(\sqrt{a}+\sqrt{x}\right)}$$

$$= \lim_{x\to a} \frac{a-x}{\sqrt{ax}\,(x-a)\left(\sqrt{a}+\sqrt{x}\right)} = \lim_{x\to a} \frac{-1}{\sqrt{ax}\left(\sqrt{a}+\sqrt{x}\right)} = \frac{-1}{\sqrt{a^2}\left(2\sqrt{a}\right)} = -\frac{1}{2a^{3/2}} \text{ or } -\frac{1}{2}a^{-3/2}$$

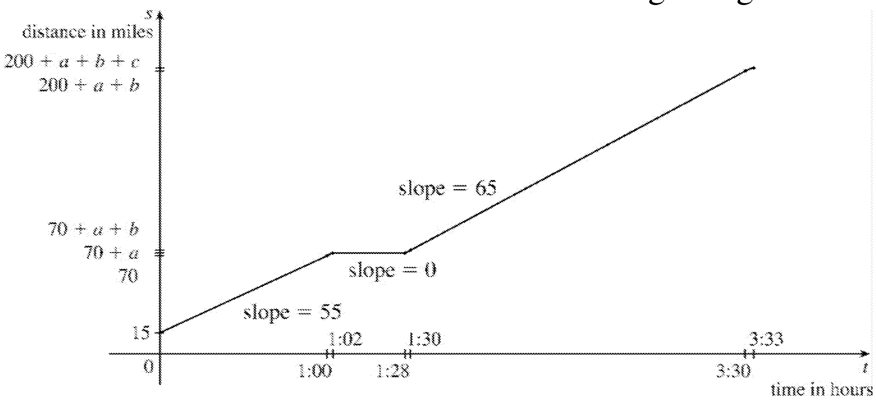
(b) At $(1,1)$: $m=-\frac{1}{2}$, so an equation of the tangent line is $y-1=-\frac{1}{2}(x-1)\Leftrightarrow y=-\frac{1}{2}x+\frac{3}{2}$.
At $\left(4,\frac{1}{2}\right)$: $m=-\frac{1}{16}$, so an equation of the tangent line is $y-\frac{1}{2}=-\frac{1}{16}(x-4)\Leftrightarrow y=-\frac{1}{16}x+\frac{3}{4}$.

(c) 

15. **(a)** Since the slope of the tangent at $t=0$ is 0 , the car's initial velocity was 0 .
(b) The slope of the tangent is greater at C than at B , so the car was going faster at C .
(c) Near A , the tangent lines are becoming steeper as x increases, so the velocity was increasing, so the car was speeding up. Near B , the tangent lines are becoming less steep, so the car was slowing down. The steepest tangent near C is the one at C , so at C the car had just finished speeding up, and was about to start slowing down.

(d) Between D and E , the slope of the tangent is 0 , so the car did not move during that time.

16. Let a denote the distance traveled from 1:00 to 1:02 , b from 1:28 to 1:30, and c from 3:30 to 3:33, where all the times are relative to $t=0$ at the beginning of the trip.



17. Let $s(t)=40t-16t^2$.

$$v(2) = \lim_{t\to 2}\frac{s(t)-s(2)}{t-2} = \lim_{t\to 2}\frac{\left(40t-16t^2\right)-16}{t-2} = \lim_{t\to 2}\frac{-16t^2+40t-16}{t-2} = \lim_{t\to 2}\frac{-8\left(2t^2-5t+2\right)}{t-2}$$

$$= \lim_{t\to 2}\frac{-8(t-2)(2t-1)}{t-2} = -8\lim_{t\to 2}(2t-1) = -8(3) = -24$$

Thus, the instantaneous velocity when $t=2$ is -24 ft / s.

18. **(a)**

$$v(1) = \lim_{h\to 0}\frac{H(1+h)-H(1)}{h}$$

$$= \lim_{h\to 0}\frac{\left(58+58h-0.83-1.66h-0.83h^2\right)-57.17}{h} = \lim_{h\to 0}(56.34-0.83h) = 56.34 \text{ m / s}$$

(b)

$$v(a) = \lim_{h\to 0}\frac{H(a+h)-H(a)}{h}$$

$$= \lim_{h\to 0}\frac{\left(58a+58h-0.83a^2-1.66ah-0.83h^2\right)-\left(58a-0.83a^2\right)}{h}$$

$$= \lim_{h\to 0}(58-1.66a-0.83h) = 58-1.66a \text{ m / s}$$

(c) The arrow strikes the moon when the height is 0 , that is,

$58t-0.83t^2=0 \Leftrightarrow t(58-0.83t)=0 \Leftrightarrow t=\frac{58}{0.83} \approx 69.9$ s (since t can't be 0).

(d) Using the time from part (c), $v\left(\frac{58}{0.83}\right)=58-1.66\left(\frac{58}{0.83}\right)=-58$ m / s. Thus, the arrow will have a velocity of -58 m / s.

19.

$$v(a)=\lim_{h\to 0}\frac{s(a+h)-s(a)}{h}=\lim_{h\to 0}\frac{4(a+h)^3+6(a+h)+2-\left(4a^3+6a+2\right)}{h}$$

$$=\lim_{h\to 0}\frac{4a^3+12a^2h+12ah^2+4h^3+6a+6h+2-4a^3-6a-2}{h}$$

$$=\lim_{h\to 0}\frac{12a^2h+12ah^2+4h^3+6h}{h}=\lim_{h\to 0}\left(12a^2+12ah+4h^2+6\right)=\left(12a^2+6\right)\text{ m / s}$$

So $v(1)=12(1)^2+6=18$ m / s, $v(2)=12(2)^2+6=54$ m / s, and $v(3)=12(3)^2+6=114$ m / s.

20. **(a)** The average velocity between times t and $t+h$ is

$$\frac{s(t+h)-s(t)}{(t+h)-t}=\frac{(t+h)^2-8(t+h)+18-\left(t^2-8t+18\right)}{h}$$

$$=\frac{t^2+2th+h^2-8t-8h+18-t^2+8t-18}{h}=\frac{2th+h^2-8h}{h}$$

$$=(2t+h-8)\text{ m/s}$$

(i) $[3,4]$: $t=3$, $h=4-3=1$, so the average velocity is $2(3)+1-8=-1$ m / s.

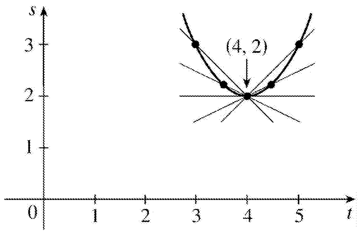
(ii) $[3.5,4]$: $t=3.5$, $h=0.5$, so the average velocity is $2(3.5)+0.5-8=-0.5$ m / s.

(iii) $[4,5]$: $t=4$, $h=1$, so the average velocity is $2(4)+1-8=1$ m / s.

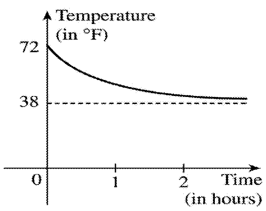
(iv) $[4,4.5]$: $t=4$, $h=0.5$, so the average velocity is $2(4)+0.5-8=0.5$ m / s.

(b) $v(t)=\lim_{h\to 0}\frac{s(t+h)-s(t)}{h}=\lim_{h\to 0}(2t+h-8)=2t-8$, so $v(4)=0$.

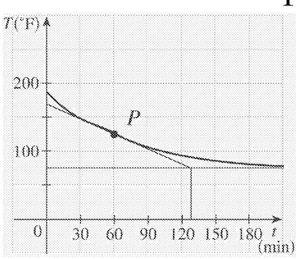
(c)



21. The sketch shows the graph for a room temperature of 72° and a refrigerator temperature of 38°. The initial rate of change is greater in magnitude than the rate of change after an hour.



22. The slope of the tangent (that is, the rate of change of temperature with respect to time) at t=1 h seems to be about $\dfrac{75-168}{132-0}\approx-0.7^{\circ}$ F / min.

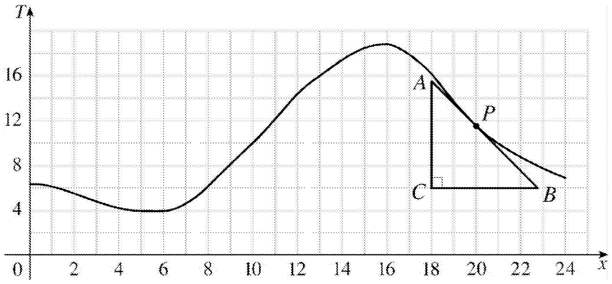


23. **(a)** (i)$[20,23]$: $\dfrac{7.9-11.5}{23-20}=-1.2^{\circ}$ C / h

(ii)$[20,22]$: $\dfrac{9.0-11.5}{22-20}=-1.25^{\circ}$ C / h

(iii)$[20,21]$: $\dfrac{10.2-11.5}{21-20}=-1.3^{\circ}$ C / h

(b) In the figure, we estimate A to be $(18,15.5)$ and B as $(23,6)$. So the slope is $\dfrac{6-15.5}{23-18}=-1.9^{\circ}$ C / h at 8:00 P.M.



24. **(a)**

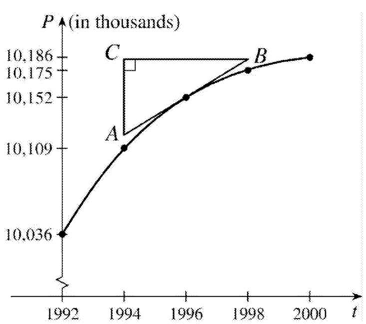
(i) $[1992,1996]:\ \dfrac{P(1996)-P(1992)}{1996-1992}=\dfrac{10{,}152-10{,}036}{4}=\dfrac{116}{4}=29$ thousand / year

(ii) $[1994,1996]:\ \dfrac{P(1996)-P(1994)}{1996-1994}=\dfrac{10{,}152-10{,}109}{2}=\dfrac{43}{2}=21.5$ thousand / year

(iii) $[1996,1998]:\ \dfrac{P(1998)-P(1996)}{1998-1996}=\dfrac{10{,}175-10{,}152}{2}=\dfrac{23}{2}=11.5$ thousand / year

(b) Using the values from (ii) and (iii), we have $\dfrac{21.5+11.5}{2}=16.5$ thousand / year.

(c) Estimating A as (1994,10 , 125) and B as $(1998,10,182)$, the slope at 1996 is
$\dfrac{10{,}182-10{,}125}{1998-1994}=\dfrac{57}{4}=14.25$ thousand / year.



25. **(a)**

(i) $[1995,1997]:\ \dfrac{N(1997)-N(1995)}{1997-1995}=\dfrac{2461-873}{2}=\dfrac{1588}{2}=794$ thousand / year

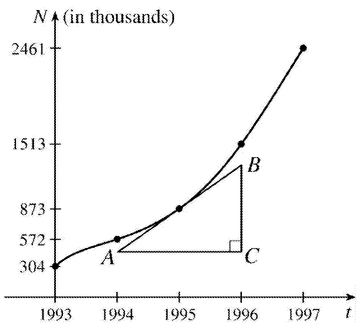
(ii) $[1995,1996]:\ \dfrac{N(1996)-N(1995)}{1996-1995}=\dfrac{1513-873}{1}=640$ thousand / year

(iii) $[1994,1995]:\ \dfrac{N(1995)-N(1994)}{1995-1994}=\dfrac{873-572}{1}=301$ thousand / year

(b) Using the values from (ii) and (iii), we have

$\frac{640+301}{2}=\frac{941}{2}$ =470.5 thousand / year.

(c) A as $(1994,420)$ and B as $(1996,1275)$, the slope at 1995 is $\frac{1275-420}{1996-1994}=\frac{855}{2}$ =427.5 thousand / year



26. **(a)**

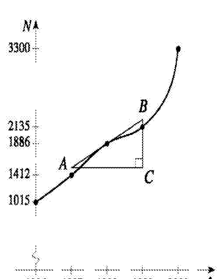
(i) $[1996,1998]$: $\frac{N(1998)-N(1996)}{1998-1996}=\frac{1886-1015}{2}=\frac{871}{2}$ =435.5 locations / year

(ii) $[1997,1998]$: $\frac{N(1998)-N(1997)}{1998-1997}=\frac{1886-1412}{1}$ =474 locations / year

(iii) $[1998,1999]$: $\frac{N(1999)-N(1998)}{1999-1998}=\frac{2135-1886}{1}$ =249 locations / year

(b) Using the values from (ii) and (iii), we have $\frac{474+249}{2}=\frac{723}{2}$ =361.5≈362 locations / year.

(c) Estimating A as $(1997,1525)$ and B as $(1999,2250)$, the slope at 1998 is $\frac{2250-1525}{1999-1997}=\frac{725}{2}$ =362.5 locations / year.



27. **(a)**

(i) $\frac{\Delta C}{\Delta x} = \frac{C(105)-C(100)}{105-100} = \frac{6601.25-6500}{5}$ =\$20.25/ unit.

(ii) $\frac{\Delta C}{\Delta x} = \frac{C(101)-C(100)}{101-100} = \frac{6520.05-6500}{1}$ =\$20.05/ unit.

(b)

$$\frac{C(100+h)-C(100)}{h} = \frac{\left[5000+10(100+h)+0.05(100+h)^2\right]-6500}{h} = \frac{20h+0.05h^2}{h}$$

$$=20+0.05h\ ,\ h\neq 0$$

So the instantaneous rate of change is $\lim_{h\to 0} \frac{C(100+h)-C(100)}{h} = \lim_{h\to 0} (20+0.05h)$=\$20/ unit.

28.

$$\Delta V = V(t+h)-V(t) = 100{,}000\left(1-\frac{t+h}{60}\right)^2 - 100{,}000\left(1-\frac{t}{60}\right)^2$$

$$=100{,}000\left[\left(1-\frac{t+h}{30}+\frac{(t+h)^2}{3600}\right)-\left(1-\frac{t}{30}+\frac{t^2}{3600}\right)\right] = 100{,}000\left(-\frac{h}{30}+\frac{2th}{3600}+\frac{h^2}{3600}\right)$$

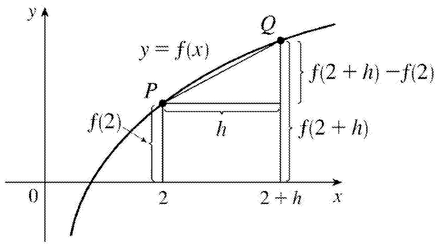
$$=\frac{100{,}000}{3600}h(-120+2t+h) = \frac{250}{9}h(-120+2t+h)$$

Dividing ΔV by h and then letting $h\to 0$, we see that the instantaneous rate of change is $\frac{500}{9}(t-60)$ gal / min.

t	*Flow rate (gal/min)*	*Water remaining $V(t)$ (gal)*
0	$-3333.\overline{3}$	100,000
10	$-2777.\overline{7}$	$69{,}444.\overline{4}$
20	$-2222.\overline{2}$	$44{,}444.\overline{4}$
30	$-1666.\overline{6}$	25,000
40	$-1111.\overline{1}$	$11{,}111.\overline{1}$
50	$-555.\overline{5}$	$2{,}777.\overline{7}$
60	0	0

The magnitude of the flow rate is greatest at the beginning and gradually decreases to 0.

1.



The line from $P(2,f(2))$ to $Q(2+h,f(2+h))$ is the line that has slope $\dfrac{f(2+h)-f(2)}{h}$

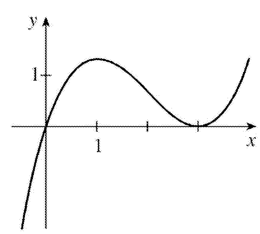
2. As h decreases, the line PQ becomes steeper, so its slope increases. So $0<\dfrac{f(4)-f(2)}{4-2}<\dfrac{f(3)-f(2)}{3-2}<\lim\limits_{x\to 2}\dfrac{f(x)-f(2)}{x-2}$. Thus, $0<\frac{1}{2}[f(4)-f(2)]<f(3)-f(2)<f'(2)$.

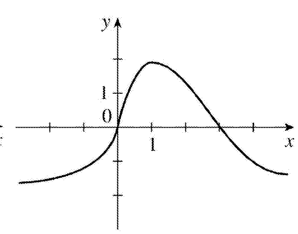
3. $g'(0)$ is the only negative value. The slope at $x=4$ is smaller than the slope at $x=2$ and both are smaller than the slope at $x=-2$. Thus, $g'(0)<0<g'(4)<g'(2)<g'(-2)$.

4. Since $(4,3)$ is on $y=f(x)$, $f(4)=3$. The slope of the tangent line between $(0,2)$ and $(4,3)$ is $\dfrac{1}{4}$, so $f'(4)=\dfrac{1}{4}$.

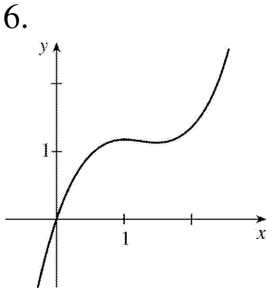
5.

We begin by drawing a curve through the origin at a slope of 3 to satisfy $f(0)=0$ and $f'(0)=3$. Since $f'(1)=0$, we will round off our figure so that there is a horizontal tangent directly over $x=1$. Lastly, we make sure that the curve has a slope of -1 as we pass over $x=2$. Two of the many possibilities are shown.





6.



7. Using Definition 2 with

$f(x)=3x^2-5x$ and the point $(2,2)$, we have

$$f'(2)=\lim_{h\to 0}\frac{f(2+h)-f(2)}{h}=\lim_{h\to 0}\frac{[3(2+h)^2-5(2+h)]-2}{h}$$

$$=\lim_{h\to 0}\frac{(12+12h+3h^2-10-5h)-2}{h}=\lim_{h\to 0}\frac{3h^2+7h}{h}=\lim_{h\to 0}(3h+7)=7\ .$$

So an equation of the tangent line at $(2,2)$ is $y-2=7(x-2)$ or $y=7x-12$.

8. Using Definition 2 with $g(x)=1-x^3$ and the point $(0,1)$, we have

$$g'(0)=\lim_{h\to 0}\frac{g(0+h)-g(0)}{h}=\lim_{h\to 0}\frac{[1-(0+h)^3]-1}{h}=\lim_{h\to 0}\frac{(1-h^3)-1}{h}=\lim_{h\to 0}(-h^2)=0\ .$$

So an equation of the tangent line is $y-1=0(x-0)$ or $y=1$.

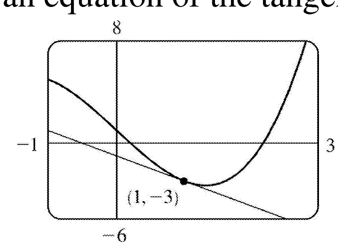
9. **(a)** Using Definition 2 with $F(x)=x^3-5x+1$ and the point $(1,-3)$, we have

$$F'(1)=\lim_{h\to 0}\frac{F(1+h)-F(1)}{h}=\lim_{h\to 0}\frac{[(1+h)^3-5(1+h)+1]-(-3)}{h}$$

$$=\lim_{h\to 0}\frac{(1+3h+3h^2+h^3-5-5h+1)+3}{h}=\lim_{h\to 0}\frac{h^3+3h^2-2h}{h}$$

$$=\lim_{h\to 0}\frac{h(h^2+3h-2)}{h}=\lim_{h\to 0}(h^2+3h-2)=-2$$

So an equation of the tangent line at $(1,-3)$ is $y-(-3)=-2(x-1)\Leftrightarrow y=-2x-1$.

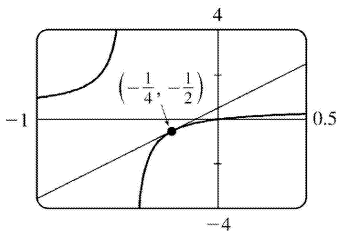
(b) 

10. **(a)**

$$G'(a)=\lim_{h\to 0}\frac{G(a+h)-G(a)}{h}=\lim_{h\to 0}\frac{\dfrac{a+h}{1+2(a+h)}-\dfrac{a}{1+2a}}{h}$$

$$=\lim_{h\to 0}\frac{a+2a^2+h+2ah-a-2a^2-2ah}{h(1+2a+2h)(1+2a)}=\lim_{h\to 0}\frac{1}{(1+2a+2h)(1+2a)}=(1+2a)^{-2}$$

So the slope of the tangent at the point $(-\frac{1}{4},-\frac{1}{2})$ is $m=\left[1+2(-\frac{1}{4})\right]^{-2}=4$, and thus an equation is $y+\frac{1}{2}=4(x+\frac{1}{4})$ or $y=4x+\frac{1}{2}$.

(b) 

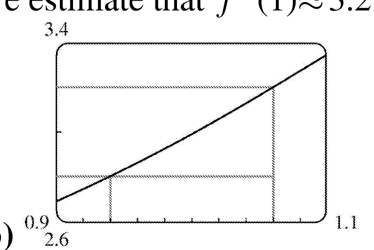
11. **(a)** $f'(1)=\lim_{h\to 0}\frac{f(1+h)-f(1)}{h}=\lim_{h\to 0}\frac{3^{1+h}-3^1}{h}$.

So let $F(h)=\frac{3^{1+h}-3}{h}$. We calculate:

h	$F(h)$
0.1	3.484
0.01	3.314
0.001	3.298
0.0001	3.296

h	$F(h)$
−0.1	3.121
−0.01	3.278
−0.001	3.294
−0.0001	3.296

We estimate that $f'(1)\approx 3.296$.

(b) 

From the graph, we estimate that the slope of the tangent is about $\frac{3.2-2.8}{1.06-0.94}=\frac{0.4}{0.12}\approx 3.3$.

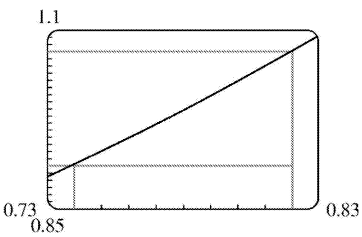
12. **(a)

$$g'\left(\frac{\pi}{4}\right)=\lim_{h\to 0}\frac{g\left(\frac{\pi}{4}+h\right)-g\left(\frac{\pi}{4}\right)}{h}=\lim_{h\to 0}\frac{\tan\left(\frac{\pi}{4}+h\right)-\tan\left(\frac{\pi}{4}\right)}{h}.$$

So let $G(h)=\dfrac{\tan\left(\frac{\pi}{4}+h\right)-1}{h}$. We calculate:

h	$G(h)$
0.1	2.2305
0.01	2.0203
0.001	2.0020
0.0001	2.0002

h	$G(h)$
–0.1	1.8237
–0.01	1.9803
–0.001	1.9980
–0.0001	1.9998

(b) 

From the graph, we estimate that the slope of the tangent is about $\dfrac{1.07-0.91}{0.82-0.74}=\dfrac{0.16}{0.08}=2$.

13. Use Definition 2 with $f(x)=3-2x+4x^2$.

$$f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}=\lim_{h\to 0}\frac{[3-2(a+h)+4(a+h)^2]-(3-2a+4a^2)}{h}$$

$$=\lim_{h\to 0}\frac{[3-2a-2h+4a^2+8ah+4h^2]-(3-2a+4a^2)}{h}=\lim_{h\to 0}\frac{-2h+8ah+4h^2}{h}$$

$$=\lim_{h\to 0}\frac{h(-2+8a+4h)}{h}=\lim_{h\to 0}(-2+8a+4h)=-2+8a$$

14.

$$f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}=\lim_{h\to 0}\frac{[(a+h)^4-5(a+h)]-(a^4-5a)}{h}$$

$$=\lim_{h\to 0}\frac{(a^4+4a^3h+6a^2h^2+4ah^3+h^4-5a-5h)-(a^4-5a)}{h}$$

$$=\lim_{h\to 0}\frac{4a^3h+6a^2h^2+4ah^3+h^4-5h}{h}=\lim_{h\to 0}\frac{h(4a^3+6a^2h+4ah^2+h^3-5)}{h}$$

$$=\lim_{h\to 0}(4a^3+6a^2h+4ah^2+h^3-5)=4a^3-5$$

15. Use Definition 2 with $f(t)=(2t+1)/(t+3)$.

$$f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}=\lim_{h\to 0}\frac{\frac{2(a+h)+1}{(a+h)+3}-\frac{2a+1}{a+3}}{h}$$

$$=\lim_{h\to 0}\frac{(2a+2h+1)(a+3)-(2a+1)(a+h+3)}{h(a+h+3)(a+3)}$$

$$=\lim_{h\to 0}\frac{(2a^2+6a+2ah+6h+a+3)-(2a^2+2ah+6a+a+h+3)}{h(a+h+3)(a+3)}$$

$$=\lim_{h\to 0}\frac{5h}{h(a+h+3)(a+3)}=\lim_{h\to 0}\frac{5}{(a+h+3)(a+3)}=\frac{5}{(a+3)^2}$$

16.

$$f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}=\lim_{h\to 0}\frac{\frac{(a+h)^2+1}{(a+h)-2}-\frac{a^2+1}{a-2}}{h}$$

$$=\lim_{h\to 0}\frac{(a^2+2ah+h^2+1))(a-2)-(a^2+1)(a+h-2)}{h(a+h-2)(a-2)}$$

$$=\lim_{h\to 0}\frac{(a^3-2a^2+2a^2h-4ah+ah^2-2h^2+a-2)-(a^3+a^2h-2a^2+a+h-2)}{h(a+h-2)(a-2)}$$

$$=\lim_{h\to 0}\frac{a^2h-4ah+ah^2-2h^2-h}{h(a+h-2)(a-2)}=\lim_{h\to 0}\frac{h(a^2-4a+ah-2h-1)}{h(a+h-2)(a-2)}$$

$$=\lim_{h\to 0}\frac{a^2-4a+ah-2h-1}{(a+h-2)(a-2)}=\frac{a^2-4a-1}{(a-2)^2}$$

17. Use Definition 2 with $f(x)=1/\sqrt{x+2}$.

$$f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}=\lim_{h\to 0}\frac{\frac{1}{\sqrt{(a+h)+2}}-\frac{1}{\sqrt{a+2}}}{h}$$

$$=\lim_{h\to 0}\frac{\frac{\sqrt{a+2}-\sqrt{a+h+2}}{\sqrt{a+h+2}\sqrt{a+2}}}{h}=\lim_{h\to 0}\left[\frac{\sqrt{a+2}-\sqrt{a+h+2}}{h\sqrt{a+h+2}\sqrt{a+2}}\cdot\frac{\sqrt{a+2}+\sqrt{a+h+2}}{\sqrt{a+2}+\sqrt{a+h+2}}\right]$$

$$=\lim_{h\to 0}\frac{(a+2)-(a+h+2)}{h\sqrt{a+h+2}\sqrt{a+2}(\sqrt{a+2}+\sqrt{a+h+2})}$$

$$=\lim_{h\to 0}\frac{-h}{h\sqrt{a+h+2}\sqrt{a+2}(\sqrt{a+2}+\sqrt{a+h+2})}$$

$$=\lim_{h\to 0}\frac{-1}{\sqrt{a+h+2}\sqrt{a+2}(\sqrt{a+2}+\sqrt{a+h+2})}$$

$$=\frac{-1}{(\sqrt{a+2})^2(2\sqrt{a+2})}=-\frac{1}{2(a+2)^{3/2}}$$

18.

$$f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}=\lim_{h\to 0}\frac{\sqrt{3(a+h)+1}-\sqrt{3a+1}}{h}$$

$$=\lim_{h\to 0}\frac{(\sqrt{3a+3h+1}-\sqrt{3a+1})(\sqrt{3a+3h+1}+\sqrt{3a+1})}{h(\sqrt{3a+3h+1}+\sqrt{3a+1})}$$

$$=\lim_{h\to 0}\frac{(3a+3h+1)-(3a+1)}{h(\sqrt{3a+3h+1}+\sqrt{3a+1})}=\lim_{h\to 0}\frac{3h}{h(\sqrt{3a+3h+1}+\sqrt{3a+1})}$$

$$=\lim_{h\to 0}\frac{3}{\sqrt{3a+3h+1}+\sqrt{3a+1}}=\frac{3}{2\sqrt{3a+1}}$$

19. By Definition 2, $\lim_{h\to 0}\frac{(1+h)^{10}-1}{h}=f'(1)$, where $f(x)=x^{10}$ and $a=1$. *Or:* By Definition 2, $\lim_{h\to 0}\frac{(1+h)^{10}-1}{h}=f'(0)$, where $f(x)=(1+x)^{10}$ and $a=0$.

20. By Definition 2,

$\lim_{h\to 0} \frac{\sqrt[4]{16+h}-2}{h} = f'(16)$, where $f(x)=\sqrt[4]{x}$ and $a=16$. *Or* : By Definition 2, $\lim_{h\to 0} \frac{\sqrt[4]{16+h}-2}{h} = f'(0)$, where $f(x)=\sqrt[4]{16+x}$ and $a=0$.

21. By Equation 3, $\lim_{x\to 5} \frac{2^x-32}{x-5} = f'(5)$, where $f(x)=2^x$ and $a=5$.

22. By Equation 3, $\lim_{x\to \pi/4} \frac{\tan x-1}{x-\pi/4} = f'(\pi/4)$, where $f(x)=\tan x$ and $a=\pi/4$.

23. By Definition 2, $\lim_{h\to 0} \frac{\cos(\pi+h)+1}{h} = f'(\pi)$, where $f(x)=\cos x$ and $a=\pi$. *Or* : By Definition 2, $\lim_{h\to 0} \frac{\cos(\pi+h)+1}{h} = f'(0)$, where $f(x)=\cos(\pi+x)$ and $a=0$.

24. By Equation 3, $\lim_{t\to 1} \frac{t^4+t-2}{t-1} = f'(1)$, where $f(t)=t^4+t$ and $a=1$.

25.

$$v(2) = f'(2)=\lim_{h\to 0} \frac{f(2+h)-f(2)}{h} = \lim_{h\to 0} \frac{[(2+h)^2-6(2+h)-5]-[2^2-6(2)-5]}{h}$$

$$= \lim_{h\to 0} \frac{(4+4h+h^2-12-6h-5)-(-13)}{h} = \lim_{h\to 0} \frac{h^2-2h}{h} = \lim_{h\to 0}(h-2) = -2\, m/s$$

26.

$$v(2) = f'(2)=\lim_{h\to 0} \frac{f(2+h)-f(2)}{h}$$

$$= \lim_{h\to 0} \frac{[2(2+h)^3-(2+h)+1]-[2(2)^3-2+1]}{h}$$

$$= \lim_{h\to 0} \frac{(2h^3+12h^2+24h+16-2-h+1)-15}{h}$$

$$=\lim_{h\to 0}\frac{2h^3+12h^2+23h}{h}=\lim_{h\to 0}(2h^2+12h+23)=23m/s$$

27. **(a)** $f'(x)$ is the rate of change of the production cost with respect to the number of ounces of gold produced. Its units are dollars per ounce.
(b) After 800 ounces of gold have been produced, the rate at which the production cost is increasing is \$17/ ounce. So the cost of producing the 800 th (or 801 st) ounce is about \$17 .
(c) In the short term, the values of $f'(x)$ will decrease because more efficient use is made of start–up costs as x increases. But eventually $f'(x)$ might increase due to large–scale operations.

28. **(a)** $f'(5)$ is the rate of growth of the bacteria population when $t=5$ hours. Its units are bacteria per hour.
(b) With unlimited space and nutrients, f' should increase as t increases; so $f'(5)<f'(10)$. If the supply of nutrients is limited, the growth rate slows down at some point in time, and the opposite may be true.

29. **(a)** $f'(v)$ is the rate at which the fuel consumption is changing with respect to the speed. Its units are $(gal / h) / (mi / h)$.
(b) The fuel consumption is decreasing by $0.05(gal / \text{h}) / (mi / h)$ as the car's speed reaches 20 mi / h . So if you increase your speed to 21 mi / h , you could expect to decrease your fuel consumption by about $0.05(gal / \text{h}) / (mi / h)$.

30. **(a)** $f'(8)$ is the rate of change of the quantity of coffee sold with respect to the price per pound when the price is \$8 per pound. The units for $f'(8)$ are pounds /(dollars / pound) .
(b) $f'(8)$ is negative since the quantity of coffee sold will decrease as the price charged for it increases. People are generally less willing to buy a product when its price increases.

31. $T'(10)$ is the rate at which the temperature is changing at 10:00 A.M. To estimate the value of $T'(10)$, we will average the difference quotients obtained using the times $t=8$ and $t=12$. Let $A=\dfrac{T(8)-T(10)}{8-10}=\dfrac{72-81}{-2}=4.5$ and $B=\dfrac{T(12)-T(10)}{12-10}=\dfrac{88-81}{2}=3.5$. Then

$$T'(10)=\lim_{t\to 10}\frac{T(t)-T(10)}{t-10}\approx\frac{A+B}{2}=\frac{4.5+3.5}{2}=4^{\circ}F/h .$$

32. **For 1910:** We will average the difference quotients obtained using the years 1900 and 1920.

Let $A=\dfrac{E(1900)-E(1910)}{1900-1910}=\dfrac{48.3-51.1}{-10}=0.28$ and
$B=\dfrac{E(1920)-E(1910)}{1920-1910}=\dfrac{55.2-51.1}{10}=0.41$. Then
$E'(1910)=\lim\limits_{t\to 1910}\dfrac{E(t)-E(1910)}{t-1910}\approx\dfrac{A+B}{2}=0.345$. This means that life expectancy at birth was increasing at about 0.345 year / year in 1910.
For 1950: Using data for 1940 and 1960 in a similar fashion, we obtain
$E'(1950)\approx[0.31+0.10]/2=0.205$. So life expectancy at birth was increasing at about 0.205 year / year in 1950.

33. **(a)** $S'(T)$ is the rate at which the oxygen solubility changes with respect to the water temperature. Its units are (mg / L)/° C .
(b) For $T=16°$ C , it appears that the tangent line to the curve goes through the points (0,14) and (32,6) . So $S'(16)\approx\dfrac{6-14}{32-0}=-\dfrac{8}{32}=-0.25$(mg / L)/° C . This means that as the temperature increases past 16° C , the oxygen solubility is decreasing at a rate of 0.25(mg / L)/° C .

34. **(a)** $S'(T)$ is the rate of change of the maximum sustainable speed of Coho salmon with respect to the temperature. Its units are (cm / s)/° C .
(b) For $T=15^{\circ}$ C, it appears the tangent line to the curve goes through the points (10,25) and (20,32) . So $S'(15)\approx\dfrac{32-25}{20-10}=0.7$(cm / s)/° C . This tells us that at $T=15^{\circ}$ C, the maximum sustainable speed of Coho salmon is changing at a rate of 0.7 (cm / s)/° C . In a similar fashion for $T=25^{\circ}$ C, we can use the points (20,35) and (25,25) to obtain $S'(25)\approx\dfrac{25-35}{25-20}=-2$ (cm / s) /° C . As it gets warmer than 20° C, the maximum sustainable speed decreases rapidly.

35. Since $f(x)=x\sin(1/x)$ when $x\neq 0$ and $f(0)=0$, we have
$f'(0)=\lim\limits_{h\to 0}\dfrac{f(0+h)-f(0)}{h}=\lim\limits_{h\to 0}\dfrac{h\sin(1/h)-0}{h}=\lim\limits_{h\to 0}(sin(1/h))$. This limit does not exist since $sin(1/h)$ takes the values -1 and 1 on any interval containing 0 . (Compare with Example 4 in Section 2.2.)

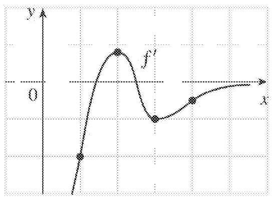
36. Since $f(x)=x^2 sin(1/x)$ when $x\neq 0$ and $f(0)=0$, we have
$f'(0)=\lim\limits_{h\to 0}\dfrac{f(0+h)-f(0)}{h}=\lim\limits_{h\to 0}\dfrac{h^2\sin(1/h)-0}{h}=\lim\limits_{h\to 0}(h\sin(1/h))$. Since $-1\leq sin\dfrac{1}{h}\leq 1$, we have

$-|h| \le |h| \sin\frac{1}{h} \le |h| \Rightarrow -|h| \le h\sin\frac{1}{h} \le |h|$. Because $\lim_{h\to 0}(-|h|)=0$ and $\lim_{h\to 0}|h|=0$, we know that $\lim_{h\to 0}(h \sin\frac{1}{h})=0$ by the Squeeze Theorem. Thus, $f'(0)=0$.

1. *Note:* Your answers may vary depending on your estimates. By estimating the slopes of tangent lines on the graph of f , it appears that

(a) $f'(1)\approx-2$ (b) $f'(2)\approx 0.8$

(c) $f'(3)\approx-1$ (d) $f'(4)\approx-0.5$

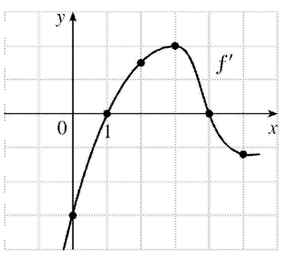


2. *Note:* Your answers may vary depending on your estimates. By estimating the slopes of tangent lines on the graph of f , it appears that

(a) $f'(0)\approx-3$ (b) $f'(1)\approx 0$

(c) $f'(2)\approx 1.5$ (d) $f'(3)\approx 2$

(e) $f'(4)\approx 0$ (f) $f'(5)\approx-1.2$



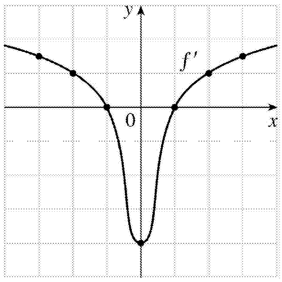
3. It appears that f is an odd function, so f' will be an even function ??? that f is, $f'(-a)=f'(a)$

(a) $f'(-3)\approx 1.5$ (b) $f'(-2)\approx 1$

(c) $f'(-1)\approx-0$ (d) $f'(0)\approx-4$

(e) $f'(1)\approx-0$ (f) $f'(2)\approx-1$

(g) $f'(3)\approx 1.5$

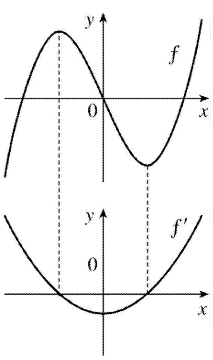


4. **(a)** $(a)'$ = II, since from left to right, the slopes of the tangents to graph (a) start out negative, become 0 , then positive, then 0 , then negative again. The actual function values in graph II follow the same pattern.

(b) $(b)'$ = IV, since from left to right, the slopes of the tangents to graph (b) start out at a fixed positive quantity, then suddenly become negative, then positive again. The discontinuities in graph IV indicate sudden changes in the slopes of the tangents.

(c) $(c)'$ = I, since the slopes of the tangents to graph (c) are negative for $x<0$ and positive for $x>0$, as are the function values of graph I.

(d) $(d)'$ = III, since from left to right, the slopes of the tangents to graph (d) are positive, then 0 , then negative, then 0 , then positive, then 0 , then negative again, and the function values in graph III follow the same pattern.



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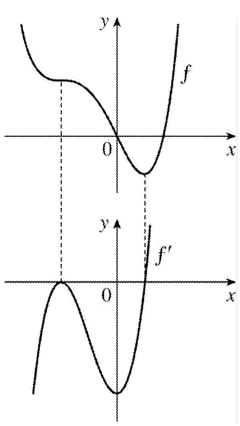

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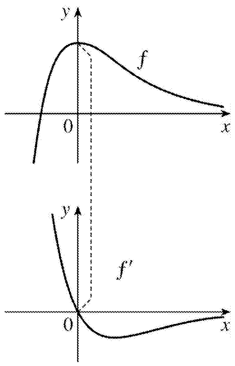

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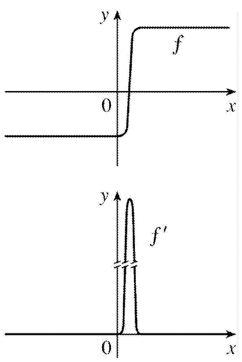

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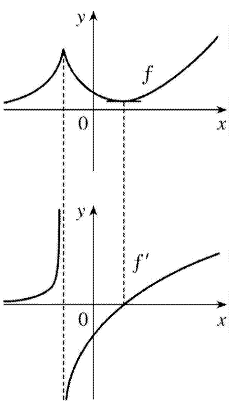
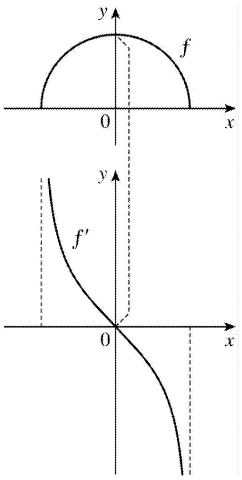
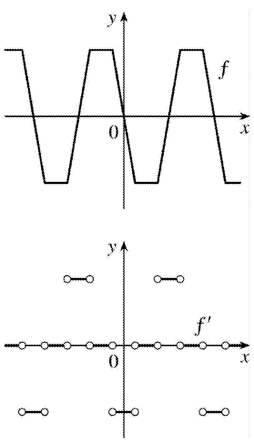
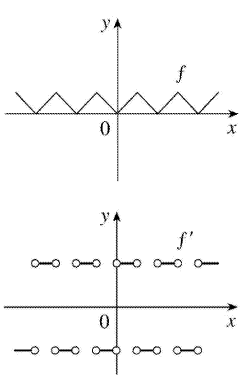


9.

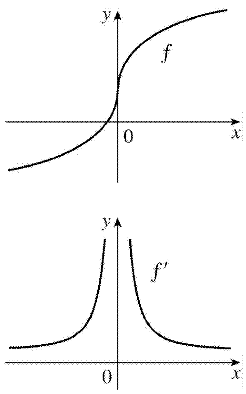
10.



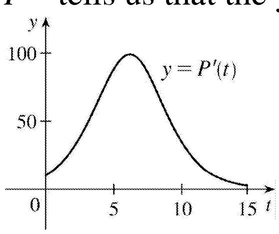
11. 

12. 

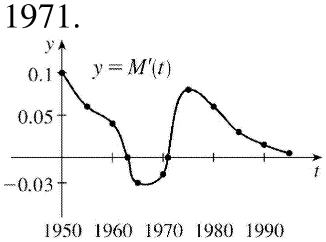
13.



14. The slopes of the tangent lines on the graph of $y=P(t)$ are always positive, so the y –values of $y=P'(t)$ are always positive. These values start out relatively small and keep increasing, reaching a maximum at about $t=6$. Then the y –values of $y=P'(t)$ decrease and get close to zero. The graph of P' tells us that the yeast culture grows most rapidly after 6 hours and then the growth rate declines.

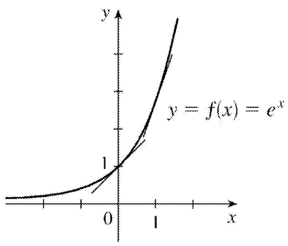


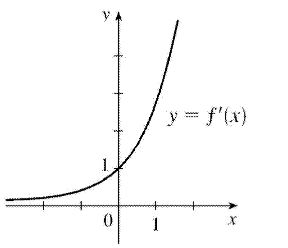
15. It appears that there are horizontal tangents on the graph of M for $t=1963$ and $t=1971$. Thus, there are zeros for those values of t on the graph of M' . The derivative is negative for the years 1963 to 1971.



16. See Figure 1 in Section 3.4.

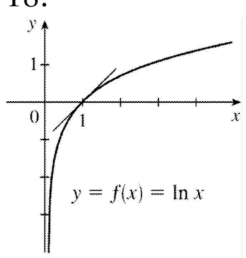
17.

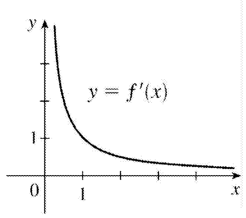




The slope at 0 appears to be 1 and the slope at 1 appears to be 2.7 . As x decreases, the slope gets closer to 0 . Since the graphs are so similar, we might guess that $f'(x)=e^x$.

18.





As x increases toward 1 , $f'(x)$ decreases from very large numbers to 1 . As x becomes large, $f'(x)$ gets closer to 0 . As a guess, $f'(x)=1/x^2$ or $f'(x)=1/x$ make sense.

19. **(a)** By zooming in, we estimate that $f'(0)=0$, $f'\left(\frac{1}{2}\right)=1$, $f'(1)=2$, and $f'(2)=4$.

(b) By symmetry, $f'(-x)=-f'(x)$. So $f'\left(-\frac{1}{2}\right)=-1$,

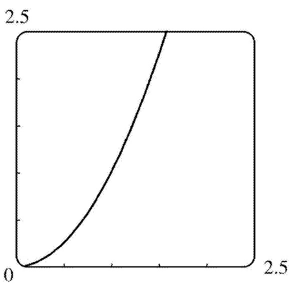
$f'(-1)=-2$, and $f'(-2)=-4$.

(c) It appears that $f'(x)$ is twice the value of x , so we guess that $f'(x)=2x$.

(d)

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{(x+h)^2-x^2}{h}$$

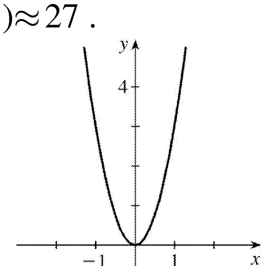
$$=\lim_{h\to 0}\frac{(x^2+2hx+h^2)-x^2}{h}=\lim_{h\to 0}\frac{2hx+h^2}{h}=\lim_{h\to 0}\frac{h(2x+h)}{h}=\lim_{h\to 0}(2x+h)=2x$$



20. **(a)** By zooming in, we estimate that $f'(0)=0$, $f'\left(\frac{1}{2}\right)\approx 0.75$, $f'(1)\approx 3$, $f'(2)\approx 12$, and $f'(3)\approx 27$.

(b) By symmetry, $f'(-x)=f'(x)$. So $f'\left(-\frac{1}{2}\right)\approx 0.75$, $f'(-1)\approx 3$, $f'(-2)\approx 12$, and $f'(-3)\approx 27$.

(c)



(d) Since $f'(0)=0$, it appears that f' may have the form $f'(x)=ax^2$. Using $f'(1)=3$, we have $a=3$, so $f'(x)=3x^2$.

(e) $$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{(x+h)^3-x^3}{h}=\lim_{h\to 0}\frac{(x^3+3x^2h+3xh^2+h^3)-x^3}{h}$$

$$=\lim_{h\to 0}\frac{3x^2h+3xh^2+h^3}{h}=\lim_{h\to 0}\frac{h(3x^2+3xh+h^2)}{h}=\lim_{h\to 0}(3x^2+3xh+h^2)=3x^2$$

21. $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{37-37}{h}=\lim_{h\to 0}\frac{0}{h}=\lim_{h\to 0}0=0$

Domain of f= domain of $f'=R$.

22.

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{[12+7(x+h)]-(12+7x)}{h}$$
$$=\lim_{h\to 0}\frac{12+7x+7h-12-7x}{h}=\lim_{h\to 0}\frac{7h}{h}=\lim_{h\to 0}7=7$$

Domain of f= domain of $f'=R$.

23.

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{[1-3(x+h)^2]-(1-3x^2)}{h}$$
$$=\lim_{h\to 0}\frac{[1-3(x^2+2xh+h^2)]-(1-3x^2)}{h}=\lim_{h\to 0}\frac{1-3x^2-6xh-3h^2-1+3x^2}{h}$$
$$=\lim_{h\to 0}\frac{-6xh-3h^2}{h}=\lim_{h\to 0}\frac{h(-6x-3h)}{h}=\lim_{h\to 0}(-6x-3h)=-6x$$

Domain of f= domain of $f'=R$.

24.

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{[5(x+h)^2+3(x+h)-2]-(5x^2+3x-2)}{h}$$
$$=\lim_{h\to 0}\frac{5x^2+10xh+5h^2+3x+3h-2-5x^2-3x+2}{h}=\lim_{h\to 0}\frac{10xh+5h^2+3h}{h}$$
$$=\lim_{h\to 0}\frac{h(10x+5h+3)}{h}=\lim_{h\to 0}(10x+5h+3)=10x+3$$

Domain of f= domain of $f'=R$.

25.

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{[(x+h)^3-3(x+h)+5]-(x^3-3x+5)}{h}$$

$$=\lim_{h\to 0}\frac{(x^3+3x^2h+3xh^2+h^3-3x-3h+5)-(x^3-3x+5)}{h}$$

$$=\lim_{h\to 0}\frac{3x^2h+3xh^2+h^3-3h}{h}=\lim_{h\to 0}\frac{h(3x^2+3xh+h^2-3)}{h}$$

$$=\lim_{h\to 0}(3x^2+3xh+h^2-3)=3x^2-3$$

Domain of f= domain of $f'=R$.

26.

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{(x+h+\sqrt{x+h})-(x+\sqrt{x})}{h}$$

$$=\lim_{h\to 0}\left(\frac{h}{h}+\frac{\sqrt{x+h}-\sqrt{x}}{h}\cdot\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}\right)=\lim_{h\to 0}\left[1+\frac{(x+h)-x}{h(\sqrt{x+h}+\sqrt{x})}\right]$$

$$=\lim_{h\to 0}\left(1+\frac{1}{\sqrt{x+h}+\sqrt{x}}\right)=1+\frac{1}{\sqrt{x}+\sqrt{x}}=1+\frac{1}{2\sqrt{x}}$$

Domain of $f=[0,\infty)$, domain of $f'=(0,\infty)$.

27.

$$g'(x)=\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}=\lim_{h\to 0}\frac{\sqrt{1+2(x+h)}-\sqrt{1+2x}}{h}\left[\frac{\sqrt{1+2(x+h)}+\sqrt{1+2x}}{\sqrt{1+2(x+h)}+\sqrt{1+2x}}\right]$$

$$=\lim_{h\to 0}\frac{(1+2x+2h)-(1+2x)}{h[\sqrt{1+2(x+h)}+\sqrt{1+2x}]}=\lim_{h\to 0}\frac{2}{\sqrt{1+2x+2h}+\sqrt{1+2x}}=\frac{2}{2\sqrt{1+2x}}=\frac{1}{\sqrt{1+2x}}$$

Domain of $g=\left[-\frac{1}{2},\infty\right)$, domain of $g'=\left(-\frac{1}{2},\infty\right)$.

28.

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\frac{3+(x+h)}{1-3(x+h)}-\frac{3+x}{1-3x}}{h}$$

$$=\lim_{h\to 0}\frac{(3+x+h)(1-3x)-(3+x)(1-3x-3h)}{h(1-3x-3h)(1-3x)}$$

$$=\lim_{h\to 0}\frac{(3-9x+x-3x^2+h-3hx)-(3-9x-9h+x-3x^2-3hx)}{h(1-3x-3h)(1-3x)}$$

$$=\lim_{h\to 0}\frac{10h}{h(1-3x-3h)(1-3x)}=\lim_{h\to 0}\frac{10}{(1-3x-3h)(1-3x)}=\frac{10}{(1-3x)^2}$$

Domain of f = domain of $f'=\left(-\infty,\frac{1}{3}\right)\cup\left(\frac{1}{3},\infty\right)$.

29.

$$G'(t)=\lim_{h\to 0}\frac{G(t+h)-G(t)}{h}=\lim_{h\to 0}\frac{\frac{4(t+h)}{(t+h)+1}-\frac{4t}{t+1}}{h}=\lim_{h\to 0}\frac{\frac{4(t+h)(t+1)-4t(t+h+1)}{(t+h+1)(t+1)}}{h}$$

$$=\lim_{h\to 0}\frac{(4t^2+4ht+4t+4h)-(4t^2+4ht+4t)}{h(t+h+1)(t+1)}$$

$$=\lim_{h\to 0}\frac{4h}{h(t+h+1)(t+1)}=\lim_{h\to 0}\frac{4}{(t+h+1)(t+1)}=\frac{4}{(t+1)^2}$$

Domain of G = domain of $G'=(-\infty,-1)\cup(-1,\infty)$.

30.

$$g'(x)=\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}=\lim_{h\to 0}\frac{\frac{1}{(x+h)^2}-\frac{1}{x^2}}{h}=\lim_{h\to 0}\frac{x^2-(x+h)^2}{h(x+h)^2x^2}$$

$$=\lim_{h\to 0}\frac{x^2-(x^2+2xh+h^2)}{h}=\lim_{h\to 0}\frac{-2xh-h^2}{h(x+h)^2x^2}=\lim_{h\to 0}\frac{-2x-h}{(x+h)^2x^2}=\frac{-2x}{x^4}=-2x^{-3}$$

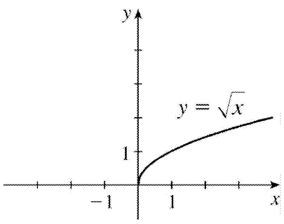
Domain of g = domain of $g'=\{x\mid x\neq 0\}$.

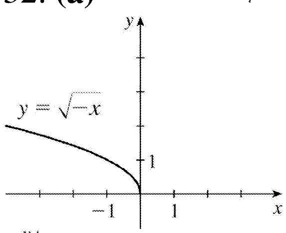
31.

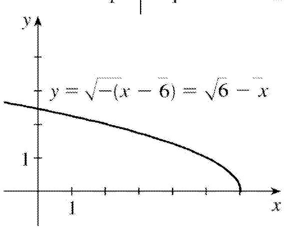
$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{(x+h)^4-x^4}{h}=\lim_{h\to 0}\frac{(x^4+4x^3h+6x^2h^2+4xh^3+h^4)-x^4}{h}$$

$$=\lim_{h\to 0}\frac{4x^3h+6x^2h^2+4xh^3+h^4}{h}=\lim_{h\to 0}(4x^3+6x^2h+4xh^2+h^3)=4x^3$$

Domain of f= domain of $f'=R$.

32. **(a)** 





(b) Note that the third graph in part (a) has small negative values for its slope, f' ; but as $x\to 6^-$, $f'\to -\infty$.
See the graph in part (d).
(c)

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\sqrt{6-(x+h)}-\sqrt{6-x}}{h}\left[\frac{\sqrt{6-(x+h)}+\sqrt{6-x}}{\sqrt{6-(x+h)}+\sqrt{6-x}}\right]$$

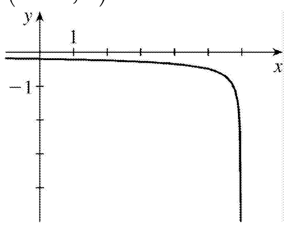
$$=\lim_{h\to 0}\frac{[6-(x+h)]-(6-x)}{h[\sqrt{6-(x+h)}+\sqrt{6-x}]}=\lim_{h\to 0}\frac{-h}{h(\sqrt{6-x-h}+\sqrt{6-x})}$$

$$=\lim_{h\to 0}\frac{-1}{\sqrt{6-x-h}+\sqrt{6-x}}=\frac{-1}{2\sqrt{6-x}}$$

Domain of $f=(-\infty,6]$, domain of

$f' = (-\infty, 6)$.

(d)



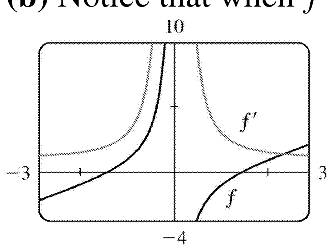
33. **(a)**

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\left[x+h-\left(\frac{2}{x+h}\right)\right]-\left[x-\left(\frac{2}{x}\right)\right]}{h}$$

$$= \lim_{h\to 0} \left[\frac{h-\frac{2}{(x+h)}+\frac{2}{x}}{h} \right] = \lim_{h\to 0} \left[1+\frac{-2x+2(x+h)}{hx(x+h)} \right] = \lim_{h\to 0} \left[1+\frac{2h}{hx(x+h)} \right]$$

$$= \lim_{h\to 0} \left[1+\frac{2}{x(x+h)} \right] = 1+\frac{2}{x^2}$$

(b) Notice that when f has steep tangent lines, $f'(x)$ is very large. When f is flatter, $f'(x)$ is smaller.

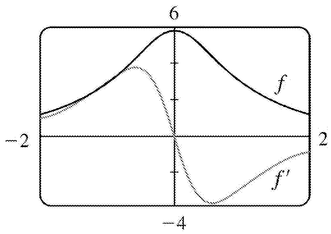


34. **(a)**

$$f'(t) = \lim_{h\to 0} \frac{f(t+h)-f(t)}{h} = \lim_{h\to 0} \frac{\frac{6}{1+(t+h)^2}-\frac{6}{1+t^2}}{h} = \lim_{h\to 0} \frac{6+6t^2-6-6(t+h)^2}{h[1+(t+h)^2](1+t^2)}$$

$$= \lim_{h\to 0} \frac{-12th-6h^2}{h[1+(t+h)^2](1+t^2)} = \lim_{h\to 0} \frac{-12t-6h}{[1+(t+h)^2](1+t^2)} = \frac{-12t}{(1+t^2)^2}$$

(b) Notice that f has a horizontal tangent when $t=0$. This corresponds to $f'(0)=0$. f' is positive when f is increasing and negative when f is decreasing.



35. **(a)** $U'(t)$ is the rate at which the unemployment rate is changing with respect to time. Its units are percent per year.

(b) To find $U'(t)$, we use $\lim_{h\to 0}\frac{U(t+h)-U(t)}{h}\approx\frac{U(t+h)-U(t)}{h}$ for small values of h.

For 1991: $U'(1991)=\frac{U(1992)-U(1991)}{1992-1991}=\frac{7.5-6.8}{1}=0.70$

For 1992: We estimate $U'(1992)$ by using $h=-1$ and $h=1$, and then average the two results to obtain a final estimate.

$h=-1\Rightarrow U'(1992)\approx\frac{U(1991)-U(1992)}{1991-1992}=\frac{6.8-7.5}{-1}=0.70$;

$h=1\Rightarrow U'(1992)\approx\frac{U(1993)-U(1992)}{1993-1992}=\frac{6.9-7.5}{1}=-0.60$.

So we estimate that $U'(1992)\approx\frac{1}{2}[0.70+(-0.60)]=0.05$.

t	1991	1992	1993	1994	1995	1996	1997	1998	1999	2000
$U'(t)$	0.70	0.05	–0.70	–0.65	–0.35	–0.35	–0.45	–0.35	–0.25	–0.20

36. **(a)** $P'(t)$ is the rate at which the percentage of Americans under the age of 18 is changing with respect to time. Its units are percent per year (%/yr).

(b) To find $P'(t)$, we use $\lim_{h\to 0}\frac{P(t+h)-P(t)}{h}\approx\frac{P(t+h)-P(t)}{h}$ for small values of h.

For 1950: $P'(1950)=\frac{P(1960)-P(1950)}{1960-1950}=\frac{35.7-31.1}{10}=0.46$

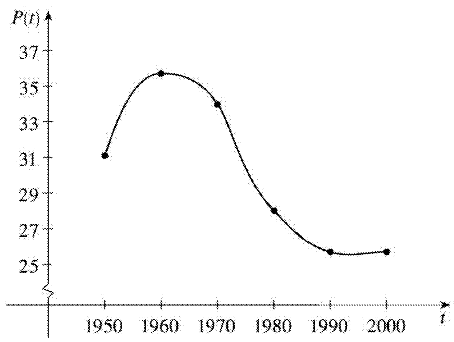
For 1960: We estimate $P'(1960)$ by using $h=-10$ and $h=10$, and then average the two results to obtain a final estimate.

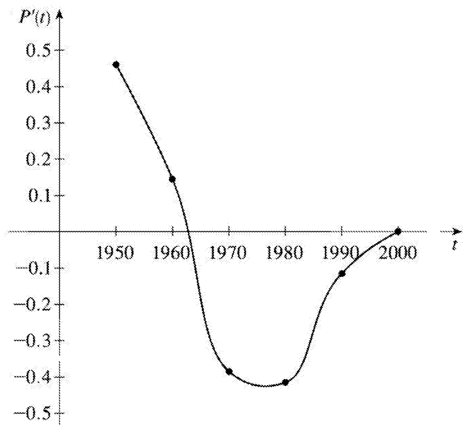
$h=-10\Rightarrow P'(1960)\approx\frac{P(1950)-P(1960)}{1950-1960}=\frac{31.1-35.7}{-10}=0.46$

$h=10 \Rightarrow P'(1960) \approx \dfrac{P(1970)-P(1960)}{1970-1960} = \dfrac{34.0-35.7}{10} = -0.17$

So we estimate that $P'(1960) \approx \frac{1}{2}[0.46+(-0.17)]=0.145$.

t	1950	1960	1970	1980	1990	2000
$P'(t)$	0.460	0.145	–0.385	–0.415	–0.115	0.000





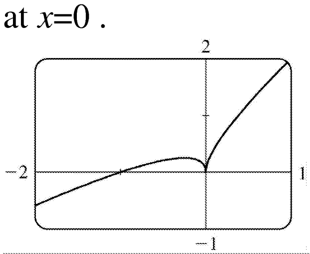
(c)

(d) We could get more accurate values for $P'(t)$ by obtaining data for the mid–decade years 1955, 1965, 1975, 1985, and 1995.

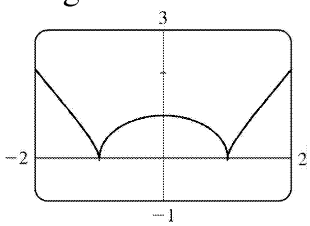
37. f is not differentiable at $x=-1$ or at $x=11$ because the graph has vertical tangents at those points; at $x=4$, because there is a discontinuity there; and at $x=8$, because the graph has a corner there.

38. **(a)** g is discontinuous at $x=-2$ (a removable discontinuity), at $x=0$ (g is not defined there), and at $x=5$ (a jump discontinuity).
(b) g is not differentiable at the points mentioned in part (a) (by Theorem 4), nor is it differentiable at $x=-1$ (corner), $x=2$ (vertical tangent), or $x=4$ (vertical tangent).

39. As we zoom in toward $(-1,0)$, the curve appears more and more like a straight line, so $f(x)=x+\sqrt{|x|}$ is differentiable at $x=-1$. But no matter how much we zoom in toward the origin, the curve doesn't straighten out—we can't eliminate the sharp point (a cusp). So f is not differentiable at $x=0$.



40. As we zoom in toward (0,1) , the curve appears more and more like a straight line, so f is differentiable at $x=0$. But no matter how much we zoom in toward (1,0) or (–1,0) , the curve doesn't straighten out—we can't eliminate the sharp point (a cusp). So f is not differentiable at $x=\pm 1$.



41. **(a)** Note that we have factored $x-a$ as the difference of two cubes in the third step.

$$f'(a)=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}=\lim_{x\to a}\frac{x^{1/3}-a^{1/3}}{x-a}=\lim_{x\to a}\frac{x^{1/3}-a^{1/3}}{(x^{1/3}-a^{1/3})(x^{2/3}+x^{1/3}a^{1/3}+a^{2/3})}$$

$$=\lim_{x\to a}\frac{1}{x^{2/3}+x^{1/3}a^{1/3}+a^{2/3}}=\frac{1}{3a^{2/3}}\text{ or }\frac{1}{3}a^{-2/3}$$

(b) $f'(0)=\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0}\frac{\sqrt[3]{h}-0}{h}=\lim_{h\to 0}\frac{1}{h^{2/3}}$. This function increases without bound, so the limit does not exist, and therefore $f'(0)$ does not exist.

(c) $\lim_{x\to 0}\left|f'(x)\right|=\lim_{x\to 0}\frac{1}{3x^{2/3}}=\infty$ and f is continuous at $x=0$ (root function), so f has a vertical tangent at $x=0$.

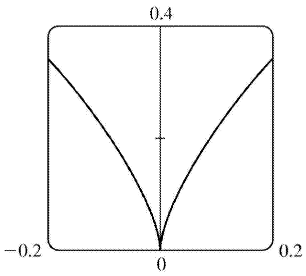
42. **(a)** $g'(0)=\lim_{x\to 0}\frac{g(x)-g(0)}{x-0}=\lim_{x\to 0}\frac{x^{2/3}-0}{x}=\lim_{x\to 0}\frac{1}{x^{1/3}}$, which does not exist.

(b)

$$g'(a)=\lim_{x\to a}\frac{g(x)-g(a)}{x-a}=\lim_{x\to a}\frac{x^{2/3}-a^{2/3}}{x-a}=\lim_{x\to a}\frac{(x^{1/3}-a^{1/3})(x^{1/3}+a^{1/3})}{(x^{1/3}-a^{1/3})(x^{2/3}+x^{1/3}a^{1/3}+a^{2/3})}$$

$$=\lim_{x\to a}\frac{x^{1/3}+a^{1/3}}{x^{2/3}+x^{1/3}a^{1/3}+a^{2/3}}=\frac{2a^{1/3}}{3a^{2/3}}=\frac{2}{3a^{1/3}}\text{ or }\frac{2}{3}a^{-1/3}$$

(c) $g(x)=x^{2/3}$ is continuous at $x=0$ and $\lim_{x\to 0}\left|g'(x)\right|=\lim_{x\to 0}\frac{2}{3|x|^{1/3}}=\infty$. This shows that g has a vertical tangent line at $x=0$.

(d) 

43. $f(x)=|x-6| = \begin{cases} -(x-6) & \text{if } x<6 \\ x-6 & \text{if } x\geq 6 \end{cases} = \begin{cases} 6-x & \text{if } x<6 \\ x-6 & \text{if } x\geq 6 \end{cases}$

$$\lim_{x\to 6^+} \frac{f(x)-f(6)}{x-6} = \lim_{x\to 6^+} \frac{|x-6|-0}{x-6} = \lim_{x\to 6^+} \frac{x-6}{x-6} = \lim_{x\to 6^+} 1=1 .$$

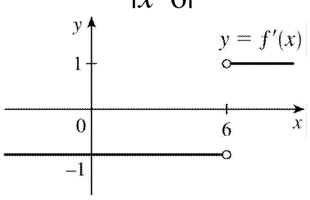
But

$$\lim_{x\to 6^-} \frac{f(x)-f(6)}{x-6} = \lim_{x\to 6^-} \frac{|x-6|-0}{x-6} = \lim_{x\to 6^-} \frac{6-x}{x-6}$$

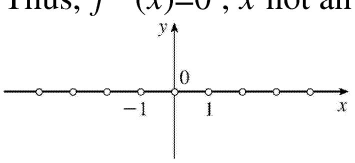
$$= \lim_{x\to 6^-} (-1)=-1$$

So, $f'(6)$ does not exist. However, $f'(x)=\begin{cases} -1 & \text{if } x<6 \\ 1 & \text{if } x\geq 6 \end{cases}$ Another way of writing the answer is

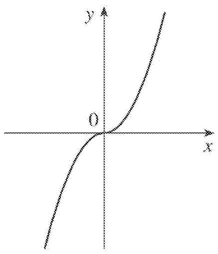
$f'(x) = \dfrac{x-6}{|x-6|}$.



44. $f(x)=[[x]]$ is not continuous at any integer n , so f is not differentiable at n by the contrapositive of Theorem 4. If a is not an integer, then f is constant on an open interval containing a , so $f'(a)=0$. Thus, $f'(x)=0$, x not an integer.



45. **(a)** $f(x)=x|x|= \begin{cases} x^2 & \text{if } x\geq 0 \\ -x^2 & \text{if } x<0 \end{cases}$



(b) Since $f(x)=x^2$ for $x\geq 0$, we have $f'(x)=2x$ for $x>0$. [See Exercise 2.9.19(d).] Similarly, since $f(x)=-x^2$ for $x<0$, we have $f'(x)=-2x$ for $x<0$. At $x=0$, we have

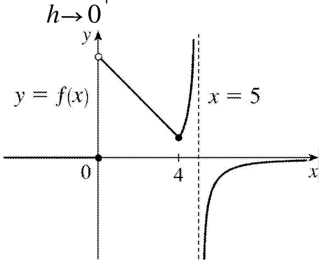
$$f'(0)=\lim_{x\to 0}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0}\frac{x|x|}{x}=\lim_{x\to 0}|x|=0.$$

So f is differentiable at 0 . Thus, f is differentiable for all x .

(c) From part (b), we have $f'(x)=\left\{\begin{array}{ll} 2x & \text{if } x\geq 0 \\ -2x & \text{if } x<0 \end{array}\right\}=2|x|$.

46. **(a)** $f'_-(4)=\lim_{h\to 0^-}\frac{f(4+h)-f(4)}{h}=\lim_{h\to 0^-}\frac{5-(4+h)-1}{h}=\lim_{h\to 0^-}\frac{-h}{h}=-1$ and

$$f'_+(4)=\lim_{h\to 0^+}\frac{f(4+h)-f(4)}{h}=\lim_{h\to 0^+}\frac{\frac{1}{5-(4+h)}-1}{h}=\lim_{h\to 0^+}\frac{1-(1-h)}{h(1-h)}=\lim_{h\to 0^+}\frac{1}{1-h}=1 .$$



(b)

(c) $f(x)=\left\{\begin{array}{ll} 0 & \text{if } x\leq 0 \\ 5-x & \text{if } 0<x<4 \\ 1/(5-x) & \text{if } x\geq 4 \end{array}\right.$ These expressions show that f is continuous on the intervals $(-\infty,0)$, $(0,4)$, $(4,5)$ and $(5,\infty)$. Since $\lim_{x\to 0^+}f(x)=\lim_{x\to 0^+}(5-x)=5\neq 0=\lim_{x\to 0^-}f(x)$, $\lim_{x\to 0}f(x)$ does not exist, so f is discontinuous (and therefore not differentiable) at 0 .

At 4 we have $\lim_{x\to 4^-}f(x)=\lim_{x\to 4^-}(5-x)=1$ and $\lim_{x\to 4^+}f(x)=\lim_{x\to 4^+}\frac{1}{5-x}=1$, so $\lim_{x\to 4}f(x)=1=f(4)$ and f is continuous at 4 . Since $f(5)$ is not defined, f is discontinuous at 5 .

(d) From (a), f is not differentiable at 4 since $f'_-(4)\neq f'_+(4)$, and from (c), f is not differentiable at 0 or 5 .

47. **(a)** If f is even, then

$$f'(-x) = \lim_{h\to 0}\frac{f(-x+h)-f(-x)}{h} = \lim_{h\to 0}\frac{f[-(x-h)]-f(-x)}{h} = \lim_{h\to 0}\frac{f(x-h)-f(x)}{h}$$

$$= -\lim_{h\to 0}\frac{f(x-h)-f(x)}{-h}\ [\text{let}\,\Delta x=-h] = -\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x} = -f'(x)$$

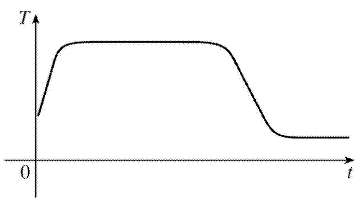
Therefore, f' is odd.

(b) If f is odd, then

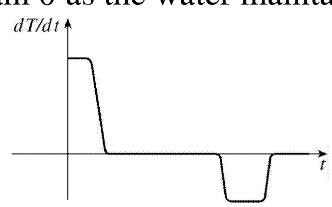
$$f'(-x) = \lim_{h\to 0}\frac{f(-x+h)-f(-x)}{h} = \lim_{h\to 0}\frac{f[-(x-h)]-f(-x)}{h} = \lim_{h\to 0}\frac{-f(x-h)+f(x)}{h}$$

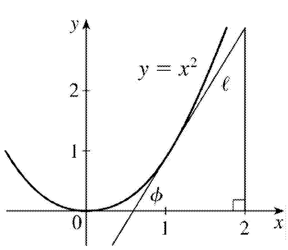
$$= \lim_{h\to 0}\frac{f(x-h)-f(x)}{-h}\ [\text{let}\,\Delta x=-h] = \lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x} = f'(x)$$

Therefore, f' is even.

48. **(a)**


(b) The initial temperature of the water is close to room temperature because of the water that was in the pipes. When the water from the hot water tank starts coming out, dT/dt is large and positive as T increases to the temperature of the water in the tank. In the next phase, $dT/dt=0$ as the water comes out at a constant, high temperature. After some time, dT/dt becomes small and negative as the contents of the hot water tank are exhausted. Finally, when the hot water has run out, dT/dt is once again 0 as the water maintains its (cold) temperature.

(c)


49.


In the right triangle in the diagram, let Δy be the side opposite angle ϕ and Δx the side adjacent angle ϕ. Then the slope of the tangent line ℓ is $m=\Delta y/\Delta x=\tan\phi$. Note that

$0<\phi<\frac{\pi}{2}$. We know (see Exercise 19) that the derivative of $f(x)=x^2$ is $f'(x)=2x$. So the slope of the tangent to the curve at the point (1,1) is 2 . Thus, ϕ is the angle between 0 and $\frac{\pi}{2}$ whose tangent is 2 ; that is, $\phi=\tan^{-1}2\approx 63^{\circ}$.

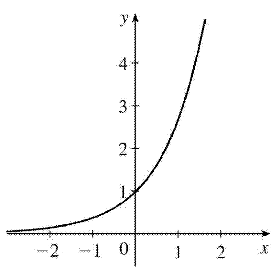
1. **(a)** e is the number such that $\lim_{h\to 0} \frac{e^h - 1}{h} = 1$.

(b)

x	$(2.7^x - 1)/x$
−0.001	0.9928
−0.0001	0.9932
0.001	0.9937
0.0001	0.9933

x	$(2.8^x - 1)/x$
−0.001	1.0291
−0.0001	1.0296
0.001	1.0301
0.0001	1.0297

From the tables (to two decimal places), $\lim_{h\to 0} \frac{2.7^h - 1}{h} = 0.99$ and $\lim_{h\to 0} \frac{2.8^h - 1}{h} = 1.03$. Since $0.99 < 1 < 1.03$, $2.7 < e < 2.8$.

2. **(a)** 

The function value at $x=0$ is 1 and the slope at $x=0$ is 1 .

(b) $f(x)=e^x$ is an exponential function and $g(x)=x^e$ is a power function. $\frac{d}{dx}(e^x)=e^x$ and $\frac{d}{dx}(x^e)=ex^{e-1}$

.

(c) $f(x)=e^x$ grows more rapidly than $g(x)=x^e$ when x is large.

3. $f(x)=186.5$ is a constant function, so its derivative is 0 , that is, $f'(x)=0$.

4. $f(x)=\sqrt{30}$ is a constant function, so its derivative is 0 , that is, $f'(x)=0$.

5. $f(x)=5x-1 \Rightarrow f'(x)=5-0=5$

6. $F(x)=-4x^{10} \Rightarrow F'(x)=-4(10x^{10-1})=-40x^9$

7. $f(x)=x^2+3x-4 \Rightarrow f'(x)=2x^{2-1}+3-0=2x+3$

8. $g(x)=5x^8-2x^5+6 \Rightarrow g'(x)=5(8x^{8-1})-2(5x^{5-1})+0=40x^7-10x^4$

9. $g(x)=5x^8-2x^5+6 \Rightarrow g'(x)=5(8x^{8-1})-2(5x^{5-1})+0=40x^7-10x^4$

10. $f(t)=\frac{1}{2}t^6-3t^4+t \Rightarrow f'(t)=\frac{1}{2}(6t^5)-3(4t^3)+1=3t^5-12t^3+1$

11. $y=x^{-2/5} \Rightarrow y'=-\frac{2}{5}x^{(-2/5)-1}=-\frac{2}{5}x^{-7/5}=-\frac{2}{5x^{7/5}}$

12. $y=5e^x+3 \Rightarrow y'=5(e^x)+0=5e^x$

13. $V(r)=\frac{4}{3}\pi r^3 \Rightarrow V'(r)=\frac{4}{3}\pi(3r^2)=4\pi r^2$

14. $R(t)=5t^{-3/5} \Rightarrow R'(t)=5\left[-\frac{3}{5}t^{(-3/5)-1}\right]=-3t^{-8/5}$

15. $Y(t)=6t^{-9} \Rightarrow Y'(t)=6(-9)t^{-10}=-54t^{-10}$

16. $R(x)=\frac{\sqrt{10}}{x^7}=\sqrt{10}\,x^{-7} \Rightarrow R'(x)=-7\sqrt{10}\,x^{-8}=-\frac{7\sqrt{10}}{x^8}$

17. $G(x)=\sqrt{x}-2e^x=x^{1/2}-2e^x \Rightarrow G'(x)=\frac{1}{2}x^{-1/2}-2e^x=\frac{1}{2\sqrt{x}}-2e^x$

18. $y=\sqrt[3]{x}=x^{1/3} \Rightarrow y'=\frac{1}{3}x^{-2/3}=\frac{1}{3x^{2/3}}$

19. $F(x)=\left(\frac{1}{2}x\right)^5=\left(\frac{1}{2}\right)^5x^5=\frac{1}{32}x^5 \Rightarrow F'(x)=\frac{1}{32}(5x^4)=\frac{5}{32}x^4$

20. $f(t)=\sqrt{t}-\frac{1}{\sqrt{t}}=t^{1/2}-t^{-1/2} \Rightarrow f'(t)=\frac{1}{2}t^{-1/2}-\left(-\frac{1}{2}t^{-3/2}\right)=\frac{1}{2\sqrt{t}}+\frac{1}{2t\sqrt{t}}$

21. $g(x)=x^2+\frac{1}{x^2}=x^2+x^{-2} \Rightarrow g'(x)=2x+(-2)x^{-3}=2x-\frac{2}{x^3}$

22. $y=\sqrt{x}\,(x-1)=x^{3/2}-x^{1/2}\Rightarrow y'=\frac{3}{2}x^{1/2}-\frac{1}{2}x^{-1/2}=\frac{1}{2}x^{-1/2}(3x-1)$ [factor out $\frac{1}{2}x^{-1/2}$] or $y'=\frac{3x-1}{2\sqrt{x}}$.

23. $y=\frac{x^2+4x+3}{\sqrt{x}}=x^{3/2}+4x^{1/2}+3x^{-1/2}\Rightarrow$
$y'=\frac{3}{2}x^{1/2}+4\left(\frac{1}{2}\right)x^{-1/2}+3\left(-\frac{1}{2}\right)x^{-3/2}=\frac{3}{2}\sqrt{x}+\frac{2}{\sqrt{x}}-\frac{3}{2x\sqrt{x}}$
$\left[\text{note that } x^{3/2}=x^{2/2}\cdot x^{1/2}=x\sqrt{x}\right]$

24. $y=\frac{x^2-2\sqrt{x}}{x}=x-2x^{-1/2}\Rightarrow\ y'=1-2\left(-\frac{1}{2}\right)x^{-3/2}=1+1/(x\sqrt{x})$

25. $y=4\pi^2\Rightarrow y'=0$ since $4\pi^2$ is a constant.

26. $g(u)=\sqrt{2}\,u+\sqrt{3u}=\sqrt{2}\,u+\sqrt{3}\,\sqrt{u}\Rightarrow\ g'(u)=\sqrt{2}\,(1)+\sqrt{3}\left(\frac{1}{2}u^{-1/2}\right)=\sqrt{2}+\sqrt{3}/(2\sqrt{u}\,)$

27. $y=ax^2+bx+c\Rightarrow\ y'=2ax+b$

28. $y=ae^v+\frac{b}{v}+\frac{c}{v^2}=ae^v+bv^{-1}+cv^{-2}\Rightarrow\ y'=ae^v-bv^{-2}-2cv^{-3}=ae^v-\frac{b}{v^2}-\frac{2c}{v^3}$

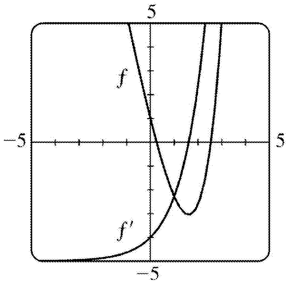
29. $v=t^2-\frac{1}{\sqrt[4]{t^3}}=t^2-t^{-3/4}\Rightarrow\ v'=2t-\left(-\frac{3}{4}\right)t^{-7/4}=2t+\frac{3}{4t^{7/4}}=2t+\frac{3}{4t\sqrt[4]{t^3}}$

30. $u=\sqrt[3]{t^2}+2\sqrt{t^3}=t^{2/3}+2t^{3/2}\Rightarrow\ u'=\frac{2}{3}t^{-1/3}+2\left(\frac{3}{2}\right)t^{1/2}=\frac{2}{3\sqrt[3]{t}}+3\sqrt{t}$

31. $z=\frac{A}{y^{10}}+Be^y=Ay^{-10}+Be^y\Rightarrow\ z'=-10Ay^{-11}+Be^y=-\frac{10A}{y^{11}}+Be^y$

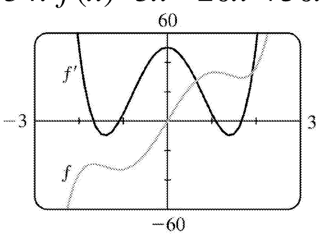
32. $y=e^{x+1}+1=e^xe^1+1=e\cdot e^x+1\Rightarrow\ y'=e\cdot e^x=e^{x+1}$

33. $f(x)=e^x-5x\Rightarrow f'(x)=e^x-5$.



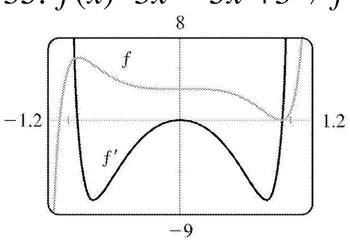
Notice that $f'(x)=0$ when f has a horizontal tangent, f' is positive when f is increasing, and f' is negative when f is decreasing.

34. $f(x)=3x^5-20x^3+50x \Rightarrow f'(x)=15x^4-60x^2+50$.



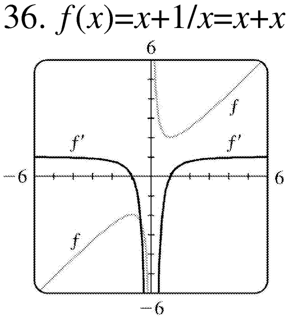
Notice that $f'(x)=0$ when f has a horizontal tangent and that f' is an even function while f is an odd function.

35. $f(x)=3x^{15}-5x^3+3 \Rightarrow f'(x)=45x^{14}-15x^2$.



Notice that $f'(x)=0$ when f has a horizontal tangent, f' is positive when f is increasing, and f' is negative when f is decreasing.

36. $f(x)=x+1/x=x+x^{-1} \Rightarrow f'(x)=1-x^{-2}=1-1/x^2$.

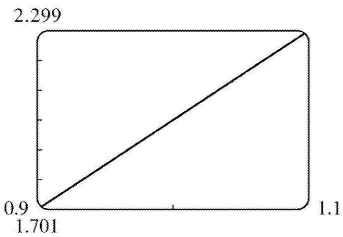


Notice that $f'(x)=0$ when f has a horizontal tangent, f' is positive when f is increasing, and f' is negative when f is decreasing.

37. To graphically estimate the value of $f'(1)$ for $f(x)=3x^2-x^3$, we'll graph f in the viewing rectangle [1–0.1,1+0.1] by [f(0.9),f(1.1)] , as shown in the figure. If we have sufficiently zoomed in on the graph of f, we should obtain a graph that looks like a diagonal line; if not, graph again with 1–0.01 and 1+0.01 , etc.

Estimated value: $f'(1)\approx\dfrac{2.299-1.701}{1.1-0.9}=\dfrac{0.589}{0.2}=2.99$.

Exact value: $f(x)=3x^2-x^3 \Rightarrow f'(x)=6x-3x^2$, so $f'(1)=6-3=3$.



38. See the previous exercise. Since f is a decreasing function, assign $Y_1(3.9)$ to Y □ and $Y_1(4.1)$ to Y_{min} .

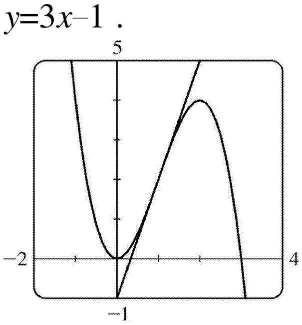
Estimated value: $f'(4)\approx\dfrac{0.49386-0.50637}{4.1-3.9}=\dfrac{-0.01251}{0.2}=-0.06255$.

Exact value: $f(x)=x^{-1/2}\Rightarrow f'(x)=-\dfrac{1}{2}x^{-3/2}$, so $f'(4)=-\dfrac{1}{2}(4^{-3/2})=-\dfrac{1}{2}\left(\dfrac{1}{8}\right)=-\dfrac{1}{16}=-0.0625$.

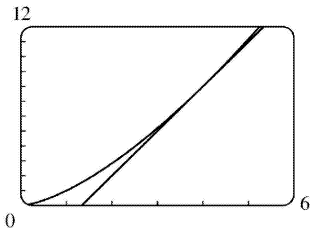
39. $y=x^4+2e^x\Rightarrow y'=4x^3+2e^x$. At (0, 2) , $y'=2$ and an equation of the tangent line is $y-2=2(x-0)$ or $y=2x+2$.

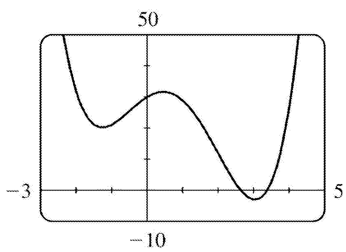
40. $y=(1+2x)^2=1+4x+4x^2\Rightarrow y'=4+8x$. At (1, 9), $y'=12$ and an equation of the tangent line is $y-9=12(x-1)$ or $y=12x-3$.

41. $y=3x^2-x^3\Rightarrow y'=6x-3x^2$. At (1, 2) , $y'=6-3=3$, so an equation of the tangent line is $y-2=3(x-1)$, or $y=3x-1$.

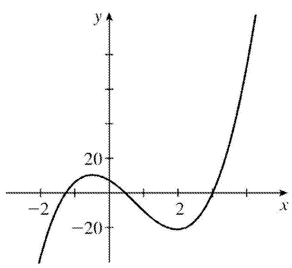


42. $y=x\sqrt{x}=x^{3/2}\Rightarrow y'=\frac{3}{2}x^{1/2}$. At (4, 8) , $y'=\frac{3}{2}(2)=3$, so an equation of the tangent line is $y-8=3(x-4)$, or $y=3x-4$.





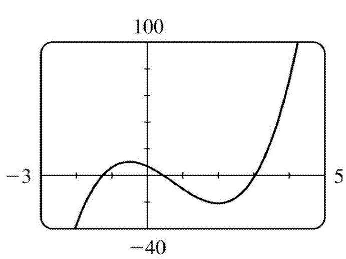
43. **(a)**



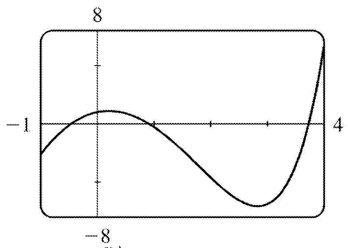
(b)

From the graph in part (a), it appears that f' is zero at $x_1\approx-1.25$, $x_2\approx0.5$, and $x_3\approx3$. The slopes are negative (so f' is negative) on $(-\infty, x_1)$ and (x_2, x_3). The slopes are positive (so f' is positive) on (x_1, x_2) and (x_3, ∞).

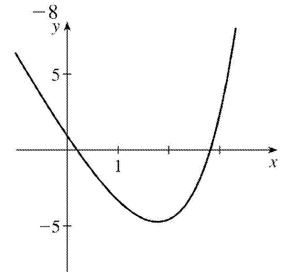
(c) $f(x)=x^4-3x^3-6x^2+7x+30\Rightarrow f'(x)=4x^3-9x^2-12x+7$



44. **(a)**

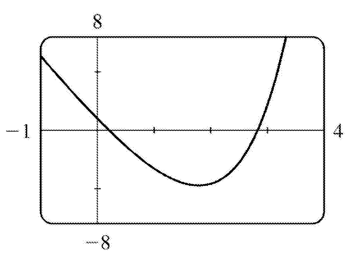


(b)



From the graph in part (a), it appears that f' is zero at $x_1 \approx 0.2$ and $x_2 \approx 2.8$. The slopes are positive (so f' is positive) on $(-\infty, x_1)$ and (x_2, ∞) . The slopes are negative (so f' is negative) on (x_1, x_2) .

(c) $g(x)=e^x-3x^2 \Rightarrow g'(x)=e^x-6x$

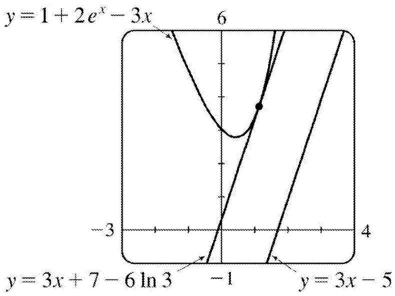


45. The curve $y=2x^3+3x^2-12x+1$ has a horizontal tangent when $y'=6x^2+6x-12=0 \Leftrightarrow$ $6(x^2+x-2)=0 \Leftrightarrow 6(x+2)(x-1)=0 \Leftrightarrow x=-2$ or $x=1$. The points on the curve are $(-2, 21)$ and $(1, -6)$.

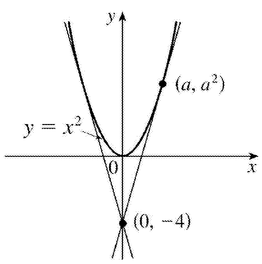
46. $f(x)=x^3+3x^2+x+3$ has a horizontal tangent when $f'(x)=3x^2+6x+1=0 \Leftrightarrow$ $x=\dfrac{-6\pm\sqrt{36-12}}{6}=-1\pm\dfrac{1}{3}\sqrt{6}$.

47. $y=6x^3+5x-3 \Rightarrow m=y'=18x^2+5$, but $x^2 \geq 0$ for all x , so $m \geq 5$ for all x .

48. The slope of $y=1+2e^x-3x$ is given by $m=y'=2e^x-3$.
The slope of $3x-y=5 \Leftrightarrow y=3x-5$ is 3 .
$m=3 \Rightarrow 2e^x-3=3 \Rightarrow e^x=3 \Rightarrow x=\ln 3$. This occurs at the point $(\ln 3, 7-3\ln 3) \approx (1.1, 3.7)$.



49.

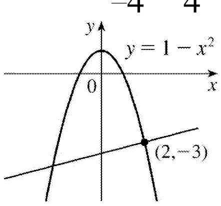


Let (a, a^2) be a point on the parabola at which the tangent line passes through the point $(0, -4)$. The tangent line has slope $2a$ and equation $y-(-4)=2a(x-0) \Leftrightarrow y=2ax-4$. Since (a, a^2) also lies on the line, $a^2=2a(a)-4$, or $a^2=4$. So $a=\pm 2$ and the points are $(2, 4)$ and $(-2, 4)$.

50. If $y=x^2+x$, then $y'=2x+1$. If the point at which a tangent meets the parabola is (a, a^2+a), then the slope of the tangent is $2a+1$. But since it passes through $(2, -3)$, the slope must also be $\frac{\Delta y}{\Delta x}=\frac{a^2+a+3}{a-2}$.

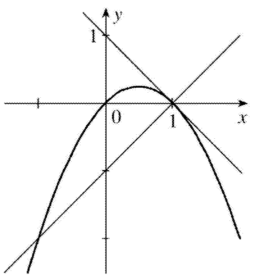
Therefore, $2a+1=\frac{a^2+a+3}{a-2}$. Solving this equation for a we get $a^2+a+3=2a^2-3a-2 \Leftrightarrow$ $a^2-4a-5=(a-5)(a+1)=0 \Leftrightarrow a=5$ or -1 . If $a=-1$, the point is $(-1, 0)$ and the slope is -1, so the equation is $y-0=(-1)(x+1)$ or $y=-x-1$. If $a=5$, the point is $(5, 30)$ and the slope is 11, so the equation is $y-30=11(x-5)$ or $y=11x-25$.

51. $y=f(x)=1-x^2 \Rightarrow f'(x)=-2x$, so the tangent line at $(2, -3)$ has slope $f'(2)=-4$. The normal line has slope $-\frac{1}{-4}=\frac{1}{4}$ and equation $y+3=\frac{1}{4}(x-2)$ or $y=\frac{1}{4}x-\frac{7}{2}$.



52. $y=f(x)=x-x^2 \Rightarrow f'(x)=1-2x$. So $f'(1)=-1$, and the slope of the normal line is the negative

reciprocal of that of the tangent line, that is, $-1/(-1)=1$. So the equation of the normal line at (1, 0) is $y-0=1(x-1)\Leftrightarrow y=x-1$. Substituting this into the equation of the parabola, we obtain $x-1=x-x^2 \Leftrightarrow x=\pm 1$. The solution $x=-1$ is the one we require. Substituting $x=-1$ into the equation of the parabola to find the y -coordinate, we have $y=-2$. So the point of intersection is $(-1, -2)$, as shown in the sketch.



53.

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\frac{1}{x+h}-\frac{1}{x}}{h}=\lim_{h\to 0}\frac{x-(x+h)}{hx(x+h)}$$

$$=\lim_{h\to 0}\frac{-h}{hx(x+h)}=\lim_{h\to 0}\frac{-1}{x(x+h)}=-\frac{1}{x^2}$$

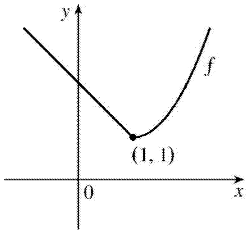
54. Substituting $x=1$ and $y=1$ into $y=ax^2+bx$ gives us $a+b=1$ **(1)** . The slope of the tangent line $y=3x-2$ is 3 and the slope of the tangent to the parabola at (x, y) is $y'=2ax+b$. At $x=1$, $y'=3\Rightarrow 3=2a+b$ **(2)** . Subtracting **(1)** from **(2)** gives us $2=a$ and it follows that $b=-1$. The parabola has equation $y=2x^2-x$.

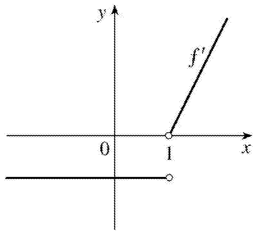
55. $f(x)=2-x$ if $x\leq 1$ and $f(x)=x^2-2x+2$ if $x>1$. Now we compute the right– and left–hand derivatives defined in Exercise :

$$f'_-(1)=\lim_{h\to 0^-}\frac{f(1+h)-f(1)}{h}=\lim_{h\to 0^-}\frac{2-(1+h)-1}{h}=\lim_{h\to 0^-}\frac{-h}{h}=\lim_{h\to 0^-}-1=-1 \text{ and}$$

$$f'_+(1)=\lim_{h\to 0^+}\frac{f(1+h)-f(1)}{h}=\lim_{h\to 0^+}\frac{(1+h)^2-2(1+h)+2-1}{h}=\lim_{h\to 0^+}\frac{h^2}{h}=\lim_{h\to 0^+}h=0 .$$

Thus, $f'(1)$ does not exist since $f'_-(1)\neq f'_+(1)$, so f is not differentiable at 1 . But $f'(x)=-1$ for $x<1$ and $f'(x)=2x-2$ if $x>1$.





56. $g(x)=\begin{cases} -1-2x & \text{if } x<-1 \\ x^2 & \text{if } -1\le x\le 1 \\ x & \text{if } x>1 \end{cases}$

$\lim\limits_{h\to 0^-}\dfrac{g(-1+h)-g(-1)}{h}=\lim\limits_{h\to 0^-}\dfrac{[-1-2(-1+h)]-1}{h}=\lim\limits_{h\to 0^-}\dfrac{-2h}{h}=\lim\limits_{h\to 0^-}(-2)=-2$ and

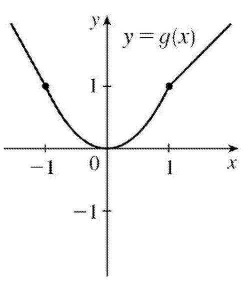
$\lim\limits_{h\to 0^+}\dfrac{g(-1+h)-g(-1)}{h}=\lim\limits_{h\to 0^+}\dfrac{(-1+h)^2-1}{h}=\lim\limits_{h\to 0^+}\dfrac{-2h+h^2}{h}=\lim\limits_{h\to 0^+}(-2+h)=-2$,

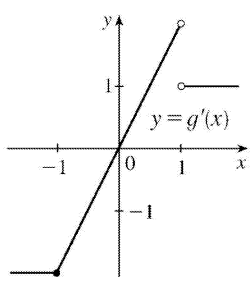
so g is differentiable at -1 and $g'(-1)=-2$.

$\lim\limits_{h\to 0^-}\dfrac{g(1)-g(1)}{h}=\lim\limits_{h\to 0^-}\dfrac{(1)^2-1}{h}=\lim\limits_{h\to 0^-}\dfrac{2h+h^2}{h}=\lim\limits_{h\to 0^-}(2+h)=2$ and

$\lim\limits_{h\to 0^+}\dfrac{g(1)-g(1)}{h}=\lim\limits_{h\to 0^+}\dfrac{(1)-1}{h}=\lim\limits_{h\to 0^+}\dfrac{h}{h}=\lim\limits_{h\to 0^+}1=1$, so $g'(1)$ does not exist.

Thus, g is differentiable except when $x=1$, and $g'(x)=\begin{cases} -2 & \text{if } x<-1 \\ 2x & \text{if } -1\le x<1 \\ 1 & \text{if } x>1 \end{cases}$





57. **(a)** Note that $x^2-9<0$ for $x^2<9 \Leftrightarrow |x|<3 \Leftrightarrow -3<x<3$. So

$$f(x)=\begin{cases} x^2-9 & \text{if } x\le -3 \\ -x^2+9 & \text{if } -3<x<3 \\ x^2-9 & \text{if } x\ge 3 \end{cases} \Rightarrow f'(x)=\begin{cases} 2x & \text{if } x<-3 \\ -2x & \text{if } -3<x<3 \\ 2x & \text{if } x>3 \end{cases} = \begin{cases} 2x & \text{if } |x|>3 \\ -2x & \text{if } |x|<3 \end{cases}$$

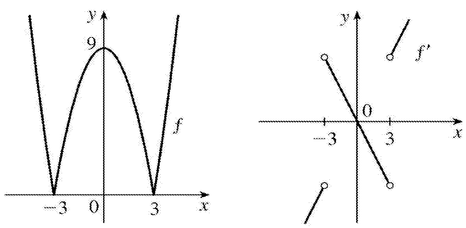
To show that

$f'(3)$ does not exist we investigate $\lim_{h\to 0}\dfrac{f(3+h)-f(3)}{h}$ by computing the left– and right–hand derivatives defined in Exercise .

$$f'_{-}(3)=\lim_{h\to 0^-}\frac{f(3+h)-f(3)}{h}=\lim_{h\to 0^-}\frac{[-(3+h)^2+9]-0}{h}=\lim_{h\to 0^-}(-6-h)=-6 \text{ and}$$

$$f'_{+}(3)=\lim_{h\to 0^+}\frac{f(3+h)-f(3)}{h}=\lim_{h\to 0^+}\frac{\left[(3+h)^2-9\right]-0}{h}=\lim_{h\to 0^+}\frac{6h+h^2}{h}=\lim_{h\to 0^+}(6+h)=6 \ .$$

Since the left and right limits are different, $\lim_{h\to 0}\dfrac{f(3+h)-f(3)}{h}$ does not exist, that is, $f'(3)$ does not exist. Similarly, $f'(-3)$ does not exist. Therefore, f is not differentiable at 3 or at -3 .

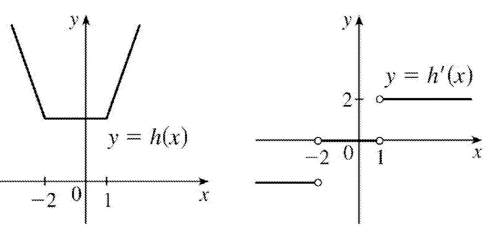


(b)

58. If $x\geq 1$, then $h(x)=|x-1|+|x+2|=x-1+x+2=2x+1$.
If $-2<x<1$, then $h(x)=-(x-1)+x+2=3$.
If $x\leq -2$, then $h(x)=-(x-1)-(x+2)=-2x-1$. Therefore,

$$h(x)=\begin{cases}-2x-1 & \text{if } x\leq -2\\ 3 & \text{if } -2<x<1\\ 2x+1 & \text{if } x\geq 1\end{cases} \Rightarrow h'(x)=\begin{cases}-2 & \text{if } x<-2\\ 0 & \text{if } -2<x<1\\ 2 & \text{if } x>1\end{cases}$$

To see that $h'(1)=\lim_{x\to 1}\dfrac{h(x)-h(1)}{x-1}$ does not exist, observe that $\lim_{x\to 1^-}\dfrac{h(x)-h(1)}{x-1}=\lim_{x\to 1^-}\dfrac{3-3}{3-1}=0$ but $\lim_{x\to 1^+}\dfrac{h(x)-h(1)}{x-1}=\lim_{x\to 1^+}\dfrac{2x-2}{x-1}=2$. Similarly, $h'(-2)$ does not exist.



59. $y=f(x)=ax^2 \Rightarrow f'(x)=2ax$. So the slope of the tangent to the parabola at $x=2$ is $m=2a(2)=4a$. The slope of the given line, $2x+y=b \Leftrightarrow y=-2x+b$, is seen to be -2, so we must have $4a=-2 \Leftrightarrow a=-\frac{1}{2}$. So when $x=2$, the point in question has y-coordinate $-\frac{1}{2}\cdot 2^2=-2$. Now we simply require that the given line, whose equation is $2x+y=b$, pass through the point $(2, -2)$: $2(2)+(-2)=b \Leftrightarrow b=2$. So we must have $a=-\frac{1}{2}$ and $b=2$.

60. f is clearly differentiable for $x<2$ and for $x>2$. For $x<2$, $f'(x)=2x$, so $f'_-(2)=4$. For $x>2$, $f'(x)=m$, so $f'_+(2)=m$. For f to be differentiable at $x=2$, we need $4=f'_-(2)=f'_+(2)=m$. So $f(x)=4x+b$. We must also have continuity at $x=2$, so $4=f(2)=\lim_{x\to 2^+} f(x)=\lim_{x\to 2^+}(4x+b)=8+b$. Hence, $b=-4$.

61. $y=f(x)=ax^3+bx^2+cx+d \Rightarrow f'(x)=3ax^2+2bx+c$. The point $(-2, 6)$ is on f, so $f(-2)=6 \Rightarrow$ $-8a+4b-2c+d=6$ **(1)** . The point $(2, 0)$ is on f, so $f(2)=0 \Rightarrow 8a+4b+2c+d=0$ **(2)** . Since there are horizontal tangents at $(-2, 6)$ and $(2, 0)$, $f'(\pm 2)=0$. $f'(-2)=0 \Rightarrow 12a-4b+c=0$ **(3)** and $f'(2)=0 \Rightarrow 12a+4b+c=0$ **(4)** . Subtracting equation **(3)** from **(4)** gives $8b=0 \Rightarrow b=0$. Adding **(1)** and **(2)** gives $8b+2d=6$, so $d=3$ since $b=0$. From **(3)** we have $c=-12a$, so **(2)** becomes $8a+4(0)+2(-12a)+3=0$ $\Rightarrow 3=16a \Rightarrow a=\frac{3}{16}$. Now $c=-12a=-12\left(\frac{3}{16}\right)=-\frac{9}{4}$ and the desired cubic function is $y=\frac{3}{16}x^3-\frac{9}{4}x+3$.

62. **(a)** $xy=c \Rightarrow y=\frac{c}{x}$. Let $P=\left(a, \frac{c}{a}\right)$. The slope of the tangent line at $x=a$ is $y'(a)=-\frac{c}{a^2}$. Its equation is $y-\frac{c}{a}=-\frac{c}{a^2}(x-a)$ or $y=-\frac{c}{a^2}x+\frac{2c}{a}$. so its y-intercept is $\frac{2c}{a}$. Setting $y=0$ gives $x=2a$, so the x-intercept is $2a$. The midpoint of the line segment joining $\left(0, \frac{2c}{a}\right)$ and $(2a, 0)$ is $\left(a, \frac{c}{a}\right)=P$.

(b) We know the x- and y-intercepts of the tangent line from part (a), so the area of the triangle bounded by the axes and the tangent is $\frac{1}{2}$(base)(height)$=\frac{1}{2}xy=\frac{1}{2}(2a)(2c/a)=2c$, a constant.

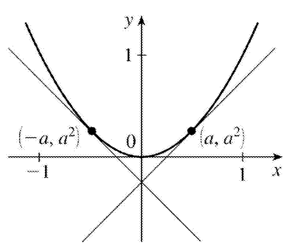
63. *Solution 1:* Let $f(x)=x^{1000}$. Then, by the definition of a derivative, $f'(1)=\lim_{x\to 1}\frac{f(x)-f(1)}{x-1}=\lim_{x\to 1}\frac{x^{1000}-1}{x-1}$. But this is just the limit we want to find, and we know (from the Power Rule) that $f'(x)=1000x^{999}$, so $f'(1)=1000(1)^{999}=1000$. So $\lim_{x\to 1}\frac{x^{1000}-1}{x-1}=1000$.

Solution 2: Note that $(x^{1000}-1)=(x-1)(x^{999}+x^{998}+x^{997}+\cdots+x^2+x+1)$. So

$$\lim_{x\to 1}\frac{x^{1000}-1}{x-1}=\lim_{x\to 1}\frac{(x-1)(x^{999}+x^{998}+x^{997}+\cdots+x^2+x+1)}{x-1}$$

$$=\lim_{x\to 1}(x^{999}+x^{998}+x^{997}+\cdots+x^2+x+1)=\underbrace{1+1+1+\ldots+1+1+1}_{1000\text{ ones}}$$

$$=1000,\ \text{as above.}$$



64.
In order for the two tangents to intersect on the y –axis, the points of tangency must be at equal distances from the y –axis, since the parabola $y=x^2$ is symmetric about the y –axis. Say the points of tangency are (a, a^2) and $(-a, a^2)$, for some $a>0$. Then since the derivative of $y=x^2$ is $dy/dx=2x$, the left–hand tangent has slope $-2a$ and equation $y-a^2=-2a(x+a)$, or $y=-2ax-a^2$, and similarly the right–hand tangent line has equation
$y-a^2=2a(x-a)$, or $y=2ax-a^2$. So the two lines intersect at $(0, -a^2)$. Now if the lines are perpendicular, then the product of their slopes is -1, so $(-2a)(2a)=-1\Leftrightarrow a^2=\frac{1}{4}\Leftrightarrow a=\frac{1}{2}$. So the lines intersect at $\left(0, -\frac{1}{4}\right)$.

1. $V=x^3 \Rightarrow \dfrac{dV}{dt}=\dfrac{dV}{dx}\dfrac{dx}{dt}=3x^2\dfrac{dx}{dt}$

2. **(a)** $A=\pi r^2 \Rightarrow \dfrac{dA}{dt}=\dfrac{dA}{dr}\dfrac{dr}{dt}=2\pi r\dfrac{dr}{dt}$

(b) $\dfrac{dA}{dt}=2\pi r\dfrac{dr}{dt}=2\pi\,(30\text{m})(1\text{m/s})=60\pi\ \text{m}^2/\text{s}$

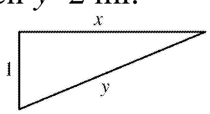
3. $y=x^3+2x \Rightarrow \dfrac{dy}{dt}=\dfrac{dy}{dx}\dfrac{dx}{dt}=(3x^2+2)(5)=5(3x^2+2)$. When $x=2$, $\dfrac{dy}{dt}=5(14)=70$.

4. $x^2+y^2=25 \Rightarrow 2x\dfrac{dx}{dt}+2y\dfrac{dy}{dt}=0 \Rightarrow x\dfrac{dx}{dt}=-y\dfrac{dy}{dt} \Rightarrow \dfrac{dx}{dt}=-\dfrac{y}{x}\dfrac{dy}{dt}$.
When $y=4$, $x^2+4^2=25 \Rightarrow x=\pm 3$. For $\dfrac{dy}{dt}=6$, $\dfrac{dx}{dt}=-\dfrac{4}{\pm 3}(6)=\mp 8$.

5. $z^2=x^2+y^2 \Rightarrow 2z\dfrac{dz}{dt}=2x\dfrac{dx}{dt}+2y\dfrac{dy}{dt} \Rightarrow \dfrac{dz}{dt}=\dfrac{1}{z}\left(x\dfrac{dx}{dt}+y\dfrac{dy}{dt}\right)$. When $x=5$ and $y=12$, $z^2=5^2+12^2 \Rightarrow$ $z^2=169 \Rightarrow z=\pm 13$. For $\dfrac{dx}{dt}=2$ and $\dfrac{dy}{dt}=3$, $\dfrac{dz}{dt}=\dfrac{1}{\pm 13}(5\cdot 2+12\cdot 3)=\pm\dfrac{46}{13}$.

6. $y=\sqrt{1+x^3} \Rightarrow \dfrac{dy}{dt}=\dfrac{dy}{dx}\dfrac{dx}{dt}=\dfrac{1}{2}(1+x^3)^{-1/2}(3x^2)\dfrac{dx}{dt}=\dfrac{3x^2}{2\sqrt{1+x^3}}\dfrac{dx}{dt}$. With $\dfrac{dy}{dt}=4$ when $x=2$ and $y=3$, we have $4=\dfrac{3(4)}{2(3)}\dfrac{dx}{dt} \Rightarrow \dfrac{dx}{dt}=2$ cm / s.

7. **(a)** Given: a plane flying horizontally at an altitude of 1 mi and a speed of 500 mi / h passes directly over a radar station. If we let t be time (in hours) and x be the horizontal distance traveled by the plane (in mi), then we are given that $dx/dt=500$ mi / h.
(b) Unknown: the rate at which the distance from the plane to the station is increasing when it is 2 mi from the station. If we let y be the distance from the plane to the station, then we want to find dy/dt when $y=2$ mi.

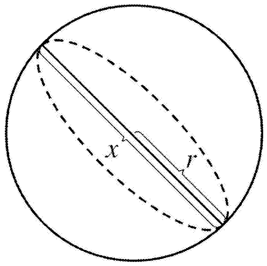
(c) 

(d) By the Pythagorean Theorem, $y^2=x^2+1 \Rightarrow 2y(dy/dt)=2x(dx/dt)$.

(e) $\dfrac{dy}{dt}=\dfrac{x}{y}\dfrac{dx}{dt}=\dfrac{x}{y}(500)$. Since $y^2=x^2+1$, when $y=2$, $x=\sqrt{3}$, so $\dfrac{dy}{dt}=\dfrac{\sqrt{3}}{2}(500)=250\sqrt{3}\approx 433$ mi / h.

8. **(a)** Given: the rate of decrease of the surface area is 1 cm

^{2}/ min. If we let t be time (in minutes) and S be the surface area (in cm^2), then we are given that $dS/dt=-1$ cm^2/ s.
(b) Unknown: the rate of decrease of the diameter when the diameter is 10 cm. If we let x be the diameter, then we want to find dx/dt when $x=10$ cm.

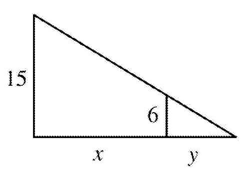


(c)

(d) If the radius is r and the diameter $x=2r$, then $r=\frac{1}{2}x$ and $S=4\pi r^2=4\pi\left(\frac{1}{2}x\right)^2=\pi x^2 \Rightarrow$
$\frac{dS}{dt}=\frac{dS}{dx}\frac{dx}{dt}=2\pi x\frac{dx}{dt}$.

(e) $-1=\frac{dS}{dt}=2\pi x\frac{dx}{dt} \Rightarrow \frac{dx}{dt}=-\frac{1}{2\pi x}$. When $x=10$, $\frac{dx}{dt}=-\frac{1}{20\pi}$. So the rate of decrease is $\frac{1}{20\pi}$ cm / min.

9. **(a)** Given: a man 6 ft tall walks away from a street light mounted on a 15 –ft–tall pole at a rate of 5 ft / s. If we let t be time (in s) and x be the distance from the pole to the man (in ft), then we are given that $dx/dt=5$ ft / s.
(b) Unknown: the rate at which the tip of his shadow is moving when he is 40 ft from the pole. If we let y be the distance from the man to the tip of
his shadow (in ft), then we want to find $\frac{d}{dt}(x+y)$ when $x=40$ ft.

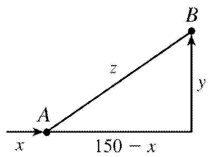


(c)

(d) By similar triangles, $\frac{15}{6}=\frac{x+y}{y} \Rightarrow 15y=6x+6y \Rightarrow 9y=6x \Rightarrow y=\frac{2}{3}x$.

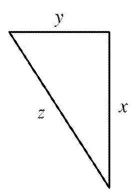
(e) The tip of the shadow moves at a rate of $\frac{d}{dt}(x+y)=\frac{d}{dt}\left(x+\frac{2}{3}x\right)=\frac{5}{3}\frac{dx}{dt}=\frac{5}{3}(5)=\frac{25}{3}$ ft / s.

10. **(a)** Given: at noon, ship A is 150 km west of ship B; ship A is sailing east at 35 km / h, and ship B is sailing north at 25 km / h. If we let t be time (in hours), x be the distance traveled by ship A (in km), and y be the distance traveled by ship B (in km), then we are given that $dx/dt=35$ km / h and $dy/dt=25$ km / h.
(b) Unknown: the rate at which the distance between the ships is changing at 4:00 P.M. If we let z be the distance between the ships, then we want to find dz/dt when $t=4$ h.

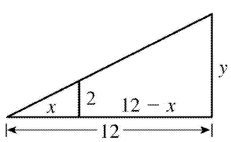
(c) 

(d) $z^2=(150-x)^2+y^2 \Rightarrow 2z\frac{dz}{dt}=2(150-x)\left(-\frac{dx}{dt}\right)+2y\frac{dy}{dt}$

(e) At 4:00 P.M., $x=4(35)=140$ and $y=4(25)=100 \Rightarrow z=\sqrt{(150-140)^2+100^2}=\sqrt{10{,}100}$. So
$\frac{dz}{dt}=\frac{1}{z}\left[(x-150)\frac{dx}{dt}+y\frac{dy}{dt}\right]=\frac{-10(35)+100(25)}{\sqrt{10{,}100}}=\frac{215}{\sqrt{101}}\approx 21.4$ km / h.

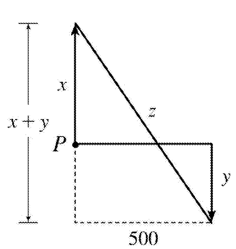
11. 

We are given that $\frac{dx}{dt}=60$ mi / h and $\frac{dy}{dt}=25$ mi / h. $z^2=x^2+y^2 \Rightarrow 2z\frac{dz}{dt}=2x\frac{dx}{dt}+2y\frac{dy}{dt} \Rightarrow$
$z\frac{dz}{dt}=x\frac{dx}{dt}+y\frac{dy}{dt} \Rightarrow \frac{dz}{dt}=\frac{1}{z}\left(x\frac{dx}{dt}+y\frac{dy}{dt}\right)$.
After 2 hours, $x=2(60)=120$ and $y=2(25)=50 \Rightarrow z=\sqrt{120^2+50^2}=130$, so
$\frac{dz}{dt}=\frac{1}{z}\left(x\frac{dx}{dt}+y\frac{dy}{dt}\right)=\frac{120(60)+50(25)}{130}=65$ mi / h.

12. 

We are given that $\frac{dx}{dt}=1.6$ m / s. By similar triangles, $\frac{y}{12}=\frac{2}{x} \Rightarrow y=\frac{24}{x} \Rightarrow \frac{dy}{dt}=-\frac{24}{x^2}\frac{dx}{dt}=-\frac{24}{x^2}(1.6)$.

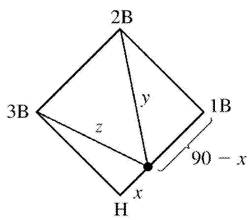
When $x=8$, $\frac{dy}{dt}=-\frac{24(1.6)}{64}=-0.6$ m / s, so the shadow is decreasing at a rate of 0.6 m / s.

13. 

We are given that $\frac{dx}{dt}=4$ ft / s and $\frac{dy}{dt}=5$ ft / s. $z^2=(x+y)^2+500^2 \Rightarrow 2z\frac{dz}{dt}=2(x+y)\left(\frac{dx}{dt}+\frac{dy}{dt}\right)$. 15

minutes after the woman starts, we have $x=(4\text{ft/s})(20\text{min})(60\text{s/min})=4800$ ft and $y=5\cdot 15\cdot 60=4500 \Rightarrow$
$z=\sqrt{(4800+4500)^2+500^2}=\sqrt{86{,}740{,}000}$, so
$\frac{dz}{dt}=\frac{x+y}{z}\left(\frac{dx}{dt}+\frac{dy}{dt}\right)=\frac{4800+4500}{\sqrt{86{,}740{,}000}}(4+5)=\frac{837}{\sqrt{8674}}\approx 8.99$ ft / s .

14. We are given that $\frac{dx}{dt}=24$ ft / s.

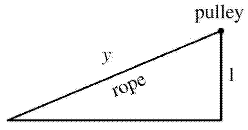


(a)
$y^2=(90-x)^2+90^2 \Rightarrow 2y\frac{dy}{dt}=2(90-x)\left(-\frac{dx}{dt}\right)$.
When $x=45$, $y=\sqrt{45^2+90^2}=45\sqrt{5}$, so $\frac{dy}{dt}=\frac{90-x}{y}\left(-\frac{dx}{dt}\right)=\frac{45}{45\sqrt{5}}(-24)=-\frac{24}{\sqrt{5}}$,
so the distance from second base is decreasing at a rate of $\frac{24}{\sqrt{5}}\approx 10.7$ ft / s.

(b) Due to the symmetric nature of the problem in part (a), we expect to get the same answer and we do. $z^2=x^2+90^2 \Rightarrow 2z\frac{dz}{dt}=2x\frac{dx}{dt}$. When $x=45$, $z=45\sqrt{5}$, so $\frac{dz}{dt}=\frac{45}{45\sqrt{5}}(24)=\frac{24}{\sqrt{5}}\approx 10.7$ ft / s.

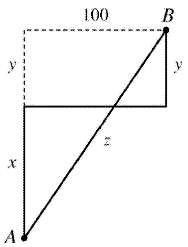
15. $A=\frac{1}{2}bh$, where b is the base and h is the altitude. We are given that $\frac{dh}{dt}=1$ cm / min and $\frac{dA}{dt}=2$ cm ^{2}/ min. Using the Product Rule, we have $\frac{dA}{dt}=\frac{1}{2}\left(b\frac{dh}{dt}+h\frac{db}{dt}\right)$. When $h=10$ and $A=100$, we have $100=\frac{1}{2}b(10) \Rightarrow \frac{1}{2}b=10 \Rightarrow b=20$, so $2=\frac{1}{2}\left(20\cdot 1+10\frac{db}{dt}\right) \Rightarrow 4=20+10\frac{db}{dt} \Rightarrow$
$\frac{db}{dt}=\frac{4-20}{10}=-1.6$ cm / min.

16.



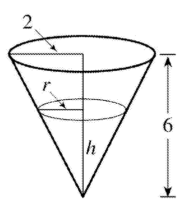
Given $\frac{dy}{dt}=-1$ m / s, find $\frac{dx}{dt}$ when $x=8$ m. $y^2=x^2+1 \Rightarrow 2y\frac{dy}{dt}=2x\frac{dx}{dt} \Rightarrow \frac{dx}{dt}=\frac{y}{x}\frac{dy}{dt}=-\frac{y}{x}$. When $x=8$, $y=\sqrt{65}$, so

$\frac{dx}{dt}=-\frac{\sqrt{65}}{8}$. Thus, the boat approaches the dock at $\frac{\sqrt{65}}{8}\approx 1.01$ m / s.

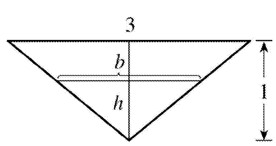
17. 

We are given that $\frac{dx}{dt}=35$ km / h and $\frac{dy}{dt}=25$ km / h. $z^2=(x+y)^2+100^2\Rightarrow 2z\frac{dz}{dt}=2(x+y)\left(\frac{dx}{dt}+\frac{dy}{dt}\right)$.
At 4:00 P.M., $x=4(35)=140$ and $y=4(25)=100\Rightarrow z=\sqrt{(140+100)^2+100^2}=\sqrt{67{,}600}=260$, so
$\frac{dz}{dt}=\frac{x+y}{z}\left(\frac{dx}{dt}+\frac{dy}{dt}\right)=\frac{140+100}{260}(35+25)=\frac{720}{13}\approx 55.4$ km / h.

18. Let D denote the distance from the origin (0,0) to the point on the curve $y=\sqrt{x}$.
$D=\sqrt{(x-0)^2+(y-0)^2}=\sqrt{x^2+(\sqrt{x})^2}=\sqrt{x^2+x}\Rightarrow \frac{dD}{dt}=\frac{1}{2}(x^2+x)^{-1/2}(2x+1)\frac{dx}{dt}=\frac{2x+1}{2\sqrt{x^2+x}}\frac{dx}{dt}$. With
$\frac{dx}{dt}=3$ when $x=4$, $\frac{dD}{dt}=\frac{9}{2\sqrt{20}}(3)=\frac{27}{4\sqrt{5}}\approx 3.02$ cm / s.

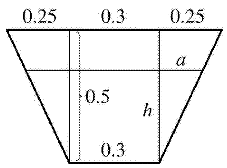
19. 

If $C=$ the rate at which water is pumped in, then $\frac{dV}{dt}=C-10{,}000$, where $V=\frac{1}{3}\pi r^2h$ is the volume at time t . By similar triangles, $\frac{r}{2}=\frac{h}{6}\Rightarrow r=\frac{1}{3}h\Rightarrow V=\frac{1}{3}\pi\left(\frac{1}{3}h\right)^2h=\frac{\pi}{27}h^3\Rightarrow\frac{dV}{dt}=\frac{\pi}{9}h^2\frac{dh}{dt}$.
When $h=200$cm , $\frac{dh}{dt}=20$cm/min , so $C-10{,}000=\frac{\pi}{9}(200)^2(20)\Rightarrow C=10{,}000+\frac{800{,}000}{9}\pi\approx 289{,}253$ cm^3/ min.

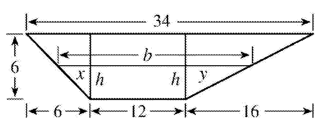
20. 

By similar triangles,

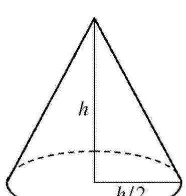
$\frac{3}{1}=\frac{b}{h}$, so $b=3h$. The trough has volume $V=\frac{1}{2}bh(10)=5(3h)h=15h^2 \Rightarrow 12=\frac{dV}{dt}=30h\frac{dh}{dt} \Rightarrow \frac{dh}{dt}=\frac{2}{5h}$. When $h=\frac{1}{2}$, $\frac{dh}{dt}=\frac{2}{5\cdot\frac{1}{2}}=\frac{4}{5}$ ft / min.

21. 

The figure is labeled in meters. The area A of a trapezoid is $\frac{1}{2}(\text{ base }_1+\text{ base }_2)(\text{ height })$, and the volume V of the 10–meter–long trough is $10A$. Thus, the volume of the trapezoid with height h is $V=(10)\frac{1}{2}[0.3+(0.3+2a)]h$. By similar triangles, $\frac{a}{h}=\frac{0.25}{0.5}=\frac{1}{2}$, so $2a=h \Rightarrow V=5(0.6+h)h=3h+5h^2$. Now $\frac{dV}{dt}=\frac{dV}{dh}\frac{dh}{dt} \Rightarrow 0.2=(3+10h)\frac{dh}{dt} \Rightarrow \frac{dh}{dt}=\frac{0.2}{3+10h}$. When $h=0.3$, $\frac{dh}{dt}=\frac{0.2}{3+10(0.3)}=\frac{0.2}{6}$ m / min $=\frac{1}{30}$ m / min or $\frac{10}{3}$ cm / min.

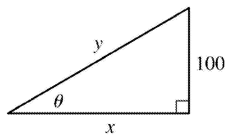
22. 

The figure is drawn without the top 3 feet. $V=\frac{1}{2}(b+12)h(20)=10(b+12)h$ and, from similar triangles, $\frac{x}{h}=\frac{6}{6}$ and $\frac{y}{h}=\frac{16}{6}=\frac{8}{3}$, so $b=x+12+y=h+12+\frac{8h}{3}=12+\frac{11h}{3}$. Thus, $V=10\left(24+\frac{11h}{3}\right)h=240h+\frac{110h^2}{3}$ and so $0.8=\frac{dV}{dt}=\left(240+\frac{220}{3}h\right)\frac{dh}{dt}$. When $h=5$, $\frac{dh}{dt}=\frac{0.8}{240+5(220/3)}=\frac{3}{2275}\approx 0.00132$ ft / min.

23. 

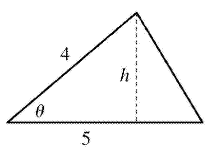
We are given that $\frac{dV}{dt}=30\text{ ft}^3/\text{ min}$. $V=\frac{1}{3}\pi r^2h=\frac{1}{3}\pi\left(\frac{h}{2}\right)^2h=\frac{\pi h^3}{12} \Rightarrow$

$\frac{dV}{dt}=\frac{dV}{dh}\frac{dh}{dt}\Rightarrow 30=\frac{\pi h^2}{4}\frac{dh}{dt}\Rightarrow\frac{dh}{dt}=\frac{120}{\pi h^2}$. When h=10 ft, $\frac{dh}{dt}=\frac{120}{10^2\pi}=\frac{6}{5\pi}\approx 0.38$ ft / min.

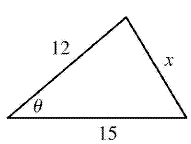
24. 

We are given dx/dt=8 ft / s. $\cot\theta=\frac{x}{100}\Rightarrow x=100\cot\theta\Rightarrow\frac{dx}{dt}=-100\csc^2\theta\frac{d\theta}{dt}\Rightarrow\frac{d\theta}{dt}=-\frac{\sin^2\theta}{100}\cdot 8$.

When $y=200$, $\sin\theta=\frac{100}{200}=\frac{1}{2}\Rightarrow\frac{d\theta}{dt}=-\frac{(1/2)^2}{100}\cdot 8=-\frac{1}{50}$ rad / s. The angle is decreasing at a rate of $\frac{1}{50}$ rad / s.

25. 

$A=\frac{1}{2}bh$, but b=5 m and $\sin\theta=\frac{h}{4}\Rightarrow h=4\sin\theta$, so $A=\frac{1}{2}(5)(4\sin\theta)=10\sin\theta$. We are given $\frac{d\theta}{dt}=0.06$ rad / s, so $\frac{dA}{dt}=\frac{dA}{d\theta}\frac{d\theta}{dt}=(10\cos\theta)(0.06)=0.6\cos\theta$. When $\theta=\frac{\pi}{3}$,
$\frac{dA}{dt}=0.6\left(\cos\frac{\pi}{3}\right)=(0.6)\left(\frac{1}{2}\right)=0.3\ \text{m}^2/\text{s}$.

26. 

We are given $d\theta/dt=2^\circ$ / min $=\frac{\pi}{90}$ rad / min. By the Law of Cosines,
$x^2=12^2+15^2-2(12)(15)\cos\theta=369-360\cos\theta\Rightarrow 2x\frac{dx}{dt}=360\sin\theta\frac{d\theta}{dt}\Rightarrow\frac{dx}{dt}=\frac{180\sin\theta}{x}\frac{d\theta}{dt}$. When $\theta=60^\circ$, $x=\sqrt{369-360\cos 60^\circ}=\sqrt{189}=3\sqrt{21}$, so $\frac{dx}{dt}=\frac{180\sin 60^\circ}{3\sqrt{21}}\frac{\pi}{90}=\frac{\pi\sqrt{3}}{3\sqrt{21}}=\frac{\sqrt{7}\pi}{21}\approx 0.396$ m / min.

27. Differentiating both sides of $PV=C$ with respect to t and using the Product Rule gives us $P\frac{dV}{dt}+V\frac{dP}{dt}=0\Rightarrow\frac{dV}{dt}=-\frac{V}{P}\frac{dP}{dt}$. When $V=600$, $P=150$ and $\frac{dP}{dt}=20$, so we have

$\frac{dV}{dt}=-\frac{600}{150}(20)=-80$. Thus, the volume is decreasing at a rate of 80 cm^3/ min.

28. $PV^{1.4}=C \Rightarrow P\cdot 1.4V^{0.4}\frac{dV}{dt}+V^{1.4}\frac{dP}{dt}=0 \Rightarrow \frac{dV}{dt}=-\frac{V^{1.4}}{P\cdot 1.4V^{0.4}}\frac{dP}{dt}=-\frac{V}{1.4P}\frac{dP}{dt}$. When $V=400$, $P=80$ and $\frac{dP}{dt}=-10$, so we have $\frac{dV}{dt}=-\frac{400}{1.4(80)}(-10)=\frac{250}{7}$. Thus, the volume is increasing at a rate of $\frac{250}{7}\approx 36$ cm^3/ min.

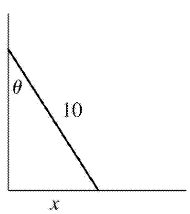
29. With $R_1=80$ and $R_2=100$, $\frac{1}{R}=\frac{1}{R_1}+\frac{1}{R_2}=\frac{1}{80}+\frac{1}{100}=\frac{180}{8000}=\frac{9}{400}$, so $R=\frac{400}{9}$. Differentiating $\frac{1}{R}=\frac{1}{R_1}+\frac{1}{R_2}$ with respect to t , we have $-\frac{1}{R^2}\frac{dR}{dt}=-\frac{1}{R_1^2}\frac{dR_1}{dt}-\frac{1}{R_2^2}\frac{dR_2}{dt} \Rightarrow$

$\frac{dR}{dt}=R^2\left(\frac{1}{R_1^2}\frac{dR_1}{dt}+\frac{1}{R_2^2}\frac{dR_2}{dt}\right)$. When $R_1=80$ and $R_2=100$,

$\frac{dR}{dt}=\frac{400^2}{9^2}\left[\frac{1}{80^2}(0.3)+\frac{1}{100^2}(0.2)\right]=\frac{107}{810}\approx 0.132\Omega/\text{ s}.$

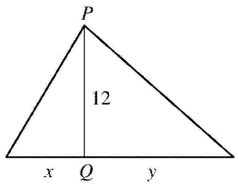
30. We want to find $\frac{dB}{dt}$ when $L=18$ using $B=0.007W^{2/3}$ and $W=0.12L^{2.53}$.

$$\frac{dB}{dt}=\frac{dB}{dW}\frac{dW}{dL}\frac{dL}{dt}=\left(0.007\cdot\frac{2}{3}W^{-1/3}\right)(0.12\cdot 2.53\cdot L^{1.53})\left(\frac{20-15}{10{,}000{,}000}\right)$$
$$=\left[0.007\cdot\frac{2}{3}(0.12\cdot 18^{2.53})^{-1/3}\right](0.12\cdot 2.53\cdot 18^{1.53})\left(\frac{5}{10^7}\right)\approx 1.045\times 10^{-8}\text{g/yr}$$

31. 

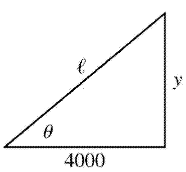
We are given that $\frac{dx}{dt}=2$ ft / s. $\sin\theta=\frac{x}{10} \Rightarrow x=10\sin\theta \Rightarrow \frac{dx}{dt}=10\cos\theta\frac{d\theta}{dt}$. When $\theta=\frac{\pi}{4}$,

$2=10\cos\frac{\pi}{4}\frac{d\theta}{dt} \Rightarrow \frac{d\theta}{dt}=\frac{2}{10(1/\sqrt{2})}=\frac{\sqrt{2}}{5}$ rad / s.

32. 

Using Q for the origin, we are given $\frac{dx}{dt}=-2$ ft / s and need to find $\frac{dy}{dt}$ when $x=-5$. Using the Pythagorean Theorem twice, we have $\sqrt{x^2+12^2}+\sqrt{y^2+12^2}=39$, the total length of the rope. Differentiating with respect to t , we get $\frac{x}{\sqrt{x^2+12^2}}\frac{dx}{dt}+\frac{y}{\sqrt{y^2+12^2}}\frac{dy}{dt}=0$, so

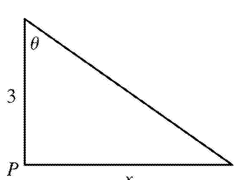
$\frac{dy}{dt}=-\frac{x\sqrt{y^2+12^2}}{y\sqrt{x^2+12^2}}\frac{dx}{dt}$. Now when $x=-5$, $39=\sqrt{(-5)^2+12^2}+\sqrt{y^2+12^2}=13+\sqrt{y^2+12^2}\Leftrightarrow\sqrt{y^2+12^2}=26$, and $y=\sqrt{26^2-12^2}=\sqrt{532}$. So when $x=-5$, $\frac{dy}{dt}=-\frac{(-5)(26)}{\sqrt{532}(13)}(-2)=-\frac{10}{\sqrt{133}}\approx-0.87$ ft / s. So cart B is moving towards Q at about 0.87 ft / s.

33. **(a)** 

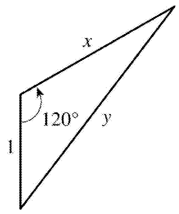
By the Pythagorean Theorem, $4000^2+y^2=\ell^2$. Differentiating with respect to t , we obtain $2y\frac{dy}{dt}=2\ell\frac{d\ell}{dt}$. We know that $\frac{dy}{dt}=600$ ft / s, so when $y=3000$ ft, $\ell=\sqrt{4000^2+3000^2}=\sqrt{25{,}000{,}000}=5000$ ft and $\frac{d\ell}{dt}=\frac{y}{\ell}\frac{dy}{dt}=\frac{3000}{5000}(600)=\frac{1800}{5}=360$ ft / s.

(b) Here $\tan\theta=\frac{y}{4000}\Rightarrow\frac{d}{dt}(\tan\theta)=\frac{d}{dt}\left(\frac{y}{4000}\right)\Rightarrow\sec^2\theta\frac{d\theta}{dt}=\frac{1}{4000}\frac{dy}{dt}\Rightarrow\frac{d\theta}{dt}=\frac{\cos^2\theta}{4000}\frac{dy}{dt}$.

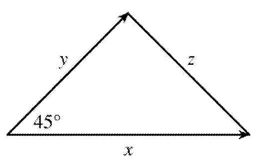
When $y=3000$ ft, $\frac{dy}{dt}=600$ ft / s, $\ell=5000$ and $\cos\theta=\frac{4000}{\ell}=\frac{4000}{5000}=\frac{4}{5}$, so $\frac{d\theta}{dt}=\frac{(4/5)^2}{4000}(600)=0.096$ rad / s.

34. 

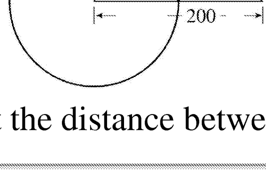
We are given that $\frac{d\theta}{dt}=4(2\pi)=8\pi$ rad / min. $x=3\tan\theta \Rightarrow \frac{dx}{dt}=3\sec^2\theta\,\frac{d\theta}{dt}$. When $x=1$, $\tan\theta=\frac{1}{3}$, so $\sec^2\theta=1+\left(\frac{1}{3}\right)^2=\frac{10}{9}$ and $\frac{dx}{dt}=3\left(\frac{10}{9}\right)(8\pi)=\frac{80\pi}{3}\approx 83.8$ km / min.

35. 

We are given that $\frac{dx}{dt}=300$ km / h. By the Law of Cosines,
$y^2=x^2+1^2-2(1)(x)\cos 120^\circ=x^2+1-2x\left(-\frac{1}{2}\right)=x^2+x+1$, so $2y\frac{dy}{dt}=2x\frac{dx}{dt}+\frac{dx}{dt} \Rightarrow \frac{dy}{dt}=\frac{2x+1}{2y}\frac{dx}{dt}$. After 1 minute,
$x=\frac{300}{60}=5$ km $\Rightarrow y=\sqrt{5^2+5+1}=\sqrt{31}$ km $\Rightarrow \frac{dy}{dt}=\frac{2(5)+1}{2\sqrt{31}}(300)=\frac{1650}{\sqrt{31}}\approx 296$ km / h.

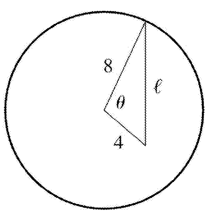
36. 

We are given that $\frac{dx}{dt}=3$ mi / h and $\frac{dy}{dt}=2$ mi / h. By the Law of Cosines,
$z^2=x^2+y^2-2xy\cos 45^\circ=x^2+y^2-\sqrt{2}\,xy \Rightarrow 2z\frac{dz}{dt}=2x\frac{dx}{dt}+2y\frac{dy}{dt}-\sqrt{2}\,x\frac{dy}{dt}-\sqrt{2}\,y\frac{dx}{dt}$. After 15 minutes $\left[=\frac{1}{4}\text{ h}\right]$,
we have $x=\frac{3}{4}$ and $y=\frac{2}{4}=\frac{1}{2} \Rightarrow z^2=\left(\frac{3}{4}\right)^2+\left(\frac{2}{4}\right)^2-\sqrt{2}\left(\frac{3}{4}\right)\left(\frac{2}{4}\right) \Rightarrow z=\frac{\sqrt{13-6\sqrt{2}}}{4}$ and
$\frac{dz}{dt}=\frac{2}{\sqrt{13-6\sqrt{2}}}\left[2\left(\frac{3}{4}\right)3+2\left(\frac{1}{2}\right)2-\sqrt{2}\left(\frac{3}{4}\right)2-\sqrt{2}\left(\frac{1}{2}\right)3\right]=\frac{2}{\sqrt{13-6\sqrt{2}}}\,\frac{13-6\sqrt{2}}{2}=\sqrt{13-6\sqrt{2}}$
≈ 2.125mi/h.

37. 

Let the distance between the runner and the friend be ℓ . Then by the Law of Cosines,

$\ell^2=200^2+100^2-2\cdot 200\cdot 100\cdot \cos\theta=50{,}000-40{,}000\cos\theta$ (*). Differentiating implicitly with respect to t, we obtain $2\ell\frac{d\ell}{dt}=-40{,}000(-\sin\theta)\frac{d\theta}{dt}$. Now if D is the distance run when the angle is θ radians, then by the formula for the length of an arc on a circle, $s=r\theta$, we have $D=100\theta$, so $\theta=\frac{1}{100}D\Rightarrow\frac{d\theta}{dt}=\frac{1}{100}\frac{dD}{dt}=\frac{7}{100}$. To substitute into the expression for $\frac{d\ell}{dt}$, we must know $\sin\theta$ at the time when $\ell=200$, which we find from (*): $200^2=50{,}000-40{,}000\cos\theta\Leftrightarrow\cos\theta=\frac{1}{4}$ $\Rightarrow\sin\theta=\sqrt{1-\left(\frac{1}{4}\right)^2}=\frac{\sqrt{15}}{4}$. Substituting, we get $2(200)\frac{d\ell}{dt}=40{,}000\frac{\sqrt{15}}{4}\left(\frac{7}{100}\right)\Rightarrow$ $d\ell/dt=\frac{7\sqrt{15}}{4}\approx 6.78$ m / s. Whether the distance between them is increasing or decreasing depends on the direction in which the runner is running.

38. 

The hour hand of a clock goes around once every 12 hours or, in radians per hour, $\frac{2\pi}{12}=\frac{\pi}{6}$ rad / h. The minute hand goes around once an hour, or at the rate of 2π rad / h. So the angle θ between them (measuring clockwise from the minute hand to the hour hand) is changing at the rate of $d\theta/dt=\frac{\pi}{6}-2\pi=-\frac{11\pi}{6}$ rad / h. Now, to relate θ to ℓ, we use the Law of Cosines: $\ell^2=4^2+8^2-2\cdot 4\cdot 8\cdot\cos\theta=80-64\cos\theta$ (*).

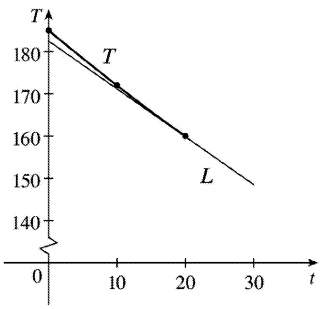
Differentiating implicitly with respect to t, we get $2\ell\frac{d\ell}{dt}=-64(-\sin\theta)\frac{d\theta}{dt}$. At 1:00, the angle between the two hands is one–twelfth of the circle, that is, $\frac{2\pi}{12}=\frac{\pi}{6}$ radians. We use (*) to find ℓ at 1:00: $\ell=\sqrt{80-64\cos\frac{\pi}{6}}=\sqrt{80-32\sqrt{3}}$. Substituting, we get $2\ell\frac{d\ell}{dt}=64\sin\frac{\pi}{6}\left(-\frac{11\pi}{6}\right)\Rightarrow$

$$\frac{d\ell}{dt}=\frac{64\left(\frac{1}{2}\right)\left(-\frac{11\pi}{6}\right)}{2\sqrt{80-32\sqrt{3}}}=-\frac{88\pi}{3\sqrt{80-32\sqrt{3}}}\approx-18.6$$

So at 1:00, the distance between the tips of the hands is decreasing at a rate of 18.6 mm / h $\approx$ 0.005 mm / s.

1. As in Example 1, $T(0)=185$, $T(10)=172$, $T(20)=160$, and
$T^{/}(20)\approx \frac{T(10)-T(20)}{10-20}=\frac{172-160}{-10}=-1.2^{\circ}$ F / min. $T(30)\approx T(20)+T^{/}(20)(30-20)\approx 160-1.2(10)=148^{\circ}$ F.

We would expect the temperature of the turkey to get closer to 75° F as time increases. Since the temperature decreased 13° F in the first 10 minutes and 12° F in the second 10 minutes, we can assume that the slopes of the tangent line are increasing through negative values: $-1.3,-1.2,\ldots$. Hence, the tangent lines are under the curve and 148° F



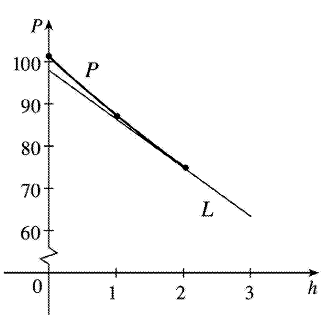
is an underestimate. From the figure, we estimate the slope of the tangent line at $t=20$ to be $\frac{184-147}{0-30}=-\frac{37}{30}$.

Then the linear approximation becomes $T(30)\approx T(20)+T^{/}(20)\cdot 10\approx 160-\frac{37}{30}(10)=147\frac{2}{3}\approx 147.7$.

2. $P^{/}(2)\approx \frac{P(1)-P(2)}{1-2}=\frac{87.1-74.9}{-1}=-12.2$ kilopascals / km.

$P(3)\approx P(2)+P^{/}(2)(3-2)\approx 74.9-12.2(1)=62.7$ kPa.

From the figure, we estimate the slope of the tangent line at $h=2$ to be $\frac{98-63}{0-3}=-\frac{35}{3}$. Then the linear approximation becomes $P(3)\approx P(2)+P^{/}(2)\cdot 1\approx 74.9-\frac{35}{3}\approx 63.23$ kPa.



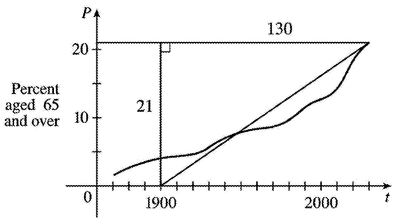
3. Extend the tangent line at the point (2030,21) to the t –axis. Answers will vary based on this approximation—we'll use $t=1900$ as our t –intercept. The linearization is then

$P(t) \approx P(2030)+P^{/}(2030)(t-2030)$

$\approx 21+\frac{21}{130}(t-2030)$

$P(2040)=21+\frac{21}{130}(2040-2030)\approx 22.6\%$

$P(2050)=21+\frac{21}{130}(2050-2030)\approx 24.2\%$



These predictions are probably too high since the tangent line lies above the graph at $t=2030$.

4. Let $A=\frac{N(1980)-N(1985)}{1980-1985}=\frac{15.0-17.0}{-5}=0.4$ and $B=\frac{N(1990)-N(1985)}{1990-1985}=\frac{19.3-17.0}{5}=0.46$. Then $N'(1985)=\lim_{t\to 1985}\frac{N(t)-N(1985)}{t-1985}\approx\frac{A+B}{2}=0.43$ million / year. So

$N(1984)\approx N(1985)+N'(1985)(1984-1985)\approx 17.0+0.43(-1)=16.57$ million.

$N'(2000)\approx\frac{N(1995)-N(2000)}{1995-2000}=\frac{22.0-24.9}{-5}=0.58$ million / year.

$N(2006)\approx N(2000)+N'(2000)(2006-2000)\approx 24.9+0.58(6)=28.38$ million.

5. $f(x)=x^3\Rightarrow f'(x)=3x^2$, so $f(1)=1$ and $f'(1)=3$. With $a=1$, $L(x)=f(a)+f'(a)(x-a)$ becomes $L(x)=f(1)+f'(1)(x-1)=1+3(x-1)=3x-2$.

6. $f(x)=\ln x\Rightarrow f'(x)=1/x$, so $f(1)=0$ and $f'(1)=1$. Thus, $L(x)=f(1)+f'(1)(x-1)=0+1(x-1)=x-1$.

7. $f(x)=\cos x\Rightarrow f'(x)=-\sin x$, so $f\left(\frac{\pi}{2}\right)=0$ and $f'\left(\frac{\pi}{2}\right)=-1$. Thus,

$L(x)=f\left(\frac{\pi}{2}\right)+f'\left(\frac{\pi}{2}\right)\left(x-\frac{\pi}{2}\right)=0-1\left(x-\frac{\pi}{2}\right)=-x+\frac{\pi}{2}$.

8. $f(x)=\sqrt[3]{x}=x^{1/3}\Rightarrow f'(x)=\frac{1}{3}x^{-2/3}$, so $f(-8)=-2$ and $f'(-8)=\frac{1}{12}$. Thus,

$L(x)=f(-8)+f'(-8)(x+8)=-2+\frac{1}{12}(x+8)=\frac{1}{12}x-\frac{4}{3}$.

9. $f(x)=\sqrt{1-x}\Rightarrow$

$f'(x)=\dfrac{-1}{2\sqrt{1-x}}$, so $f(0)=1$ and $f'(0)=-\dfrac{1}{2}$. Therefore,

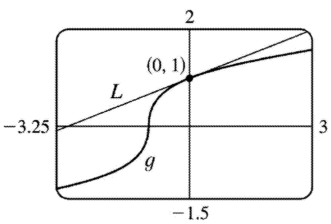
$$\begin{aligned}\sqrt{1-x}=f(x)&\approx f(0)+f'(0)(x-0)\\ &= 1+\left(-\frac{1}{2}\right)(x-0)=1-\frac{1}{2}x\end{aligned}$$

So $\sqrt{0.9}=\sqrt{1-0.1}\approx 1-\frac{1}{2}(0.1)=0.95$ and $\sqrt{0.99}=\sqrt{1-0.01}\approx 1-\frac{1}{2}(0.01)=0.995$.

10. $g(x)=\sqrt[3]{1+x}=(1+x)^{1/3}\Rightarrow g'(x)=\frac{1}{3}(1+x)^{-2/3}$,

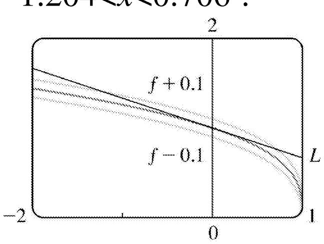
so $g(0)=1$ and $g'(0)=\frac{1}{3}$. Therefore, $\sqrt[3]{1+x}=g(x)\approx g(0)+g'(0)(x-0)=1+\frac{1}{3}x$. So

$\sqrt[3]{0.95}=\sqrt[3]{1+(-0.05)}\approx 1+\frac{1}{3}(-0.05)=0.98\overline{3}$, and $\sqrt[3]{1.1}=\sqrt[3]{1+0.1}\approx 1+\frac{1}{3}(0.1)=1.0\overline{3}$.

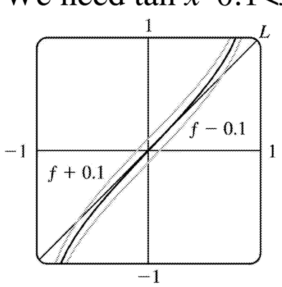


11. $f(x)=\sqrt[3]{1-x}=(1-x)^{1/3}\Rightarrow f'(x)=-\frac{1}{3}(1-x)^{-2/3}$, so $f(0)=1$ and $f'(0)=-\frac{1}{3}$. Thus,

$f(x)\approx f(0)+f'(0)(x-0)=1-\frac{1}{3}x$. We need $\sqrt[3]{1-x}-0.1<1-\frac{1}{3}x<\sqrt[3]{1-x}+0.1$, which is true when $-1.204<x<0.706$.

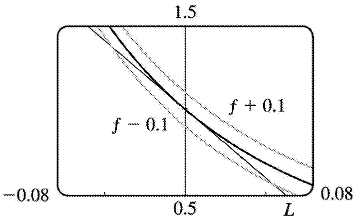


12. $f(x)=\tan x\Rightarrow f'(x)=\sec^2 x$, so $f(0)=0$ and $f'(0)=1$. Thus, $f(x)\approx f(0)+f'(0)(x-0)=0+1(x-0)=x$. We need $\tan x-0.1<x<\tan x+0.1$, which is true when $-0.63<x<0.63$.



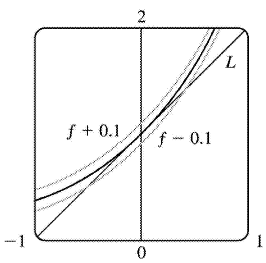
13. $f(x)=\dfrac{1}{(1+2x)^4}=(1+2x)^{-4}\Rightarrow f'(x)=-4(1+2x)^{-5}(2)=\dfrac{-8}{(1+2x)^5}$, so $f(0)=1$ and $f'(0)=-8$. Thus, $f(x)\approx f(0)+f'(0)(x-0)=1+(-8)(x-0)=1-8x$.

We need $1/(1+2x)^4-0.1<1-8x<1/(1+2x)^4+0.1$, which is true when $-0.045<x<0.055$.



14. $f(x)=e^x\Rightarrow f'(x)=e^x$, so $f(0)=1$ and $f'(0)=1$. Thus, $f(x)\approx f(0)+f'(0)(x-0)=1+1(x-0)=1+x$.

We need $e^x-0.1<1+x<e^x+0.1$, which is true when $-0.483<x<0.416$.



15. If $y=f(x)$, then the differential dy is equal to $f'(x)dx$. $y=x^4+5x\Rightarrow dy=(4x^3+5)dx$.

16. $y=\cos\pi x\Rightarrow dy=-\sin\pi x\cdot\pi\,dx=-\pi\sin\pi x\,dx$

17. $y=x\ln x\Rightarrow dy=\left(x\cdot\dfrac{1}{x}+\ln x\cdot 1\right)dx=(1+\ln x)dx$

18. $y=\sqrt{1+t^2}\Rightarrow dy=\dfrac{1}{2}(1+t^2)^{-1/2}(2t)dt=\dfrac{t}{\sqrt{1+t^2}}\,dt$

19. $y=\dfrac{u+1}{u-1}\Rightarrow dy=\dfrac{(u-1)(1)-(u+1)(1)}{(u-1)^2}\,du=\dfrac{-2}{(u-1)^2}\,du$

20. $y=(1+2r)^{-4}\Rightarrow dy=-4(1+2r)^{-5}\cdot 2\,dr=-8(1+2r)^{-5}dr$

21. **(a)** $y=x^2+2x\Rightarrow dy=(2x+2)dx$

(b) When $x=3$ and

$dx=\frac{1}{2}$, $dy=[2(3)+2]\left(\frac{1}{2}\right)=4$.

22. **(a)** $y=e^{x/4} \Rightarrow dy=\frac{1}{4}e^{x/4}dx$

(b) When $x=0$ and $dx=0.1$, $dy=\left(\frac{1}{4}e^{0}\right)(0.1)=0.025$.

23. **(a)** $y=\sqrt{4+5x} \Rightarrow dy=\frac{1}{2}(4+5x)^{-1/2}\cdot 5dx=\frac{5}{2\sqrt{4+5x}}dx$

(b) When $x=0$ and $dx=0.04$, $dy=\frac{5}{2\sqrt{4}}(0.04)=\frac{5}{4}\cdot\frac{1}{25}=\frac{1}{20}=0.05$.

24. **(a)** $y=1/(x+1) \Rightarrow dy=-\frac{1}{(x+1)^2}dx$

(b) When $x=1$ and $dx=-0.01$, $dy=-\frac{1}{2^2}(-0.01)=\frac{1}{4}\cdot\frac{1}{100}=\frac{1}{400}=0.0025$.

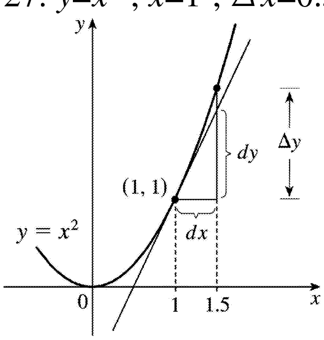
25. **(a)** $y=\tan x \Rightarrow dy=\sec^2 x\,dx$

(b) When $x=\pi/4$ and $dx=-0.1$, $dy=[\sec(\pi/4)]^2(-0.1)=(\sqrt{2})^2(-0.1)=-0.2$.

26. **(a)** $y=\cos x \Rightarrow dy=-\sin x\,dx$

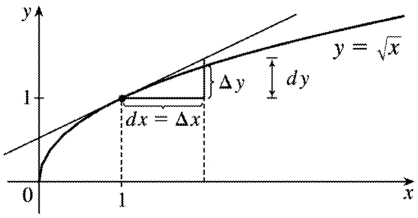
(b) When $x=\pi/3$ and $dx=0.05$, $dy=-\sin(\pi/3)(0.05)=-0.5\sqrt{3}(0.05)=-0.025\sqrt{3}\approx-0.043$.

27. $y=x^2$, $x=1$, $\Delta x=0.5 \Rightarrow \Delta y=(1.5)^2-1^2=1.25$. $dy=2x\,dx=2(1)(0.5)=1$

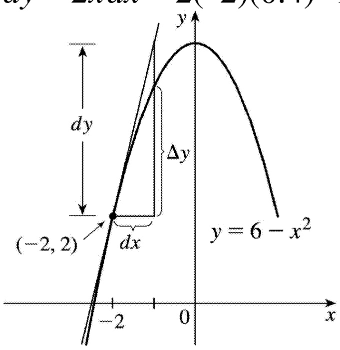


28. $y=\sqrt{x}$, $x=1$, $\Delta x=1 \Rightarrow \Delta y=\sqrt{2}-\sqrt{1}=\sqrt{2}-1\approx 0.414$

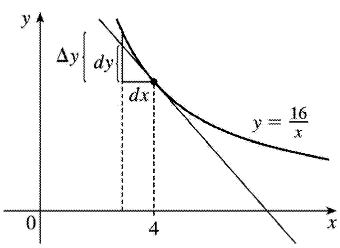
$dy=\frac{1}{2\sqrt{x}}dx=\frac{1}{2}(1)=0.5$



29. $y=6-x^2$, $x=-2$, $\Delta x=0.4 \Rightarrow \Delta y=(6-(-1.6)^2)-(6-(-2)^2)=1.44$
$dy=-2x\,dx=-2(-2)(0.4)=1.6$



30. $y=\frac{16}{x}$, $x=4$, $\Delta x=-1 \Rightarrow \Delta y=\frac{16}{3}-\frac{16}{4}=\frac{4}{3}$. $dy=-\left(\frac{16}{x^2}\right)dx=-\left(\frac{16}{4^2}\right)(-1)=1$



31. $y=f(x)=x^5 \Rightarrow dy=5x^4\,dx$. When $x=2$ and $dx=0.001$, $dy=5(2)^4(0.001)=0.08$, so
$(2.001)^5=f(2.001)\approx f(2)+dy=32+0.08=32.08$.

32. $y=f(x)=\sqrt{x} \Rightarrow dy=\frac{1}{2\sqrt{x}}\,dx$. When $x=100$ and $dx=-0.2$, $dy=\frac{1}{2\sqrt{100}}(-0.2)=-0.01$, so
$\sqrt{99.8}=f(99.8)\approx f(100)+dy=10-0.01=9.99$.

33. $y=f(x)=x^{2/3} \Rightarrow dy=\frac{2}{3\sqrt[3]{x}}\,dx$. When $x=8$ and $dx=0.06$, $dy=\frac{2}{3\sqrt[3]{8}}(0.06)=0.02$, so
$(8.06)^{2/3}=f(8.06)\approx f(8)+dy=4+0.02=4.02$.

34. $y=f(x)=1/x \Rightarrow dy=(-1/x^2)dx$. When $x=1000$ and $dx=2, dy=[-1/(1000)^2](2)=-0.000002$, so
$1/1002=f(1002)\approx f(1000)+dy=1/1000-0.000002=0.000998$

35. $y=f(x)=\tan x \Rightarrow dy=\sec^2 x\,dx$. When $x=45^\circ$ and $dx=-1^\circ$, $dy=\sec^2 45^\circ(-\pi/180)=(\sqrt{2})^2(-\pi/180)=-\pi/90$, so $\tan 44^\circ=f(44^\circ)\approx f(45^\circ)+dy=1-\pi/90\approx 0.965$.

36. $y=f(x)=\ln x \Rightarrow dy=\frac{1}{x}\,dx$. When $x=1$ and $dx=0.07$, $dy=\frac{1}{1}(0.07)=0.07$, so $\ln 1.07=f(1.07)\approx f(1)+dy=0+0.07=0.07$.

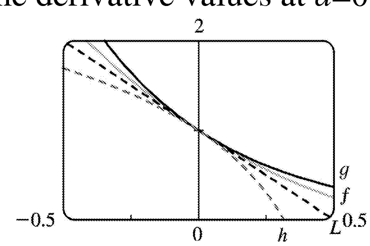
37. $y=f(x)=\sec x \Rightarrow f'(x)=\sec x\tan x$, so $f(0)=1$ and $f'(0)=1\cdot 0=0$. The linear approximation of f at 0 is $f(0)+f'(0)(x-0)=1+0(x)=1$. Since 0.08 is close to 0 , approximating sec 0.08 with 1 is reasonable.

38. If $y=x^6$, $y'=6x^5$ and the tangent line approximation at (1,1) has slope 6 . If the change in x is 0.01 , the change in y on the tangent line is 0.06 , and approximating $(1.01)^6$ with 1.06 is reasonable.

39. $y=f(x)=\ln x \Rightarrow f'(x)=1/x$, so $f(1)=0$ and $f'(1)=1$. The linear approximation of f at 1 is $f(1)+f'(1)(x-1)=0+1(x-1)=x-1$. Now $f(1.05)=\ln 1.05\approx 1.05-1=0.05$, so the approximation is reasonable.

40. **(a)** $f(x)=(x-1)^2 \Rightarrow f'(x)=2(x-1)$, so $f(0)=1$ and $f'(0)=-2$.
Thus, $f(x)\approx L_f(x)=f(0)+f'(0)(x-0)=1-2x$.
$g(x)=e^{-2x} \Rightarrow g'(x)=-2e^{-2x}$, so $g(0)=1$ and $g'(0)=-2$.
Thus, $g(x)\approx L_g(x)=g(0)+g'(0)(x-0)=1-2x$.
$h(x)=1+\ln(1-2x) \Rightarrow h'(x)=\frac{-2}{1-2x}$, so $h(0)=1$ and $h'(0)=-2$.
Thus, $h(x)\approx L_h(x)=h(0)+h'(0)(x-0)=1-2x$.
Notice that $L_f=L_g=L_h$. This happens because f , g , and h have the same function values and the same derivative values at $a=0$.

(b)



The linear approximation appears to be the best for the function f since it is closer to f for a larger domain than it is to g and h . The approximation looks worst for h since h moves away from L faster than f and g do.

41. **(a)** If x is the edge length, then $V=x^3 \Rightarrow dV=3x^2 dx$. When $x=30$ and $dx=0.1$, $dV=3(30)^2(0.1)=270$, so the maximum possible error in computing the volume of the cube is about 270 cm^3 . The relative error is calculated by dividing the change in V , ΔV , by V . We approximate ΔV with dV .

Relative error = $\dfrac{\Delta V}{V} \approx \dfrac{dV}{V} = \dfrac{3x^2 dx}{x^3} = 3\dfrac{dx}{x} = 3\left(\dfrac{0.1}{30}\right) = 0.01$.

Percentage error = relative error$\times$ 100%=0.01$\times$ 100%=1% .

(b) $S=6x^2 \Rightarrow dS=12x\,dx$. When $x=30$ and $dx=0.1$, $dS=12(30)(0.1)=36$, so the maximum possible error in computing the surface area of the cube is about 36 cm^2 .

Relative error = $\dfrac{\Delta S}{S} \approx \dfrac{dS}{S} = \dfrac{12x\,dx}{6x^2} = 2\dfrac{dx}{x} = 2\left(\dfrac{0.1}{30}\right) = 0.00\overline{6}$.

Percentage error = relative error$\times$ 100%=$0.00\overline{6}\times$ 100%=$0.\overline{6}$% .

42. **(a)** $A=\pi r^2 \Rightarrow dA=2\pi r\,dr$. When $r=24$ and $dr=0.2$, $dA=2\pi(24)(0.2)=9.6\pi$, so the maximum possible error in the calculated area of the disk is about $9.6\pi \approx 30$ cm^2 .

(b) Relative error = $\dfrac{\Delta A}{A} \approx \dfrac{dA}{A} = \dfrac{2\pi r\,dr}{\pi r^2} = \dfrac{2dr}{r} = \dfrac{2(0.2)}{24} = \dfrac{0.2}{12} = \dfrac{1}{60} = 0.01\overline{6}$.

Percentage error = relative error$\times$ 100%=$0.01\overline{6}\times$ 100%=$1.\overline{6}$% .

43. **(a)** For a sphere of radius r , the circumference is $C=2\pi r$ and the surface area is $S=4\pi r^2$, so $r=C/(2\pi) \Rightarrow S=4\pi(C/2\pi)^2=C^2/\pi \Rightarrow dS=(2/\pi)C\,dC$. When $C=84$ and $dC=0.5$, $dS=\dfrac{2}{\pi}(84)(0.5)=\dfrac{84}{\pi}$, so the maximum error is about $\dfrac{84}{\pi} \approx 27$ cm^2 . Relative error $\approx \dfrac{dS}{S} = \dfrac{84/\pi}{84^2/\pi} = \dfrac{1}{84} \approx 0.012$

(b) $V=\dfrac{4}{3}\pi r^3=\dfrac{4}{3}\pi\left(\dfrac{C}{2\pi}\right)^3=\dfrac{C^3}{6\pi^2} \Rightarrow dV=\dfrac{1}{2\pi^2}C^2\,dC$. When $C=84$ and $dC=0.5$, $dV=\dfrac{1}{2\pi^2}(84)^2(0.5)=\dfrac{1764}{\pi^2}$, so the maximum error is about $\dfrac{1764}{\pi^2} \approx 179$ cm^3 . The relative error is approximately $\dfrac{dV}{V} = \dfrac{1764/\pi^2}{(84)^3/(6\pi^2)} = \dfrac{1}{56} \approx 0.018$.

44. For a hemispherical dome,

$V=\frac{2}{3}\pi r^3 \Rightarrow dV=2\pi r^2 dr$. When $r=\frac{1}{2}(50)=25$ m and $dr=0.05$ cm $=0.0005$ m, $dV=2\pi(25)^2(0.0005)=\frac{5\pi}{8}$, so the amount of paint needed is about $\frac{5\pi}{8}\approx 2$ m ^{3} .

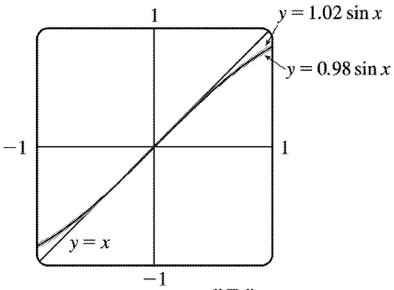
45. **(a)** $V=\pi r^2 h \Rightarrow \Delta V \approx dV=2\pi rh\,dr=2\pi rh\,\Delta r$
(b) The error is

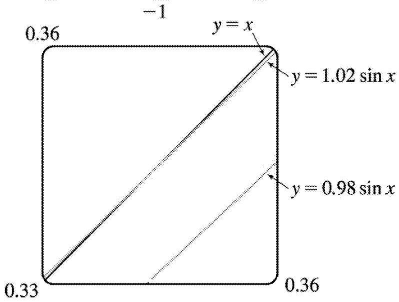
$$\begin{aligned}\Delta V-dV &=[\pi(r+\Delta r)^2 h-\pi r^2 h]-2\pi rh\,\Delta r=\pi r^2 h+2\pi rh\,\Delta r+\pi(\Delta r)^2 h-\pi r^2 h-2\pi rh\Delta r\\ &=\pi(\Delta r)^2 h\end{aligned}$$

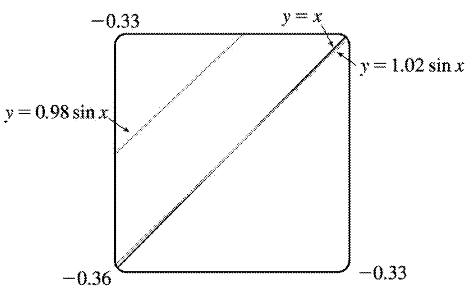
46. $F=kR^4 \Rightarrow dF=4kR^3\,dR \Rightarrow \frac{dF}{F}=\frac{4kR^3\,dR}{kR^4}=4\left(\frac{dR}{R}\right)$. Thus, the relative change in F is about 4 times the relative change in R . So a 5% increase in the radius corresponds to a 20% increase in blood flow.

47. **(a)** $dc=\frac{dc}{dx}\,dx=0\,dx=0$
(b) $d(cu)=\frac{d}{dx}(cu)\,dx=c\frac{du}{dx}\,dx=c\,du$
(c) $d(u+v)=\frac{d}{dx}(u+v)\,dx=\left(\frac{du}{dx}+\frac{dv}{dx}\right)dx=\frac{du}{dx}\,dx+\frac{dv}{dx}\,dx=du+dv$
(d) $d(uv)=\frac{d}{dx}(uv)\,dx=\left(u\frac{dv}{dx}+v\frac{du}{dx}\right)dx=u\frac{dv}{dx}\,dx+v\frac{du}{dx}\,dx=u\,dv+v\,du$
(e) $d\left(\frac{u}{v}\right)=\frac{d}{dx}\left(\frac{u}{v}\right)dx=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}\,dx=\frac{v\frac{du}{dx}\,dx-u\frac{dv}{dx}\,dx}{v^2}=\frac{v\,du-u\,dv}{v^2}$
(f) $d(x^n)=\frac{d}{dx}(x^n)dx=nx^{n-1}\,dx$

48. **(a)** $f(x)=\sin x \Rightarrow f'(x)=\cos x$, so $f(0)=0$ and $f'(0)=1$. Thus, $f(x)\approx f(0)+f'(0)(x-0)=0+1(x-0)=x$.
(b)







We want to know the values of x for which $y=x$ approximates $y=\sin x$ with less than a 2% difference; that is, the values of x for which

$$\left| \frac{x-\sin x}{\sin x} \right| <0.02 \Leftrightarrow -0.02< \frac{x-\sin x}{\sin x} <0.02 \Leftrightarrow$$

$$\begin{cases} -0.02\sin x<x-\sin x<0.02\sin x & \text{if } \sin x>0 \\ -0.02\sin x>x-\sin x>0.02\sin x & \text{if } \sin x<0 \end{cases} \Leftrightarrow \begin{cases} 0.98\sin x<x<1.02\sin x & \text{if } \sin x>0 \\ 1.02\sin x<x<0.98\sin x & \text{if } \sin x<0 \end{cases}$$

In the first figure, we see that the graphs are very close to each other near $x=0$. Changing the viewing rectangle and using an intersect feature (see the second figure) we find that $y=x$ intersects $y=1.02\sin x$ at $x\approx 0.344$. By symmetry, they also intersect at $x\approx -0.344$ (see the third figure.). Converting 0.344 radians to degrees, we get $0.344 \left(\frac{180^\circ}{\pi} \right) \approx 19.7^\circ \approx 20^\circ$, which verifies the statement.

49. **(a)** The graph shows that $f'(1)=2$, so $L(x)=f(1)+f'(1)(x-1)=5+2(x-1)=2x+3$. $f(0.9)\approx L(0.9)=4.8$ and $f(1.1)\approx L(1.1)=5.2$.

(b) From the graph, we see that $f'(x)$ is positive and decreasing. This means that the slopes of the tangent lines are positive, but the tangents are becoming less steep. So the tangent lines lie *above* the curve. Thus, the estimates in part (a) are too large.

50. **(a)** $g'(x)=\sqrt{x^2+5} \Rightarrow g'(2)=\sqrt{9}=3$. $g(1.95) \approx g(2)+g'(2)(1.95-2)=-4+3(-0.05)=-4.15$. $g(2.05) \approx g(2)+g'(2)(2.05-2)=-4+3(0.05)=-3.85$.

(b) The formula $g'(x)=\sqrt{x^2+5}$ shows that $g'(x)$ is positive and increasing. This means that the slopes of the tangent lines are positive and the tangents are getting steeper. So the tangent lines lie *below* the graph of g . Hence, the estimates in part (a) are too small.

1. Product Rule:
$y=(x^2+1)(x^3+1)\Rightarrow$

$y'=(x^2+1)(3x^2)+(x^3+1)(2x)=3x^4+3x^2+2x^4+2x=5x^4+3x^2+2x$.
Multiplying first: $y=(x^2+1)(x^3+1)=x^5+x^3+x^2+1\Rightarrow y'=5x^4+3x^2+2x$ (equivalent).

2. Quotient Rule: $F(x)=\dfrac{x-3x\sqrt{x}}{\sqrt{x}}=\dfrac{x-3x^{3/2}}{x^{1/2}}\Rightarrow$

$$F'(x)=\frac{x^{1/2}\left(1-\frac{9}{2}x^{1/2}\right)-\left(x-3x^{3/2}\right)\left(\frac{1}{2}x^{-1/2}\right)}{\left(x^{1/2}\right)^2}$$

$$=\frac{x^{1/2}-\frac{9}{2}x-\frac{1}{2}x^{1/2}+\frac{3}{2}x}{x}=\frac{\frac{1}{2}x^{1/2}-3x}{x}=\frac{1}{2}x^{-1/2}-3$$

Simplifying first: $F(x)=\dfrac{x-3x\sqrt{x}}{\sqrt{x}}=\sqrt{x}-3x=x^{1/2}-3x\Rightarrow F'(x)=\dfrac{1}{2}x^{-1/2}-3$ (equivalent).
For this problem, simplifying first seems to be the better method.

3. By the Product Rule, $f(x)=x^2e^x\Rightarrow f'(x)=x^2\dfrac{d}{dx}(e^x)+e^x\dfrac{d}{dx}(x^2)=x^2e^x+e^x(2x)=xe^x(x+2)$.

4. By the Product Rule, $g(x)=\sqrt{x}\,e^x=x^{1/2}e^x\Rightarrow g'(x)=x^{1/2}(e^x)+e^x\left(\dfrac{1}{2}x^{-1/2}\right)=\dfrac{1}{2}x^{-1/2}e^x(2x+1)$.

5. By the Quotient Rule, $y=\dfrac{e^x}{x^2}\Rightarrow y'=\dfrac{x^2\frac{d}{dx}(e^x)-e^x\frac{d}{dx}(x^2)}{(x^2)^2}=\dfrac{x^2(e^x)-e^x(2x)}{x^4}=\dfrac{xe^x(x-2)}{x^4}=\dfrac{e^x(x-2)}{x^3}$

6. By the Quotient Rule, $y=\dfrac{e^x}{1+x}\Rightarrow y'=\dfrac{(1+x)e^x-e^x(1)}{(1+x)^2}=\dfrac{e^x+xe^x-e^x}{(x+1)^2}=\dfrac{xe^x}{(x+1)^2}$.

7. $g(x)=\dfrac{3x-1}{2x+1}\overset{QR}{\Rightarrow}g'(x)=\dfrac{(2x+1)(3)-(3x-1)(2)}{(2x+1)^2}=\dfrac{6x+3-6x+2}{(2x+1)^2}=\dfrac{5}{(2x+1)^2}$

8. $f(t)=\dfrac{2t}{4+t^2}\overset{QR}{\Rightarrow}f'(t)=\dfrac{(4+t^2)(2)-(2t)(2t)}{(4+t^2)^2}=\dfrac{8+2t^2-4t^2}{(4+t^2)^2}=\dfrac{8-2t^2}{(4+t^2)^2}$

9. $V(x)=(2x^3+3)(x^4-2x) \overset{\text{PR}}{\Rightarrow}$

$V'(x)=(2x^3+3)(4x^3-2)+(x^4-2x)(6x^2)=(8x^6+8x^3-6)+(6x^6-12x^3)=14x^6-4x^3-6$

10. $Y(u)=(u^{-2}+u^{-3})(u^5-2u^2)\overset{\text{PR}}{\Rightarrow}$

$Y'(u)=(u^{-2}+u^{-3})(5u^4-4u)+(u^5-2u^2)(-2u^{-3}-3u^{-4})$

$=(5u^2-4u^{-1}+5u-4u^{-2})+(-2u^2-3u+4u^{-1}+6u^{-2})=3u^2+2u+2u^{-2}$

11. $F(y)=\left(\frac{1}{y^2}-\frac{3}{y^4}\right)(y+5y^3)=\left(y^{-2}-3y^{-4}\right)\left(y+5y^3\right)\overset{\text{PR}}{\Rightarrow}$

$F'(y)=\left(y^{-2}-3y^{-4}\right)\left(1+15y^2\right)+\left(y+5y^3\right)\left(-2y^{-3}+12y^{-5}\right)$

$=\left(y^{-2}+15-3y^{-4}-45y^{-2}\right)+(-2y^{-2}+12y^{-4}-10+60y^{-2})$

$=5+14y^{-2}+9y^{-4}$ or $5+14/y^2+9/y^4$

12. $R(t)=(t+e^t)(3-\sqrt{t})=$

$R'(t)=(t+e^t)(-\frac{1}{2}t^{-1/2})+(3-\sqrt{t})(1+e^t)$

$=\left(-\frac{1}{2}t^{1/2}-\frac{1}{2}t^{-1/2}e^t\right)+(3+3e^t-\sqrt{t}-\sqrt{t}\,e^t)=3+3e^t-\frac{3}{2}\sqrt{t}-\sqrt{t}\,e^t-e^t/(2\sqrt{t})$

13. $y=\dfrac{t^2}{3t^2-2t+1}\overset{\text{QR}}{\Rightarrow}$

$y'=\dfrac{(3t^2-2t+1)(2t)-t^2(6t-2)}{(3t^2-2t+1)^2}=\dfrac{2t[3t^2-2t+1-t(3t-1)]}{(3t^2-2t+1)^2}$

$=\dfrac{2t(3t^2-2t+1-3t^2+t)}{(3t^2-2t+1)^2}=\dfrac{2t(1-t)}{(3t^2-2t+1)^2}$

14. $y=\dfrac{t^3+t}{t^4-2}\overset{\text{QR}}{\Rightarrow}y'=\dfrac{(t^4-2)(3t^2+1)-(t^3+t)(4t^3)}{(t^4-2)^2}=\dfrac{(3t^6+t^4-6t^2-2)-(4t^6+4t^4)}{(t^4-2)^2}$

$=\dfrac{-t^6-3t^4-6t^2-2}{(t^4-2)^2}=-\dfrac{t^6+3t^4+6t^2+2}{(t^4-2)^2}$

15. $y=(r^2-2r)e^r = y'=(r^2-2r)(e^r)+e^r(2r-2)=e^r(r^2-2r+2r-2)=e^r(r^2-2)$

16. $y=\dfrac{1}{s+ke^s} = y' = \dfrac{(s+ke^s)(0)-(1)(1+ke^s)}{(s+ke^s)^2} = -\dfrac{1+ke^s}{(s+ke^s)^2}$

17. $y=\dfrac{v^3-2v\sqrt{v}}{v}=v^2-2\sqrt{v}=v^2-2v^{1/2}\Rightarrow y'=2v-2\left(\dfrac{1}{2}\right)v^{-1/2}=2v-v^{-1/2}$.

We can change the form of the answer as follows: $2v-v^{-1/2}=2v-\dfrac{1}{\sqrt{v}}=\dfrac{2v\sqrt{v}-1}{\sqrt{v}}=\dfrac{2v^{3/2}-1}{\sqrt{v}}$

18. $z=w^{3/2}(w+ce^w)=w^{5/2}+cw^{3/2}e^w \Rightarrow z'=\dfrac{5}{2}w^{3/2}+c\left(w^{3/2}\cdot e^w+e^w\cdot\dfrac{3}{2}w^{1/2}\right)=\dfrac{5}{2}w^{3/2}+\dfrac{1}{2}cw^{1/2}e^w(2w+3)$

19. $y=\dfrac{1}{x^4+x^2+1}\Rightarrow y'=\dfrac{(x^4+x^2+1)(0)-1(4x^3+2x)}{(x^4+x^2+1)^2}=-\dfrac{2x(2x^2+1)}{(x^4+x^2+1)^2}$

20. $y=\dfrac{\sqrt{x}-1}{\sqrt{x}+1}\Rightarrow y'=\dfrac{(\sqrt{x}+1)\left(\dfrac{1}{2\sqrt{x}}\right)-(\sqrt{x}-1)\left(\dfrac{1}{2\sqrt{x}}\right)}{(\sqrt{x}+1)^2}=\dfrac{\dfrac{1}{2}+\dfrac{1}{2\sqrt{x}}-\dfrac{1}{2}+\dfrac{1}{2\sqrt{x}}}{(\sqrt{x}+1)^2}=\dfrac{1}{\sqrt{x}(\sqrt{x}+1)^2}$

21. $f(x)=\dfrac{x}{x+c/x}\Rightarrow f'(x)=\dfrac{(x+c/x)(1)-x(1-c/x^2)}{\left(x+\dfrac{c}{x}\right)^2}=\dfrac{x+c/x-x+c/x}{\left(\dfrac{x^2+c}{x}\right)^2}=\dfrac{2c/x}{\dfrac{(x^2+c)^2}{x^2}}\cdot\dfrac{x^2}{x^2}=\dfrac{2cx}{(x^2+c)^2}$

22. $f(x)=\dfrac{ax+b}{cx+d}\Rightarrow f'(x)=\dfrac{(cx+d)(a)-(ax+b)(c)}{(cx+d)^2}=\dfrac{acx+ad-acx-bc}{(cx+d)^2}=\dfrac{ad-bc}{(cx+d)^2}$

23. $y=\dfrac{2x}{x+1}\Rightarrow y'=\dfrac{(x+1)(2)-(2x)(1)}{(x+1)^2}=\dfrac{2}{(x+1)^2}$. At $(1,1)$, $y'=\dfrac{1}{2}$, and an equation of the tangent line is $y-1=\dfrac{1}{2}(x-1)$, or $y=\dfrac{1}{2}x+\dfrac{1}{2}$.

24. $y=\dfrac{\sqrt{x}}{x+1}\Rightarrow$

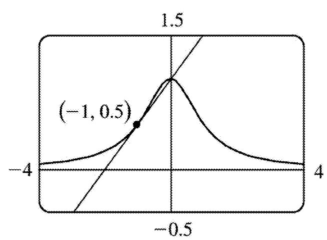
$y' = \dfrac{(x+1)\left(\dfrac{1}{2\sqrt{x}}\right) - \sqrt{x}\,(1)}{(x+1)^2} = \dfrac{(x+1)-(2x)}{2\sqrt{x}\,(x+1)^2} = \dfrac{1-x}{2\sqrt{x}\,(x+1)^2}$. At (4,0.4) , $y' = \dfrac{-3}{100} = -0.03$, and an equation of the tangent line is $y-0.4=-0.03(x-4)$, or $y=-0.03x+0.52$.

25. $y=2xe^x \Rightarrow y' = 2(x\cdot e^x + e^x\cdot 1) = 2e^x(x+1)$. At (0, 0) , $y' = 2e^0(0+1) = 2\cdot 1\cdot 1 = 2$, and an equation of the tangent line is $y-0=2(x-0)$, or $y=2x$.

26. $y=\dfrac{e^x}{x} \Rightarrow y' = \dfrac{x\cdot e^x - e^x\cdot 1}{x^2} = \dfrac{e^x(x-1)}{x^2}$. At $(1, e)$, $y'=0$, and an equation of the tangent line is $y-e=0(x-1)$, or $y=e$.

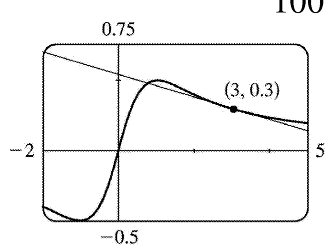
27. **(a)** $y=f(x)=\dfrac{1}{1+x^2} \Rightarrow f'(x)=\dfrac{(1+x^2)(0)-1(2x)}{(1+x^2)^2} = \dfrac{-2x}{(1+x^2)^2}$. So the slope of the tangent line at the point $\left(-1, \dfrac{1}{2}\right)$ is $f'(-1)=\dfrac{2}{2^2}=\dfrac{1}{2}$ and its equation is $y-\dfrac{1}{2}=\dfrac{1}{2}(x+1)$ or $y=\dfrac{1}{2}x+1$.

(b)

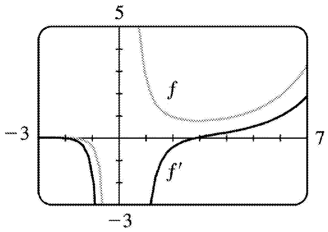


28. **(a)** $y=f(x)=\dfrac{x}{1+x^2} \Rightarrow f'(x)=\dfrac{(1+x^2)1-x(2x)}{(1+x^2)^2} = \dfrac{1-x^2}{(1+x^2)^2}$. So the slope of the tangent line at the point (3,0.3) is $f'(3)=\dfrac{-8}{100}$ and its equation is $y-0.3=-0.08(x-3)$ or $y=-0.08x+0.54$.

(b)

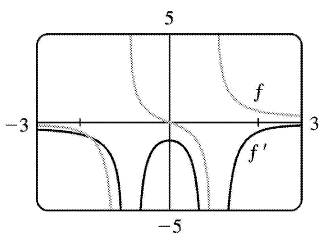


29. **(a)** $f(x)=\dfrac{e^x}{x^3} \Rightarrow f'(x)=\dfrac{x^3(e^x)-e^x(3x^2)}{(x^3)^2} = \dfrac{x^2e^x(x-3)}{x^6} = \dfrac{e^x(x-3)}{x^4}$

(b) 

$f'=0$ when f has a horizontal tangent line, f' is negative when f is decreasing, and f' is positive when f is increasing.

30. $f(x)=\dfrac{x}{x^2-1} \Rightarrow f'(x)=\dfrac{(x^2-1)1-x(2x)}{(x^2-1)^2}=\dfrac{-x^2-1}{(x^2-1)^2}=-\dfrac{x^2+1}{(x^2-1)^2}$ Notice that the slopes of all tangents to f are negative and $f'(x)<0$ always.



31. We are given that $f(5)=1$, $f'(5)=6$, $g(5)=-3$, and $g'(5)=2$.

(a) $(fg)'(5)=f(5)g'(5)+g(5)f'(5)=(1)(2)+(-3)(6)=2-18=-16$

(b) $\left(\dfrac{f}{g}\right)'(5)=\dfrac{g(5)f'(5)-f(5)g'(5)}{[g(5)]^2}=\dfrac{(-3)(6)-(1)(2)}{(-3)^2}=-\dfrac{20}{9}$

(c) $\left(\dfrac{g}{f}\right)'(5)=\dfrac{f(5)g'(5)-g(5)f'(5)}{[f(5)]^2}=\dfrac{(1)(2)-(-3)(6)}{(1)^2}=20$

32. We are given that $f(3)=4$, $g(3)=2$, $f'(3)=-6$, and $g'(3)=5$.

(a) $(f+g)'(3)=f'(3)+g'(3)=-6+5=-1$

(b) $(fg)'(3)=f(3)g'(3)+g(3)f'(3)=(4)(5)+(2)(-6)=20-12=8$

(c) $\left(\dfrac{f}{g}\right)'(3)=\dfrac{g(3)f'(3)-f(3)g'(3)}{[g(3)]^2}=\dfrac{(2)(-6)-(4)(5)}{(2)^2}=\dfrac{-32}{4}=-8$

(d)

$$\left(\frac{f}{f-g}\right)'(3)=\frac{[f(3)-g(3)]f'(3)-f(3)[f'(3)-g'(3)]}{[f(3)-g(3)]^2}$$
$$=\frac{(4-2)(-6)-4(-6-5)}{(4-2)^2}=\frac{-12+44}{2^2}=8$$

33. $f(x)=e^x g(x) \Rightarrow f'(x)=e^x g'(x)+g(x)e^x=e^x\left[g'(x)+g(x)\right]$.
$f'(0)=e^0\left[g'(0)+g(0)\right]=1(5+2)=7$

34. $\frac{d}{dx}\left[\frac{h(x)}{x}\right]=\frac{xh'(x)-h(x)\cdot 1}{x^2} \Rightarrow \frac{d}{dx}\left[\frac{h(x)}{x}\right]_{x=2}=\frac{2h'(2)-h(2)}{2^2}=\frac{2(-3)-(4)}{4}=\frac{-10}{4}=-2.5$

35. **(a)** From the graphs of f and g, we obtain the following values: $f(1)=2$ since the point $(1,2)$ is on the graph of f; $g(1)=1$ since the point $(1,1)$ is on the graph of g; $f'(1)=2$ since the slope of the line segment between $(0,0)$ and $(2,4)$ is $\frac{4-0}{2-0}=2$; $g'(1)=-1$ since the slope of the line segment between $(-2,4)$ and $(2,0)$ is $\frac{0-4}{2-(-2)}=-1$. Now $u(x)=f(x)g(x)$, so $u'(1)=f(1)g'(1)+g(1)f'(1)=2\cdot(-1)+1\cdot 2=0$.

(b) $v(x)=f(x)/g(x)$, so $v'(5)=\frac{g(5)f'(5)-f(5)g'(5)}{[g(5)]^2}=\frac{2\left(-\frac{1}{3}\right)-3\cdot\frac{2}{3}}{2^2}=\frac{-\frac{8}{3}}{4}=-\frac{2}{3}$

36. **(a)** $P(x)=F(x)G(x)$, so $P'(2)=F(2)G'(2)+G(2)F'(2)=3\cdot\frac{2}{4}+2\cdot 0=\frac{3}{2}$.

(b) $Q(x)=F(x)/G(x)$, so $Q'(7)=\frac{G(7)F'(7)-F(7)G'(7)}{[G(7)]^2}=\frac{1\cdot\frac{1}{4}-5\cdot\left(-\frac{2}{3}\right)}{1^2}=\frac{1}{4}+\frac{10}{3}=\frac{43}{12}$

37. **(a)** $y=xg(x) \Rightarrow y'=xg'(x)+g(x)\cdot 1=xg'(x)+g(x)$

(b) $y=\frac{x}{g(x)} \Rightarrow y'=\frac{g(x)\cdot 1-xg'(x)}{[g(x)]^2}=\frac{g(x)-xg'(x)}{[g(x)]^2}$

(c) $y=\frac{g(x)}{x} \Rightarrow y'=\frac{xg'(x)-g(x)\cdot 1}{(x)^2}=\frac{xg'(x)-g(x)}{x^2}$

38. **(a)**

$y=x^2 f(x) \Rightarrow y' = x^2 f'(x) + f(x)(2x)$

(b) $y = \dfrac{f(x)}{x^2} \Rightarrow y' = \dfrac{x^2 f'(x) - f(x)(2x)}{(x^2)^2} = \dfrac{xf'(x) - 2f(x)}{x^3}$

(c) $y = \dfrac{x^2}{f(x)} \Rightarrow y' = \dfrac{f(x)(2x) - x^2 f'(x)}{[f(x)]^2}$

(d) $y = \dfrac{1+xf(x)}{\sqrt{x}} \Rightarrow$

$$y' = \frac{\sqrt{x}[xf'(x)+f(x)] - [1+xf(x)]\dfrac{1}{2\sqrt{x}}}{(\sqrt{x})^2}$$

$$= \frac{x^{3/2} f'(x) + x^{1/2} f(x) - \frac{1}{2} x^{-1/2} - \frac{1}{2} x^{1/2} f(x)}{x} \cdot \frac{2x^{1/2}}{2x^{1/2}} = \frac{xf(x) + 2x^2 f'(x) - 1}{2x^{3/2}}$$

39. If $P(t)$ denotes the population at time t and $A(t)$ the average annual income, then $T(t)=P(t)A(t)$ is the total personal income. The rate at which $T(t)$ is rising is given by $T'(t) = P(t)A'(t) + A(t)P'(t) \Rightarrow$

$$T'(1999) = P(1999)A'(1999) + A(1999)P'(1999) = (961{,}400)(\$1400/\text{yr}) + (\$30{,}593)(9200/\text{yr})$$
$$= \$1{,}345{,}960{,}000/\text{yr} + \$281{,}455{,}600/\text{yr} = \$1{,}627{,}415{,}600/\text{yr}$$

So the total personal income was rising by about \$ 1.627 billion per year in 1999.
The term $P(t)A'(t) \approx \$1.346$ billion represents the portion of the rate of change of total income due to the existing population's increasing income. The term $A(t)P'(t) \approx \$281$ million represents the portion of the rate of change of total income due to increasing population.

40. **(a)** $f(20)=10,000$ means that when the price of the fabric is \$20/ yard, 10 , 000 yards will be sold.
$f'(20)=-350$ means that as the price of the fabric increases past \$20/ yard, the amount of fabric which will be sold is decreasing at a rate of 350 yards per (dollar per yard).

(b) $R(p)=pf(p) \Rightarrow R'(p) = pf'(p) + f(p)\cdot 1 \Rightarrow R'(20) = 20f'(20) + f(20)\cdot 1 = 20(-350) + 10{,}000 = 3000$.
This means that as the price of the fabric increases past \$20/ yard, the total revenue is increasing at \$3000/(\$/yard). Note that the Product Rule indicates that we will lose \$7000/(\$/yard) due to selling less fabric, but that that loss is more than made up for by the additional revenue due to the increase in price.

41. If $y=f(x)=\frac{x}{x+1}$, then $f'(x)=\frac{(x+1)(1)-x(1)}{(x+1)^2}=\frac{1}{(x+1)^2}$. When $x=a$, the equation of the tangent line is $y-\frac{a}{a+1}=\frac{1}{(a+1)^2}(x-a)$. This line passes through (1,2) when $2-\frac{a}{a+1}=\frac{1}{(a+1)^2}(1-a)\Leftrightarrow$ $2(a+1)^2-a(a+1)=1-a\Leftrightarrow 2a^2+4a+2-a^2-a-1+a=0\Leftrightarrow a^2+4a+1=0$.

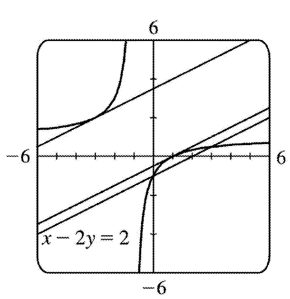
The quadratic formula gives the roots of this equation as $a=\frac{-4\pm\sqrt{4^2-4(1)(1)}}{2(1)}=\frac{-4\pm\sqrt{12}}{2}=-2\pm\sqrt{3}$, so there are two such tangent lines. Since

$$f(-2\pm\sqrt{3})=\frac{-2\pm\sqrt{3}}{-2\pm\sqrt{3}+1}=\frac{-2\pm\sqrt{3}}{-1\pm\sqrt{3}}\cdot\frac{-1\mp\sqrt{3}}{-1\mp\sqrt{3}}$$
$$=\frac{2\pm2\sqrt{3}\mp\sqrt{3}-3}{1-3}=\frac{-1\pm\sqrt{3}}{-2}=\frac{1\mp\sqrt{3}}{2},$$

the lines touch the curve at $A\left(-2+\sqrt{3},\frac{1-\sqrt{3}}{2}\right)\approx(-0.27,-0.37)$ and $B\left(-2-\sqrt{3},\frac{1+\sqrt{3}}{2}\right)\approx(-3.73,1.37)$.

42. $y=\frac{x-1}{x+1}\Rightarrow y'=\frac{(x+1)(1)-(x-1)(1)}{(x+1)^2}=\frac{2}{(x+1)^2}$. If the tangent intersects the curve when $x=a$, then its slope is $2/(a+1)^2$. But if the tangent is parallel to $x-2y=2$, that is, $y=\frac{1}{2}x-1$, then its slope is $\frac{1}{2}$. Thus, $\frac{2}{(a+1)^2}=\frac{1}{2}\Rightarrow(a+1)^2=4\Rightarrow a+1=\pm2\Rightarrow a=1$ or -3. When $a=1$, $y=0$ and the equation of the tangent is $y-0=\frac{1}{2}(x-1)$ or $y=\frac{1}{2}x-\frac{1}{2}$.

When $a=-3$, $y=2$ and the equation of the tangent is $y-2=\frac{1}{2}(x+3)$ or $y=\frac{1}{2}x+\frac{7}{2}$.



43. **(a)** $(fgh)'=[(fg)h]'=(fg)'h+(fg)h'=(f'g+fg')h+(fg)h'=f'gh+fg'h+fgh'$

(b) Putting $f=g=h$ in part (a) , we have

$$\frac{d}{dx}[f(x)]^3=(fff)'=f'ff+ff'f+fff'=3fff'=3[f(x)]^2f'(x).$$

(c) $\frac{d}{dx}(e^{3x})=\frac{d}{dx}(e^x)^3=3(e^x)^2e^x=3e^{2x}e^x=3e^{3x}$

44. **(a)**

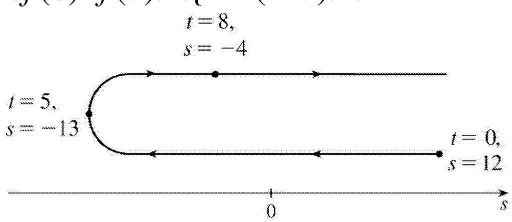
$$\frac{d}{dx}\left(\frac{1}{g(x)}\right)=\frac{g(x)\cdot\frac{d}{dx}(1)-1\cdot\frac{d}{dx}[g(x)]}{[g(x)]^2}\quad\text{[Quotient Rule]}$$

$$=\frac{g(x)\cdot 0-1\cdot g'(x)}{[g(x)]^2}=\frac{0-g'(x)}{[g(x)]^2}=-\frac{g'(x)}{[g(x)]^2}$$

(b) $y=\frac{1}{x^4+x^2+1}\Rightarrow y'=-\frac{4x^3+2x}{(x^4+x^2+1)^2}$ or $\frac{-2x(2x^2+1)}{(x^4+x^2+1)^2}$

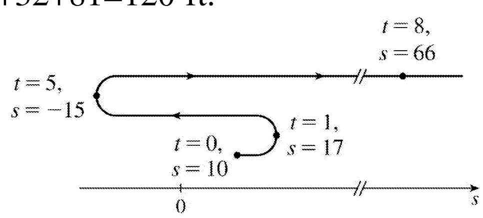
(c) $\frac{d}{dx}(x^{-n})=\frac{d}{dx}\left(\frac{1}{x^n}\right)=-\frac{(x^n)'}{(x^n)^2}$ [by the Reciprocal Rule] $=-\frac{nx^{n-1}}{x^{2n}}=-nx^{n-1-2n}=-nx^{-n-1}$

1. **(a)** $s=f(t)=t^{2}-10t+12 \Rightarrow v(t)=f'(t)=2t-10$
(b) $v(3)=2(3)-10=-4$ ft / s
(c) The particle is at rest when $v(t)=0 \Leftrightarrow 2t-10=0 \Leftrightarrow t=5$ s.
(d) The particle is moving in the positive direction when $v(t)>0 \Leftrightarrow 2t-10>0 \Leftrightarrow 2t>10 \Leftrightarrow t>5$.
(e) Since the particle is moving in the positive direction and in the negative direction, we need to calculate the distance traveled in the intervals [0,5] and [5,8] separately. $|f(5)-f(0)|=|-13-12|=25$ ft and $|f(8)-f(5)|=|\{-4-(-13)|=9$ ft. The total distance traveled during the first 8 s is $25+9=34$ ft.

(f) 

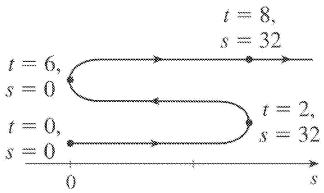
2. **(a)** $s=f(t)=t^{3}-9t^{2}+15t+10 \Rightarrow v(t)=f'(t)=3t^{2}-18t+15=3(t-1)(t-5)$

(b) $v(3)=3(2)(-2)=-12$ ft / s
(c) $v(t)=0 \Leftrightarrow t=1$ s or 5 s
(d) $v(t)>0 \Leftrightarrow 0 \leq t<1$ or $t>5$
(e) $|f(1)-f(0)|=|17-10|=7$, $|f(5)-f(1)|=|-15-17|=32$, and $|f(8)-f(5)|=|66-(-15)|=81$. Total distance $=7+32+81=120$ ft.

(f) 

3. **(a)** $s=f(t)=t^{3}-12t^{2}+36t \Rightarrow v(t)=f'(t)=3t^{2}-24t+36$
(b) $v(3)=27-72+36=-9$ ft / s

(c) The particle is at rest when $v(t)=0$. $3t^{2}-24t+36=0 \Rightarrow 3(t-2)(t-6)=0 \Rightarrow t=2$ s or 6 s.
(d) The particle is moving in the positive direction when $v(t)>0$. $3(t-2)(t-6)>0 \Leftrightarrow 0 \leq t<2$ or $t>6$.
(e) Since the particle is moving in the positive direction and in the negative direction, we need to calculate the distance traveled in the intervals (0,2) , (2,6) , and [6,8] separately.
$|f(2)-f(0)|=|32-0|=32$.
$|f(6)-f(2)|=|0-32|=32$.
$|f(8)-f(6)|=|32-0|=32$.
The total distance is $32+32+32=96$ ft.
(f)



4. **(a)** $s=f(t)=t^4-4t+1 \Rightarrow v(t)=f'(t)=4t^3-4$

(b) $v(3)=4(3)^3-4=104$ ft / s

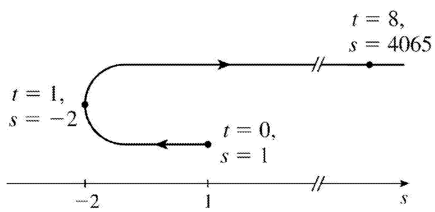
(c) It is at rest when $v(t)=4(t^3-1)=4(t-1)(t^2+t+1)=0 \Leftrightarrow t=1$ s.

(d) It moves in the positive direction when $4(t^3-1)>0 \Leftrightarrow t>1$.

(e) Distance in positive direction $=|f(8)-f(1)|=|4065-(-2)|=4067$ ft

Distance in negative direction $=|f(1)-f(0)|=|-2-1|=3$ ft

Total distance traveled $=4067+3=4070$ ft

(f) 

5. **(a)** $s=\dfrac{t}{t^2+1} \Rightarrow v(t)=s'(t)=\dfrac{(t^2+1)(1)-t(2t)}{(t^2+1)^2}=\dfrac{1-t^2}{(t^2+1)^2}$

(b) $v(3)=\dfrac{1-(3)^2}{(3^2+1)^2}=\dfrac{1-9}{10^2}=\dfrac{-8}{100}=-\dfrac{2}{25}$ ft / s

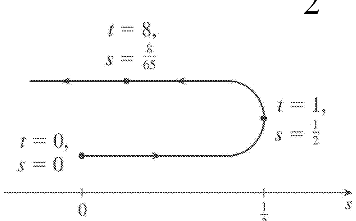
(c) It is at rest when $v=0 \Leftrightarrow 1-t^2=0 \Leftrightarrow t=1$ s [$t\neq -1$ since $t\geq 0$].

(d) It moves in the positive direction when $v>0 \Leftrightarrow 1-t^2>0 \Leftrightarrow t^2<1 \Leftrightarrow 0\leq t<1$.

(e) Distance in positive direction $=|s(1)-s(0)|=\left|\dfrac{1}{2}-0\right|=\dfrac{1}{2}$ ft

Distance in negative direction $=|s(8)-s(1)|=\left|\dfrac{8}{65}-\dfrac{1}{2}\right|=\dfrac{49}{130}$ ft

Total distance traveled $=\dfrac{1}{2}+\dfrac{49}{130}=\dfrac{57}{65}$ ft

(f) 

6. **(a)** $s=\sqrt{t}\,(3t^2-35t+90)=3t^{5/2}-35t^{3/2}+90t^{1/2}\Rightarrow$

$v(t)=s'(t)=\frac{15}{2}t^{3/2}-\frac{105}{2}t^{1/2}+45t^{-1/2}=\frac{15}{2}t^{-1/2}(t^2-7t+6)=\frac{15}{2\sqrt{t}}(t-1)(t-6)$

(b) $v(3)=\frac{15}{2\sqrt{3}}(2)(-3)=-15\sqrt{3}$ ft / s

(c) It is at rest when $v=0\Leftrightarrow t=1$ s or 6 s.

(d) It moves in the positive direction when $v>0\Leftrightarrow(t-1)(t-6)>0\Leftrightarrow 0\le t<1$ or $t>6$.

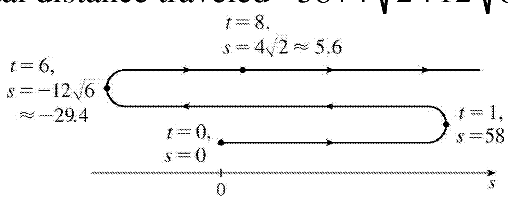
(e)

Distance in positive direction $=|s(1)-s(0)|+|s(8)-s(6)|=|58-0|+|4\sqrt{2}-(-12\sqrt{6})$

$=58+4\sqrt{2}+12\sqrt{6}\approx 93.05$ ft

Distance in negative direction $=|s(6)-s(1)|=|-12\sqrt{6}-58|=58+12\sqrt{6}\approx 87.39$ ft

Total distance traveled $=58+4\sqrt{2}+12\sqrt{6}+58+12\sqrt{6}=116+4\sqrt{2}+24\sqrt{6}\approx 180.44$ ft

(f) 

7. $s(t)=t^3-4.5t^2-7t\Rightarrow v(t)=s'(t)=3t^2-9t-7=5\Leftrightarrow 3t^2-9t-12=0\Leftrightarrow 3(t-4)(t+1)=0\Leftrightarrow t=4$ or -1 . Since $t\ge 0$, the particle reaches a velocity of 5 m / s at $t=4$ s.

8. **(a)** $s=5t+3t^2\Rightarrow v(t)=\frac{ds}{dt}=5+6t$, so $v(2)=5+6(2)=17$ m / s.

(b) $v(t)=35\Rightarrow 5+6t=35\Rightarrow 6t=30\Rightarrow t=5$ s.

9. **(a)** $h=10t-0.83t^2\Rightarrow v(t)=\frac{dh}{dt}=10-1.66t$, so $v(3)=10-1.66(3)=5.02$ m / s.

(b) $h=25\Rightarrow 10t-0.83t^2=25\Rightarrow 0.83t^2-10t+25=0\Rightarrow t=\frac{10\pm\sqrt{17}}{1.66}\approx 3.54$ or 8.51 .

The value $t_1=(10-\sqrt{17})/1.66$ corresponds to the time it takes for the stone to rise 25 m and $t_2=(10+\sqrt{17})/1.66$ corresponds to the time when the stone is 25 m high on the way down. Thus, $v(t_1)=10-1.66[(10-\sqrt{17})/1.66]=\sqrt{17}\approx 4.12$ m / s.

10. **(a)** At maximum height the velocity of the ball is 0 ft / s. $v(t)=s'(t)=80-32t=0\Leftrightarrow 32t=80\Leftrightarrow t=\frac{5}{2}$.

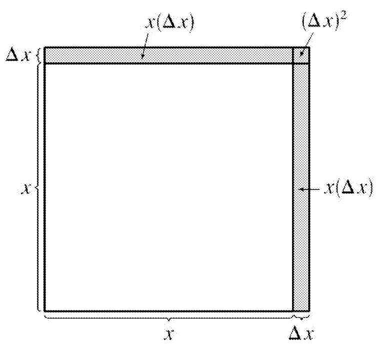
So the maximum height is

$s\left(\frac{5}{2}\right)=80\left(\frac{5}{2}\right)-16\left(\frac{5}{2}\right)^2=200-100=100$ ft.

(b) $s(t)=80t-16t^2=96 \Leftrightarrow 16t^2-80t+96=0 \Leftrightarrow 16(t^2-5t+6)=0 \Leftrightarrow 16(t-3)(t-2)=0$.
So the ball has a height of 96 ft on the way up at $t=2$ and on the way down at $t=3$. At these times the velocities are $v(2)=80-32(2)=16$ ft / s and $v(3)=80-32(3)=-16$ ft / s, respectively.

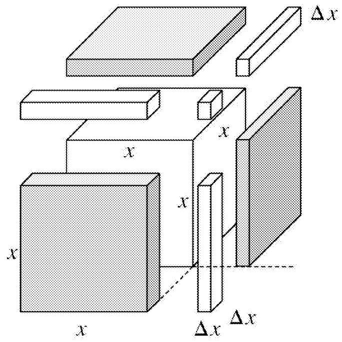
11. **(a)** $A(x)=x^2 \Rightarrow A'(x)=2x$. $A'(15)=30$ mm ^{2}/ mm is the rate at which the area is increasing with respect to the side length as x reaches 15 mm.

(b) The perimeter is $P(x)=4x$, so $A'(x)=2x=\frac{1}{2}(4x)=\frac{1}{2}P(x)$. The figure suggests that if Δx is small, then the change in the area of the square is approximately half of its perimeter (2 of the 4 sides) times Δx . From the figure, $\Delta A=2x(\Delta x)+(\Delta x)^2$. If Δx is small, then $\Delta A\approx 2x(\Delta x)$ and so $\Delta A/\Delta x\approx 2x$.



12. **(a)** $V(x)=x^3 \Rightarrow \frac{dV}{dx}=3x^2$. $\left.\frac{dV}{dx}\right|_{x=3}=3(3)^2=27$ mm ^{3}/ mm is the rate at which the volume is increasing as x increases past 3 mm.

(b) The surface area is $S(x)=6x^2$, so $V'(x)=3x^2=\frac{1}{2}(6x^2)=\frac{1}{2}S(x)$. The figure suggests that if Δx is small, then the change in the volume of the cube is approximately half of its surface area (the area of 3 of the 6 faces) times Δx . From the figure, $\Delta V=3x^2(\Delta x)+3x(\Delta x)^2+(\Delta x)^3$. If Δx is small, then $\Delta V\approx 3x^2(\Delta x)$ and so $\Delta V/\Delta x\approx 3x^2$.



13. **(a)**

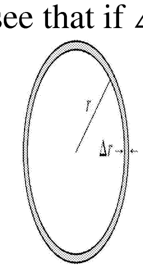
(i) $\dfrac{A(3)-A(2)}{3-2}=\dfrac{9\pi-4\pi}{1}=5\pi$

(ii) $\dfrac{A(2.5)-A(2)}{2.5-2}=\dfrac{6.25\pi-4\pi}{0.5}=4.5\pi$

(iii) $\dfrac{A(2.1)-A(2)}{2.1-2}=\dfrac{4.41\pi-4\pi}{0.1}=4.1\pi$

(b) $A(r)=\pi r^2 \Rightarrow A'(r)=2\pi r$, so $A'(2)=4\pi$.

(c) The circumference is $C(r)=2\pi r=A'(r)$. The figure suggests that if Δr is small, then the change in the area of the circle (a ring around the outside) is approximately equal to its circumference times Δr . Straightening out this ring gives us a shape that is approximately rectangular with length $2\pi r$ and width Δr , so $\Delta A\approx 2\pi r(\Delta r)$. Algebraically, $\Delta A=A(r+\Delta r)-A(r)=\pi(r+\Delta r)^2-\pi r^2=2\pi r(\Delta r)+\pi(\Delta r)^2$. So we see that if Δr is small, then $\Delta A\approx 2\pi r(\Delta r)$ and therefore, $\Delta A/\Delta r\approx 2\pi r$.



14. **(a)** $A'(1)=7200\pi\ \text{cm}^2/\text{s}$

(b) $A'(3)=21,600\pi\ \text{cm}^2/\text{s}$

(c) $A'(5)=36,000\pi\ \text{cm}^2/\text{s}$

15. **(a)** $S'(1)=8\pi\ \text{ft}^2/\text{ft}$

(b) $S'(2)=16\pi\ \text{ft}^2/\text{ft}$

(c) $S'(3)=24\pi\ \text{ft}^2/\text{ft}$

16. **(a)**

(a)
$$\frac{V(8)-V(5)}{8-5}=\frac{\frac{4}{3}\pi(512)-\frac{4}{3}\pi(125)}{3}=172\pi\,\mu\text{ m}^3/\mu\text{ m}$$

(b)
$$\frac{V(8)-V(5)}{8-5}=\frac{\frac{4}{3}\pi(512)-\frac{4}{3}\pi(125)}{3}=172\pi\,\mu\text{ m}^3/\mu\text{ m}$$

(c)
$$\frac{V(6)-V(5)}{6-5}=\frac{\frac{4}{3}\pi(216)-\frac{4}{3}\pi(125)}{1}=121.\overline{3}\pi\,\mu\text{ m}^3/\mu\text{ m}$$

(d)
$$\frac{V(6)-V(5)}{6-5}=\frac{\frac{4}{3}\pi(216)-\frac{4}{3}\pi(125)}{1}=121.\overline{3}\pi\,\mu\text{ m}^3/\mu\text{ m}$$

(e)
$$\frac{V(5.1)-V(5)}{5.1-5}=\frac{\frac{4}{3}\pi(5.1)^3-\frac{4}{3}\pi(5)^3}{0.1}=102.01\overline{3}\pi\,\mu\text{ m}^3/\mu\text{ m}$$

(f)
$$\frac{V(5.1)-V(5)}{5.1-5}=\frac{\frac{4}{3}\pi(5.1)^3-\frac{4}{3}\pi(5)^3}{0.1}=102.01\overline{3}\pi\,\mu\text{ m}^3/\mu\text{ m}$$

(b) $V'(r)=4\pi r^2$, so $V'(5)=100\pi\,\mu\text{ m}^3/\mu\text{ m}$.

(c) $V(r)=\frac{4}{3}\pi r^3\Rightarrow V'(r)=4\pi r^2=S(r)$. By analogy with Exercise 13(c) , we can say that the change in the volume of the spherical shell, ΔV , is approximately equal to its thickness, Δr , times the surface area of the inner sphere. Thus, $\Delta V\approx 4\pi r^2(\Delta r)$ and so $\Delta V/\Delta r\approx 4\pi r^2$.

17. **(a)** $\rho(1)=6$ kg / m
(b) $\rho(2)=12$ kg / m
(c) $\rho(3)=18$ kg / m

18. **(a)** $V'(5)=-250\left(1-\frac{5}{40}\right)=-218.75$ gal / min
(b) $V'(10)=-250\left(1-\frac{10}{40}\right)=-187.5$ gal / min
(c) $V'(20)=-250\left(1-\frac{20}{40}\right)=-125$ gal / min
(d)

$V'(40)=-250\left(1-\frac{40}{40}\right)=0$ gal / min

19. **(a)** $Q'(0.5)=3(0.5)^2-4(0.5)+6=4.75$ A
(b) $Q'(1)=3(1)^2-4(1)+6=5$ A

20. **(a)** $F=\frac{GmM}{r^2}=(GmM)r^{-2}\Rightarrow \frac{dF}{dr}=-2(GmM)r^{-3}=-\frac{2GmM}{r^3}$, which is the rate of change of the force with respect to the distance between the bodies. The minus sign indicates that as the distance r between the bodies increases, the magnitude of the force F exerted by the body of mass m on the body of mass M is decreasing.
(b) Given $F'(20,000)=-2$, find $F'(10,000)$. $-2=-\frac{2GmM}{20{,}000^3}\Rightarrow GmM=20,000^3$.

$F'(10{,}000)=-\frac{2\left(20{,}000^3\right)}{10{,}000^3}=-2\cdot 2^3=-16$ N / km

21. **(a)** To find the rate of change of volume with respect to pressure, we first solve for V in terms of P .
$PV=C\Rightarrow V=\frac{C}{P}\Rightarrow \frac{dV}{dP}=-\frac{C}{P^2}$.
(b) From the formula for dV/dP in part (a), we see that as P increases, the absolute value of dV/dP decreases. Thus, the volume is decreasing more rapidly at the beginning.
(c) $\beta=-\frac{1}{V}\frac{dV}{dP}=-\frac{1}{V}\left(-\frac{C}{P^2}\right)=\frac{C}{(PV)P}=\frac{C}{CP}=\frac{1}{P}$

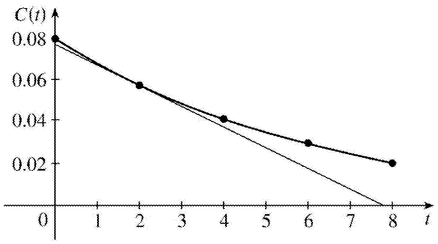
22. **(a)**

(i)
$$\frac{C(6)-C(2)}{6-2}=\frac{0.0295-0.0570}{4}=-0.006875\,\text{(moles/L)/ min}$$
$$\frac{C(4)-C(2)}{4-2}=\frac{0.0408-0.0570}{2}=-0.008\,\text{(moles/L)/ min}$$

(iii) $\frac{C(2)-C(0)}{2-0} = \frac{0.0570-0.0800}{2}$

$= -0.0115 \,(\text{moles/L}) / \text{ min}$

(b) Slope $= \frac{\Delta C}{\Delta t} \approx -\frac{0.077}{7.8} \approx -0.01 \,(\text{moles/L}) / \text{min}$



23. **(a)** **1920:** $m_1 = \frac{1860-1750}{1920-1910} = \frac{110}{10} = 11$, $m_2 = \frac{2070-1860}{1930-1920} = \frac{210}{10} = 21$,
$(m_1 + m_2)2 = (11+21)/2 = 16$ million / year

1980: $m_1 = \frac{4450-3710}{1980-1970} = \frac{740}{10} = 74$, $m_2 = \frac{5280-4450}{1990-1980} = \frac{830}{10} = 83$,
$(m_1 + m_2)2 = (74+83)/2 = 78.5$ million / year

(b) $P(t) = at^3 + bt^2 + ct + d$ (in millions of people), where $a \approx 0.0012937063$, $b \approx -7.061421911$, $c \approx 12{,}822.97902$, and $d \approx -7{,}743{,}770.396$.

(c) $P(t) = at^3 + bt^2 + ct + d \Rightarrow P'(t) = 3at^2 + 2bt + c$ (in millions of people per year)

(d)

$P'(1920) = 3(0.0012937063)(1920)^2 + 2(-7.061421911)(1920) + 12{,}822.97902$

≈ 14.48 million / year

$P'(1980) \approx 75.29$ million / year (smaller, but close)

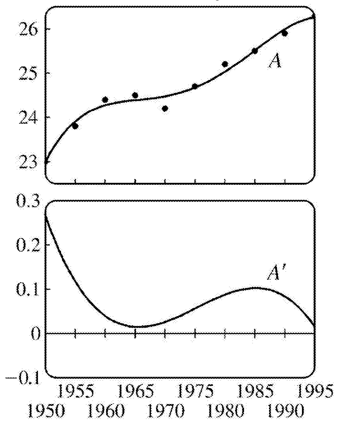
(e) $P'(1985) \approx 81.62$ million / year, so the rate of growth in 1985 was about 81.62 million / year.

24. **(a)** $A(t) = at^4 + bt^3 + ct^2 + dt + e$, where $a = -5.8275058275396 \times 10^{-6}$, $b = 0.0460458430461$, $c = -136.43277039706$, $d = 179{,}661.02676871$, and $e = -88{,}717{,}597.060767$.

(b) $A(t) = at^4 + bt^3 + ct^2 + dt + e \Rightarrow A'(t) = 4at^3 + 3bt^2 + 2ct + d$

(c)

$A'(1990) \approx 0.0833$ years of age per year



(d)

25. **(a)** $[\mathrm{C}]=\dfrac{a^2kt}{akt+1} \Rightarrow$ rate of reaction

$$=\frac{d[\mathrm{C}]}{dt}=\frac{(akt+1)(a^2k)-(a^2kt)(ak)}{(akt+1)^2}=\frac{a^2k(akt+1-akt)}{(akt+1)^2}=\frac{a^2k}{(akt+1)^2}$$

(b) If $x=[\mathrm{C}]$, then $a-x=a-\dfrac{a^2kt}{akt+1}=\dfrac{a^2kt+a-a^2kt}{akt+1}=\dfrac{a}{akt+1}$.

So $k(a-x)^2=k\left(\dfrac{a}{akt+1}\right)^2=\dfrac{a^2k}{(akt+1)^2}=\dfrac{d[\mathrm{C}]}{dt}=\dfrac{dx}{dt}$.

26. **(a)** After an hour the population is $n(1)=3\cdot 500$; after two hours it is $n(2)=3(3\cdot 500)=3^2\cdot 500$; after three hours, $n(3)=3(3^2\cdot 500)=3^3\cdot 500$; after four hours, $n(4)=3^4\cdot 500$. From this pattern, we see that the population after t hours is $n(t)=3^t\cdot 500=500\cdot 3^t$.

(b) From (5) in Section 3.1, we have $\dfrac{d}{dx}(3^x)\approx(1.10)3^x$. Thus, for $n(t)=500\cdot 3^t$,

$\dfrac{dn}{dt}=500\dfrac{d}{dt}(3^t)\approx 500(1.10)3^t \Rightarrow \left.\dfrac{dn}{dt}\right|_{t=6}\approx 500(1.10)3^6\approx 400 , 950$ bacteria / hour.

27. **(a)** Using $v=\dfrac{P}{4\eta l}(R^2-r^2$ with $R=0.01$, $l=3$, $P=3000$, and $\eta=0.027$, we have v as a function of r :

$v(r)=\dfrac{3000}{4(0.027)3}(0.01^2-r^2)$. $v(0)=0.\overline{925}$ cm / s, $v(0.005)=0.69\overline{4}$ cm / s, $v(0.01)=0$.

(b) $v(r)=\dfrac{P}{4\eta l}(R^2-r^2)\Rightarrow v'(r)=\dfrac{P}{4\eta l}(-2r)=-\dfrac{Pr}{2\eta l}$. When $l=3$, $P=3000$, and $\eta=0.027$, we have

$v'(r) = -\frac{3000r}{2(0.027)3}$. $v'(0)=0$, $v'(0.005)=-92.\overline{592}$(cm/s)/ cm, and $v'(0.01)=-185.\overline{185}$(cm/s)/ cm.

(c) The velocity is greatest where $r=0$ (at the center) and the velocity is changing most where $r=R=0.01$ cm (at the edge).

28. **(a)**

(a) $f=\frac{1}{2L}\sqrt{\frac{T}{\rho}}=\left(\frac{1}{2}\sqrt{\frac{T}{\rho}}\right)L^{-1}\Rightarrow \frac{df}{dL}=-\left(\frac{1}{2}\sqrt{\frac{T}{\rho}}\right)L^{-2}=-\frac{1}{2L^2}\sqrt{\frac{T}{\rho}}$

(b) $f=\frac{1}{2L}\sqrt{\frac{T}{\rho}}=\left(\frac{1}{2}\sqrt{\frac{T}{\rho}}\right)L^{-1}\Rightarrow \frac{df}{dL}=-\left(\frac{1}{2}\sqrt{\frac{T}{\rho}}\right)L^{-2}=-\frac{1}{2L^2}\sqrt{\frac{T}{\rho}}$

(c) $f=\frac{1}{2L}\sqrt{\frac{T}{\rho}}=\left(\frac{1}{2L\sqrt{\rho}}\right)T^{1/2}\Rightarrow \frac{df}{dT}=\frac{1}{2}\left(\frac{1}{2L\sqrt{\rho}}\right)T^{-1/2}=\frac{1}{4L\sqrt{T\rho}}$

(d) $f=\frac{1}{2L}\sqrt{\frac{T}{\rho}}=\left(\frac{1}{2L\sqrt{\rho}}\right)T^{1/2}\Rightarrow \frac{df}{dT}=\frac{1}{2}\left(\frac{1}{2L\sqrt{\rho}}\right)T^{-1/2}=\frac{1}{4L\sqrt{T\rho}}$

(e) $f=\frac{1}{2L}\sqrt{\frac{T}{\rho}}=\left(\frac{\sqrt{T}}{2L}\right)\rho^{-1/2}\Rightarrow \frac{df}{d\rho}=-\frac{1}{2}\left(\frac{\sqrt{T}}{2L}\right)\rho^{-3/2}=-\frac{\sqrt{T}}{4L\rho^{3/2}}$

(f) $f=\frac{1}{2L}\sqrt{\frac{T}{\rho}}=\left(\frac{\sqrt{T}}{2L}\right)\rho^{-1/2}\Rightarrow \frac{df}{d\rho}=-\frac{1}{2}\left(\frac{\sqrt{T}}{2L}\right)\rho^{-3/2}=-\frac{\sqrt{T}}{4L\rho^{3/2}}$

(b)

(i) $\frac{df}{dL}<0$ and L is decreasing $\Rightarrow f$ is increasing $\Rightarrow$ higher note

(ii) $\frac{df}{dT}>0$ and T is increasing $\Rightarrow f$ is increasing $\Rightarrow$ higher note

(iii) $\frac{df}{d\rho}<0$ and ρ is increasing $\Rightarrow f$ is decreasing $\Rightarrow$ lower note

29. **(a)** $C(x)=2000+3x+0.01x^2+0.0002x^3 \Rightarrow C'(x)=3+0.02x+0.0006x^2$

(b) $C'(100)=3+0.02(100)+0.0006(10,000)=3+2+6=\$11$/ pair. $C'(100)$ is the rate at which the cost is increasing as the 100 th pair of jeans is produced. It predicts the cost of the 101 st pair.

(c) The cost of manufacturing the 101 st pair of jeans is

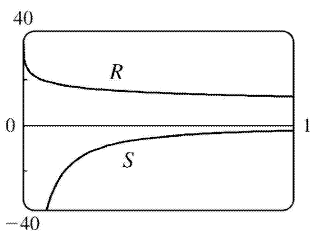
$$C(101)-C(100) = (2000+303+102.01+206.0602)-(2000+300+100+200) = 11.0702\approx \$11.07$$

30. **(a)** $C(x)=84+0.16x-0.0006x^2+0.000003x^3 \Rightarrow C'(x)=0.16-0.0012x+0.000009x^2 \Rightarrow C'(100)=0.13$. This is the rate at which the cost is increasing as the 100 th item is produced.
(b) $C(101)-C(100)=97.13030299-97\approx \$0.13$.

31. **(a)** $A(x)=\dfrac{p(x)}{x} \Rightarrow A'(x)=\dfrac{xp'(x)-p(x)\cdot 1}{x^2}=\dfrac{xp'(x)-p(x)}{x^2}$. $A'(x)>0\Rightarrow A(x)$ is increasing; that is, the average productivity increases as the size of the workforce increases.
(b) $p'(x)$ is greater than the average productivity $\Rightarrow p'(x)>A(x)\Rightarrow p'(x)>\dfrac{p(x)}{x} \Rightarrow$
$xp'(x)>p(x)\Rightarrow xp'(x)-p(x)>0\Rightarrow \dfrac{xp'(x)-p(x)}{x^2}>0\Rightarrow A'(x)>0$.

32. **(a)**

$$S=\frac{dR}{dx}=\frac{\left(1+4x^{0.4}\right)\left(9.6x^{-0.6}\right)-\left(40+24x^{0.4}\right)\left(1.6x^{-0.6}\right)}{\left(1+4x^{0.4}\right)^2}$$
$$=\frac{9.6x^{-0.6}+38.4x^{-0.2}-64x^{-0.6}-38.4x^{-0.2}}{\left(1+4x^{0.4}\right)^2}=-\frac{54.4x^{-0.6}}{\left(1+4x^{0.4}\right)^2}$$



(b)
At low levels of brightness, R is quite large and is quickly decreasing, that is, S is negative with large absolute value. This is to be expected: at low levels of brightness, the eye is more sensitive to slight changes than it is at higher levels of brightness.

33. $PV=nRT\Rightarrow T=\dfrac{PV}{nR}=\dfrac{PV}{(10)(0.0821)}=\dfrac{1}{0.821}(PV)$. Using the Product Rule, we have
$\dfrac{dT}{dt}=\dfrac{1}{0.821}[P(t)V'(t)+V(t)P'(t)]=\dfrac{1}{0.821}[(8)(-0.15)+(10)(0.10)]\approx -0.2436$ K / min.

34. **(a)** If $dP/dt=0$, the population is stable (it is constant).

(b) $\frac{dP}{dt}=0 \Rightarrow \beta P=r_0\left(1-\frac{P}{P_c}\right)P \Rightarrow \frac{\beta}{r_0}=1-\frac{P}{P_c} \Rightarrow \frac{P}{P_c}=1-\frac{\beta}{r_0} \Rightarrow P=P_c\left(1-\frac{\beta}{r_0}\right)$.

If $P_c=10,000$, $r_0=5\%=0.05$, and $\beta=4\%=0.04$, then $P=10,000\left(1-\frac{4}{5}\right)=2000$.

(c) If $\beta=0.05$, then $P=10,000\left(1-\frac{5}{5}\right)=0$. There is no stable population.

35. **(a)** If the populations are stable, then the growth rates are neither positive nor negative; that is, $\frac{dC}{dt}=0$ and $\frac{dW}{dt}=0$.

(b) ''The caribou go extinct'' means that the population is zero, or mathematically, $C=0$.

(c) We have the equations $\frac{dC}{dt}=aC-bCW$ and $\frac{dW}{dt}=-cW+dCW$. Let $dC/dt=dW/dt=0$, $a=0.05$, $b=0.001$, $c=0.05$, and $d=0.0001$ to obtain $0.05C-0.001CW=0$ **(1)** and $-0.05W+0.0001CW=0$ **(2)** . Adding 10 times **(2)** to **(1)** eliminates the CW –terms and gives us $0.05C-0.5W=0 \Rightarrow C=10W$. Substituting $C=10W$ into **(1)** results in

$0.05(10W)-0.001(10W)W=0 \Leftrightarrow 0.5W-0.01W^2=0 \Leftrightarrow 50W-W^2=0 \Leftrightarrow W(50-W)=0 \Leftrightarrow W=0$ or 50 . Since $C=10W$, $C=0$ or 500 . Thus, the population pairs (C,W) that lead to stable populations are (0,0) and (500,50) . So it is possible for the two species to live in harmony.

1. $f(x)=x-3\sin x \Rightarrow f'(x)=1-3\cos x$

2. $f(x)=x\sin x \Rightarrow f'(x)=x\cdot \cos x+(\sin x)\cdot 1=x\cos x+\sin x$

3. $y=\sin x+10\tan x \Rightarrow y'=\cos x+10\sec^2 x$

4. $y=2\ \ x+5\cos x \Rightarrow y'=-2\ \ x\cot x-5\sin x$

5. $g(t)=t^3\cos t \Rightarrow g'(t)=t^3(-\sin t)+(\cos t)\cdot 3t^2=3t^2\cos t-t^3\sin t$ or $t^2(3\cos t-t\sin t)$

6. $g(t)=4\sec t+\tan t \Rightarrow g'(t)=4\sec t\tan t+\sec^2 t$

7. $h(\theta)=\csc\theta+e^\theta\cot\theta \Rightarrow$
$h'(\theta)=-\csc\theta\cot\theta+e^\theta(-\csc^2\theta)+(\cot\theta)e^\theta=-\csc\theta\cot\theta+e^\theta(\cot\theta-\csc^2\theta)$

8. $y=e^u(\cos u+cu) \Rightarrow y'=e^u(-\sin u+c)+(\cos u+cu)e^u=e^u(\cos u-\sin u+cu+c)$

9. $y=\dfrac{x}{\cos x} \Rightarrow y'=\dfrac{(\cos x)(1)-(x)(-\sin x)}{(\cos x)^2}=\dfrac{\cos x+x\sin x}{\cos^2 x}$

10. $y=\dfrac{1+\sin x}{x+\cos x} \Rightarrow$

$$y'=\frac{(x+\cos x)(\cos x)-(1+\sin x)(1-\sin x)}{(x+\cos x)^2}=\frac{x\cos x+\cos^2 x-(1-\sin^2 x)}{(x+\cos x)^2}$$
$$=\frac{x\cos x+\cos^2 x-(\cos^2 x)}{(x+\cos x)^2}=\frac{x\cos x}{(x+\cos x)^2}$$

11. $f(\theta)=\dfrac{\sec\theta}{1+\sec\theta} \Rightarrow$

$$f'(\theta)=\frac{(1+\sec\theta)(\sec\theta\tan\theta)-(\sec\theta)(\sec\theta\tan\theta)}{(1+\sec\theta)^2}=\frac{(\sec\theta\tan\theta)[(1+\sec\theta)-\sec\theta]}{(1+\sec\theta)^2}=\frac{\sec\theta\tan\theta}{(1+\sec\theta)^2}$$

12. $y=\dfrac{\tan x-1}{\sec x} \Rightarrow$

$$\frac{dy}{dx}=\frac{\sec x\sec^2 x-(\tan x-1)\sec x\tan x}{\sec^2 x}=\frac{\sec x\left(\sec^2 x-\tan^2 x+\tan x\right)}{\sec^2 x}=\frac{1+\tan x}{\sec x}$$

Another method: Simplify y first: $y=\sin x-\cos x \Rightarrow y'=\cos x+\sin x$.

13. $y=\dfrac{\sin x}{x^2} \Rightarrow y'=\dfrac{x^2\cos x-(\sin x)(2x)}{\left(x^2\right)^2}=\dfrac{x(x\cos x-2\sin x)}{x^4}=\dfrac{x\cos x-2\sin x}{x^3}$

14. $y=\theta(\theta+\cot\theta)\Rightarrow$

$$\begin{aligned} y' &=\theta(1-\csc^2\theta)+(\theta+\cot\theta)(-\ \theta\cot\theta)=\theta(1-\csc^2\theta-\theta\cot\theta-\cot^2\theta)\\ &=\theta(-\cot^2\theta-\theta\cot\theta-\cot^2\theta) \qquad \{1+\cot^2\theta=\csc^2\theta\}\\ &=\theta(-\theta\cot\theta-2\cot^2\theta)=-\theta\cot\theta(\theta+2\cot\theta) \end{aligned}$$

15. $y=\sec\theta\tan\theta \Rightarrow y'=\sec\theta(\sec^2\theta)+\tan\theta(\sec\theta\tan\theta)=\sec\theta(\sec^2\theta+\tan^2\theta)$

Using the identity $1+\tan^2\theta=\sec^2\theta$, we can write alternative forms of the answer as $\sec\theta(1+2\tan^2\theta)$ or $\sec\theta(2\sec^2\theta-1)$

16. Recall that if $y=fgh$, then $y'=f'gh+fg'h+fgh'$. $y=x\sin x\cos x\Rightarrow$

$$\frac{dy}{dx}=\sin x\cos x+x\cos x\cos x+x\sin x(-\sin x)=\sin x\cos x+x\cos^2 x-x\sin^2 x$$

17. $\dfrac{d}{dx}(\csc(x))=\dfrac{d}{dx}\left(\dfrac{1}{\sin x}\right)=\dfrac{(\sin x)(0)-1(\cos x)}{\sin^2 x}=\dfrac{-\cos x}{\sin^2 x}=-\dfrac{1}{\sin x}\cdot\dfrac{\cos x}{\sin x}=-\ x\cot x$

18. $\dfrac{d}{dx}(\sec x)=\dfrac{d}{dx}\left(\dfrac{1}{\cos x}\right)=\dfrac{(\cos x)(0)-1(-\sin x)}{\cos^2 x}=\dfrac{\sin x}{\cos^2 x}=\dfrac{1}{\cos x}\cdot\dfrac{\sin x}{\cos x}=\sec x\tan x$

19. $\dfrac{d}{dx}(\cot x)=\dfrac{d}{dx}\left(\dfrac{\cos x}{\sin x}\right)=\dfrac{(\sin x)(-\sin x)-(\cos x)(\cos x)}{\sin^2 x}=-\dfrac{\sin^2 x+\cos^2 x}{\sin^2 x}=-\dfrac{1}{\sin^2 x}=-\ ^2 x$

20. $f(x)=\cos x\Rightarrow$

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\cos(x+h)-\cos x}{h}=\lim_{h\to 0}\frac{\cos x\cos h-\sin x\sin h-\cos x}{h}$$

$$=\lim_{h\to 0}\left(\cos x\,\frac{\cos h-1}{h}-\sin x\,\frac{\sin h}{h}\right)=\cos x\lim_{h\to 0}\frac{\cos h-1}{h}-\sin x\lim_{h\to 0}\frac{\sin h}{h}$$
$$=(\cos x)(0)-(\sin x)(1)=-\sin x$$

21. $y=\tan x \Rightarrow y'=\sec^2 x \Rightarrow$ the slope of the tangent line at $\left(\frac{\pi}{4},1\right)$ is $\sec^2\frac{\pi}{4}=(\sqrt{2})^2=2$ and an equation of the tangent line is $y-1=2\left(x-\frac{\pi}{4}\right)$ or $y=2x+1-\frac{\pi}{2}$.

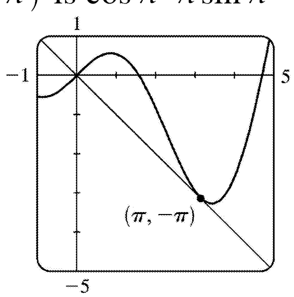
22. $y=e^x\cos x \Rightarrow y'=e^x(-\sin x)+(\cos x)e^x=e^x(\cos x-\sin x) \Rightarrow$ the slope of the tangent line at (0, 1) is $e^0(\cos 0-\sin 0)=1(1-0)=1$ and an equation is $y-1=1(x-0)$ or $y=x+1$.

23. $y=x+\cos x \Rightarrow y'=1-\sin x$. At $(0,1)$, $y'=1$, and an equation of the tangent line is $y-1=1(x-0)$, or $y=x+1$.

24. $y=\frac{1}{\sin x+\cos x} \Rightarrow y'=-\frac{\cos x-\sin x}{(\sin x+\cos x)^2}$ [Reciprocal Rule]. At $(0,1)$, $y'=-\frac{1-0}{(0+1)^2}=-1$, and an equation of the tangent line is $y-1=-1(x-0)$, or $y=-x+1$.

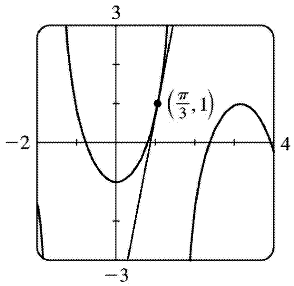
25. **(a)** $y=x\cos x \Rightarrow y'=x(-\sin x)+\cos x(1)=\cos x-x\sin x$. So the slope of the tangent at the point $(\pi,-\pi)$ is $\cos\pi-\pi\sin\pi=-1-\pi(0)=-1$, and an equation is $y+\pi=-(x-\pi)$ or $y=-x$.

(b)



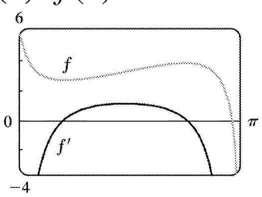
26. **(a)** $y=\sec x-2\cos x \Rightarrow y'=\sec x\tan x+2\sin x \Rightarrow$
the slope of the tangent line at $\left(\frac{\pi}{3},1\right)$ is $\sec\frac{\pi}{3}\tan\frac{\pi}{3}+2\sin\frac{\pi}{3}=2\cdot\sqrt{3}+2\cdot\frac{\sqrt{3}}{2}=3\sqrt{3}$ and an equation is $y-1=3\sqrt{3}\left(x-\frac{\pi}{3}\right)$ or $y=3\sqrt{3}x+1-\pi\sqrt{3}$.

(b)



27. **(a)** $f(x)=2x+\cot x \Rightarrow f'(x)=2-\csc^2 x$

(b)

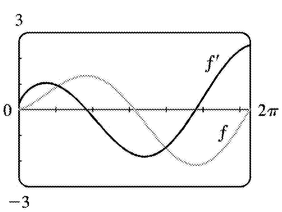


Notice that $f'(x)=0$ when f has a horizontal tangent.

f' is positive when f is increasing and f' is negative when f is decreasing. Also, $f'(x)$ is large negative when the graph of f is steep.

28. **(a)** $f(x)=\sqrt{x}\sin x \Rightarrow f'(x)=\sqrt{x}\cos x+(\sin x)\left(\frac{1}{2}x^{-1/2}\right)=\sqrt{x}\cos x+\frac{\sin x}{2\sqrt{x}}$

(b)



Notice that $f'(x)=0$ when f has a horizontal tangent.

f' is positive when f is increasing and f' is negative when f is decreasing.

29. $f(x)=x+2\sin x$ has a horizontal tangent when $f'(x)=0 \Leftrightarrow 1+2\cos x=0 \Leftrightarrow \cos x=-\frac{1}{2} \Leftrightarrow$ $x=\frac{2\pi}{3}+2\pi n$ or $\frac{4\pi}{3}+2\pi n$, where n is an integer. Note that $\frac{4\pi}{3}$ and $\frac{2\pi}{3}$ are $\pm\frac{\pi}{3}$ units from π . This allows us to write the solutions in the more compact equivalent form $(2n+1)\pi \pm \frac{\pi}{3}$, n an integer.

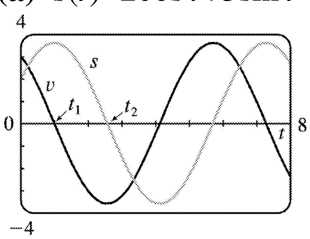
30. $y=\frac{\cos x}{2+\sin x} \Rightarrow y'=\frac{(2+\sin x)(-\sin x)-\cos x\cos x}{(2+\sin x)^2}=\frac{-2\sin x-\sin^2 x-\cos^2 x}{(2+\sin x)^2}=\frac{-2\sin x-1}{(2+\sin x)^2}=0$ when

$-2\sin x-1=0 \Leftrightarrow \sin x=-\frac{1}{2} \Leftrightarrow x=\frac{11\pi}{6}+2\pi n$ or $x=\frac{7\pi}{6}+2\pi n$, n an integer. So $y=\frac{1}{\sqrt{3}}$ or $y=-\frac{1}{\sqrt{3}}$ and the points on the curve with horizontal tangents are: $\left(\frac{11\pi}{6}+2\pi n, \frac{1}{\sqrt{3}}\right)$, $\left(\frac{7\pi}{6}+2\pi n, -\frac{1}{\sqrt{3}}\right)$, n an integer.

31. **(a)** $x(t)=8\sin t \Rightarrow v(t)=x'(t)=8\cos t$

(b) The mass at time $t=\frac{2\pi}{3}$ has position $x\left(\frac{2\pi}{3}\right)=8\sin\frac{2\pi}{3}=8\left(\frac{\sqrt{3}}{2}\right)=4\sqrt{3}$ and velocity $v\left(\frac{2\pi}{3}\right)=8\cos\frac{2\pi}{3}=8\left(-\frac{1}{2}\right)=-4$. Since $v\left(\frac{2\pi}{3}\right)<0$, the particle is moving to the left.

32. **(a)** $s(t)=2\cos t+3\sin t \Rightarrow v(t)=-2\sin t+3\cos t$

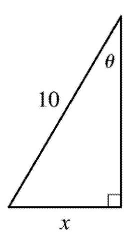


(b)

(c) $s=0 \Rightarrow t_2 \approx 2.55$. So the mass passes through the equilibrium position for the first time when $t \approx 2.55$ s.

(d) $v=0 \Rightarrow t_1 \approx 0.98$, $s(t_1) \approx 3.61$ cm. So the mass travels a maximum of about 3.6 cm (upward and downward) from its equilibrium position.

(e) The speed $|v|$ is greatest when $s=0$; that is, when $t=t_2+n\pi$, n a positive integer.

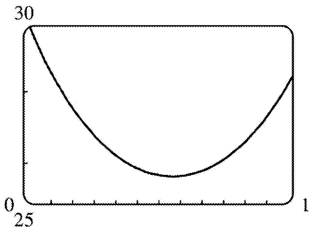


33.
From the diagram we can see that $\sin\theta=x/10 \Leftrightarrow x=10\sin\theta$. We want to find the rate of change of x with respect to θ ; that is, $dx/d\theta$. Taking the derivative of the above expression, $dx/d\theta=10(\cos\theta)$. So when $\theta=\frac{\pi}{3}$, $\frac{dx}{d\theta}=10\cos\left(\frac{\pi}{3}\right)=10\cdot\left(\frac{1}{2}\right)=5$ ft/rad.

34. **(a)** $F=\frac{\mu W}{\mu\sin\theta+\cos\theta} \Rightarrow \frac{dF}{d\theta}=\frac{(\mu\sin\theta+\cos\theta)(0)-\mu W(\mu\cos\theta-\sin\theta)}{(\mu\sin\theta+\cos\theta)^2}=\frac{\mu W(\sin\theta-\mu\cos\theta)}{(\mu\sin\theta+\cos\theta)^2}$

(b)

$$\frac{dF}{d\theta}=0 \Rightarrow \mu W(\sin\theta-\mu\cos\theta)=0 \Rightarrow \sin\theta=\mu\cos\theta \Rightarrow \tan\theta=\mu \Rightarrow \theta=\tan^{-1}\mu$$



(c)

From the graph of $F=\dfrac{0.6(50)}{0.6\sin\theta+\cos\theta}$ for $0\le\theta\le 1$, we see that $\dfrac{dF}{d\theta}=0 \Rightarrow \theta\approx 0.54$. Checking this with part (b) and $\mu=0.6$, we calculate $\theta=\tan^{-1}0.6\approx 0.54$. So the value from the graph is consistent with the value in part (b).

35.

$$\begin{aligned}
\lim_{x\to 0}\frac{\sin 3x}{x} &= \lim_{x\to 0}\frac{3\sin 3x}{3x} && \text{[multiply numerator and denominator by 3]}\\
&= 3\lim_{3x\to 0}\frac{\sin 3x}{3x} && \text{[as } x\to 0 \text{ , } 3x\to 0 \text{]}\\
&= 3\lim_{\theta\to 0}\frac{\sin\theta}{\theta} && \text{[let } \theta=3x \text{]}\\
&= 3(1) && \text{[Equation 2]}\\
&= 3
\end{aligned}$$

36.

$$\begin{aligned}
\lim_{x\to 0}\frac{\sin 4x}{\sin 6x} &= \lim_{x\to 0}\left(\frac{\sin 4x}{x}\cdot\frac{x}{\sin 6x}\right)=\lim_{x\to 0}\frac{4\sin 4x}{4x}\cdot\lim_{x\to 0}\frac{6x}{6\sin 6x}\\
&= 4\lim_{x\to 0}\frac{\sin 4x}{4x}\cdot\frac{1}{6}\lim_{x\to 0}\frac{6x}{\sin 6x}=4(1)\cdot\frac{1}{6}(1)=\frac{2}{3}
\end{aligned}$$

37.

$$\begin{aligned}
\lim_{t\to 0}\frac{\tan 6t}{\sin 2t} &= \lim_{t\to 0}\left(\frac{\sin 6t}{t}\cdot\frac{1}{\cos 6t}\cdot\frac{t}{\sin 2t}\right)=\lim_{t\to 0}\frac{6\sin 6t}{6t}\cdot\lim_{t\to 0}\frac{1}{\cos 6t}\cdot\lim_{t\to 0}\frac{2t}{2\sin 2t}\\
&= 6\lim_{t\to 0}\frac{\sin 6t}{6t}\cdot\lim_{t\to 0}\frac{1}{\cos 6t}\cdot\frac{1}{2}\lim_{t\to 0}\frac{2t}{\sin 2t}=6(1)\cdot\frac{1}{1}\cdot\frac{1}{2}(1)=3
\end{aligned}$$

38.

$$\lim_{\theta\to 0}\frac{\cos\theta-1}{\sin\theta}=\lim_{\theta\to 0}\frac{\frac{\cos\theta-1}{\theta}}{\frac{\sin\theta}{\theta}}=\frac{\lim_{\theta\to 0}\frac{\cos\theta-1}{\theta}}{\lim_{\theta\to 0}\frac{\sin\theta}{\theta}}=\frac{0}{1}=0$$

39. $\lim_{\theta\to 0}\frac{\sin(\cos\theta)}{\sec\theta}=\frac{\sin\left(\lim_{\theta\to 0}(\cos\theta)\right)}{\lim_{\theta\to 0}(\sec\theta)}=\frac{\sin 1}{1}=\sin 1$

40.

$$\lim_{t\to 0}\frac{\sin^2 3t}{t^2}=\lim_{t\to 0}\left(\frac{\sin 3t}{t}\cdot\frac{\sin 3t}{t}\right)=\lim_{t\to 0}\frac{\sin 3t}{t}\cdot\lim_{t\to 0}\frac{\sin 3t}{t}$$

$$=\left(\lim_{t\to 0}\frac{\sin 3t}{t}\right)^2=\left(3\lim_{t\to 0}\frac{\sin 3t}{3t}\right)^2=(3\cdot 1)^2=9$$

41.

$$\lim_{x\to 0}\frac{\cot 2x}{x}=\lim_{x\to 0}\frac{\cos 2x\cdot\sin x}{\sin 2x}=\lim_{x\to 0}\left(\cos 2x\left[\frac{(\sin x)/x}{(\sin 2x)/x}\right]\right)=\lim_{x\to 0}\left(\cos 2x\left[\frac{\lim_{x\to 0}[(\sin x)/x]}{2\lim_{x\to 0}[(\sin 2x)/2x]}\right]\right)$$

$$=1\cdot\frac{1}{2\cdot 1}=\frac{1}{2}$$

42.

$$\lim_{x\to\pi/4}\frac{\sin x-\cos x}{\cos 2x}=\lim_{x\to\pi/4}\frac{\sin x-\cos x}{\cos^2 x-\sin^2 x}=\lim_{x\to\pi/4}\frac{\sin x-\cos x}{(\cos x+\sin x)(\cos x-\sin x)}$$

$$=\lim_{x\to\pi/4}\frac{-1}{\cos x+\sin x}=\frac{-1}{\cos\frac{\pi}{4}+\sin\frac{\pi}{4}}=\frac{-1}{\sqrt{2}}$$

43. Divide numerator and denominator by θ . ($\sin(\theta)$ also works.)

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta+\tan\theta}=\lim_{\theta\to 0}\frac{\frac{\sin\theta}{\theta}}{1+\frac{\sin\theta}{\theta}\cdot\frac{1}{\cos\theta}}=\frac{\lim_{\theta\to 0}\frac{\sin\theta}{\theta}}{1+\lim_{\theta\to 0}\frac{\sin\theta}{\theta}\lim_{\theta\to 0}\frac{1}{\cos\theta}}=\frac{1}{1+1\cdot 1}=\frac{1}{2}$$

44.

$$\lim_{x\to 1}\frac{\sin(x-1)}{x^2+x-2}=\lim_{x\to 1}\frac{\sin(x-1)}{(x+2)(x-1)}=\lim_{x\to 1}\frac{1}{x+2}\lim_{x\to 1}\frac{\sin(x-1)}{x-1}=\frac{1}{3}\cdot 1=\frac{1}{3}$$

45. **(a)** $\frac{d}{dx}\tan x=\frac{d}{dx}\frac{\sin x}{\cos x}\Rightarrow \sec^2 x=\frac{\cos x\cos x-\sin x(-\sin x)}{\cos^2 x}=\frac{\cos^2 x+\sin^2 x}{\cos^2 x}$. So $\sec^2 x=\frac{1}{\cos^2 x}$.

(b) $\frac{d}{dx}\sec x=\frac{d}{dx}\frac{1}{\cos x}\Rightarrow \sec x\tan x=\frac{(\cos x)(0)-1(-\sin x)}{\cos^2 x}$. So $\sec x\tan x=\frac{\sin x}{\cos^2 x}$.

(c) $\frac{d}{dx}(\sin x+\cos x)=\frac{d}{dx}\frac{1+\cot x}{x}\Rightarrow$

$$\cos x-\sin x=\frac{x(-\csc^2 x)-(1+\cot x)(-\ x\cot x)}{\csc^2 x}=\frac{x[-\csc^2 x+(1+\cot x)\cot x]}{\csc^2 x}$$

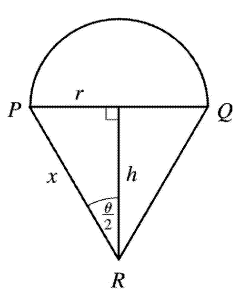
$$=\frac{-\csc^2 x+\cot^2 x+\cot x}{x}=\frac{-1+\cot x}{x}$$

So $\cos x-\sin x=\frac{\cot x-1}{x}$.

46. Let $|PR|=x$. Then we get the following formulas for r and h in terms of θ and x : $\sin\frac{\theta}{2}=\frac{r}{x}\Rightarrow r=x\sin\frac{\theta}{2}$ and $\cos\frac{\theta}{2}=\frac{h}{x}\Rightarrow h=x\cos\frac{\theta}{2}$. Now $A(\theta)=\frac{1}{2}\pi r^2$ and $B(\theta)=\frac{1}{2}(2r)h=rh$. So

$$\lim_{\theta\to 0^+}\frac{A(\theta)}{B(\theta)}=\lim_{\theta\to 0^+}\frac{\frac{1}{2}\pi r^2}{rh}=\frac{1}{2}\pi\lim_{\theta\to 0^+}\frac{r}{h}=\frac{1}{2}\pi\lim_{\theta\to 0^+}\frac{x\sin(\theta/2)}{x\cos(\theta/2)}$$

$$=\frac{1}{2}\pi\lim_{\theta\to 0^+}\tan(\theta/2)=0 .$$



47. By the definition of radian measure, $s=r\theta$, where r is the radius of the circle.

By drawing the bisector of the angle θ , we can see that $\sin \frac{\theta}{2} = \frac{d/2}{r} \Rightarrow d = 2r\sin \frac{\theta}{2}$.

So $\lim_{\theta \to 0^+} \frac{s}{d} = \lim_{\theta \to 0^+} \frac{r\theta}{2r\sin(\theta/2)} = \lim_{\theta \to 0^+} \frac{2\cdot(\theta/2)}{2\sin(\theta/2)} = \lim_{\theta \to 0} \frac{\theta/2}{\sin(\theta/2)} = 1$.

1. Let $u=g(x)=4x$ and $y=f(u)=\sin u$. Then $\frac{dy}{dx}=\frac{dy}{du}\ \frac{du}{dx}=(\cos u)(4)=4\cos 4x$.

2. Let $u=g(x)=4+3x$ and $y=f(u)=\sqrt{u}=u^{1/2}$. Then $\frac{dy}{dx}=\frac{dy}{du}\ \frac{du}{dx}=\frac{1}{2}\ u^{-1/2}(3)=\frac{3}{2\sqrt{u}}=\frac{3}{2\sqrt{4+3x}}$.

3. Let $u=g(x)=1-x^2$ and $y=f(u)=u^{10}$. Then $\frac{dy}{dx}=\frac{dy}{du}\ \frac{du}{dx}=(10u^9)(-2x)=-20x(1-x^2)^9$.

4. Let $u=g(x)=\sin x$ and $y=f(u)=\tan u$. Then $\frac{dy}{dx}=\frac{dy}{du}\ \frac{du}{dx}=(\sec^2 u)(\cos x)=(\sec^2 \text{u})(\sin x)\cdot\cos x$, or equivalently, $[\sec(\sin x)]^2\cos x$.

5. Let $u=g(x)=\sin x$ and $y=f(u)=\sqrt{u}$. Then $\frac{dy}{dx}=\frac{dy}{du}\ \frac{du}{dx}=\frac{1}{2}\ u^{-1/2}\cos x=\frac{\cos x}{2\sqrt{u}}=\frac{\cos x}{2\sqrt{\sin x}}$.

6. Let $u=g(x)=e^x$ and $y=f(u)=\sin u$. Then $\frac{dy}{dx}=\frac{dy}{du}\ \frac{du}{dx}=(\cos u)(e^x)=e^x\cos e^x$.

7. $F(x)=(x^3+4x)^7\Rightarrow F'(x)=7(x^3+4x)^6(3x^2+4)$ [or $7x^6(x^2+4)^6(3x^2+4)$]

8. $F(x)=(x^2-x+1)^3\Rightarrow F'(x)=3(x^2-x+1)^2(2x-1)$

9. $F(x)=\sqrt[4]{1+2x+x^3}=(1+2x+x^3)^{1/4}\Rightarrow$

$$F'(x)=\frac{1}{4}(1+2x+x^3)^{-3/4}\cdot\frac{d}{dx}(1+2x+x^3)=\frac{1}{4(1+2x+x^3)^{3/4}}\cdot(2+3x^2)$$

$$=\frac{2+3x^2}{4(1+2x+x^3)^{3/4}}=\frac{2+3x^2}{4\sqrt[4]{(1+2x+x^3)^3}}$$

10. $f(x)=(1+x^4)^{2/3}\Rightarrow f'(x)=\frac{2}{3}(1+x^4)^{-1/3}(4x^3)=\frac{8x^3}{3\sqrt[3]{1+x^4}}$

11. $g(t)=\frac{1}{(t^4+1)^3}=(t^4+1)^{-3}\Rightarrow g'(t)=-3(t^4+1)^{-4}(4t^3)=-12t^3(t^4+1)^{-4}=\frac{-12t^3}{(t^4+1)^4}$

12.

$f(t)=\sqrt[3]{1+\tan t}=(1+\tan t)^{1/3} \Rightarrow f'(t)=\frac{1}{3}(1+\tan t)^{-2/3}\sec^2 t=\frac{\sec^2 t}{3\sqrt[3]{(1+\tan t)^2}}$

13. $y=\cos(a^3+x^3) \Rightarrow y'=-\sin(a^3+x^3)\cdot 3x^2$ [a^3 is just a constant] $=-3x^2\sin(a^3+x^3)$

14. $y=a^3+\cos^3 x \Rightarrow y'=3(\cos x)^2(-\sin x)$ [a^3 is just a constant] $=-3\sin x\cos^2 x$

15. $y=e^{-mx} \Rightarrow y'=e^{-mx}\frac{d}{dx}(-mx)=e^{-mx}(-m)=-me^{-mx}$

16. $y=4\sec 5x \Rightarrow y'=4\sec 5x\tan 5x(5)=20\sec 5x\tan 5x$

17. $g(x)=(1+4x)^5(3+x-x^2)^8 \Rightarrow$

$$
\begin{aligned}
g'(x) &=(1+4x)^5\cdot 8(3+x-x^2)^7(1-2x)+(3+x-x^2)^8\cdot 5(1+4x)^4\cdot 4 \\
&=4(1+4x)^4(3+x-x^2)^7[2(1+4x)(1-2x)+5(3+x-x^2)] \\
&=4(1+4x)^4(3+x-x^2)^7[(2+4x-16x^2)+(15+5x-5x^2)] \\
&=4(1+4x)^4(3+x-x^2)^7(17+9x-21x^2)
\end{aligned}
$$

18. $h(t)=(t^4-1)^3(t^3+1)^4 \Rightarrow$

$$
\begin{aligned}
h'(t) &=(t^4-1)^3\cdot 4(t^3+1)^3(3t^2)+(t^3+1)^4\cdot 3(t^4-1)^2(4t^3) \\
&=12t^2(t^4-1)^2(t^3+1)^3[(t^4-1)+t(t^3+1)]=12t^2(t^4-1)^2(t^3+1)^3(2t^4+t-1)
\end{aligned}
$$

19. $y=(2x-5)^4(8x^2-5)^{-3} \Rightarrow$

$$
\begin{aligned}
y' &=4(2x-5)^3(2)(8x^2-5)^{-3}+(2x-5)^4(-3)(8x^2-5)^{-4}(16x) \\
&=8(2x-5)^3(8x^2-5)^{-3}-48x(2x-5)^4(8x^2-5)^{-4}
\end{aligned}
$$

20. $y=(x^2+1)(x^2+2)^{1/3} \Rightarrow y'=2x(x^2+2)^{1/3}+(x^2+1)\left(\frac{1}{3}\right)(x^2+2)^{-2/3}(2x)=2x(x^2+2)^{1/3}\left[1+\frac{x^2+1}{3(x^2+2)}\right]$

21. $y=xe^{-x^2} \Rightarrow y'=xe^{-x^2}(-2x)+e^{-x^2}\cdot 1=e^{-x^2}(-2x^2+1)=e^{-x^2}(1-2x^2)$

22. $y=e^{-5x}\cos 3x \Rightarrow y'=e^{-5x}(-3\sin 3x)+(\cos 3x)(-5e^{-5x})=-e^{-5x}(3\sin 3x+5\cos 3x)$

23. $y=e^{x\cos x} \Rightarrow y'=e^{x\cos x}\cdot \frac{d}{dx}(x\cos x)=e^{x\cos x}\left[x(-\sin x)+(\cos x)\cdot 1\right]=e^{x\cos x}(\cos x-x\sin x)$

24. Using Formula 5 and the Chain Rule, $y=10^{1-x^2} \Rightarrow y'=10^{1-x^2}(\ln 10)\cdot \frac{d}{dx}(1-x^2)=-2x(\ln 10)10^{1-x^2}$.

25. $F(z)=\sqrt{\frac{z-1}{z+1}}=\left(\frac{z-1}{z+1}\right)^{1/2} \Rightarrow$

$$F'(z)=\frac{1}{2}\left(\frac{z-1}{z+1}\right)^{-1/2}\cdot \frac{d}{dz}\left(\frac{z-1}{z+1}\right)=\frac{1}{2}\left(\frac{z+1}{z-1}\right)^{1/2}\cdot \frac{(z+1)(1)-(z-1)(1)}{(z+1)^2}$$

$$=\frac{1}{2}\frac{(z+1)^{1/2}}{(z-1)^{1/2}}\cdot \frac{z+1-z+1}{(z+1)^2}=\frac{1}{2}\frac{(z+1)^{1/2}}{(z-1)^{1/2}}\cdot \frac{2}{(z+1)^2}=\frac{1}{(z-1)^{1/2}(z+1)^{3/2}}$$

26. $G(y)=\frac{(y-1)^4}{(y^2+2y)^5} \Rightarrow$

$$G'(y)=\frac{(y^2+2y)^5\cdot 4(y-1)^3\cdot 1-(y-1)^4\cdot 5(y^2+2y)^4(2y+2)}{[(y^2+2y)^5]^2}$$

$$=\frac{2(y^2+2y)^4(y-1)^3[2(y^2+2y)-5(y-1)(y+1)]}{(y^2+2y)^{10}}$$

$$=\frac{2(y-1)^3[(2y^2+4y)+(-5y^2+5)]}{(y^2+2y)^6}=\frac{2(y-1)^3(-3y^2+4y+5)}{(y^2+2y)^6}$$

27. $y=\frac{r}{\sqrt{r^2+1}} \Rightarrow$

$$y'=\frac{\sqrt{r^2+1}\,(1)-r\cdot \frac{1}{2}(r^2+1)^{-1/2}(2r)}{\left(\sqrt{r^2+1}\right)^2}=\frac{\sqrt{r^2+1}-\frac{r^2}{\sqrt{r^2+1}}}{\left(\sqrt{r^2+1}\right)^2}=\frac{\frac{\sqrt{r^2+1}\sqrt{r^2+1}-r^2}{\sqrt{r^2+1}}}{\left(\sqrt{r^2+1}\right)^2}$$

$$=\frac{(r^2+1)-r^2}{\left(\sqrt{r^2+1}\right)^3}=\frac{1}{(r^2+1)^{3/2}}\ \text{ or } (r^2+1)^{-3/2}$$

Another solution: Write y as a product and make use of the Product Rule. $y=r(r^2+1)^{-1/2}\Rightarrow$

$$y'=r\cdot -\frac{1}{2}(r^2+1)^{-3/2}(2r)+r^2+1)^{-1/2}\cdot 1$$

$$=(r^2+1)^{-3/2}[-r^2+(r^2+1)^1]=(r^2+1)^{-3/2}(1)=(r^2+1)^{-3/2}$$

The step that students usually have trouble with is factoring out $(r^2+1)^{-3/2}$. But this is no different than factoring out x^2 from x^2+x^5; that is, we are just factoring out a factor with the *smallest* exponent that appears on it. In this case, $-\frac{3}{2}$ is smaller than $-\frac{1}{2}$.

28. $y=\dfrac{e^{2u}}{e^u+e^{-u}}\Rightarrow$

$$y'=\frac{(e^u+e^{-u})(e^{2u}\cdot 2)-e^{2u}(e^u-e^{-u})}{(e^u+e^{-u})^2}=\frac{e^{2u}(2e^u+2e^{-u}-e^u+e^{-u})}{(e^u+e^{-u})^2}=\frac{e^{2u}(e^u+3e^{-u})}{(e^u+e^{-u})^2}$$

Another solution: Eliminate negative exponents by first changing the form of y.

$$y=\frac{e^{2u}}{e^u+e^{-u}}\cdot\frac{e^u}{e^u}=\frac{e^{3u}}{e^{2u}+1}\Rightarrow$$

$$y'=\frac{(e^{2u}+1)(3e^{3u})-e^{3u}(2e^{2u})}{(e^{2u}+1)^2}=\frac{e^{3u}(3e^{2u}+3-2e^{2u})}{(e^{2u}+1)^2}=\frac{e^{3u}(e^{2u}+3)}{(e^{2u}+1)^2}$$

29. $y=\tan(\cos x)\Rightarrow y'=\sec^2(\cos x)\cdot(-\sin x)=-\sin x\cdot\sec^2(\cos x)$

30. $y=\dfrac{\sin^2 x}{\cos x}\Rightarrow$

$$y'=\frac{\cos x(2\sin x\cdot\cos x)-\sin^2 x(-\sin x)}{\cos^2 x}=\frac{\sin x(2\cos^2 x+\sin^2 x)}{\cos^2 x}=\frac{\sin x(1+\cos^2 x)}{\cos^2 x}$$

$$=\sin x(1+\sec^2 x)$$

Another method: $y=\tan x\cdot\sin x\Rightarrow$

$y'=\sec^2 x\cdot \sin x+\tan x\cdot \cos x=\sec^2 x\cdot \sin x+\sin x$

31. Using Formula 5 and the Chain Rule, $y=2^{\sin \pi x}\Rightarrow$
$y'=2^{\sin \pi x}(\ln 2)\cdot \frac{d}{dx}(\sin \pi x)=2^{\sin \pi x}(\ln 2)\cdot \cos \pi x\cdot \pi=2^{\sin \pi x}(\pi \ln 2)\cos \pi x$

32. $y=\tan^2(3\theta)=(\tan 3\theta)^2\Rightarrow y'=2(\tan 3\theta)\cdot \frac{d}{d\theta}(\tan 3\theta)=2\tan 3\theta\cdot \sec^2 3\theta\cdot 3=6\tan 3\theta \sec^2 3\theta$

33. $y=(1+\cos^2 x)^6\Rightarrow y'=6(1+\cos^2 x)^5\cdot 2\cos x(-\sin x)=-12\cos x\sin x(1+\cos^2 x)^5$

34. $y=x\sin \frac{1}{x}\Rightarrow y'=\sin \frac{1}{x}+x\cos \frac{1}{x}\left(-\frac{1}{x^2}\right)=\sin \frac{1}{x}-\frac{1}{x}\cos \frac{1}{x}$

35. $y=\sec^2 x+\tan^2 x=(\sec x)^2+(\tan x)^2\Rightarrow$
$y'=2(\sec x)\cdot (\sec x\tan x)+2(\tan x)(\sec^2 x)=2\sec^2 x\cdot \tan x+2\sec^2 x\cdot \tan x=4\sec^2 x\cdot \tan x$

36. $y=e^{k\tan \sqrt{x}}\Rightarrow y'=e^{k\tan \sqrt{x}}\cdot \frac{d}{dx}(k\tan \sqrt{x})=e^{k\tan \sqrt{x}}\left(k\sec^2\sqrt{x}\cdot \frac{1}{2}x^{-1/2}\right)=\frac{k\sec^2\sqrt{x}}{2\sqrt{x}}e^{k\tan \sqrt{x}}$

37. $y=\cot^2(\sin \theta)=[\cot (\sin \theta)]^2\Rightarrow$
$y'=2[\cot (\sin \theta)]\cdot \frac{d}{d\theta}[\cot (\sin \theta)]=2\cot (\sin \theta)\cdot [-\csc^2(\sin \theta)\cdot \cos \theta]=-2\cos \theta\cdot \cot (\sin \theta)\cdot \csc^2(\sin \theta)$

38. $y=\sin (\sin (\sin x))\Rightarrow y'=\cos (\sin (\sin x))\frac{d}{dx}(\sin (\sin x))=\cos (\sin (\sin x))\cdot \cos (\sin x)\cdot \cos x$

39. $y=\sqrt{x+\sqrt{x}}\Rightarrow y'=\frac{1}{2}(x+\sqrt{x})^{-1/2}\left(1+\frac{1}{2}x^{-1/2}\right)=\frac{1}{2\sqrt{x+\sqrt{x}}}\left(1+\frac{1}{2\sqrt{x}}\right)$

40. $y=\sqrt{x+\sqrt{x+\sqrt{x}}}\Rightarrow y'=\frac{1}{2}\left(x+\sqrt{x+\sqrt{x}}\right)^{-1/2}\left[1+\frac{1}{2}(x+\sqrt{x})^{-1/2}\left(1+\frac{1}{2}x^{-1/2}\right)\right]$

41. $y=\sin (\tan \sqrt{\sin x})\Rightarrow$
$y'=\cos (\tan \sqrt{\sin x})\cdot \frac{d}{dx}(\tan \sqrt{\sin x})=\cos (\tan \sqrt{\sin x})\cdot \sec^2\sqrt{\sin x}\cdot \frac{d}{dx}(\sin x)^{1/2}$

$=\cos (\tan \sqrt{\sin x})\sec^2\sqrt{\sin x}\cdot \frac{1}{2}(\sin x)^{-1/2}\cdot \cos x$

$$=\cos(\tan\sqrt{\sin x})\left(\sec^2\sqrt{\sin x}\right)\left(\frac{1}{2\sqrt{\sin x}}\right)(\cos x)$$

42. $y=2^{3^{x^2}} \Rightarrow y'=2^{3^{x^2}}(\ln 2)\frac{d}{dx}\left(3^{x^2}\right)=2^{3^{x^2}}(\ln 2)3^{x^2}(\ln 3)(2x)$

43. $y=(1+2x)^{10} \Rightarrow y'=10(1+2x)^9\cdot 2=20(1+2x)^9$. At $(0,1)$, $y'=20(1+0)^9=20$, and an equation of the tangent line is $y-1=20(x-0)$, or $y=20x+1$.

44. $y=\sin x+\sin^2 x \Rightarrow y'=\cos x+2\sin x\cos x$. At $(0,0)$, $y'=1$, and an equation of the tangent line is $y-0=1(x-0)$, or $y=x$.

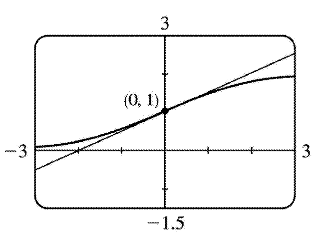
45. $y=\sin(\sin x) \Rightarrow y'=\cos(\sin x)\cdot\cos x$. At $(\pi,0)$, $y'=\cos(\sin\pi)\cdot\cos\pi=\cos(0)\cdot(-1)=1(-1)=-1$, and an equation of the tangent line is $y-0=-1(x-\pi)$, or $y=-x+\pi$.

46. $y=x^2e^{-x} \Rightarrow y'=x^2(-e^{-x})+e^{-x}(2x)=2xe^{-x}-x^2e^{-x}$. At $\left(1,\ \frac{1}{e}\right)$, $y'=2e^{-1}-e^{-1}=\frac{1}{e}$. So an equation of the tangent line is $y-\frac{1}{e}=\frac{1}{e}(x-1)$ or $y=\frac{1}{e}x$.

47. **(a)** $y=\dfrac{2}{1+e^{-x}} \Rightarrow y'=\dfrac{(1+e^{-x})(0)-2(-e^{-x})}{(1+e^{-x})^2}=\dfrac{2e^{-x}}{(1+e^{-x})^2}$.

At $(0,\ 1)$, $y'=\dfrac{2e^0}{(1+e^0)^2}=\dfrac{2(1)}{(1+1)^2}=\dfrac{2}{2^2}=\dfrac{1}{2}$.

So an equation of the tangent line is $y-1=\frac{1}{2}(x-0)$ or $y=\frac{1}{2}x+1$.

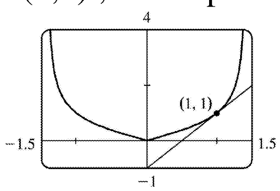
(b) 

48. **(a)** For $x>0$, $|x|=x$, and $y=f(x)=\dfrac{x}{\sqrt{2-x^2}} \Rightarrow$

$$f'(x) = \frac{\sqrt{2-x^2}\,(1)-x\left(\frac{1}{2}\right)(2-x^2)^{-1/2}(-2x)}{\left(\sqrt{2-x^2}\right)^2} \cdot \frac{(2-x^2)^{1/2}}{(2-x^2)^{1/2}}$$

$$= \frac{(2-x^2)+x^2}{(2-x^2)^{3/2}} = \frac{2}{(2-x^2)^{3/2}}$$

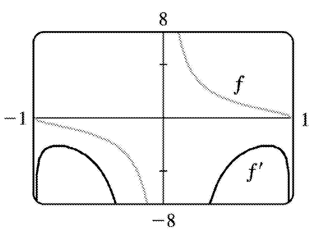
So at $(1,1)$, the slope of the tangent line is $f'(1)=2$ and its equation is $y-1=2(x-1)$ or $y=2x-1$.

(b) 

49. **(a)** $f(x)=\dfrac{\sqrt{1-x^2}}{x} \Rightarrow$

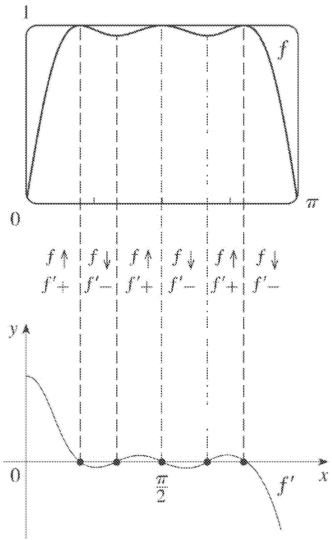
$$f'(x) = \frac{x\cdot\frac{1}{2}(1-x^2)^{-1/2}(-2x)-\sqrt{1-x^2}\,(1)}{x^2} \cdot \frac{\sqrt{1-x^2}}{\sqrt{1-x^2}}$$

$$= \frac{-x^2-(1-x^2)}{x^2\sqrt{1-x^2}} = \frac{-1}{x^2\sqrt{1-x^2}}$$

(b) 

Notice that all tangents to the graph of f have negative slopes and $f'(x)<0$ always.

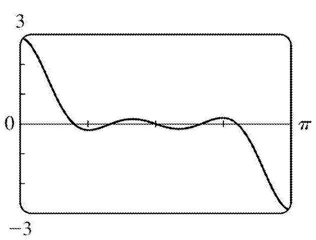
50. **(a)**



From the graph of f , we see that there are 5 horizontal tangents, so there must be 5 zeros on the graph of f' . From the symmetry of the graph of f , we must have the graph of f' as high at $x=0$ as it is low at $x=\pi$. The intervals of increase and decrease as well as the signs of f' are indicated in the figure.

(b)

$$\begin{aligned} f(x) &= \sin(x+\sin 2x) \Rightarrow \\ f'(x) &= \cos(x+\sin 2x)\cdot \frac{d}{dx}(x+\sin 2x) \\ &= \cos(x+\sin 2x)(1+2\cos 2x) \end{aligned}$$



51. For the tangent line to be horizontal, $f'(x)=0$. $f(x)=2\sin x+\sin^2 x \Rightarrow f'(x)=2\cos x+2\sin x\cos x=0$ $\Leftrightarrow 2\cos x(1+\sin x)=0 \Leftrightarrow \cos x=0$ or $\sin x=-1$, so $x=\frac{\pi}{2}+2n\pi$ or $\frac{3\pi}{2}+2n\pi$, where n is any integer. Now $f\left(\frac{\pi}{2}\right)=3$ and $f\left(\frac{3\pi}{2}\right)=-1$, so the points on the curve with a horizontal tangent are $\left(\frac{\pi}{2}+2n\pi,3\right)$ and $\left(\frac{3\pi}{2}+2n\pi,-1\right)$, where n is any integer.

52. $f(x)=\sin 2x-2\sin x \Rightarrow$

$f'(x)=2\cos 2x-2\cos x=4\cos^2 x-2\cos x-2$, and $4\cos^2 x-2\cos x-2=0 \Leftrightarrow (\cos x-1)(4\cos x+2)=0 \Leftrightarrow$ $\cos x=1$ or $\cos x=-\frac{1}{2}$. So $x=2n\pi$ or $(2n+1)\pi \pm \frac{\pi}{3}$, n any integer.

53. $F(x)=f(g(x)) \Rightarrow F'(x)=f'(g(x))\cdot g'(x)$,
so $F'(3)=f'(g(3))\cdot g'(3)=f'(6)\cdot g'(3)=7\cdot 4=28$. Notice that we did not use $f'(3)=2$.

54. $w=u\circ v \Rightarrow w(x)=u(v(x)) \Rightarrow w'(x)=u'(v(x))\cdot v'(x)$, so
$w'(0)=u'(v(0))\cdot v'(0)=u'(2)\cdot v'(0)=4\cdot 5=20$. The other pieces of information, $u(0)=1$, $u'(0)=3$, and $v'(2)=6$, were not needed.

55. **(a)** $h(x)=f(g(x)) \Rightarrow h'(x)=f'(g(x))\cdot g'(x)$, so $h'(1)=f'(g(1))\cdot g'(1)=f'(2)\cdot 6=5\cdot 6=30$.
(b) $H(x)=g(f(x)) \Rightarrow H'(x)=g'(f(x))\cdot f'(x)$, so $H'(1)=g'(f(1))\cdot f'(1)=g'(3)\cdot 4=9\cdot 4=36$.

56. **(a)** $F(x)=f(f(x)) \Rightarrow F'(x)=f'(f(x))\cdot f'(x)$, so $F'(2)=f'(f(2))\cdot f'(2)=f'(1)\cdot 5=4\cdot 5=20$.
(b) $G(x)=g(g(x)) \Rightarrow G'(x)=g'(g(x))\cdot g'(x)$, so $G'(3)=g'(g(3))\cdot g'(3)=g'(2)\cdot 9=7\cdot 9=63$.

57. **(a)** $u(x)=f(g(x)) \Rightarrow u'(x)=f'(g(x))g'(x)$. So $u'(1)=f'(g(1))g'(1)=f'(3)g'(1)$. To find $f'(3)$, note that f is linear from $(2,4)$ to $(6,3)$, so its slope is $\frac{3-4}{6-2}=-\frac{1}{4}$. To find $g'(1)$, note that g is linear from $(0,6)$ to $(2,0)$, so its slope is $\frac{0-6}{2-0}=-3$. Thus, $f'(3)g'(1)=\left(-\frac{1}{4}\right)(-3)=\frac{3}{4}$.
(b) $v(x)=g(f(x)) \Rightarrow v'(x)=g'(f(x))f'(x)$. So $v'(1)=g'(f(1))f'(1)=g'(2)f'(1)$, which does not exist since $g'(2)$ does not exist.
(c) $w(x)=g(g(x)) \Rightarrow w'(x)=g'(g(x))g'(x)$. So $w'(1)=g'(g(1))g'(1)=g'(3)g'(1)$. To find $g'(3)$, note that g is linear from $(2,0)$ to $(5,2)$, so its slope is $\frac{2-0}{5-2}=\frac{2}{3}$. Thus, $g'(3)\cdot g'(1)=\left(\frac{2}{3}\right)(-3)=-2$.

58. **(a)** $h(x)=f(f(x)) \Rightarrow h'(x)=f'(f(x))f'(x)$.
So $h'(2)=f'(f(2))f'(2)=f'(1)f'(2)\approx(-1)(-1)=1$.
(b) $g(x)=f(x^2) \Rightarrow g'(x)=f'(x^2)\cdot \frac{d}{dx}(x^2)=f'(x^2)(2x)$. So $g'(2)=f'(2^2)(2\cdot 2)=4f'(4)\approx 4(1.5)=6$.

59. $h(x)=f(g(x))\Rightarrow h'(x)=f'(g(x))g'(x)$. So $h'(0.5)=f'(g(0.5))g'(0.5)=f'(0.1)g'(0.5)$. We can estimate the derivatives by taking the average of two secant slopes.

For $f'(0.1)$: $m_1=\dfrac{14.8-12.6}{0.1-0}=22$, $m_2=\dfrac{18.4-14.8}{0.2-0.1}=36$. So $f'(0.1)\approx\dfrac{m_1+m_2}{2}=\dfrac{22+36}{2}=29$.

For $g'(0.5)$: $m_1=\dfrac{0.10-0.17}{0.5-0.4}=-0.7$, $m_2=\dfrac{0.05-0.10}{0.6-0.5}=-0.5$. So $g'(0.5)\approx\dfrac{m_1+m_2}{2}=-0.6$.

Hence, $h'(0.5)=f'(0.1)g'(0.5)\approx(29)(-0.6)=-17.4$.

60. $g(x)=f(f(x))\Rightarrow g'(x)=f'(f(x))f'(x)$. So $g'(1)=f'(f(1))f'(1)=f'(2)f'(1)$.

For $f'(2)$: $m_1=\dfrac{3.1-2.4}{2.0-1.5}=1.4$, $m_2=\dfrac{4.4-3.1}{2.5-2.0}=2.6$. So $f'(2)\approx\dfrac{m_1+m_2}{2}=2$.

For $f'(1)$: $m_1=\dfrac{2.0-1.8}{1.0-0.5}=0.4$, $m_2=\dfrac{2.4-2.0}{1.5-1.0}=0.8$. So $f'(1)\approx\dfrac{m_1+m_2}{2}=0.6$.

Hence, $g'(1)=f'(2)f'(1)\approx(2)(0.6)=1.2$.

61. **(a)** $F(x)=f(e^x)\Rightarrow F'(x)=f'(e^x)\dfrac{d}{dx}(e^x)=f'(e^x)e^x$

(b) $G(x)=e^{f(x)}\Rightarrow G'(x)=e^{f(x)}\dfrac{d}{dx}f(x)=e^{f(x)}f'(x)$

62. **(a)** $F(x)=f(x^\alpha)\Rightarrow F'(x)=f'(x^\alpha)\dfrac{d}{dx}(x^\alpha)=f'(x^\alpha)\alpha x^{\alpha-1}$

(b) $G(x)=[f(x)]^\alpha\Rightarrow G'(x)=\alpha[f(x)]^{\alpha-1}f'(x)$

63. **(a)** $f(x)=L(x^4)\Rightarrow f'(x)=L'(x^4)\cdot 4x^3=(1/x^4)\cdot 4x^3=4/x$ for $x>0$.

(b) $g(x)=L(4x)\Rightarrow g'(x)=L'(4x)\cdot 4=(1/(4x))\cdot 4=1/x$ for $x>0$.

(c) $F(x)=[L(x)]^4\Rightarrow F'(x)=4[L(x)]^3\cdot L'(x)=4[L(x)]^3\cdot(1/x)=4[L(x)]^3/x$

(d) $G(x)=L(1/x)\Rightarrow G'(x)=L'(1/x)\cdot(-1/x^2)=(1/(1/x))\cdot(-1/x^2)=x\cdot(-1/x^2)=-1/x$ for $x>0$.

64. $r(x)=f(g(h(x)))\Rightarrow r'(x)=f'(g(h(x)))\cdot g'(h(x))\cdot h'(x)$, so
$r'(1)=f'(g(h(1)))\cdot g'(h(1))\cdot h'(1)=f'(g(2))\cdot g'(2)\cdot 4=f'(3)\cdot 5\cdot 4=6\cdot 5\cdot 4=120$

65. $s(t)=10+\dfrac{1}{4}\sin(10\pi t)\Rightarrow$ the velocity after t seconds is

$v(t)=s'(t)=\frac{1}{4}\cos(10\pi t)(10\pi)=\frac{5\pi}{2}\cos(10\pi t)$ cm / s.

66. **(a)** $s=A\cos(\omega t+\delta)\Rightarrow$ velocity $=s'=-\omega A\sin(\omega t+\delta)$.

(b) If $A\neq 0$ and $\omega\neq 0$, then $s'=0\Leftrightarrow\sin(\omega t+\delta)=0\Leftrightarrow\omega t+\delta=n\pi\Leftrightarrow t=\frac{n\pi-\delta}{\omega}$, n an integer.

67. **(a)** $B(t)=4.0+0.35\sin\frac{2\pi t}{5.4}\Rightarrow\frac{dB}{dt}=\left(0.35\cos\frac{2\pi t}{5.4}\right)\left(\frac{2\pi}{5.4}\right)=\frac{0.7\pi}{5.4}\cos\frac{2\pi t}{5.4}=\frac{7\pi}{54}\cos\frac{2\pi t}{5.4}$

(b) At $t=1$, $\frac{dB}{dt}=\frac{7\pi}{54}\cos\frac{2\pi}{5.4}\approx 0.16$.

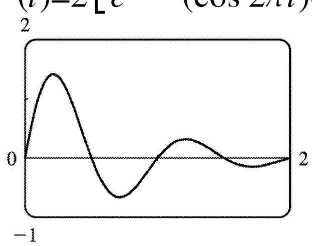
68. $L(t)=12+2.8\sin\left(\frac{2\pi}{365}(t-80)\right)\Rightarrow L'(t)=2.8\cos\left(\frac{2\pi}{365}(t-80)\right)\left(\frac{2\pi}{365}\right)$.

On March 21, $t=80$, and $L'(80)\approx 0.0482$ hours per day. On May 21, $t=141$, and $L'(141)\approx 0.02398$, which is approximately one–half of $L'(80)$.

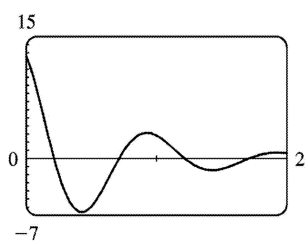
69. $s(t)=2e^{-1.5t}\sin 2\pi t\Rightarrow$

$v(t)=s'(t)=2\left[e^{-1.5t}(\cos 2\pi t)(2\pi)+(\sin 2\pi t)e^{-1.5t}(-1.5)\right]=2e^{-1.5t}(2\pi\cos 2\pi t-1.5\sin 2\pi t)$

Graph of position



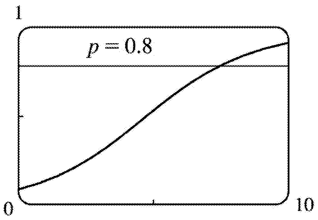
Graph of velocity



70. **(a)** $\lim_{t\to\infty}p(t)=\lim_{t\to\infty}\frac{1}{1+ae^{-kt}}=\frac{1}{1+a\cdot 0}=1$, since $k>0\Rightarrow -kt\to-\infty\Rightarrow e^{-kt}\to 0$.

(b) $p(t)=(1+ae^{-kt})^{-1}\Rightarrow\frac{dp}{dt}=-(1+ae^{-kt})^{-2}(-kae^{-kt})=\frac{kae^{-kt}}{(1+ae^{-kt})^2}$

(c)

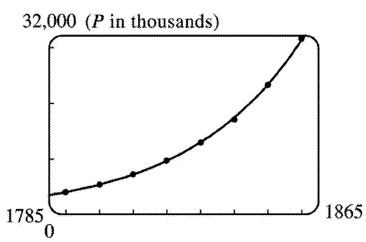


From the graph of $p(t)=(1+10e^{-0.5t})^{-1}$, it seems that $p(t)=0.8$ (indicating that 80% of the population has heard the rumor) when $t\approx 7.4$ hours.

71. **(a)** Using a calculator or CAS, we obtain the model $Q=ab^{t}$ with $a=100.0124369$ and $b=0.000045145933$. We can change this model to one with base e and exponent $\ln b$ [$b^{t}=e^{t\ln b}$ from precalculus mathematics or from Section 7.3]: $Q=ae^{t\ln b}=100.012437e^{-10.005531t}$.

(b) Use $Q'(t)=ab^{t}\ln b$ or the calculator command nDeriv(Y_1, X, .04) with $Y_1=ab^{x}$ to get $Q'(0.04)\approx -670.63\mu A$. The result of Example 2 in Section 2.1 was $-670\mu A$.

72. **(a)** $P=ab^{t}$ with $a=4.502714\times 10^{-20}$ and $b=1.029953851$,
where P is measured in thousands of people. The fit appears to be very good.



(b) For 1800: $m_1=\dfrac{5308-3929}{1800-1790}=137.9$, $m_2=\dfrac{7240-5308}{1810-1800}=193.2$.

So $P'(1800)\approx (m_1+m_2)/2=165.55$ thousand people / year.

For 1850: $m_1=\dfrac{23{,}192-17{,}063}{1850-1840}=612.9$, $m_2=\dfrac{31{,}443-23{,}192}{1860-1850}=825.1$.

So $P'(1850)\approx (m_1+m_2)/2=719$ thousand people / year.

(c) Use the calculator command nDeriv(Y_1, X, .04) with $Y_1=ab^{x}$ to get

$P'(1800)\approx 156.85$ and $P'(1850)\approx 686.07$. These estimates are somewhat less than the ones in part (b).

(d) $P(1870)\approx 41{,}946.56$. The difference of 3.4 million people is most likely due to the Civil War (1861—1865).

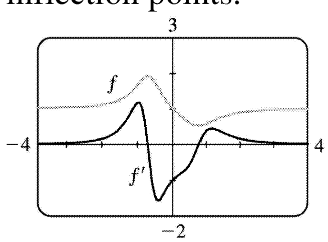
73. **(a)** Derive gives $g'(t)=\dfrac{45(t-2)^8}{(2t+1)^{10}}$ without simplifying. With either Maple or Mathematica, we first get $g'(t)=9\dfrac{(t-2)^8}{(2t+1)^9}-18\dfrac{(t-2)^9}{(2t+1)^{10}}$, and the simplification command results in the above expression.

(b) Derive gives $y'=2(x^3-x+1)^3(2x+1)^4(17x^3+6x^2-9x+3)$ without simplifying.
With either Maple or Mathematica, we first get $y'=10(2x+1)^4(x^3-x+1)^4+4(2x+1)^5(x^3-x+1)^3(3x^2-1)$.
If we use Mathematica's **Factor** or **Simplify**, or Maple's **factor**, we get the above expression, but Maple's **simplify** gives the polynomial expansion instead. For locating horizontal tangents, the factored form is the most helpful.

74. **(a)** $f(x)=\left(\dfrac{x^4-x+1}{x^4+x+1}\right)^{1/2}$. Derive gives $f'(x)=\dfrac{(3x^4-1)\sqrt{\dfrac{x^4-x+1}{x^4+x+1}}}{(x^4+x+1)(x^4-x+1)}$ whereas either Maple or Mathematica give $f'(x)=\dfrac{3x^4-1}{\sqrt{\dfrac{x^4-x+1}{x^4+x+1}}\,(x^4+x+1)^2}$ after simplification.

(b) $f'(x)=0\Leftrightarrow 3x^4-1=0\Leftrightarrow x=\pm\sqrt[4]{\dfrac{1}{3}}\approx\pm 0.7598$.

(c) $f'(x)=0$ where f has horizontal tangents. f' has two maxima and one minimum where f has inflection points.



75. **(a)** If f is even, then $f(x)=f(-x)$. Using the Chain Rule to differentiate this equation, we get $f'(x)=f'(-x)\dfrac{d}{dx}(-x)=-f'(-x)$. Thus, $f'(-x)=-f'(x)$, so f' is odd.

(b) If f is odd, then $f(x)=-f(-x)$. Differentiating this equation, we get $f'(x)=-f'(-x)(-1)=f'(-x)$, so f' is even.

76.

$$\left[\frac{f(x)}{g(x)}\right]' = \left\{f(x)[g(x)]^{-1}\right\}' = f'(x)[g(x)]^{-1} + (-1)[g(x)]^{-2}g'(x)f(x)$$

$$= \frac{f'(x)}{g(x)} - \frac{f(x)g'(x)}{[g(x)]^2} = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

77. **(a)**

$$\frac{d}{dx}\left(\sin^n x\cos nx\right) = n\sin^{n-1}x\cos x\cos nx + \sin^n x(-n\sin nx) \quad \text{[Product Rule]}$$

$$= n\sin^{n-1}x(\cos nx\cos x - \sin nx\sin x) \quad \text{[factor out } n\sin^{n-1}x\text{]}$$

$$= n\sin^{n-1}x\cos(nx+x) \quad \text{[Addition Formula for cosine]}$$

$$= n\sin^{n-1}x\cos[(n+1)x] \quad \text{[factor out } x\text{]}$$

(b)

$$\frac{d}{dx}\left(\cos^n x\cos nx\right) = n\cos^{n-1}x(-\sin x)\cos nx + \cos^n x(-n\sin nx) \quad \text{[Product Rule]}$$

$$= -n\cos^{n-1}x(\cos nx\sin x + \sin nx\cos x) \quad \text{[factor out } n\cos^{n-1}x\text{]}$$

$$= -n\cos^{n-1}x\sin(nx+x) \quad \text{[Addition Formula for sine]}$$

$$= -n\cos^{n-1}x\sin[(n+1)x] \quad \text{[factor out } x\text{]}$$

78. ‘‘The rate of change of y^5 with respect to x is eighty times the rate of change of y with respect to x ’’ $\Leftrightarrow \frac{d}{dx}y^5 = 80\frac{dy}{dx} \Leftrightarrow 5y^4\frac{dy}{dx} = 80\frac{dy}{dx} \Leftrightarrow 5y^4 = 80$ (Note that $\frac{dy}{dx} \neq 0$ since the curve never has a horizontal tangent) $\Leftrightarrow y^4 = 16 \Leftrightarrow y = 2$ (since $y > 0$ for all x)

79. Since $\theta^\circ = \left(\frac{\pi}{180}\right)\theta$ rad, we have

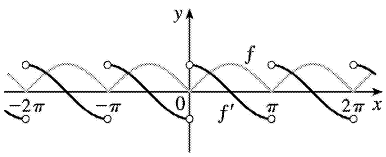
$$\frac{d}{d\theta}\left(\sin\theta^\circ\right) = \frac{d}{d\theta}\left(\sin\frac{\pi}{180}\theta\right) = \frac{\pi}{180}\cos\frac{\pi}{180}\theta = \frac{\pi}{180}\cos\theta^\circ .$$

80. **(a)** $f(x) = |x| = \sqrt{x^2} = (x^2)^{1/2} \Rightarrow f'(x) = \frac{1}{2}(x^2)^{-1/2}(2x) = \frac{x}{\sqrt{x^2}} = \frac{x}{|x|}$ for $x \neq 0$.

f is not differentiable at $x = 0$.

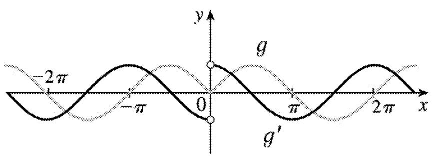
(b) $f(x)=|\sin x|=\sqrt{\sin^2 x} \Rightarrow$

$$f'(x)=\frac{1}{2}(\sin^2 x)^{-1/2}\cdot 2\sin x\cdot\cos x=\frac{\sin x}{|\sin x|}\cos x=\begin{cases}\cos x & \text{if } \sin x>0\\ -\cos x & \text{if } \sin x<0\end{cases}$$



f is not differentiable when $x=n\pi$, n an integer.

(c) $g(x)=\sin|x|=\sin\sqrt{x^2} \Rightarrow g'(x)=\cos|x|\cdot\frac{x}{|x|}=\frac{x}{|x|}\cos x=\begin{cases}\cos x & \text{if } x>0\\ -\cos x & \text{if } x<0\end{cases}$



g is not differentiable at 0 .

81. First note that products and differences of polynomials are polynomials and that the derivative of a polynomial is also a polynomial. When $n=1$,

$$f^{(1)}(x)=\left(\frac{P(x)}{Q(x)}\right)'=\frac{Q(x)P'(x)-P(x)Q'(x)}{[Q(x)]^2}=\frac{A_1(x)}{[Q(x)]^{1+1}},$$ where $A_1(x)=Q(x)P'(x)-P(x)Q'(x)$.

Suppose the result is true for $n=k$, where $k\geq 1$. Then $f^{(k)}(x)=\dfrac{A_k(x)}{[Q(x)]^{k+1}}$, so

$$f^{(k+1)}(x)=\left(\frac{A_k(x)}{[Q(x)]^{k+1}}\right)'=\frac{[Q(x)]^{k+1}A_k'(x)-A_k(x)\cdot(k+1)[Q(x)]^k\cdot Q'(x)}{\{[Q(x)]^{k+1}\}^2}$$

$$=\frac{[Q(x)]^{k+1}A_k'(x)-(k+1)A_k(x)[Q(x)]^kQ'(x)}{[Q(x)]^{2k+2}}$$

$$=\frac{[Q(x)]^{k\,1}\{^1A_k'(x)-(k+1)A_k(x)Q'(x)\}}{[Q(x)]^k[Q(x)]^{k+2}}=\frac{Q(x)A_k'(x)-(k+1)A_k(x)Q'(x)}{[Q(x)]^{k+2}}$$

$$=A_{k+1}(x)/[Q(x)]^{k+2},\ \text{where } A_{k+1}(x)=Q(x)A_k'(x)-(k+1)A_k(x)Q'(x)\ .$$

We have shown that the formula holds for $n=1$, and that when it holds for $n=k$ it also holds for $n=k+1$. Thus, by mathematical induction, the formula holds for all positive integers n.

1. **(a)** $\frac{d}{dx}(xy+2x+3x^2)=\frac{d}{dx}(4)\Rightarrow(x\cdot y'+y\cdot 1)+2+6x=0\Rightarrow xy'=-y-2-6x\Rightarrow y'=\frac{-y-2-6x}{x}$ or $y'=-6-\frac{y+2}{x}$.

(b) $xy+2x+3x^2=4\Rightarrow xy=4-2x-3x^2\Rightarrow y=\frac{4-2x-3x^2}{x}=\frac{4}{x}-2-3x$, so $y'=-\frac{4}{x^2}-3$.

(c) From part (a), $y'=\frac{-y-2-6x}{x}=\frac{-(4/x-2-3x)-2-6x}{x}=\frac{-4/x-3x}{x}=-\frac{4}{x^2}-3$.

2. **(a)** $\frac{d}{dx}(4x^2+9y^2)=\frac{d}{dx}(36)\Rightarrow 8x+18y\cdot y'=0\Rightarrow y'=-\frac{8x}{18y}=-\frac{4x}{9y}$

(b) $4x^2+9y^2=36\Rightarrow 9y^2=36-4x^2\Rightarrow y^2=\frac{4}{9}(9-x^2)\Rightarrow y=\pm\frac{2}{3}\sqrt{9-x^2}$, so

$y'=\pm\frac{2}{3}\cdot\frac{1}{2}(9-x^2)^{-1/2}(-2x)=\mp\frac{2x}{3\sqrt{9-x^2}}$

(c) From part (a), $y'=-\frac{4x}{9y}=-\frac{4x}{9\left(\pm\frac{2}{3}\sqrt{9-x^2}\right)}=\mp\frac{2x}{3\sqrt{9-x^2}}$.

3. **(a)** $\frac{d}{dx}\left(\frac{1}{x}+\frac{1}{y}\right)=\frac{d}{dx}(1)\Rightarrow-\frac{1}{x^2}-\frac{1}{y^2}y'=0\Rightarrow-\frac{1}{y^2}y'=\frac{1}{x^2}\Rightarrow y'=-\frac{y^2}{x^2}$

(b) $\frac{1}{x}+\frac{1}{y}=1\Rightarrow\frac{1}{y}=1-\frac{1}{x}=\frac{x-1}{x}\Rightarrow y=\frac{x}{x-1}$, so $y'=\frac{(x-1)(1)-(x)(1)}{(x-1)^2}=\frac{-1}{(x-1)^2}$.

(c) $y'=-\frac{y^2}{x^2}=-\frac{[x/(x-1)]^2}{x^2}=-\frac{x^2}{x^2(x-1)^2}=-\frac{1}{(x-1)^2}$

4. **(a)** $\frac{d}{dx}(\sqrt{x}+\sqrt{y})=\frac{d}{dx}(4)\Rightarrow\frac{1}{2\sqrt{x}}+\frac{1}{2\sqrt{y}}y'=0\Rightarrow y'=-\frac{\sqrt{y}}{\sqrt{x}}$

(b) $\sqrt{y}=4-\sqrt{x}\Rightarrow y=(4-\sqrt{x})^2=16-8\sqrt{x}+x\Rightarrow y'=-\frac{4}{\sqrt{x}}+1$

(c) $y'=-\frac{\sqrt{y}}{\sqrt{x}}=-\frac{4-\sqrt{x}}{\sqrt{x}}=-\frac{4}{\sqrt{x}}+1$

5. $\frac{d}{dx}(x^2+y^2)=\frac{d}{dx}(1)\Rightarrow 2x+2yy'=0\Rightarrow 2yy'=-2x\Rightarrow y'=-\frac{x}{y}$

6. $\frac{d}{dx}(x^2-y^2)=\frac{d}{dx}(1)\Rightarrow 2x-2yy'=0\Rightarrow 2x=2yy'\Rightarrow y'=\frac{x}{y}$

7. $\frac{d}{dx}(x^3+x^2y+4y^2)=\frac{d}{dx}(6)\Rightarrow 3x^2+(x^2y'+y\cdot 2x)+8yy'=0\Rightarrow x^2y'+8yy'=-3x^2-2xy\Rightarrow$

$(x^2+8y)y'=-3x^2-2xy\Rightarrow y'=-\frac{3x^2+2xy}{x^2+8y}=-\frac{x(3x+2y)}{x^2+8y}$

8. $\frac{d}{dx}(x^2-2xy+y^3)=\frac{d}{dx}(c)\Rightarrow 2x-2(xy'+y\cdot 1)+3y^2y'=0\Rightarrow 2x-2y=2xy'-3y^2y'\Rightarrow 2x-2y=y'(2x-3y^2)\Rightarrow$
$y'=\frac{2x-2y}{2x-3y^2}$

9. $\frac{d}{dx}(x^2y+xy^2)=\frac{d}{dx}(3x)\Rightarrow (x^2y'+y\cdot 2x)+(x\cdot 2yy'+y^2\cdot 1)=3\Rightarrow x^2y'+2xyy'=3-2xy-y^2\Rightarrow$

$y'(x^2+2xy)=3-2xy-y^2\Rightarrow y'=\frac{3-2xy-y^2}{x^2+2xy}$

10. $\frac{d}{dx}(y^5+x^2y^3)=\frac{d}{dx}(1+x^4y)\Rightarrow 5y^4y'+x^2\cdot 3y^2y'+y^3\cdot 2x=0+x^4y'+y\cdot 4x^3\Rightarrow$

$y'(5y^4+3x^2y^2-x^4)=4x^3y-2xy^3\Rightarrow y'=\frac{4x^3y-2xy^3}{5y^4+3x^2y^2-x^4}$

11. $\frac{d}{dx}(x^2y^2+x\sin y)=\frac{d}{dx}(4)\Rightarrow x^2\cdot 2yy'+y^2\cdot 2x+x\cos y\cdot y'+\sin y\cdot 1=0\Rightarrow$

$2x^2yy'+x\cos y\cdot y'=-2xy^2-\sin y\Rightarrow (2x^2y+x\cos y)y'=-2xy^2-\sin y\Rightarrow y'=\frac{-2xy^2-\sin y}{2x^2y+x\cos y}$

12. $\frac{d}{dx}(1+x)=\frac{d}{dx}[\sin(xy^2)]\Rightarrow 1=[\cos(xy^2)](x\cdot 2yy'+y^2\cdot 1)\Rightarrow 1=2xy\cos(xy^2)y'+y^2\cos(xy^2)\Rightarrow$

$1-y^2\cos(xy^2)=2xy\cos(xy^2)y'\Rightarrow y'=\frac{1-y^2\cos(xy^2)}{2xy\cos(xy^2)}$

13. $\frac{d}{dx}(4\cos x\sin y)=\frac{d}{dx}(1)\Rightarrow 4[\cos x\cdot\cos y\cdot y'+\sin y\cdot(-\sin x)]=0\Rightarrow$

$y'(4\cos x\cos y)=4\sin x\sin y \Rightarrow y'=\dfrac{4\sin x\sin y}{4\cos x\cos y}=\tan x\tan y$

14. $\dfrac{d}{dx}[y\sin(x^2)]=\dfrac{d}{dx}[x\sin(y^2)] \Rightarrow y\cos(x^2)\cdot 2x+\sin(x^2)\cdot y'=x\cos(y^2)\cdot 2yy'+\sin(y^2)\cdot 1 \Rightarrow$

$y'[\sin(x^2)-2xy\cos(y^2)]=\sin(y^2)-2xy\cos(x^2) \Rightarrow y'=\dfrac{\sin(y^2)-2xy\cos(x^2)}{\sin(x^2)-2xy\cos(y^2)}$

15. $\dfrac{d}{dx}\left(e^{x^2y}\right)=\dfrac{d}{dx}(x+y) \Rightarrow e^{x^2y}(x^2y'+y\cdot 2x)=1+y' \Rightarrow x^2e^{x^2y}y'+2xye^{x^2y}=1+y' \Rightarrow$

$x^2e^{x^2y}y'-y'=1-2xye^{x^2y} \Rightarrow y'(x^2e^{x^2y}-1)=1-2xye^{x^2y} \Rightarrow y'=\dfrac{1-2xye^{x^2y}}{x^2e^{x^2y}-1}$

16. $\dfrac{d}{dx}(\sqrt{x+y})=\dfrac{d}{dx}(1+x^2y^2) \Rightarrow \dfrac{1}{2}(x+y)^{-1/2}(1+y')=x^2\cdot 2yy'+y^2\cdot 2x \Rightarrow$

$\dfrac{1}{2\sqrt{x+y}}+\dfrac{y'}{2\sqrt{x+y}}=2x^2yy'+2xy^2 \Rightarrow 1+y'=4x^2y\sqrt{x+y}\,y'+4xy^2\sqrt{x+y} \Rightarrow$

$y'-4x^2y\sqrt{x+y}\,y'=4xy^2\sqrt{x+y}-1 \Rightarrow y'(1-4x^2y\sqrt{x+y})=4xy^2\sqrt{x+y}-1 \Rightarrow y'=\dfrac{4xy^2\sqrt{x+y}-1}{1-4x^2y\sqrt{x+y}}$

17. $\sqrt{xy}=1+x^2y \Rightarrow \dfrac{1}{2}(xy)^{-1/2}(xy'+y\cdot 1)=0+x^2y'+y\cdot 2x \Rightarrow \dfrac{x}{2\sqrt{xy}}y'+\dfrac{y}{2\sqrt{xy}}=x^2y'+2xy \Rightarrow$

$y'\left(\dfrac{x}{2\sqrt{xy}}-x^2\right)=2xy-\dfrac{y}{2\sqrt{xy}} \Rightarrow y'\left(\dfrac{x-2x^2\sqrt{xy}}{2\sqrt{xy}}\right)=\dfrac{4xy\sqrt{xy}-y}{2\sqrt{xy}} \Rightarrow y'=\dfrac{4xy\sqrt{xy}-y}{x-2x^2\sqrt{xy}}$

18. $\tan(x-y)=\dfrac{y}{1+x^2} \Rightarrow (1+x^2)\tan(x-y)=y \Rightarrow (1+x^2)\sec^2(x-y)\cdot(1-y')+\tan(x-y)\cdot 2x=y' \Rightarrow$

$(1+x^2)\sec^2(x-y)-(1+x^2)\sec^2(x-y)\cdot y'+2x\tan(x-y)=y' \Rightarrow$

$(1+x^2)\sec^2(x-y)+2x\tan(x-y)=[1+(1+x^2)]\sec^2(x-y)\cdot y' \Rightarrow y'=\dfrac{(1+x^2)\sec^2(x-y)+2x\tan(x-y)}{1+(1+x^2)\sec^2(x-y)}$

19. $xy=\cot(xy) \Rightarrow y+xy'=-\csc^2(xy)(y+xy') \Rightarrow (y+xy')[1+\csc^2(xy)]=0 \Rightarrow y+xy'=0 \Rightarrow y'=-y/x$

20. $\sin x+\cos y=\sin x\cos y \Rightarrow$

$\cos x - \sin y \cdot y' = \sin x(-\sin y \cdot y') + \cos y \cos x \Rightarrow (\sin x \sin y - \sin y)\, y' = \cos x \cos y - \cos x \Rightarrow$
$y' = \dfrac{\cos x(\cos y - 1)}{\sin y(\sin x - 1)}$

21. $\dfrac{d}{dx}\{1+f(x)+x^2[f(x)]^3\} = \dfrac{d}{dx}(0) \Rightarrow f'(x)+x^2 \cdot 3[f(x)]^2 \cdot f'(x)+[f(x)]^3 \cdot 2x=0$. If $x=1$, we have $f'(1)+1^2 \cdot 3[f(1)]^2 \cdot f'(1)+[f(1)]^3 \cdot 2(1)=0 \Rightarrow f'(1)+1 \cdot 3 \cdot 2^2 \cdot f'(1)+2^3 \cdot 2=0 \Rightarrow f'(1)+12f'(1)=-16 \Rightarrow$
$13f'(1)=-16 \Rightarrow f'(1)=-\dfrac{16}{13}$.

22. $\dfrac{d}{dx} = \dfrac{d}{dx}(x^2) \Rightarrow g'(x)+x\cos g(x) \cdot g'(x)+\sin g(x) \cdot 1=2x$. If $x=1$, we have
$g'(1)+1\cos g(1) \cdot g'(1)+\sin g(1)=2(1) \Rightarrow g'(1)+\cos 0 \cdot g'(1)+\sin 0=2 \Rightarrow g'(1)+g'(1)=2 \Rightarrow 2g'(1)=2 \Rightarrow$
$g'(1)=1$.

23. $y^4+x^2y^2+yx^4=y+1 \Rightarrow 4y^3+\left(x^2 \cdot 2y+y^2 \cdot 2x\dfrac{dx}{dy}\right)+\left(y \cdot 4x^3\dfrac{dx}{dy}+x^4 \cdot 1\right)=1 \Rightarrow 2xy^2$
$\dfrac{dx}{dy}+4x^3y\dfrac{dx}{dy}=1-4y^3-2x^2y-x^4 \Rightarrow \dfrac{dx}{dy}=\dfrac{1-4y^3-2x^2y-x^4}{2xy^2+4x^3y}$

24. $(x^2+y^2)^2=ax^2y \Rightarrow 2(x^2+y^2)\left(2x\dfrac{dx}{dy}+2y\right)=2ayx\dfrac{dx}{dy}+ax^2 \Rightarrow \dfrac{dx}{dy}=\dfrac{ax^2-4y(x^2+y^2)}{4x(x^2+y^2)-2axy}$

25. $x^2+xy+y^2=3 \Rightarrow 2x+xy'+y \cdot 1+2yy'=0 \Rightarrow xy'+2yy'=-2x-y \Rightarrow y'(x+2y)=-2x-y \Rightarrow y'=\dfrac{-2x-y}{x+2y}$. When $x=1$ and $y=1$, we have $y'=\dfrac{-2-1}{1+2 \cdot 1}=\dfrac{-3}{3}=-1$, so an equation of the tangent line is $y-1=-1(x-1)$ or $y=-x+2$.

26. $x^2+2xy-y^2+x=2 \Rightarrow 2x+2(xy'+y \cdot 1)-2yy'+1=0 \Rightarrow 2xy'-2yy'=-2x-2y-1 \Rightarrow y'(2x-2y)=-2x-2y-1 \Rightarrow$
$y'=\dfrac{-2x-2y-1}{2x-2y}$. When $x=1$ and $y=2$, we have $y'=\dfrac{-2-4-1}{2-4}=\dfrac{-7}{-2}=\dfrac{7}{2}$, so an equation of the tangent line is $y-2=\dfrac{7}{2}(x-1)$ or $y=\dfrac{7}{2}x-\dfrac{3}{2}$.

27. $x^2+y^2=(2x^2+2y^2-x)^2 \Rightarrow 2x+2yy'=2(2x^2+2y^2-x)(4x+4yy'-1)$. When $x=0$ and $y=\dfrac{1}{2}$, we have

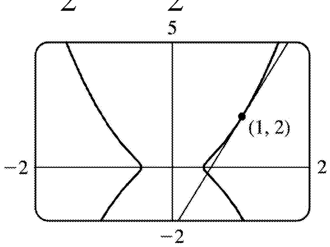
$0+y'=2\left(\frac{1}{2}\right)(2y'-1)\Rightarrow y'=2y'-1\Rightarrow y'=1$, so an equation of the tangent line is $y-\frac{1}{2}=1(x-0)$ or $y=x+\frac{1}{2}$.

28. $x^{2/3}+y^{2/3}=4\Rightarrow \frac{2}{3}x^{-1/3}+\frac{2}{3}y^{-1/3}y'=0\Rightarrow \frac{1}{\sqrt[3]{x}}+\frac{y'}{\sqrt[3]{y}}=0\Rightarrow y'=-\frac{\sqrt[3]{y}}{\sqrt[3]{x}}$. When $x=-3\sqrt{3}$ and $y=1$, we have $y'=-\frac{1}{(-3\sqrt{3})^{1/3}}=-\frac{(-3\sqrt{3})^{2/3}}{-3\sqrt{3}}=\frac{3}{3\sqrt{3}}=\frac{1}{\sqrt{3}}$, so an equation of the tangent line is $y-1=\frac{1}{\sqrt{3}}(x+3\sqrt{3})$ or $y=\frac{1}{\sqrt{3}}x+4$.

29. $2(x^2+y^2)^2=25(x^2-y^2)\Rightarrow 4(x^2+y^2)(2x+2yy')=25(2x-2yy')\Rightarrow$
$4(x+yy')(x^2+y^2)=25(x-yy')\Rightarrow 4yy'(x^2+y^2)+25yy'=25x-4x(x^2+y^2)\Rightarrow y'=\frac{25x-4x(x^2+y^2)}{25y+4y(x^2+y^2)}$. When $x=3$ and $y=1$, we have $y'=\frac{75-120}{25+40}=-\frac{45}{65}=-\frac{9}{13}$, so an equation of the tangent line is $y-1=-\frac{9}{13}(x-3)$ or $y=-\frac{9}{13}x+\frac{40}{13}$.

30. $y^2(y^2-4)=x^2(x^2-5)\Rightarrow y^4-4y^2=x^4-5x^2\Rightarrow 4y^3y'-8yy'=4x^3-10x$. When $x=0$ and $y=-2$, we have $-32y'+16y'=0\Rightarrow -16y'=0\Rightarrow y'=0$, so an equation of the tangent line is $y+2=0(x-0)$ or $y=-2$.

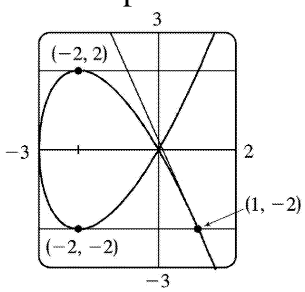
31. **(a)** $y^2=5x^4-x^2\Rightarrow 2yy'=5(4x^3)-2x\Rightarrow y'=\frac{10x^3-x}{y}$. So at the point (1,2) we have $y'=\frac{10(1)^3-1}{2}=\frac{9}{2}$, and an equation of the tangent line is $y-2=\frac{9}{2}(x-1)$ or $y=\frac{9}{2}x-\frac{5}{2}$.

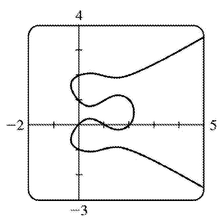
(b) 

32. **(a)** $y^2=x^3+3x^2\Rightarrow 2yy'=3x^2+3(2x)\Rightarrow y'=\frac{3x^2+6x}{2y}$. So at the point (1,–2) we have

$y' = \dfrac{3(1)^2+6(1)}{2(-2)} = -\dfrac{9}{4}$, and an equation of the tangent line is $y+2=-\dfrac{9}{4}(x-1)$ or $y=-\dfrac{9}{4}x+\dfrac{1}{4}$.

(b) The curve has a horizontal tangent where $y'=0 \Leftrightarrow 3x^2+6x=0 \Leftrightarrow 3x(x+2)=0 \Leftrightarrow x=0$ or $x=-2$. But note that at $x=0$, $y=0$ also, so the derivative does not exist. At $x=-2$, $y^2=(-2)^3+3(-2)^2=-8+12=4$, so $y=\pm 2$. So the two points at which the curve has a horizontal tangent are $(-2,-2)$ and $(-2,2)$.

(c) 

33. **(a)** 

There are eight points with horizontal tangents: four at $x \approx 1.57735$ and four at $x \approx 0.42265$.

(b) $y' = \dfrac{3x^2-6x+2}{2(2y^3-3y^2-y+1)} \Rightarrow y'=-1$ at (0, 1) and $y'=\dfrac{1}{3}$ at (0, 2)

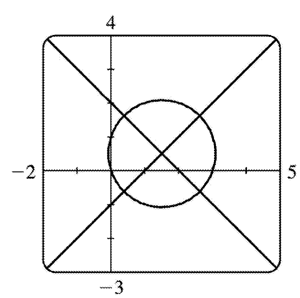
Equations of the tangent lines are $y=-x+1$ and $y=\dfrac{1}{3}x+2$.

(c) $y'=0 \Rightarrow 3x^2-6x+2=0 \Rightarrow x=1\pm\dfrac{1}{3}\sqrt{3}$

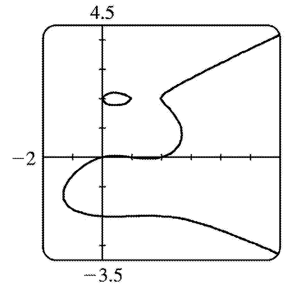
(d)

By multiplying the right side of the equation by $x-3$, we obtain the first graph.

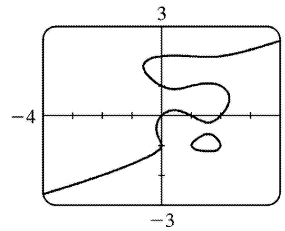
By modifying the equation in other ways, we can generate the other graphs.



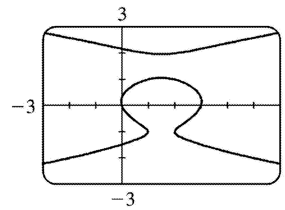
$y(y^2-1)(y-2)=x(x-1)(x-2)(x-3)$



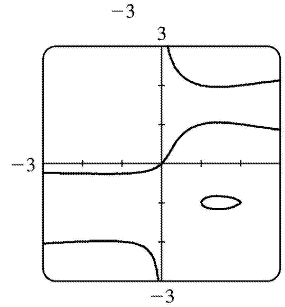
$y(y^2-4)(y-2)=x(x-1)(x-2)$



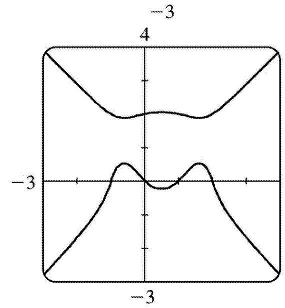
$y(y+1)(y^2-1)(y-2)=(x-1)(x-2)$



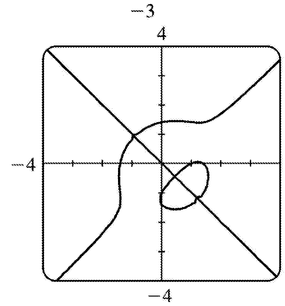
$(y+1)(y^2-1)(y-2)=x(x-1)(x-2)$



$x(y+1)(y^2-1)(y-2)=y(x-1)(x-2)$

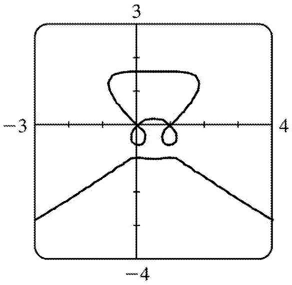


$y(y^2+1)(y-2)=x(x^2-1)(x-2)$



$y(y+1)(y^2-2)=x(x-1)(x^2-2)$

34. **(a)**



(b) There are 9 points with horizontal tangents: 3 at $x=0$, 3 at $x=\frac{1}{2}$, and 3 at $x=1$. The three horizontal tangents along the top of the wagon are hard to find, but by limiting the y –range of the graph (to $[1.6,1.7]$, for example) they are distinguishable.

35. From Exercise 29, a tangent to the lemniscate will be horizontal if $y'=0 \Rightarrow 25x-4x(x^2+y^2)=0 \Rightarrow x[25-4(x^2+y^2)]=0 \Rightarrow x^2+y^2=\frac{25}{4}$ (**1**). (Note that when x is 0 , y is also 0 , and there is no horizontal tangent at the origin.) Substituting $\frac{25}{4}$ for x^2+y^2 in the equation of the lemniscate, $2(x^2+y^2)^2=25(x^2-y^2)$, we get $x^2-y^2=\frac{25}{8}$ (**2**). Solving (**1**) and (**2**), we have $x^2=\frac{75}{16}$ and $y^2=\frac{25}{16}$, so the four points are $\left(\pm\frac{5\sqrt{3}}{4},\pm\frac{5}{4}\right)$.

36. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \Rightarrow \frac{2x}{a^2}+\frac{2yy'}{b^2}=0 \Rightarrow y'=-\frac{b^2x}{a^2y} \Rightarrow$ an equation of the tangent line at (x_0,y_0) is $y-y_0=\frac{-b^2x_0}{a^2y_0}(x-x_0)$. Multiplying both sides by $\frac{y_0}{b^2}$ gives $\frac{y_0y}{b^2}-\frac{y_0^2}{b^2}=-\frac{x_0x}{a^2}+\frac{x_0^2}{a^2}$. Since (x_0,y_0) lies on the ellipse, we have $\frac{x_0x}{a^2}+\frac{y_0y}{b^2}=\frac{x_0^2}{a^2}+\frac{y_0^2}{b^2}=1$.

37. $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \Rightarrow \frac{2x}{a^2}-\frac{2yy'}{b^2}=0 \Rightarrow y'=\frac{b^2x}{a^2y} \Rightarrow$ an equation of the tangent line at (x_0,y_0) is $y-y_0=\frac{b^2x_0}{a^2y_0}(x-x_0)$. Multiplying both sides by $\frac{y_0}{b^2}$ gives $\frac{y_0y}{b^2}-\frac{y_0^2}{b^2}=\frac{x_0x}{a^2}-\frac{x_0^2}{a^2}$. Since (x_0,y_0) lies on the hyperbola, we have

$\frac{x_0x}{a^2}-\frac{y_0y}{b^2}=\frac{x_0^2}{a^2}-\frac{y_0^2}{b^2}=1$.

38. $\sqrt{x}+\sqrt{y}=\sqrt{c}\Rightarrow \frac{1}{2\sqrt{x}}+\frac{y'}{2\sqrt{y}}=0\Rightarrow y'=-\frac{\sqrt{y}}{\sqrt{x}}\Rightarrow$ an equation of the tangent line at (x_0,y_0) is $y-y_0=-\frac{\sqrt{y_0}}{\sqrt{x_0}}(x-x_0)$. Now $x=0\Rightarrow y=y_0-\frac{\sqrt{y_0}}{\sqrt{x_0}}(-x_0)=y_0+\sqrt{x_0}\sqrt{y_0}$, so the y –intercept is $y_0+\sqrt{x_0}\sqrt{y_0}$. And $y=0\Rightarrow -y_0=-\frac{\sqrt{y_0}}{\sqrt{x_0}}(x-x_0)\Rightarrow x-x_0=\frac{y_0\sqrt{x_0}}{\sqrt{y_0}}\Rightarrow x=x_0+\sqrt{x_0}\sqrt{y_0}$, so the x –intercept is $x_0+\sqrt{x_0}\sqrt{y_0}$. The sum of the intercepts is
$\left(y_0+\sqrt{x_0}\sqrt{y_0}\right)+\left(x_0+\sqrt{x_0}\sqrt{y_0}\right)=x_0+2\sqrt{x_0}\sqrt{y_0}+y_0=\left(\sqrt{x_0}+\sqrt{y_0}\right)^2=(\sqrt{c})^2=c$.

39. If the circle has radius r , its equation is $x^2+y^2=r^2\Rightarrow 2x+2yy'=0\Rightarrow y'=-\frac{x}{y}$, so the slope of the tangent line at $P(x_0,y_0)$ is $-\frac{x_0}{y_0}$. The negative reciprocal of that slope is $\frac{-1}{-x_0/y_0}=\frac{y_0}{x_0}$, which is the slope of OP , so the tangent line at P is perpendicular to the radius OP .

40. $y^q=x^p\Rightarrow qy^{q-1}y'=px^{p-1}\Rightarrow y'=\frac{px^{p-1}}{qy^{q-1}}=\frac{px^{p-1}y}{qy^q}=\frac{px^{p-1}x^{p/q}}{qx^p}=\frac{p}{q}x^{(p/q)-1}$

41. $y=\tan^{-1}\sqrt{x}\Rightarrow y'=\frac{1}{1+(\sqrt{x})^2}\cdot\frac{d}{dx}(\sqrt{x})=\frac{1}{1+x}\left(\frac{1}{2}x^{-1/2}\right)=\frac{1}{2\sqrt{x}(1+x)}$

42. $y=\sqrt{\tan^{-1}x}=(\tan^{-1}x)^{1/2}\Rightarrow$
$y'=\frac{1}{2}(\tan^{-1}x)^{-1/2}\cdot\frac{d}{dx}(\tan^{-1}x)=\frac{1}{2\sqrt{\tan^{-1}x}}\cdot\frac{1}{1+x^2}=\frac{1}{2\sqrt{\tan^{-1}x}(1+x^2)}$

43. $y=\sin^{-1}(2x+1)\Rightarrow$

$y' = \frac{1}{\sqrt{1-(2x+1)^2}} \cdot \frac{d}{dx}(2x+1) = \frac{1}{\sqrt{1-(4x^2+4x+1)}} \cdot 2 = \frac{2}{\sqrt{-4x^2-4x}} = \frac{1}{\sqrt{-x^2-x}}$

44. $h(x)=\sqrt{1-x^2}\arcsin x \Rightarrow h'(x)=\sqrt{1-x^2}\cdot\frac{1}{\sqrt{1-x^2}}+\arcsin x\left[\frac{1}{2}(1-x^2)^{-1/2}(-2x)\right]=1-\frac{x\arcsin x}{\sqrt{1-x^2}}$

45. $H(x)=(1+x^2)\arctan x \Rightarrow H'(x)=(1+x^2)\frac{1}{1+x^2}+(\arctan x)(2x)=1+2x\arctan x$

46. $y=\tan^{-1}\left(x-\sqrt{x^2+1}\right)\Rightarrow$

$$y' = \frac{1}{1+\left(x-\sqrt{x^2+1}\right)^2}\left(1-\frac{x}{\sqrt{x^2+1}}\right)=\frac{1}{1+x^2-2x\sqrt{x^2+1}+x^2+1}\left(\frac{\sqrt{x^2+1}-x}{\sqrt{x^2+1}}\right)$$

$$=\frac{\sqrt{x^2+1}-x}{2\left(1+x^2-x\sqrt{x^2+1}\right)\sqrt{x^2+1}}=\frac{\sqrt{x^2+1}-x}{2\left[\sqrt{x^2+1}\,(1+x^2)-x(x^2+1)\right]}$$

$$=\frac{\sqrt{x^2+1}-x}{2\left[(1+x^2)\left(\sqrt{x^2+1}-x\right)\right]}=\frac{1}{2(1+x^2)}$$

47. $h(t)=\cot^{-1}(t)+\cot^{-1}(1/t)\Rightarrow$

$h'(t)=-\frac{1}{1+t^2}-\frac{1}{1+(1/t)^2}\cdot\frac{d}{dt}\frac{1}{t}=-\frac{1}{1+t^2}-\frac{t^2}{t^2+1}\cdot\left(-\frac{1}{t^2}\right)=-\frac{1}{1+t^2}+\frac{1}{t^2+1}=0$.

Note that this makes sense because $h(t)=\frac{\pi}{2}$ for $t>0$ and $h(t)=-\frac{\pi}{2}$ for $t<0$.

48. $y=x\cos^{-1}x-\sqrt{1-x^2}\Rightarrow y'=\cos^{-1}x-\frac{x}{\sqrt{1-x^2}}+\frac{x}{\sqrt{1-x^2}}=\cos^{-1}x$

49. $y=\cos^{-1}(e^{2x})\Rightarrow y'=-\frac{1}{\sqrt{1-(e^{2x})^2}}\cdot\frac{d}{dx}(e^{2x})=-\frac{2e^{2x}}{\sqrt{1-e^{4x}}}$

50. $y=\arctan(\cos\theta)\Rightarrow$

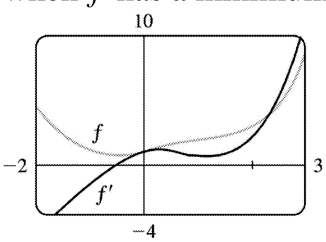
$$y' = \frac{1}{1+(\cos\theta)^2}(-\sin\theta) = -\frac{\sin\theta}{1+\cos^2\theta}$$

51. $f(x) = e^x - x^2 \arctan x \Rightarrow$

$$f'(x) = e^x - \left[x^2\left(\frac{1}{1+x^2}\right) + (\arctan x)(2x)\right]$$

$$= e^x - \frac{x^2}{1+x^2} - 2x\arctan x$$

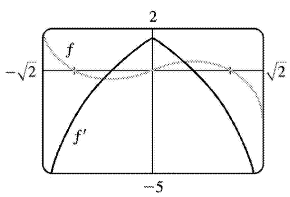
This is reasonable because the graphs show that f is increasing when f' is positive, and f' is zero when f has a minimum.



52. $f(x) = x\arcsin(1-x^2) \Rightarrow$

$$f'(x) = x\left[\frac{-2x}{\sqrt{1-(1-x^2)^2}}\right] + \arcsin(1-x^2)\cdot 1$$

$$= \arcsin(1-x^2) - \frac{2x^2}{\sqrt{2x^2-x^4}}$$



This is reasonable because the graphs show that f is increasing when f' is positive, and that f has an inflection point when f' changes from increasing to decreasing.

53. Let $y = \cos^{-1}x$. Then $\cos y = x$ and $0 \le y \le \pi \Rightarrow -\sin y \frac{dy}{dx} = 1 \Rightarrow$

$\frac{dy}{dx} = -\frac{1}{\sin y} = -\frac{1}{\sqrt{1-\cos^2 y}} = -\frac{1}{\sqrt{1-x^2}}$. (Note that $\sin y \ge 0$ for $0 \le y \le \pi$).

54. **(a)** Let $y=\sec^{-1}x$. Then $\sec y=x$ and $y\in\left(0,\ \frac{\pi}{2}\right]\cup\left(\pi,\ \frac{3\pi}{2}\right]$. Differentiate with respect to x: $\sec y\tan y\left(\frac{dy}{dx}\right)=1\Rightarrow\frac{dy}{dx}=\frac{1}{\sec y\tan y}=\frac{1}{\sec y\sqrt{\sec^2 y-1}}=\frac{1}{x\sqrt{x^2-1}}$. Note that $\tan^2 y=\sec^2 y-1\Rightarrow$ $\tan y=\sqrt{\sec^2 y-1}$ since $\tan y>0$ when $0<y<\frac{\pi}{2}$ or $\pi<y<\frac{3\pi}{2}$.

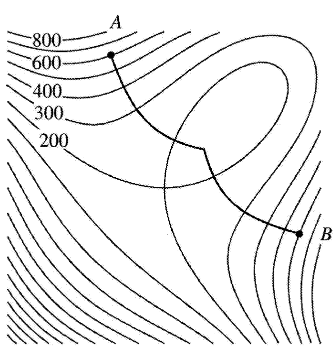
(b) $y=\sec^{-1}x\Rightarrow\sec y=x\Rightarrow\sec y\tan y\frac{dy}{dx}=1\Rightarrow\frac{dy}{dx}=\frac{1}{\sec y\tan y}$. Now $\tan^2 y=\sec^2 y-1=x^2-1$, so $\tan y=\pm\sqrt{x^2-1}$. For $y\in\left[0,\ \frac{\pi}{2}\right)$, $x\geq 1$, so $\sec y=x=|x|$ and $\tan y\geq 0\Rightarrow$

$\frac{dy}{dx}=\frac{1}{x\sqrt{x^2-1}}=\frac{1}{|x|\sqrt{x^2-1}}$. For $y\in\left(\frac{\pi}{2},\pi\right]$, $x\leq -1$, so $|x|=-x$ and $\tan y=-\sqrt{x^2-1}\Rightarrow$

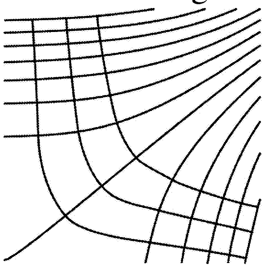
$$\frac{dy}{dx}=\frac{1}{\sec y\tan y}=\frac{1}{x\left(-\sqrt{x^2-1}\right)}=\frac{1}{(-x)\sqrt{x^2-1}}=\frac{1}{|x|\sqrt{x^2-1}}.$$

55. $2x^2+y^2=3$ and $x=y^2$ intersect when $2x^2+x-3=0\Leftrightarrow(2x+3)(x-1)=0\Leftrightarrow x=-\frac{3}{2}$ or 1 , but $-\frac{3}{2}$ is extraneous since $x=y^2$ is nonnegative. When $x=1$, $1=y^2\Rightarrow y=\pm 1$, so there are two points of intersection: $(1,\pm 1)$. $2x^2+y^2=3\Rightarrow 4x+2yy'=0\Rightarrow y'=-2x/y$, and $x=y^2\Rightarrow 1=2yy'\Rightarrow y'=1/(2y)$. At $(1,1)$, the slopes are $m_1=-2(1)/1=-2$ and $m_2=1/(2\cdot 1)=\frac{1}{2}$, so the curves are orthogonal (since m_1 and m_2 are negative reciprocals of each other). By symmetry, the curves are also orthogonal at $(1,-1)$.

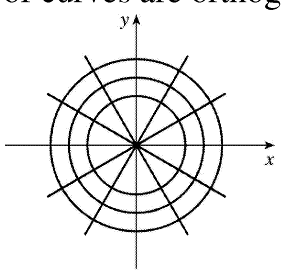
56. $x^2-y^2=5$ and $4x^2+9y^2=72$ intersect when $4x^2+9(x^2-5)=72\Leftrightarrow 13x^2=117\Leftrightarrow x=\pm 3$, so there are four points of intersection: $(\pm 3,\pm 2)$. $x^2-y^2=5\Rightarrow 2x-2yy'=0\Rightarrow y'=x/y$, and $4x^2+9y^2=72\Rightarrow 8x+18yy'=0\Leftrightarrow$ $y'=-4x/9y$. At $(3,2)$, the slopes are $m_1=\frac{3}{2}$ and $m_2=-\frac{2}{3}$, so the curves are orthogonal. By symmetry, the curves are also orthogonal at $(3,-2)$, $(-3,2)$ and $(-3,-2)$.

57. 

58. The orthogonal family represents the direction of the wind.

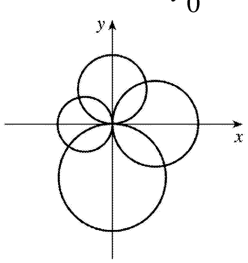


59. $x^2+y^2=r^2$ is a circle with center O and $ax+by=0$ is a line through O . $x^2+y^2=r^2 \Rightarrow 2x+2yy'=0 \Rightarrow y'=-x/y$, so the slope of the tangent line at $P_0(x_0,y_0)$ is $-x_0/y_0$. The slope of the line OP_0 is y_0/x_0 , which is the negative reciprocal of $-x_0/y_0$. Hence, the curves are orthogonal, and the families of curves are orthogonal trajectories of each other.

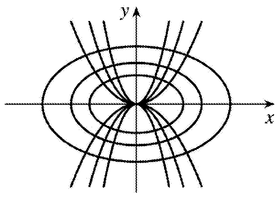


60. The circles $x^2+y^2=ax$ and $x^2+y^2=by$ intersect at the origin where the tangents are vertical and horizontal. If (x_0,y_0) is the other point of intersection, then $x_0^2+y_0^2=ax_0$ (**1**) and $x_0^2+y_0^2=by_0$ (**2**). Now $x^2+y^2=ax \Rightarrow 2x+2yy'=a \Rightarrow y'=\dfrac{a-2x}{2y}$ and $x^2+y^2=by \Rightarrow 2x+2yy'=by' \Rightarrow y'=\dfrac{2x}{b-2y}$. Thus, the curves are orthogonal at

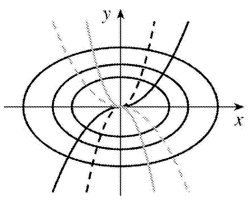
$(x_0,y_0) \Leftrightarrow \dfrac{a-2x_0}{2y_0}=-\dfrac{b-2y_0}{2x_0} \Leftrightarrow 2ax_0-4x_0^2=4y_0^2-2by_0 \Leftrightarrow ax_0+by_0=2(x_0^2+y_0^2)$, which is true by (**1**) and (**2**).



61. $y=cx^2 \Rightarrow y'=2cx$ and $x^2+2y^2=k \Rightarrow 2x+4yy'=0 \Rightarrow 2yy'=-x \Rightarrow y'=-\dfrac{x}{2(y)}=-\dfrac{x}{2(cx^2)}=-\dfrac{1}{2cx}$, so the curves are orthogonal.

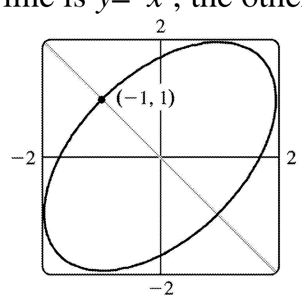


62. $y=ax^3 \Rightarrow y'=3ax^2$ and $x^2+3y^2=b \Rightarrow 2x+6yy'=0 \Rightarrow 3yy'=-x \Rightarrow y'=-\frac{x}{3(y)}=-\frac{x}{3(ax^3)}=-\frac{1}{3ax^2}$, so the curves are orthogonal.



63. To find the points at which the ellipse $x^2-xy+y^2=3$ crosses the x –axis, let $y=0$ and solve for x . $y=0 \Rightarrow x^2-x(0)+0^2=3 \Leftrightarrow x=\pm\sqrt{3}$. So the graph of the ellipse crosses the x –axis at the points $(\pm\sqrt{3},0)$. Using implicit differentiation to find y' , we get $2x-xy'-y+2yy'=0 \Rightarrow y'(2y-x)=y-2x \Leftrightarrow y'=\frac{y-2x}{2y-x}$. So y' at $(\sqrt{3},0)$ is $\frac{0-2\sqrt{3}}{2(0)-\sqrt{3}}=2$ and y' at $(-\sqrt{3},0)$ is $\frac{0+2\sqrt{3}}{2(0)+\sqrt{3}}=2$. Thus, the tangent lines at these points are parallel.

64. **(a)** We use implicit differentiation to find $y'=\frac{y-2x}{2y-x}$ as in Exercise 49. The slope of the tangent line at $(-1,1)$ is $m=\frac{1-2(-1)}{2(1)-(-1)}=\frac{3}{3}=1$, so the slope of the normal line is $-\frac{1}{m}=-1$, and its equation is $y-1=-1(x+1) \Leftrightarrow y=-x$. Substituting $-x$ for y in the equation of the ellipse, we get $x^2-x(-x)+(-x)^2=3 \Rightarrow 3x^2=3 \Leftrightarrow x=\pm 1$. So the normal line must intersect the ellipse again at $x=1$, and since the equation of the line is $y=-x$, the other point of intersection must be $(1,-1)$.

(b) 

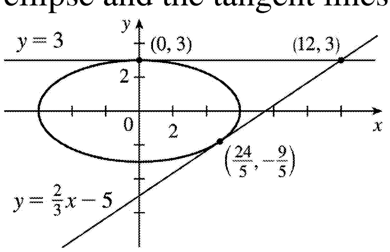
65. $x^2y^2+xy=2 \Rightarrow x^2\cdot 2yy'+y^2\cdot 2x+x\cdot y'+y\cdot 1=0 \Leftrightarrow y'(2x^2y+x)=-2xy^2-y \Leftrightarrow$

$y'=-\dfrac{2xy^2+y}{2x^2y+x}$. So $-\dfrac{2xy^2+y}{2x^2y+x}=-1\Leftrightarrow 2xy^2+y=2x^2y+x\Leftrightarrow y(2xy+1)=x(2xy+1)\Leftrightarrow$

$y(2xy+1)-x(2xy+1)=0\Leftrightarrow (2xy+1)(y-x)=0\Leftrightarrow xy=-\dfrac{1}{2}$ or $y=x$. But $xy=-\dfrac{1}{2}\Rightarrow x^2y^2+xy=\dfrac{1}{4}-\dfrac{1}{2}\neq 2$, so we must have $x=y$. Then $x^2y^2+xy=2\Rightarrow x^4+x^2=2\Leftrightarrow x^4+x^2-2=0\Leftrightarrow (x^2+2)(x^2-1)=0$. So $x^2=-2$, which is impossible, or $x^2=1\Leftrightarrow x=\pm 1$. Since $x=y$, the points on the curve where the tangent line has a slope of -1 are $(-1,-1)$ and $(1,1)$.

66. $x^2+4y^2=36\Rightarrow 2x+8yy'=0\Rightarrow y'=-\dfrac{x}{4y}$. Let (a,b) be a point on $x^2+4y^2=36$ whose tangent line passes through (12,3) . The tangent line is then $y-3=-\dfrac{a}{4b}(x-12)$, so $b-3=-\dfrac{a}{4b}(a-12)$. Multiplying both sides by $4b$ gives $4b^2-12b=-a^2+12a$, so $4b^2+a^2=12(a+b)$. But $4b^2+a^2=36$, so $36=12(a+b)\Rightarrow$ $a+b=3\Rightarrow b=3-a$. Substituting $3-a$ for b into $a^2+4b^2=36$ gives $a^2+4(3-a)^2=36\Leftrightarrow a^2+36-24a+4a^2=36\Leftrightarrow$ $5a^2-24a=0\Leftrightarrow a(5a-24)=0$, so $a=0$ or $a=\dfrac{24}{5}$. If $a=0$, $b=3-0=3$, and if $a=\dfrac{24}{5}$, $b=3-\dfrac{24}{5}=-\dfrac{9}{5}$. So the two points on the ellipse are (0,3) and $\left(\dfrac{24}{5},-\dfrac{9}{5}\right)$.

Using $y-3=-\dfrac{a}{4b}(x-12)$ with (a,b)=(0,3) gives us the tangent line $y-3=0$ or $y=3$. With $(a,b)=\left(\dfrac{24}{5},-\dfrac{9}{5}\right)$, we have $y-3=-\dfrac{24/5}{4(-9/5)}(x-12)\Leftrightarrow y-3=\dfrac{2}{3}(x-12)\Leftrightarrow y=\dfrac{2}{3}x-5$. A graph of the ellipse and the tangent lines confirms our results.



67. **(a)** If $y=f^{-1}(x)$, then $f(y)=x$. Differentiating implicitly with respect to x and remembering that y is a function of x, we get $f'(y)\dfrac{dy}{dx}=1$, so $\dfrac{dy}{dx}=\dfrac{1}{f'(y)}\Rightarrow (f^{-1})'(x)=\dfrac{1}{f'(f^{-1}(x))}$.

(b) $f(4)=5\Rightarrow f^{-1}(5)=4$. By part (a), $(f^{-1})'(5)=1/f'(f^{-1}(5))=1/f'(4)=1/\left(\dfrac{2}{3}\right)=\dfrac{3}{2}$.

68. **(a)** $f(x)=2x+\cos x\Rightarrow f'(x)=2-\sin x>0$ for all x . Thus, f is increasing for all x and is therefore one–to–one.

(b) Since f is one–to–one, $f^{-1}(1)=k \Leftrightarrow f(k)=1$. By inspection, we see that $f(0)=2(0)+\cos 0=1$, so $k=f^{-1}(1)=0$.

(c) $(f^{-1})'(1)=1/f'(f^{-1}(1))=1/f'(0)=1/(2-\sin 0)=\frac{1}{2}$

69. $x^2+4y^2=5 \Rightarrow 2x+4(2yy')=0 \Rightarrow y'=-\frac{x}{4y}$. Now let h be the height of the lamp, and let (a,b) be the point of tangency of the line passing through the points (3,h) and (–5,0) . This line has slope $\frac{h-0}{3-(-5)}=\frac{1}{8}h$. But the slope of the tangent line through the point (a,b) can be expressed as $y'=-\frac{a}{4b}$, or as $\frac{b-0}{a-(-5)}=\frac{b}{a+5}$, so $-\frac{a}{4b}=\frac{b}{a+5} \Leftrightarrow 4b^2=-a^2-5a \Leftrightarrow a^2+4b^2=-5a$. But $a^2+4b^2=5$, so $5=-5a \Leftrightarrow a=-1$. Then $4b^2=-a^2-5a=-1-5(-1)=4 \Rightarrow b=1$, since the point is on the top half of the ellipse. So $\frac{h}{8}=\frac{b}{a+5}=\frac{1}{-1+5}=\frac{1}{4} \Rightarrow h=2$. So the lamp is located 2 units above the x –axis.

1. $a=f$, $b=f'$, $c=f''$. We can see this because where a has a horizontal tangent, $b=0$, and where b has a horizontal tangent, $c=0$. We can immediately see that c can be neither f nor f' , since at the points where c has a horizontal tangent, neither a nor b is equal to 0 .

2. Where d has horizontal tangents, only c is 0 , so $d'=c$. c has negative tangents for $x<0$ and b is the only graph that is negative for $x<0$, so $c'=b$. b has positive tangents on R (except at $x=0$), and the only graph that is positive on the same domain is a , so $b'=a$. We conclude that $d=f$, $c=f'$, $b=f''$ and $a=f'''$.

3. We can immediately see that a is the graph of the acceleration function, since at the points where a has a horizontal tangent, neither c nor b is equal to 0 . Next, we note that $a=0$ at the point where b has a horizontal tangent, so b must be the graph of the velocity function, and hence, $b'=a$. We conclude that c is the graph of the position function.

4. a must be the jerk since none of the graphs are 0 at its high and low points. a is 0 where b has a maximum, so $b'=a$. b is 0 where c has a maximum, so $c'=b$. We conclude that d is the position function, c is the velocity, b is the acceleration, and a is the jerk.

5. $f(x)=x^5+6x^2-7x \Rightarrow f'(x)=5x^4+12x-7 \Rightarrow f''(x)=20x^3+12$

6. $f(t)=t^8-7t^6+2t^4 \Rightarrow f'(t)=8t^7-42t^5+8t^3 \Rightarrow f''(t)=56t^6-210t^4+24t^2$

7. $y=\cos 2\theta \Rightarrow y'=-2\sin 2\theta \Rightarrow y''=-4\cos 2\theta$

8. $y=\theta\sin\theta \Rightarrow y'=\theta\cos\theta+\sin\theta \Rightarrow y''=\theta(-\sin\theta)+\cos\theta\cdot 1+\cos\theta=2\cos\theta-\theta\sin\theta$

9. $F(t)=(1-7t)^6 \Rightarrow F'(t)=6(1-7t)^5(-7)=-42(1-7t)^5 \Rightarrow F''(t)=-42\cdot 5(1-7t)^4(-7)=1470(1-7t)^4$

10. $g(x)=\dfrac{2x+1}{x-1} \Rightarrow g'(x)=\dfrac{(x-1)(2)-(2x+1)(1)}{(x-1)^2}=\dfrac{2x-2-2x-1}{(x-1)^2}=\dfrac{-3}{(x-1)^2}$ or $-3(x-1)^{-2}$

$\Rightarrow g''(x)=-3(-2)(x-1)^{-3}=6(x-1)^{-3}$ or $\dfrac{6}{(x-1)^3}$

11. $h(u)=\dfrac{1-4u}{1+3u} \Rightarrow h'(u)=\dfrac{(1+3u)(-4)-(1-4u)(3)}{(1+3u)^2}=\dfrac{-4-12u-3+12u}{(1+3u)^2}=\dfrac{-7}{(1+3u)^2}$ or $-7(1+3u)^{-2} \Rightarrow$

$h''(u)=-7(-2)(1+3u)^{-3}(3)=42(1+3u)^{-3}$ or

$$\frac{42}{(1+3u)^{3}}$$

12.

$$H(s) = a\sqrt{s}+\frac{b}{\sqrt{s}} = as^{1/2}+bs^{-1/2} \Rightarrow$$

$$H'(s) = a\cdot\frac{1}{2}s^{-1/2}+b\left(-\frac{1}{2}s^{-3/2}\right)=\frac{1}{2}as^{-1/2}-\frac{1}{2}bs^{-3/2} \Rightarrow$$

$$H''(s)=\frac{1}{2}a\left(-\frac{1}{2}s^{-3/2}\right)-\frac{1}{2}b\left(-\frac{3}{2}s^{-5/2}\right)=-\frac{1}{4}as^{-3/2}+\frac{3}{4}bs^{-5/2}$$

13. $h(x)=\sqrt{x^{2}+1} \Rightarrow h'(x)=\frac{1}{2}(x^{2}+1)^{-1/2}(2x)=\frac{x}{\sqrt{x^{2}+1}} \Rightarrow$

$$h''(x)=\frac{\sqrt{x^{2}+1}\cdot 1-x\left[\frac{1}{2}(x^{2}+1)^{-1/2}(2x)\right]}{\left(\sqrt{x^{2}+1}\right)^{2}}=\frac{(x^{2}+1)^{-1/2}[(x^{2}+1)-x^{2}]}{(x^{2}+1)^{1}}=\frac{1}{(x^{2}+1)^{3/2}}$$

14. $y=xe^{cx} \Rightarrow y'=x\cdot e^{cx}\cdot c+e^{cx}\cdot 1=e^{cx}(cx+1) \Rightarrow y''=e^{cx}(c)+(cx+1)e^{cx}\cdot c=ce^{cx}(1+cx+1)=ce^{cx}(cx+2)$

15. $y=(x^{3}+1)^{2/3} \Rightarrow y'=\frac{2}{3}(x^{3}+1)^{-1/3}(3x^{2})=2x^{2}(x^{3}+1)^{-1/3} \Rightarrow$

$$y''=2x^{2}\left(-\frac{1}{3}\right)(x^{3}+1)^{-4/3}(3x^{2})+(x^{3}+1)^{-1/3}(4x)=4x(x^{3}+1)^{-1/3}-2x^{4}(x^{3}+1)^{-4/3}$$

16.

$$y = \frac{4x}{\sqrt{x+1}} \Rightarrow$$

$$y' = \frac{\sqrt{x+1}\cdot 4-4x\cdot\frac{1}{2}(x+1)^{-1/2}}{(\sqrt{x+1})^{2}}=\frac{4\sqrt{x+1}-2x/\sqrt{x+1}}{x+1}=\frac{4(x+1)-2x}{(x+1)^{3/2}}=\frac{2x+4}{(x+1)^{3/2}} \Rightarrow$$

$$y'' = \frac{(x+1)^{3/2}\cdot 2-(2x+4)\cdot\frac{3}{2}(x+1)^{1/2}}{[(x+1)^{3/2}]^{2}}=\frac{(x+1)^{1/2}[2(x+1)-3(x+2)]}{(x+1)^{3}}$$

$$=\frac{2x+2-3x-6}{(x+1)^{5/2}}=\frac{-x-4}{(x+1)^{5/2}}$$

17. $H(t)=\tan 3t \Rightarrow$

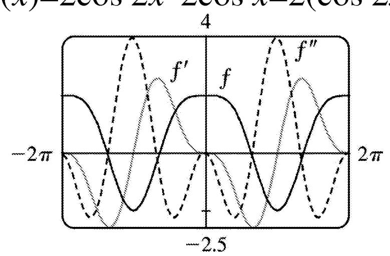
$H'(t)=3\sec^2 3t \Rightarrow H''(t)=2\cdot 3\sec 3t\ \dfrac{d}{dt}(\sec 3t)=6\sec 3t(3\sec 3t\cdot\tan 3t)=18\sec^2 3t\cdot\tan 3t$

18. $g(s)=s^2\cos s \Rightarrow g'(s)=2s\cdot\cos s-s^2\sin s\Rightarrow$
$g''(s)=2\cos s-2s\cdot\sin s-2s\cdot\sin s-s^2\cos s=(2-s^2)\cos s-4s\cdot\sin s$

19. $g(t)=t^3e^{5t}\Rightarrow g'(t)=t^3e^{5t}\cdot 5+e^{5t}\cdot 3t^2=t^2e^{5t}(5t+3)\Rightarrow$
$g''(t)=(2t)e^{5t}(5t+3)+t^2(5e^{5t})(5t+3)+t^2e^{5t}(5)$
$=te^{5t}[2(5t+3)+5t(5t+3)+5t]=te^{5t}(25t^2+30t+6)$

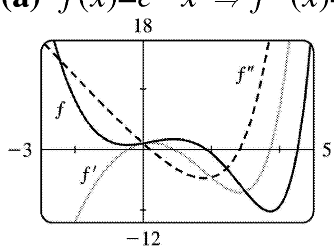
20. $h(x)=\tan^{-1}(x^2)\Rightarrow h'(x)=\dfrac{1}{1+(x^2)^2}\cdot 2x=\dfrac{2x}{1+x^4}\Rightarrow h''(x)=\dfrac{(1+x^4)(2)-(2x)(4x^3)}{(1+x^4)^2}=\dfrac{2-6x^4}{(1+x^4)^2}$

21. **(a)** $f(x)=2\cos x+\sin^2 x\Rightarrow f'(x)=2(-\sin x)+2\sin x(\cos x)=\sin 2x-2\sin x\Rightarrow$
$f''(x)=2\cos 2x-2\cos x=2(\cos 2x-\cos x)$

(b) 

We can see that our answers are plausible, since f has horizontal tangents where $f'(x)=0$, and f' has horizontal tangents where $f''(x)=0$.

22. **(a)** $f(x)=e^x-x^3\Rightarrow f'(x)=e^x-3x^2\Rightarrow f''(x)=e^x-6x$

(b) 

The graphs seem reasonable since f has horizontal tangents where f' is zero, f' is positive where f is increasing, and f' is negative where f is decreasing; and the same relationships exist between f' and f''.

23. $y=\sqrt{2x+3}=(2x+3)^{1/2} \Rightarrow y' = \frac{1}{2}(2x+3)^{-1/2} \cdot 2=(2x+3)^{-1/2} \Rightarrow y'' = -\frac{1}{2}(2x+3)^{-3/2} \cdot 2=-(2x+3)^{-3/2} \Rightarrow$
$y''' = \frac{3}{2}(2x+3)^{-5/2} \cdot 2=3(2x+3)^{-5/2}$

24. $y=\frac{x}{2x-1} \Rightarrow y' = \frac{(2x-1)(1)-x(2)}{(2x-1)^2} = \frac{-1}{(2x-1)^2}$ or $-1(2x-1)^{-2} \Rightarrow$
$y''=-1(-2)(2x-1)^{-3}(2)=4(2x-1)^{-3} \Rightarrow$
$y'''=4(-3)(2x-1)^{-4}(2)=-24(2x-1)^{-4}$ or $-24/(2x-1)^4$

25. $f(t)=t\cos t \Rightarrow f'(t)=t(-\sin t)+\cos t \cdot 1 \Rightarrow f''(t)=t(-\cos t)-\sin t \cdot 1-\sin t \Rightarrow$
$f'''(t)=t\sin t-\cos t \cdot 1-\cos t-\cos t=t\sin t-3\cos t$, so $f'''(0)=0-3=-3$.

26. $g(x)=\sqrt{5-2x} \Rightarrow g'(x)=\frac{1}{2}(5-2x)^{-1/2}(-2)=-(5-2x)^{-1/2} \Rightarrow g''(x)=\frac{1}{2}(5-2x)^{-3/2}(-2)=-(5-2x)^{-3/2} \Rightarrow$
$g'''(x)=\frac{3}{2}(5-2x)^{-5/2}(-2)=-3(5-2x)^{-5/2}$, so $g'''(2)=-3(1)^{-5/2}=-3$.

27. $f(\theta)=\cot\theta \Rightarrow f'(\theta)=-\csc^2\theta \Rightarrow f''(\theta)=-2\csc\theta(-\csc\theta \cdot \cot\theta)=2\csc^2\theta \cdot \cot\theta \Rightarrow$
$f'''(\theta)=2(-2\csc^2\theta \cdot \cot\theta)\cot\theta+2\csc^2\theta(-\csc^2\theta)=-2\csc^2\theta(2\cot^2\theta+\csc^2\theta) \Rightarrow$
$f'''\left(\frac{\pi}{6}\right)=-2(2)^2[2(\sqrt{3})^2+(2)^2]=-80$

28. $g(x)=\sec x \Rightarrow g'(x)=\sec x \cdot \tan x \Rightarrow$
$g''(x)=\sec x \cdot \sec^2 x+\tan x(\sec x \cdot \tan x)=\sec^3 x+\sec x \cdot \tan^2 x=\sec^3 x+\sec x(\sec^2 x-1)=2\sec^3 x-\sec x \Rightarrow$
$g'''(x)=6\sec^2 x(\sec x \cdot \tan x)-\sec x \cdot \tan x=\sec x \cdot \tan x(6\sec^2 x-1) \Rightarrow g'''\left(\frac{\pi}{4}\right)=\sqrt{2}(1)(6 \cdot 2-1)=11\sqrt{2}$

29. $9x^2+y^2=9 \Rightarrow 18x+2yy'=0 \Rightarrow 2yy'=-18x \Rightarrow y'=-9x/y \Rightarrow$
$y''=-9\left(\frac{y \cdot 1-x \cdot y'}{y^2}\right)=-9\left(\frac{y-x(-9x/y)}{y^2}\right)=-9 \cdot \frac{y^2+9x^2}{y^3}=-9 \cdot \frac{9}{y^3}$ [since x and y must satisfy the original equation, $9x^2+y^2=9$] . Thus, $y''=-81/y^3$.

30.

$\sqrt{x}+\sqrt{y}=1 \Rightarrow \dfrac{1}{2\sqrt{x}}+\dfrac{y'}{2\sqrt{y}}=0 \Rightarrow y'=-\dfrac{\sqrt{y}}{\sqrt{x}} \Rightarrow$

$$y''=-\frac{\sqrt{x}\,\frac{1}{(2\sqrt{y})}\,y'-\sqrt{y}\,\frac{1}{(2\sqrt{x})}}{x}=-\frac{\sqrt{x}\left(\frac{1}{\sqrt{y}}\right)\left(-\frac{\sqrt{y}}{\sqrt{x}}\right)-\sqrt{y}\left(\frac{1}{\sqrt{x}}\right)}{2x}=\frac{1+\frac{\sqrt{y}}{\sqrt{x}}}{2x}$$

$=\dfrac{\sqrt{x}+\sqrt{y}}{2x\sqrt{x}}=\dfrac{1}{2x\sqrt{x}}$ since x and y must satisfy the original equation, $\sqrt{x}+\sqrt{y}=1$.

31. $x^3+y^3=1 \Rightarrow 3x^2+3y^2y'=0 \Rightarrow y'=-\dfrac{x^2}{y^2} \Rightarrow$

$$y''=-\frac{y^2(2x)-x^2\cdot 2yy'}{(y^2)^2}=-\frac{2xy^2-2x^2y\left(-\frac{x^2}{y^2}\right)}{y^4}=-\frac{2xy^4+2x^4y}{y^6}=-\frac{2xy(y^3+x^3)}{y^6}=-\frac{2x}{y^5},$$

since x and y must satisfy the original equation, $x^3+y^3=1$.

32. $x^4+y^4=a^4 \Rightarrow 4x^3+4y^3y'=0 \Rightarrow 4y^3y'=-4x^3 \Rightarrow y'=-x^3/y^3 \Rightarrow$

$$y''=-\left(\frac{y^3\cdot 3x^2-x^3\cdot 3y^2y'}{(y^3)^2}\right)=-3x^2y^2\cdot\frac{y-x\left(-\frac{x^3}{y^3}\right)}{y^6}=-3x^2\cdot\frac{y^4+x^4}{y^4y^3}=-3x^2\cdot\frac{a^4}{y^7}=\frac{-3a^4x^2}{y^7}$$

33. $f(x)=x^n \Rightarrow f'(x)=nx^{n-1} \Rightarrow f''(x)=n(n-1)x^{n-2} \Rightarrow \cdots \Rightarrow f^{(n)}(x)=n(n-1)(n-2)\cdots 2\cdot 1x^{n-n}=\text{n}!$

34. $f(x)=\dfrac{1}{5x-1}=(5x-1)^{-1} \Rightarrow f'(x)=-1(5x-1)^{-2}\cdot 5 \Rightarrow f''(x)=(-1)(-2)(5x-1)^{-3}\cdot 5^2 \Rightarrow$
$f'''(x)=(-1)(-2)(-3)(5x-1)^{-4}\cdot 5^3 \Rightarrow \cdots \Rightarrow f^{(n)}(x)=(-1)^n\text{n}!\,5^n(5x-1)^{-(n+1)}$

35. $f(x)=e^{2x} \Rightarrow f'(x)=2e^{2x} \Rightarrow f''(x)=2\cdot 2e^{2x}=2^2e^{2x} \Rightarrow$
$f'''(x)=2^2\cdot 2e^{2x}=2^3e^{2x} \Rightarrow \cdots \Rightarrow f^{(n)}(x)=2^ne^{2x}$

36. $f(x)=\sqrt{x}=x^{1/2} \Rightarrow f'(x)=\dfrac{1}{2}x^{-1/2} \Rightarrow$
$f''(x)=\dfrac{1}{2}\left(-\dfrac{1}{2}\right)x^{-3/2} \Rightarrow f'''(x)=\dfrac{1}{2}\left(-\dfrac{1}{2}\right)\left(-\dfrac{3}{2}\right)x^{-5/2} \Rightarrow$

$f^{(4)}(x)=\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)x^{-7/2}=-\frac{1\cdot 3\cdot 5}{2^4}x^{-7/2}\Rightarrow$

$f^{(5)}(x)=\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(-\frac{7}{2}\right)x^{-9/2}=\frac{1\cdot 3\cdot 5\cdot 7}{2^5}x^{-9/2}\Rightarrow\cdots\Rightarrow$

$f^{(n)}(x)=\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\cdots\left(\frac{1}{2}-n+1\right)x^{-(2n-1)/2}=(-1)^{n-1}\frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-3)}{2^n}x^{-(2n-1)/2}$

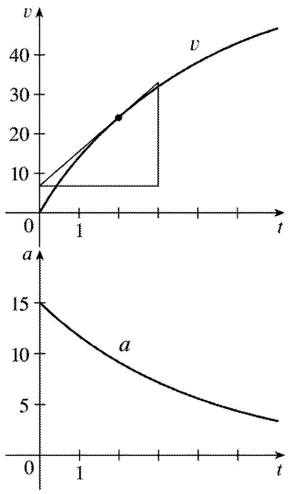
37. $f(x)=1/(3x^3)=\frac{1}{3}x^{-3}\Rightarrow f'(x)=\frac{1}{3}(-3)x^{-4}\Rightarrow f''(x)=\frac{1}{3}(-3)(-4)x^{-5}\Rightarrow$
$f'''(x)=\frac{1}{3}(-3)(-4)(-5)x^{-6}\Rightarrow\cdots\Rightarrow$
$f^{(n)}(x)=\frac{1}{3}(-3)(-4)\cdots[-(n+2)]x^{-(n+3)}=\frac{(-1)^n\cdot 3\cdot 4\cdot 5\cdot\cdots\cdot(n+2)}{3x^{n+3}}\cdot\frac{2}{2}=\frac{(-1)^n(n+2)!}{6x^{n+3}}$

38. $D\sin x=\cos x\Rightarrow D^2\sin x=-\sin x\Rightarrow D^3\sin x=-\cos x\Rightarrow D^4\sin x=\sin x$. The derivatives of $\sin x$ occur in a cycle of four. Since $74=4(18)+2$, we have $D^{74}\sin x=D^2\sin x=-\sin x$.

39. Let $f(x)=\cos x$. Then $Df(2x)=2f'(2x)$, $D^2f(2x)=2^2f''(2x)$, $D^3f(2x)=2^3f'''(2x)$, … , $D^{(n)}f(2x)=2^nf^{(n)}(2x)$. Since the derivatives of $\cos x$ occur in a cycle of four, and since $103=4(25)+3$, we have $f^{(103)}(x)=f^{(3)}(x)=\sin x$ and $D^{103}\cos 2x=2^{103}f^{(103)}(2x)=2^{103}\sin 2x$.

40. $f(x)=xe^{-x}\Rightarrow f'(x)=x(-e^{-x})+e^{-x}=(1-x)e^{-x}\Rightarrow f''(x)=(1-x)(-e^{-x})+e^{-x}(-1)=(x-2)e^{-x}\Rightarrow$
$f'''(x)=(x-2)(-e^{-x})+e^{-x}=(3-x)e^{-x}\Rightarrow f^{(4)}(x)=(3-x)(-e^{-x})+e^{-x}(-1)=(x-4)e^{-x}\Rightarrow\cdots\Rightarrow$
$f^{(n)}(x)=(-1)^n(x-n)e^{-x}$.
So $D^{1000}xe^{-x}=(x-1000)e^{-x}$.

41. By measuring the slope of the graph of $s=f(t)$ at $t=0$, 1 , 2 , 3 , 4 , and 5 , and using the method of Example 1 in Section 3.2, we plot the graph of the velocity function $v=f'(t)$ in the first figure. The acceleration when $t=2$ s is $a=f''(2)$, the slope of the tangent line to the graph of f' when $t=2$. We estimate the slope of this tangent line to be $a(2)=f''(2)=v'(2)\approx\frac{27}{3}=9\text{ ft / s}^2$. Similar measurements enable us to graph the acceleration function in the second figure.



42. **(a)** Since we estimate the velocity to be a maximum at $t=10$, the acceleration is 0 at $t=10$.

(b) Drawing a tangent line at $t=10$ on the graph of a, a appears to decrease by 10 $\mathrm{ft/s^2}$ over a period of 20 s. So at $t=10$ s, the jerk is approximately $-10/20=-0.5\left(\mathrm{ft/s^2}\right)/\mathrm{s}$ or $\mathrm{ft/s^3}$.

43. **(a)** $s=2t^3-15t^2+36t+2 \Rightarrow v(t)=s'(t)=6t^2-30t+36 \Rightarrow a(t)=v'(t)=12t-30$

(b) $a(1)=12\cdot 1-30=-18\ \mathrm{m/s^2}$

(c) $v(t)=6(t^2-5t+6)=6(t-2)(t-3)=0$ when $t=2$ or 3 and $a(2)=24-30=-6\ \mathrm{m/s^2}$, $a(3)=36-30=6\ \mathrm{m/s^2}$.

44. **(a)** $s=2t^3-3t^2-12t \Rightarrow v(t)=s'(t)=6t^2-6t-12 \Rightarrow a(t)=v'(t)=12t-6$

(b) $a(1)=12\cdot 1-6=6\ \mathrm{m/s^2}$

(c) $v(t)=6(t^2-t-2)=6(t+1)(t-2)=0$ when $t=-1$ or 2. Since $t\geq 0$, $t\neq -1$ and $a(2)=24-6=18\ \mathrm{m/s^2}$.

45. **(a)** $s=\sin\left(\frac{\pi}{6}t\right)+\cos\left(\frac{\pi}{6}t\right)$, $0\leq t\leq 2$.

$v(t)=s'(t)=\cos\left(\frac{\pi}{6}t\right)\cdot\frac{\pi}{6}-\sin\left(\frac{\pi}{6}t\right)\cdot\frac{\pi}{6}=\frac{\pi}{6}\left[\cos\left(\frac{\pi}{6}t\right)-\sin\left(\frac{\pi}{6}t\right)\right]\Rightarrow$

$a(t)=v'(t)=\frac{\pi}{6}\left[-\sin\left(\frac{\pi}{6}t\right)\cdot\frac{\pi}{6}-\cos\left(\frac{\pi}{6}t\right)\cdot\frac{\pi}{6}\right]=-\frac{\pi^2}{36}\left[\sin\left(\frac{\pi}{6}t\right)+\cos\left(\frac{\pi}{6}t\right)\right]$

(b) $a(1)=-\frac{\pi^2}{36}\left[\sin\left(\frac{\pi}{6}\cdot 1\right)+\cos\left(\frac{\pi}{6}\cdot 1\right)\right]=-\frac{\pi^2}{36}\left[\frac{1}{2}+\frac{\sqrt{3}}{2}\right]=-\frac{\pi^2}{72}(1+\sqrt{3})\approx -0.3745\ \mathrm{m/s^2}$

(c) $v(t)=0$ for $0\leq t\leq 2 \Rightarrow \cos\left(\frac{\pi}{6}t\right)=\sin\left(\frac{\pi}{6}t\right)\Rightarrow$

$1=\dfrac{\sin\left(\frac{\pi}{6}t\right)}{\cos\left(\frac{\pi}{6}t\right)}\Rightarrow$

$\tan\left(\frac{\pi}{6}t\right)=1\Rightarrow\frac{\pi}{6}t=\tan^{-1}1\Rightarrow t=\frac{6}{\pi}\cdot\frac{\pi}{4}=\frac{3}{2}=1.5$ s. Thus,

$a(\frac{3}{2})=-\frac{\pi^2}{36}\left[\sin\left(\frac{\pi}{6}\cdot\frac{3}{2}\right)+\cos\left(\frac{\pi}{6}\cdot\frac{3}{2}\right)\right]=-\frac{\pi^2}{36}\left[\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}\right]=-\frac{\pi^2}{36}\sqrt{2}\approx-0.3877\text{ m / s}^2$.

46. **(a)** $s=2t^3-7t^2+4t+1\Rightarrow v(t)=s'(t)=6t^2-14t+4\Rightarrow a(t)=v'(t)=12t-14$

(b) $a(1)=12-14=-2\text{ m / s}^2$

(c) $v(t)=2\left(3t^2-7t+2\right)=2(3t-1)(t-2)=0$ when $t=\frac{1}{3}$ or 2 and $a\left(\frac{1}{3}\right)=12\left(\frac{1}{3}\right)-14=-10\text{ m / s}^2$,

$a(2)=12(2)-14=10\text{ m / s}^2$.

47. **(a)** $s(t)=t^4-4t^3+2\Rightarrow v(t)=s'(t)=4t^3-12t^2\Rightarrow a(t)=v'(t)=12t^2-24t=12t(t-2)=0$ when $t=0$ or 2 .

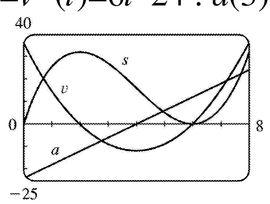
(b) $s(0)=2$ m, $v(0)=0$ m / s, $s(2)=-14$ m, $v(2)=-16$ m / s

48. **(a)** $s(t)=2t^3-9t^2\Rightarrow v(t)=s'(t)=6t^2-18t\Rightarrow a(t)=v'(t)=12t-18=0$ when $t=1.5$.

(b) $s(1.5)=-13.5$ m, $v(1.5)=-13.5$ m / s

49. **(a)** $s=f(t)=t^3-12t^2+36t$, $t\geq 0\Rightarrow v(t)=f'(t)=3t^2-24t+36$.

$a(t)=v'(t)=6t-24$. $a(3)=6(3)-24=-6$ (m/s) / s or m / s^2 .

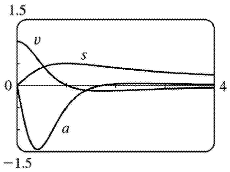
(b) 

(c) The particle is speeding up when v and a have the same sign. This occurs when $2<t<4$ and when $t>6$. It is slowing down when v and a have opposite signs; that is, when $0\leq t<2$ and when $4<t<6$.

50. **(a)** $x(t)=\dfrac{t}{1+t^2}\Rightarrow v(t)=x'(t)=\dfrac{(1+t^2)(1)-t(2t)}{(1+t^2)^2}=\dfrac{1-t^2}{(1+t^2)^2}$.

$a(t)=v'(t)=\dfrac{2t(t^2-3)}{(1+t^2)^3}$. $a(t)=0\Rightarrow 2t(t^2-3)=0\Rightarrow t=0$ or $\sqrt{3}$

(b)



(c) v and a have the same sign and the particle is speeding up when $1<t<\sqrt{3}$. The particle is slowing down and v and a have opposite signs when $0<t<1$ and when $t>\sqrt{3}$.

51. **(a)** $y(t)=A\sin\omega t \Rightarrow v(t)=y'(t)=A\omega\cos\omega t \Rightarrow a(t)=v'(t)=-A\omega^2\sin\omega t$

(b) $a(t)=-A\omega^2\sin\omega t=-\omega^2(A\sin\omega t)=-\omega^2 y(t)$, so $a(t)$ is proportional to $y(t)$.

(c) speed $=|v(t)|=A\omega\,|\cos\omega t|$ is a maximum when $\cos\omega t=\pm1$. But when $\cos\omega t=\pm1$, we have $\sin\omega t=0$, and $a(t)=-A\omega^2\sin\omega t=-A\omega^2(0)=0$.

52. By the Chain Rule, $a(t)=\dfrac{dv}{dt}=\dfrac{dv}{ds}\,\dfrac{ds}{dt}=\dfrac{dv}{ds}\,v(t)=v(t)\,\dfrac{dv}{ds}$. The derivative dv/dt is the rate of change of the velocity with respect to time (in other words, the acceleration) whereas the derivative dv/ds is the rate of change of the velocity with respect to the displacement.

53. Let $P(x)=ax^2+bx+c$. Then $P'(x)=2ax+b$ and $P''(x)=2a$. $P''(2)=2 \Rightarrow 2a=2 \Rightarrow a=1$.
$P'(2)=3 \Rightarrow 2(1)(2)+b=3 \Rightarrow 4+b=3 \Rightarrow b=-1$.
$P(2)=5 \Rightarrow 1(2)^2+(-1)(2)+c=5 \Rightarrow 2+c=5 \Rightarrow c=3$. So $P(x)=x^2-x+3$.

54. Let $Q(x)=ax^3+bx^2+cx+d$. Then $Q'(x)=3ax^2+2bx+c$, $Q''(x)=6ax+2b$ and $Q'''(x)=6a$. Thus, $Q(1)=a+b+c+d=1$, $Q'(1)=3a+2b+c=3$, $Q''(1)=6a+2b=6$ and $Q'''(1)=6a=12$. Solving these four equations in four unknowns a , b , c and d we get $a=2$, $b=-3$, $c=3$ and $d=-1$, so $Q(x)=2x^3-3x^2+3x-1$.

55. $y=A\sin x+B\cos x \Rightarrow y'=A\cos x-B\sin x \Rightarrow y''=-A\sin x-B\cos x$. Substituting into $y''+y'-2y=\sin x$ gives us $(-3A-B)\sin x+(A-3B)\cos x=1\sin x$, so we must have $-3A-B=1$ and $A-3B=0$. Solving for A and B , we add the first equation to three times the second to get $B=-\dfrac{1}{10}$ and $A=-\dfrac{3}{10}$.

56. $y=Ax^2+Bx+C \Rightarrow y'=2Ax+B \Rightarrow y''=2A$. We substitute these expressions into the equation $y''+y'-2y=x^2$ to get

$$\begin{aligned}(2A)+(2Ax+B)-2(Ax^2+Bx+C) &= x^2\\ 2A+2Ax+B-2Ax^2-2Bx-2C &= x^2\end{aligned}$$

$(-2A)x^2+(2A-2B)x+(2A+B-2C) = (1)x^2+(0)x+(0)$

The coefficients of x^2 on each side must be equal, so $-2A=1 \Rightarrow A=-\frac{1}{2}$. Similarly, $2A-2B=0 \Rightarrow$ $A=B=-\frac{1}{2}$ and $2A+B-2C=0 \Rightarrow -1-\frac{1}{2}-2C=0 \Rightarrow C=-\frac{3}{4}$.

57. $y=e^{rx} \Rightarrow y'=re^{rx} \Rightarrow y''=r^2e^{rx}$, so $y''+5y'-6y=r^2e^{rx}+5re^{rx}-6e^{rx}=e^{rx}(r^2+5r-6)=e^{rx}(r+6)(r-1)=0 \Rightarrow$ $(r+6)(r-1)=0 \Rightarrow r=1$ or -6.

58. $y=e^{\lambda x} \Rightarrow y'=\lambda e^{\lambda x} \Rightarrow y''=\lambda^2 e^{\lambda x}$. Thus, $y+y'=y'' \Leftrightarrow e^{\lambda x}+\lambda e^{\lambda x}=\lambda^2 e^{\lambda x} \Leftrightarrow e^{\lambda x}(\lambda^2-\lambda-1)=0 \Leftrightarrow$ $\lambda=\frac{1\pm\sqrt{5}}{2}$, since $e^{\lambda x}\neq 0$.

59. $f(x)=xg(x^2) \Rightarrow f'(x)=x\cdot g'(x^2)\cdot 2x+g(x^2)\cdot 1=g(x^2)+2x^2g'(x^2) \Rightarrow$
$f''(x)=g'(x^2)\cdot 2x+2x^2\cdot g''(x^2)\cdot 2x+g'(x^2)\cdot 4x=6xg'(x^2)+4x^3g''(x^2)$

60. $f(x)=\frac{g(x)}{x} \Rightarrow f'(x)=\frac{xg'(x)-g(x)}{x^2} \Rightarrow$

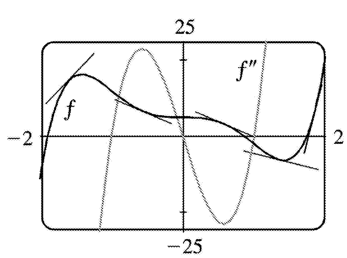
$$f''(x)=\frac{x^2[g'(x)+xg''(x)-g'(x)]-2x[xg'(x)-g(x)]}{x^4}=\frac{x^2g''(x)-2xg'(x)+2g(x)}{x^3}$$

61.

$$f(x) = g(\sqrt{x}) \Rightarrow f'(x)=g'(\sqrt{x})\cdot \frac{1}{2}x^{-1/2}=\frac{g'(\sqrt{x})}{2\sqrt{x}} \Rightarrow$$

$$f''(x)=\frac{2\sqrt{x}\cdot g''(\sqrt{x})\cdot \frac{1}{2}x^{-1/2}-g'(\sqrt{x})\cdot 2\cdot \frac{1}{2}x^{-1/2}}{(2\sqrt{x})^2}=\frac{x^{-1/2}[\sqrt{x}\,g''(\sqrt{x})-g'(\sqrt{x})]}{4x}$$

$$=\frac{\sqrt{x}\,g''(\sqrt{x})-g'(\sqrt{x})}{4x\sqrt{x}}$$

62. 

$f(x)=3x^5-10x^3+5 \Rightarrow f'(x)=15x^4-30x^2 \Rightarrow f''(x)=60x^3-60x=60x(x^2-1)=60x(x+1)(x-1)$

So $f''(x)>0$ when $-1<x<0$ or $x>1$, and on these intervals the graph of f lies above its tangent lines; and $f''(x)<0$ when $x<-1$ or $0<x<1$, and on these intervals the graph of f lies below its tangent lines.

63. **(a)**

$$f(x) = \frac{1}{x^2+x} \Rightarrow f'(x)=\frac{-(2x+1)}{(x^2+x)^2} \Rightarrow$$

$$f''(x) = \frac{(x^2+x)^2(-2)+(2x+1)(2)(x^2+x)(2x+1)}{(x^2+x)^4} = \frac{2(3x^2+3x+1)}{(x^2+x)^3} \Rightarrow$$

$$f'''(x)=\frac{(x^2+x)^3(2)(6x+3)-2(3x^2+3x+1)(3)(x^2+x)^2(2x+1)}{(x^2+x)^6}$$

$$= \frac{-6(4x^3+6x^2+4x+1)}{(x^2+x)^4} \Rightarrow$$

$$f^{(4)}(x) = \frac{(x^2+x)^4(-6)(12x^2+12x+4)+6(4x^3+6x^2+4x+1)(4)(x^2+x)^3(2x+1)}{(x^2+x)^8}$$

$$= \frac{24(5x^4+10x^3+10x^2+5x+1)}{(x^2+x)^5}$$

$f^{(5)}(x) =?$

(b) $f(x)=\frac{1}{x(x+1)} = \frac{1}{x} - \frac{1}{x+1} \Rightarrow f'(x)=-x^{-2}+(x+1)^{-2} \Rightarrow f''(x)=2x^{-3}-2(x+1)^{-3} \Rightarrow$
$f'''(x)=(-3)(2)x^{-4}+(3)(2)(x+1)^{-4} \Rightarrow \cdots \Rightarrow f^{(n)}(x)=(-1)^n n!\left[x^{-(n+1)}-(x+1)^{-(n+1)}\right]$

64. **(a)** For $f(x)=\frac{7x+17}{2x^2-7x-4}$, a CAS gives us $f'''(x)=\frac{-6(56x^4+544x^3-2184x^2+6184x-6139)}{(2x^2-7x-4)^4}$

(b) Using a CAS we get $f(x)=\frac{7x+17}{2x^2-7x-4} = \frac{-3}{2x+1} + \frac{5}{x-4}$. Now we differentiate three times to obtain
$f'''(x)=\frac{144}{(2x+1)^4} - \frac{30}{(x-4)^4}$.

65.

For $f(x)=x^2e^x$, $f'(x)=x^2e^x+e^x(2x) =e^x(x^2+2x)$. Similarly, we have

$$f''(x)=e^{x}(x^{2}+4x+2)$$
$$f'''(x)=e^{x}(x^{2}+6x+6)$$
$$f^{(4)}(x)=e^{x}(x^{2}+8x+12)$$
$$f^{(5)}(x)=e^{x}(x^{2}+10x+20)$$

It appears that the coefficient of x in the quadratic term increases by 2 with each differentiation. The pattern for the constant terms seems to be $0=1\cdot 0$, $2=2\cdot 1$, $6=3\cdot 2$, $12=4\cdot 3$, $20=5\cdot 4$. So a reasonable guess is that $f^{(n)}(x)=e^{x}\left[x^{2}+2nx+n(n-1)\right]$.

Proof: Let S_n be the statement that $f^{(n)}(x)=e^{x}\left[x^{2}+2nx+n(n-1)\right]$. 1. S_1 is true because $f'(x)=e^{x}(x^{2}+2x)$.

2. Assume that S_k is true; that is, $f^{(k)}(x)=e^{x}\left[x^{2}+2kx+k(k-1)\right]$. Then

$$f^{(k+1)}(x)=\frac{d}{dx}\left[f^{(k)}(x)\right]=e^{x}(2x+2k)+\left[x^{2}+2kx+k(k-1)\right]e^{x}$$
$$=e^{x}\left[x^{2}+(2k+2)x+(k^{2}+k)\right]=e^{x}\left[x^{2}+2(k+1)x+(k+1)k\right]$$

This shows that S_{k+1} is true.

3. Therefore, by mathematical induction, S_n is true for all n; that is, $f^{(n)}(x)=e^{x}\left[x^{2}+2nx+n(n-1)\right]$ for every positive integer n.

66. **(a)** Use the Product Rule repeatedly: $F=fg\Rightarrow F'=f'g+fg'\Rightarrow$
$F''=(f''g+f'g')+(f'g'+fg'')=f''g+2f'g'+fg''$.

(b) $F'''=f'''g+f''g'+2(f''g'+f'g'')+f'g''+fg'''=f'''g+3f''g'+3f'g''+fg'''\Rightarrow$
$$F^{(4)}=f^{(4)}g+f'''g'+3(f'''g'+f''g'')+3(f''g''+f'g''')+f'g'''+fg^{(4)}$$
$$=f^{(4)}g+4f'''g'+6f''g''+4f'g'''+fg^{(4)}$$

(c) By analogy with the Binomial Theorem, we make the guess:

$$F^{(n)}=f^{(n)}g+nf^{(n-1)}g'+\binom{n}{2}f^{(n-2)}g''+\cdots+\binom{n}{k}f^{(n-k)}g^{(k)}+\cdots+nf'g^{(n-1)}+fg^{(n)}$$

where $\binom{n}{k}=\dfrac{n!}{k!(n-k)!}=\dfrac{n(n-1)(n-2)\cdots(n-k+1)}{k!}$.

67. The Chain Rule says that

$\frac{dy}{dx}=\frac{dy}{du}\,\frac{du}{dx}$, so

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dx}\left(\frac{dy}{du}\,\frac{du}{dx}\right)=\left[\frac{d}{dx}\left(\frac{dy}{du}\right)\right]\frac{du}{dx}+\frac{dy}{du}\,\frac{d}{dx}\left(\frac{du}{dx}\right)\text{ [Product Rule]}$$

$$=\left[\frac{d}{du}\left(\frac{dy}{du}\right)\frac{du}{dx}\right]\frac{du}{dx}+\frac{dy}{du}\,\frac{d^2u}{dx^2}=\frac{d^2y}{du^2}\left(\frac{du}{dx}\right)^2+\frac{dy}{du}\,\frac{d^2u}{dx^2}$$

68. From Exercise 65, $\frac{d^2y}{dx^2}=\frac{d^2y}{du^2}\left(\frac{du}{dx}\right)^2+\frac{dy}{du}\,\frac{d^2u}{dx^2}\Rightarrow$

$$\frac{d^3y}{dx^3}=\frac{d}{dx}\,\frac{d^2y}{dx^2}=\frac{d}{dx}\left[\frac{d^2y}{du^2}\left(\frac{du}{dx}\right)^2\right]+\frac{d}{dx}\left[\frac{dy}{du}\,\frac{d^2u}{dx^2}\right]$$

$$=\left[\frac{d}{dx}\left(\frac{d^2y}{du^2}\right)\right]\left(\frac{du}{dx}\right)^2+\left[\frac{d}{dx}\left(\frac{du}{dx}\right)^2\right]\frac{d^2y}{du^2}+\left[\frac{d}{dx}\left(\frac{dy}{du}\right)\right]\frac{d^2u}{dx^2}+\left[\frac{d}{dx}\left(\frac{d^2u}{dx^2}\right)\right]\frac{dy}{du}$$

$$=\left[\frac{d}{du}\left(\frac{d^2y}{du^2}\right)\frac{du}{dx}\right]\left(\frac{du}{dx}\right)^2+2\,\frac{du}{dx}\,\frac{d^2u}{dx^2}\,\frac{d^2y}{du^2}+\left[\frac{d}{du}\left(\frac{dy}{du}\right)\frac{du}{dx}\right]\left(\frac{d^2u}{dx^2}\right)+\frac{d^3u}{dx^3}\,\frac{dy}{du}$$

$$=\frac{d^3y}{du^3}\left(\frac{du}{dx}\right)^3+3\,\frac{du}{dx}\,\frac{d^2u}{dx^2}\,\frac{d^2y}{du^2}+\frac{dy}{du}\,\frac{d^3u}{dx^3}$$

69. We will show that for each positive integer n , the n th derivative $f^{(n)}$ exists and equals one of f , f' , f'' , f''' , … , $f^{(p-1)}$. Since $f^{(p)}=f$, the first p derivatives of f are f' , f'' , f''' , … , $f^{(p-1)}$, and f. In particular, our statement is true for $n=1$. Suppose that k is an integer, $k\geq 1$, for which f is k –times differentiable with $f^{(k)}$ in the set $S=\{f , f' , f'' , \dots , f^{(p-1)}\}$. Since f is p –times differentiable, every member of S is differentiable, so $f^{(k+1)}$ exists and equals the derivative of some member of S . Thus, $f^{(k+1)}$ is in the set $\{f' , f'' , f''' , \dots , f^{(p)}\}$, which equals S since $f^{(p)}=f$. We have shown that the statement is true for $n=1$ and that its truth for $n=k$ implies its truth for $n=k+1$. By mathematical induction, the statement is true for all positive integers n.

1. The differentiation formula for logarithmic functions, $\frac{d}{dx}(\log_a x)=\frac{1}{x\ln a}$, is simplest when $a=e$ because $\ln e=1$.

2. $f(x)=\ln(x^2+10)\Rightarrow f'(x)=\frac{1}{x^2+10}\frac{d}{dx}(x^2+10)=\frac{2x}{x^2+10}$

3. $f(\theta)=\ln(\cos\theta)\Rightarrow f'(\theta)=\frac{1}{\cos\theta}\frac{d}{d\theta}(\cos\theta)=\frac{-\sin\theta}{\cos\theta}=-\tan\theta$

4. $f(x)=\cos(\ln x)\Rightarrow f'(x)=-\sin(\ln x)\cdot\frac{1}{x}=\frac{-\sin(\ln x)}{x}$

5. $f(x)=\log_2(1-3x)\Rightarrow f'(x)=\frac{1}{(1-3x)\ln 2}\frac{d}{dx}(1-3x)=\frac{-3}{(1-3x)\ln 2}$ or $\frac{3}{(3x-1)\ln 2}$

6. $f(x)=\log_{10}\left(\frac{x}{x-1}\right)=\log_{10}x-\log_{10}(x-1)\Rightarrow f'(x)=\frac{1}{x\ln 10}-\frac{1}{(x-1)\ln 10}$ or $-\frac{1}{x(x-1)\ln 10}$

7. $f(x)=\sqrt[5]{\ln x}=(\ln x)^{1/5}\Rightarrow f'(x)=\frac{1}{5}(\ln x)^{-4/5}\frac{d}{dx}(\ln x)=\frac{1}{5(\ln x)^{4/5}}\cdot\frac{1}{x}=\frac{1}{5x\sqrt[5]{(\ln x)^4}}$

8. $f(x)=\ln\sqrt[5]{x}=\ln x^{1/5}=\frac{1}{5}\ln x\Rightarrow f'(x)=\frac{1}{5}\cdot\frac{1}{x}=\frac{1}{5x}$

9. $f(x)=\sqrt{x}\ln x\Rightarrow f'(x)=\sqrt{x}\left(\frac{1}{x}\right)+(\ln x)\cdot\frac{1}{2\sqrt{x}}=\frac{1}{\sqrt{x}}+\frac{\ln x}{2\sqrt{x}}=\frac{2+\ln x}{2\sqrt{x}}$

10. $f(t)=\frac{1+\ln t}{1-\ln t}\Rightarrow$

$$f'(t)=\frac{(1-\ln t)(1/t)-(1+\ln t)(-1/t)}{(1-\ln t)^2}=\frac{(1/t)\left[(1-\ln t)+(1+\ln t)\right]}{(1-\ln t)^2}=\frac{2}{t(1-\ln t)^2}$$

11. $F(t)=\ln\frac{(2t+1)^3}{(3t-1)^4}=\ln(2t+1)^3-\ln(3t-1)^4=3\ln(2t+1)-4\ln(3t-1)\Rightarrow$

$F'(t)=3\cdot\frac{1}{2t+1}\cdot 2-4\cdot\frac{1}{3t-1}\cdot 3=\frac{6}{2t+1}-\frac{12}{3t-1}$, or combined, $\frac{-6(t+3)}{(2t+1)(3t-1)}$.

12.

$h(x)=\ln\left(x+\sqrt{x^2-1}\right)\Rightarrow h'(x)=\frac{1}{x+\sqrt{x^2-1}}\left(1+\frac{x}{\sqrt{x^2-1}}\right)=\frac{1}{x+\sqrt{x^2-1}}\cdot\frac{\sqrt{x^2-1}+x}{\sqrt{x^2-1}}=\frac{1}{\sqrt{x^2-1}}$

13. $g(x)=\ln\frac{a-x}{a+x}=\ln(a-x)-\ln(a+x)\Rightarrow$
$g'(x)=\frac{1}{a-x}(-1)-\frac{1}{a+x}=\frac{-(a+x)-(a-x)}{(a-x)(a+x)}=\frac{-2a}{a^2-x^2}$

14. $F(y)=y\ln(1+e^y)\Rightarrow F'(y)=y\cdot\frac{1}{1+e^y}\cdot e^y+\ln(1+e^y)\cdot 1=\frac{ye^y}{1+e^y}+\ln(1+e^y)$

15. $f(u)=\frac{\ln u}{1+\ln(2u)}\Rightarrow$

$$f'(u)=\frac{[1+\ln(2u)]\cdot\frac{1}{u}-\ln u\cdot\frac{1}{2u}\cdot 2}{[1+\ln(2u)]^2}=\frac{\frac{1}{u}[1+\ln(2u)-\ln u]}{[1+\ln(2u)]^2}$$
$$=\frac{1+(\ln 2+\ln u)-\ln u}{u[1+\ln(2u)]^2}=\frac{1+\ln 2}{u[1+\ln(2u)]^2}$$

16. $y=\ln(x^4\sin^2x)=\ln x^4+\ln(\sin x)^2=4\ln x+2\ln\sin x\Rightarrow y'=4\cdot\frac{1}{x}+2\cdot\frac{1}{\sin x}\cdot\cos x=\frac{4}{x}+2\cot x$

17. $y=\ln|2-x-5x^2|\Rightarrow y'=\frac{1}{2-x-5x^2}\cdot(-1-10x)=\frac{-10x-1}{2-x-5x^2}$ or $\frac{10x+1}{5x^2+x-2}$

18. $G(u)=\ln\sqrt{\frac{3u+2}{3u-2}}=\frac{1}{2}[\ln(3u+2)-\ln(3u-2)]\Rightarrow G'(u)=\frac{1}{2}\left(\frac{3}{3u+2}-\frac{3}{3u-2}\right)=\frac{-6}{9u^2-4}$

19. $y=\ln(e^{-x}+xe^{-x})=\ln(e^{-x}(1+x))=\ln(e^{-x})+\ln(1+x)=-x+\ln(1+x)\Rightarrow y'=-1+\frac{1}{1+x}=\frac{-1-x+1}{1+x}=-\frac{x}{1+x}$

20. $y=[\ln(1+e^x)]^2\Rightarrow y'=2[\ln(1+e^x)]\cdot\frac{1}{1+e^x}\cdot e^x=\frac{2e^x\ln(1+e^x)}{1+e^x}$

21. $y=x\ln x\Rightarrow y'=x(1/x)+(\ln x)\cdot 1=1+\ln x\Rightarrow y''=1/x$

22. $y=\dfrac{\ln x}{x^2} \Rightarrow y' = \dfrac{x^2(1/x)-(\ln x)(2x)}{\left(x^2\right)^2} = \dfrac{x(1-2\ln x)}{x^4} = \dfrac{1-2\ln x}{x^3} \Rightarrow$

$y'' = \dfrac{x^3(-2/x)-(1-2\ln x)(3x^2)}{(x^3)^2} = \dfrac{x^2(-2-3+6\ln x)}{x^6} = \dfrac{6\ln x-5}{x^4}$

23. $y=\log_{10}x \Rightarrow y' = \dfrac{1}{x\ln 10} = \dfrac{1}{\ln 10}\left(\dfrac{1}{x}\right) \Rightarrow y'' = \dfrac{1}{\ln 10}\left(-\dfrac{1}{x^2}\right) = -\dfrac{1}{x^2\ln 10}$

24. $y=\ln(\sec x+\tan x) \Rightarrow y' = \dfrac{\sec x\tan x+\sec^2 x}{\sec x+\tan x} = \sec x \Rightarrow y'' = \sec x\tan x$

25. $f(x)=\dfrac{x}{1-\ln(x-1)} \Rightarrow$

$$f'(x) = \frac{[1-\ln(x-1)]\cdot 1 - x\cdot\dfrac{-1}{x-1}}{[1-\ln(x-1)]^2} = \frac{\dfrac{(x-1)[1-\ln(x-1)]+x}{x-1}}{[1-\ln(x-1)]^2} = \frac{x-1-(x-1)\ln(x-1)+x}{(x-1)^2}$$

$$= \frac{2x-1-(x-1)\ln(x-1)}{(x-1)[1-\ln(x-1)]^2}$$

$\mathrm{Dom}(f) = \{x \mid x-1>0 \text{ and } 1-\ln(x-1)\neq 0\} = \{x \mid x>1 \text{ and } \ln(x-1)\neq 1\}$

$= \{x \mid x>1 \text{ and } x-1\neq e^1\} = \{x \mid x>1 \text{ and } x\neq 1+e\} = (1, 1+e)\cup(1+e, \infty)$

26. $f(x) = \dfrac{1}{1+\ln x} \Rightarrow f'(x) = -\dfrac{1/x}{(1+\ln x)^2} = -\dfrac{1}{x(1+\ln x)^2}$. $\mathrm{Dom}(f) = \{x \mid x>0 \text{ and } \ln x\neq -1\} = \{x \mid x>0 \text{ and } x\neq 1/e\} = (0, 1/e)\cup(1/e, \infty)$.

27. $f(x) = x^2\ln(1-x^2) \Rightarrow f'(x) = 2x\ln(1-x^2) + \dfrac{x^2(-2x)}{1-x^2} = 2x\ln(1-x^2) - \dfrac{2x^3}{1-x^2}$.

$\mathrm{Dom}(f) = \{x \mid 1-x^2>0\} = \{x \mid |x|<1\} = (-1, 1)$.

28. $f(x) = \ln\ln\ln x \Rightarrow f'(x) = \dfrac{1}{\ln\ln x}\cdot\dfrac{1}{\ln x}\cdot\dfrac{1}{x}$.

$\mathrm{Dom}(f) = \{x \mid \ln\ln x>0\} = \{x \mid \ln x>1\} = \{x \mid x>e\} = (e, \infty)$.

29. $f(x) =$

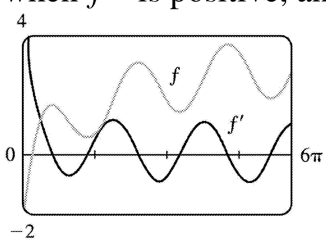
$\frac{x}{\ln x} \Rightarrow f'(x) = \frac{\ln x - x(1/x)}{(\ln x)^2} = \frac{\ln x - 1}{(\ln x)^2} \Rightarrow f'(e) = \frac{1-1}{1^2} = 0$

30. $f(x) = x^2 \ln x \Rightarrow f'(x) = 2x\ln x + x^2\left(\frac{1}{x}\right) = 2x\ln x + x \Rightarrow f'(1) = 2\ln 1 + 1 = 1$

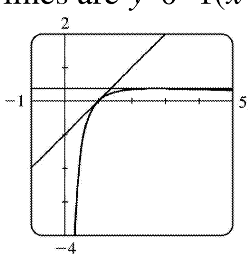
31. $y = f(x) = \ln\ln x \Rightarrow f'(x) = \frac{1}{\ln x}\left(\frac{1}{x}\right) \Rightarrow f'(e) = \frac{1}{e}$, so an equation of the tangent line at $(e, 0)$ is $y-0 = \frac{1}{e}(x-e)$, or $y = \frac{1}{e}x - 1$, or $x - ey = e$.

32. $y = \ln(x^3-7) \Rightarrow y' = \frac{1}{x^3-7}\cdot 3x^2 \Rightarrow y'(2) = \frac{12}{8-7} = 12$, so an equation of a tangent line at $(2, 0)$ is $y-0 = 12(x-2)$ or $y = 12x-24$.

33. $f(x) = \sin x + \ln x \Rightarrow f'(x) = \cos x + 1/x$. This is reasonable, because the graph shows that f increases when f' is positive, and $f'(x) = 0$ when f has a horizontal tangent.



34. $y = \frac{\ln x}{x} \Rightarrow y' = \frac{x(1/x) - \ln x}{x^2} = \frac{1-\ln x}{x^2}$. $y'(1) = \frac{1-0}{1^2} = 1$ and $y'(e) = \frac{1-1}{e^2} = 0 \Rightarrow$ equations of tangent lines are $y-0 = 1(x-1)$ or $y = x-1$ and $y - 1/e = 0(x-e)$ or $y = 1/e$.



35. $y = (2x+1)^5(x^4-3)^6 \Rightarrow \ln y = \ln\left((2x+1)^5(x^4-3)^6\right) \Rightarrow$

$\ln y = 5\ln(2x+1) + 6\ln(x^4-3) \Rightarrow \frac{1}{y}y' = 5\cdot\frac{1}{2x+1}\cdot 2 + 6\cdot\frac{1}{x^4-3}\cdot 4x^3 \Rightarrow$

$y' = y\left(\frac{10}{2x+1} + \frac{24x^3}{x^4-3}\right) = (2x+1)^5(x^4-3)^6\left(\frac{10}{2x+1} + \frac{24x^3}{x^4-3}\right).$

36. $y=\sqrt{x}\,e^{x^2}(x^2+1)^{10} \Rightarrow \ln y=\ln\sqrt{x}+\ln e^{x^2}+\ln(x^2+1)^{10} \Rightarrow \ln y=\frac{1}{2}\ln x+x^2+10\ln(x^2+1)\Rightarrow$

$\frac{1}{y}y'=\frac{1}{2}\cdot\frac{1}{x}+2x+10\cdot\frac{1}{x^2+1}\cdot 2x\Rightarrow y'=\sqrt{x}\,e^{x^2}(x^2+1)^{10}\left(\frac{1}{2x}+2x+\frac{20x}{x^2+1}\right)$

37. $y=\frac{\sin^2 x\tan^4 x}{(x^2+1)^2}\Rightarrow \ln y=\ln(\sin^2 x\tan^4 x)-\ln(x^2+1)^2\Rightarrow$

$\ln y=\ln(\sin x)^2+\ln(\tan x)^4-\ln(x^2+1)^2\Rightarrow \ln y=2\ln|\sin x|+4\ln|\tan x|-2\ln(x^2+1)\Rightarrow$

$\frac{1}{y}y'=2\cdot\frac{1}{\sin x}\cdot\cos x+4\cdot\frac{1}{\tan x}\cdot\sec^2 x-2\cdot\frac{1}{x^2+1}\cdot 2x\Rightarrow$

$y'=\frac{\sin^2 x\tan^4 x}{(x^2+1)^2}\left(2\cot x+\frac{4\sec^2 x}{\tan x}-\frac{4x}{x^2+1}\right)$

38. $y=\sqrt[4]{\frac{x^2+1}{x^2-1}}\Rightarrow \ln y=\frac{1}{4}\ln(x^2+1)-\frac{1}{4}\ln(x^2-1)\Rightarrow \frac{1}{y}y'=\frac{1}{4}\cdot\frac{1}{x^2+1}\cdot 2x-\frac{1}{4}\cdot\frac{1}{x^2-1}\cdot 2x\Rightarrow$

$y'=\sqrt[4]{\frac{x^2+1}{x^2-1}}\cdot\frac{1}{2}\left(\frac{x}{x^2+1}-\frac{x}{x^2-1}\right)=\frac{1}{2}\sqrt[4]{\frac{x^2+1}{x^2-1}}\left(\frac{-2x}{x^4-1}\right)=\frac{x}{1-x^4}\sqrt[4]{\frac{x^2+1}{x^2-1}}$

39. $y=x^x\Rightarrow \ln y=\ln x^x\Rightarrow \ln y=x\ln x\Rightarrow y'/y=x(1/x)+(\ln x)\cdot 1\Rightarrow y'=y(1+\ln x)\Rightarrow y'=x^x(1+\ln x)$

40. $y=x^{1/x}\Rightarrow \ln y=\frac{1}{x}\ln x\Rightarrow \frac{y'}{y}=\frac{1}{x}\left(\frac{1}{x}\right)+(\ln x)\left(-\frac{1}{x^2}\right)\Rightarrow y'=x^{1/x}\,\frac{1-\ln x}{x^2}$

41. $y=x^{\sin x}\Rightarrow \ln y=\ln x^{\sin x}\Rightarrow \ln y=\sin x\ln x\Rightarrow \frac{y'}{y}=(\sin x)\cdot\frac{1}{x}+(\ln x)(\cos x)\Rightarrow$

$y'=y\left(\frac{\sin x}{x}+\ln x\cos x\right)\Rightarrow y'=x^{\sin x}\left(\frac{\sin x}{x}+\ln x\cos x\right)$

42. $y=(\sin x)^x\Rightarrow \ln y=x\ln(\sin x)\Rightarrow \frac{y'}{y}=x\cdot\frac{1}{\sin x}\cdot\cos x+[\ln(\sin x)]\cdot 1\Rightarrow y'=(\sin x)^x[x\cot x+\ln(\sin x)]$

43. $y=(\ln x)^x\Rightarrow \ln y=\ln(\ln x)^x\Rightarrow \ln y=x\ln\ln x\Rightarrow \frac{y'}{y}=x\cdot\frac{1}{\ln x}\cdot\frac{1}{x}+(\ln\ln x)\cdot 1\Rightarrow$

$y'=y\left(\frac{x}{x\ln x}+\ln\ln x\right)\Rightarrow\ y'=(\ln x)^x\left(\frac{1}{\ln x}+\ln\ln x\right)$

44. $y=x^{\ln x}\Rightarrow \ln y=\ln x\ln x=(\ln x)^2\Rightarrow\ \frac{y'}{y}=2\ln x\left(\frac{1}{x}\right)\Rightarrow\ y'=x^{\ln x}\left(\frac{2\ln x}{x}\right)$

45. $y=x^{e^x}\Rightarrow \ln y=e^x\ln x\Rightarrow\ \frac{y'}{y}=e^x\cdot\frac{1}{x}+(\ln x)\cdot e^x\Rightarrow\ y'=x^{e^x}e^x\left(\ln x+\frac{1}{x}\right)$

46. $y=(\ln x)^{\cos x}\Rightarrow \ln y=\cos x\ln(\ln x)\Rightarrow\ \frac{y'}{y}=\cos x\cdot\frac{1}{\ln x}\cdot\frac{1}{x}+(\ln\ln x)(-\sin x)\Rightarrow$
$y'=(\ln x)^{\cos x}\left(\frac{\cos x}{x\ln x}-\sin x\ln\ln x\right)$

47. $y=\ln(x^2+y^2)\Rightarrow y'=\frac{1}{x^2+y^2}\frac{d}{dx}(x^2+y^2)\Rightarrow y'=\frac{2x+2yy'}{x^2+y^2}\Rightarrow x^2y'+y^2y'=2x+2yy'\Rightarrow$
$x^2y'+y^2y'-2yy'=2x\Rightarrow(x^2+y^2-2y)y'=2x\Rightarrow\ y'=\frac{2x}{x^2+y^2-2y}$

48. $x^y=y^x\Rightarrow y\ln x=x\ln y\Rightarrow y\cdot\frac{1}{x}+(\ln x)\cdot y'=x\cdot\frac{1}{y}\cdot y'+\ln y\Rightarrow y'\ln x-\frac{x}{y}y'=\ln y-\frac{y}{x}\Rightarrow\ y'=\frac{\ln y-y/x}{\ln x-x/y}$

49. $f(x)=\ln(x-1)\Rightarrow f'(x)=1/(x-1)=(x-1)^{-1}\Rightarrow f''(x)=-(x-1)^{-2}\Rightarrow$
$f'''(x)=2(x-1)^{-3}\Rightarrow f^{(4)}(x)=-2\cdot 3(x-1)^{-4}\Rightarrow\cdots\Rightarrow$
$f^{(n)}(x)=(-1)^{n-1}\cdot 2\cdot 3\cdot 4\cdot\ldots\cdot(n-1)(x-1)^{-n}=(-1)^{n-1}\frac{(n-1)!}{(x-1)^n}$

50. $y=x^8\ln x$, so $D^9y=D^8y'=D^8(8x^7\ln x+x^7)$. But the eighth derivative of x^7 is 0 , so we now have
$D^8(8x^7\ln x)=D^7(8\cdot 7x^6\ln x+8x^6)=D^7(8\cdot 7x^6\ln x)$
$=D^6(8\cdot 7\cdot 6x^5\ln x)=\ldots=D(8!x^0\ln x)=8!/x$.

51. If $f(x)=\ln(1+x)$, then $f'(x)=\frac{1}{1+x}$, so $f'(0)=1$. Thus,
$\lim_{x\to 0}\frac{\ln(1+x)}{x}=\lim_{x\to 0}\frac{f(x)}{x}=\lim_{x\to 0}\frac{f(x)-f(0)}{x-0}=f'(0)=1.$

52. Let $m=n/x$. Then $n=xm$, and as $n\to\infty$, $m\to\infty$. Therefore,

$$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^{n}=\lim_{m\to\infty}\left(1+\frac{1}{m}\right)^{mx}=\left[\lim_{m\to\infty}\left(1+\frac{1}{m}\right)^{m}\right]^{x}=e^{x}$$ by Equation 6.

1. **(a)** $\sinh 0=\frac{1}{2}\left(e^{0}-e^{0}\right)=0$

(b) $\cosh 0=\frac{1}{2}\left(e^{0}+e^{0}\right)=\frac{1}{2}(1+1)=1$

2. **(a)** $\tanh 0=\dfrac{\left(e^{0}-e^{-0}\right)/2}{\left(e^{0}+e^{-0}\right)/2}=0$

(b) $\tanh 1=\dfrac{e^{1}-e^{-1}}{e^{1}+e^{-1}}=\dfrac{e^{2}-1}{e^{2}+1}\approx 0.76159$

3. **(a)** $\sinh(\ln 2)=\dfrac{e^{\ln 2}-e^{-\ln 2}}{2}=\dfrac{e^{\ln 2}-\left(e^{\ln 2}\right)^{-1}}{2}=\dfrac{2-2^{-1}}{2}=\dfrac{2-\frac{1}{2}}{2}=\dfrac{3}{4}$

(b) $\sinh 2=\frac{1}{2}\left(e^{2}-e^{-2}\right)\approx 3.62686$

4. **(a)** $\cosh 3=\frac{1}{2}\left(e^{3}+e^{-3}\right)\approx 10.06766$

(b) $\cosh(\ln 3)=\dfrac{e^{\ln 3}+e^{-\ln 3}}{2}=\dfrac{3+\frac{1}{3}}{2}=\dfrac{5}{3}$

5. **(a)** $\operatorname{sech} 0=\dfrac{1}{\cosh 0}=\dfrac{1}{1}=1$

(b) $\cosh^{-1} 1=0$ because $\cosh 0=1$.

6. **(a)** $\sinh 1=\frac{1}{2}\left(e^{1}-e^{-1}\right)\approx 1.17520$

(b) Using Equation 3, we have $\sinh^{-1} 1=\ln\left(1+\sqrt{1^{2}+1}\right)=\ln(1+\sqrt{2})\approx 0.88137$.

7. $\sinh(-x)=\frac{1}{2}\left[e^{-x}-e^{-(-x)}\right]=\frac{1}{2}\left(e^{-x}-e^{x}\right)=-\frac{1}{2}\left(e^{x}-e^{-x}\right)=-\sinh x$

8. $\cosh(-x)=\frac{1}{2}\left[e^{-x}+e^{-(-x)}\right]=\frac{1}{2}\left(e^{-x}+e^{x}\right)=\frac{1}{2}\left(e^{x}+e^{-x}\right)=\cosh x$

9. $\cosh x+\sinh x=\frac{1}{2}\left(e^{x}+e^{-x}\right)+\frac{1}{2}\left(e^{x}-e^{-x}\right)=\frac{1}{2}\left(2e^{x}\right)=e^{x}$

10. $\cosh x - \sinh x = \frac{1}{2}(e^{x}+e^{-x}) - \frac{1}{2}(e^{x}-e^{-x}) = \frac{1}{2}(2e^{-x}) = e^{-x}$

11. $\sinh x \cosh y + \cosh x \sinh y = \left[\frac{1}{2}(e^{x}-e^{-x})\right]\left[\frac{1}{2}(e^{y}+e^{-y})\right] + \left[\frac{1}{2}(e^{x}+e^{-x})\right]\left[\frac{1}{2}(e^{y}-e^{-y})\right]$
$= \frac{1}{4}\left[(e^{x+y}+e^{x-y}-e^{-x+y}-e^{-x-y})+(e^{x+y}-e^{x-y}+e^{-x+y}-e^{-x-y})\right]$
$= \frac{1}{4}(2e^{x+y}-2e^{-x-y}) = \frac{1}{2}\left[e^{x+y}-e^{-(x+y)}\right] = \sinh(x+y)$

12. $\cosh x \cosh y + \sinh x \sinh y = \left[\frac{1}{2}(e^{x}+e^{-x})\right]\left[\frac{1}{2}(e^{y}+e^{-y})\right] + \left[\frac{1}{2}(e^{x}-e^{-x})\right]\left[\frac{1}{2}(e^{y}-e^{-y})\right]$
$= \frac{1}{4}\left[(e^{x+y}+e^{x-y}+e^{-x+y}+e^{-x-y})+(e^{x+y}-e^{x-y}-e^{-x+y}+e^{-x-y})\right]$
$= \frac{1}{4}(2e^{x+y}+2e^{-x-y}) = \frac{1}{2}\left[e^{x+y}+e^{-(x+y)}\right] = \cosh(x+y)$

13. Divide both sides of the identity $\cosh^2 x - \sinh^2 x = 1$ by $\sinh^2 x$:
$\frac{\cosh^2 x}{\sinh^2 x} - \frac{\sinh^2 x}{\sinh^2 x} = \frac{1}{\sinh^2 x} \Leftrightarrow \coth^2 x - 1 = \operatorname{csch}^2 x.$

14.

$$\tanh(x+y) = \frac{\sinh(x+y)}{\cosh(x+y)} = \frac{\sinh x \cosh y + \cosh x \sinh y}{\cosh x \cosh y + \sinh x \sinh y} = \frac{\frac{\sinh x \cosh y}{\cosh x \cosh y} + \frac{\cosh x \sinh y}{\cosh x \cosh y}}{\frac{\cosh x \cosh y}{\cosh x \cosh y} + \frac{\sinh x \sinh y}{\cosh x \cosh y}}$$
$$= \frac{\tanh x + \tanh y}{1 + \tanh x \tanh y}$$

15. Putting $y=x$ in the result from Exercise 11, we have
$\sinh 2x = \sinh(x+x) = \sinh x \cosh x + \cosh x \sinh x = 2\sinh x \cosh x.$

16. Putting $y=x$ in the result from Exercise 12, we have
$\cosh 2x = \cosh(x+x) = \cosh x \cosh x + \sinh x \sinh x = \cosh^2 x + \sinh^2 x$.

17.

$$\tanh(\ln x) = \frac{\sinh(\ln x)}{\cosh(\ln x)} = \frac{(e^{\ln x} - e^{-\ln x})/2}{(e^{\ln x} + e^{-\ln x})/2} = \frac{x - (e^{\ln x})^{-1}}{x + (e^{\ln x})^{-1}} = \frac{x - x^{-1}}{x + x^{-1}}$$

$$=\frac{x-1/x}{x+1/x}=\frac{(x^2-1)/x}{(x^2+1)/x}=\frac{x^2-1}{x^2+1}$$

18.

$$\frac{1+\tanh x}{1-\tanh x}=\frac{1+(\sinh x)/\cosh x}{1-(\sinh x)/\cosh x}=\frac{\cosh x+\sinh x}{\cosh x-\sinh x}=\frac{\frac{1}{2}(e^x+e^{-x})+\frac{1}{2}(e^x-e^{-x})}{\frac{1}{2}(e^x+e^{-x})-\frac{1}{2}(e^x-e^{-x})}$$

$$=\frac{e^x+e^{-x}+e^x-e^{-x}}{e^x+e^{-x}-e^x+e^{-x}}=\frac{2e^x}{2e^{-x}}=e^{2x}$$

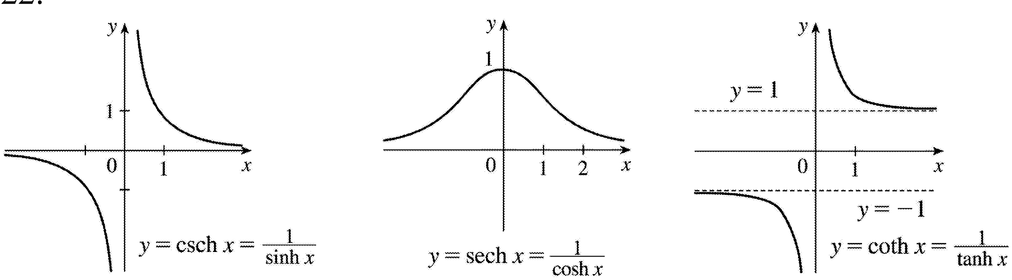
Or: Using the results of Exercises 9 and 10, $\frac{\cosh x+\sinh x}{\cosh x-\sinh x}=\frac{e^x}{e^{-x}}=e^{2x}$

19. By Exercise 9, $(\cosh x+\sinh x)^n=(e^x)^n=e^{nx}=\cosh nx+\sinh nx$.

20. $\sinh x=\frac{3}{4}\Rightarrow \operatorname{csch} x=1/\sinh x=\frac{4}{3}$. $\cosh^2 x=\sinh^2 x+1=\frac{9}{16}+1=\frac{25}{16}\Rightarrow \cosh x=\frac{5}{4}$ (since $\cosh x>0$). $\coth x=1/\cosh x=\frac{4}{5}$, $\tanh x=\sinh x/\cosh x=\frac{3/4}{5/4}=\frac{3}{5}$, and $\coth x=1/\tanh x=\frac{5}{3}$.

21. $\tanh x=\frac{4}{5}>0$, so $x>0$. $\coth x=1/\tanh x=\frac{5}{4}$, $\operatorname{sech}^2 x=1-\tanh^2 x=1-\left(\frac{4}{5}\right)^2=\frac{9}{25}\Rightarrow \operatorname{sech} x=\frac{3}{5}$ (since $\operatorname{sech} x>0$), $\cosh x=1/\operatorname{sech} x=\frac{5}{3}$, $\sinh x=\tanh x\cosh x=\frac{4}{5}\cdot\frac{5}{3}=\frac{4}{3}$, and $\operatorname{csch} x=1/\sinh x=\frac{3}{4}$.

22.



23. **(a)**

$\lim_{x\to\infty} \tanh x=\lim_{x\to\infty} \frac{e^{x}-e^{-x}}{e^{x}+e^{-x}} \cdot \frac{e^{-x}}{e^{-x}}=\lim_{x\to\infty} \frac{1-e^{-2x}}{1+e^{-2x}}=\frac{1-0}{1+0}=1$

(b) $\lim_{x\to-\infty} \tanh x=\lim_{x\to-\infty} \frac{e^{x}-e^{-x}}{e^{x}+e^{-x}} \cdot \frac{e^{x}}{e^{x}}=\lim_{x\to-\infty} \frac{e^{2x}-1}{e^{2x}+1}=\frac{0-1}{0+1}=-1$

(c) $\lim_{x\to\infty} \sinh x=\lim_{x\to\infty} \frac{e^{x}-e^{-x}}{2}=\infty$

(d) $\lim_{x\to-\infty} \sinh x=\lim_{x\to-\infty} \frac{e^{x}-e^{-x}}{2}=-\infty$

(e) $\lim_{x\to\infty} \operatorname{sech} x=\lim_{x\to\infty} \frac{2}{e^{x}+e^{-x}}=0$

(f) $\lim_{x\to\infty} \coth x=\lim_{x\to\infty} \frac{e^{x}+e^{-x}}{e^{x}-e^{-x}} \cdot \frac{e^{-x}}{e^{-x}}=\lim_{x\to\infty} \frac{1+e^{-2x}}{1-e^{-2x}}=\frac{1+0}{1-0}=1$

(g) $\lim_{x\to 0^{+}} \coth x=\lim_{x\to 0^{+}} \frac{\cosh x}{\sinh x}=\infty$, since $\sinh x\to 0$ through positive values and $\cosh x\to 1$.

(h) $\lim_{x\to 0^{-}} \coth x=\lim_{x\to 0^{-}} \frac{\cosh x}{\sinh x}=-\infty$, since $\sinh x\to 0$ through negative values and $\cosh x\to 1$.

(i) $\lim_{x\to-\infty} \operatorname{csch} x=\lim_{x\to-\infty} \frac{2}{e^{x}-e^{-x}}=0$

24. **(a)** $\frac{d}{dx}\cosh x=\frac{d}{dx}\left[\frac{1}{2}(e^{x}+e^{-x})\right]=\frac{1}{2}(e^{x}-e^{-x})=\sinh x$

(b) $\frac{d}{dx}\tanh x=\frac{d}{dx}\left[\frac{\sinh x}{\cosh x}\right]=\frac{\cosh x\cosh x-\sinh x\sinh x}{\cosh^{2} x}=\frac{\cosh^{2} x-\sinh^{2} x}{\cosh^{2} x}=\frac{1}{\cosh^{2} x}=\operatorname{sech}^{2} x$

(c) $\frac{d}{dx}\operatorname{csch} x=\frac{d}{dx}\left[\frac{1}{\sinh x}\right]=-\frac{\cosh x}{\sinh^{2} x}=-\frac{1}{\sinh x}\cdot\frac{\cosh x}{\sinh x}=-\operatorname{csch} x\coth x$

(d) $\frac{d}{dx}\operatorname{sech} x=\frac{d}{dx}\left[\frac{1}{\cosh x}\right]=-\frac{\sinh x}{\cosh^{2} x}=-\frac{1}{\cosh x}\cdot\frac{\sinh x}{\cosh x}=-\operatorname{sech} x\tanh x$

(e)

$$\frac{d}{dx}\coth x=\frac{d}{dx}\left[\frac{\cosh x}{\sinh x}\right]=\frac{\sinh x\sinh x-\cosh x\cosh x}{\sinh^{2} x}=\frac{\sinh^{2} x-\cosh^{2} x}{\sinh^{2} x}=-\frac{1}{\sinh^{2} x}$$

$$=-\operatorname{csch}^{2} x$$

25. Let $y=\sinh^{-1}x$. Then $\sinh y=x$ and, by Example 1(a), $\cosh^2 y-\sinh^2 y=1\Rightarrow$ [with $\cosh y>0$] $\cosh y=\sqrt{1+\sinh^2 y}=\sqrt{1+x^2}$. So by Exercise 9, $e^y=\sinh y+\cosh y=x+\sqrt{1+x^2}\Rightarrow y=\ln\left(x+\sqrt{1+x^2}\right)$.

26. Let $y=\cosh^{-1}x$. Then $\cosh y=x$ and $y\geq 0$, so $\sinh y=\sqrt{\cosh^2 y-1}=\sqrt{x^2-1}$. So, by Exercise 9, $e^y=\cosh y+\sinh y=x+\sqrt{x^2-1}\Rightarrow y=\ln\left(x+\sqrt{x^2-1}\right)$.

Another method: Write $x=\cosh y=\frac{1}{2}(e^y+e^{-y})$ and solve a quadratic, as in Example 3.

27. **(a)** Let $y=\tanh^{-1}x$. Then $x=\tanh y=\dfrac{\sinh y}{\cosh y}=\dfrac{(e^y-e^{-y})/2}{(e^y+e^{-y})/2}\cdot\dfrac{e^y}{e^y}=\dfrac{e^{2y}-1}{e^{2y}+1}\Rightarrow$

$xe^{2y}+x=e^{2y}-1\Rightarrow 1+x=e^{2y}-xe^{2y}\Rightarrow 1+x=e^{2y}(1-x)\Rightarrow$

$e^{2y}=\dfrac{1+x}{1-x}\Rightarrow 2y=\ln\left(\dfrac{1+x}{1-x}\right)\Rightarrow y=\dfrac{1}{2}\ln\left(\dfrac{1+x}{1-x}\right)$.

(b) Let $y=\tanh^{-1}x$. Then $x=\tanh y$, so from Exercise 18 we have

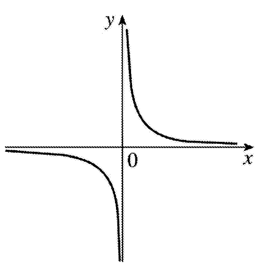
$e^{2y}=\dfrac{1+\tanh y}{1-\tanh y}=\dfrac{1+x}{1-x}\Rightarrow 2y=\ln\left(\dfrac{1+x}{1-x}\right)\Rightarrow y=\dfrac{1}{2}\ln\left(\dfrac{1+x}{1-x}\right)$.

28. **(a)**

(i) $y=\operatorname{csch}^{-1}x\Leftrightarrow \operatorname{csch} y=x$ ($x\neq 0$)

(ii) We sketch the graph of csch^{-1} by reflecting the graph of csch (see Exercise 22) about the line y=x.

(iii) Let $y=\operatorname{csch}^{-1}x$. Then $x=\operatorname{csch} y=\dfrac{2}{e^y-e^{-y}}\Rightarrow xe^y-xe^{-y}=2\Rightarrow x(e^y)^2-2e^y-x=0\Rightarrow e^y=\dfrac{1\pm\sqrt{x^2+1}}{x}$.



But $e^y>0$, so for $x>0$, $e^y=\dfrac{1+\sqrt{x^2+1}}{x}$ and for $x<0$, $e^y=\dfrac{1-\sqrt{x^2+1}}{x}$. Thus,

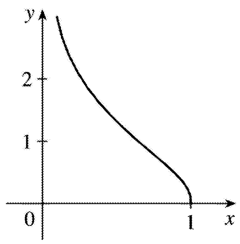
$\operatorname{csch}^{-1}x=\ln\left(\dfrac{1}{x}+\dfrac{\sqrt{x^2+1}}{|x|}\right)$.

(b)

(i) $y=\text{sech}^{-1}x \Leftrightarrow \text{sech } y=x$ and $y>0$.

(ii) We sketch the graph of coth^{-1} by reflecting the graph of coth (see Exercise 22) about the line $y=x$.

(iii) Let $y=\coth^{-1}x$, so $x=\text{sech } y=\dfrac{2}{e^{y}+e^{-y}} \Rightarrow xe^{y}+xe^{-y}=2 \Rightarrow x(e^{y})^{2}-2e^{y}+x=0 \Leftrightarrow e^{y}=\dfrac{1\pm\sqrt{1-x^{2}}}{x}$.



But $y>0 \Rightarrow e^{y}>1$. This rules out the minus sign because $\dfrac{1-\sqrt{1-x^{2}}}{x}>1 \Leftrightarrow 1-\sqrt{1-x^{2}}>x \Leftrightarrow$

$1-x>\sqrt{1-x^{2}} \Leftrightarrow 1-2x+x^{2}>1-x^{2} \Leftrightarrow x^{2}>x \Leftrightarrow x>1$, but $x=\text{sech } y\leq 1$. Thus, $e^{y}=\dfrac{1+\sqrt{1-x^{2}}}{x} \Rightarrow$

$\text{sech}^{-1}x=\ln\left(\dfrac{1+\sqrt{1-x^{2}}}{x}\right)$.

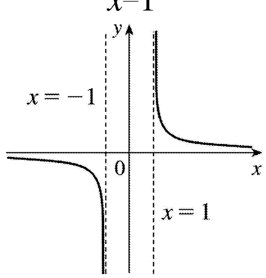
(c)

(i) $y=\coth^{-1}x \Leftrightarrow \coth y=x$

(ii) We sketch the graph of coth^{-1} by reflecting the graph of coth (see Exercise 22) about the line $y=x$.

(iii) Let $y=\coth^{-1}x$. Then $x=\coth y=\dfrac{e^{y}+e^{-y}}{e^{y}-e^{-y}} \Rightarrow xe^{y}-xe^{-y}=e^{y}+e^{-y} \Rightarrow (x-1)e^{y}=(x+1)e^{-y} \Rightarrow e^{2y}=\dfrac{x+1}{x-1} \Rightarrow$

$2y=\ln\dfrac{x+1}{x-1} \Rightarrow \coth^{-1}x=\dfrac{1}{2}\ln\dfrac{x+1}{x-1}$



29. **(a)** Let $y=\cosh^{-1}x$. Then $\cosh y=x$ and $y\geq 0 \Rightarrow \sinh y\,\frac{dy}{dx}=1\Rightarrow$

$\frac{dy}{dx}=\frac{1}{\sinh y}=\frac{1}{\sqrt{\cosh^2 y-1}}=\frac{1}{\sqrt{x^2-1}}$ (since $\sinh y\geq 0$ for $y\geq 0$). *Or:* Use Formula 4.

(b) Let $y=\tanh^{-1}x$. Then $\tanh y=x\Rightarrow \operatorname{sech}^2 y\,\frac{dy}{dx}=1\Rightarrow \frac{dy}{dx}=\frac{1}{\operatorname{sech}^2 y}=\frac{1}{1-\tanh^2 y}=\frac{1}{1-x^2}$.

Or: Use Formula 5.

(c) Let $y=\operatorname{csch}^{-1}x$. Then $\operatorname{csch} y=x\Rightarrow -\operatorname{csch} y\coth y\,\frac{dy}{dx}=1\Rightarrow \frac{dy}{dx}=-\frac{1}{\operatorname{csch} y\coth y}$. By Exercise 13, $\coth y=\pm\sqrt{\operatorname{csch}^2 y+1}=\pm\sqrt{x^2+1}$. If $x>0$, then $\coth y>0$, so $\coth y=\sqrt{x^2+1}$. If $x<0$, then $\coth y<0$, so $\coth y=-\sqrt{x^2+1}$. In either case we have $\frac{dy}{dx}=-\frac{1}{\operatorname{csch} y\coth y}=-\frac{1}{|x|\sqrt{x^2+1}}$.

(d) Let $y=\operatorname{sech}^{-1}x$. Then $\operatorname{sech} y=x\Rightarrow -\operatorname{sech} y\tanh y\,\frac{dy}{dx}=1\Rightarrow$

$\frac{dy}{dx}=-\frac{1}{\operatorname{sech} y\tanh y}=-\frac{1}{\operatorname{sech} y\sqrt{1-\operatorname{sech}^2 y}}=-\frac{1}{x\sqrt{1-x^2}}$. (Note that $y>0$ and so $\tanh y>0$.)

(e) Let $y=\coth^{-1}x$. Then $\coth y=x\Rightarrow -\operatorname{csch}^2 y\,\frac{dy}{dx}=1\Rightarrow \frac{dy}{dx}=-\frac{1}{\operatorname{csch}^2 y}=\frac{1}{1-\coth^2 y}=\frac{1}{1-x^2}$ by Exercise 13.

30. $f(x)=\tanh 4x\Rightarrow f'(x)=4\operatorname{sech}^2 4x$

31. $f(x)=x\cosh x\Rightarrow f'(x)=x(\cosh x)'+(\cosh x)(x)'=x\sinh x+\cosh x$

32. $g(x)=\sinh^2 x\Rightarrow g'(x)=2\sinh x\cosh x$

33. $h(x)=\sinh(x^2)\Rightarrow h'(x)=\cosh(x^2)\cdot 2x=2x\cosh(x^2)$

34. $F(x)=\sinh x\tanh x\Rightarrow F'(x)=\sinh x\operatorname{sech}^2 x+\tanh x\cosh x$

35. $G(x)=\frac{1-\cosh x}{1+\cosh x}\Rightarrow$

$$G'(x)=\frac{(1+\cosh x)(-\sinh x)-(1-\cosh x)(\sinh x)}{(1+\cosh x)^2}$$

$$=\frac{-\sinh x-\sinh x\cosh x-\sinh x+\sinh x\cosh x}{(1+\cosh x)^2}=\frac{-2\sinh x}{(1+\cosh x)^2}$$

36. $f(t)=e^t\operatorname{sech} t\Rightarrow f'(t)=e^t(-\operatorname{sech} t\tanh t)+(\operatorname{sech} t)e^t=e^t\operatorname{sech} t(1-\tanh t)$

37. $h(t)=\coth\sqrt{1+t^2}\Rightarrow h'(t)=-\operatorname{csch}^2\sqrt{1+t^2}\cdot\frac{1}{2}(1+t^2)^{-1/2}(2t)=-\frac{t\operatorname{csch}^2\sqrt{1+t^2}}{\sqrt{1+t^2}}$

38. $f(t)=\ln(\sinh t)\Rightarrow f'(t)=\frac{1}{\sinh t}\cosh t=\coth t$

39. $H(t)=\tanh(e^t)\Rightarrow H'(t)=\operatorname{sech}^2(e^t)\cdot e^t=e^t\operatorname{sech}^2(e^t)$

40. $y=\sinh(\cosh x)\Rightarrow y'=\cosh(\cosh x)\cdot\sinh x$

41. $y=e^{\cosh 3x}\Rightarrow y'=e^{\cosh 3x}\cdot\sinh 3x\cdot 3=3e^{\cosh 3x}\sinh 3x$

42. $y=x^2\sinh^{-1}(2x)\Rightarrow y'=x^2\cdot\frac{1}{\sqrt{1+(2x)^2}}\cdot 2+\sinh^{-1}(2x)\cdot 2x=2x\left[\frac{x}{\sqrt{1+4x^2}}+\sinh^{-1}(2x)\right]$

43. $y=\tanh^{-1}\sqrt{x}\Rightarrow y'=\frac{1}{1-(\sqrt{x})^2}\cdot\frac{1}{2}x^{-1/2}=\frac{1}{2\sqrt{x}(1-x)}$

44. $y=x\tanh^{-1}x+\ln\sqrt{1-x^2}=x\tanh^{-1}x+\frac{1}{2}\ln(1-x^2)\Rightarrow$

$y'=\tanh^{-1}x+\frac{x}{1-x^2}+\frac{1}{2}\left(\frac{1}{1-x^2}\right)(-2x)=\tanh^{-1}x$

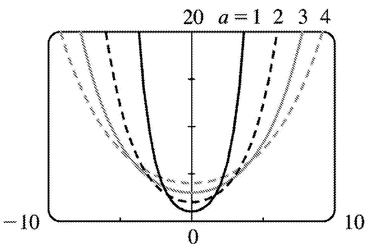
45. $y=x\sinh^{-1}(x/3)-\sqrt{9+x^2}\Rightarrow$

$y'=\sinh^{-1}\left(\frac{x}{3}\right)+x\frac{1/3}{\sqrt{1+(x/3)^2}}-\frac{2x}{2\sqrt{9+x^2}}=\sinh^{-1}\left(\frac{x}{3}\right)+\frac{x}{\sqrt{9+x^2}}-\frac{x}{\sqrt{9+x^2}}=\sinh^{-1}\left(\frac{x}{3}\right)$

46. $y=\operatorname{sech}^{-1}\sqrt{1-x^2}\Rightarrow$

$y' = -\frac{1}{\sqrt{1-x^2}\sqrt{1-(1-x^2)}} \frac{-2x}{2\sqrt{1-x^2}} = \frac{x}{(1-x^2)|x|}$

47. $y = \coth^{-1}\sqrt{x^2+1} \Rightarrow y' = \frac{1}{1-(x^2+1)} \frac{2x}{2\sqrt{x^2+1}} = -\frac{1}{x\sqrt{x^2+1}}$

48. 

For $y = a\cosh(x/a)$ with $a > 0$, we have the y-intercept equal to a. As a increases, the graph flattens.

49. **(a)** $y = 20\cosh(x/20) - 15 \Rightarrow y' = 20\sinh(x/20) \cdot \frac{1}{20} = \sinh(x/20)$. Since the right pole is positioned at $x = 7$, we have $y'(7) = \sinh\frac{7}{20} \approx 0.3572$.

(b) If α is the angle between the tangent line and the x-axis, then $\tan\alpha$ = slope of the line $= \sinh\frac{7}{20}$, so $\alpha = \tan^{-1}\left(\sinh\frac{7}{20}\right) \approx 0.343 \text{ rad} \approx 19.66^\circ$. Thus, the angle between the line and the pole is $\theta = 90^\circ - \alpha \approx 70.34^\circ$.

50. We differentiate the function twice, then substitute into the differential equation: $y = \frac{T}{\rho g}\cosh\frac{\rho g x}{T}$
$\Rightarrow \frac{dy}{dx} = \frac{T}{\rho g}\sinh\left(\frac{\rho g x}{T}\right)\frac{\rho g}{T} = \sinh\frac{\rho g x}{T} \Rightarrow \frac{d^2y}{dx^2} = \cosh\left(\frac{\rho g x}{T}\right)\frac{\rho g}{T} = \frac{\rho g}{T}\cosh\frac{\rho g x}{T}$.

We evaluate the two sides separately: LHS $= \frac{d^2y}{dx^2} = \frac{\rho g}{T}\cosh\frac{\rho g x}{T}$,

RHS $= \frac{\rho g}{T}\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \frac{\rho g}{T}\sqrt{1+\sinh^2\frac{\rho g x}{T}} = \frac{\rho g}{T}\frac{\rho g x}{T}$ by the identity proved in Example 1(a).

51. **(a)** $y = A\sinh mx + B\cosh mx \Rightarrow y' = mA\cosh mx + mB\sinh mx \Rightarrow$
$y'' = m^2A\sinh mx + m^2B\cosh mx = m^2(A\sinh mx + B\cosh mx) = m^2y$

(b) From part (a), a solution of $y'' = 9y$ is $y(x) = A\sinh 3x + B\cosh 3x$. So $-4 = y(0) = A\sinh 0 + B\cosh 0 = B$, so $B=-4$. Now $y'(x)=3A\cosh 3x-12\sinh 3x \Rightarrow 6=y'(0)=3A \Rightarrow A=2$, so $y=2\sinh 3x-4\cosh 3x$.

52. $\lim_{x\to\infty} \frac{\sinh x}{e^x} = \lim_{x\to\infty} \frac{e^x-e^{-x}}{2e^x} = \lim_{x\to\infty} \frac{1-e^{-2x}}{2} = \frac{1-0}{2} = \frac{1}{2}$

53. The tangent to $y=\cosh x$ has slope 1 when $y'=\sinh x=1 \Rightarrow x=\sinh^{-1}1=\ln(1+\sqrt{2})$, by Equation 3. Since $\sinh x=1$ and $y=\cosh x=\sqrt{1+\sinh^2 x}$, we have $\cosh x=\sqrt{2}$. The point is $(\ln(1+\sqrt{2}), \sqrt{2})$.

54.

$$\cosh x = \cosh[\ln(\sec\theta+\tan\theta)] = \frac{1}{2}\left[e^{\ln(\sec\theta+\tan\theta)}+e^{-\ln(\sec\theta+\tan\theta)}\right]$$
$$= \frac{1}{2}\left[\sec\theta+\tan\theta+\frac{1}{\sec\theta+\tan\theta}\right] = \frac{1}{2}\left[\sec\theta+\tan\theta+\frac{\sec\theta-\tan\theta}{(\sec\theta+\tan\theta)(\sec\theta-\tan\theta)}\right]$$
$$= \frac{1}{2}\left[\sec\theta+\tan\theta+\frac{\sec\theta-\tan\theta}{\sec^2\theta-\tan^2\theta}\right] = \frac{1}{2}(\sec\theta+\tan\theta+\sec\theta-\tan\theta)=\sec\theta$$

55. If $ae^x+be^{-x}=\alpha\cosh(x+\beta)$ [or $\alpha\sinh(x+\beta)$], then
$ae^x+be^{-x}=\frac{\alpha}{2}(e^{x+\beta}\pm e^{-x-\beta})=\frac{\alpha}{2}(e^x e^\beta \pm e^{-x}e^{-\beta})=\left(\frac{\alpha}{2}e^\beta\right)e^x \pm \left(\frac{\alpha}{2}e^{-\beta}\right)e^{-x}$. Comparing coefficients of e^x and e^{-x} , we have $a=\frac{\alpha}{2}e^\beta$ **(1)** and $b=\pm\frac{\alpha}{2}e^{-\beta}$ **(2)** . We need to find α and β . Dividing equation **(1)** by
equation **(2)** gives us $\frac{a}{b}=\pm e^{2\beta} \Rightarrow (*) 2\beta=\ln(\pm\frac{a}{b}) \Rightarrow \beta=\frac{1}{2}\ln\left(\pm\frac{a}{b}\right)$. Solving equations **(1)** and **(2)** for e^β gives us $e^\beta=\frac{2a}{\alpha}$ and $e^\beta=\pm\frac{\alpha}{2b}$, so $\frac{2a}{\alpha}=\pm\frac{\alpha}{2b} \Rightarrow \alpha^2=\pm 4ab \Rightarrow \alpha=2\sqrt{\pm ab}$.

(*) If $\frac{a}{b}>0$, we use the + sign and obtain a cosh function, whereas if $\frac{a}{b}<0$, we use the – sign and obtain a sinh function.

In summary, if a and b have the same sign, we have $ae^x+be^{-x}=2\sqrt{ab}\cosh\left(x+\frac{1}{2}\ln\frac{a}{b}\right)$, whereas, if a and b have the opposite sign, then $ae^x+be^{-x}=2\sqrt{-ab}\sinh\left(x+\frac{1}{2}\ln\left(-\frac{a}{b}\right)\right)$.

1. A function f has an **absolute minimum** at $x=c$ if $f(c)$ is the smallest function value on the entire domain of f , whereas f has a **local minimum** at c if $f(c)$ is the smallest function value when x is near c .

2. **(a)** The Extreme Value Theorem
(b) See the Closed Interval Method.

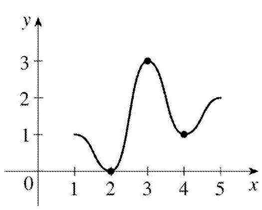
3. Absolute maximum at b ; absolute minimum at d ; local maxima at b and e ; local minima at d and s ;
neither a maximum nor a minimum at a , c , r , and t .

4. Absolute maximum at e ; absolute minimum at t ; local maxima at c , e , and s ; local minima at b , c , d , and r ;
neither a maximum nor a minimum at a .

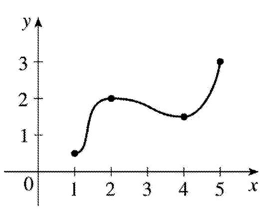
5. Absolute maximum value is $f(4)=4$; absolute minimum value is $f(7)=0$; local maximum values are $f(4)=4$ and $f(6)=3$; local minimum values are $f(2)=1$ and $f(5)=2$.

6. Absolute maximum value is $f(8)=5$; absolute minimum value is $f(2)=0$; local maximum values are $f(1)=2$, $f(4)=4$, and $f(6)=3$; local minimum values are $f(2)=0$, $f(5)=2$, and $f(7)=1$.

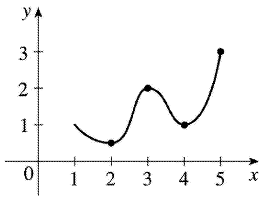
7. Absolute minimum at 2, absolute maximum at 3, local minimum at 4



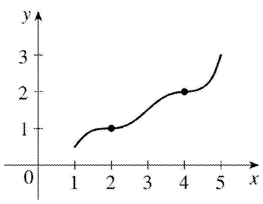
8. Absolute minimum at 1, absolute maximum at 5, local maximum at 2, local minimum at 4



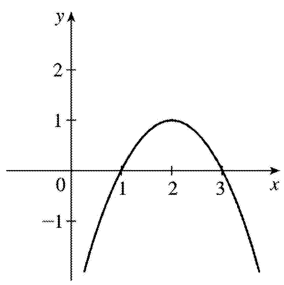
9. Absolute maximum at 5, absolute minimum at 2, local maximum at 3, local minima at 2 and 4



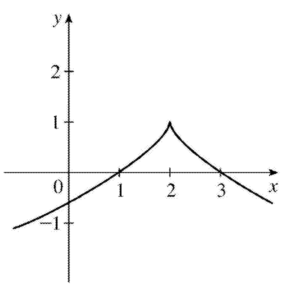
10. f has no local maximum or minimum, but 2 and 4 are critical numbers



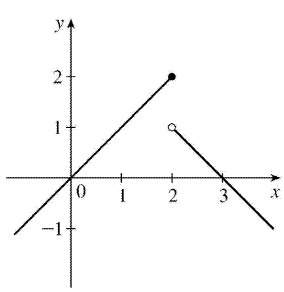
11. **(a)**



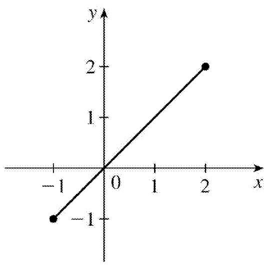
(b)



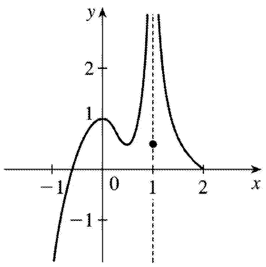
(c)

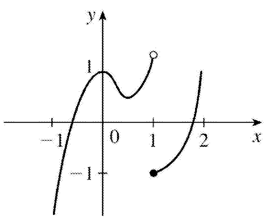


12. **(a)** Note that a local maximum cannot occur at an endpoint.



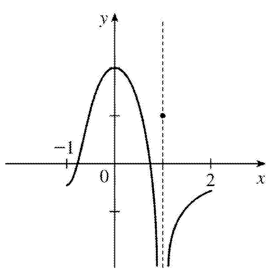
(b)



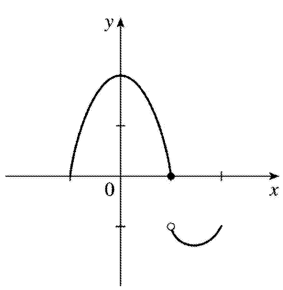


Note: By the Extreme Value Theorem, f must *not* be continuous.

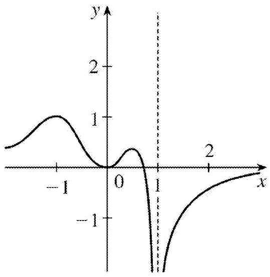
13. **(a)** *Note:* By the Extreme Value Theorem, f must *not* be continuous; because if it were, it would attain an absolute minimum.

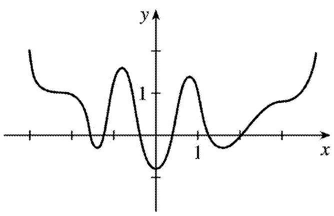


(b)



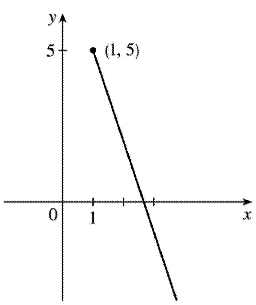
14. **(a)**



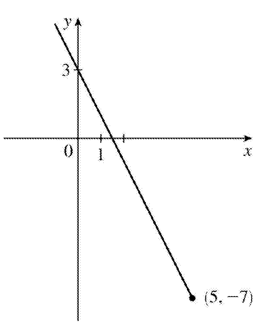


(b)

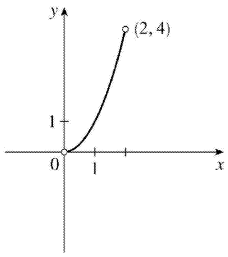
15. $f(x)=8-3x$, $x\geq 1$. Absolute maximum $f(1)=5$; no local maximum. No absolute or local minimum.



16. $f(x)=3-2x$, $x\leq 5$. Absolute minimum $f(5)=-7$; no local minimum. No absolute or local maximum.

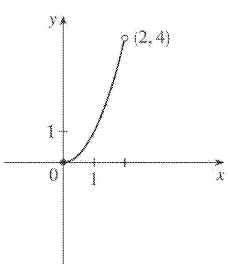


17. $f(x)=x^2$, $0<x<2$. No absolute or local maximum or minimum value.

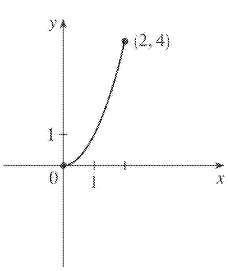


18. $f(x)=x^2$, $0<x\leq 2$. Absolute maximum $f(2)=4$; no local maximum. No absolute or local minimum.

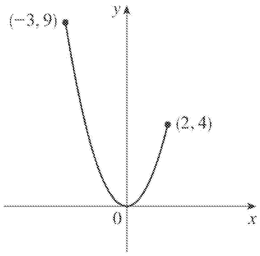
19. $f(x)=x^2$, $0\leq x<2$. Absolute minimum $f(0)=0$; no local minimum. No absolute or local maximum.



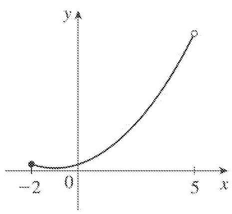
20. $f(x)=x^2$, $0\leq x\leq 2$. Absolute maximum $f(2)=4$. Absolute minimum $f(0)=0$. No local maximum or minimum.



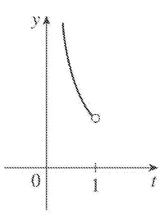
21. $f(x)=x^2$, $-3\leq x\leq 2$. Absolute maximum $f(-3)=9$. No local maximum. Absolute and local minimum $f(0)=0$.



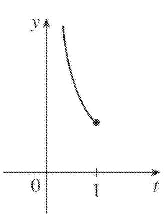
22. $f(x)=1+(x+1)^2$, $-2\leq x<5$. No absolute or local maximum. Absolute and local minimum $f(-1)=1$.



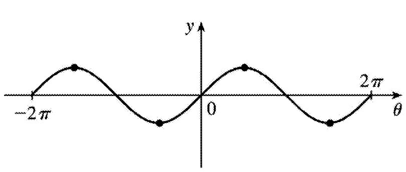
23. $f(t)=1/t$, $0<t<1$. No maximum or minimum.



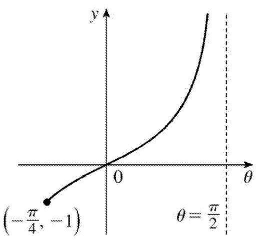
24. $f(t)=1/t$, $0<t\leq 1$. Absolute minimum $f(1)=1$; no local minimum. No local or absolute maximum.



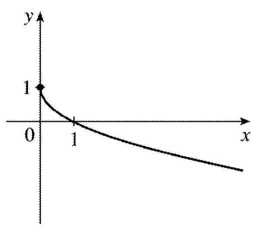
25. $f(\theta)=\sin\theta$, $-2\pi\leq\theta\leq 2\pi$. Absolute and local maxima $f\left(-\frac{3\pi}{2}\right)=f\left(\frac{\pi}{2}\right)=1$. Absolute and local minima $f\left(-\frac{\pi}{2}\right)=f\left(\frac{3\pi}{2}\right)=-1$.



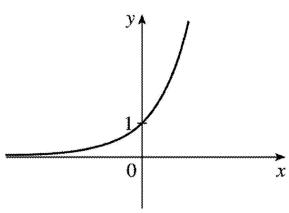
26. $f(\theta)=\tan\theta$, $-\frac{\pi}{4}\leq\theta<\frac{\pi}{2}$. Absolute minimum $f\left(-\frac{\pi}{4}\right)=-1$; no local minimum. No absolute or local maximum.



27. $f(x)=1-\sqrt{x}$. Absolute maximum $f(0)=1$; no local maximum. No absolute or local minimum.

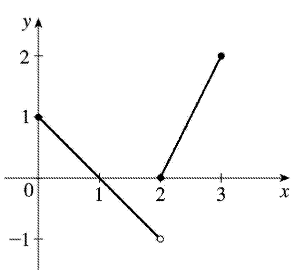


28. $f(x)=e^{x}$. No absolute or local maximum or minimum value.



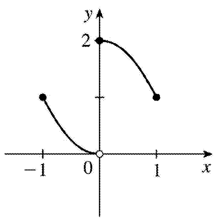
29. $f(x)=\begin{cases} 1-x & \text{if } 0\leq x<2 \\ 2x-4 & \text{if } 2\leq x\leq 3 \end{cases}$

Absolute maximum $f(3)=2$; no local maximum. No absolute or local minimum.



30. $f(x)=\begin{cases} x^2 & \text{if} -1\leq x<0 \\ 2-x^2 & \text{if} 0\leq x\leq 1 \end{cases}$

Absolute and local maximum $f(0)=2$.
No absolute or local minimum.



31. $f(x)=5x^2+4x \Rightarrow f'(x)=10x+4$. $f'(x)=0 \Rightarrow x=-\frac{2}{5}$, so $-\frac{2}{5}$ is the only critical number.

32. $f(x)=x^3+x^2-x \Rightarrow f'(x)=3x^2+2x-1$. $f'(x)=0 \Rightarrow (x+1)(3x-1)=0 \Rightarrow x=-1$, $\frac{1}{3}$. These are the only critical numbers.

33. $f(x)=x^3+3x^2-24x \Rightarrow f'(x)=3x^2+6x-24=3\left(x^2+2x-8\right)$.
$f'(x)=0 \Rightarrow 3(x+4)(x-2)=0 \Rightarrow x=-4$, 2 . These are the only critical numbers.
34. $f(x)=x^3+x^2+x \Rightarrow f'(x)=3x^2+2x+1$. $f'(x)=0 \Rightarrow 3x^2+2x+1=0 \Rightarrow x=\frac{-2\pm\sqrt{4-12}}{6}$. Neither of these is a real number. Thus, there are no critical numbers.

35. $s(t)=3t^4+4t^3-6t^2 \Rightarrow s'(t)=12t^3+12t^2-12t$. $s'(t)=0 \Rightarrow 12t\left(t^2+t-1\right) \Rightarrow t=0$ or $t^2+t-1=0$. Using the quadratic formula to solve the latter equation gives us $t=\frac{-1\pm\sqrt{1^2-4(1)(-1)}}{2(1)}=\frac{-1\pm\sqrt{5}}{2}\approx 0.618$, -1.618 . The three critical numbers are 0 , $\frac{-1\pm\sqrt{5}}{2}$.

36. $f(z)=\frac{z+1}{z^2+z+1} \Rightarrow f'(z)=\frac{\left(z^2+z+1\right)1-(z+1)(2z+1)}{\left(z^2+z+1\right)^2}=\frac{-z^2-2z}{\left(z^2+z+1\right)^2}=0 \Leftrightarrow z(z+2)=0 \Rightarrow z=0$, -2 are the critical numbers. (Note that $z^2+z+1\neq 0$ since the discriminant <0 .)

37. $g(x)=|2x+3|=\begin{cases} 2x+3 & \text{if } 2x+3\geq 0 \\ -(2x+3) & \text{if } 2x+3<0 \end{cases} \Rightarrow g'(x)=\begin{cases} 2 & \text{if } x>-\frac{3}{2} \\ -2 & \text{if } x<-\frac{3}{2} \end{cases}$ $g'(x)$ is never 0 , but $g'(x)$ does not exist for

$x=-\frac{3}{2}$, so $-\frac{3}{2}$ is the only critical number.

38. $g(x)=x^{1/3}-x^{-2/3}\Rightarrow g'(x)=\frac{1}{3}x^{-2/3}+\frac{2}{3}x^{-5/3}=\frac{1}{3}x^{-5/3}(x+2)=\frac{x+2}{3x^{5/3}}$.

$g'(-2)=0$ and $g'(0)$ does not exist, but 0 is not in the domain of g , so the only critical number is -2 .

39. $g(t)=5t^{2/3}+t^{5/3}\Rightarrow g'(t)=\frac{10}{3}t^{-1/3}+\frac{5}{3}t^{2/3}$. $g'(0)$ does not exist, so $t=0$ is a critical number.
$g'(t)=\frac{5}{3}t^{-1/3}(2+t)=0\Leftrightarrow t=-2$, so $t=-2$ is also a critical number.

40. $g(t)=\sqrt{t}\,(1-t)=t^{1/2}-t^{3/2}\Rightarrow g'(t)=\frac{1}{2\sqrt{t}}-\frac{3}{2}\sqrt{t}$. $g'(0)$ does not exist, so $t=0$ is a critical number.
$0=g'(t)=\frac{1-3t}{2\sqrt{t}}\Rightarrow t=\frac{1}{3}$, so $t=\frac{1}{3}$ is also a critical number.

41. $F(x)=x^{4/5}(x-4)^2\Rightarrow$
$F'(x)=x^{4/5}\cdot 2(x-4)+(x-4)^2\cdot\frac{4}{5}x^{-1/5}=\frac{1}{5}x^{-1/5}(x-4)[5\cdot x\cdot 2+(x-4)\cdot 4]$
$=\frac{(x-4)(14x-16)}{5x^{1/5}}=\frac{2(x-4)(7x-8)}{5x^{1/5}}=0$ when $x=4$, $\frac{8}{7}$; and $F'(0)$ does not exist.

Critical numbers are 0 , $\frac{8}{7}$, 4 .

42. $G(x)=\sqrt[3]{x^2-x}\Rightarrow G'(x)=\frac{1}{3}\left(x^2-x\right)^{-2/3}(2x-1)$. $G'(x)$ does not exist when $x^2-x=0$, that is, when $x=0$ or 1 . $G'(x)=0\Leftrightarrow 2x-1=0\Leftrightarrow x=\frac{1}{2}$. So the critical numbers are $x=0$, $\frac{1}{2}$, 1 .

43. $f(\theta)=2\cos\theta+\sin^2\theta\Rightarrow f'(\theta)=-2\sin\theta+2\sin\theta\cos\theta$. $f'(\theta)=0\Rightarrow 2\sin\theta(\cos\theta-1)=0\Rightarrow\sin\theta=0$ or $\cos\theta=1\Rightarrow\theta=n\pi$ (n an integer) or $\theta=2n\pi$. The solutions $\theta=n\pi$ include the solutions $\theta=2n\pi$, so the critical numbers are $\theta=n\pi$.

44. $g(\theta)=4\theta-\tan\theta\Rightarrow g'(\theta)=4-\sec^2\theta$. $g'(\theta)=0\Rightarrow\sec^2\theta=4\Rightarrow\sec\theta=\pm 2\Rightarrow\cos\theta=\pm\frac{1}{2}\Rightarrow\theta=\frac{\pi}{3}+2n\pi$,

$\frac{5\pi}{3}+2n\pi$, $\frac{2\pi}{3}+2n\pi$, and $\frac{4\pi}{3}+2n\pi$ are critical numbers.

Note: The values of θ that make $g'(\theta)$ undefined are not in the domain of g .

45. $f(x)=x\ln x \Rightarrow f'(x)=x(1/x)+(\ln x)\cdot 1=\ln x+1$. $f'(x)=0 \Leftrightarrow \ln x=-1 \Leftrightarrow x=e^{-1}=1/e$. Therefore, the only critical number is $x=1/e$.

46. $f(x)=xe^{2x} \Rightarrow f'(x)=x(2e^{2x})+e^{2x}=e^{2x}(2x+1)$. Since e^{2x} is never 0 , we have $f'(x)=0$ only when $2x+1=0 \Leftrightarrow x=-\frac{1}{2}$. So $-\frac{1}{2}$ is the only critical number.

47. $f(x)=3x^2-12x+5$, $[0,3]$. $f'(x)=6x-12=0 \Leftrightarrow x=2$. Applying the Closed Interval Method, we find that $f(0)=5$, $f(2)=-7$, and $f(3)=-4$. So $f(0)=5$ is the absolute maximum value and $f(2)=-7$ is the absolute minimum value.

48. $f(x)=x^3-3x+1$, $[0,3]$. $f'(x)=3x^2-3=0 \Leftrightarrow x=\pm 1$, but -1 is not in $[0,3]$. $f(0)=1$, $f(1)=-1$, and $f(3)=19$. So $f(3)=19$ is the absolute maximum value and $f(1)=-1$ is the absolute minimum value.

49. $f(x)=2x^3-3x^2-12x+1$, $[-2,3]$. $f'(x)=6x^2-6x-12=6\left(x^2-x-2\right)=6(x-2)(x+1)=0 \Leftrightarrow x=2,-1$. $f(-2)=-3$, $f(-1)=8$, $f(2)=-19$, and $f(3)=-8$. So $f(-1)=8$ is the absolute maximum value and $f(2)=-19$ is the absolute minimum value.

50. $f(x)=x^3-6x^2+9x+2$, $[-1,4]$. $f'(x)=3x^2-12x+9=3\left(x^2-4x+3\right)=3(x-1)(x-3)=0 \Leftrightarrow x=1,3$. $f(-1)=-14$, $f(1)=6$, $f(3)=2$, and $f(4)=6$. So $f(1)=f(4)=6$ is the absolute maximum value and $f(-1)=-14$ is the absolute minimum value.

51. $f(x)=x^4-2x^2+3$, $[-2,3]$. $f'(x)=4x^3-4x=4x\left(x^2-1\right)=4x(x+1)(x-1)=0 \Leftrightarrow x=-1$, 0 , 1 . $f(-2)=11$, $f(-1)=2$, $f(0)=3$, $f(1)=2$, $f(3)=66$. So $f(3)=66$ is the absolute maximum value and $f(\pm 1)=2$ is the absolute minimum value.

52. $f(x)=\left(x^2-1\right)^3$, $[-1,2]$. $f'(x)=3\left(x^2-1\right)^2(2x)=6x(x+1)^2(x-1)^2=0 \Leftrightarrow x=-1,0,1$. $f(\pm 1)=0$, $f(0)=-1$, and $f(2)=27$. So $f(2)=27$ is the absolute maximum value and $f(0)=-1$ is the absolute minimum value.

53. $f(x)=\frac{x}{x^2+1}$, $[0,2]$. $f'(x)=\frac{\left(x^2+1\right)-x(2x)}{\left(x^2+1\right)^2}=\frac{1-x^2}{\left(x^2+1\right)^2}=0 \Leftrightarrow x=\pm 1$, but -1 is not in $[0,2]$. $f(0)=0$, $f(1)=\frac{1}{2}$, $f(2)=\frac{2}{5}$. So $f(1)=\frac{1}{2}$ is the absolute maximum value and $f(0)=0$ is the absolute

minimum value.

54. $f(x)=\frac{x^2-4}{x^2+4}$, $[-4,4]$. $f'(x)=\frac{(x^2+4)(2x)-(x^2-4)(2x)}{(x^2+4)^2}=\frac{16x}{(x^2+4)^2}=0\Leftrightarrow x=0$. $f(\pm 4)=\frac{12}{20}=\frac{3}{5}$ and $f(0)=-1$. So $f(\pm 4)=\frac{3}{5}$ is the absolute maximum value and $f(0)=-1$ is the absolute minimum value.

55. $f(t)=t\sqrt{4-t^2}$, $[-1,2]$.
$f'(t)=t\cdot\frac{1}{2}\left(4-t^2\right)^{-1/2}(-2t)+\left(4-t^2\right)^{1/2}\cdot 1=\frac{-t^2}{\sqrt{4-t^2}}+\sqrt{4-t^2}=\frac{-t^2+\left(4-t^2\right)}{\sqrt{4-t^2}}=\frac{4-2t^2}{\sqrt{4-t^2}}$. $f'(t)=0\Rightarrow$ $4-2t^2=0\Rightarrow t^2=2\Rightarrow t=\pm\sqrt{2}$, but $t=-\sqrt{2}$ is not in the given interval, $[-1,2]$. $f'(t)$ does not exist if $4-t^2=0\Rightarrow t=\pm 2$, but -2 is not in the given interval. $f(-1)=-\sqrt{3}$, $f\left(\sqrt{2}\right)=2$, and $f(2)=0$. So $f\left(\sqrt{2}\right)=2$ is the absolute maximum value and $f(-1)=-\sqrt{3}$ is the absolute minimum value.

56. $f(t)=\sqrt[3]{t}\,(8-t)$, $[0,8]$. $f(t)=8t^{1/3}-t^{4/3}\Rightarrow f'(t)=\frac{8}{3}t^{-2/3}-\frac{4}{3}t^{1/3}=\frac{4}{3}t^{-2/3}(2-t)=\frac{4(2-t)}{3\sqrt[3]{t^2}}$. $f'(t)=0\Rightarrow t=2$. $f'(t)$ does not exist if $t=0$. $f(0)=0$, $f(2)=6\sqrt[3]{2}\approx 7.56$, and $f(8)=0$.
So $f(2)=6\sqrt[3]{2}$ is the absolute maximum value and $f(0)=f(8)=0$ is the absolute minimum value.

57. $f(x)=\sin x+\cos x$, $\left[0,\frac{\pi}{3}\right]$. $f'(x)=\cos x-\sin x=0\Leftrightarrow \sin x=\cos x\Rightarrow \frac{\sin x}{\cos x}=1\Rightarrow \tan x=1\Rightarrow$ $x=\frac{\pi}{4}$. $f(0)=1$, $f\left(\frac{\pi}{4}\right)=\sqrt{2}\approx 1.41$, $f\left(\frac{\pi}{3}\right)=\frac{\sqrt{3}+1}{2}\approx 1.37$. So $f\left(\frac{\pi}{4}\right)=\sqrt{2}$ is the absolute maximum value and $f(0)=1$ is the absolute minimum value.

58. $f(x)=x-2\cos x$, $[-\pi,\pi]$. $f'(x)=1+2\sin x=0\Leftrightarrow \sin x=-\frac{1}{2}\Leftrightarrow x=-\frac{5\pi}{6}, -\frac{\pi}{6}$. $f(-\pi)=2-\pi\approx -1.14$, $f\left(-\frac{5\pi}{6}\right)=\sqrt{3}-\frac{5\pi}{6}\approx -0.886$, $f\left(-\frac{\pi}{6}\right)=-\frac{\pi}{6}-\sqrt{3}\approx -2.26$, $f(\pi)=\pi+2\approx 5.14$. So $f(\pi)=\pi+2$ is the absolute maximum value and $f\left(-\frac{\pi}{6}\right)=-\frac{\pi}{6}-\sqrt{3}$ is the absolute minimum value.

59. $f(x)=xe^{-x}$, $[0,2]$. $f'(x)=x(-e^{-x})+e^{-x}=e^{-x}(1-x)=0\Leftrightarrow x=1$.
$f(0)=0$, $f(1)=e^{-1}=1/e\approx 0.37$, $f(2)=2/e^2\approx 0.27$. So $f(1)=1/e$ is the absolute maximum value and $f(0)=0$ is the absolute minimum value.

60. $f(x)=\dfrac{\ln x}{x}$, $[1,3]$. $f'(x)=\dfrac{x(1/x)-\ln x}{x^2}=\dfrac{1-\ln x}{x^2}=0\Leftrightarrow 1-\ln x=0\Leftrightarrow \ln x=1\Leftrightarrow x=e$. $f(1)=0/1=0$, $f(e)=1/e\approx 0.368$, $f(3)=(\ln 3)/3\approx 0.366$. So $f(e)=1/e$ is the absolute maximum value and $f(1)=0$ is the absolute minimum value.

61. $f(x)=x-3\ln x$, $[1,4]$. $f'(x)=1-\dfrac{3}{x}=\dfrac{x-3}{x}=0\Leftrightarrow x=3$. f' does not exist for $x=0$, but 0 is not in the domain of f . $f(1)=1$, $f(3)=3-3\ln 3\approx -0.296$, $f(4)=4-3\ln 4\approx -0.159$. So $f(1)=1$ is the absolute maximum value and $f(3)=3-3\ln 3\approx -0.296$ is the absolute minimum value.

62. $f(x)=e^{-x}-e^{-2x}$, $[0,1]$. $f'(x)=e^{-x}(-1)-e^{-2x}(-2)=\dfrac{2}{e^{2x}}-\dfrac{1}{e^{x}}=\dfrac{2-e^{x}}{e^{2x}}=0\Leftrightarrow e^{x}=2\Leftrightarrow x=\ln 2\approx 0.69$. $f(0)=0$, $f(\ln 2)=e^{-\ln 2}-e^{-2\ln 2}=\left(e^{\ln 2}\right)^{-1}-\left(e^{\ln 2}\right)^{-2}=2^{-1}-2^{-2}=\dfrac{1}{2}-\dfrac{1}{4}=\dfrac{1}{4}$, $f(1)=e^{-1}-e^{-2}\approx 0.233$. So $f(\ln 2)=\dfrac{1}{4}$ is the absolute maximum value and $f(0)=0$ is the absolute minimum value.

63. $f(x)=x^{a}(1-x)^{b}$, $0\le x\le 1$, $a>0$, $b>0$.

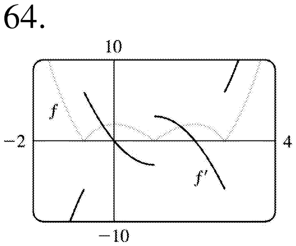
$$\begin{aligned} f'(x) &= x^{a}\cdot b(1-x)^{b-1}(-1)+(1-x)^{b}\cdot ax^{a-1}=x^{a-1}(1-x)^{b-1}\left[x\cdot b(-1)+(1-x)\cdot a\right] \\ &= x^{a-1}(1-x)^{b-1}(a-ax-bx) \end{aligned}$$

At the endpoints, we have $f(0)=f(1)=0$ [the minimum value of f]. In the interval $(0,1)$, $f'(x)=0\Leftrightarrow x=\dfrac{a}{a+b}$.

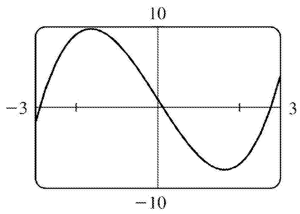
$$f\left(\frac{a}{a+b}\right)=\left(\frac{a}{a+b}\right)^{a}\left(1-\frac{a}{a+b}\right)^{b}=\frac{a^{a}}{(a+b)^{a}}\left(\frac{a+b-a}{a+b}\right)^{b}=\frac{a^{a}}{(a+b)^{a}}\cdot\frac{b^{b}}{(a+b)^{b}}=\frac{a^{a}b^{b}}{(a+b)^{a+b}} .$$

So $f\left(\dfrac{a}{a+b}\right)=\dfrac{a^{a}b^{b}}{(a+b)^{a+b}}$ is the absolute maximum value.

64.



We see that $f'(x)=0$ at about $x=0.0$ and 2.0 , and that $f'(x)$ does not exist at about $x=-0.7$, 1.0 , and 2.7 , so the critical numbers of f are about -0.7 , 0.0 , 1.0 , 2.0 , and 2.7 .

65. **(a)** 

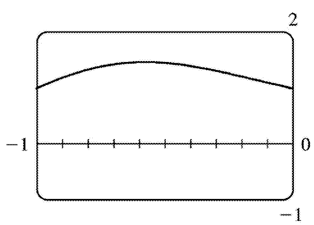
From the graph, it appears that the absolute maximum value is about $f(-1.63)=9.71$, and the absolute minimum value is about $f(1.63)=-7.71$. These values make sense because the graph is symmetric about the point $(0,1)$. ($y=x^3-8x$ is symmetric about the origin.)

(b) $f(x)=x^3-8x+1 \Rightarrow f'(x)=3x^2-8$. So $f'(x)=0 \Rightarrow x=\pm\sqrt{\frac{8}{3}}$.

$$f\left(\pm\sqrt{\frac{8}{3}}\right)=\left(\pm\sqrt{\frac{8}{3}}\right)^3-8\left(\pm\sqrt{\frac{8}{3}}\right)+1=\pm\frac{8}{3}\sqrt{\frac{8}{3}}\mp 8\sqrt{\frac{8}{3}}+1$$

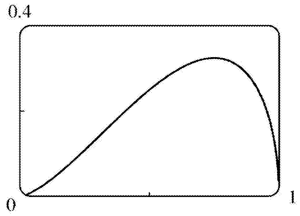
$$=-\frac{16}{3}\sqrt{\frac{8}{3}}+1=1-\frac{32\sqrt{6}}{9} \text{ or } \frac{16}{3}\sqrt{\frac{8}{3}}+1=1+\frac{32\sqrt{6}}{9}$$

(From the graph, we see that the extreme values do not occur at the endpoints.)

66. **(a)** 

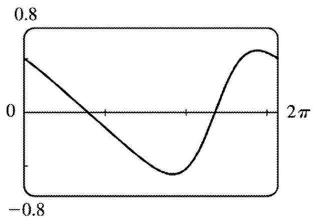
From the graph, it appears that the absolute maximum value is about $f(-0.58)=1.47$, and the absolute minimum value is about $f(-1)=f(0)=1.00$; that is, at both endpoints.

(b) $f(x)=e^{x^3-x} \Rightarrow f'(x)=e^{x^3-x}\left(3x^2-1\right)$. So $f'(x)=0$ on $[-1,0] \Rightarrow x=-\sqrt{1/3}$. $f(-1)=f(0)=1$ (minima) and $f\left(-\sqrt{1/3}\right)=e^{-\sqrt{3}/9+\sqrt{3}/3}=e^{2\sqrt{3}/9}$ (maximum).

67. **(a)** 

From the graph, it appears that the absolute maximum value is about $f(0.75)=0.32$, and the absolute minimum value is $f(0)=f(1)=0$; that is, at both endpoints.

(b) $f(x)=x\sqrt{x-x^2} \Rightarrow f'(x)=x\cdot \dfrac{1-2x}{2\sqrt{x-x^2}}+\sqrt{x-x^2}=\dfrac{\left(x-2x^2\right)+\left(2x-2x^2\right)}{2\sqrt{x-x^2}}=\dfrac{3x-4x^2}{2\sqrt{x-x^2}}$. So $f'(x)=0\Rightarrow$ $3x-4x^2=0\Rightarrow x(3-4x)=0\Rightarrow x=0$ or $\dfrac{3}{4}$. $f(0)=f(1)=0$, and $f\left(\dfrac{3}{4}\right)=\dfrac{3}{4}\sqrt{\dfrac{3}{4}-\left(\dfrac{3}{4}\right)^2}=\dfrac{3\sqrt{3}}{16}$.

68. **(a)** 

From the graph, it appears that the absolute maximum value is about $f(5.76)=0.58$, and the absolute minimum value is about $f(3.67)=-0.58$.

(b) $f(x)=\dfrac{\cos x}{2+\sin x}\Rightarrow f'(x)=\dfrac{(2+\sin x)(-\sin x)-(\cos x)(\cos x)}{(2+\sin x)^2}=\dfrac{-1-2\sin x}{(2+\sin x)^2}$.

So $f'(x)=0\Rightarrow \sin x=-\dfrac{1}{2}\Rightarrow x=\dfrac{7\pi}{6}$ or $\dfrac{11\pi}{6}$. Now $f\left(\dfrac{7\pi}{6}\right)=\dfrac{-\sqrt{3}/2}{3/2}=-\dfrac{1}{\sqrt{3}}$, and $f\left(\dfrac{11\pi}{6}\right)=\dfrac{\sqrt{3}/2}{3/2}=\dfrac{1}{\sqrt{3}}$.

69. The density is defined as $\rho=\dfrac{\text{mass}}{\text{volume}}=\dfrac{1000}{V(T)}$ (in g / cm^3). But a critical point of ρ will also be a critical point of V since [$\dfrac{d\rho}{dT}=-1000V^{-2}\dfrac{dV}{dT}$ and V is never 0], and V is easier to differentiate than ρ .

$V(T)=999.87-0.06426T+0.0085043T^2-0.0000679T^3\Rightarrow$

$V'(T)=-0.06426+0.0170086T-0.0002037T^2$. Setting this equal to 0 and using the quadratic formula to

find T , we get $T=\dfrac{-0.0170086\pm\sqrt{0.0170086^2-4\cdot 0.0002037\cdot 0.06426}}{2(-0.0002037)}\approx 3.9665^\circ$ C or 79.5318° C.

Since we are only interested in the region 0° C $\le T\le 30^\circ$ C, we check the density ρ at the endpoints and at 3.9665° C: $\rho(0)\approx\dfrac{1000}{999.87}\approx 1.00013$; $\rho(30)\approx\dfrac{1000}{1003.7628}\approx 0.99625$;

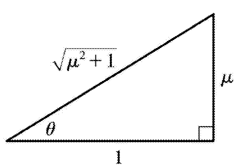
$\rho(3.9665) \approx \dfrac{1000}{999.7447} \approx 1.000255$. So water has its maximum density at about 3.9665° C.

70. $F = \dfrac{\mu W}{\mu \sin\theta + \cos\theta} \Rightarrow \dfrac{dF}{d\theta} = \dfrac{(\mu \sin\theta + \cos\theta)(0) - \mu W(\mu \cos\theta - \sin\theta)}{(\mu \sin\theta + \cos\theta)^2} = \dfrac{-\mu W(\mu \cos\theta - \sin\theta)}{(\mu \sin\theta + \cos\theta)^2}$.

So $\dfrac{dF}{d\theta} = 0 \Rightarrow \mu \cos\theta - \sin\theta = 0 \Rightarrow \mu = \dfrac{\sin\theta}{\cos\theta} = \tan\theta$. Substituting $\tan\theta$ for μ in F gives us

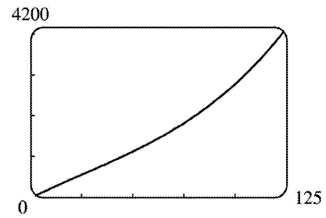
$F = \dfrac{(\tan\theta)W}{(\tan\theta)\sin\theta + \cos\theta} = \dfrac{W \tan\theta}{\dfrac{\sin^2\theta}{\cos\theta} + \cos\theta} = \dfrac{W \tan\theta \cos\theta}{\sin^2\theta + \cos^2\theta} = \dfrac{W \sin\theta}{1} = W \sin\theta$.

If $\tan\theta = \mu$, then $\sin\theta = \dfrac{\mu}{\sqrt{\mu^2+1}}$ (see the figure), so $F = \dfrac{\mu}{\sqrt{\mu^2+1}} W$. We compare this with the value of F at the endpoints: $F(0) = \mu W$ and $F\left(\dfrac{\pi}{2}\right) = W$. Now because $\dfrac{\mu}{\sqrt{\mu^2+1}} \le 1$ and $\dfrac{\mu}{\sqrt{\mu^2+1}} \le \mu$, we have that $\dfrac{\mu}{\sqrt{\mu^2+1}} W$



is less than or equal to each of $F(0)$ and $F\left(\dfrac{\pi}{2}\right)$. Hence, $\dfrac{\mu}{\sqrt{\mu^2+1}} W$ is the absolute minimum value of $F(\theta)$, and it occurs when $\tan\theta = \mu$.

71. We apply the Closed Interval Method to the continuous function $I(t) = 0.00009045t^5 + 0.001438t^4 - 0.06561t^3 + 0.4598t^2 - 0.6270t + 99.33$ on $[0,10]$. Its derivative is $I'(t) = 0.00045225t^4 + 0.005752t^3 - 0.19683t^2 + 0.9196t - 0.6270$. Since I' exists for all t , the only critical numbers of I occur when $I'(t) = 0$. We use a root–finder on a computer algebra system (or a graphing device) to find that $I'(t) = 0$ when $t \approx -29.7186$, 0.8231 , 5.1309 , or 11.0459 , but only the second and third roots lie in the interval $[0,10]$. The values of I at these critical numbers are $I(0.8231) \approx 99.09$ and $I(5.1309) \approx 100.67$. The values of I at the endpoints of the interval are $I(0) = 99.33$ and $I(10) \approx 96.86$. Comparing these four numbers, we see that food was most expensive at $t \approx 5.1309$ (corresponding roughly to August, 1989) and cheapest at $t = 10$ (midyear 1994).

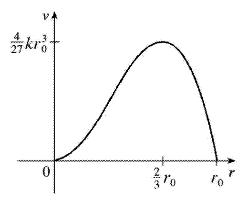
72. **(a)** 

The equation of the graph in the figure is
$v(t)=0.00146t^3-0.11553t^2+24.98169t-21.26872$.

(b) $a(t)=v'(t)=0.00438t^2-0.23106t+24.98169 \Rightarrow a'(t)=0.00876t-0.23106$. $a'(t)=0 \Rightarrow$
$t_1=\dfrac{0.23106}{0.00876}\approx 26.4$. $a(0)\approx 24.98$, $a(t_1)\approx 21.93$, and $a(125)\approx 64.54$. The maximum acceleration is about 64.5 ft / s^2 and the minimum acceleration is about 21.93 ft / s^2 .

73. **(a)** $v(r)=k(r_0-r)r^2=kr_0r^2-kr^3 \Rightarrow v'(r)=2kr_0r-3kr^2$. $v'(r)=0 \Rightarrow kr(2r_0-3r)=0 \Rightarrow r=0$ or $\frac{2}{3}r_0$ (but 0 is not in the interval). Evaluating v at $\frac{1}{2}r_0$, $\frac{2}{3}r_0$, and r_0 , we get $v\left(\frac{1}{2}r_0\right)=\frac{1}{8}kr_0^3$,
$v\left(\frac{2}{3}r_0\right)=\frac{4}{27}kr_0^3$, and $v(r_0)=0$. Since $\frac{4}{27}>\frac{1}{8}$, v attains its maximum value at $r=\frac{2}{3}r_0$. This supports the statement in the text.

(b) From part (a), the maximum value of v is $\frac{4}{27}kr_0^3$.

(c) 

74. $g(x)=2+(x-5)^3 \Rightarrow g'(x)=3(x-5)^2 \Rightarrow g'(5)=0$, so 5 is a critical number. But $g(5)=2$ and g takes on values >2 and values <2 in any open interval containing 5 , so g does not have a local maximum or minimum at 5 .

75. $f(x)=x^{101}+x^{51}+x+1 \Rightarrow f'(x)=101x^{100}+51x^{50}+1 \geq 1$ for all x , so $f'(x)=0$ has no solution. Thus, $f(x)$ has no critical number, so $f(x)$ can have no local maximum or minimum.

76. Suppose that f has a minimum value at c , so $f(x)\geq f(c)$ for all x near c . Then $g(x)=-f(x)\leq -f(c)=g(c)$ for all x near c , so $g(x)$ has a maximum value at c .

77. If f has a local minimum at c , then $g(x)=-f(x)$ has a local maximum at c , so $g'(c)=0$ by the case

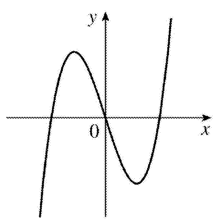
of Fermat's Theorem proved in the text. Thus, $f'(c)=-g'(c)=0$.

78.

(a) $f(x)=ax^3+bx^2+cx+d$, $a\neq 0$. So $f'(x)=3ax^2+2bx+c$ is a quadratic and hence has either 2 , 1 , or 0 real roots, so $f(x)$ has either 2 , 1 or 0 critical numbers.

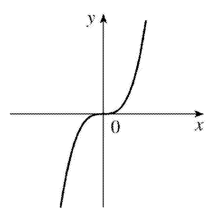
Case (i) (2 critical numbers):

$f(x)=x^3-3x\Rightarrow$
$f'(x)=3x^2-3$, so $x=-1,1$
are critical numbers.



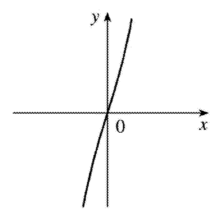
Case (ii) (1 critical number):

f(x)=x $\Rightarrow$
$f'(x)=3x$, *so* x=0
is the only critical number.



Case (iii) (no critical number):

$f(x)=x^3+3x\Rightarrow$
$f'(x)=3x^2+3$,
so there are no real roots.



(b) Since there are at most two critical numbers, it can have at most two local extreme values and by (i) this can occur. By (iii) it can have no local extreme value. However, if there is only one critical number, then there is no local extreme value.

1. $f(x)=6x^2-8x+3 \Rightarrow F(x)=6\dfrac{x^{2+1}}{2+1}-8\dfrac{x^{1+1}}{1+1}+3x+C=2x^3-4x^2+3x+C$

Check: $F'(x)=2\cdot 3x^2-4\cdot 2x+3+0=6x^2-8x+3=f(x)$

2. $f(x)=4+x^2-5x^3 \Rightarrow F(x)=4x+\dfrac{1}{3}x^3-\dfrac{5}{4}x^4+C$

3. $f(x)=1-x^3+5x^5-3x^7 \Rightarrow F(x)=x-\dfrac{x^{3+1}}{3+1}+5\dfrac{x^{5+1}}{5+1}-3\dfrac{x^{7+1}}{7+1}+C=x-\dfrac{1}{4}x^4+\dfrac{5}{6}x^6-\dfrac{3}{8}x^8+C$

4. $f(x)=x^{20}+4x^{10}+8 \Rightarrow F(x)=\dfrac{1}{21}x^{21}+\dfrac{4}{11}x^{11}+8x+C$

5. $f(x)=5x^{1/4}-7x^{3/4} \Rightarrow F(x)=5\dfrac{x^{1/4+1}}{\frac{1}{4}+1}-7\dfrac{x^{3/4+1}}{\frac{3}{4}+1}+C=5\dfrac{x^{5/4}}{5/4}-7\dfrac{x^{7/4}}{7/4}+C=4x^{5/4}-4x^{7/4}+C$

6. $f(x)=2x+3x^{1.7} \Rightarrow F(x)=x^2+\dfrac{3}{2.7}x^{2.7}+C=x^2+\dfrac{10}{9}x^{2.7}+C$

7. $f(x)=6\sqrt{x}-\sqrt[6]{x}=6x^{1/2}-x^{1/6} \Rightarrow$

$F(x)=6\dfrac{x^{1/2+1}}{\frac{1}{2}+1}-\dfrac{x^{1/6+1}}{\frac{1}{6}+1}+C=6\dfrac{x^{3/2}}{3/2}-\dfrac{x^{7/6}}{7/6}+C=4x^{3/2}-\dfrac{6}{7}x^{7/6}+C$

8. $f(x)=\sqrt[4]{x^3}+\sqrt[3]{x^4}=x^{3/4}+x^{4/3} \Rightarrow F(x)=\dfrac{x^{7/4}}{7/4}+\dfrac{x^{7/3}}{7/3}+C=\dfrac{4}{7}x^{7/4}+\dfrac{3}{7}x^{7/3}+C$

9. $f(x)=\dfrac{10}{x^9}=10x^{-9}$ has domain $(-\infty,0)\cup(0,\infty)$, so

$$F(x)=\begin{cases} \dfrac{10x^{-8}}{-8}+C_1=-\dfrac{5}{4x^8}+C_1 & \text{if } x<0 \\ -\dfrac{5}{4x^8}+C_2 & \text{if } x>0 \end{cases}$$

See Example 1 for a similar problem.

10.

$g(x)=\dfrac{5-4x^3+2x^6}{x^6}=5x^{-6}-4x^{-3}+2$ has domain $(-\infty,0)\cup(0,\infty)$, so

$$G(x)=\begin{cases} 5\dfrac{x^{-5}}{-5}-4\dfrac{x^{-2}}{-2}+2x+C_1=-\dfrac{1}{x^5}+\dfrac{2}{x^2}+2x+C_1 & \text{if } x<0 \\ -\dfrac{1}{x^5}+\dfrac{2}{x^2}+2x+C_2 & \text{if } x>0 \end{cases}$$

11. $f(u)=\dfrac{u^4+3\sqrt{u}}{u^2}=\dfrac{u^4}{u^2}+\dfrac{3u^{1/2}}{u^2}=u^2+3u^{-3/2}\Rightarrow$

$F(u)=\dfrac{u^3}{3}+3\dfrac{u^{-3/2+1}}{-3/2+1}+C=\dfrac{1}{3}u^3+3\dfrac{u^{-1/2}}{-1/2}+C=\dfrac{1}{3}u^3-\dfrac{6}{\sqrt{u}}+C$

12. $f(x)=3e^x+7\sec^2 x\Rightarrow F(x)=3e^x+7\tan x+C_n$ on the interval $\left(n\pi-\dfrac{\pi}{2},n\pi+\dfrac{\pi}{2}\right)$.

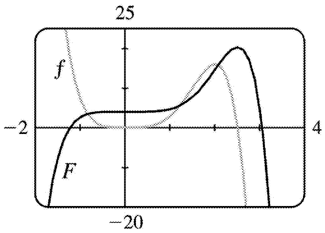
13. $g(\theta)=\cos\theta-5\sin\theta\Rightarrow G(\theta)=\sin\theta-5(-\cos\theta)+C=\sin\theta+5\cos\theta+C$

14. $h(\theta)=\dfrac{\sin\theta}{\cos^2\theta}=\dfrac{1}{\cos\theta}\cdot\dfrac{\sin\theta}{\cos\theta}=\sec\theta\tan\theta\Rightarrow H(\theta)=\sec\theta+C_n$ on the interval $\left(n\pi-\dfrac{\pi}{2},n\pi+\dfrac{\pi}{2}\right)$.

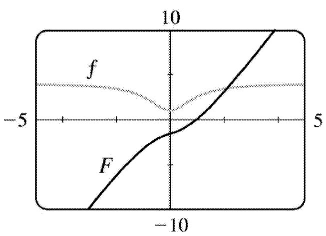
15. $f(x)=2x+5\left(1-x^2\right)^{-1/2}=2x+\dfrac{5}{\sqrt{1-x^2}}\Rightarrow F(x)=x^2+5\sin^{-1}x+C$

16. $f(x)=\dfrac{x^2+x+1}{x}=x+1+\dfrac{1}{x}\Rightarrow F(x)=\begin{cases}\dfrac{1}{2}x^2+x+\ln|x|+C_1 & \text{if } x<0\\ \dfrac{1}{2}x^2+x+\ln|x|+C_2 & \text{if } x>0\end{cases}$

17. $f(x)=5x^4-2x^5\Rightarrow F(x)=5\cdot\dfrac{x^5}{5}-2\cdot\dfrac{x^6}{6}+C=x^5-\dfrac{1}{3}x^6+C$. $F(0)=4\Rightarrow 0^5-\dfrac{1}{3}\cdot 0^6+C=4\Rightarrow C=4$, so $F(x)=x^5-\dfrac{1}{3}x^6+4$. The graph confirms our answer since $f(x)=0$ when F has a local maximum, f is positive when F is increasing, and f is negative when F is decreasing.



18. $f(x)=4-3\left(1+x^2\right)^{-1}=4-\dfrac{3}{1+x^2}\Rightarrow F(x)=4x-3\tan^{-1}x+C$. $F(1)=0\Rightarrow 4-3\left(\dfrac{\pi}{4}\right)+C=0\Rightarrow C=\dfrac{3\pi}{4}-4$, so $F(x)=4x-3\tan^{-1}x+\dfrac{3\pi}{4}-4$. Note that f is positive and F is increasing on R . Also, f has smaller values where the slopes of the tangent lines of F are smaller.



19. $f''(x)=6x+12x^2\Rightarrow f'(x)=6\cdot\dfrac{x^2}{2}+12\cdot\dfrac{x^3}{3}+C=3x^2+4x^3+C\Rightarrow$
$f(x)=3\cdot\dfrac{x^3}{3}+4\cdot\dfrac{x^4}{4}+Cx+D=x^3+x^4+Cx+D$

20. $f''(x)=2+x^3+x^6\Rightarrow f'(x)=2x+\dfrac{1}{4}x^4+\dfrac{1}{7}x^7+C\Rightarrow\ f(x)=x^2+\dfrac{1}{20}x^5+\dfrac{1}{56}x^8+Cx+D$

21. $f''(x)=1+x^{4/5}\Rightarrow f'(x)=x+\dfrac{5}{9}x^{9/5}+C\Rightarrow\ f(x)=\dfrac{1}{2}x^2+\dfrac{5}{9}\cdot\dfrac{5}{14}x^{14/5}+Cx+D=\dfrac{1}{2}x^2+\dfrac{25}{126}x^{14/5}+Cx+D$

22. $f''(x)=\cos x\Rightarrow f'(x)=\sin x+C\Rightarrow\ f(x)=-\cos x+Cx+D$

23. $f'''(t)=60t^2\Rightarrow f''(t)=20t^3+C\Rightarrow f'(t)=5t^4+Ct+D\Rightarrow\ f(t)=t^5+\dfrac{1}{2}Ct^2+Dt+E$

24. $f'''(t)=t-\sqrt{t}\Rightarrow f''(t)=\dfrac{1}{2}t^2-\dfrac{2}{3}t^{3/2}+C\Rightarrow f'(t)=\dfrac{1}{6}t^3-\dfrac{4}{15}t^{5/2}+Ct+D\Rightarrow$
$f(t)=\dfrac{1}{24}t^4-\dfrac{8}{105}t^{7/2}+\dfrac{1}{2}Ct^2+Dt+E$

25. $f'(x)=1-6x\Rightarrow f(x)=x-3x^2+C$. $f(0)=C$ and $f(0)=8\Rightarrow C=8$, so $f(x)=x-3x^2+8$.

26. $f'(x)=8x^3+12x+3 \Rightarrow f(x)=2x^4+6x^2+3x+C$. $f(1)=11+C$ and $f(1)=6 \Rightarrow$
$11+C=6 \Rightarrow C=-5$, so $f(x)=2x^4+6x^2+3x-5$.
27. $f'(x)=\sqrt{x}(6+5x)=6x^{1/2}+5x^{3/2} \Rightarrow f(x)=4x^{3/2}+2x^{5/2}+C$.
$f(1)=6+C$ and $f(1)=10 \Rightarrow C=4$, so $f(x)=4x^{3/2}+2x^{5/2}+4$.

28. $f'(x)=2x-3/x^4=2x-3x^{-4} \Rightarrow f(x)=x^2+x^{-3}+C$ because we're given that $x>0$.
$f(1)=2+C$ and $f(1)=3 \Rightarrow C=1$, so $f(x)=x^2+1/x^3+1$.

29. $f'(t)=2\cos t+\sec^2 t \Rightarrow f(t)=2\sin t+\tan t+C$ because $-\pi/2<t<\pi/2$.
$f\left(\frac{\pi}{3}\right)=2\left(\sqrt{3}/2\right)+\sqrt{3}+C=2\sqrt{3}+C$ and $f\left(\frac{\pi}{3}\right)=4 \Rightarrow C=4-2\sqrt{3}$, so $f(t)=2\sin t+\tan t+4-2\sqrt{3}$.

30. $f'(x)=3x^{-2} \Rightarrow f(x)=\begin{cases} -3/x+C_1 & \text{if } x>0 \\ -3/x+C_2 & \text{if } x<0 \end{cases}$ $f(1)=-3+C_1=0 \Rightarrow C_1=3$,
$f(-1)=3+C_2=0 \Rightarrow C_2=-3$. So $f(x)=\begin{cases} -3/x+3 & \text{if } x>0 \\ -3/x-3 & \text{if } x<0 \end{cases}$
31. $f'(x)=2/x \Rightarrow f(x)=2\ln|x|+C=2\ln(-x)+C$ (since $x<0$). Now $f(-1)=2\ln 1+C=2(0)+C=7 \Rightarrow C=7$.
Therefore, $f(x)=2\ln(-x)+7$, $x<0$.

32. $f'(x)=4/\sqrt{1-x^2} \Rightarrow f(x)=4\sin^{-1}x+C$. $f\left(\frac{1}{2}\right)=4\sin^{-1}\left(\frac{1}{2}\right)+C=4\cdot\frac{\pi}{6}+C$ and $f\left(\frac{1}{2}\right)=1 \Rightarrow$
$\frac{2\pi}{3}+C=1 \Rightarrow C=1-\frac{2\pi}{3}$, so $f(x)=4\sin^{-1}x+1-\frac{2\pi}{3}$.

33. $f''(x)=24x^2+2x+10 \Rightarrow f'(x)=8x^3+x^2+10x+C$. $f'(1)=8+1+10+C$ and $f'(1)=-3 \Rightarrow 19+C=-3 \Rightarrow$
$C=-22$, so $f'(x)=8x^3+x^2+10x-22$ and hence, $f(x)=2x^4+\frac{1}{3}x^3+5x^2-22x+D$. $f(1)=2+\frac{1}{3}+5-22+D$ and
$f(1)=5 \Rightarrow D=22-\frac{7}{3}=\frac{59}{3}$, so $f(x)=2x^4+\frac{1}{3}x^3+5x^2-22x+\frac{59}{3}$.

34. $f''(x)=4-6x-40x^3 \Rightarrow f'(x)=4x-3x^2-10x^4+C$. $f'(0)=C$ and $f'(0)=1 \Rightarrow C=1$, so
$f'(x)=4x-3x^2-10x^4+1$ and hence, $f(x)=2x^2-x^3-2x^5+x+D$. $f(0)=D$ and $f(0)=2 \Rightarrow D=2$, so
$f(x)=2x^2-x^3-2x^5+x+2$.

35.

$f''(\theta)=\sin\theta+\cos\theta \Rightarrow f'(\theta)=-\cos\theta+\sin\theta+C$. $f'(0)=-1+C$ and $f'(0)=4\Rightarrow C=5$, so $f'(\theta)=-\cos\theta+\sin\theta+5$ and hence, $f(\theta)=-\sin\theta-\cos\theta+5\theta+D$. $f(0)=-1+D$ and $f(0)=3\Rightarrow D=4$, so $f(\theta)=-\sin\theta-\cos\theta+5\theta+4$.

36. $f''(t)=3/\sqrt{t}=3t^{-1/2}\Rightarrow f'(t)=6t^{1/2}+C$. $f'(4)=12+C$ and $f'(4)=7\Rightarrow C=-5$, so $f'(t)=6t^{1/2}-5$ and hence, $f(t)=4t^{3/2}-5t+D$. $f(4)=32-20+D$ and $f(4)=20\Rightarrow D=8$, so $f(t)=4t^{3/2}-5t+8$.

37. $f''(x)=2-12x\Rightarrow f'(x)=2x-6x^2+C\Rightarrow f(x)=x^2-2x^3+Cx+D$.
$f(0)=D$ and $f(0)=9\Rightarrow D=9$. $f(2)=4-16+2C+9=2C-3$ and $f(2)=15\Rightarrow 2C=18\Rightarrow C=9$, so $f(x)=x^2-2x^3+9x+9$.

38. $f''(x)=20x^3+12x^2+4\Rightarrow f'(x)=5x^4+4x^3+4x+C\Rightarrow f(x)=x^5+x^4+2x^2+Cx+D$. $f(0)=D$ and $f(0)=8\Rightarrow D=8$. $f(1)=1+1+2+C+8=C+12$ and $f(1)=5\Rightarrow C=-7$, so $f(x)=x^5+x^4+2x^2-7x+8$.

39. $f''(x)=2+\cos x\Rightarrow f'(x)=2x+\sin x+C\Rightarrow f(x)=x^2-\cos x+Cx+D$. $f(0)=-1+D$ and $f(0)=-1\Rightarrow D=0$. $f\left(\frac{\pi}{2}\right)=\pi^2/4+\left(\frac{\pi}{2}\right)C$ and $f\left(\frac{\pi}{2}\right)=0\Rightarrow \left(\frac{\pi}{2}\right)C=-\pi^2/4\Rightarrow C=-\frac{\pi}{2}$, so $f(x)=x^2-\cos x-\left(\frac{\pi}{2}\right)x$.

40. $f''(t)=2e^t+3\sin t\Rightarrow f'(t)=2e^t-3\cos t+C\Rightarrow f(t)=2e^t-3\sin t+Ct+D$. $f(0)=2+D$ and $f(0)=0\Rightarrow D=-2$. $f(\pi)=2e^\pi+\pi C-2$ and $f(\pi)=0\Rightarrow \pi C=2-2e^\pi\Rightarrow C=\dfrac{2-2e^\pi}{\pi}$, so $f(t)=2e^t-3\sin t+\dfrac{2-2e^\pi}{\pi}t-2$.

41. $f''(x)=x^{-2}$, $x>0\Rightarrow f'(x)=-1/x+C\Rightarrow f(x)=-\ln|x|+Cx+D=-\ln x+Cx+D$ (since $x>0$). $f(1)=0\Rightarrow C+D=0$ and $f(2)=0\Rightarrow -\ln 2+2C+D=0\Rightarrow$ $-\ln 2+2C-C=0$ [since $D=-C$] $\Rightarrow -\ln 2+C=0\Rightarrow C=\ln 2$ and $D=-\ln 2$.
So $f(x)=-\ln x+(\ln 2)x-\ln 2$.

42. $f'''(x)=\sin x\Rightarrow f''(x)=-\cos x+C\Rightarrow 1=f''(0)=-1+C\Rightarrow C=2$, so $f''(x)=-\cos x+2\Rightarrow f'(x)=-\sin x+2x+D\Rightarrow 1=f'(0)=D\Rightarrow f'(x)=-\sin x+2x+1\Rightarrow f(x)=\cos x+x^2+x+E$ $\Rightarrow 1=f(0)=1+E\Rightarrow E=0$, so $f(x)=\cos x+x^2+x$.

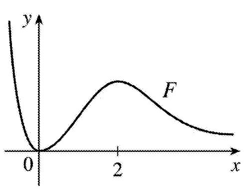
43. Given $f'(x)=2x+1$, we have $f(x)=x^2+x+C$. Since f passes through $(1,6)$, $f(1)=6\Rightarrow 1^2+1+C=6$ $\Rightarrow C=4$. Therefore, $f(x)=x^2+x+4$ and $f(2)=2^2+2+4=10$.

44. $f'(x)=x^3 \Rightarrow f(x)=\frac{1}{4}x^4+C$. $x+y=0 \Rightarrow y=-x \Rightarrow m=-1$. Now $m=f'(x) \Rightarrow -1=x^3 \Rightarrow x=-1 \Rightarrow y=1$ (from the equation of the tangent line), so $(-1,1)$ is a point on the graph of f . From f , $1=\frac{1}{4}(-1)^4+C \Rightarrow C=\frac{3}{4}$. Therefore, the function is $f(x)=\frac{1}{4}x^4+\frac{3}{4}$.

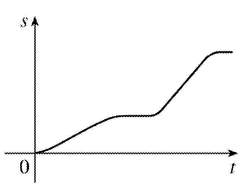
45. b is the antiderivative of f . For small x , f is negative, so the graph of its antiderivative must be decreasing. But both a and c are increasing for small x , so only b can be f 's antiderivative. Also, f is positive where b is increasing, which supports our conclusion.

46. We know right away that c cannot be f 's antiderivative, since the slope of c is not zero at the x–value where $f=0$. Now f is positive when a is increasing and negative when a is decreasing, so a is the antiderivative of f .

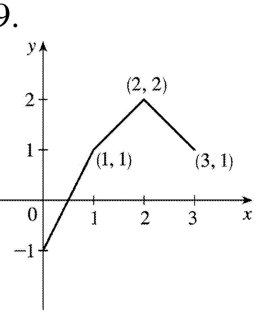
47. The graph of F will have a minimum at 0 and a maximum at 2 , since $f=F'$ goes from negative to positive at $x=0$, and from positive to negative at $x=2$.



48. The position function is the antiderivative of the velocity function, so its graph has to be horizontal where the velocity function is equal to 0 .



49.

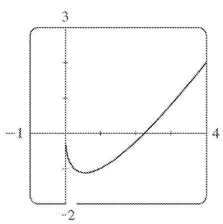


$$f'(x)=\begin{cases} 2 & \text{if } 0\le x<1 \\ 1 & \text{if } 1<x<2 \\ -1 & \text{if } 2<x\le 3 \end{cases} \Rightarrow f(x)=\begin{cases} 2x+C & \text{if } 0\le x<1 \\ x+D & \text{if } 1<x<2 \\ -x+E & \text{if } 2<x\le 3 \end{cases} \quad f(0)=-1\Rightarrow 2(0)+C=-1\Rightarrow C=-1 .$$

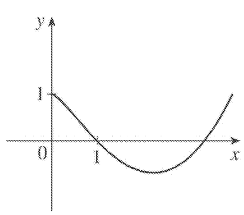
Starting at the point $(0,-1)$ and moving to the right on a line with slope 2 gets us to the point $(1,1)$. The slope for $1<x<2$ is 1, so we get to the point $(2,2)$. Here we have used the fact that f is continuous. We can include the point $x=1$ on either the first or the second part of f. The line connecting $(1,1)$ to $(2,2)$ is $y=x$, so $D=0$. The slope for $2<x\le 3$ is -1, so we get to $(3,1)$. $f(3)=1\Rightarrow -3+E=1\Rightarrow E=4$. Thus,

$$f(x)=\begin{cases} 2x-1 & \text{if } 0\le x\le 1 \\ x & \text{if } 1<x<2 \\ -x+4 & \text{if } 2\le x\le 3 \end{cases}$$

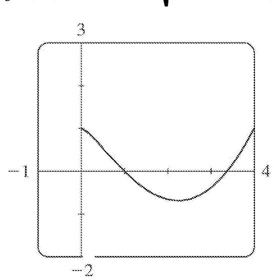
Note that $f'(x)$ does not exist at $x=1$ or at $x=2$.

50. **(a)** 

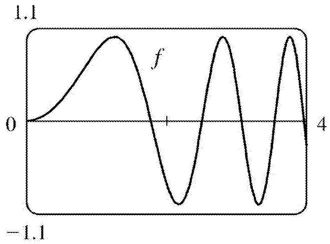
(b) Since $F(0)=1$, we can start our graph at $(0,1)$. f has a minimum at about $x=0.5$, so its derivative is zero there. f is decreasing on $(0,0.5)$, so its derivative is negative and hence, F is CD on $(0,0.5)$ and has an IP at $x\approx 0.5$. On $(0.5,2.2)$, f is negative and increasing (f' is positive), so F is decreasing and CU. On $(2.2,\infty)$, f is positive and increasing, so F is increasing and CU.

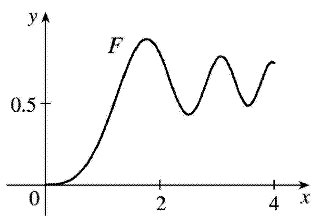


(c) $f(x)=2x-3\sqrt{x}\Rightarrow F(x)=x^2-3\cdot\frac{2}{3}x^{3/2}+C$. $F(0)=C$ and $F(0)=1\Rightarrow C=1$, so $F(x)=x^2-2x^{3/2}+1$.

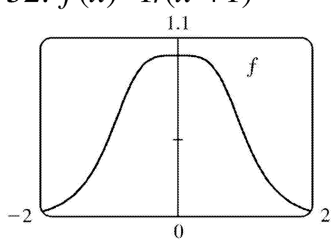
(d) 

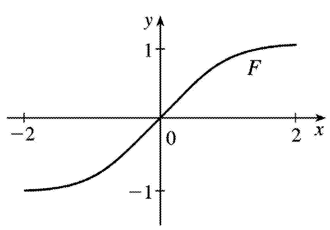
51. $f(x)=\sin(x^2)$, $0\leq x\leq 4$



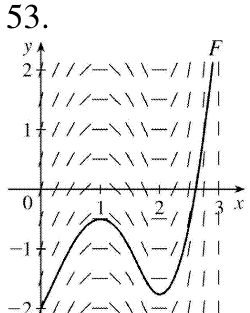


52. $f(x)=1/(x^4+1)$

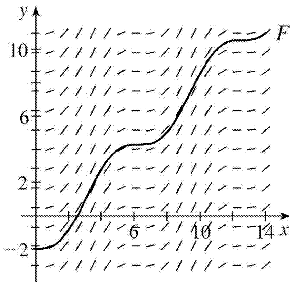




53.



54.

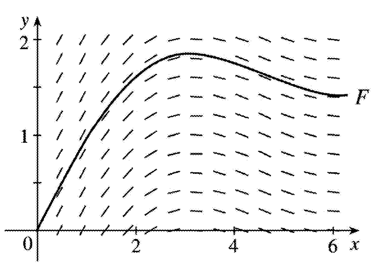


55.

x	$f(x)$
0	1
0.5	0.959
1.0	0.841
1.5	0.665
2.0	0.455
2.5	0.239
3.0	0.047

x	$f(x)$
3.5	−0.100
4.0	−0.189
4.5	−0.217
5.0	−0.192
5.5	−0.128
6.0	−0.047

We compute slopes [values of $f(x)=(\sin x)/x$ for $0 < x < 2\pi$] as in the table $\left[\lim_{x \to 0^+} f(x)=1\right]$ and draw a direction field as in Example 6. Then we use the direction field to graph F starting at $(0,0)$

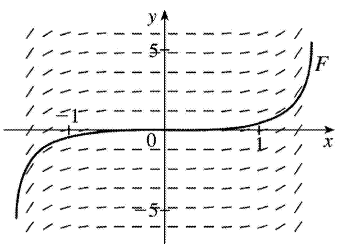


56.

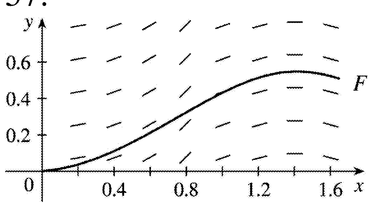
x	$f(x)$
0	0
± 0.2	0.041
± 0.4	0.169
± 0.6	0.410
± 0.8	0.824
± 1.0	1.557
± 1.2	3.087

± 1.4	8.117
± 1.5	21.152

We compute slopes [values of $f(x) = x \tan x$ for $-\pi/2 < x \;\; \pi/2$] as in the table and draw a direction field as in Example 6. Then we use the direction field to graph F starting at $(0,0)$ and extending in both directions. Note that if f is an even function, then the antiderivative F that passes through the origin is an odd function.]

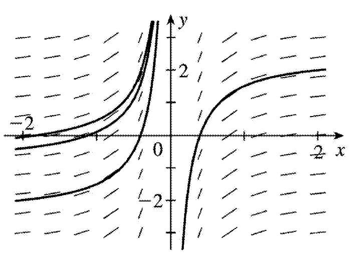


57.



Remember that the given table values of f are the slopes of F at any x . For example, at $x=1.4$, the slope of F is $f(1.4)=0$.

58. **(a)**



(b) The general antiderivative of $f(x)=x^{-2}$ is $F(x)=\begin{cases} -1/x+C_1 & \text{if } x<0 \\ -1/x+C_2 & \text{if } x>0 \end{cases}$ since $f(x)$ is not defined at $x=0$. The graph of the general antiderivatives of $f(x)$ looks like the graph in part (a), as expected.

59. $v(t)=s'(t)=\sin t-\cos t \Rightarrow s(t)=-\cos t-\sin t+C$. $s(0)=-1+C$ and $s(0)=0 \Rightarrow C=1$, so $s(t)=-\cos t-\sin t+1$.

60. $v(t)=s'(t)=1.5\sqrt{t} \Rightarrow s(t)=t^{3/2}+C$. $s(4)=8+C$ and $s(4)=10 \Rightarrow C=2$, so $s(t)=t^{3/2}+2$.

61. $a(t)=v'(t)=t-2 \Rightarrow v(t)=\frac{1}{2}t^2-2t+C$. $v(0)=C$ and $v(0)=3 \Rightarrow C=3$, so $v(t)=\frac{1}{2}t^2-2t+3$ and $s(t)=\frac{1}{6}t^3-t^2+3t+D$. $s(0)=D$ and $s(0)=1 \Rightarrow D=1$, and $s(t)=\frac{1}{6}t^3-t^2+3t+1$.

62. $a(t)=v'(t)=\cos t+\sin t \Rightarrow v(t)=\sin t-\cos t+C \Rightarrow 5=v(0)=-1+C \Rightarrow C=6$, so $v(t)=\sin t-\cos t+6 \Rightarrow$ $s(t)=-\cos t-\sin t+6t+D \Rightarrow 0=s(0)=-1+D \Rightarrow D=1$, so $s(t)=-\cos t-\sin t+6t+1$.

63. $a(t)=v'(t)=10\sin t+3\cos t \Rightarrow v(t)=-10\cos t+3\sin t+C \Rightarrow s(t)=-10\sin t-3\cos t+Ct+D$. $s(0)=-3+D=0$ and $s(2\pi)=-3+2\pi C+D=12 \Rightarrow D=3$ and $C=\frac{6}{\pi}$. Thus, $s(t)=-10\sin t-3\cos t+\frac{6}{\pi}t+3$.

64. $a(t)=v'(t)=10+3t-3t^2 \Rightarrow v(t)=10t+\frac{3}{2}t^2-t^3+C \Rightarrow s(t)=5t^2+\frac{1}{2}t^3-\frac{1}{4}t^4+Ct+D \Rightarrow 0=s(0)=D$ and $10=s(2)=20+4-4+2C \Rightarrow C=-5$, so $s(t)=-5t+5t^2+\frac{1}{2}t^3-\frac{1}{4}t^4$.

65. **(a)** We first observe that since the stone is dropped 450 m above the ground, $v(0)=0$ and $s(0)=450$.
$v'(t)=a(t)=-9.8 \Rightarrow v(t)=-9.8t+C$. Now $v(0)=0 \Rightarrow C=0$, so $v(t)=-9.8t \Rightarrow s(t)=-4.9t^2+D$. Last, $s(0)=450$ $\Rightarrow D=450 \Rightarrow s(t)=450-4.9t^2$.
(b) The stone reaches the ground when $s(t)=0$. $450-4.9t^2=0 \Rightarrow t^2=450/4.9 \Rightarrow t_1=\sqrt{450/4.9} \approx 9.58$ s.
(c) The velocity with which the stone strikes the ground is $v\left(t_1\right)=-9.8\sqrt{450/4.9} \approx -93.9$ m / s.
(d) This is just reworking parts (a) and (b) with $v(0)=-5$. Using $v(t)=-9.8t+C$, $v(0)=-5 \Rightarrow 0+C=-5 \Rightarrow$ $v(t)=-9.8t-5$. So $s(t)=-4.9t^2-5t+D$ and $s(0)=450 \Rightarrow D=450 \Rightarrow s(t)=-4.9t^2-5t+450$. Solving $s(t)=0$ by using the quadratic formula gives us $t=\left(5 \pm \sqrt{8845}\right) / (-9.8) \Rightarrow t_1 \approx 9.09$ s.

66. $v'(t)=a(t)=a \Rightarrow v(t)=at+C$ and $v_0=v(0)=C \Rightarrow v(t)=at+v_0 \Rightarrow$
$s(t)=\frac{1}{2}at^2+v_0t+D \Rightarrow s_0=s(0)=D \Rightarrow$ $s(t)=\frac{1}{2}at^2+v_0t+s_0$

67. By Exercise with $a=-9.8$, $s(t)=-4.9t^2+v_0t+s_0$ and $v(t)=s'(t)=-9.8t+v_0$. So $[v(t)]^2=\left(-9.8t+v_0\right)^2=(9.8)^2t^2-19.6v_0t+v_0^2=v_0^2+96.04t^2-19.6v_0t=v_0^2-19.6\left(-4.9t^2+v_0t\right)$. But $-4.9t^2+v_0t$ is just $s(t)$ without the s_0 term; that is, $s(t)-s_0$. Thus, $[v(t)]^2=v_0^2-19.6\left[s(t)-s_0\right]$.

68. For the first ball, $s_1(t)=-16t^2+48t+432$ from Example 8. For the second ball, $a(t)=-32 \Rightarrow$ $v(t)=-32t+C$, but $v(1)=-32(1)+C=24 \Rightarrow C=56$, so $v(t)=-32t+56 \Rightarrow s(t)=-16t^2+56t+D$, but $s(1)=-16(1)^2+56(1)+D=432 \Rightarrow D=392$, and $s_2(t)=-16t^2+56t+392$. The balls pass each other when $s_1(t)=s_2(t) \Rightarrow -16t^2+48t+432=-16t^2+56t+392 \Leftrightarrow 8t=40 \Leftrightarrow t=5$ s.

Another solution: From Exercise , we have $s_1(t)=-16t^2+48t+432$ and $s_2(t)=-16t^2+24t+432$. We now want to solve $s_1(t)=s_2(t-1) \Rightarrow -16t^2+48t+432=-16(t-1)^2+24(t-1)+432 \Rightarrow 48t=32t-16+24t-24 \Rightarrow 40=8t$ $\Rightarrow t=5$ s.

69. Using Exercise with $a=-32$, $v_0=0$, and $s_0=h$ (the height of the cliff), we know that the height at time t is $s(t)=-16t^2+h$. $v(t)=s'(t)=-32t$ and $v(t)=-120 \Rightarrow -32t=-120 \Rightarrow t=3.75$, so $0=s(3.75)=-16(3.75)^2+h \Rightarrow h=16(3.75)^2=225$ ft.

70. **(a)** $EIy''=mg(L-x)+\frac{1}{2}\rho g(L-x)^2 \Rightarrow EIy'=-\frac{1}{2}mg(L-x)^2-\frac{1}{6}\rho g(L-x)^3+C \Rightarrow$ $EIy=\frac{1}{6}mg(L-x)^3+\frac{1}{24}\rho g(L-x)^4+Cx+D$. Since the left end of the board is fixed, we must have $y=y'=0$ when $x=0$. Thus, $0=-\frac{1}{2}mgL^2-\frac{1}{6}\rho gL^3+C$ and $0=\frac{1}{6}mgL^3+\frac{1}{24}\rho gL^4+D$. It follows that $EIy=\frac{1}{6}mg(L-x)^3+\frac{1}{24}\rho g(L-x)^4+\left(\frac{1}{2}mgL^2+\frac{1}{6}\rho gL^3\right)x-\left(\frac{1}{6}mgL^3+\frac{1}{24}\rho gL^4\right)$ and

$$f(x)=y=\frac{1}{EI}\left[\frac{1}{6}mg(L-x)^3+\frac{1}{24}\rho g(L-x)^4+\left(\frac{1}{2}mgL^2+\frac{1}{6}\rho gL^3\right)x-\left(\frac{1}{6}mgL^3+\frac{1}{24}\rho gL^4\right)\right]$$

(b) $f(L)<0$, so the end of the board is a *distance* approximately $-f(L)$ below the horizontal. From our result in (a), we calculate

$$\begin{aligned}-f(L) &= \frac{-1}{EI}\left[\frac{1}{2}mgL^3+\frac{1}{6}\rho gL^4-\frac{1}{6}mgL^3-\frac{1}{24}\rho gL^4\right]\\ &=\frac{-1}{EI}\left(\frac{1}{3}mgL^3+\frac{1}{8}\rho gL^4\right)=-\frac{gL^3}{EI}\left(\frac{m}{3}+\frac{\rho L}{8}\right)\end{aligned}$$

Note: This is positive because g is negative.

71. Marginal cost $=1.92-0.002x=C'(x) \Rightarrow C(x)=1.92x-0.001x^2+K$. But $C(1)=1.92-0.001+K=562 \Rightarrow$ $K=560.081$. Therefore, $C(x)=1.92x-0.001x^2+560.081 \Rightarrow C(100)=742.081$, so the cost of producing

100 items is \$742.08 .

72. Let the mass, measured from one end, be $m(x)$. Then $m(0)=0$ and $\rho=\frac{dm}{dx}=x^{-1/2}\Rightarrow m(x)=2x^{1/2}+C$ and $m(0)=C=0$, so $m(x)=2\sqrt{x}$. Thus, the mass of the 100–centimeter rod is $m(100)=2\sqrt{100}=20$ g.

73. Taking the upward direction to be positive we have that for $0\le t\le 10$ (using the subscript 1 to refer to $0\le t\le 10$), $a_1(t)=-(9-0.9t)=v_1'(t)\Rightarrow v_1(t)=-9t+0.45t^2+v_0$, but $v_1(0)=v_0=-10\Rightarrow$ $v_1(t)=-9t+0.45t^2-10=s_1'(t)\Rightarrow s_1(t)=-\frac{9}{2}t^2+0.15t^3-10t+s_0$. But $s_1(0)=500=s_0\Rightarrow$ $s_1(t)=-\frac{9}{2}t^2+0.15t^3-10t+500$. $s_1(10)=-450+150-100+500=100$, so it takes more than 10 seconds for the raindrop to fall. Now for $t>10$, $a(t)=0=v'(t)\Rightarrow v(t)=$ constant $=v_1(10)=-9(10)+0.45(10)^2-10=-55$ $\Rightarrow v(t)=-55$. At 55 ft / s, it will take $100/55\approx 1.8$ s to fall the last 100 ft. Hence, the total time is $10+\frac{100}{55}=\frac{130}{11}\approx 11.8$ s.

74. $v'(t)=a(t)=-22$. The initial velocity is 50 mi / h $=\frac{50\cdot 5280}{3600}=\frac{220}{3}$ ft / s, so $v(t)=-22t+\frac{220}{3}$. The car stops when $v(t)=0\Leftrightarrow t=\frac{220}{3\cdot 22}=\frac{10}{3}$. Since $s(t)=-11t^2+\frac{220}{3}t$, the distance covered is $s\left(\frac{10}{3}\right)=-11\left(\frac{10}{3}\right)^2+\frac{220}{3}\cdot\frac{10}{3}=\frac{1100}{9}=122.\overline{2}$ ft.

75. $a(t)=k$, the initial velocity is 30 mi / h $=30\cdot\frac{5280}{3600}=44$ ft / s, and the final velocity (after 5 seconds) is 50 mi / h $=50\cdot\frac{5280}{3600}=\frac{220}{3}$ ft / s . So $v(t)=kt+C$ and $v(0)=44\Rightarrow C=44$. Thus, $v(t)=kt+44$ $\Rightarrow v(5)=5k+44$. But $v(5)=\frac{220}{3}$, so $5k+44=\frac{220}{3}\Rightarrow 5k=\frac{88}{3}\Rightarrow k=\frac{88}{15}\approx 5.87$ ft / s ^{2} .

76. $a(t)=-16\Rightarrow v(t)=-16t+v_0$ where v_0 is the car’s speed (in ft / s) when the brakes were applied. The car stops when $-16t+v_0=0\Leftrightarrow t=\frac{1}{16}v_0$. Now $s(t)=\frac{1}{2}(-16)t^2+v_0t=-8t^2+v_0t$. The car travels 200 ft in the time that it takes to stop, so $s\left(\frac{1}{16}v_0\right)=200\Rightarrow 200=-8\left(\frac{1}{16}v_0\right)^2+v_0\left(\frac{1}{16}v_0\right)=\frac{1}{32}v_0^2\Rightarrow$ $v_0^2=32\cdot 200=6400\Rightarrow v_0=80$ ft / s ($54.\overline{54}$ mi / h).

77. Let the acceleration be $a(t)=k\ \mathrm{km/h}^2$. We have $v(0)=100$ km / h and we can take the initial position $s(0)$ to be 0 . We want the time t_f for which $v(t)=0$ to satisfy $s(t)<0.08$ km. In general, $v'(t)=a(t)=k$, so $v(t)=kt+C$, where $C=v(0)=100$. Now $s'(t)=v(t)=kt+100$, so $s(t)=\frac{1}{2}kt^2+100t+D$, where $D=s(0)=0$. Thus, $s(t)=\frac{1}{2}kt^2+100t$. Since $v(t_f)=0$, we have $kt_f+100=0$ or $t_f=-100/k$, so $s(t_f)=\frac{1}{2}k\left(-\frac{100}{k}\right)^2+100\left(-\frac{100}{k}\right)=10{,}000\left(\frac{1}{2k}-\frac{1}{k}\right)=-\frac{5{,}000}{k}$. The condition $s(t_f)$ must satisfy is $-\frac{5{,}000}{k}<0.08\Rightarrow-\frac{5{,}000}{0.08}>k$ [k is negative] $\Rightarrow k<-62{,}500\ \mathrm{km/h}^2$, or equivalently, $k<-\frac{3125}{648}\approx-4.82\ \mathrm{m/s}^2$.

78. **(a)** For $0\le t\le 3$ we have $a(t)=60t\Rightarrow v(t)=30t^2+C\Rightarrow v(0)=0=C\Rightarrow v(t)=30t^2$, so $s(t)=10t^3+C\Rightarrow s(0)=0=C\Rightarrow s(t)=10t^3$. Note that $v(3)=270$ and $s(3)=270$.

For $3<t\le 17$: $a(t)=-g=-32$ ft / s $\Rightarrow v(t)=-32(t-3)+C\Rightarrow v(3)=270=C\Rightarrow v(t)=-32(t-3)+270\Rightarrow s(t)=-16(t-3)^2+270(t-3)+C\Rightarrow s(3)=270=C\Rightarrow s(t)=-16(t-3)^2+270(t-3)+270$. Note that $v(17)=-178$ and $s(17)=914$.

For $17<t\le 22$: The velocity increases linearly from -178 ft / s to -18 ft / s during this period, so $\frac{\Delta v}{\Delta t}=\frac{-18-(-178)}{22-17}=\frac{160}{5}=32$. Thus, $v(t)=32(t-17)-178\Rightarrow s(t)=16(t-17)^2-178(t-17)+914$ and $s(22)=424$ ft.

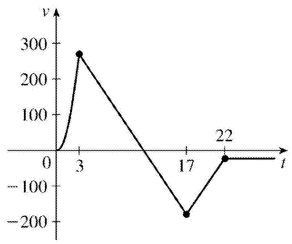
For $t>22$: $v(t)=-18\Rightarrow s(t)=-18(t-22)+C$. But $s(22)=424=C\Rightarrow s(t)=-18(t-22)+424$.

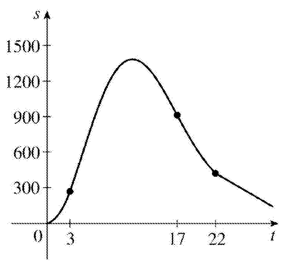
Therefore, until the rocket lands, we have

$$v(t)=\begin{cases} 30t^2 & \text{if } 0\le t\le 3\\ -32(t-3)+270 & \text{if } 3<t\le 17\\ 32(t-17)-178 & \text{if } 17<t\le 22\\ -18 & \text{if } t>22\end{cases}$$

and

$$s(t)=\begin{cases} 10t^3 & \text{if } 0\le t\le 3\\ -16(t-3)^2+270(t-3)+270 & \text{if } 3<t\le 17\\ 16(t-17)^2-178(t-17)+914 & \text{if } 17<t\le 22\\ -18(t-22)+424 & \text{if } t>22\end{cases}$$





(b) To find the maximum height, set $v(t)$ on $3<t\leq 17$ equal to 0 . $-32(t-3)+270=0\Rightarrow t_1=11.4375$ s and the maximum height is $s\left(t_1\right)=-16\left(t_1-3\right)^2+270\left(t_1-3\right)+270=1409.0625$ ft.

(c) To find the time to land, set $s(t)=-18(t-22)+424=0$. Then $t-22=\dfrac{424}{18}=23.\overline{5}$, so $t\approx 45.6$ s.

79. **(a)** First note that 90 mi / h $=90\times\dfrac{5280}{3600}$ ft / s =132 ft / s. Then $a(t)=4$ ft / s $^2\Rightarrow v(t)=4t+C$, but $v(0)=0\Rightarrow C=0$. Now $4t=132$ when $t=\dfrac{132}{4}=33$ s, so it takes 33 s to reach 132 ft / s. Therefore, taking $s(0)=0$, we have $s(t)=2t^2$, $0\leq t\leq 33$. So $s(33)=2178$ ft. 15 minutes =15(60)=900 s, so for $33<t\leq 933$ we have $v(t)=132$ ft / s $\Rightarrow s(933)=132(900)+2178=120,978$ ft =22.9125 mi.
(b) As in part (a), the train accelerates for 33 s and travels 2178 ft while doing so. Similarly, it decelerates for 33 s and travels 2178 ft at the end of its trip. During the remaining 900−66=834 s it travels at 132 ft / s, so the distance traveled is $132\cdot 834=110,088$ ft. Thus, the total distance is 2178+110 , 088+2178=114 , 444 ft =21.675 mi.
(c) 45 mi =45(5280)=237 , 600 ft. Subtract 2(2178) to take care of the speeding up and slowing down, and we have 233 , 244 ft at 132 ft / s for a trip of 233 , 244/132=1767 s at 90 mi / h. The total time is 1767+2(33)=1833s=30 min 33s=30.55 min.
(d) 37.5(60)=2250 s. 2250−2(33)=2184 s at maximum speed. 2184(132)+2(2178)=292 , 644 total feet or 292 , 644/5280=55.425 mi.

1. $f(x)=x^2-4x+1$, $[0,4]$. Since f is a polynomial, it is continuous and differentiable on R , so it is continuous on $[0,4]$ and differentiable on $(0,4)$. Also, $f(0)=1=f(4)$. $f'(c)=0 \Leftrightarrow 2c-4=0 \Leftrightarrow c=2$, which is in the open interval $(0,4)$, so $c=2$ satisfies the conclusion of Rolle's Theorem.

2. $f(x)=x^3-3x^2+2x+5$, $[0,2]$. f is continuous on $[0,2]$ and differentiable on $(0,2)$. Also, $f(0)=5=f(2)$. $f'(c)=0 \Leftrightarrow 3c^2-6c+2=0 \Leftrightarrow c=\dfrac{6\pm\sqrt{36-24}}{6}=1\pm\dfrac{1}{3}\sqrt{3}$, both in $(0,2)$.

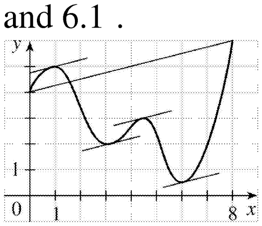
3. $f(x)=\sin 2\pi x$, $[-1,1]$. f , being the composite of the sine function and the polynomial $2\pi x$, is continuous and differentiable on R , so it is continuous on $[-1,1]$ and differentiable on $(-1,1)$. Also, $f(-1)=0=f(1)$. $f'(c)=0 \Leftrightarrow 2\pi \cos 2\pi c=0 \Leftrightarrow \cos 2\pi c=0 \Leftrightarrow 2\pi c=\pm\dfrac{\pi}{2}+2\pi n \Leftrightarrow c=\pm\dfrac{1}{4}+n$. If $n=0$ or ± 1 , then $c=\pm\dfrac{1}{4}$, $\pm\dfrac{3}{4}$ is in $(-1,1)$.

4. $f(x)=x\sqrt{x+6}$, $[-6,0]$. f is continuous on its domain, $[-6,\infty)$, and differentiable on $(-6,\infty)$, so it is continuous on $[-6,0]$ and differentiable on $(-6,0)$. Also, $f(-6)=0=f(0)$. $f'(c)=0 \Leftrightarrow \dfrac{3c+12}{2\sqrt{c+6}}=0 \Leftrightarrow c=-4$, which is in $(-6,0)$.

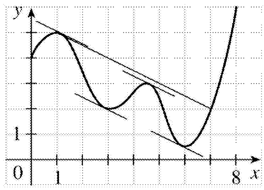
5. $f(x)=1-x^{2/3}$. $f(-1)=1-(-1)^{2/3}=1-1=0=f(1)$. $f'(x)=-\dfrac{2}{3}x^{-1/3}$, so $f'(c)=0$ has no solution. This does not contradict Rolle's Theorem, since $f'(0)$ does not exist, and so f is not differentiable on $(-1,1)$.

6. $f(x)=(x-1)^{-2}$. $f(0)=(0-1)^{-2}=1=(2-1)^{-2}=f(2)$. $f'(x)=-2(x-1)^{-3} \Rightarrow f'(x)$ is never 0 . This does not contradict Rolle's Theorem since $f'(1)$ does not exist.

7. $\dfrac{f(8)-f(0)}{8-0}=\dfrac{6-4}{8}=\dfrac{1}{4}$. The values of c which satisfy $f'(c)=\dfrac{1}{4}$ seem to be about $c=0.8$, 3.2 , 4.4 , and 6.1 .

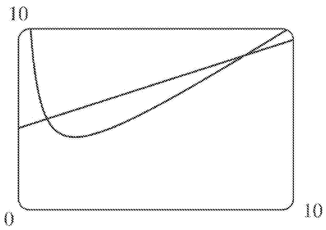


8. $\dfrac{f(7)-f(1)}{7-1}=\dfrac{2-5}{6}=-\dfrac{1}{2}$. The values of c which satisfy $f'(c)=-\dfrac{1}{2}$ seem to be about $c=1.1$, 2.8 , 4.6 , and 5.8 .



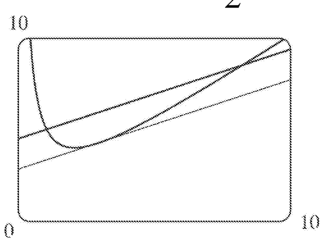
9.

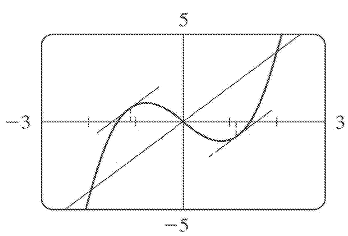
(a), (b) The equation of the secant line is $y-5=\frac{8.5-5}{8-1}(x-1)\Leftrightarrow y=\frac{1}{2}x+\frac{9}{2}$.



(c) $f(x)=x+4/x\Rightarrow f'(x)=1-4/x^2$.

So $f'(c)=\frac{1}{2}\Rightarrow c^2=8\Rightarrow c=2\sqrt{2}$, and $f(c)=2\sqrt{2}+\frac{4}{2\sqrt{2}}=3\sqrt{2}$. Thus, an equation of the tangent line is $y-3\sqrt{2}=\frac{1}{2}\left(x-2\sqrt{2}\right)\Leftrightarrow y=\frac{1}{2}x+2\sqrt{2}$.





10. **(a)**
It seems that the tangent lines are parallel to the secant at $x\approx\pm 1.2$.

(b) The slope of the secant line is 2 , and its equation is $y=2x$. $f(x)=x^3-2x\Rightarrow f'(x)=3x^2-2$, so we solve $f'(c)=2\Rightarrow 3c^2=4\Rightarrow c=\pm\frac{2\sqrt{3}}{3}\approx 1.155$. Our estimates were off by about 0.045 in each case.

11. $f(x)=3x^2+2x+5$, $[-1,1]$. f is continuous on $[-1,1]$ and differentiable on $(-1,1)$ since polynomials are continuous and differentiable on R . $f'(c)=\frac{f(b)-f(a)}{b-a}\Leftrightarrow$

$6c+2=\frac{f(1)-f(-1)}{1-(-1)}=\frac{10-6}{2}=2 \Leftrightarrow 6c=0 \Leftrightarrow c=0$, which is in $(-1,1)$.

12. $f(x)=x^3+x-1$, $[0,2]$. f is continuous on $[0,2]$ and differentiable on $(0,2)$. $f'(c)=\frac{f(2)-f(0)}{2-0} \Leftrightarrow$ $3c^2+1=\frac{9-(-1)}{2} \Leftrightarrow 3c^2=5-1 \Leftrightarrow c^2=\frac{4}{3} \Leftrightarrow c=\pm\frac{2}{\sqrt{3}}$, but only $\frac{2}{\sqrt{3}}$ is in $(0,2)$.

13. $f(x)=e^{-2x}$, $[0,3]$. f is continuous and differentiable on R , so it is continuous on $[0,3]$ and differentiable on $(0,3)$. $f'(c)=\frac{f(b)-f(a)}{b-a} \Leftrightarrow -2e^{-2c}=\frac{e^{-6}-e^{0}}{3-0} \Leftrightarrow e^{-2c}=\frac{1-e^{-6}}{6} \Leftrightarrow -2c=\ln\left(\frac{1-e^{-6}}{6}\right) \Leftrightarrow$ $c=-\frac{1}{2}\ln\left(\frac{1-e^{-6}}{6}\right) \approx 0.897$, which is in $(0,3)$.

14. $f(x)=\frac{x}{x+2}$, $[1,4]$. f is continuous on $[1,4]$ and differentiable on $(1,4)$. $f'(c)=\frac{f(b)-f(a)}{b-a} \Leftrightarrow$ $\frac{2}{(c+2)^2}=\frac{\frac{2}{3}-\frac{1}{3}}{4-1} \Leftrightarrow (c+2)^2=18 \Leftrightarrow c=-2\pm 3\sqrt{2}$. $-2+3\sqrt{2} \approx 2.24$ is in $(1,4)$.

15. $f(x)=|x-1|$. $f(3)-f(0)=|3-1|-|0-1|=1$. Since $f'(c)=-1$ if $c<1$ and $f'(c)=1$ if $c>1$, $f'(c)(3-0)=\pm 3$ and so is never equal to 1 . This does not contradict the Mean Value Theorem since $f'(1)$ does not exist.

16. $f(x)=\frac{x+1}{x-1}$. $f(2)-f(0)=3-(-1)=4$. $f'(x)=\frac{1(x-1)-1(x+1)}{(x-1)^2}=\frac{-2}{(x-1)^2}$. Since $f'(x)<0$ for all x (except $x=1$), $f'(c)(2-0)$ is always <0 and hence cannot equal 4 . This does not contradict the Mean Value Theorem since f is not continuous at $x=1$.

17. Let $f(x)=1+2x+x^3+4x^5$. Then $f(-1)=-6<0$ and $f(0)=1>0$. Since f is a polynomial, it is continuous, so the Intermediate Value Theorem says that there is a number c between -1 and 0 such that $f(c)=0$. Thus, the given equation has a real root. Suppose the equation has distinct real roots a and b with $a<b$. Then $f(a)=f(b)=0$. Since f is a polynomial, it is differentiable on (a,b) and continuous on $[a,b]$. By Rolle's Theorem, there is a number r in (a,b) such that $f'(r)=0$. But $f'(x)=2+3x^2+20x^4 \geq 2$ for all x , so $f'(x)$ can never be 0 . This contradiction shows that the equation can't have two distinct real roots. Hence, it has exactly one real root.

18. Let $f(x)=2x-1-\sin x$. Then $f(0)=-1<0$ and $f(\pi/2)=\pi-2>0$. f is the sum of the polynomial $2x-1$ and the scalar multiple $(-1)\cdot\sin x$ of the trigonometric function $\sin x$, so f is continuous (and differentiable) for all x . By the Intermediate Value Theorem, there is a number c in $(0,\pi/2)$ such that $f(c)=0$. Thus, the given equation has at least one real root. If the equation has distinct real roots a and b with $a<b$, then $f(a)=f(b)=0$. Since f is continuous on $[a,b]$ and differentiable on (a,b) , Rolle's Theorem implies that there is a number r in (a,b) such that $f'(r)=0$. But $f'(r)=2-\cos r>0$ since $\cos r\le 1$. This contradiction shows that the given equation can't have two distinct real roots, so it has exactly one real root.

19. Let $f(x)=x^3-15x+c$ for x in $[-2,2]$. If f has two real roots a and b in $[-2,2]$, with $a<b$, then $f(a)=f(b)=0$. Since the polynomial f is continuous on $[a,b]$ and differentiable on (a,b) , Rolle's Theorem implies that there is a number r in (a,b) such that $f'(r)=0$. Now $f'(r)=3r^2-15$. Since r is in (a,b) , which is contained in $[-2,2]$, we have $|r|<2$, so $r^2<4$. It follows that $3r^2-15<3\cdot 4-15=-3<0$. This contradicts $f'(r)=0$, so the given equation can't have two real roots in $[-2,2]$. Hence, it has at most one real root in $[-2,2]$.

20. $f(x)=x^4+4x+c$. Suppose that $f(x)=0$ has three distinct real roots a , b , d where $a<b<d$. Then $f(a)=f(b)=f(d)=0$. By Rolle's Theorem there are numbers c_1 and c_2 with $a<c_1<b$ and $b<c_2<d$ and $0=f'\left(c_1\right)=f'\left(c_2\right)$, so $f'(x)=0$ must have at least two real solutions. However $0=f'(x)=4x^3+4=4\left(x^3+1\right)=4(x+1)\left(x^2-x+1\right)$ has as its only real solution $x=-1$. Thus, $f(x)$ can have at most two real roots.

21. **(a)** Suppose that a cubic polynomial $P(x)$ has roots $a_1<a_2<a_3<a_4$, so $P\left(a_1\right)=P\left(a_2\right)=P\left(a_3\right)=P\left(a_4\right)$. By Rolle's Theorem there are numbers c_1 , c_2 , c_3 with $a_1<c_1<a_2$, $a_2<c_2<a_3$ and $a_3<c_3<a_4$ and $P'\left(c_1\right)=P'\left(c_2\right)=P'\left(c_3\right)=0$. Thus, the second-degree polynomial $P'(x)$ has three distinct real roots, which is impossible.

(b) We prove by induction that a polynomial of degree n has at most n real roots. This is certainly true for $n=1$. Suppose that the result is true for all polynomials of degree n and let $P(x)$ be a polynomial of degree $n+1$. Suppose that $P(x)$ has more than $n+1$ real roots, say $a_1<a_2<a_3<\cdots<a_{n+1}<a_{n+2}$. Then $P\left(a_1\right)=P\left(a_2\right)=\cdots=P\left(a_{n+2}\right)=0$. By Rolle's Theorem there are real numbers $c_1,\cdots,c_{n+1}$ with $a_1<c_1<a_2,\cdots,a_{n+1}<c_{n+1}<a_{n+2}$ and $P'\left(c_1\right)=\cdots=P'\left(c_{n+1}\right)=0$ Thus, the n th degree polynomial $P'(x)$ has at least $n+1$ roots. This contradiction shows that $P(x)$ has at most $n+1$ real roots.

22. **(a)** Suppose that $f(a)=f(b)=0$ where $a<b$. By Rolle's Theorem applied to f on $[a,b]$ there is a number c such that $a<c<b$ and $f'(c)=0$.
(b) Suppose that $f(a)=f(b)=f(c)=0$ where $a<b<c$. By Rolle's Theorem applied to $f(x)$ on $[a,b]$ and $[b,c]$ there are numbers $a<d<b$ and $b<e<c$ with $f'(d)=0$ and $f'(e)=0$. By Rolle's Theorem applied to $f'(x)$ on $[d,e]$ there is a number g with $d<g<e$ such that $f''(g)=0$.
(c) Suppose that f is n times differentiable on R and has $n+1$ distinct real roots. Then $f^{(n)}$ has at least one real root.

23. By the Mean Value Theorem, $f(4)-f(1)=f'(c)(4-1)$ for some $c\in(1,4)$. But for every $c\in(1,4)$ we have $f'(c)\geq 2$. Putting $f'(c)\geq 2$ into the above equation and substituting $f(1)=10$, we get $f(4)=f(1)+f'(c)(4-1)=10+3f'(c)\geq 10+3\cdot 2=16$. So the smallest possible value of $f(4)$ is 16 .

24. If $3\leq f'(x)\leq 5$ for all x , then by the Mean Value Theorem, $f(8)-f(2)=f'(c)\cdot(8-2)$ for some c in $[2,8]$.
(f is differentiable for all x , so, in particular, f is differentiable on $(2,8)$ and continuous on $[2,8]$. Thus, the hypotheses of the Mean Value Theorem are satisfied.) Since $f(8)-f(2)=6f'(c)$ and $3\leq f'(c)\leq 5$, it follows that $6\cdot 3\leq 6f'(c)\leq 6\cdot 5\Rightarrow 18\leq f(8)-f(2)\leq 30$.

25. Suppose that such a function f exists. By the Mean Value Theorem there is a number $0<c<2$ with $f'(c)=\dfrac{f(2)-f(0)}{2-0}=\dfrac{5}{2}$. But this is impossible since $f'(x)\leq 2<\dfrac{5}{2}$ for all x , so no such function can exist.

26. Let $h=f-g$. Then since f and g are continuous on $[a,b]$ and differentiable on (a,b) , so is h , and thus h satisfies the assumptions of the Mean Value Theorem. Therefore, there is a number c with $a<c<b$ such that $h(b)=h(b)-h(a)=h'(c)(b-a)$. Since $h'(c)<0$, $h'(c)(b-a)<0$, so $f(b)-g(b)=h(b)<0$ and hence $f(b)<g(b)$.

27. We use Exercise 26 with $f(x)=\sqrt{1+x}$, $g(x)=1+\frac{1}{2}x$, and $a=0$. Notice that $f(0)=1=g(0)$ and $f'(x)=\dfrac{1}{2\sqrt{1+x}}<\dfrac{1}{2}=g'(x)$ for $x>0$. So by Exercise 26, $f(b)<g(b)\Rightarrow\sqrt{1+b}<1+\frac{1}{2}b$ for $b>0$.
Another method: Apply the Mean Value Theorem directly to either $f(x)=1+\frac{1}{2}x-\sqrt{1+x}$ or $g(x)=\sqrt{1+x}$ on $[0,b]$.

28. f satisfies the conditions for the Mean Value Theorem, so we use this theorem on the interval $[-b,b]$: $\frac{f(b)-f(-b)}{b-(-b)}=f'(c)$ for some $c\in(-b,b)$. But since f is odd, $f(-b)=-f(b)$. Substituting this into the above equation, we get $\frac{f(b)+f(b)}{2b}=f'(c)\Rightarrow\frac{f(b)}{b}=f'(c)$.

29. Let $f(x)=\sin x$ and let $b<a$. Then $f(x)$ is continuous on $[b,a]$ and differentiable on (b,a) . By the Mean Value Theorem, there is a number $c\in(b,a)$ with
$\sin a-\sin b=f(a)-f(b)=f'(c)(a-b)=(\cos c)(a-b)$. Thus, $|\sin a-\sin b|\le|\cos c|\,|b-a|\le|a-b|$. If $a<b$, then $|\sin a-\sin b|=|\sin b-\sin a|\le|b-a|=|a-b|$. If $a=b$, both sides of the inequality are 0 .

30. Suppose that $f'(x)=c$. Let $g(x)=cx$, so $g'(x)=c$. Then, by Corollary 7, $f(x)=g(x)+d$, where d is a constant, so $f(x)=cx+d$.

31. For $x>0$, $f(x)=g(x)$, so $f'(x)=g'(x)$. For $x<0$, $f'(x)=(1/x)'=-1/x^2$ and $g'(x)=(1+1/x)'=-1/x^2$, so again $f'(x)=g'(x)$. However, the domain of $g(x)$ is not an interval so we cannot conclude that $f-g$ is constant (in fact it is not).

32. Let $f(x)=2\sin^{-1}x-\cos^{-1}\left(1-2x^2\right)$. Then
$f'(x)=\frac{2}{\sqrt{1-x^2}}-\frac{4x}{\sqrt{1-\left(1-2x^2\right)^2}}=\frac{2}{\sqrt{1-x^2}}-\frac{4x}{2x\sqrt{1-x^2}}=0$ (since $x\ge0$). Thus, $f'(x)=0$ for all $x\in(0,1)$. Thus, $f(x)=C$ on $(0,1)$. To find C , let $x=0.5$. Thus,
$2\sin^{-1}(0.5)-\cos^{-1}(0.5)=2\left(\frac{\pi}{6}\right)-\frac{\pi}{3}=0=C$. We conclude that $f(x)=0$ for x in $(0,1)$. By continuity of f , $f(x)=0$ on $[0,1]$. Therefore, we see that $f(x)=2\sin^{-1}x-\cos^{-1}\left(1-2x^2\right)=0\Rightarrow$
$2\sin^{-1}x=\cos^{-1}\left(1-2x^2\right)$.

33. Let $f(x)=\arcsin\left(\frac{x-1}{x+1}\right)-2\arctan\sqrt{x}+\frac{\pi}{2}$. Note that the domain of f is $[0,\infty)$. Thus,
$f'(x)=\frac{1}{\sqrt{1-\left(\frac{x-1}{x+1}\right)^2}}\frac{(x+1)-(x-1)}{(x+1)^2}-\frac{2}{1+x}\cdot\frac{1}{2\sqrt{x}}=\frac{1}{\sqrt{x}\,(x+1)}-\frac{1}{\sqrt{x}\,(x+1)}=0$. Then
$f(x)=C$ on $(0,\infty)$ by Theorem 5. By continuity of f , $f(x)=C$ on $[0,\infty)$. To find C , we let $x=0\Rightarrow$ $\arcsin(-1)-2\arctan(0)+\frac{\pi}{2}=C\Rightarrow-\frac{\pi}{2}-0+\frac{\pi}{2}=0=C$. Thus, $f(x)=0\Rightarrow\arcsin\left(\frac{x-1}{x+1}\right)=2\arctan\sqrt{x}-\frac{\pi}{2}$.

34. Let $v(t)$ be the velocity of the car t hours after 2:00 P.M. Then

$\frac{v(1/6)-v(0)}{1/6-0}=\frac{50-30}{1/6}=120$. By the Mean Value Theorem, there is a number c such that $0<c<\frac{1}{6}$ with $v'(c)=120$. Since $v'(t)$ is the acceleration at time t , the acceleration c hours after 2:00 P.M. is exactly 120 mi / h^2 .

35. Let $g(t)$ and $h(t)$ be the position functions of the two runners and let $f(t)=g(t)-h(t)$. By hypothesis, $f(0)=g(0)-h(0)=0$ and $f(b)=g(b)-h(b)=0$, where b is the finishing time. Then by the Mean Value Theorem, there is a time c , with $0<c<b$, such that $f'(c)=\frac{f(b)-f(0)}{b-0}$. But $f(b)=f(0)=0$, so $f'(c)=0$. Since $f'(c)=g'(c)-h'(c)=0$, we have $g'(c)=h'(c)$. So at time c , both runners have the same speed $g'(c)=h'(c)$.

36. Assume that f is differentiable (and hence continuous) on R and that $f'(x)\neq 1$ for all x . Suppose f has more than one fixed point. Then there are numbers a and b such that $a<b$, $f(a)=a$, and $f(b)=b$. Applying the Mean Value Theorem to the function f on $[a,b]$, we find that there is a number c in (a,b) such that $f'(c)=\frac{f(b)-f(a)}{b-a}$. But then $f'(c)=\frac{b-a}{b-a}=1$, contradicting our assumption that $f'(x)\neq 1$ for every real number x . This shows that our supposition was wrong, that is, that f cannot have more than one fixed point.

1. **(a)** f is increasing on $(0,6)$ and $(8,9)$.
(b) f is decreasing on $(6,8)$.
(c) f is concave upward on $(2,4)$ and $(7,9)$.
(d) f is concave downward on $(0,2)$ and $(4,7)$.
(e) The points of inflection are $(2,3)$, $(4,4.5)$ and $(7,4)$ (where the concavity changes).

2. **(a)** f is increasing on $(1,\approx 3.8)$ and $(5,\approx 6.5)$.
(b) f is decreasing on $(0,1)$, $(\approx 3.8,5)$, $(\approx 6.5,8)$, and $(8,9)$.
(c) f is concave upward on $(0,3)$ and $(8,9)$.
(d) f is concave downward on $(3,5)$ and $(5,8)$.
(e) The point of inflection is $(3,\approx 1.8)$ (where the concavity changes).

3. **(a)** Use the Increasing/Decreasing (I/D) Test.
(b) Use the Concavity Test.
(c) At any value of x where the concavity changes, we have an inflection point at $(x,f(x))$.

4. **(a)** See the First Derivative Test.
(b) See the Second Derivative Test and the note that precedes Example 7.

5. **(a)** Since $f'(x)>0$ on $(1,5)$, f is increasing on this interval. Since $f'(x)<0$ on $(0,1)$ and $(5,6)$, f is decreasing on these intervals.
(b) Since $f'(x)=0$ at $x=1$ and f' changes from negative to positive there, f changes from decreasing to increasing and has a local minimum at $x=1$. Since $f'(x)=0$ at $x=5$ and f' changes from positive to negative there, f changes from increasing to decreasing and has a local maximum at $x=5$.

6. **(a)** $f'(x)>0$ and f is increasing on $(0,1)$ and $(3,5)$. $f'(x)<0$ and f is decreasing on $(1,3)$ and $(5,6)$.
(b) Since $f'(x)=0$ at $x=1$ and $x=5$ and f' changes from positive to negative at both values, f changes from increasing to decreasing and has local maxima at $x=1$ and $x=5$. Since $f'(x)=0$ at $x=3$ and f' changes from negative to positive there, f changes from decreasing to increasing and has a local minimum at $x=3$.

7. There is an inflection point at $x=1$ because $f''(x)$ changes from negative to positive there, and so the graph of f changes from concave downward to concave upward. There is an inflection point at $x=7$ because $f''(x)$ changes from positive to negative there, and so the graph of f changes from concave upward to concave downward.

8. **(a)** f is increasing on the intervals where

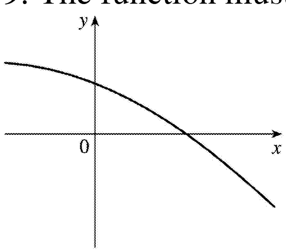
$f'(x)>0$, namely, $(2,4)$ and $(6,9)$.

(b) f has a local maximum where it changes from increasing to decreasing, that is, where f' changes from positive to negative (at $x=4$). Similarly, where f' changes from negative to positive, f has a local minimum (at $x=2$ and at $x=6$).

(c) When f' is increasing, its derivative f'' is positive and hence, f is concave upward. This happens on $(1,3)$, $(5,7)$, and $(8,9)$. Similarly, f is concave downward when f' is decreasing — that is, on $(0,1)$, $(3,5)$, and $(7,8)$.

(d) f has inflection points at $x=1$, 3 , 5 , 7 , and 8 , since the direction of concavity changes at each of these values.

9. The function must be always decreasing and concave downward.



10. **(a)** The rate of increase of the population is initially very small, then gets larger until it reaches a maximum at about $t=8$ hours, and decreases toward 0 as the population begins to level off.

(b) The rate of increase has its maximum value at $t=8$ hours.

(c) The population function is concave upward on $(0,8)$ and concave downward on $(8,18)$.

(d) At $t=8$, the population is about 350 , so the inflection point is about $(8,350)$.

11. **(a)** $f(x)=x^3-12x+1 \Rightarrow f'(x)=3x^2-12=3(x+2)(x-2)$.

We don’t need to include "3" in the chart to determine the sign of $f'(x)$.

Interval	$x+2$	$x-2$	$f'(x)$	f
$x<-2$	–	–	+	increasing on $(-\infty,-2)$
$-2<x<2$	+	–	–	decreasing on $(-2,2)$
$x>2$	+	+	+	increasing on $(2,\infty)$

So f is increasing on $(-\infty,-2)$ and $(2,\infty)$ and f is decreasing on $(-2,2)$.

(b) f changes from increasing to decreasing at $x=-2$ and from decreasing to increasing at $x=2$. Thus, $f(-2)=17$ is a local maximum value and $f(2)=-15$ is a local minimum value.

(c) $f''(x)=6x$. $f''(x)>0 \Leftrightarrow x>0$ and $f''(x)<0 \Leftrightarrow x<0$. Thus, f is concave upward on $(0,\infty)$ and concave downward on $(-\infty,0)$. There is an inflection point where the concavity changes, at $(0,f(0))=(0,1)$.

12. **(a)** $f(x)=5-3x^{2}+x^{3} \Rightarrow f'(x)=-6x+3x^{2}=3x(x-2)$. Thus, $f'(x)>0 \Leftrightarrow x<0$ or $x>2$ and $f'(x)<0 \Leftrightarrow 0<x<2$. So f is increasing on $(-\infty,0)$ and $(2,\infty)$ and f is decreasing on $(0,2)$.
(b) f changes from increasing to decreasing at $x=0$ and from decreasing to increasing at $x=2$. Thus, $f(0)=5$ is a local maximum value and $f(2)=1$ is a local minimum value.
(c) $f''(x)=-6+6x=6(x-1)$. $f''(x)>0 \Leftrightarrow x>1$ and $f''(x)<0 \Leftrightarrow x<1$. Thus, f is concave upward on $(1,\infty)$ and concave downward on $(-\infty,1)$. There is an inflection point at $(1,3)$.

13. **(a)** $f(x)=x^{4}-2x^{2}+3 \Rightarrow f'(x)=4x^{3}-4x=4x\left(x^{2}-1\right)=4x(x+1)(x-1)$.

Interval	$x+1$	x	$x-1$	$f'(x)$	f
$x<-1$	–	–	–	–	decreasing on $(-\infty,-1)$
$-1<x<0$	+	–	–	+	increasing on $(-1,0)$
$0<x<1$	+	+	–	–	decreasing on $(0,1)$
$x>1$	+	+	+	+	increasing on $(1,\infty)$

So f is increasing on $(-1,0)$ and $(1,\infty)$ and f is decreasing on $(-\infty,-1)$ and $(0,1)$.
(b) f changes from increasing to decreasing at $x=0$ and from decreasing to increasing at $x=-1$ and $x=1$. Thus, $f(0)=3$ is a local maximum value and $f(\pm 1)=2$ are local minimum values.
(c) $f''(x)=12x^{2}-4=12\left(x^{2}-\frac{1}{3}\right)=12\left(x+1/\sqrt{3}\right)\left(x-1/\sqrt{3}\right)$. $f''(x)>0 \Leftrightarrow x<-1/\sqrt{3}$ or $x>1/\sqrt{3}$ and $f''(x)<0 \Leftrightarrow -1/\sqrt{3}<x<1/\sqrt{3}$. Thus, f is concave upward on $\left(-\infty,-\sqrt{3}/3\right)$ and $\left(\sqrt{3}/3,\infty\right)$ and concave downward on $\left(-\sqrt{3}/3,\sqrt{3}/3\right)$. There are inflection points at $\left(\pm\sqrt{3}/3,\frac{22}{9}\right)$.

14. **(a)** $f(x)=\frac{x^{2}}{x^{2}+3} \Rightarrow f'(x)=\frac{\left(x^{2}+3\right)(2x)-x^{2}(2x)}{\left(x^{2}+3\right)^{2}}=\frac{6x}{\left(x^{2}+3\right)^{2}}$. The denominator is positive so the sign of $f'(x)$ is determined by the sign of x . Thus, $f'(x)>0 \Leftrightarrow x>0$ and $f'(x)<0 \Leftrightarrow x<0$. So f is increasing on $(0,\infty)$ and f is decreasing on $(-\infty,0)$.
(b) f changes from decreasing to increasing at $x=0$. Thus, $f(0)=0$ is a local minimum value.
(c)

$$f''(x)=\frac{x^{2}+3^{\;2}(6)-6x\cdot 2\left(x^{2}+3\right)(2x)}{\left[\left(x^{2}+3\right)^{2}\right]^{2}}=\frac{6\left(x^{2}+3\right)\left[x^{2}+3-4x^{2}\right]}{\left(x^{2}+3\right)^{4}}$$

$$=\frac{6\left(3-3x^2\right)}{\left(x^2+3\right)^3}=\frac{-18(x+1)(x-1)}{\left(x^2+3\right)^3}\ .$$

$f''(x)>0 \Leftrightarrow -1<x<1$ and $f''(x)<0 \Leftrightarrow x<-1$ or $x>1$. Thus, f is concave upward on $(-1,1)$ and concave downward on $(-\infty,-1)$ and $(1,\infty)$. There are inflection points at $\left(\pm 1, \frac{1}{4}\right)$.

15. **(a)** $f(x)=x-2\sin x$ on $(0,3\pi) \Rightarrow f'(x)=1-2\cos x$. $f'(x)>0 \Leftrightarrow 1-2\cos x>0 \Leftrightarrow \cos x<\frac{1}{2} \Leftrightarrow \frac{\pi}{3}<x<\frac{5\pi}{3}$ or $\frac{7\pi}{3}<x<3\pi$. $f'(x)<0 \Leftrightarrow \cos x>\frac{1}{2} \Leftrightarrow 0<x<\frac{\pi}{3}$ or $\frac{5\pi}{3}<x<\frac{7\pi}{3}$. So f is increasing on $\left(\frac{\pi}{3},\frac{5\pi}{3}\right)$ and $\left(\frac{7\pi}{3},3\pi\right)$, and f is decreasing on $\left(0,\frac{\pi}{3}\right)$ and $\left(\frac{5\pi}{3},\frac{7\pi}{3}\right)$.

(b) f changes from increasing to decreasing at $x=\frac{5\pi}{3}$, and from decreasing to increasing at $x=\frac{\pi}{3}$ and at $x=\frac{7\pi}{3}$. Thus, $f\left(\frac{5\pi}{3}\right)=\frac{5\pi}{3}+\sqrt{3}\approx 6.97$ is a local maximum value and $f\left(\frac{\pi}{3}\right)=\frac{\pi}{3}-\sqrt{3}\approx -0.68$ and $f\left(\frac{7\pi}{3}\right)=\frac{7\pi}{3}-\sqrt{3}\approx 5.60$ are local minimum values.

(c) $f''(x)=2\sin x>0 \Leftrightarrow 0<x<\pi$ and $2\pi<x<3\pi$, $f''(x)<0 \Leftrightarrow \pi<x<2\pi$. Thus, f is concave upward on $(0,\pi)$ and $(2\pi,3\pi)$, and f is concave downward on $(\pi,2\pi)$. There are inflection points at (π,π) and $(2\pi,2\pi)$.

16. **(a)** $f(x)=\cos^2 x-2\sin x$, $0\le x\le 2\pi$. $f'(x)=-2\cos x\sin x-2\cos x=-2\cos x(1+\sin x)$. Note that $1+\sin x\ge 0$ [since $\sin x\ge -1$], with equality $\Leftrightarrow \sin x=-1 \Leftrightarrow x=3\pi/2$ [since $0\le x\le 2\pi$] $\Rightarrow \cos x=0$. Thus, $f'(x)>0 \Leftrightarrow \cos x<0 \Leftrightarrow \pi/2<x<3\pi/2$ and $f'(x)<0 \Leftrightarrow \cos x>0 \Leftrightarrow 0<x<\pi/2$ or $3\pi/2<x<2\pi$. Thus, f is increasing on $(\pi/2,3\pi/2)$ and f is decreasing on $(0,\pi/2)$ and $(3\pi/2,2\pi)$.

(b) f changes from decreasing to increasing at $x=\pi/2$ and from increasing to decreasing at $x=3\pi/2$. Thus, $f(\pi/2)=-2$ is a local minimum value and $f(3\pi/2)=2$ is a local maximum value.

(c)

$$\begin{aligned} f''(x) &=2\sin x(1+\sin x)-2\cos^2 x=2\sin x+2\sin^2 x-2\left(1-\sin^2 x\right) \\ &=4\sin^2 x+2\sin x-2=2(2\sin x-1)(\sin x+1) \end{aligned}$$

so $f''(x)>0 \Leftrightarrow \sin x>\frac{1}{2} \Leftrightarrow \frac{\pi}{6}<x<\frac{5\pi}{6}$, and $f''(x)<0 \Leftrightarrow \sin x<\frac{1}{2}$ and $\sin x\neq -1 \Leftrightarrow 0<x<\frac{\pi}{6}$ or $\frac{5\pi}{6}<x<\frac{3\pi}{2}$ or $\frac{3\pi}{2}<x<2\pi$. Thus, f is concave upward on $\left(\frac{\pi}{6},\frac{5\pi}{6}\right)$ and concave downward on

$\left(0,\frac{\pi}{6}\right)$, $\left(\frac{5\pi}{6},\frac{3\pi}{2}\right)$, and $\left(\frac{3\pi}{2},2\pi\right)$. There are inflection points at $\left(\frac{\pi}{6},-\frac{1}{4}\right)$ and $\left(\frac{5\pi}{6},-\frac{1}{4}\right)$.

17. **(a)** $y=f(x)=xe^x \Rightarrow f'(x)=xe^x+e^x=e^x(x+1)$. So $f'(x)>0 \Leftrightarrow x+1>0 \Leftrightarrow x>-1$. Thus, f is increasing on $(-1,\infty)$ and decreasing on $(-\infty,-1)$.

(b) f changes from decreasing to increasing at its only critical number, $x=-1$. Thus, $f(-1)=-e^{-1}$ is a local minimum value.

(c) $f'(x)=e^x(x+1) \Rightarrow f''(x)=e^x(1)+(x+1)e^x=e^x(x+2)$. So $f''(x)>0 \Leftrightarrow x+2>0 \Leftrightarrow x>-2$. Thus, f is concave upward on $(-2,\infty)$ and concave downward on $(-\infty,-2)$. Since the concavity changes direction at $x=-2$, the point $\left(-2,-2e^{-2}\right)$ is an inflection point.

18. **(a)** $y=f(x)=x^2e^x \Rightarrow f'(x)=x^2e^x+2xe^x=x(x+2)e^x$. So $f'(x)>0 \Leftrightarrow x(x+2)>0 \Leftrightarrow$ either $x<-2$ or $x>0$. Therefore f is increasing on $(-\infty,-2)$ and $(0,\infty)$, and decreasing on $(-2,0)$.

(b) f changes from increasing to decreasing at $x=-2$, so $f(-2)=4e^{-2}$ is a local maximum value. f changes from decreasing to increasing at $x=0$, so $f(0)=0$ is a local minimum value.

(c) $f'(x)=\left(x^2+2x\right)e^x \Rightarrow f''(x)=\left(x^2+2x\right)e^x+e^x(2x+2)=e^x\left(x^2+4x+2\right)$. $f''(x)=0 \Leftrightarrow x^2+4x+2=0 \Leftrightarrow x=-2\pm\sqrt{2}$. $f''(x)<0 \Leftrightarrow -2-\sqrt{2}<x<-2+\sqrt{2}$, so f is concave downward on $\left(-2-\sqrt{2},-2+\sqrt{2}\right)$ and concave upward on $\left(-\infty,-2-\sqrt{2}\right)$ and $\left(-2+\sqrt{2},\infty\right)$. There are inflection points at $\left(-2-\sqrt{2},\mathrm{f}\left(-2-\sqrt{2}\right)\right)\approx(-3.41,0.38)$ and $\left(-2+\sqrt{2},\mathrm{f}\left(-2+\sqrt{2}\right)\right)\approx(-0.59,0.19)$.

19. **(a)** $y=f(x)=\frac{\ln x}{\sqrt{x}}$. (Note that f is only defined for $x>0$.)

$$f'(x)=\frac{\sqrt{x}(1/x)-\ln x\left(\frac{1}{2}x^{-1/2}\right)}{x}=\frac{\frac{1}{\sqrt{x}}-\frac{\ln x}{2\sqrt{x}}}{x}\cdot\frac{2\sqrt{x}}{2\sqrt{x}}=\frac{2-\ln x}{2x^{3/2}}>0 \Leftrightarrow 2-\ln x>0 \Leftrightarrow \ln x<2 \Leftrightarrow x<e^2 .$$

Therefore f is increasing on $\left(0,\mathrm{e}^2\right)$ and decreasing on $\left(e^2,\infty\right)$.

(b) f changes from increasing to decreasing at $x=e^2$, so $f\left(e^2\right)=\frac{\ln e^2}{\sqrt{e^2}}=\frac{2}{e}$ is a local maximum value.

(c)

$$f''(x)=\frac{2x^{3/2}(-1/x)-(2-\ln x)\left(3x^{1/2}\right)}{\left(2x^{3/2}\right)^2}=\frac{-2x^{1/2}+3x^{1/2}(\ln x-2)}{4x^3}$$
$$=\frac{x^{1/2}(-2+3\ln x-6)}{4x^3}=\frac{3\ln x-8}{4x^{5/2}}$$

$f''(x)=0 \Leftrightarrow \ln x=\frac{8}{3} \Leftrightarrow x=e^{8/3}$. $f''(x)>0 \Leftrightarrow x>e^{8/3}$, so f is concave upward on $(e^{8/3},\infty)$ and concave downward on $(0,e^{8/3})$. There is an inflection point at $\left(e^{8/3},\frac{8}{3}e^{-4/3}\right)\approx(14.39,0.70)$.

20. **(a)** $y=f(x)=x\ln x$. (Note that f is only defined for $x>0$.)
$f'(x)=x(1/x)+\ln x=1+\ln x$. $f'(x)>0 \Leftrightarrow \ln x+1>0 \Leftrightarrow \ln x>-1 \Leftrightarrow x>e^{-1}$. Therefore f is increasing on $(1/e,\infty)$ and decreasing on $(0,1/e)$.
(b) f changes from decreasing to increasing at $x=1/e$, so $f(1/e)=-1/e$ is a local minimum value.
(c) $f''(x)=1/x>0$ for $x>0$. So f is concave upward on its entire domain, and has no inflection point.

21. $f(x)=x^5-5x+3 \Rightarrow f'(x)=5x^4-5=5\left(x^2+1\right)(x+1)(x-1)$.
First Derivative Test: $f'(x)<0 \Rightarrow -1<x<1$ and $f'(x)>0 \Rightarrow x>1$ or $x<-1$. Since f' changes from positive to negative at $x=-1$, $f(-1)=7$ is a local maximum value; and since f' changes from negative to positive at $x=1$, $f(1)=-1$ is a local minimum value.
Second Derivative Test: $f''(x)=20x^3$. $f'(x)=0 \Leftrightarrow x=\pm 1$. $f''(-1)=-20<0 \Rightarrow f(-1)=7$ is a local maximum value. $f''(1)=20>0 \Rightarrow f(1)=-1$ is a local minimum value.
Preference: For this function, the two tests are equally easy.

22. $f(x)=\frac{x}{x^2+4} \Rightarrow f'(x)=\frac{\left(x^2+4\right)\cdot 1-x(2x)}{x^2+4\ ^2}=\frac{4-x^2}{\frac{()}{x^2+4}\ ^2}=\frac{(2+x)(2-x)}{\left(x^2+4\right)^2}$.

First Derivative Test: $f'(x)>0 \Rightarrow -2<x<2$ and $f'(x)<0 \Rightarrow x>2$ or $x<-2$. Since f' changes from positive to negative at $x=2$, $f(2)=\frac{1}{4}$ is a local maximum value; and since f' changes from negative to positive at $x=-2$, $f(-2)=-\frac{1}{4}$ is a local minimum value. *Second Derivative Test:*

$$f''(x) = \frac{\left(x^2+4\right)^2(-2x)-\left(4-x^2\right)\cdot 2\left(x^2+4\right)(2x)}{\left[\left(x^2+4\right)^2\right]^2}$$

$$= \frac{-2x\left(x^2+4\right)\left[\left(x^2+4\right)+2\left(4-x^2\right)\right]}{\left(x^2+4\right)^4} = \frac{-2x\left(12-x^2\right)}{\left(x^2+4\right)^3}$$

$f'(x)=0 \Leftrightarrow x=\pm 2$. $f''(-2)=\frac{1}{16}>0 \Rightarrow f(-2)=-\frac{1}{4}$ is a local minimum value.

$f''(2)=-\frac{1}{16}<0 \Rightarrow f(2)=\frac{1}{4}$ is a local maximum value.

Preference: Since calculating the second derivative is fairly difficult, the First Derivative Test is easier to use for this function.

23. $f(x)=x+\sqrt{1-x} \Rightarrow f'(x)=1+\frac{1}{2}(1-x)^{-1/2}(-1)=1-\frac{1}{2\sqrt{1-x}}$. Note that f is defined for $1-x\geq 0$; that is, for $x\leq 1$. $f'(x)=0 \Rightarrow 2\sqrt{1-x}=1 \Rightarrow \sqrt{1-x}=\frac{1}{2} \Rightarrow 1-x=\frac{1}{4} \Rightarrow x=\frac{3}{4}$. f' does not exist at $x=1$, but we can't have a local maximum or minimum at an endpoint.

First Derivative Test: $f'(x)>0 \Rightarrow x<\frac{3}{4}$ and $f'(x)<0 \Rightarrow \frac{3}{4}<x<1$. Since f' changes from positive to negative at $x=\frac{3}{4}$, $f\left(\frac{3}{4}\right)=\frac{5}{4}$ is a local maximum value.

Second Derivative Test: $f''(x)=-\frac{1}{2}\left(-\frac{1}{2}\right)(1-x)^{-3/2}(-1)=-\frac{1}{4\left(\sqrt{1-x}\right)^3}$. $f''\left(\frac{3}{4}\right)=-2<0 \Rightarrow$ $f\left(\frac{3}{4}\right)=\frac{5}{4}$ is a local maximum value.

Preference: The First Derivative Test may be slightly easier to apply in this case.

24. **(a)** $f(x)=x^4(x-1)^3 \Rightarrow f'(x)=x^4\cdot 3(x-1)^2+(x-1)^3\cdot 4x^3=x^3(x-1)^2\left[3x+4(x-1)\right]=x^3(x-1)^2(7x-4)$

The critical numbers are 0 , 1 , and $\frac{4}{7}$.

(b)

$$f''(x) = 3x^2(x-1)^2(7x-4)+x^3\cdot 2(x-1)(7x-4)+x^3(x-1)^2\cdot 7$$
$$= x^2(x-1)\left[3(x-1)(7x-4)+2x(7x-4)+7x(x-1)\right]$$

Now $f''(0)=f''(1)=0$, so the Second Derivative Test gives no information for $x=0$ or $x=1$.

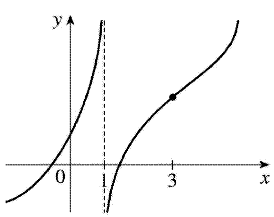
$f''\left(\frac{4}{7}\right)=\left(\frac{4}{7}\right)^2\left(\frac{4}{7}-1\right)\left[0+0+7\left(\frac{4}{7}\right)\left(\frac{4}{7}-1\right)\right]=\left(\frac{4}{7}\right)^2\left(-\frac{3}{7}\right)(4)\left(-\frac{3}{7}\right)>0$, so there is a local minimum at $x=\frac{4}{7}$.

(c) f' is positive on $(-\infty,0)$, negative on $\left(0,\frac{4}{7}\right)$, positive on $\left(\frac{4}{7},1\right)$, and positive on $(1,\infty)$. So f has a local maximum at $x=0$, a local minimum at $x=\frac{4}{7}$, and no local maximum or minimum at $x=1$.

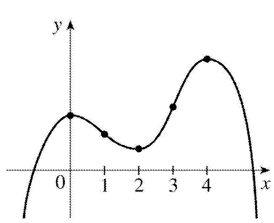
25. **(a)** By the Second Derivative Test, if $f'(2)=0$ and $f''(2)=-5<0$, f has a local maximum at $x=2$.

(b) If $f'(6)=0$, we know that f has a horizontal tangent at $x=6$. Knowing that $f''(6)=0$ does not provide any additional information since the Second Derivative Test fails. For example, the first and second derivatives of $y=(x-6)^4$, $y=-(x-6)^4$, and $y=(x-6)^3$ all equal zero for $x=6$, but the first has a local minimum at $x=6$, the second has a local maximum at $x=6$, and the third has an inflection point at $x=6$.

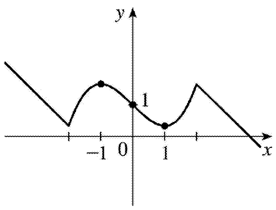
26. $f'(x)>0$ for all $x\neq 1$ with vertical asymptote $x=1$, so f is increasing on $(-\infty,1)$ and $(1,\infty)$. $f''(x)>0$ if $x<1$ or $x>3$, and $f''(x)<0$ if $1<x<3$, so f is concave upward on $(-\infty,1)$ and $(3,\infty)$, and concave downward on $(1,3)$. There is an inflection point when $x=3$.



27. $f'(0)=f'(2)=f'(4)=0\Leftrightarrow$ horizontal tangents at $x=0$, 2 , 4 . $f'(x)>0$ if $x<0$ or $2<x<4\Rightarrow f$ is increasing on $(-\infty,0)$ and $(2,4)$. $f'(x)<0$ if $0<x<2$ or $x>4\Rightarrow f$ is decreasing on $(0,2)$ and $(4,\infty)$. $f''(x)>0$ if $1<x<3\Rightarrow f$ is concave upward on $(1,3)$. $f''(x)<0$ if $x<1$ or $x>3\Rightarrow f$ is concave downward on $(-\infty,1)$ and $(3,\infty)$. There are inflection points when $x=1$ and 3 .

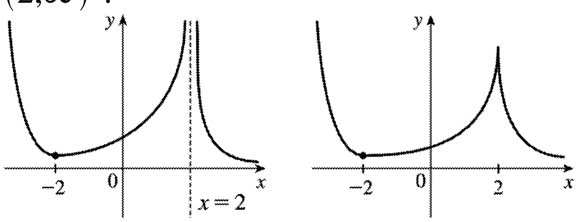


28.

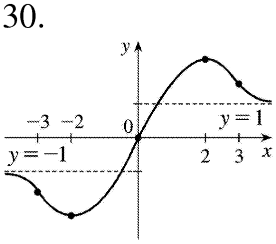


$f'(1)=f'(-1)=0 \Rightarrow$ horizontal tangents at $x=\pm 1$. $f'(x)<0$ if $|x|<1 \Rightarrow f$ is decreasing on $(-1,1)$. $f'(x)>0$ if $1<|x|<2 \Rightarrow f$ is increasing on $(-2,-1)$ and $(1,2)$. $f'(x)=-1$ if $|x|>2 \Rightarrow$ the graph of f has constant slope -1 on $(-\infty,-2)$ and $(2,\infty)$. $f''(x)<0$ if $-2<x<0 \Rightarrow f$ is concave downward on $(-2,0)$. Inflection point $(0,1)$.

29. $f'(x)>0$ if $|x|<2 \Rightarrow f$ is increasing on $(-2,2)$. $f'(x)<0$ if $|x|>2 \Rightarrow f$ is decreasing on $(-\infty,-2)$ and $(2,\infty)$. $f'(-2)=0 \Rightarrow$ horizontal tangent at $x=-2$. $\lim_{x\to 2} \left|f'(x)\right|=\infty \Rightarrow$ there is a vertical asymptote or vertical tangent (cusp) at $x=2$. $f''(x)>0$ if $x\neq 2 \Rightarrow f$ is concave upward on $(-\infty,2)$ and $(2,\infty)$.



30.



$f'(x)>0$ if $|x|<2 \Rightarrow f$ is increasing on $(-2,2)$. $f'(x)<0$ if $|x|>2 \Rightarrow f$ is decreasing on $(-\infty,-2)$ and $(2,\infty)$. $f'(2)=0$, so f has a horizontal tangent (and local maximum) at $x=2$. $\lim_{x\to\infty} f(x)=1 \Rightarrow y=1$ is a horizontal asymptote. $f(-x)=-f(x) \Rightarrow f$ is an odd function (its graph is symmetric about the origin). Finally, $f''(x)<0$ if $0<x<3$ and $f''(x)>0$ if $x>3$, so f is CD on $(0,3)$ and CU on $(3,\infty)$.

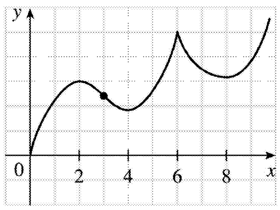
31. **(a)** f is increasing where f' is positive, that is, on $(0,2)$, $(4,6)$, and $(8,\infty)$; and decreasing where f' is negative, that is, on $(2,4)$ and $(6,8)$.

(b) f has local maxima where f' changes from positive to negative, at $x=2$ and at $x=6$, and local minima where f' changes from negative to positive, at $x=4$ and at $x=8$.

(c) f is concave upward (CU) where f' is increasing, that is, on $(3,6)$ and $(6,\infty)$, and concave downward (CD) where f' is decreasing, that is, on $(0,3)$.

(d) There is a point of in.ection where f changes from being CD to being CU, that is, at $x = 3$.

(e)



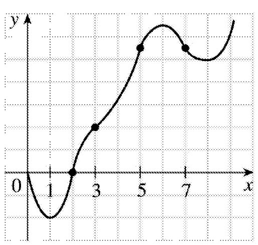
32. **(a)** f is increasing where f' is positive, on $(1,6)$ and $(8,\infty)$, and decreasing where f' is negative, on $(0,1)$ and $(6,8)$.

(b) f has a local maximum where f' changes from positive to negative, at $x=6$, and local minima where f' changes from negative to positive, at $x=1$ and at $x=8$.

(c) f is concave upward where f' is increasing, that is, on $(0\ 2)$, $(3,\ 5)$; and $(7,\ \infty)$ and concave downward where f' is decreasing, that is, on $(2\ 3)$, $(5,\ 7)$.

(d) There are points of inflection where f changes its direction of concavity, at $x=2$, $x=3$, $x=5$ and $x=7$.

(e)

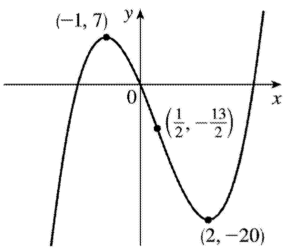


33. **(a)** $f(x)=2x^3-3x^2-12x \Rightarrow f'(x)=6x^2-6x-12=6\left(x^2-x-2\right)=6(x-2)(x+1)$. $f'(x)>0 \Leftrightarrow x<-1$ or $x>2$ and $f'(x)<0 \Leftrightarrow -1<x<2$. So f is increasing on $(-\infty,-1)$ and $(2,\infty)$, and f is decreasing on $(-1,2)$.

(b) Since f changes from increasing to decreasing at $x=-1$, $f(-1)=7$ is a local maximum value. Since f changes from decreasing to increasing at $x=2$, $f(2)=-20$ is a local minimum value.

(c) $f''(x)=6(2x-1) \Rightarrow f''(x)>0$ on $\left(\frac{1}{2},\infty\right)$ and $f''(x)<0$ on $\left(-\infty,\frac{1}{2}\right)$. So f is concave upward on $\left(\frac{1}{2},\infty\right)$ and concave downward on $\left(-\infty,\frac{1}{2}\right)$. There is a change in concavity at $x=\frac{1}{2}$, and we have an inflection point at $\left(\frac{1}{2},-\frac{13}{2}\right)$.

(d)

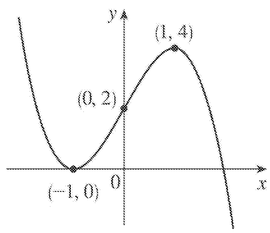


34. **(a)** $f(x)=2+3x-x^3 \Rightarrow f'(x)=3-3x^2=-3\left(x^2-1\right)=-3(x+1)(x-1)$. $f'(x)>0 \Leftrightarrow -1<x<1$ and $f'(x)<0 \Leftrightarrow x<-1$ or $x>1$. So f is increasing on $(-1,1)$ and f is decreasing on $(-\infty,-1)$ and $(1,\infty)$.

(b) $f(-1)=0$ is a local minimum value and $f(1)=4$ is a local maximum value.

(c) $f''(x)=-6x \Rightarrow f''(x)>0$ on $(-\infty,0)$ and $f''(x)<0$ on $(0,\infty)$. So f is concave upward on $(-\infty,0)$ and concave downward on $(0,\infty)$. There is an inflection point at $(0,2)$.

(d)



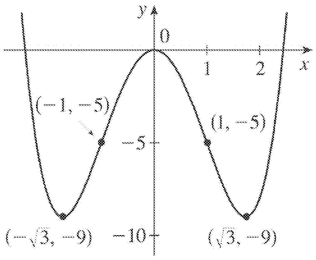
35. **(a)** $f(x)=x^4-6x^2 \Rightarrow f'(x)=4x^3-12x=4x\left(x^2-3\right)=0$ when $x=0$, $\pm\sqrt{3}$.

Interval	$4x$	x^2-3	$f'(x)$	f
$x<-\sqrt{3}$	−	+	−	decreasing on $\left(-\infty,-\sqrt{3}\right)$
$-\sqrt{3}<x<0$	−	−	+	increasing on $\left(-\sqrt{3},0\right)$
$0<x<\sqrt{3}$	+	−	−	decreasing on $\left(0,\sqrt{3}\right)$
$x>\sqrt{3}$	+	+	+	increasing on $\left(\sqrt{3},\infty\right)$

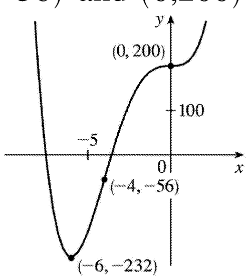
(b) Local minimum values $f\left(\pm\sqrt{3}\right)=-9$, local maximum value $f(0)=0$

(c) $f''(x)=12x^2-12=12\left(x^2-1\right)>0 \Leftrightarrow x^2>1 \Leftrightarrow |x|>1 \Leftrightarrow x>1$ or $x<-1$, so f is CU on $(-\infty,-1)$, $(1,\infty)$ and CD on $(-1,1)$. Inflection points at $(\pm 1,-5)$

(d)



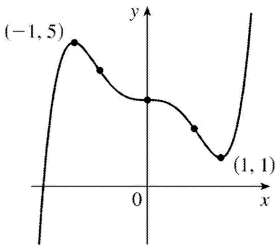
36. **(a)** $g(x)=200+8x^3+x^4 \Rightarrow g'(x)=24x^2+4x^3=4x^2(6+x)=0$ when $x=-6$ and when $x=0$. $g'(x)>0 \Leftrightarrow x>-6$ ($x\neq 0$) and $g'(x)<0 \Leftrightarrow x<-6$, so g is decreasing on $(-\infty,-6)$ and g is increasing on $(-6,\infty)$, with a horizontal tangent at $x=0$.
(b) $g(-6)=-232$ is a local minimum value. There is no local maximum value.
(c) $g''(x)=48x+12x^2=12x(4+x)=0$ when $x=-4$ and when $x=0$. $g''(x)>0 \Leftrightarrow x<-4$ or $x>0$ and $g''(x)<0 \Leftrightarrow -4<x<0$, so g is CU on $(-\infty,-4)$ and $(0,\infty)$, and g is CD on $(-4,0)$. Inflection points at $(-4,-56)$ and $(0,200)$

(d) 

37. **(a)** $h(x)=3x^5-5x^3+3 \Rightarrow h'(x)=15x^4-15x^2=15x^2\left(x^2-1\right)=0$ when $x=0$, ± 1 . Since $15x^2$ is nonnegative, $h'(x)>0 \Leftrightarrow x^2>1 \Leftrightarrow |x|>1 \Leftrightarrow x>1$ or $x<-1$, so h is increasing on $(-\infty,-1)$ and $(1,\infty)$ and decreasing on $(-1,1)$, with a horizontal tangent at $x=0$.
(b) Local maximum value $h(-1)=5$, local minimum value $h(1)=1$
(c)

$$\begin{aligned} h''(x) &= 60x^3-30x=30x\left(2x^2-1\right) \\ &= 60x\left(x+\frac{1}{\sqrt{2}}\right)\left(x-\frac{1}{\sqrt{2}}\right) \Rightarrow \end{aligned}$$

$h''(x)>0$ when $x>\frac{1}{\sqrt{2}}$ or $-\frac{1}{\sqrt{2}}<x<0$, so h is CU on $\left(-\frac{1}{\sqrt{2}},0\right)$ and $\left(\frac{1}{\sqrt{2}},\infty\right)$ and CD on $\left(-\infty,-\frac{1}{\sqrt{2}}\right)$ and $\left(0,\frac{1}{\sqrt{2}}\right)$. Inflection points at $(0,3)$ and $\left(\pm\frac{1}{\sqrt{2}},3\mp\frac{7}{8}\sqrt{2}\right)$.
(d)

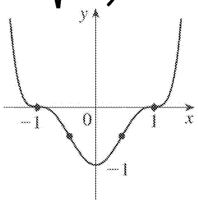


38. **(a)** $h(x)=\left(x^2-1\right)^3 \Rightarrow h'(x)=6x\left(x^2-1\right)^2 \geq 0 \Leftrightarrow x>0$ ($x\neq 1$), so h is increasing on $(0,\infty)$ and decreasing on $(-\infty,0)$.
(b) $h(0)=-1$ is a local minimum value.
(c) $h''(x)=6\left(x^2-1\right)^2+24x^2\left(x^2-1\right)=6\left(x^2-1\right)\left(5x^2-1\right)$. The roots ± 1 and $\pm\dfrac{1}{\sqrt{5}}$ divide R into five intervals.

Interval	x^2	$5x^2-1$	$h''(x)$	Concavity
$x<-1$	+	+	+	upward
$-1<x<-\dfrac{1}{\sqrt{5}}$	–	+	–	downward
$-\dfrac{1}{\sqrt{5}}<x<\dfrac{1}{\sqrt{5}}$	–	–	+	upward
$\dfrac{1}{\sqrt{5}}<x<1$	–	+	–	downward
$x>1$	+	+	+	upward

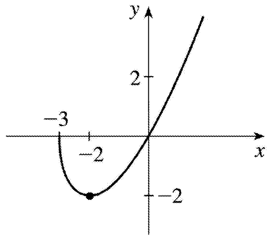
From the table, we see that h is CU on $(-\infty,-1)$, $\left(-\dfrac{1}{\sqrt{5}},\dfrac{1}{\sqrt{5}}\right)$ and $(1,\infty)$, and CD on $\left(-1,-\dfrac{1}{\sqrt{5}}\right)$ and $\left(\dfrac{1}{\sqrt{5}},1\right)$. Inflection points at $(\pm 1,0)$ and $\left(\pm\dfrac{1}{\sqrt{5}},-\dfrac{64}{125}\right)$

(d)



39. **(a)** $A(x)=x\sqrt{x+3} \Rightarrow A'(x)=x\cdot\dfrac{1}{2}(x+3)^{-1/2}+\sqrt{x+3}\cdot 1=\dfrac{x}{2\sqrt{x+3}}+\sqrt{x+3}=\dfrac{x+2(x+3)}{2\sqrt{x+3}}=\dfrac{3x+6}{2\sqrt{x+3}}$.
The domain of A is $[-3,\infty)$. $A'(x)>0$ for $x>-2$ and $A'(x)<0$ for $-3<x<-2$, so A is increasing on

$(-2,\infty)$ and decreasing on $(-3,-2)$.
(b) $A(-2)=-2$ is a local minimum value.
(c) $A''(x)=\dfrac{2\sqrt{x+3}\cdot 3-(3x+6)\cdot\dfrac{1}{\sqrt{x+3}}}{\left(2\sqrt{x+3}\right)^2}=\dfrac{6(x+3)-(3x+6)}{4(x+3)^{3/2}}=\dfrac{3x+12}{4(x+3)^{3/2}}=\dfrac{3(x+4)}{4(x+3)^{3/2}}$ $A''(x)>0$ for all $x>-3$, so A is concave upward on $(-3,\infty)$. There is no inflection point.



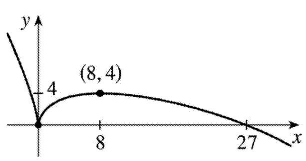
(d)

40. **(a)** $B(x)=3x^{2/3}-x\Rightarrow B'(x)=2x^{-1/3}-1=\dfrac{2}{\sqrt[3]{x}}-1=\dfrac{2-\sqrt[3]{x}}{\sqrt[3]{x}}$. $B'(x)>0$ if $0<x<8$ and $B'(x)<0$ if $x<0$ or $x>8$, so B is decreasing on $(-\infty,0)$ and $(8,\infty)$, and B is increasing on $(0,8)$.

(b) $B(0)=0$ is a local minimum value.
$B(8)=4$ is a local maximum value.

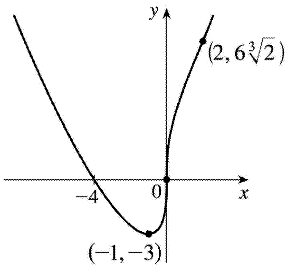
(c) $B''(x)=-\dfrac{2}{3}x^{-4/3}=\dfrac{-2}{3x^{4/3}}$, so $B''(x)<0$ for all $x\neq 0$. B is concave downward on $(-\infty,0)$ and $(0,\infty)$. There is no inflection point.

(d)



41. **(a)** $C(x)=x^{1/3}(x+4)=x^{4/3}+4x^{1/3}\Rightarrow C'(x)=\dfrac{4}{3}x^{1/3}+\dfrac{4}{3}x^{-2/3}=\dfrac{4}{3}x^{-2/3}(x+1)=\dfrac{4(x+1)}{3\sqrt[3]{x^2}}$. $C'(x)>0$ if $-1<x<0$ or $x>0$ and $C'(x)<0$ for $x<-1$, so C is increasing on $(-1,\infty)$ and C is decreasing on $(-\infty,-1)$.

(b) $C(-1)=-3$ is a local minimum value.
(c) $C''(x)=\dfrac{4}{9}x^{-2/3}-\dfrac{8}{9}x^{-5/3}=\dfrac{4}{9}x^{-5/3}(x-2)=\dfrac{4(x-2)}{9\sqrt[3]{x^5}}$. $C''(x)<0$ for $0<x<2$ and $C''(x)>0$ for $x<0$ and $x>2$, so C is concave downward on $(0,2)$ and concave upward on $(-\infty,0)$ and $(2,\infty)$. There are inflection points at $(0,0)$ and $\left(2,6\sqrt[3]{2}\right)\approx(2,7.56)$.
(d)



42. **(a)** $f(x)=\ln\left(x^4+27\right) \Rightarrow f'(x)=\dfrac{4x^3}{x^4+27}$. $f'(x)>0$ if $x>0$ and $f'(x)<0$ if $x<0$, so f is increasing on $(0,\infty)$ and f is decreasing on $(-\infty,0)$.

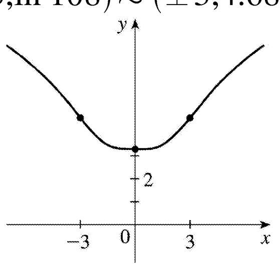
(b) $f(0)=\ln 27\approx 3.3$ is a local minimum value.

(c)

$$f''(x)=\frac{\left(x^4+27\right)\left(12x^2\right)-4x^3\left(4x^3\right)}{\left(x^4+27\right)^2}=\frac{4x^2\left[3\left(x^4+27\right)-4x^4\right]}{\left(x^4+27\right)^2}$$

$$=\frac{4x^2\left(81-x^4\right)}{\left(x^4+27\right)^2}=\frac{-4x^2\left(x^2+9\right)(x+3)(x-3)}{\left(x^4+27\right)^2}$$

$f''(x)>0$ if $-3<x<0$ and $0<x<3$, and $f''(x)<0$ if $x<-3$ or $x>3$. Thus, f is concave upward on $(-3,0)$ and $(0,3)$ and f is concave downward on $(-\infty,-3)$ and $(3,\infty)$. There are inflection points at $(\pm 3,\ln 108)\approx(\pm 3,4.68)$.



(d)

43. **(a)** $f(\theta)=2\cos\theta-\cos 2\theta$, $0\leq\theta\leq 2\pi$.

$f'(\theta)=-2\sin\theta+2\sin 2\theta=-2\sin\theta+2(2\sin\theta\cos\theta)=2\sin\theta(2\cos\theta-1)$.

Interval	$\sin\theta$	$2\cos\theta-1$	$f'(\theta)$	f
$0<\theta<\frac{\pi}{3}$	+	+	+	increasing on $(0,\frac{\pi}{3})$
$\frac{\pi}{3}<\theta<\pi$	+	–	–	decreasing on $(\frac{\pi}{3},\pi)$

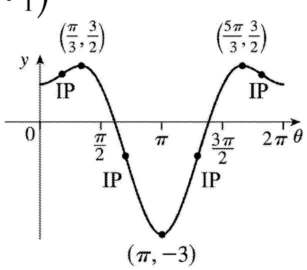
$\pi < \theta < \frac{5\pi}{3}$	–	–	+	increasing on $(\pi, \frac{5\pi}{3})$
$\frac{5\pi}{3} < \theta < 2\pi$	–	+	–	decreasing on $(\frac{5\pi}{3}, 2\pi)$

(b) $f\left(\frac{\pi}{3}\right)=\frac{3}{2}$ and $f\left(\frac{5\pi}{3}\right)=\frac{3}{2}$ are local maximum values and $f(\pi)=-3$ is a local minimum value.

(c) $f'(\theta)=-2\sin\theta+2\sin 2\theta \Rightarrow$

$$f''(\theta) = -2\cos\theta+4\cos 2\theta = -2\cos\theta+4(2\cos^2\theta-1)$$
$$=2(4\cos^2\theta-\cos\theta-2)$$

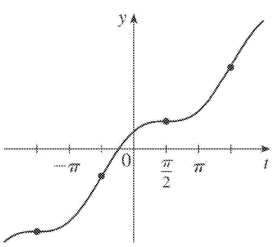
$f''(\theta)=0 \Leftrightarrow \cos\theta=\frac{1\pm\sqrt{33}}{8} \Leftrightarrow \theta=\cos^{-1}\left(\frac{1\pm\sqrt{33}}{8}\right) \Leftrightarrow \theta=\cos^{-1}\left(\frac{1+\sqrt{33}}{8}\right)\approx 0.5678$, $2\pi-\cos^{-1}\left(\frac{1+\sqrt{33}}{8}\right)\approx 5.7154$, $\cos^{-1}\left(\frac{1-\sqrt{33}}{8}\right)\approx 2.2057$, or $2\pi-\cos^{-1}\left(\frac{1-\sqrt{33}}{8}\right)\approx 4.0775$. Denote these four values of θ by θ_1, θ_4, θ_2, and θ_3, respectively. Then f is CU on $(0,\theta_1)$, CD on (θ_1,θ_2),
CU on (θ_2,θ_3), CD on (θ_3,θ_4), and CU on $(\theta_4,2\pi)$. To find the *exact* y-coordinate for $\theta=\theta_1$, we have $f(\theta_1)=2\cos\theta_1-\cos 2\theta_1=2\cos\theta_1-\left(2\cos^2\theta_1-1\right)=2\left(\frac{1+\sqrt{33}}{8}\right)-2\left(\frac{1+\sqrt{33}}{8}\right)^2+1$
$=\frac{1}{4}+\frac{1}{4}\sqrt{33}-\frac{1}{32}-\frac{1}{16}\sqrt{33}-\frac{33}{32}+1=\frac{3}{16}+\frac{3}{16}\sqrt{33}=\frac{3}{16}(1+\sqrt{33})=y_1\approx 1.26$. Similarly, $f(\theta_2)=\frac{3}{16}(1-\sqrt{33})=y_2\approx -0.89$. So f has inflection points at (θ_1,y_1), (θ_2,y_2), (θ_3,y_2), and (θ_4,y_1).

(d) 

44. **(a)** $f(t)=t+\cos t$, $-2\pi \le t \le 2\pi \Rightarrow f'(t)=1-\sin t \ge 0$ for all t and $f'(t)=0$ when $\sin t=1 \Leftrightarrow t=-\frac{3\pi}{2}$ or $\frac{\pi}{2}$, so f is increasing on $(-2\pi, 2\pi)$.

(b) No maximum or minimum

(c) $f''(t) = -\cos t > 0 \Leftrightarrow t \in \left(-\frac{3\pi}{2}, -\frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$, so f is CU on these intervals and CD 0n $\left(-2\pi, -\frac{3\pi}{2}\right)$, $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, and $\left(\frac{3\pi}{2}, 2\pi\right)$. Points of inflection at $\left(\pm\frac{3\pi}{2}, \pm\frac{3\pi}{2}\right)$ and $\left(\pm\frac{\pi}{2}, \pm\frac{\pi}{2}\right)$

(d) 

45. $f(x) = \frac{x^2}{x^2-1} = \frac{x^2}{(x+1)(x-1)}$ has domain $(-\infty,-1)\cup(-1,1)\cup(1,\infty)$.

(a) $\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} \frac{x^2/x^2}{\left(x^2-1\right)/x^2} = \lim_{x\to\pm\infty} \frac{1}{1-1/x^2} = \frac{1}{1-0} = 1$, so $y=1$ is a HA.

$\lim_{x\to-1^-} \frac{x^2}{x^2-1} = \infty$ since $x^2 \to 1$ and $\left(x^2-1\right)\to 0^+$ as $x\to -1^-$, so $x=-1$ is a VA.

$\lim_{x\to 1^+} \frac{x^2}{x^2-1} = \infty$ since $x^2 \to 1$ and $\left(x^2-1\right)\to 0^+$ as $x\to 1^+$, so $x=1$ is a VA.

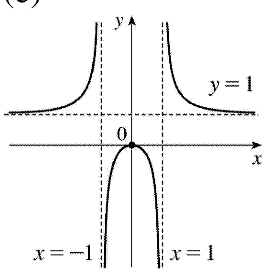
(b) $f(x) = \frac{x^2}{x^2-1} \Rightarrow f'(x) = \frac{\left(x^2-1\right)(2x)-x^2(2x)}{\left(x^2-1\right)^2} = \frac{2x\left[\left(x^2-1\right)-x^2\right]}{\left(x^2-1\right)^2} = \frac{-2x}{\left(x^2-1\right)^2}$. Since $\left(x^2-1\right)^2$ is positive for all x in the domain of f , the sign of the derivative is determined by the sign of $-2x$. Thus, $f'(x)>0$ if $x<0\,(x\neq -1)$ and $f'(x)<0$ if $x>0\,(x\neq 1)$. So f is increasing on $(-\infty,-1)$ and $(-1,0)$, and f is decreasing on $(0,1)$ and $(1,\infty)$.

(c) $f'(x)=0 \Rightarrow x=0$ and $f(0)=0$ is a local maximum value.

(d)
$$f''(x)=\frac{\left(x^2-1\right)^2(-2)-(-2x)\cdot 2\left(x^2-1\right)(2x)}{\left(\left(x^2-1\right)^2\right)^2}$$
$$=\frac{2\left(x^2-1\right)\left(-\left(x^2-1\right)+4x^2\right)}{\left(x^2-1\right)^4}=\frac{2\left(3x^2+1\right)}{\left(x^2-1\right)^3}\ .$$

The sign of $f''(x)$ is determined by the denominator; that is, $f''(x)>0$ if $|x|>1$ and $f''(x)<0$ if $|x|<1$. Thus, f is CU on $(-\infty,-1)$ and $(1,\infty)$, and f is CD on $(-1,1)$. There are no inflection points.

(e)



46. $f(x)=\dfrac{x^2}{(x-2)^2}$ has domain $(-\infty,2)\cup(2,\infty)$.

(a) $\displaystyle\lim_{x\to\pm\infty}\frac{x^2}{x^2-4x+4}=\lim_{x\to\pm\infty}\frac{x^2/x^2}{\left(x^2-4x+4\right)/x^2}=\lim_{x\to\pm\infty}\frac{1}{1-4/x+4/x^2}=\frac{1}{1-0+0}=1$,

so $y=1$ is a HA. $\displaystyle\lim_{x\to 2^+}\frac{x^2}{(x-2)^2}=\infty$ since $x^2\to 4$ and $(x-2)^2\to 0^+$ as $x\to 2^+$, so $x=2$ is a VA.

(b) $f(x)=\dfrac{x^2}{(x-2)^2}\Rightarrow f'(x)=\dfrac{(x-2)^2(2x)-x^2\cdot 2(x-2)}{\left[(x-2)^2\right]^2}=\dfrac{2x(x-2)[(x-2)-x]}{(x-2)^4}=\dfrac{-4x}{(x-2)^3}$. $f'(x)>0$ if $0<x<2$ and $f'(x)<0$ if $x<0$ or $x>2$, so f is increasing on $(0,2)$ and f is decreasing on $(-\infty,0)$ and $(2,\infty)$.

(c) $f(0)=0$ is a local minimum value.

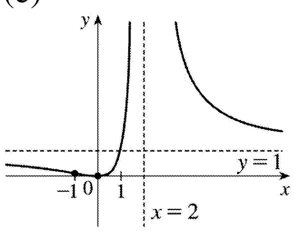
(d)

$$f''(x)=\frac{(x-2)^3(-4)-(-4x)\cdot 3(x-2)^2}{\left[(x-2)^3\right]^2}$$

$$=\frac{4(x-2)^2[-(x-2)+3x]}{(x-2)^6}=\frac{8(x+1)}{(x-2)^4}$$

$f''(x)>0$ if $x>-1$ ($x\neq 2$) and $f''(x)<0$ if $x<-1$. Thus, f is CU on $(-1,2)$ and $(2,\infty)$, and f is CD on $(-\infty,-1)$. There is an inflection point at $\left(-1,\frac{1}{9}\right)$.

(e)



47. **(a)** $\lim_{x\to-\infty}\left(\sqrt{x^2+1}-x\right)=\infty$ and

$\lim_{x\to\infty}\left(\sqrt{x^2+1}-x\right)=\lim_{x\to\infty}\left(\sqrt{x^2+1}-x\right)\frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}+x}=\lim_{x\to\infty}\frac{1}{\sqrt{x^2+1}+x}=0$, so $y=0$ is a HA.

(b) $f(x)=\sqrt{x^2+1}-x\Rightarrow f'(x)=\frac{x}{\sqrt{x^2+1}}-1$. Since $\frac{x}{\sqrt{x^2+1}}<1$ for all x , $f'(x)<0$, so f is decreasing on R .

(c) No minimum or maximum

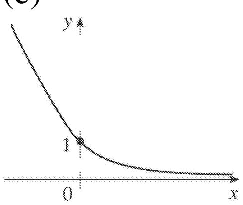
(d)

$$f''(x) = \frac{\left(x^2+1\right)^{1/2}(1) - x\cdot\frac{1}{2}\left(x^2+1\right)^{-1/2}(2x)}{\left(\sqrt{x^2+1}\right)^2}$$

$$= \frac{\left(x^2+1\right)^{1/2} - \frac{x^2}{\left(x^2+1\right)^{1/2}}}{x^2+1} = \frac{\left(x^2+1\right) - x^2}{\left(x^2+1\right)^{3/2}}$$

$$= \frac{1}{\left(x^2+1\right)^{3/2}} > 0$$, so f is CU on R . No IP

(e)



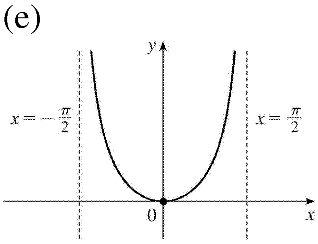
48. **(a)** $\lim_{x\to\pi/2^-} x\tan x = \infty$ and $\lim_{x\to-\pi/2^+} x\tan x = \infty$, so $x = \frac{\pi}{2}$ and $x = -\frac{\pi}{2}$ are VA.

(b) $f(x) = x\tan x$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$. $f'(x) = x\sec^2 x + \tan x > 0 \Leftrightarrow 0 < x < \frac{\pi}{2}$, so f increases on $\left(0, \frac{\pi}{2}\right)$ and decreases on $\left(-\frac{\pi}{2}, 0\right)$.

(c) $f(0)=0$ is a local minimum value.

(d) $f''(x) = 2\sec^2 x + 2x\tan x\sec^2 x > 0$ for $-\frac{\pi}{2} < x < \frac{\pi}{2}$, so f is CU on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. No IP

(e)



49. $f(x) = \ln(1 - \ln x)$ is defined when x>0 (so that ln\:\italic{x} is defined) and $1 - \ln x > 0$ [so that $\ln(1 - \ln x)$ is defined]. The second condition is equivalent to $1 > \ln x \Leftrightarrow x < e$, so f has domain $(0, e)$.

(a) As $x \to 0^+, \ln x \to -\infty$, so $1 - \ln x \to \infty$ and $f(x) \to \infty$. As $x \to e^-, \ln x \to 1^-$, so $1 - \ln x \to 0^+$ and

$f(x) \to -\infty$. Thus, $x=0$ and $x=e$ are vertical asymptotes. There is no horizontal asymptote.

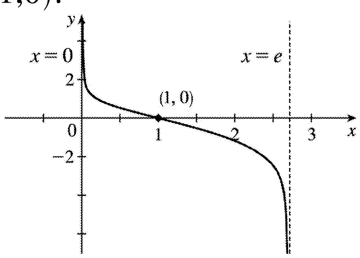
(b) $f'(x)=\dfrac{1}{1-\ln x}\left(-\dfrac{1}{x}\right)=-\dfrac{1}{x(1-\ln x)}<0$ on $(0,e)$. Thus, f is decreasing on its domain, $(0,e)$.

(c) $f'(x)\neq 0$ on $(0,e)$, so f has no local maximum or minimum value.

(d)

$$f''(x)=-\frac{-[x(1-\ln x)]'}{[x(1-\ln x)]^2}=\frac{x(-1/x)+(1-\ln x)}{x^2(1-\ln x)^2}$$
$$=-\frac{\ln x}{x^2(1-\ln x)^2}$$

so $f''(x)>0 \Leftrightarrow \ln x<0 \Leftrightarrow 0<x<1$. Thus, f is CU on $(0,1)$ and CD on $(1,e)$. There is an inflection point at $(1,0)$.

(e) 

50. $f(x)=\dfrac{e^x}{1+e^x}$ has domain R .

(a) $\lim\limits_{x\to\infty} f(x)=\lim\limits_{x\to\infty}\dfrac{e^x/e^x}{\left(1+e^x\right)/e^x}=\lim\limits_{x\to\infty}\dfrac{1}{e^{-x}+1}=\dfrac{1}{0+1}=1$, so $y=1$ is a HA.

$\lim\limits_{x\to-\infty} f(x)=\lim\limits_{x\to-\infty}\dfrac{e^x}{1+e^x}=\dfrac{0}{1+0}=0$, so $y=0$ is a HA. No VA.

(b) $f'(x)=\dfrac{\left(1+e^x\right)e^x-e^x\cdot e^x}{\left(1+e^x\right)^2}=\dfrac{e^x}{\left(1+e^x\right)^2}>0$ for all x . Thus, f is increasing on R .

(c) There is no local maximum or minimum.

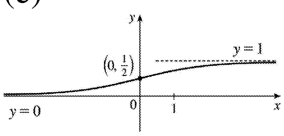
(d)

$$f''(x) = \frac{\left(1+e^x\right)^2 e^x - e^x \cdot 2\left(1+e^x\right)e^x}{\left[\left(1+e^x\right)^2\right]^2}$$

$$= \frac{e^x\left(1+e^x\right)\left[\left(1+e^x\right)-2e^x\right]}{\left(1+e^x\right)^4} = \frac{e^x\left(1-e^x\right)}{\left(1+e^x\right)^3}$$

$f''(x)>0 \Leftrightarrow 1-e^x>0 \Leftrightarrow x<0$, so f is CU on $(-\infty,0)$ and CD on $(0,\infty)$. There is an inflection point at $\left(0,\frac{1}{2}\right)$.

(e)



51. **(a)** $\lim_{x\to\pm\infty} e^{-1/(x+1)}=1$ since $-1/(x+1)\to 0$, so $y=1$ is a HA. $\lim_{x\to -1^+} e^{-1/(x+1)}=0$ since $-1/(x+1)\to -\infty$, $\lim_{x\to -1^-} e^{-1/(x+1)}=\infty$ since $-1/(x+1)\to \infty$, so $x=-1$ is a VA.

(b) $f(x)=e^{-1/(x+1)} \Rightarrow f'(x)=e^{-1/(x+1)}\left[-(-1)\frac{1}{(x+1)^2}\right]$ [Reciprocal Rule] $=e^{-1/(x+1)}/(x+1)^2 \Rightarrow f'(x)>0$ for all x except -1 , so f is increasing on $(-\infty,-1)$ and $(-1,\infty)$.

(c) No local maximum or minimum

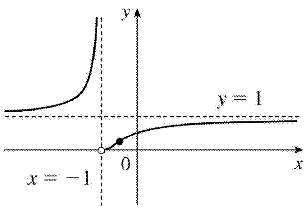
(d)

$$f''(x) = \frac{(x+1)^2 e^{-1/(x+1)}\left[1/(x+1)^2\right] - e^{-1/(x+1)}\left[2(x+1)\right]}{\left[(x+1)^2\right]^2}$$

$$= \frac{e^{-1/(x+1)}\left[1-(2x+2)\right]}{(x+1)^4} = -\frac{e^{-1/(x+1)}(2x+1)}{(x+1)^4} \Rightarrow$$

$f''(x)>0 \Leftrightarrow 2x+1<0 \Leftrightarrow x<-\frac{1}{2}$, so f is CU on $(-\infty,-1)$ and $\left(-1,-\frac{1}{2}\right)$, and CD on $\left(-\frac{1}{2},\infty\right)$. f has an IP at $\left(-\frac{1}{2},e^{-2}\right)$.

(e)



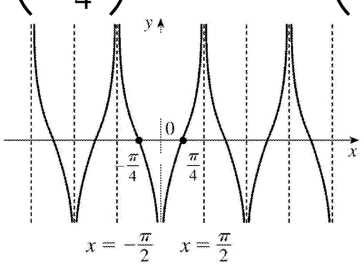
52. **(a)** f is periodic with period π, so we consider only $-\frac{\pi}{2}<x<\frac{\pi}{2}$. $\lim_{x\to 0} \ln\left(\tan^2 x\right)=-\infty$, $\lim_{x\to(\pi/2)^-} \ln\left(\tan^2 x\right)=\infty$, and $\lim_{x\to(-\pi/2)^+} \ln\left(\tan^2 x\right)=\infty$, so $x=0$, $x=\pm\frac{\pi}{2}$ are VA.

(b) $f(x)=\ln\left(\tan^2 x\right)\Rightarrow f'(x)=\frac{2\tan x\sec^2 x}{\tan^2 x}=2\frac{\sec^2 x}{\tan x}>0\Leftrightarrow\tan x>0\Leftrightarrow 0<x<\frac{\pi}{2}$, so f is increasing on $\left(0,\frac{\pi}{2}\right)$ and decreasing on $\left(-\frac{\pi}{2},0\right)$.

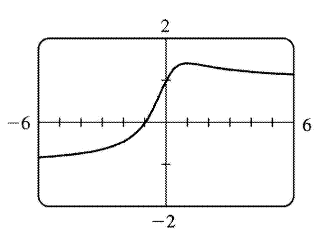
(c) No maximum or minimum

(d) $f'(x)=\frac{2}{\sin x\cos x}=\frac{4}{\sin 2x}\Rightarrow f''(x)=\frac{-8\cos 2x}{\sin^2 2x}<0$ $\Leftrightarrow\cos 2x>0\Leftrightarrow-\frac{\pi}{4}<x<\frac{\pi}{4}$, so f is CD on $\left(-\frac{\pi}{4},0\right)$ and $\left(0,\frac{\pi}{4}\right)$, and CU on $\left(-\frac{\pi}{2},-\frac{\pi}{4}\right)$ and $\left(\frac{\pi}{4},\frac{\pi}{2}\right)$. IP are $\left(\pm\frac{\pi}{4},0\right)$.

(e)



53. **(a)**

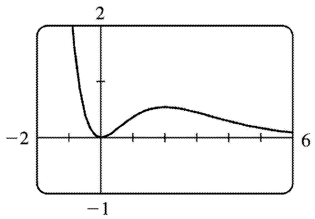


From the graph, we get an estimate of $f(1)\approx 1.41$ as a local maximum value, and no local minimum value.

$f(x)=\frac{x+1}{\sqrt{x^2+1}}\Rightarrow f'(x)=\frac{1-x}{\left(x^2+1\right)^{3/2}}$.

$f'(x)=0 \Leftrightarrow x=1$. $f(1)=\frac{2}{\sqrt{2}}=\sqrt{2}$ is the exact value.

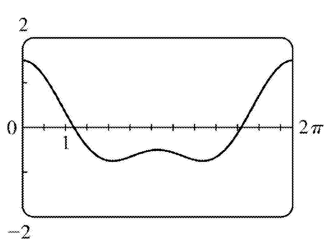
(b) From the graph in part (a), f increases most rapidly somewhere between $x=-\frac{1}{2}$ and $x=-\frac{1}{4}$. To find the exact value, we need to find the maximum value of f' , which we can do by finding the critical numbers of f' . $f''(x)=\frac{2x^2-3x-1}{\left(x^2+1\right)^{5/2}}=0 \Leftrightarrow x=\frac{3\pm\sqrt{17}}{4}$. $x=\frac{3+\sqrt{17}}{4}$ corresponds to the *minimum* value of f' . The maximum value of f' is at $\left(\frac{3-\sqrt{17}}{4},\sqrt{\frac{7}{6}-\frac{\sqrt{17}}{6}}\right)\approx(-0.28,0.69)$.



54. **(a)**
Tracing the graph gives us estimates of $f(0)=0$ for a local minimum value and $f(2)=0.54$ for a local maximum value.
$f(x)=x^2e^{-x} \Rightarrow f'(x)=xe^{-x}(2-x)$. $f'(x)=0 \Leftrightarrow x=0$ or 2 . $f(0)=0$ and $f(2)=4e^{-2}$ are the exact values.

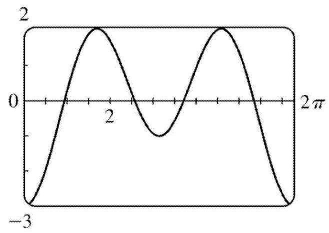
(b) From the graph in part (a), f increases most rapidly around $x=\frac{3}{4}$. To find the exact value, we need to find the maximum value of f' , which we can do by finding the critical numbers of f' .
$f''(x)=e^{-x}\left(x^2-4x+2\right)=0 \Rightarrow x=2\pm\sqrt{2}$. $x=2+\sqrt{2}$ corresponds to the *minimum* value of f' . The maximum value of f' is at $\left(2-\sqrt{2},(2-\sqrt{2})^2e^{-2+\sqrt{2}}\right)\approx(0.59,0.19)$.

55. $f(x)=\cos x+\frac{1}{2}\cos 2x \Rightarrow f'(x)=-\sin x-\sin 2x \Rightarrow f''(x)=-\cos x-2\cos 2x$



(a)
From the graph of f , it seems that f is CD on $(0,1)$, CU on $(1,2.5)$, CD on $(2.5,3.7)$, CU on $(3.7,5.3)$, and CD on $(5.3,2\pi)$. The points of inflection appear to be at $(1,0.4)$, $(2.5,-0.6)$,

$(3.7,-0.6)$, and $(5.3,0.4)$.

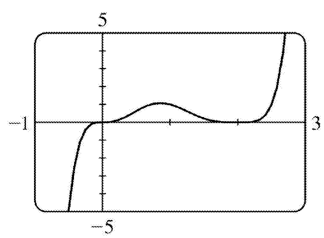


(b)

From the graph of f'' (and zooming in near the zeros), it seems that f is CD on $(0,0.94)$, CU on $(0.94,2.57)$, CD on $(2.57,3.71)$, CU on $(3.71,5.35)$, and CD on $(5.35,2\pi)$. Refined estimates of the inflection points are $(0.94,0.44)$, $(2.57,-0.63)$, $(3.71,-0.63)$, and $(5.35,0.44)$.

56. $f(x)=x^3(x-2)^4 \Rightarrow f'(x)=x^3\cdot 4(x-2)^3+(x-2)^4\cdot 3x^2=x^2(x-2)^3[4x+3(x-2)]=x^2(x-2)^3(7x-6)\Rightarrow$

$$f''(x)=(2x)(x-2)^3(7x-6)+x^2\cdot 3(x-2)^2(7x-6)+x^2(x-2)^3(7)$$
$$=x(x-2)^2[2(x-2)(7x-6)+3x(7x-6)+7x(x-2)]$$
$$=x(x-2)^2\left[42x^2-72x+24\right]=6x(x-2)^2\left(7x^2-12x+4\right)$$



(a)

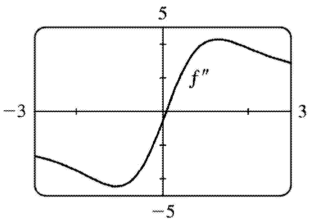
From the graph of f , it seems that f is CD on $(-\infty,0)$, CU on $(0,0.5)$, CD on $(0.5,1.3)$, and CU on $(1.3,\infty)$. The points of inflection appear to be at $(0,0)$, $(0.5,0.5)$, and $(1.3,0.6)$.

(b)

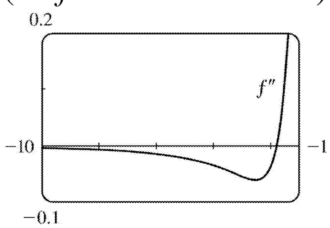
From the graph of f'' (and zooming in near the zeros), it seems that f is CD on $(-\infty,0)$, CU on $(0,0.45)$, CD on $(0.45,1.26)$, and CU on $(1.26,\infty)$. Refined estimates of the inflection points are $(0,0)$, $(0.45,0.53)$, and $(1.26,0.60)$.

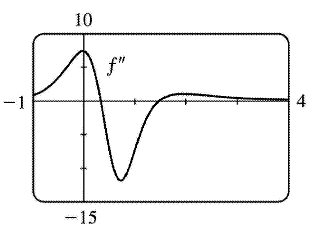
57. In Maple, we define f and then use the command plot (diff (diff (f, x), x), x= –3..3);. In Mathematica, we define f and then use Plot [Dt [Dt [f, x], x], {x, –3, 3}]. We see that $f''>0$ for

$x > 0.1$ and $f'' < 0$ for $x < 0.1$. So f is concave up on $(0.1, \infty)$ and concave down on $(-\infty, 0.1)$.



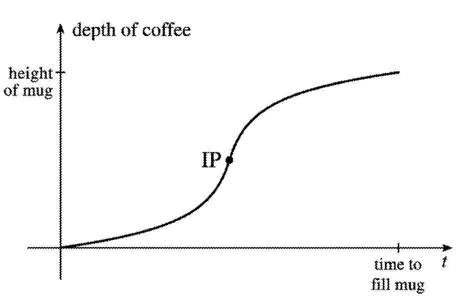
58. It appears that f'' is positive (and thus f is concave up) on $(-1.8, 0.3)$ and $(1.5, \infty)$ and negative (so f is concave down) on $(-\infty, -1.8)$ and $(0.3, 1.5)$.



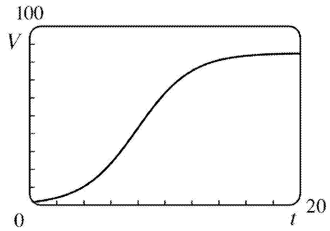


59. Most students learn more in the third hour of studying than in the eighth hour, so $K(3) - K(2)$ is larger than $K(8) - K(7)$. In other words, as you begin studying for a test, the rate of knowledge gain is large and then starts to taper off, so $K'(t)$ decreases and the graph of K is concave downward.

60. At first the depth increases slowly because the base of the mug is wide. But as the mug narrows, the coffee rises more quickly. Thus, the depth d increases at an increasing rate and its graph is concave upward. The rate of increase of d has a maximum where the mug is narrowest; that is, when the mug is half full. It is there that the inflection point (IP) occurs. Then the rate of increase of d starts to decrease as the mug widens and the graph becomes concave down.



61.



From the graph, we estimate that the most rapid increase in the percentage of households in the United States with at least one VCR occurs at about $t=8$. To maximize the first derivative, we need to determine the values for which the second derivative is 0. We'll use $V(t)=\dfrac{a}{1+be^{ct}}$, and substitute $a=85$, $b=53$, and $c=-0.5$ later.

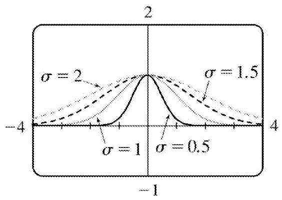
$V'(t)=-\dfrac{a\left(bce^{ct}\right)}{\left(1+be^{ct}\right)^2}$ [by the Reciprocal Rule] and

$$V''(t)=-abc\cdot\frac{\left(1+be^{ct}\right)^2\cdot ce^{ct}-e^{ct}\cdot 2\left(1+be^{ct}\right)\cdot bce^{ct}}{\left[\left(1+be^{ct}\right)^2\right]^2}$$

$$=\frac{-abc\cdot ce^{ct}\left(1+be^{ct}\right)\left[\left(1+be^{ct}\right)-2be^{ct}\right]}{\left(1+be^{ct}\right)^4}=\frac{-abc^2e^{ct}\left(1-be^{ct}\right)}{\left(1+be^{ct}\right)^3}$$

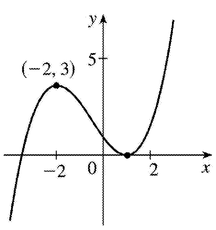
So $V''(t)=0\Leftrightarrow 1=be^{ct}\Leftrightarrow e^{ct}=1/b$. Now graph $y=e^{-0.5t}$ and $y=\dfrac{1}{53}$. These graphs intersect at $t\approx 7.94$ years, which corresponds to roughly midyear 1988.

62. **(a)** As $|x|\to\infty$, $t=-x^2/\left(2\sigma^2\right)\to-\infty$, and $e^t\to 0$. The HA is $y=0$. Since t takes on its maximum value at $x=0$, so does e^t . Showing this result using derivatives, we have $f(x)=e^{-x^2/\left(2\sigma^2\right)}\Rightarrow$ $f'(x)=e^{-x^2/\left(2\sigma^2\right)}\left(-x/\sigma^2\right)$. $f'(x)=0\Leftrightarrow x=0$. Because f' changes from positive to negative at $x=0$, $f(0)=1$ is a local maximum. For inflection points, we find
$f''(x)=-\dfrac{1}{\sigma^2}\left[e^{-x^2/\left(2\sigma^2\right)}\cdot 1+xe^{-x^2/\left(2\sigma^2\right)}\left(-x/\sigma^2\right)\right]=\dfrac{-1}{\sigma^2}e^{-x^2/\left(2\sigma^2\right)}\left(1-x^2/\sigma^2\right)$.
$f''(x)=0\Leftrightarrow x^2=\sigma^2\Leftrightarrow x=\pm\sigma$. $f''(x)<0\Leftrightarrow x^2<\sigma^2\Leftrightarrow-\sigma<x<\sigma$. So f is CD on $(-\sigma,\sigma)$ and CU on $(-\infty,-\sigma)$ and (σ,∞) . IP at $\left(\pm\sigma,e^{-1/2}\right)$.
(b) Since we have IP at $x=\pm\sigma$, the inflection points move away from the y -axis as σ increases.



(c)
From the graph, we see that as σ increases, the graph tends to spread out and there is more area between the curve and the x –axis.

63. $f(x)=ax^3+bx^2+cx+d \Rightarrow f'(x)=3ax^2+2bx+c$. We are given that $f(1)=0$ and $f(-2)=3$, so $f(1)=a+b+c+d=0$ and $f(-2)=-8a+4b-2c+d=3$. Also $f'(1)=3a+2b+c=0$ and $f'(-2)=12a-4b+c=0$ by Fermat's Theorem. Solving these four equations, we get $a=\frac{2}{9}$, $b=\frac{1}{3}$, $c=-\frac{4}{3}$, $d=\frac{7}{9}$, so the function is $f(x)=\frac{1}{9}\left(2x^3+3x^2-12x+7\right)$.



64. $f(x)=axe^{bx^2} \Rightarrow f'(x)=a\left[xe^{bx^2}\cdot 2bx+e^{bx^2}\cdot 1\right]=ae^{bx^2}\left(2bx^2+1\right)$. For $f(2)=1$ to be a maximum value, we must have $f'(2)=0$. $f(2)=1 \Rightarrow 1=2ae^{4b}$ and $f'(2)=0 \Rightarrow 0=(8b+1)ae^{4b}$. So $8b+1=0 \Rightarrow b=-\frac{1}{8}$ and now $1=2ae^{-1/2} \Rightarrow a=\sqrt{e}/2$.

65. Suppose that f is differentiable on an interval I and $f'(x)>0$ for all x in I except $x=c$. To show that f is increasing on I , let x_1 , x_2 be two numbers in I with $x_1<x_2$.

Case 2 $x_1<x_2<c$. Let J be the interval $\{x\in I \mid x<c\}$. By applying the Increasing/Decreasing Test to f on J , we see that f is increasing on J , so $f(x_1)<f(x_2)$.

Case 2 $c<x_1<x_2$. Apply the Increasing/Decreasing Test to f on $K=\{x\in I \mid x>c\}$.

Case 3 $x_1<x_2=c$. Apply the proof of the Increasing/Decreasing Test, using the Mean Value Theorem (MVT) on the interval $\left[x_1,x_2\right]$ and noting that the MVT does not require f to be differentiable at the endpoints of $\left[x_1,x_2\right]$.

Case 4 $c=x_1<x_2$. Same proof as in Case 3.

Case 5 $x_1<c<x_2$. By Cases 3 and 4, f is increasing on $\left[x_1,c\right]$ and on $\left[c,x_2\right]$, so $f(x_1)<f(c)<f(x_2)$. In all cases, we have shown that $f(x_1)<f(x_2)$. Since x_1 , x_2 were any numbers in I with $x_1<x_2$, we have shown that f is increasing on I .

66. **(a)** We will make use of the converse of the Concavity Test (along with the stated assumptions); that is, if f is concave upward on I , then $f''>0$ on I . If f and g are CU on I , then $f''>0$ and $g''>0$ on I , so $(f+g)''=f''+g''>0$ on $I\Rightarrow f+g$ is CU on I .
(b) Since f is positive and CU on I , $f>0$ and $f''>0$ on I . So $g(x)=[f(x)]^2\Rightarrow g'=2ff'\Rightarrow$ $g''=2f'f'+2ff''=2\left(f'\right)^2+2ff''>0\Rightarrow g$ is CU on I .

67. **(a)** Since f and g are positive, increasing, and CU on I with f'' and g'' never equal to 0 , we have $f>0$, $f'\geq 0$, $f''>0$, $g>0$, $g'\geq 0$, $g''>0$ on I . Then $(fg)'=f'g+fg'\Rightarrow$ $(fg)''=f''g+2f'g'+fg''\geq f''g+fg''>0$ on $I\Rightarrow fg$ is CU on I .
(b) In part (a), if f and g are both decreasing instead of increasing, then $f'\leq 0$ and $g'\leq 0$ on I , so we still have $2f'g'\geq 0$ on I . Thus, $(fg)''=f''g+2f'g'+fg''\geq f''g+fg''>0$ on $I\Rightarrow fg$ is CU on I as in part (a).
(c) Suppose f is increasing and g is decreasing . Then $f'\geq 0$ and $g'\leq 0$ on I , so $2f'g'\leq 0$ on I and the argument in parts (a) and (b) fails.
Example 1.
$I=(0,\infty)$, $f(x)=x^3$, $g(x)=1/x$. Then $(fg)(x)=x^2$, so $(fg)'(x)=2x$ and $(fg)''(x)=2>0$ on I . Thus, fg is CU on I .

Example 2.

$I=(0,\infty)$, $f(x)=4x\sqrt{x}$, $g(x)=1/x$. Then $(fg)(x)=4\sqrt{x}$, so $(fg)'(x)=2/\sqrt{x}$ and $(fg)''(x)=-1/\sqrt{x^3}<0$ on I . Thus, fg is CD on I .

Example 3.

$I=(0,\infty)$, $f(x)=x^2$, $g(x)=1/x$. Thus, $(fg)(x)=x$, so fg is linear on I .

68. Since f and g are CU on $(-\infty,\infty)$, $f''>0$ and $g''>0$ on $(-\infty,\infty)$.
$h(x)=f(g(x))\Rightarrow h'(x)=f'(g(x))g'(x)\Rightarrow$

$h''(x)=f''(g(x))g'(x)g'(x)+f'(g(x))g''(x)=f''(g(x))\left[g'(x)\right]^2+f'(g(x))g''(x)>0$ if $f'>0$. So h is CU if f is increasing.

69. $f(x)=\tan x-x\Rightarrow f'(x)=\sec^2 x-1>0$ for $0<x<\frac{\pi}{2}$ since $\sec^2 x>1$ for $0<x<\frac{\pi}{2}$. So f is increasing on $\left(0,\frac{\pi}{2}\right)$. Thus, $f(x)>f(0)=0$ for $0<x<\frac{\pi}{2}\Rightarrow\tan x-x>0\Rightarrow\tan x>x$ for $0<x<\frac{\pi}{2}$.

70. **(a)** Let $f(x)=e^x-1-x$. Now $f(0)=e^0-1=0$, and for $x\geq 0$, we have $f'(x)=e^x-1\geq 0$. Now, since $f(0)=0$ and f is increasing on $[0,\infty)$, $f(x)\geq 0$ for $x\geq 0\Rightarrow e^x-1-x\geq 0\Rightarrow e^x\geq 1+x$.

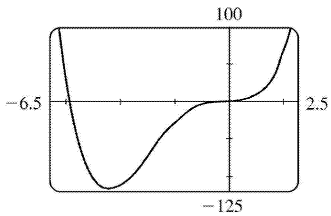
(b) Let $f(x)=e^x-1-x-\frac{1}{2}x^2$. Thus, $f'(x)=e^x-1-x$, which is positive for $x\geq 0$ by part (a). Thus, $f(x)$ is increasing on $(0,\infty)$, so on that interval, $0=f(0)\leq f(x)=e^x-1-x-\frac{1}{2}x^2\Rightarrow e^x\geq 1+x+\frac{1}{2}x^2$.

(c) By part (a), the result holds for $n=1$. Suppose that $e^x\geq 1+x+\frac{x^2}{2!}+\cdots+\frac{x^k}{k!}$ for $x\geq 0$. Let $f(x)=e^x-1-x-\frac{x^2}{2!}-\cdots-\frac{x^k}{k!}-\frac{x^{k+1}}{(k+1)!}$. Then $f'(x)=e^x-1-x-\cdots-\frac{x^k}{k!}\geq 0$ by assumption. Hence, $f(x)$ is increasing on $(0,\infty)$. So $0\leq x$ implies that $0=f(0)\leq f(x)=e^x-1-x-\cdots-\frac{x^k}{k!}-\frac{x^{k+1}}{(k+1)!}$, and hence $e^x\geq 1+x+\cdots+\frac{x^k}{k!}+\frac{x^{k+1}}{(k+1)!}$ for $x\geq 0$. Therefore, for $x\geq 0$, $e^x\geq 1+x+\frac{x^2}{2!}+\cdots+\frac{x^n}{n!}$ for every positive integer n , by mathematical induction.

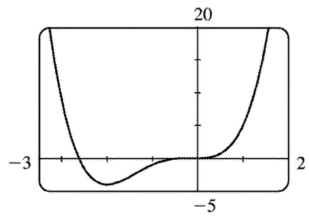
71. Let the cubic function be $f(x)=ax^3+bx^2+cx+d\Rightarrow f'(x)=3ax^2+2bx+c\Rightarrow f''(x)=6ax+2b$. So f is CU when $6ax+2b>0\Leftrightarrow x>-b/(3a)$, CD when $x<-b/(3a)$, and so the only point of inflection occurs when $x=-b/(3a)$. If the graph has three x –intercepts x_1 , x_2 and x_3 , then the expression for $f(x)$ must factor as $f(x)=a\left(x-x_1\right)\left(x-x_2\right)\left(x-x_3\right)$. Multiplying these factors together gives us $f(x)=a\left[x^3-\left(x_1+x_2+x_3\right)x^2+\left(x_1x_2+x_1x_3+x_2x_3\right)x-x_1x_2x_3\right]$. \ Equating the coefficients of the x^2 –terms for the two forms of f gives us $b=-a\left(x_1+x_2+x_3\right)$. Hence, the x– coordinate of the point of inflection is $-\frac{b}{3a}=-\frac{-a\left(x_1+x_2+x_3\right)}{3a}=\frac{x_1+x_2+x_3}{3}$.

72. $P(x)=x^4+cx^3+x^2\Rightarrow P'(x)=4x^3+3cx^2+2x\Rightarrow P''(x)=12x^2+6cx+2$. The graph of $P''(x)$ is a parabola. \ If $P''(x)$ has two roots, then it changes sign twice and so has two inflection points. This

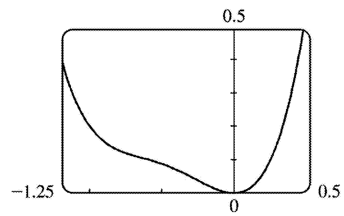
happens when the discriminant of $P''(x)$ is positive, that is, $(6c)^2-4\cdot 12\cdot 2>0 \Leftrightarrow 36c^2-96>0 \Leftrightarrow$ $|c|>\dfrac{2\sqrt{6}}{3}\approx 1.63$. If $36c^2-96=0 \Leftrightarrow c=\pm\dfrac{2\sqrt{6}}{3}$, $P''(x)$ is 0 at one point, but there is still no inflection point since $P''(x)$ never changes sign, and if $36c^2-96<0 \Leftrightarrow |c|<\dfrac{2\sqrt{6}}{3}$, then $P''(x)$ never changes sign, and so there is no inflection point.



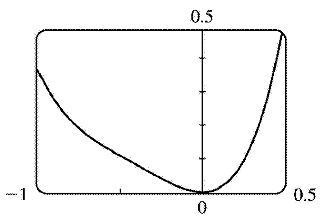
$c=6$



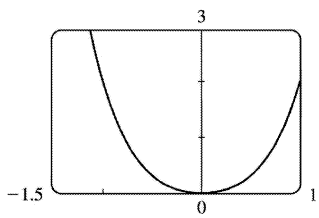
$c=3$



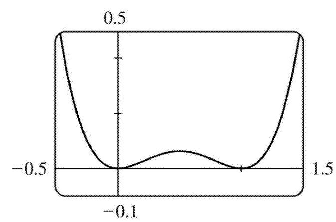
$c=1.8$



$c=\dfrac{2\sqrt{6}}{3}$



$c=0$



$c=-2$

For large positive c , the graph of f has two inflection points and a large dip to the left of the $y-$ axis. As c decreases, the graph of f becomes flatter for $x<0$, and eventually the dip rises above the $x-$ axis, and then disappears entirely, along with the inflection points. As c continues to decrease, the dip and the inflection points reappear, to the right of the origin.

73. By hypothesis $g=f'$ is differentiable on an open interval containing c . Since $(c,f(c))$ is a point of inflection, the concavity changes at $x=c$, so $f''(x)$ changes signs at $x=c$. Hence, by the First Derivative Test, f' has a local extremum at $x=c$. Thus, by Fermat's Theorem $f''(c)=0$.

74. $f(x)=x^4 \Rightarrow f'(x)=4x^3 \Rightarrow f''(x)=12x^2 \Rightarrow f''(0)=0$. For $x<0$, $f''(x)>0$, so f is CU on $(-\infty,0)$; for $x>0$, $f''(x)>0$, so f is also CU on $(0,\infty)$. Since f does not change concavity at 0 , $(0,0)$ is not an inflection point.

75. Using the fact that $|x|=\sqrt{x^2}$, we have that $g(x)=x\sqrt{x^2} \Rightarrow g'(x)=\sqrt{x^2}+\sqrt{x^2}=2\sqrt{x^2}=2|x| \Rightarrow$ $g''(x)=2x\left(x^2\right)^{-1/2}=\dfrac{2x}{|x|}<0$ for $x<0$ and $g''(x)>0$ for $x>0$, so $(0,0)$ is an inflection point. But

$g''(0)$ does not exist.

76. There must exist some interval containing c on which f''' is positive, since $f'''(c)$ is positive and f''' is continuous. On this interval, f'' is increasing (since f''' is positive), so $f''=(f')'$ changes from negative to positive at c . So by the First Derivative Test, f' has a local minimum at $x=c$ and thus cannot change sign there, so f has no maximum or minimum at c . But since f'' changes from negative to positive at c , f has a point of inflection at c (it changes from concave down to concave up).

1. **(a)** $\lim_{x\to a} \frac{f(x)}{g(x)}$ is an indeterminate form of type $\frac{0}{0}$.

(b) $\lim_{x\to a} \frac{f(x)}{p(x)}=0$ because the numerator approaches 0 while the denominator becomes large.

(c) $\lim_{x\to a} \frac{h(x)}{p(x)}=0$ because the numerator approaches a finite number while the denominator becomes large.

(d) If $\lim_{x\to a} p(x)=\infty$ and $f(x)\to 0$ through positive values, then $\lim_{x\to a} \frac{p(x)}{f(x)}=\infty$. If $f(x)\to 0$ through negative values, then $\lim_{x\to a} \frac{p(x)}{f(x)}=-\infty$. If $f(x)\to 0$ through both positive and negative values, then the limit might not exist.

(e) $\lim_{x\to a} \frac{p(x)}{q(x)}$ is an indeterminate form of type $\frac{\infty}{\infty}$.

2. **(a)** $\lim_{x\to a} [f(x)p(x)]$ is an indeterminate form of type $0\cdot\infty$.

(b) When x is near a , $p(x)$ is large and $h(x)$ is near 1 , so $h(x)p(x)$ is large. Thus, $\lim_{x\to a} [h(x)p(x)]=\infty$.

(c) When x is near a , $p(x)$ and $q(x)$ are both large, so $p(x)q(x)$ is large. Thus, $\lim_{x\to a} [p(x)q(x)]=\infty$.

3. **(a)** When x is near a , $f(x)$ is near 0 and $p(x)$ is large, so $f(x)-p(x)$ is large negative. Thus, $\lim_{x\to a} [f(x)-p(x)]=-\infty$.

(b) $\lim_{x\to a} [p(x)-q(x)]$ is an indeterminate form of type $\infty-\infty$.

(c) When x is near a , $p(x)$ and $q(x)$ are both large, so $p(x)+q(x)$ is large. Thus, $\lim_{x\to a} [p(x)+q(x)]=\infty$.

4. **(a)** $\lim_{x\to a} [f(x)]^{g(x)}$ is an indeterminate form of type 0^0 .

(b) If $y=[f(x)]^{p(x)}$, then $\ln y=p(x)\ln f(x)$. When x is near a , $p(x)\to\infty$ and $\ln f(x)\to -\infty$, so $\ln y\to-\infty$. Therefore, $\lim_{x\to a} [f(x)]^{p(x)}=\lim_{x\to a} y=\lim_{x\to a} e^{\ln y}=0$, provided f^p is defined.

(c) $\lim_{x\to a} [h(x)]^{p(x)}$ is an indeterminate form of type 1^∞ .

(d) $\lim_{x\to a} [p(x)]^{f(x)}$ is an indeterminate form of type ∞^0 .

(e) If $y=[p(x)]^{q(x)}$, then $\ln y=q(x)\ln p(x)$. When x is near a , $q(x)\to\infty$ and $\ln p(x)\to\infty$, so

$\ln y \to \infty$. Therefore, $\lim_{x\to a} [p(x)]^{q(x)} = \lim_{x\to a} y = \lim_{x\to a} e^{\ln y} = \infty$.

(f) $\lim_{x\to a} \sqrt[q(x)]{p(x)} = \lim_{x\to a} [p(x)]^{1/q(x)}$ is an indeterminate form of type ∞^0 .

5. This limit has the form $\frac{0}{0}$. We can simply factor the numerator to evaluate this limit.

$$\lim_{x\to -1} \frac{x^2-1}{x+1} = \lim_{x\to -1} \frac{(x+1)(x-1)}{x+1} = \lim_{x\to -1} (x-1) = -2$$

6. $\lim_{x\to -2} \frac{x+2}{x^2+3x+2} = \lim_{x\to -2} \frac{x+2}{(x+1)(x+2)} = \lim_{x\to -2} \frac{1}{x+1} = -1$

7. This limit has the form $\frac{0}{0}$. $\lim_{x\to 1} \frac{x^9-1}{x^5-1} = \lim_{x\to 1} \frac{9x^8}{5x^4} = \frac{9}{5} \lim_{x\to 1} x^4 = \frac{9}{5}(1) = \frac{9}{5}$

8. $\lim_{x\to 1} \frac{x^a-1}{x^b-1} = \lim_{x\to 1} \frac{ax^{a-1}}{bx^{b-1}} = \frac{a}{b}$

9. This limit has the form $\frac{0}{0}$. $\lim_{x\to (\pi/2)^+} \frac{\cos x}{1-\sin x} = \lim_{x\to (\pi/2)^+} \frac{-\sin x}{-\cos x} = \lim_{x\to (\pi/2)^+} \tan x = -\infty$.

10. $\lim_{x\to 0} \frac{x+\tan x}{\sin x} = \lim_{x\to 0} \frac{1+\sec^2 x}{\cos x} = \frac{1+1^2}{1} = 2$

11. This limit has the form $\frac{0}{0}$. $\lim_{t\to 0} \frac{e^t-1}{t^3} = \lim_{t\to 0} \frac{e^t}{3t^2} = \infty$ since $e^t \to 1$ and $3t^2 \to 0^+$ as $t \to 0$.

12. $\lim_{t\to 0} \frac{e^{3t}-1}{t} = \lim_{t\to 0} \frac{3e^{3t}}{1} = 3$

13. This limit has the form $\frac{0}{0}$. $\lim_{x\to 0} \frac{\tan px}{\tan qx} = \lim_{x\to 0} \frac{p\sec^2 px}{q\sec^2 qx} = \frac{p(1)^2}{q(1)^2} = \frac{p}{q}$

14. $\lim_{\theta\to\pi/2} \frac{1-\sin\theta}{\csc\theta} = \frac{0}{1} = 0$. L'Hospital's Rule does not apply.

15. This limit has the form $\frac{\infty}{\infty}$. $\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1/x}{1} = 0$

16. $\lim_{x\to\infty} \frac{e^x}{x} = \lim_{x\to\infty} \frac{e^x}{1} = \lim_{x\to\infty} e^x = \infty$

17. $\lim_{x\to 0^+} [(\ln x)/x] = -\infty$ since $\ln x \to -\infty$ as $x\to 0^+$ and dividing by small values of x just increases the magnitude of the quotient $(\ln x)/x$. L'Hospital's Rule does not apply.

18. $\lim_{x\to\infty} \frac{\ln\ln x}{x} = \lim_{x\to\infty} \frac{\frac{1}{\ln x}\cdot\frac{1}{x}}{1} = \lim_{x\to\infty} \frac{1}{x\ln x} = 0$

19. This limit has the form $\frac{0}{0}$. $\lim_{t\to 0} \frac{5^t-3^t}{t} = \lim_{t\to 0} \frac{5^t\ln 5-3^t\ln 3}{1} = \ln 5-\ln 3 = \ln\frac{5}{3}$

20. $\lim_{x\to 1} \frac{\ln x}{\sin\pi x} = \lim_{x\to 1} \frac{1/x}{\pi\cos\pi x} = \frac{1}{\pi(-1)} = -\frac{1}{\pi}$

21. This limit has the form $\frac{0}{0}$. $\lim_{x\to 0} \frac{e^x-1-x}{x^2} = \lim_{x\to 0} \frac{e^x-1}{2x} = \lim_{x\to 0} \frac{e^x}{2} = \frac{1}{2}$

22. $\lim_{x\to 0} \frac{e^x-1-x-x^2/2}{x^3} = \lim_{x\to 0} \frac{e^x-1-x}{3x^2} = \lim_{x\to 0} \frac{e^x-1}{6x} = \lim_{x\to 0} \frac{e^x}{6} = \frac{1}{6}$

23. This limit has the form $\frac{\infty}{\infty}$. $\lim_{x\to\infty} \frac{e^x}{x^3} = \lim_{x\to\infty} \frac{e^x}{3x^2} = \lim_{x\to\infty} \frac{e^x}{6x} = \lim_{x\to\infty} \frac{e^x}{6} = \infty$

24. $\lim_{x\to 0} \frac{\sin x}{\sinh x} = \lim_{x\to 0} \frac{\cos x}{\cosh x} = \frac{1}{1} = 1$

25. This limit has the form $\frac{0}{0}$. $\lim_{x\to 0}\frac{\sin^{-1}x}{x}=\lim_{x\to 0}\frac{1/\sqrt{1-x^2}}{1}=\lim_{x\to 0}\frac{1}{\sqrt{1-x^2}}=\frac{1}{1}=1$

26. $\lim_{x\to 0}\frac{\sin x-x}{x^3}=\lim_{x\to 0}\frac{\cos x-1}{3x^2}=\lim_{x\to 0}\frac{-\sin x}{6x}=\lim_{x\to 0}\frac{-\cos x}{6}=-\frac{1}{6}$

27. This limit has the form $\frac{0}{0}$. $\lim_{x\to 0}\frac{1-\cos x}{x^2}=\lim_{x\to 0}\frac{\sin x}{2x}=\lim_{x\to 0}\frac{\cos x}{2}=\frac{1}{2}$

28. $\lim_{x\to\infty}\frac{(\ln x)^2}{x}=\lim_{x\to\infty}\frac{2(\ln x)(1/x)}{1}=2\lim_{x\to\infty}\frac{\ln x}{x}=2\lim_{x\to\infty}\frac{1/x}{1}=2(0)=0$

29. $\lim_{x\to 0}\frac{x+\sin x}{x+\cos x}=\frac{0+0}{0+1}=\frac{0}{1}=0$. L'Hospital's Rule does not apply.

30. $\lim_{x\to 0}\frac{\cos mx-\cos nx}{x^2}=\lim_{x\to 0}\frac{-m\sin mx+n\sin nx}{2x}=\lim_{x\to 0}\frac{-m^2\cos mx+n^2\cos nx}{2}=\frac{1}{2}\left(n^2-m^2\right)$

31. This limit has the form $\frac{\infty}{\infty}$. $\lim_{x\to\infty}\frac{x}{\ln\left(1+2e^x\right)}=\lim_{x\to\infty}\frac{1}{\frac{1}{1+2e^x}\cdot 2e^x}=\lim_{x\to\infty}\frac{1+2e^x}{2e^x}=\lim_{x\to\infty}\frac{2e^x}{2e^x}=1$

32. $\lim_{x\to 0}\frac{x}{\tan^{-1}(4x)}=\lim_{x\to 0}\frac{1}{\frac{1}{1+(4x)^2}\cdot 4}=\lim_{x\to 0}\frac{1+16x^2}{4}=\frac{1}{4}$

33. This limit has the form $\frac{0}{0}$. $\lim_{x\to 1}\frac{1-x+\ln x}{1+\cos\pi x}=\lim_{x\to 1}\frac{-1+1/x}{-\pi\sin\pi x}=\lim_{x\to 1}\frac{-1/x^2}{-\pi^2\cos\pi x}=\frac{-1}{-\pi^2(-1)}=-\frac{1}{\pi^2}$

34. $\lim_{x\to\infty}\frac{\sqrt{x^2+2}}{\sqrt{2x^2+1}}=\lim_{x\to\infty}\sqrt{\frac{x^2+2}{2x^2+1}}=\sqrt{\lim_{x\to\infty}\frac{x^2+2}{2x^2+1}}=\sqrt{\lim_{x\to\infty}\frac{1+2/x^2}{2+1/x^2}}=\sqrt{\frac{1}{2}}$

35. This limit has the form $\frac{0}{0}$. $\lim_{x\to 1}\frac{x^a-ax+a-1}{(x-1)^2}=\lim_{x\to 1}\frac{ax^{a-1}-a}{2(x-1)}=\lim_{x\to 1}\frac{a(a-1)x^{a-2}}{2}=\frac{a(a-1)}{2}$

36. $\lim_{x\to 0}\frac{1-e^{-2x}}{\sec x}=\frac{1-1}{1}=0$. L'Hospital's Rule does not apply.

37. This limit has the form $0\cdot(-\infty)$. We need to write this product as a quotient, but keep in mind that we will have to differentiate both the numerator and the denominator. If we differentiate $\frac{1}{\ln x}$, we get a complicated expression that results in a more difficult limit. Instead we write the quotient as $\frac{\ln x}{x^{-1/2}}$.

$$\lim_{x\to 0^+}\sqrt{x}\,\ln x=\lim_{x\to 0^+}\frac{\ln x}{x^{-1/2}}=\lim_{x\to 0^+}\frac{1/x}{-\frac{1}{2}x^{-3/2}}\cdot\frac{-2x^{3/2}}{-2x^{3/2}}=\lim_{x\to 0^+}\left(-2\sqrt{x}\right)=0$$

38. $\lim_{x\to -\infty}x^2e^x=\lim_{x\to -\infty}\frac{x^2}{e^{-x}}=\lim_{x\to -\infty}\frac{2x}{-e^{-x}}=\lim_{x\to -\infty}\frac{2}{e^{-x}}=\lim_{x\to -\infty}2e^x=0$

39. This limit has the form $\infty\cdot 0$. We'll change it to the form $\frac{0}{0}$.

$$\lim_{x\to 0}\cot 2x\sin 6x=\lim_{x\to 0}\frac{\sin 6x}{\tan 2x}=\lim_{x\to 0}\frac{6\cos 6x}{2\sec^2 2x}=\frac{6(1)}{2(1)^2}=3$$

40.

$$\lim_{x\to 0^+}\sin x\ln x=\lim_{x\to 0^+}\frac{\ln x}{\csc x}=\lim_{x\to 0^+}\frac{1/x}{-\csc x\cot x}=-\lim_{x\to 0^+}\left(\frac{\sin x}{x}\cdot\tan x\right)$$
$$=-\left(\lim_{x\to 0^+}\frac{\sin x}{x}\right)\left(\lim_{x\to 0^+}\tan x\right)=-1\cdot 0=0$$

41. This limit has the form $\infty\cdot 0$. $\lim_{x\to\infty}x^3e^{-x^2}=\lim_{x\to\infty}\frac{x^3}{e^{x^2}}=\lim_{x\to\infty}\frac{3x^2}{2xe^{x^2}}=\lim_{x\to\infty}\frac{3x}{2e^{x^2}}=\lim_{x\to\infty}\frac{3}{4xe^{x^2}}=0$

42.

$\lim_{x\to\pi/4} (1-\tan x)\sec x=(1-1)\sqrt{2}=0$. L'Hospital's Rule does not apply.

43. This limit has the form $0\cdot(-\infty)$.

$$\lim_{x\to 1^+} \ln x\tan(\pi x/2)=\lim_{x\to 1^+}\frac{\ln x}{\cot(\pi x/2)}=\lim_{x\to 1^+}\frac{1/x}{(-\pi/2)\csc^2(\pi x/2)}=\frac{1}{(-\pi/2)(1)^2}=-\frac{2}{\pi}$$

44. $$\lim_{x\to\infty} x\tan(1/x)=\lim_{x\to\infty}\frac{\tan(1/x)}{1/x}=\lim_{x\to\infty}\frac{\sec^2(1/x)\left(-1/x^2\right)}{-1/x^2}=\lim_{x\to\infty}\sec^2(1/x)=1^2=1$$

45.

$$\lim_{x\to 0}\left(\frac{1}{x}-\csc x\right)=\lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{\sin x}\right)=\lim_{x\to 0}\frac{\sin x-x}{x\sin x}$$

$$=\lim_{x\to 0}\frac{\cos x-1}{x\cos x+\sin x}=\lim_{x\to 0}\frac{-\sin x}{2\cos x-x\sin x}=\frac{0}{2}=0$$

46. $$\lim_{x\to 0}(\csc x-\cot x)=\lim_{x\to 0}\left(\frac{1}{\sin x}-\frac{\cos x}{\sin x}\right)=\lim_{x\to 0}\frac{1-\cos x}{\sin x}=\lim_{x\to 0}\frac{\sin x}{\cos x}=0$$

47. We will multiply and divide by the conjugate of the expression to change the form of the expression.

$$\lim_{x\to\infty}\left(\sqrt{x^2+x}-x\right)=\lim_{x\to\infty}\left(\frac{\sqrt{x^2+x}-x}{1}\cdot\frac{\sqrt{x^2+x}+x}{\sqrt{x^2+x}+x}\right)=\lim_{x\to\infty}\frac{\left(x^2+x\right)-x^2}{\sqrt{x^2+x}+x}$$

$$=\lim_{x\to\infty}\frac{x}{\sqrt{x^2+x}+x}=\lim_{x\to\infty}\frac{1}{\sqrt{1+1/x}+1}=\frac{1}{\sqrt{1}+1}=\frac{1}{2}.$$

As an alternate solution, write $\sqrt{x^2+x}-x$ as $\sqrt{x^2+x}-\sqrt{x^2}$, factor out $\sqrt{x^2}$, rewrite as $\left(\sqrt{1+1/x}-1\right)/(1/x)$, and apply l'Hospital's Rule.

48.

$$\lim_{x\to 1}\left(\frac{1}{\ln x}-\frac{1}{x-1}\right)=\lim_{x\to 1}\frac{x-1-\ln x}{(x-1)\ln x}=\lim_{x\to 1}\frac{1-1/x}{(x-1)(1/x)+\ln x}\cdot\frac{x}{x}$$

$$=\lim_{x\to 1}\frac{x-1}{x-1+x\ln x}=\lim_{x\to 1}\frac{1}{1+1+\ln x}=\frac{1}{2+0}=\frac{1}{2}$$

49. The limit has the form $\infty-\infty$ and we will change the form to a product by factoring out x.

$$\lim_{x\to\infty}(x-\ln x)=\lim_{x\to\infty}x\left(1-\frac{\ln x}{x}\right)=\infty \text{ since } \lim_{x\to\infty}\frac{\ln x}{x}=\lim_{x\to\infty}\frac{1/x}{1}=0 .$$

50. As $x\to\infty$, $1/x\to 0$, and $e^{1/x}\to 1$. So the limit has the form $\infty-\infty$ and we will change the form to a product by factoring out x.

$$\lim_{x\to\infty}\left(xe^{1/x}-x\right)=\lim_{x\to\infty}x\left(e^{1/x}-1\right)=\lim_{x\to\infty}\frac{e^{1/x}-1}{1/x}=\lim_{x\to\infty}\frac{e^{1/x}\left(-1/x^2\right)}{-1/x^2}=\lim_{x\to\infty}e^{1/x}=e^0=1$$

51. $y=x^{x^2}\Rightarrow \ln y=x^2\ln x$, so $\lim_{x\to0^+}\ln y=\lim_{x\to0^+}x^2\ln x=\lim_{x\to0^+}\frac{\ln x}{1/x^2}=\lim_{x\to0^+}\frac{1/x}{-2/x^3}=\lim_{x\to0^+}\left(-\frac{1}{2}x^2\right)=0\Rightarrow$

$\lim_{x\to0^+}x^{x^2}=\lim_{x\to0^+}e^{\ln y}=e^0=1$.

52. $y=(\tan 2x)^x\Rightarrow \ln y=x\cdot\ln\tan 2x$, so

$$\lim_{x\to0^+}\ln y=\lim_{x\to0^+}x\cdot\ln\tan 2x=\lim_{x\to0^+}\frac{\ln\tan 2x}{1/x}$$

$$=\lim_{x\to0^+}\frac{(1/\tan 2x)(2\sec^2 2x)}{-1/x^2}=\lim_{x\to0^+}\frac{-2x^2\cos 2x}{\sin 2x\cos^2 2x}=\lim_{x\to0^+}\frac{2x}{\sin 2x}\cdot\lim_{x\to0^+}\frac{-x}{\cos 2x}=1\cdot0=0\Rightarrow$$

$$\lim_{x\to0^+}(\tan 2x)^x=\lim_{x\to0^+}e^{\ln y}=e^0=1 .$$

53. $y=(1-2x)^{1/x}\Rightarrow \ln y=\frac{1}{x}\ln(1-2x)$, so $\lim_{x\to0}\ln y=\lim_{x\to0}\frac{\ln(1-2x)}{x}=\lim_{x\to0}\frac{-2/(1-2x)}{1}=-2\Rightarrow$

$\lim_{x\to0}(1-2x)^{1/x}=\lim_{x\to0}e^{\ln y}=e^{-2}$.

54. $y=\left(1+\frac{a}{x}\right)^{bx}\Rightarrow \ln y=bx\ln\left(1+\frac{a}{x}\right)$, so

$$\lim_{x\to\infty}\ln y=\lim_{x\to\infty}\frac{b\ln(1+a/x)}{1/x}=\lim_{x\to\infty}\frac{b\left(\frac{1}{1+a/x}\right)\left(-\frac{a}{x^2}\right)}{-1/x^2}=\lim_{x\to\infty}\frac{ab}{1+a/x}=ab\Rightarrow$$

$\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^{bx}=\lim_{x\to\infty}e^{\ln y}=e^{ab}$.

55. $y=\left(1+\frac{3}{x}+\frac{5}{x^2}\right)^x \Rightarrow \ln y=x\ln\left(1+\frac{3}{x}+\frac{5}{x^2}\right)\Rightarrow$

$$\lim_{x\to\infty}\ln y=\lim_{x\to\infty}\frac{\ln\left(1+\frac{3}{x}+\frac{5}{x^2}\right)}{1/x}=\lim_{x\to\infty}\frac{\left(-\frac{3}{x^2}-\frac{10}{x^3}\right)\Big/\left(1+\frac{3}{x}+\frac{5}{x^2}\right)}{-1/x^2}=\lim_{x\to\infty}\frac{3+\frac{10}{x}}{1+\frac{3}{x}+\frac{5}{x^2}}=3,$$

so $\lim_{x\to\infty}\left(1+\frac{3}{x}+\frac{5}{x^2}\right)^x=\lim_{x\to\infty}e^{\ln y}=e^3$.

56. $y=x^{(\ln 2)/(1+\ln x)}\Rightarrow \ln y=\frac{\ln 2}{1+\ln x}\ln x\Rightarrow$

$$\lim_{x\to\infty}\ln y=\lim_{x\to\infty}\frac{(\ln 2)(\ln x)}{1+\ln x}=\lim_{x\to\infty}\frac{(\ln 2)(1/x)}{1/x}=\lim_{x\to\infty}\ln 2=\ln 2,$$

so $\lim_{x\to\infty}x^{(\ln 2)/(1+\ln x)}=\lim_{x\to\infty}e^{\ln y}=e^{\ln 2}=2$.

57. $y=x^{1/x}\Rightarrow \ln y=(1/x)\ln x\Rightarrow \lim_{x\to\infty}\ln y=\lim_{x\to\infty}\frac{\ln x}{x}=\lim_{x\to\infty}\frac{1/x}{1}=0\Rightarrow \lim_{x\to\infty}x^{1/x}=\lim_{x\to\infty}e^{\ln y}=e^0=1$

58. $y=\left(e^x+x\right)^{1/x}\Rightarrow \ln y=\frac{1}{x}\ln\left(e^x+x\right)$, so

$$\lim_{x\to\infty}\ln y=\lim_{x\to\infty}\frac{\ln\left(e^x+x\right)}{x}=\lim_{x\to\infty}\frac{e^x+1}{e^x+x}=\lim_{x\to\infty}\frac{e^x}{e^x+1}=\lim_{x\to\infty}\frac{e^x}{e^x}=1\Rightarrow \lim_{x\to\infty}\left(e^x+x\right)^{1/x}=\lim_{x\to\infty}e^{\ln y}=e^1=e.$$

59. $y=\left(\frac{x}{x+1}\right)^x\Rightarrow \ln y=x\ln\left(\frac{x}{x+1}\right)\Rightarrow$

$$\lim_{x\to\infty}\ln y=\lim_{x\to\infty}x\ln\left(\frac{x}{x+1}\right)=\lim_{x\to\infty}\frac{\ln x-\ln(x+1)}{1/x}=\lim_{x\to\infty}\frac{1/x-1/(x+1)}{-1/x^2}$$

$$=\lim_{x\to\infty}\left(-x+\frac{x^2}{x+1}\right)=\lim_{x\to\infty}\frac{-x}{x+1}=-1$$

so

$\lim\limits_{x\to\infty}\left(\frac{x}{x+1}\right)^{x}=\lim\limits_{x\to\infty}e^{\ln y}=e^{-1}$

Or: $\lim\limits_{x\to\infty}\left(\frac{x}{x+1}\right)^{x}=\lim\limits_{x\to\infty}\left[\left(\frac{x+1}{x}\right)^{-1}\right]^{x}=\left[\lim\limits_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}\right]^{-1}=e^{-1}$

60. $y=(\cos 3x)^{5/x}\Rightarrow\ln y=\frac{5}{x}\ln(\cos 3x)\Rightarrow\lim\limits_{x\to 0}\ln y=5\lim\limits_{x\to 0}\frac{\ln(\cos 3x)}{x}=5\lim\limits_{x\to 0}\frac{-3\tan 3x}{1}=0$,

so $\lim\limits_{x\to 0}(\cos 3x)^{5/x}=e^{0}=1$.

61. $y=(\cos x)^{1/x^2}\Rightarrow\ln y=\frac{1}{x^2}\ln\cos x\Rightarrow\lim\limits_{x\to 0^+}\ln y=\lim\limits_{x\to 0^+}\frac{\ln\cos x}{x^2}=\lim\limits_{x\to 0^+}\frac{-\tan x}{2x}=\lim\limits_{x\to 0^+}\frac{-\sec^2 x}{2}=-\frac{1}{2}\Rightarrow$

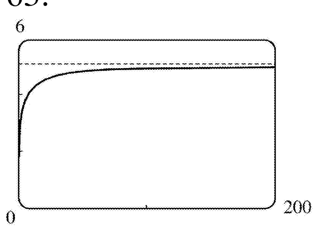
$\lim\limits_{x\to 0^+}(\cos x)^{1/x^2}=\lim\limits_{x\to 0^+}e^{\ln y}=e^{-1/2}=1/\sqrt{e}$

62. $y=\left(\frac{2x-3}{2x+5}\right)^{2x+1}\Rightarrow\ln y=(2x+1)\ln\left(\frac{2x-3}{2x+5}\right)\Rightarrow$

$\lim\limits_{x\to\infty}\ln y=\lim\limits_{x\to\infty}\frac{\ln(2x-3)-\ln(2x+5)}{1/(2x+1)}=\lim\limits_{x\to\infty}\frac{2/(2x-3)-2/(2x+5)}{-2/(2x+1)^2}=\lim\limits_{x\to\infty}\frac{-8(2x+1)^2}{(2x-3)(2x+5)}$

$=\lim\limits_{x\to\infty}\frac{-8(2+1/x)^2}{(2-3/x)(2+5/x)}=-8\Rightarrow\lim\limits_{x\to\infty}\left(\frac{2x-3}{2x+5}\right)^{2x+1}=e^{-8}$

63.



From the graph, it appears that $\lim\limits_{x\to\infty}x[\ln(x+5)-\ln x]=5$.

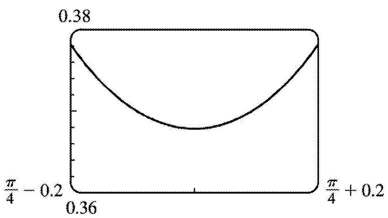
To prove this, we first note that

$\ln(x+5)-\ln x=\ln\frac{x+5}{x}=\ln\left(1+\frac{5}{x}\right)\to\ln 1=0$ as $x\to\infty$. Thus,

$$\lim_{x\to\infty} x[\ln(x+5)-\ln x] = \lim_{x\to\infty} \frac{\ln(x+5)-\ln x}{1/x} = \lim_{x\to\infty} \frac{\frac{1}{x+5}-\frac{1}{x}}{-1/x^2}$$

$$= \lim_{x\to\infty} \left[\frac{x-(x+5)}{x(x+5)} \cdot \frac{-x^2}{1} \right] = \lim_{x\to\infty} \frac{5x^2}{x^2+5x} = 5$$

64.



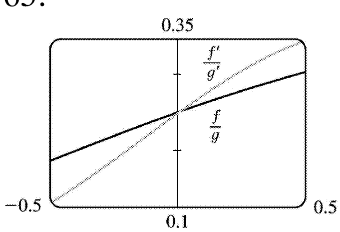
From the graph, it appears that $\lim_{x\to\pi/4} (\tan x)^{\tan 2x} \approx 0.368$.

The limit has the form 1^{∞} . Now $y=(\tan x)^{\tan 2x} \Rightarrow$
$\ln y = \tan 2x \ln(\tan x)$, so

$$\lim_{x\to\pi/4} \ln y = \lim_{x\to\pi/4} \frac{\ln(\tan x)}{\cot 2x} = \lim_{x\to\pi/4} \frac{\sec^2 x/\tan x}{-2\csc^2 2x} = \frac{2/1}{-2(1)} = -1 \Rightarrow$$

$$\lim_{x\to\pi/4} (\tan x)^{\tan 2x} = \lim_{x\to\pi/4} e^{\ln y} = e^{-1} = 1/e \approx 0.3679 .$$

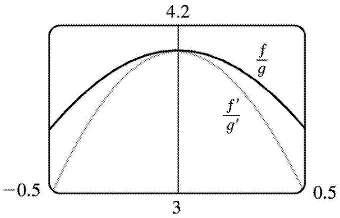
65.



From the graph, it appears that $\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{f'(x)}{g'(x)} = 0.25$. We calculate

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{e^x-1}{x^3+4x} = \lim_{x\to 0} \frac{e^x}{3x^2+4} = \frac{1}{4} .$$

66.



From the graph, it appears that $\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{f'(x)}{g'(x)} = 4$. We calculate

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{2x\sin x}{\sec x - 1} = \lim_{x\to 0} \frac{2(x\cos x + \sin x)}{\sec x\tan x}$$

$$= \lim_{x\to 0} \frac{2(-x\sin x + \cos x + \cos x)}{\sec x\left(\sec^2 x\right) + \tan x(\sec x\tan x)} = \frac{4}{1} = 4$$

67. $\lim_{x\to\infty} \frac{e^x}{x^n} = \lim_{x\to\infty} \frac{e^x}{nx^{n-1}} = \lim_{x\to\infty} \frac{e^x}{n(n-1)x^{n-2}} = \cdots = \lim_{x\to\infty} \frac{e^x}{n!} = \infty$

68. $\lim_{x\to\infty} \frac{\ln x}{x^p} = \lim_{x\to\infty} \frac{1/x}{px^{p-1}} = \lim_{x\to\infty} \frac{1}{px^p} = 0$ since $p>0$.

69. First we will find $\lim_{n\to\infty} \left(1+\frac{i}{n}\right)^{nt}$, which is of the form 1^∞ . $y=\left(1+\frac{i}{n}\right)^{nt} \Rightarrow$

$\ln y = nt\ln\left(1+\frac{i}{n}\right)$, so

$$\lim_{n\to\infty} \ln y = \lim_{n\to\infty} nt\ln\left(1+\frac{i}{n}\right) = t\lim_{n\to\infty} \frac{\ln(1+i/n)}{1/n} = t\lim_{n\to\infty} \frac{\left(-i/n^2\right)}{(1+i/n)\left(-1/n^2\right)} = t\lim_{n\to\infty} \frac{i}{1+i/n} = ti \Rightarrow \lim_{n\to\infty} y = e^{it}$$

. Thus, as $n\to\infty$, $A = A_0\left(1+\frac{i}{n}\right)^{nt} \to A_0 e^{it}$.

70. **(a)**

$$\lim_{t\to\infty} v = \lim_{t\to\infty} \frac{mg}{c}\left(1-e^{-ct/m}\right) = \frac{mg}{c}\lim_{t\to\infty}\left(1-e^{-ct/m}\right)$$

$$= \frac{mg}{c}(1-0) = \frac{mg}{c} ,$$

which is the speed the object approaches as time goes on, the so–called limiting velocity.

(b) $\lim_{m\to\infty} v$

$$=\lim_{m\to\infty}\frac{mg}{c}\left(1-e^{-ct/m}\right)=\frac{g}{c}\lim_{m\to\infty}\frac{1-e^{-ct/m}}{1/m}=\frac{g}{c}\lim_{m\to\infty}\frac{-e^{-ct/m}\left(ct/m^2\right)}{-1/m^2}$$

$$=\frac{g}{c}(ct)\lim_{m\to\infty}e^{-ct/m}=gt(1)$$

The speed of a very heavy falling object is approximately proportional to the elapsed time t , provided it can fall for time t in an environment where the given model continues to hold. .

71. We see that both numerator and denominator approach 0 , so we can use l'Hospital's Rule:

$$\lim_{x\to a}\frac{\sqrt{2a^3x-x^4}-a\sqrt[3]{aax}}{a-\sqrt[4]{ax^3}}=\lim_{x\to a}\frac{\frac{1}{2}\left(2a^3x-x^4\right)^{-1/2}\left(2a^3-4x^3\right)-a\left(\frac{1}{3}\right)(aax)^{-2/3}a^2}{-\frac{1}{4}\left(ax^3\right)^{-3/4}\left(3ax^2\right)}$$

$$=\frac{\frac{1}{2}\left(2a^3a-a^4\right)^{-1/2}\left(2a^3-4a^3\right)-\frac{1}{3}a^3\left(a^2a\right)^{-2/3}}{-\frac{1}{4}\left(aa^3\right)^{-3/4}\left(3aa^2\right)}$$

$$=\frac{\left(a^4\right)^{-1/2}\left(-a^3\right)-\frac{1}{3}a^3\left(a^3\right)^{-2/3}}{-\frac{3}{4}a^3\left(a^4\right)^{-3/4}}=\frac{-a-\frac{1}{3}a}{-\frac{3}{4}}=\frac{4}{3}\left(\frac{4}{3}a\right)=\frac{16}{9}a$$

72. Let the radius of the circle be r . We see that $A(\theta)$ is the area of the whole figure (a sector of the circle with

radius 1), minus the area of $\triangle OPR$. But the area of the sector of the circle is $\frac{1}{2}r^2\theta$ (see Reference

Page 1), and the area of the triangle is $\frac{1}{2}r|PQ|=\frac{1}{2}r(r\sin\theta)=\frac{1}{2}r^2\sin\theta$. So we have

$A(\theta)=\frac{1}{2}r^2\theta-\frac{1}{2}r^2\sin\theta=\frac{1}{2}r^2(\theta-\sin\theta)$. Now by elementary trigonometry,

$B(\theta)=\frac{1}{2}|QR|\,|PQ|=\frac{1}{2}(r-|OQ|)|PQ|=\frac{1}{2}(r-r\cos\theta)(r\sin\theta)=\frac{1}{2}r^2(1-\cos\theta)\sin\theta$.

So the limit we want is

$$\lim_{\theta\to 0^+}\frac{A(\theta)}{B(\theta)}=\lim_{\theta\to 0^+}\frac{\frac{1}{2}r^2(\theta-\sin\theta)}{\frac{1}{2}r^2(1-\cos\theta)\sin\theta}=\lim_{\theta\to 0^+}\frac{1-\cos\theta}{(1-\cos\theta)\cos\theta+\sin\theta(\sin\theta)}$$

$$= \lim_{\theta \to 0^+} \frac{1-\cos\theta}{\cos\theta - \cos^2\theta + \sin^2\theta} = \lim_{\theta \to 0^+} \frac{\sin\theta}{-\sin\theta - 2\cos\theta(-\sin\theta) + 2\sin\theta(\cos\theta)}$$

$$= \lim_{\theta \to 0^+} \frac{\sin\theta}{-\sin\theta + 4\sin\theta\cos\theta} = \lim_{\theta \to 0^+} \frac{1}{-1+4\cos\theta} = \frac{1}{-1+4\cos 0} = \frac{1}{3}$$

73. Since $f(2)=0$, the given limit has the form $\frac{0}{0}$.

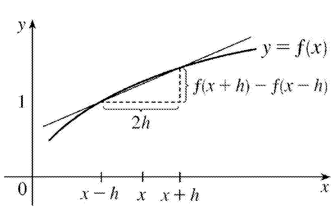
$$\lim_{x\to 0} \frac{f(2+3x)+f(2+5x)}{x} = \lim_{x\to 0} \frac{f'(2+3x)\cdot 3+f'(2+5x)\cdot 5}{1} = f'(2)\cdot 3+f'(2)\cdot 5=8f'(2)=8\cdot 7=56$$

74. $L=\lim_{x\to 0}\left(\frac{\sin 2x}{x^3}+a+\frac{b}{x^2}\right)=\lim_{x\to 0}\frac{\sin 2x+ax^3+bx}{x^3}=\lim_{x\to 0}\frac{2\cos 2x+3ax^2+b}{3x^2}$. As $x\to 0$, $3x^2\to 0$, and $\left(2\cos 2x+3ax^2+b\right)\to b+2$, so the last limit exists only if $b+2=0$, that is, $b=-2$. Thus, $\lim_{x\to 0}\frac{2\cos 2x+3ax^2-2}{3x^2}=\lim_{x\to 0}\frac{-4\sin 2x+6ax}{6x}=\lim_{x\to 0}\frac{-8\cos 2x+6a}{6}=\frac{6a-8}{6}$, which is equal to 0 if and only if $a=\frac{4}{3}$. Hence, $L=0$ if and only if $b=-2$ and $a=\frac{4}{3}$.

75. Since $\lim_{h\to 0}[f(x+h)-f(x-h)]=f(x)-f(x)=0$ (f is differentiable and hence continuous) and $\lim_{h\to 0} 2h=0$, we use l'Hospital's Rule:

$$\lim_{h\to 0}\frac{f(x+h)-f(x-h)}{2h}=\lim_{h\to 0}\frac{f'(x+h)(1)-f'(x-h)(-1)}{2}=\frac{f'(x)+f'(x)}{2}=\frac{2f'(x)}{2}=f'(x)$$

$\frac{f(x+h)-f(x-h)}{2h}$ is the slope of the secant line between $(x-h, f(x-h))$ and $(x+h, f(x+h))$. As $h\to 0$, this line gets closer to the tangent line and its slope approaches $f'(x)$.



76. Since $\lim_{h\to 0}\left[f(x+h)-2f(x)+f(x-h)\right]=f(x)-2f(x)+f(x)=0$ (f is differentiable and hence continuous) and

$\lim_{h\to 0} h^2=0$, we can apply l'Hospital's Rule:

$$\lim_{h\to 0}\frac{f(x+h)-2f(x)+f(x-h)}{h^2}=\lim_{h\to 0}\frac{f'(x+h)-f'(x-h)}{2h}=f''(x)$$

At the last step, we have applied the result of Exercise to $f'(x)$.

77. **(a)** We show that $\lim_{x\to 0}\frac{f(x)}{x^n}=0$ for every integer $n\geq 0$. Let $y=\frac{1}{x^2}$. Then

$$\lim_{x\to 0}\frac{f(x)}{x^{2n}}=\lim_{x\to 0}\frac{e^{-1/x^2}}{\left(x^2\right)^n}=\lim_{y\to\infty}\frac{y^n}{e^y}=\lim_{y\to\infty}\frac{ny^{n-1}}{e^y}=\cdots=\lim_{y\to\infty}\frac{n!}{e^y}=0\Rightarrow$$

$\lim_{x\to 0}\frac{f(x)}{x^n}=\lim_{x\to 0} x^n\frac{f(x)}{x^{2n}}=\lim_{x\to 0} x^n\lim_{x\to 0}\frac{f(x)}{x^{2n}}=0$. Thus, $f'(0)=\lim_{x\to 0}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0}\frac{f(x)}{x}=0$.

(b) Using the Chain Rule and the Quotient Rule we see that $f^{(n)}(x)$ exists for $x\neq 0$. In fact, we prove by induction that for each $n\geq 0$, there is a polynomial p_n and a non–negative integer k_n with $f^{(n)}(x)=p_n(x)f(x)/x^{k_n}$ for $x\neq 0$. This is true for $n=0$; suppose it is true for the n th derivative. Then $f'(x)=f(x)(2/x^3)$, so

$$\begin{aligned}
f^{(n+1)}(x) &= \left[x^{k_n}\left[p_n'(x)f(x)+p_n(x)f'(x)\right]-k_n x^{k_n-1}p_n(x)f(x)\right]x^{-2k_n}\\
&= \left[x^{k_n}p_n'(x)+p_n(x)\left(2/x^3\right)-k_n x^{k_n-1}p_n(x)\right]f(x)x^{-2k_n}\\
&= \left[x^{k_n+3}p_n'(x)+2p_n(x)-k_n x^{k_n+2}p_n(x)\right]f(x)x^{-\left(2k_n+3\right)}
\end{aligned}$$

which has the desired form.

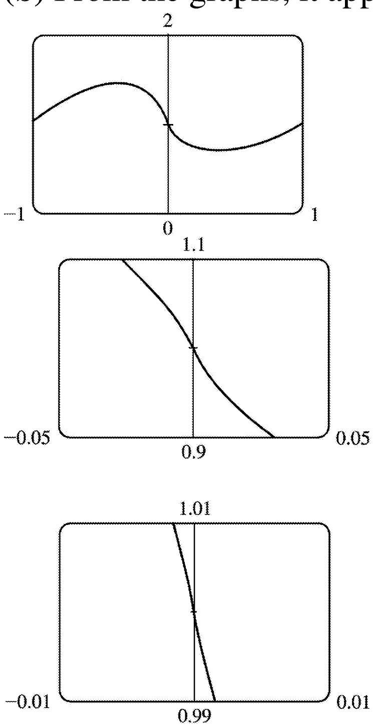
Now we show by induction that $f^{(n)}(0)=0$ for all n . By part (a), $f'(0)=0$. Suppose that $f^{(n)}(0)=0$. Then

$$f^{(n+1)}(0)=\lim_{x\to 0}\frac{f^{(n)}(x)-f^{(n)}(0)}{x-0}=\lim_{x\to 0}\frac{f^{(n)}(x)}{x}=\lim_{x\to 0}\frac{p_n(x)f(x)/x^{k_n}}{x}=\lim_{x\to 0}\frac{p_n(x)f(x)}{x^{k_n+1}}$$

$$= \lim_{x\to 0} p_n(x) \lim_{x\to 0} \frac{f(x)}{x^{k_n+1}} = p_n(0)\cdot 0 = 0$$

78. **(a)** For f to be continuous, we need $\lim_{x\to 0} f(x)=f(0)=1$. We note that for $x\neq 0$, $\ln f(x)=\ln |x|^x = x\ln |x|$. So $\lim_{x\to 0} \ln f(x)=\lim_{x\to 0} x\ln |x| = \lim_{x\to 0} \frac{\ln |x|}{1/x} = \lim_{x\to 0} \frac{1/x}{-1/x^2} = 0$. Therefore, $\lim_{x\to 0} f(x)=\lim_{x\to 0} e^{\ln f(x)} = e^0 = 1$. So f is continuous at 0 .

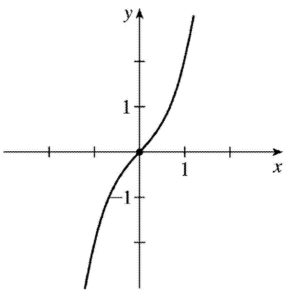
(b) From the graphs, it appears that f is differentiable at 0 .



(c) To find f' , we use logarithmic differentiation: $\ln f(x)=x\ln |x| \Rightarrow \frac{f'(x)}{f(x)} = x\left(\frac{1}{x}\right)+\ln |x| \Rightarrow$ $f'(x)=f(x)\left(1+\ln |x|\right)=|x|^x\left(1+\ln |x|\right)$, $x\neq 0$. Now $f'(x)\to -\infty$ as $x\to 0$, so the curve has a vertical tangent at $(0,1)$ and is therefore not differentiable there. The fact cannot be seen in the graphs in part (b) because $\ln |x|\to -\infty$ very slowly as $x\to 0$.

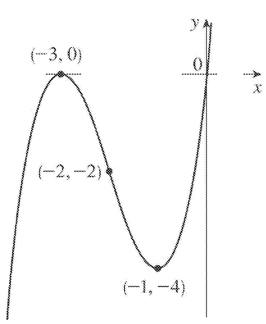
1. $y=f(x)=x^3+x=x\left(x^2+1\right)$ **A.** f is a polynomial, so $D=R$. **B.** x –intercept =0 , y– intercept $=f(0)=0$ **C.** $f(-x)=-f(x)$, so f is odd; the curve is symmetric about the origin. **D.** f is a polynomial, so there is no asymptote. **E.** $f'(x)=3x^2+1>0$, so f is increasing on $(-\infty,\infty)$. **F.** There is no critical number and hence, no local maximum or minimum value. **G.** $f''(x)=6x>0$ on $(0,\infty)$ and $f''(x)<0$ on $(-\infty,0)$, so f is CU on $(0,\infty)$ and CD on $(-\infty,0)$. Since the concavity changes at $x=0$, there is an inflection point at $(0,0)$.

H.



2. $y=f(x)=x^3+6x^2+9x=x(x+3)^2$ **A.** $D=R$ **B.** x –intercepts are –3 and 0 , y –intercept =0 **C.** No symmetry **D.** No asymptote **E.** $f'(x)=3x^2+12x+9=3(x+1)(x+3)<0\Leftrightarrow -3<x<-1$, so f is decreasing on $(-3,-1)$ and increasing on $(-\infty,-3)$ and $(-1,\infty)$. **F.** Local maximum value $f(-3)=0$, local minimum value $f(-1)=-4$ **G.** $f''(x)=6x+12=6(x+2)>0\Leftrightarrow x>-2$, so f is CU on $(-2,\infty)$ and CD on $(-\infty,-2)$. IP at $(-2,-2)$

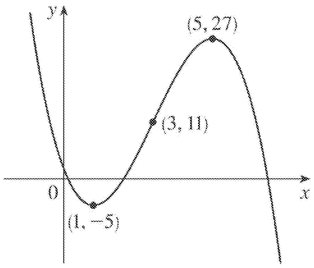
H.



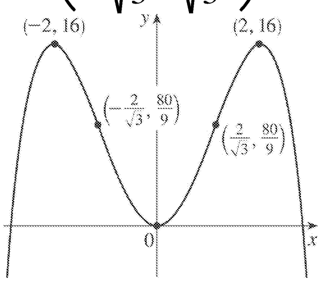
3. $y=f(x)=2-15x+9x^2-x^3=-(x-2)\left(x^2-7x+1\right)$ **A.** $D=R$ **B.** y –intercept: $f(0)=2$; x– intercepts: $f(x)=0$ $\Rightarrow x=2$ or (by the quadratic formula) $x=\dfrac{7\pm\sqrt{45}}{2}\approx 0.15$, 6.85

C. No symmetry **D.** No asymptote **E.** $f'(x)=-15+18x-3x^2=-3\left(x^2-6x+5\right)=-3(x-1)(x-5)>0\Leftrightarrow 1<x<5$ so f is increasing on $(1,5)$ and decreasing on $(-\infty,1)$ and $(5,\infty)$. **F.** Local maximum value $f(5)=27$, local minimum value $f(1)=-5$ **G.** $f''(x)=18-6x=-6(x-3)>0\Leftrightarrow x<3$, so f is CU on $(-\infty,3)$ and CD on $(3,\infty)$. IP at $(3,11)$

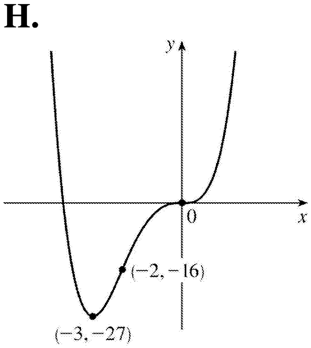
H.



4. $y=f(x)=8x^2-x^4=x^2\left(8-x^2\right)$ **A.** $D=R$ **B.** y –intercept: $f(0)=0$; x– intercepts: $f(x)=0\Rightarrow x=0$, $\pm 2\sqrt{2}$ ($\approx\pm 2.83$) **C.** $f(-x)=f(x)$, so f is even and symmetric about the y –axis. **D.** No asymptote **E.** $f'(x)=16x-4x^3=4x\left(4-x^2\right)=4x(2+x)(2-x)>0\Leftrightarrow x<-2$ or $0<x<2$, so f is increasing on $(-\infty,-2)$ and $(0,2)$ and decreasing on $(-2,0)$ and $(2,\infty)$. **F.** Local maximum value $f(\pm 2)=16$, local minimum value $f(0)=0$ **G.** $f''(x)=16-12x^2=4\left(4-3x^2\right)=0\Leftrightarrow x=\pm\dfrac{2}{\sqrt{3}}$. $f''(x)>0\Leftrightarrow -\dfrac{2}{\sqrt{3}}<x<\dfrac{2}{\sqrt{3}}$, so f is CU on $\left(-\dfrac{2}{\sqrt{3}},\dfrac{2}{\sqrt{3}}\right)$ and CD on $\left(-\infty,-\dfrac{2}{\sqrt{3}}\right)$ and $\left(\dfrac{2}{\sqrt{3}},\infty\right)$. IP at $\left(\pm\dfrac{2}{\sqrt{3}},\dfrac{80}{9}\right)$ **H.**



5. $y=f(x)=x^4+4x^3=x^3(x+4)$ **A.** $D=R$ **B.** y –intercept: $f(0)=0$; x –intercepts: $f(x)=0\Leftrightarrow x=-4,0$ **C.** No symmetry **D.** No asymptote **E.** $f'(x)=4x^3+12x^2=4x^2(x+3)>0\Leftrightarrow x>-3$, so f is increasing on $(-3,\infty)$ and decreasing on $(-\infty,-3)$. **F.** Local minimum value $f(-3)=-27$, no local maximum **G.**
$f''(x)=12x^2+24x=12x(x+2)<0\Leftrightarrow -2<x<0$,
so f is CD on $(-2,0)$ and CU on $(-\infty,-2)$ and $(0,\infty)$.
IP at $(0,0)$ and $(-2,-16)$
H.



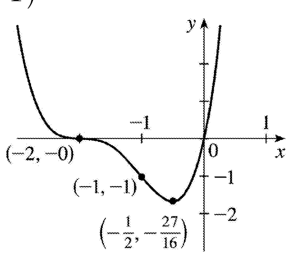
6. $y=f(x)=x(x+2)^3$ **A.** $D=R$ **B.** $y-$ intercept: $f(0)=0$; x –intercepts: $f(x)=0 \Leftrightarrow x=-2,0$ **C.** No symmetry **D.** No asymptote **E.** $f'(x)=3x(x+2)^2+(x+2)^3=(x+2)^2[3x+(x+2)]=(x+2)^2(4x+2)$. $f'(x)>0 \Leftrightarrow x>-\frac{1}{2}$, and $f'(x)<0 \Leftrightarrow x<-2$ or $-2<x<-\frac{1}{2}$, so f is increasing on $\left(-\frac{1}{2},\infty\right)$ and decreasing on $(-\infty,-2)$ **F.** Local minimum value $f\left(-\frac{1}{2}\right)=-\frac{27}{16}$, no local maximum

G.

$$\begin{aligned} f''(x) &=(x+2)^2(4)+(4x+2)(2)(x+2) \\ &=2(x+2)[(x+2)(2)+4x+2] \\ &=2(x+2)(6x+6)=12(x+1)(x+2) \end{aligned}$$

$f''(x)<0 \Leftrightarrow -2<x<-1$, so f is CD on $(-2,-1)$ and CU on $(-\infty,-2)$ and $(-1,\infty)$. IP at $(-2,0)$ and $(-1,-1)$

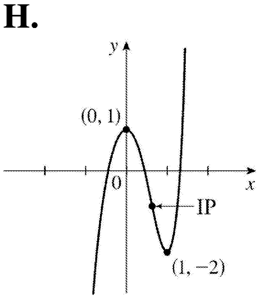
H.



7. $y=f(x)=2x^5-5x^2+1$ **A.** $D=R$ **B.** $y-$ intercept: $f(0)=1$ **C.** No symmetry **D.** No asymptote **E.** $f'(x)=10x^4-10x=10x(x^3-1)=10x(x-1)(x^2+x+1)$, so $f'(x)<0 \Leftrightarrow 0<x<1$ and $f'(x)>0 \Leftrightarrow x<0$ or $x>1$. Thus, f is increasing on $(-\infty,0)$ and $(1,\infty)$ and decreasing on $(0,1)$.

F. Local maximum value $f(0)=1$, local minimum value $f(1)=-2$ **G.** $f''(x)=40x^3-10=10(4x^3-1)$ so $f''(x)=0 \Leftrightarrow x=1/\sqrt[3]{4}$. $f''(x)>0 \Leftrightarrow x>1/\sqrt[3]{4}$ and $f''(x)<0 \Leftrightarrow x<1/\sqrt[3]{4}$, < so f is CD on $\left(-\infty,1/\sqrt[3]{4}\right)$ and CU on $\left(1/\sqrt[3]{4},\infty\right)$. IP at $\left(\frac{1}{\sqrt[3]{4}},1-\frac{9}{2\left(\sqrt[3]{4}\right)^2}\right) \approx (0.630,-0.786)$

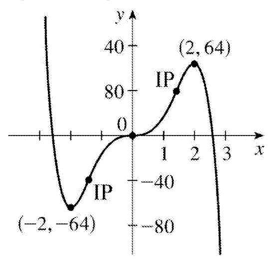
H.



8.

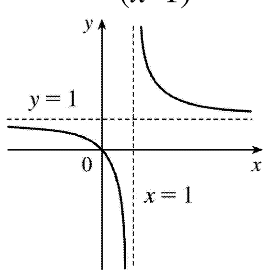
$y=f(x)=20x^3-3x^5$ **A.** $D=R$ **B.** $y-$ intercept: $f(0)=0$; x –intercepts: $f(x)=0\Leftrightarrow -3x^3\left(x^2-\dfrac{20}{3}\right)=0\Leftrightarrow x=0$ or $\pm\sqrt{20/3}\approx\pm 2.582$ **C.** $f(-x)=-f(x)$, so f is odd;
the curve is symmetric about the origin. **D.** No asymptote **E.**
$f'(x)=60x^2-15x^4=-15x^2(x^2-4)=-15x^2(x+2)(x-2)$, so $f'(x)>0\Leftrightarrow -2<x<0$ or $0<x<2$ and $f'(x)<0\Leftrightarrow x<-2$ or $x>2$. Thus, f is increasing on $(-2,0)$ and $(0,2)$ and f is decreasing on $(-\infty,-2)$ and $(2,\infty)$. **F.** Local minimum value $f(-2)=-64$, local maximum value $f(2)=64$ **G.**
$f''(x)=120x-60x^3=-60x(x^2-2)$. $f''(x)>0\Leftrightarrow x<-\sqrt{2}$ or $0<x<\sqrt{2}$; $f''(x)<0\Leftrightarrow -\sqrt{2}<x<0$ or $x>\sqrt{2}$. Thus, f is CU on $(-\infty,-\sqrt{2})$
and $(0,\sqrt{2})$, and f is CD on $(-\sqrt{2},0)$ and $(\sqrt{2},\infty)$. IP at $(-\sqrt{2},-28\sqrt{2})\approx(-1.414,-39.598)$, $(0,0)$, and $(\sqrt{2},28\sqrt{2})$

H.



9. $y=f(x)=x/(x-1)$ **A.** $D=\{x\mid x\neq 1\}=(-\infty,1)\cup(1,\infty)$ **B.** x –intercept $=0$, y –intercept $=f(0)=0$ **C.** No symmetry **D.** $\lim\limits_{x\to\pm\infty}\dfrac{x}{x-1}=1$, so $y=1$ is a HA. $\lim\limits_{x\to 1^-}\dfrac{x}{x-1}=-\infty$, $\lim\limits_{x\to 1^+}\dfrac{x}{x-1}=\infty$, so $x=1$ is a VA. **E.**
$f'(x)=\dfrac{(x-1)-x}{(x-1)^2}=\dfrac{-1}{(x-1)^2}<0$ for $x\neq 1$, so f is decreasing on $(-\infty,1)$ and $(1,\infty)$. **F.** No extreme values
G. $f''(x)=\dfrac{2}{(x-1)^3}>0\Leftrightarrow x>1$, so f is CU on $(1,\infty)$ and CD on $(-\infty,1)$. No IP

H.

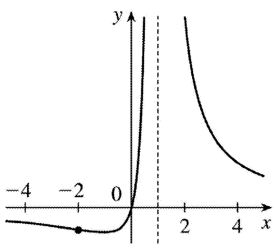


10. $y=x/(x-1)^2$ **A.** $D=\{x\mid x\neq 1\}=(-\infty,1)\cup(1,\infty)$ **B.** $x-$ intercept $=0$, y –intercept $=f(0)=0$ **C.** No symmetry **D.** $\lim\limits_{x\to\pm\infty}\dfrac{x}{(x-1)^2}=0$, so $y=0$ is a HA. $\lim\limits_{x\to 1}\dfrac{x}{(x-1)^2}=\infty$, so $x=1$ is a VA. **E.**

$f'(x)=\dfrac{(x-1)^2(1)-x(2)(x-1)}{(x-1)^4}=\dfrac{-x-1}{(x-1)^3}$. This is negative on $(-\infty,-1)$ and $(1,\infty)$ and positive on $(-1,1)$, so $f(x)$ is decreasing on $(-\infty,-1)$ and $(1,\infty)$ and increasing on $(-1,1)$.

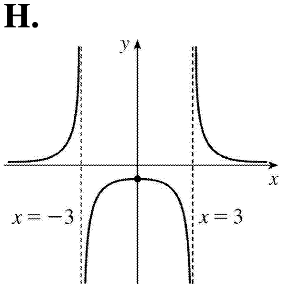
F. Local minimum value $f(-1)=-\dfrac{1}{4}$, no local maximum. **G.** $f''(x)=\dfrac{(x-1)^3(-1)+(x+1)(3)(x-1)^2}{(x-1)^6}=\dfrac{2(x+2)}{(x-1)^4}$. This is negative on $(-\infty,-2)$, and positive on $(-2,1)$ and $(1,\infty)$. So f is CD on $(-\infty,-2)$ and CU on $(-2,1)$ and $(1,\infty)$. IP at $\left(-2,-\dfrac{2}{9}\right)$

H.



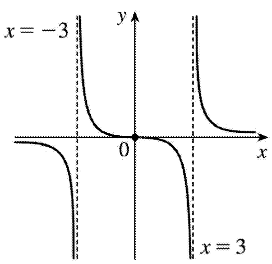
11. $y=f(x)=1/\left(x^2-9\right)$ **A.** $D=\{x\mid x\neq\pm 3\}=(-\infty,-3)\cup(-3,3)\cup(3,\infty)$ **B.** y –intercept $=f(0)=-\dfrac{1}{9}$, no x –intercept **C.** $f(-x)=f(x)\Rightarrow f$ is even; the curve is symmetric about the y– axis. **D.** $\lim\limits_{x\to\pm\infty}\dfrac{1}{x^2-9}=0$, so $y=0$ is a HA. $\lim\limits_{x\to 3^-}\dfrac{1}{x^2-9}=-\infty$, $\lim\limits_{x\to 3^+}\dfrac{1}{x^2-9}=\infty$, $\lim\limits_{x\to -3^-}\dfrac{1}{x^2-9}=\infty$, $\lim\limits_{x\to -3^+}\dfrac{1}{x^2-9}=-\infty$, so $x=3$ and $x=-3$ are VA. **E.** $f'(x)=-\dfrac{2x}{\left(x^2-9\right)^2}>0\Leftrightarrow x<0$ ($x\neq-3$) so f is increasing on $(-\infty,-3)$ and $(-3,0)$ and decreasing on $(0,3)$ and $(3,\infty)$. **F.** Local maximum value $f(0)=-\dfrac{1}{9}$. **G.** $y''=\dfrac{-2\left(x^2-9\right)^2+(2x)2\left(x^2-9\right)(2x)}{\left(x^2-9\right)^4}=\dfrac{6\left(x^2+3\right)}{\left(x^2-9\right)^3}>0\Leftrightarrow x^2>9\Leftrightarrow x>3$ or $x<-3$, so f is CU on $(-\infty,-3)$ and $(3,\infty)$ and CD on $(-3,3)$. No IP

H.



12. $y=f(x)=x/\left(x^2-9\right)$ **A.** $D=\{x \mid x\neq\pm 3\}=(-\infty,-3)\cup(-3,3)\cup(3,\infty)$ **B.** x –intercept $=0$, y– intercept $=f(0)=0$. **C.** $f(-x)=-f(x)$, so f is odd; the curve is symmetric about the origin. **D.** $\lim_{x\to\pm\infty}\frac{x}{x^2-9}=0$, so $y=0$ is a HA. $\lim_{x\to 3^+}\frac{x}{x^2-9}=\infty$, $\lim_{x\to 3^-}\frac{x}{x^2-9}=-\infty$, $\lim_{x\to -3^+}\frac{x}{x^2-9}=\infty$, $\lim_{x\to -3^-}\frac{x}{x^2-9}=-\infty$, so $x=3$ and $x=-3$ are VA. **E.** $f'(x)=\frac{\left(x^2-9\right)-x(2x)}{\left(x^2-9\right)^2}=-\frac{x^2+9}{\left(x^2-9\right)^2}<0$ ($x\neq\pm 3$) so f is decreasing on $(-\infty,-3)$, $(-3,3)$, and $(3,\infty)$. **F.** No extreme values **G.** $f''(x)=-\frac{2x\left(x^2-9\right)^2-\left(x^2+9\right)\cdot 2\left(x^2-9\right)(2x)}{\left(x^2-9\right)^4}=\frac{2x\left(x^2+27\right)}{\left(x^2-9\right)^3}>0$ when $-3<x<0$ or $x>3$, so f is CU on $(-3,0)$ and $(3,\infty)$; CD on $(-\infty,-3)$ and $(0,3)$ > IP at $(0,0)$

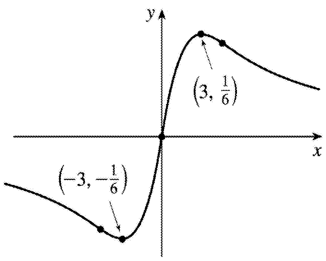
H.



13. $y=f(x)=x/\left(x^2+9\right)$ **A.** $D=R$ **B.** y –intercept: $f(0)=0$; x –intercept: $f(x)=0\Leftrightarrow x=0$ **C.** $f(-x)=-f(x)$, so f is odd and the curve is symmetric about the origin. **D.** $\lim_{x\to\pm\infty}\left[x/\left(x^2+9\right)\right]=0$, so $y=0$ is a HA; no VA **E.** $f'(x)=\frac{\left(x^2+9\right)(1)-x(2x)}{\left(x^2+9\right)^2}=\frac{9-x^2}{\left(x^2+9\right)^2}=\frac{(3+x)(3-x)}{\left(x^2+9\right)^2}>0\Leftrightarrow -3<x<3$, so f is increasing on $(-3,3)$ and decreasing on $(-\infty,-3)$ and $(3,\infty)$.

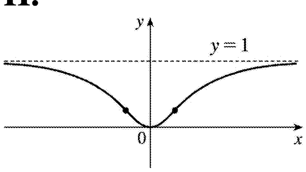
F. Local minimum value $f(-3)=-\frac{1}{6}$, local maximum value $f(3)=\frac{1}{6}$ **G.** $f''(x)$ $=\frac{\left(x^2+9\right)^2(-2x)-\left(9-x^2\right)\cdot 2\left(x^2+9\right)(2x)}{\left[\left(x^2+9\right)^2\right]^2}=\frac{(2x)\left(x^2+9\right)\left[-\left(x^2+9\right)-2\left(9-x^2\right)\right]}{\left(x^2+9\right)^4}=\frac{2x\left(x^2-27\right)}{\left(x^2+9\right)^3}=0\Leftrightarrow x=0$, $\pm\sqrt{27}=\pm 3\sqrt{3}$ $f''(x)>0\Leftrightarrow -3\sqrt{3}<x<0$ or $x>3\sqrt{3}$, so f is CU on $\left(-3\sqrt{3},0\right)$ and $\left(3\sqrt{3},\infty\right)$, and CD on $\left(-\infty,-3\sqrt{3}\right)$ and $\left(0,3\sqrt{3}\right)$. There are three inflection points: $(0,0)$ and $\left(\pm 3\sqrt{3},\pm\frac{1}{12}\sqrt{3}\right)$.

H.



14. $y=f(x)=x^2/(x^2+9)$ **A.** $D=R$ **B.** y –intercept: $f(0)=0$; x –intercept: $f(x)=0 \Leftrightarrow x=0$ **C.** $f(-x)=f(x)$, so f is even and symmetric about the y –axis. **D.** $\lim_{x\to\pm\infty}\left[x^2/(x^2+9)\right]=1$, so $y=1$ is a HA; no VA **E.** $f'(x)=\frac{(x^2+9)(2x)-x^2(2x)}{(x^2+9)^2}=\frac{18x}{(x^2+9)^2}>0 \Leftrightarrow x>0$, so f is increasing on $(0,\infty)$ and decreasing on $(-\infty,0)$. **F.** Local minimum value $f(0)=0$; no local maximum **G.** $f''(x)$ $=\frac{(x^2+9)^2(18)-18x\cdot 2(x^2+9)\cdot 2x}{\left[(x^2+9)^2\right]^2}=\frac{18(x^2+9)\left[(x^2+9)-4x^2\right]}{(x^2+9)^4}=\frac{18(9-3x^2)}{(x^2+9)^3}$ $=\frac{-54(x+\sqrt{3})(x-\sqrt{3})}{(x^2+9)^3}>0 \Leftrightarrow -\sqrt{3}<x<\sqrt{3}$ so f is CU on $(-\sqrt{3},\sqrt{3})$ and CD on $(-\infty,-\sqrt{3})$ and $(\sqrt{3},\infty)$. There are two inflection points: $\left(\pm\sqrt{3},\frac{1}{4}\right)$.

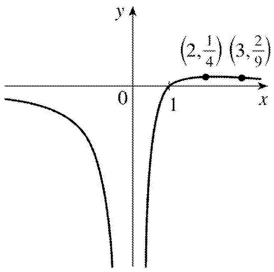
H.



15. $y=f(x)=\frac{x-1}{x^2}$ **A.** $D=\{x\,|\,x\neq 0\}=(-\infty,0)\cup(0,\infty)$ **B.** No y –intercept; x –intercept: $f(x)=0 \Leftrightarrow x=1$ **C.** No symmetry **D.** $\lim_{x\to\pm\infty}\frac{x-1}{x^2}=0$, so $y=0$ is a HA. $\lim_{x\to 0}\frac{x-1}{x^2}=-\infty$, so $x=0$ is a VA. **E.** $f'(x)=\frac{x^2\cdot 1-(x-1)\cdot 2x}{(x^2)^2}=\frac{-x^2+2x}{x^4}=\frac{-(x-2)}{x^3}$, so $f'(x)>0 \Leftrightarrow 0<x<2$ and $f'(x)<0 \Leftrightarrow x<0$ or $x>2$. Thus, f is increasing on $(0,2)$ and decreasing on $(-\infty,0)$ and $(2,\infty)$. **F.** No local minimum, local maximum value $f(2)=\frac{1}{4}$. **G.** $f''(x)=\frac{x^3\cdot(-1)-[-(x-2)]\cdot 3x^2}{(x^3)^2}=\frac{2x^3-6x^2}{x^6}=\frac{2(x-3)}{x^4}$. $f''(x)$ is negative on

$(-\infty,0)$ and $(0,3)$ and positive on $(3,\infty)$, so f is CD on $(-\infty,0)$ and $(0,3)$ and CU on $(3,\infty)$. IP at $\left(3,\frac{2}{9}\right)$

H.

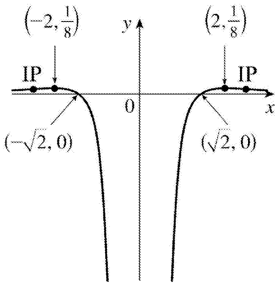


16. $y=f(x)=\dfrac{x^2-2}{x^4}$ **A.** $D=\{x\,|\,x\neq 0\}=(-\infty,0)\cup(0,\infty)$ **B.** No y –intercept; x –intercepts: $f(x)=0\Leftrightarrow$ $x=\pm\sqrt{2}$ **C.** $f(-x)=f(x)$, so f is even; the curve is symmetric about the y –axis. **D.** $\lim\limits_{x\to\pm\infty}\dfrac{x^2-2}{x^4}=0$, so $y=0$ is a HA. $\lim\limits_{x\to 0}\dfrac{x^2-2}{x^4}=-\infty$, so $x=0$ is a VA. **E.**

$f'(x)=\dfrac{x^4\cdot 2x-\left(x^2-2\right)(4x^3)}{\left(x^4\right)^2}=\dfrac{-2x^5+8x^3}{x^8}=\dfrac{-2(x^2-4)}{x^5}=\dfrac{-2(x+2)(x-2)}{x^5}$. $f'(x)$ is negative on $(-2,0)$ and $(2,\infty)$ and positive on $(-\infty,-2)$ and $(0,2)$, so f is decreasing on $(-2,0)$ and $(2,\infty)$ and increasing on $(-\infty,-2)$ and $(0,2)$. **F.** Local maximum value $f(\pm 2)=\dfrac{1}{8}$, no local minimum. G.

$f''(x)=\dfrac{x^5\cdot(-4x)+2\left(x^2-4\right)\cdot 5x^4}{\left(x^5\right)^2}=\dfrac{2x^4\left[-2x^2+5\left(x^2-4\right)\right]}{x^{10}}=\dfrac{2\left(3x^2-20\right)}{x^6}$ $f''(x)$ is positive on $\left(-\infty,-\sqrt{\frac{20}{3}}\right)$ and $\left(\sqrt{\frac{20}{3}},\infty\right)$ and negative on $\left(-\sqrt{\frac{20}{3}},0\right)$ and $\left(0,\sqrt{\frac{20}{3}}\right)$, so f is CU on $\left(-\infty,-\sqrt{\frac{20}{3}}\right)$ and $\left(\sqrt{\frac{20}{3}},\infty\right)$ and CD on $\left(-\sqrt{\frac{20}{3}},0\right)$ and $\left(0,\sqrt{\frac{20}{3}}\right)$. IP at $\left(\pm\sqrt{\frac{20}{3}},\frac{21}{200}\right)\approx(\pm 2.5820,0.105)$

H.

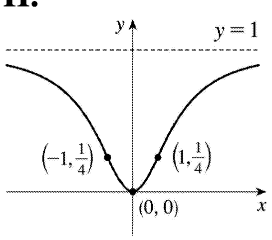


17. $y=f(x)=\dfrac{x^2}{x^2+3}=\dfrac{\left(x^2+3\right)-3}{x^2+3}=1-\dfrac{3}{x^2+3}$ **A.** $D=R$ **B.** $y-$ intercept: $f(0)=0$; x –intercepts: $f(x)=0\Leftrightarrow$ $x=0$ **C.** $f(-x)=f(x)$, so f is even; the graph is symmetric about the y –axis. **D.** $\lim\limits_{x\to\pm\infty}\dfrac{x^2}{x^2+3}=1$, so $y=1$ is a HA. No VA. **E.** Using the Reciprocal Rule, $f'(x)=-3\cdot\dfrac{-2x}{\left(x^2+3\right)^2}=\dfrac{6x}{\left(x^2+3\right)^2}$. $f'(x)>0\Leftrightarrow x>0$ and $f'(x)<0\Leftrightarrow x<0$, so f is decreasing on $(-\infty,0)$ and increasing on $(0,\infty)$. **F.** Local minimum value $f(0)=0$, no local maximum. **G.** $f''(x)=\dfrac{\left(x^2+3\right)^2\cdot 6-6x\cdot 2\left(x^2+3\right)\cdot 2x}{\left[\left(x^2+3\right)^2\right]^2}$

$=\dfrac{6\left(x^2+3\right)\left[\left(x^2+3\right)-4x^2\right]}{\left(x^2+3\right)^4}=\dfrac{6\left(3-3x^2\right)}{\left(x^2+3\right)^3}=\dfrac{-18(x+1)(x-1)}{\left(x^2+3\right)^3}$

$f''(x)$ is negative on $(-\infty,-1)$ and $(1,\infty)$ and positive on $(-1,1)$, so f is CD on $(-\infty,-1)$ and $(1,\infty)$ and CU on $(-1,1)$. IP at $\left(\pm 1,\frac{1}{4}\right)$

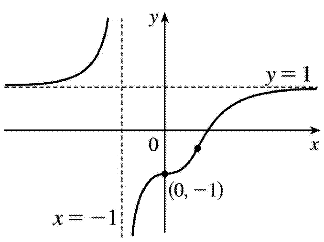
H.



18. $y=f(x)=\dfrac{x^3-1}{x^3+1}$ **A.** $D=\{x\mid x\neq -1\}=(-\infty,-1)\cup(-1,\infty)$ **B.** x –intercept $=1$, y –intercept $=f(0)=-1$ **C.** No symmetry **D.** $\lim\limits_{x\to\pm\infty}\dfrac{x^3-1}{x^3+1}=\lim\limits_{x\to\pm\infty}\dfrac{1-1/x^3}{1+1/x^3}=1$, so $y=1$ is a HA.

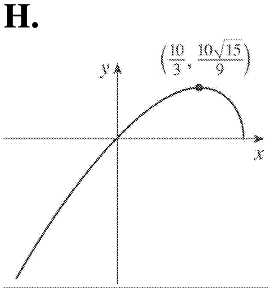
$\lim\limits_{x\to -1^-} \frac{x^3-1}{x^3+1}=\infty$ and $\lim\limits_{x\to -1^+} \frac{x^3-1}{x^3+1}=-\infty$, so $x=-1$ is a VA. **E.**
$f'(x)=\frac{\left(x^3+1\right)\left(3x^2\right)-\left(x^3-1\right)\left(3x^2\right)}{\left(x^3+1\right)^2}=\frac{6x^2}{\left(x^3+1\right)^2}>0$ ($x\neq -1$) so f is increasing on $(-\infty,-1)$ and $(-1,\infty)$. **F.** No extreme values **G.** $y''=\frac{12x\left(x^3+1\right)^2-6x^2\cdot 2\left(x^3+1\right)\cdot 3x^2}{\left(x^3+1\right)^4}=\frac{12x\left(1-2x^3\right)}{\left(x^3+1\right)^3}>0\Leftrightarrow x<-1$ or $0<x<\frac{1}{\sqrt[3]{2}}$, <so f is CU on $(-\infty,-1)$ and $\left(0,\frac{1}{\sqrt[3]{2}}\right)$ and CD on $(-1,0)$ and $\left(\frac{1}{\sqrt[3]{2}},\infty\right)$. IP at $(0,-1)$, $\left(\frac{1}{\sqrt[3]{2}},-\frac{1}{3}\right)$

H.



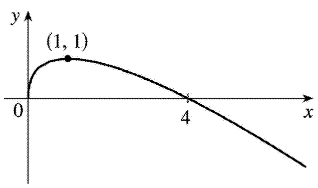
19. $y=f(x)=x\sqrt{5-x}$ **A.** The domain is $\{x|5-x\geq 0\}=(-\infty,5]$ **B.** $y-$ intercept: $f(0)=0$; x –intercepts: $f(x)=0\Leftrightarrow x=0$, 5 **C.** No symmetry **D.** No asymptote **E.**
$f'(x)=x\cdot \frac{1}{2}(5-x)^{-1/2}(-1)+(5-x)^{1/2}\cdot 1=\frac{1}{2}(5-x)^{-1/2}[-x+2(5-x)]=\frac{10-3x}{2\sqrt{5-x}}>0\Leftrightarrow x<\frac{10}{3}$, so f is increasing on $\left(-\infty,\frac{10}{3}\right)$ and decreasing on $\left(\frac{10}{3},5\right)$. **F.** Local maximum value $f\left(\frac{10}{3}\right)=\frac{10}{9}\sqrt{15}\approx 4.3$; no local minimum **G.** $f''(x)$
$=\frac{2(5-x)^{1/2}(-3)-(10-3x)\cdot 2\left(\frac{1}{2}\right)(5-x)^{-1/2}(-1)}{\left(2\sqrt{5-x}\right)^2}=\frac{(5-x)^{-1/2}[-6(5-x)+(10-3x)]}{4(5-x)}=\frac{3x-20}{4(5-x)^{3/2}}$ $f''(x)<0$ for $x<5$, so f is CD on $(-\infty,5)$. No IP

H.



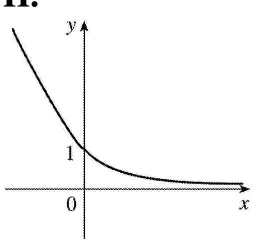
20. $y=f(x)=2\sqrt{x}-x$ **A.** $D=[0,\infty)$ **B.** $y-$ intercept: $f(0)=0$; x –intercepts: $f(x)=0 \Rightarrow 2\sqrt{x}=x \Rightarrow 4x=x^2 \Rightarrow 4x-x^2=0 \Rightarrow x(4-x)=0 \Rightarrow x=0$, 4 **C.** No symmetry **D.** No asymptote **E.** $f'(x)=\frac{1}{\sqrt{x}}-1=\frac{1}{\sqrt{x}}\left(1-\sqrt{x}\right)$. This is positive for $x<1$ and negative for $x>1$, so f is increasing on $(0,1)$ and decreasing on $(1,\infty)$. **F.** Local maximum value $f(1)=1$, no local minimum. **G.** $f''(x)=\left(x^{-1/2}-1\right)'=-\frac{1}{2}x^{-3/2}=\frac{-1}{2x^{3/2}}<0$ for $x>0$, so f is CD on $(0,\infty)$. No IP

H.



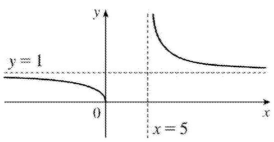
21. $y=f(x)=\sqrt{x^2+1}-x$ **A.** $D=R$ **B.** No x –intercept, y –intercept $=1$ **C.** No symmetry **D.** $\lim_{x\to-\infty}\left(\sqrt{x^2+1}-x\right)=\infty$ and $\lim_{x\to\infty}\left(\sqrt{x^2+1}-x\right)=\lim_{x\to\infty}\left(\sqrt{x^2+1}-x\right)\frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}+x}=\lim_{x\to\infty}\frac{1}{\sqrt{x^2+1}+x}=0$, so $y=0$ is a HA. **E.** $f'(x)=\frac{x}{\sqrt{x^2+1}}-1=\frac{x-\sqrt{x^2+1}}{\sqrt{x^2+1}} \Rightarrow f'(x)<0$, so f is decreasing on R. **F.** No extreme values **G.** $f''(x)=\frac{1}{\left(x^2+1\right)^{3/2}}>0$, so f is CU on R . No IP

H.



22. $y=f(x)=\sqrt{x/(x-5)}$ **A.** $D=\{x \mid x/(x-5)\geq 0\}=(-\infty,0]\cup(5,\infty)$. **B.** Intercepts are 0 . **C.** No symmetry **D.** $\lim_{x\to\pm\infty}\sqrt{\frac{x}{x-5}}=\lim_{x\to\pm\infty}\sqrt{\frac{1}{1-5/x}}=1$, so $y=1$ is a HA. $\lim_{x\to5^+}\sqrt{\frac{x}{x-5}}=\infty$, so $x=5$ is a VA. **E.** $f'(x)=\frac{1}{2}\left(\frac{x}{x-5}\right)^{-1/2}\frac{(-5)}{(x-5)^2}=-\frac{5}{2}\left[x(x-5)^3\right]^{-1/2}<0$, so f is decreasing on $(-\infty,0)$ and $(5,\infty)$. **F.** No extreme values **G.** $f''(x)=\frac{5}{4}\left[x(x-5)^3\right]^{-3/2}(x-5)^2(4x-5)>0$ for $x>5$, and $f''(x)<0$ for $x<0$, so f

is CU on $(5,\infty)$ and CD on $(-\infty,0)$. No IP
H.

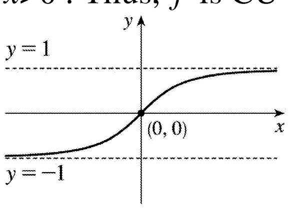


23. $y=f(x)=x/\sqrt{x^2+1}$ **A.** $D=R$ **B.** y –intercept: $f(0)=0$; x –intercepts: $f(x)=0\Rightarrow x=0$ **C.** $f(-x)=-f(x)$, so f is odd; the graph is symmetric about the origin. **D.** $\lim_{x\to\infty} f(x)$

$$=\lim_{x\to\infty}\frac{x}{\sqrt{x^2+1}}=\lim_{x\to\infty}\frac{x/x}{\sqrt{x^2+1}/x}=\lim_{x\to\infty}\frac{x/x}{\sqrt{x^2+1}/\sqrt{x^2}}=\lim_{x\to\infty}\frac{1}{\sqrt{1+1/x^2}}\quad \lim_{x\to-\infty} f(x)$$

$$=\lim_{x\to-\infty}\frac{x}{\sqrt{x^2+1}}=\lim_{x\to-\infty}\frac{x/x}{\sqrt{x^2+1}/x}=\lim_{x\to-\infty}\frac{x/x}{\sqrt{x^2+1}\Big/\left(-\sqrt{x^2}\right)}=\lim_{x\to-\infty}\frac{1}{-\sqrt{1+1/x^2}}=\frac{1}{-\sqrt{1+0}}=-1 \text{ so}$$

$y=\pm 1$ are HA. No VA. **E.** $f'(x)=\dfrac{\sqrt{x^2+1}-x\cdot\dfrac{2x}{2\sqrt{x^2+1}}}{[(x^2+1)^{1/2}]^2}=\dfrac{x^2+1-x^2}{\left(x^2+1\right)^{3/2}}=\dfrac{1}{\left(x^2+1\right)^{3/2}}>0$ for all x , so f is increasing on R . $f''(x)=-\dfrac{3}{2}\left(x^2+1\right)^{-5/2}\cdot 2x=\dfrac{-3x}{(x^2+1)^{5/2}}$, so $f''(x)>0$ for $x<0$ and $f''(x)<0$ for $x>0$. Thus, f is CU on $(-\infty,0)$ and CD on $(0,\infty)$. IP at $(0,0)$ **H.**

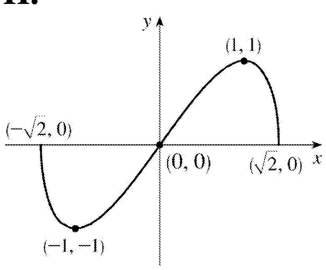


24. $y=f(x)=x\sqrt{2-x^2}$ **A.** $D=\left[-\sqrt{2},\sqrt{2}\,\right]$ **B.** y –intercept: $f(0)=0$; x –intercepts: $f(x)=0\Rightarrow x=0$, $\pm\sqrt{2}$. **C.** $f(-x)=-f(x)$, so f is odd; the graph is symmetric about the origin. **D.** No asymptote **E.** $f'(x)=x\cdot\dfrac{-x}{\sqrt{2-x^2}}+\sqrt{2-x^2}=\dfrac{-x^2+2-x^2}{\sqrt{2-x^2}}=\dfrac{2(1+x)(1-x)}{\sqrt{2-x^2}}$. $f'(x)$ is negative for $-\sqrt{2}<x<-1$ and $1<x<\sqrt{2}$, and positive for $-1<x<1$, so f is decreasing on $\left(-\sqrt{2},-1\right)$ and $\left(1,\sqrt{2}\right)$ and increasing on $(-1,1)$. **F.** Local minimum value $f(-1)=-1$, local maximum value $f(1)=1$. **G** $f''(x)$

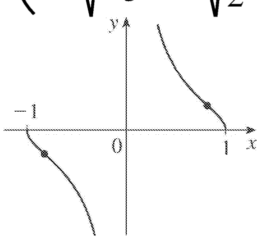
$$= \frac{\sqrt{2-x^2}(-4x)-(2-2x^2)\dfrac{-x}{\sqrt{2-x^2}}}{\left[(2-x^2)^{1/2}\right]^2} = \frac{(2-x^2)(-4x)+(2-2x^2)x}{(2-x^2)^{3/2}} = \frac{2x^3-6x}{(2-x^2)^{3/2}} = \frac{2x(x^2-3)}{(2-x^2)^{3/2}}$$

. Since $x^2-3<0$ for x in $\left[-\sqrt{2},\sqrt{2}\right]$, $f''(x)>0$ for $-\sqrt{2}<x<0$ and $f''(x)<0$ for $0<x<\sqrt{2}$. Thus, f is CU on $\left(-\sqrt{2},0\right)$ and CD on $\left(0,\sqrt{2}\right)$. The only IP is $(0,0)$.

H.



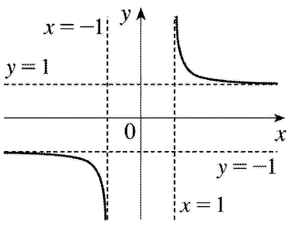
25. $y=f(x)=\sqrt{1-x^2}/x$ **A.** $D=\{x \mid |x|\le 1, x\ne 0\}=[-1,0)\cup(0,1]$ **B.** $x-$ intercepts ± 1 , no y –intercept **C.** $f(-x)=-f(x)$, so the curve is symmetric about $(0,0)$. **D.** $\lim_{x\to 0^+}\dfrac{\sqrt{1-x^2}}{x}=\infty$, $\lim_{x\to 0^-}\dfrac{\sqrt{1-x^2}}{x}=-\infty$, so $x=0$ is a VA. **E.** $f'(x)=\dfrac{\left(-x^2/\sqrt{1-x^2}\right)-\sqrt{1-x^2}}{x^2}=-\dfrac{1}{x^2\sqrt{1-x^2}}<0$, so f is decreasing on $(-1,0)$ and $(0,1)$. **F.** No extreme values **G.** $f''(x)=\dfrac{2-3x^2}{x^3\left(1-x^2\right)^{3/2}}>0 \Leftrightarrow -1<x<-\sqrt{\dfrac{2}{3}}$ or $0<x<\sqrt{\dfrac{2}{3}}$, so f is CU on $\left(-1,-\sqrt{\dfrac{2}{3}}\right)$ and $\left(0,\sqrt{\dfrac{2}{3}}\right)$ and CD on $\left(-\sqrt{\dfrac{2}{3}},0\right)$ and $\left(\sqrt{\dfrac{2}{3}},1\right)$. IP at $\left(\pm\sqrt{\dfrac{2}{3}},\pm\dfrac{1}{\sqrt{2}}\right)$ **H.**



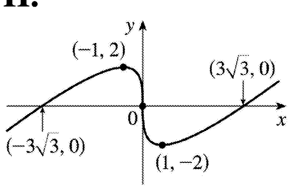
26. $y=f(x)=x/\sqrt{x^2-1}$ **A.** $D=(-\infty,-1)\cup(1,\infty)$ **B.** No intercepts **C.** $f(-x)=-f(x)$, so f is odd; the graph is symmetric about the origin. **D.**

$\lim_{x\to\infty} \frac{x}{\sqrt{x^2-1}}=1$ and $\lim_{x\to-\infty} \frac{x}{\sqrt{x^2-1}}=-1$, so $y=\pm 1$ are HA. $\lim_{x\to 1^+} f(x)=+\infty$ and $\lim_{x\to -1^-} f(x)=-\infty$, so $x=\pm 1$ are VA. **E.** $f'(x)=\frac{\sqrt{x^2-1}-x\cdot\frac{x}{\sqrt{x^2-1}}}{\left[\left(x^2-1\right)^{1/2}\right]^2}=\frac{x^2-1-x^2}{\left(x^2-1\right)^{3/2}}=\frac{-1}{\left(x^2-1\right)^{3/2}}<0$, so f is decreasing on $(-\infty,-1)$ and $(1,\infty)$. **F.** No extreme values **G.** $f''(x)=(-1)\left(-\frac{3}{2}\right)\left(x^2-1\right)^{-5/2}\cdot 2x=\frac{3x}{\left(x^2-1\right)^{5/2}}$

$f''(x)<0$ on $(-\infty,-1)$ and $f''(x)>0$ on $(1,\infty)$, so f is CD on $(-\infty,-1)$ and CU on $(1,\infty)$. No IP
H.

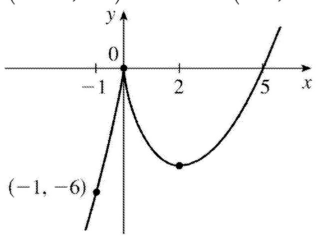


27. $y=f(x)=x-3x^{1/3}$ **A.** $D=R$ **B.** $y-$ intercept: $f(0)=0$; x –intercepts: $f(x)=0\Rightarrow x=3x^{1/3}\Rightarrow x^3=27x\Rightarrow x^3-27x=0\Rightarrow x(x^2-27)=0\Rightarrow x=0$, $\pm 3\sqrt{3}$ **C.** $f(-x)=-f(x)$, so f is odd; the graph is symmetric about the origin. **D.** No asymptote **E.** $f'(x)=1-x^{-2/3}=1-\frac{1}{x^{2/3}}=\frac{x^{2/3}-1}{x^{2/3}}$. $f'(x)>0$ when $|x|>1$ and $f'(x)<0$ when $0<|x|<1$, so f is increasing on $(-\infty,-1)$ and $(1,\infty)$, and decreasing on $(-1,0)$ and $(0,1)$. **F.** Local maximum value $f(-1)=2$, local minimum value $f(1)=-2$ **G.** $f''(x)=\frac{2}{3}x^{-5/3}<0$ when $x<0$ and $f''(x)>0$ when $x>0$, so f is CD on $(-\infty,0)$ and CU on $(0,\infty)$. IP at $(0,0)$
H.

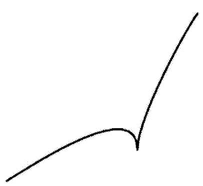


28. $y=f(x)=x^{5/3}-5x^{2/3}=x^{2/3}(x-5)$ **A.** $D=R$ **B.** x –intercepts 0 , 5 ; y –intercept 0 **C.** No symmetry **D.** $\lim_{x\to\pm\infty} x^{2/3}(x-5)=\pm\infty$, so there is no asymptote **E.** $f'(x)=\frac{5}{3}x^{2/3}-\frac{10}{3}x^{-1/3}=\frac{5}{3}x^{-1/3}(x-2)>0\Leftrightarrow x<0$ or $x>2$, so f is increasing on $(-\infty,0)$, $(2,\infty)$ and decreasing on $(0,2)$. **F.** Local maximum value $f(0)=0$, local minimum value $f(2)=-3\sqrt[3]{4}$ **G.**

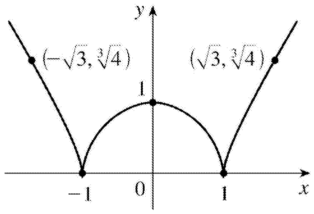
$f''(x)=\frac{10}{9}x^{-1/3}+\frac{10}{9}x^{-4/3}=\frac{10}{9}x^{-4/3}(x+1)>0\Leftrightarrow x>-1$, so f is CU on $(-1,0)$ and $(0,\infty)$, CD on $(-\infty,-1)$. IP at $(-1,-6)$ **H.**



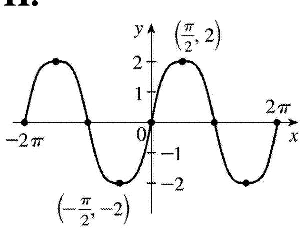
29. $y=f(x)=x+\sqrt{|x|}$ **A.** $D=R$ **B.** x –intercepts 0 , –1 ; y –intercept 0 **C.** No symmetry **D.** $\lim_{x\to\infty}\left(x+\sqrt{|x|}\right)=\infty$, $\lim_{x\to-\infty}\left(x+\sqrt{|x|}\right)=-\infty$. No asymptote **E.** For $x>0$, $f(x)=x+\sqrt{x}\Rightarrow$ $f'(x)=1+\frac{1}{2\sqrt{x}}>0$, so f increases on $(0,\infty)$. For $x<0$,f$(x)=x+\sqrt{-x}\Rightarrow f'(x)=1-\frac{1}{2\sqrt{-x}}>0\Leftrightarrow 2\sqrt{-x}>1$ $\Leftrightarrow -x>\frac{1}{4}\Leftrightarrow x<-\frac{1}{4}$, so f increases on $\left(-\infty,-\frac{1}{4}\right)$ and decreases on $\left(-\frac{1}{4},0\right)$. **F.** Local maximum value $f\left(-\frac{1}{4}\right)=\frac{1}{4}$, local minimum value $f(0)=0$ **G.** For $x>0$, $f''(x)=-\frac{1}{4}x^{-3/2}\Rightarrow f''(x)<0$, so f is CD on $(0,\infty)$. For $x<0$, $f''(x)=-\frac{1}{4}(-x)^{-3/2}\Rightarrow f''(x)<0$, so f is CD on $(-\infty,0)$. No IP
H.



30. $y=f(x)=\sqrt[3]{\left(x^2-1\right)^2}=\left(x^2-1\right)^{2/3}$ **A.** $D=R$ **B.** x –intercepts ±1 , y –intercept 1 **C.** $f(-x)=f(x)$, so the curve is symmetric about the y– axis. **D.** $\lim_{x\to\pm\infty}\left(x^2-1\right)^{2/3}=\infty$, no asymptote **E.** $f'(x)=\frac{4}{3}x\left(x^2-1\right)^{-1/3}\Rightarrow f'(x)>0\Leftrightarrow x>1$ or $-1<x<0$, $f'(x)<0\Leftrightarrow x<-1$ or $0<x<1$. So f is increasing on $(-1,0)$, $(1,\infty)$ and decreasing on $(-\infty,-1)$, $(0,1)$. **F.** Local minimum values $f(-1)=f(1)=0$, local maximum value $f(0)=1$ **G.** $f''(x)=\frac{4}{3}\left(x^2-1\right)^{-1/3}+\frac{4}{3}x\left(-\frac{1}{3}\right)\left(x^2-1\right)^{-4/3}(2x)$ $=\frac{4}{9}\left(x^2-3\right)\left(x^2-1\right)^{-4/3}>0\Leftrightarrow |x|>\sqrt{3}$ so f is CU on $\left(-\infty,-\sqrt{3}\right)$, $\left(\sqrt{3},\infty\right)$ and CD on $\left(-\sqrt{3},-1\right)$, $(-1,1)$, $\left(1,\sqrt{3}\right)$. IPs at $\left(\pm\sqrt{3},\sqrt[3]{4}\right)$ **H.**



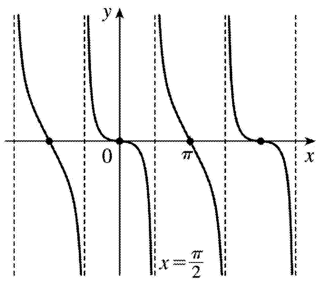
31. $y=f(x)=3\sin x-\sin^3 x$ **A.** $D=R$ **B.** $y-$ intercept: $f(0)=0$; x –intercepts: $f(x)=0 \Rightarrow \sin x(3-\sin^2 x)=0 \Rightarrow$ $\sin x=0$ [since $\sin^2 x \le 1<3$] $\Rightarrow x=n\pi$, n an integer.
C. $f(-x)=-f(x)$, so f is odd; the graph (shown for $-2\pi \le x \le 2\pi$) is symmetric about the origin and periodic with period 2π . **D.** No asymptote **E.** $f'(x)=3\cos x-3\sin^2 x\cos x=3\cos x\left(1-\sin^2 x\right)=3\cos^3 x$. $f'(x)>0 \Leftrightarrow \cos x>0 \Leftrightarrow x\in\left(2n\pi-\frac{\pi}{2}, 2n\pi+\frac{\pi}{2}\right)$ for each integer n , and $f'(x)<0 \Leftrightarrow$ $\cos x<0 \Leftrightarrow x\in\left(2n\pi+\frac{\pi}{2}, 2n\pi+\frac{3\pi}{2}\right)$ for each integer n . Thus, f is increasing on $\left(2n\pi-\frac{\pi}{2}, 2n\pi+\frac{\pi}{2}\right)$ for each integer n , and f is decreasing on $\left(2n\pi+\frac{\pi}{2}, 2n\pi+\frac{3\pi}{2}\right)$ for each integer n . **F.** f has local maximum values $f(2n\pi+\frac{\pi}{2})=2$ and local minimum values $f(2n\pi+\frac{3\pi}{2})=-2$.
G. $f''(x)=-9\sin x\cos^2 x=-9\sin x(1-\sin^2 x)=-9\sin x(1-\sin x)(1+\sin x)$. $f''(x)<0 \Leftrightarrow \sin x>0$ and $\sin x\neq \pm 1 \Leftrightarrow x\in\left(2n\pi, 2n\pi+\frac{\pi}{2}\right)\cup\left(2n\pi+\frac{\pi}{2}, 2n\pi+\pi\right)$ for some integer n . $f''(x)>0 \Leftrightarrow \sin x<0$ and $\sin x\neq \pm 1 \Leftrightarrow x\in\left((2n-1)\pi, (2n-1)\pi+\frac{\pi}{2}\right)\cup\left((2n-1)\pi+\frac{\pi}{2}, 2n\pi\right)$ for some integer n . Thus, f is CD on the intervals $\left(2n\pi, \left(2n+\frac{1}{2}\right)\pi\right)$ and $\left(\left(2n+\frac{1}{2}\right)\pi, (2n+1)\pi\right)$ for each integer n , and f is CU on the intervals $\left((2n-1)\pi, \left(2n-\frac{1}{2}\right)\pi\right)$ and $\left(\left(2n-\frac{1}{2}\right)\pi, 2n\pi\right)$ for each integer n . f has inflection points at $(n\pi, 0)$ for each integer n .
H.



32. $y=f(x)=\sin x-\tan x$ **A.** $D=\left\{ x \mid x\neq(2n+1)\frac{\pi}{2} \right\}$ **B.** $y=0 \Leftrightarrow \sin x=\tan x=\frac{\sin x}{\cos x} \Leftrightarrow \sin x=0$ or $\cos x=1$

$\Leftrightarrow x=n\pi$ ($x-$ intercepts), y –intercept $=f(0)=0$ **C.** $f(-x)=-f(x)$, so the curve is symmetric about $(0,0)$. Also periodic with period 2π **D.** $\lim\limits_{x\to(\pi/2)^-}(\sin x-\tan x)=-\infty$ and $\lim\limits_{x\to(\pi/2)^+}(\sin x-\tan x)=\infty$, so $x=n\pi+\dfrac{\pi}{2}$ are VA. **E.** $f'(x)=\cos x-\sec^2 x\leq 0$, so f decreases on each interval in its domain, that is, on $\left((2n-1)\dfrac{\pi}{2},(2n+1)\dfrac{\pi}{2}\right)$. **F.** No extreme values **G.** $f''(x)=-\sin x-2\sec^2 x\tan x=\sin x\left(1+2\sec^3 x\right)$. Note that $1+2\sec^3 x\neq 0$ since $\sec^3 x\neq-\dfrac{1}{2}$. $f''(x)>0$ for $-\dfrac{\pi}{2}<x<0$ and $\dfrac{3\pi}{2}<x<2\pi$, so f is CU on $\left(\left(n-\dfrac{1}{2}\right)\pi,n\pi\right)$ and CD on $\left(n\pi,\left(n+\dfrac{1}{2}\right)\pi\right)$. f has IPs at $(n\pi,0)$. Note also that $f'(0)=0$, but $f'(\pi)=-2$.

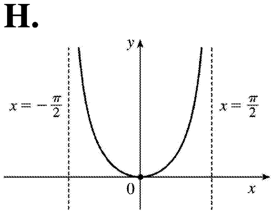
H.



33. $y=f(x)=x\tan x$, $-\dfrac{\pi}{2}<x<\dfrac{\pi}{2}$ **A.** $D=\left(-\dfrac{\pi}{2},\dfrac{\pi}{2}\right)$ **B.** Intercepts are 0 **C.** $f(-x)=f(x)$, so the curve is symmetric about the $y-$ axis. **D.** $\lim\limits_{x\to(\pi/2)^-}x\tan x=\infty$ and $\lim\limits_{x\to-(\pi/2)^+}x\tan x=\infty$, so $x=\dfrac{\pi}{2}$ and $x=-\dfrac{\pi}{2}$ are VA. **E.** $f'(x)=\tan x+x\sec^2 x>0\Leftrightarrow 0<x<\dfrac{\pi}{2}$, so f increases on $\left(0,\dfrac{\pi}{2}\right)$ and decreases on $\left(-\dfrac{\pi}{2},0\right)$.

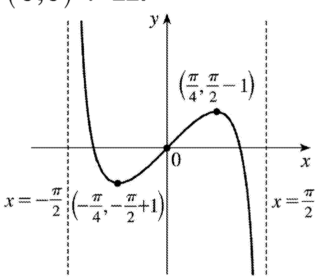
F. Absolute and local minimum value $f(0)=0$. **G.** $y''=2\sec^2 x+2x\tan x\sec^2 x>0$ for $-\dfrac{\pi}{2}<x<\dfrac{\pi}{2}$, so f is CU on $\left(-\dfrac{\pi}{2},\dfrac{\pi}{2}\right)$. No IP

H.

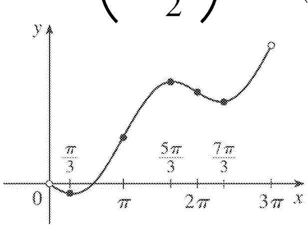


34. $y=f(x)=2x-\tan x$, $-\dfrac{\pi}{2}<x<\dfrac{\pi}{2}$ **A.** $D=\left(-\dfrac{\pi}{2},\dfrac{\pi}{2}\right)$ **B.** $y-$ intercept: $f(0)=0$; x –intercepts: $f(0)=0$

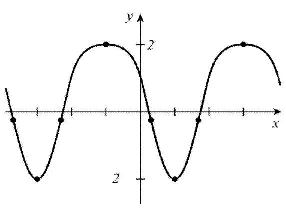
$\Leftrightarrow 2x=\tan x \Leftrightarrow x=0$ or $x\approx\pm 1.17$ **C.** $f(-x)=-f(x)$, so f is odd; the graph is symmetric about the origin. **D.** $\lim_{x\to(-\pi/2)^+}(2x-\tan x)=\infty$ and $\lim_{x\to(\pi/2)^-}(2x-\tan x)=-\infty$, so $x=\pm\frac{\pi}{2}$ are VA. No HA. **E.** $f'(x)=2-\sec^2 x<0 \Leftrightarrow |\sec x|>\sqrt{2}$ and $f'(x)>0 \Leftrightarrow |\sec x|<\sqrt{2}$, so f is decreasing on $\left(-\frac{\pi}{2},-\frac{\pi}{4}\right)$, increasing on $\left(-\frac{\pi}{4},\frac{\pi}{4}\right)$, and decreasing again on $\left(\frac{\pi}{4},\frac{\pi}{2}\right)$ **F.** Local maximum value $f\left(\frac{\pi}{4}\right)=\frac{\pi}{2}-1$, local minimum value $f\left(-\frac{\pi}{4}\right)=-\frac{\pi}{2}+1$ **G.** $f''(x)$ $=-2\sec x\cdot\sec x\tan x=-2\tan x\sec^2 x=-2\tan x(\tan^2 x+1)$ so $f''(x)>0 \Leftrightarrow \tan x<0 \Leftrightarrow -\frac{\pi}{2}<x<0$, and $f''(x)<0 \Leftrightarrow \tan x>0 \Leftrightarrow 0<x<\frac{\pi}{2}$. Thus, f is CU on $\left(-\frac{\pi}{2},0\right)$ and CD on $\left(0,\frac{\pi}{2}\right)$. f has an IP at $(0,0)$. **H.**



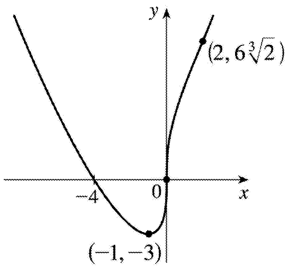
35. $y=f(x)=\frac{1}{2}x-\sin x$, $0<x<3\pi$ **A.** $D=(0,3\pi)$ **B.** No y –intercept. The x –intercept, approximately 1.9 , can be found using Newton's Method. **C.** No symmetry **D.** No asymptote **E.** $f'(x)=\frac{1}{2}-\cos\:,x>0 \Leftrightarrow$ $\cos x<\frac{1}{2} \Leftrightarrow \frac{\pi}{3}<x<\frac{5\pi}{3}$ or $\frac{7\pi}{3}<x<3\pi$, so f is increasing on $\left(\frac{\pi}{3},\frac{5\pi}{3}\right)$ and $\left(\frac{7\pi}{3},3\pi\right)$ and decreasing on $\left(0,\frac{\pi}{3}\right)$ and $\left(\frac{5\pi}{3},\frac{7\pi}{3}\right)$. **F.** Local minimum value $f\left(\frac{\pi}{3}\right)=\frac{\pi}{6}-\frac{\sqrt{3}}{2}$, local maximum value $f\left(\frac{5\pi}{3}\right)=\frac{5\pi}{6}+\frac{\sqrt{3}}{2}$, local minimum value $f\left(\frac{7\pi}{3}\right)=\frac{7\pi}{6}-\frac{\sqrt{3}}{2}$ **G.** $f''(x)=\sin x>0 \Leftrightarrow 0<x<\pi$ or $2\pi<x<3\pi$, so f is CU on $(0,\pi)$ and $(2\pi,3\pi)$ and CD on $(\pi,2\pi)$. IPs at $\left(\pi,\frac{\pi}{2}\right)$ and $(2\pi,\pi)$. **H.**



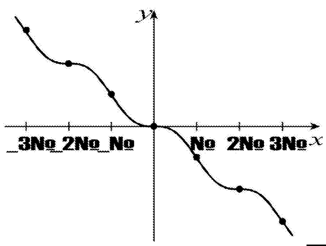
36. $y=f(x)=\cos^2 x-2\sin x$ **A.** $D=R$ **B.** $y-$ intercept: $f(0)=1$ **C.** No symmetry, but f has period 2π . **D.** No asymptote **E.** $y'=2\cos x(-\sin x)-2\cos x=-2\cos x(\sin x+1)$. $y'=0\Leftrightarrow\cos x=0$ or $\sin x=-1\Leftrightarrow$ $x=(2n+1)\frac{\pi}{2}$. $y'>0$ when $\cos x<0$ since $\sin x+1\geq 0$ for all x . So $y'>0$ and f is increasing on $\left((4n+1)\frac{\pi}{2},(4n+3)\frac{\pi}{2}\right)$; $y'<0$ and f is decreasing on $\left((4n-1)\frac{\pi}{2},(4n+1)\frac{\pi}{2}\right)$. **F.** Local maximum values $f\left((4n+3)\frac{\pi}{2}\right)=2$, local minimum values $f\left((4n+1)\frac{\pi}{2}\right)=-2$ **G.**
$y'=-2\cos x(\sin x+1)=-\sin 2x-2\cos x\Rightarrow y''=-2\cos 2x+2\sin x=-2\left(1-2\sin^2 x\right)+2\sin x$
$=4\sin^2 x+2\sin x-2=2(2\sin x-1)(\sin x+1)y''=0\Leftrightarrow\sin x=\frac{1}{2}$ or $-1\Rightarrow x=\frac{\pi}{6}+2n\pi$, $\frac{5\pi}{6}+2n\pi$, or $\frac{3\pi}{2}+2n\pi$. $y''>0$ and f is CU on $\left(\frac{\pi}{6}+2n\pi,\frac{5\pi}{6}+2n\pi\right)$; $y''\leq 0$ and f is CD on $\left(\frac{5\pi}{6}+2n\pi,\frac{\pi}{6}+2(n+1)\pi\right)$ .IPs at $\left(\frac{\pi}{6}+2n\pi,-\frac{1}{4}\right)$ and $\left(\frac{5\pi}{6}+2n\pi,-\frac{1}{4}\right)$
. **H.**



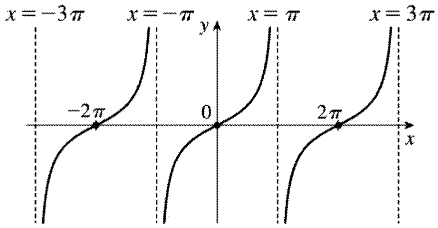
37. $y=f(x)=\sin 2x-2\sin x$ **A.** $D=R$ **B.** y –intercept $=f(0)=0$. $y=0\Leftrightarrow 2\sin x=\sin 2x=2\sin x\cos x\Leftrightarrow$ $\sin x=0$ or $\cos x=1\Leftrightarrow x=n\pi$ ($x-$ intercepts) **C.** $f(-x)=-f(x)$, so the curve is symmetric about $(0,0)$. *Note:* f is periodic with period 2π , so we determine E — G for $-\pi\leq x\leq\pi$. **D.** No asymptotes **E.** $f'(x)=2\cos 2x-2\cos x=2\left(2\cos^2 x-1-\cos x\right)=2(2\cos x+1)(\cos x-1)>0\Leftrightarrow\cos x<-\frac{1}{2}\Leftrightarrow-\pi<x<-\frac{2\pi}{3}$ or $\frac{2\pi}{3}<x<\pi$, so f is increasing on $\left(-\pi,-\frac{2\pi}{3}\right)$, $\left(\frac{2\pi}{3},\pi\right)$ and decreasing on $\left(-\frac{2\pi}{3},\frac{2\pi}{3}\right)$. **F.** Local maximum value $f\left(-\frac{2\pi}{3}\right)=\frac{3\sqrt{3}}{2}$, local minimum value $f\left(\frac{2\pi}{3}\right)=-\frac{3\sqrt{3}}{2}$ **G.**
$f''(x)=-4\sin 2x+2\sin x=2\sin x(1-4\cos x)=0$ when $x=0$, $\pm\pi$ or $\cos x=\frac{1}{4}$. If $\alpha=\cos^{-1}\frac{1}{4}$, then f is CU on $(-\alpha,0)$ and (α,π) and CD on $(-\pi,-\alpha)$ and $(0,\alpha)$. IPs at $(0,0)$, $(\pm\pi,0)$, $\left(\alpha,-\frac{3\sqrt{15}}{8}\right)$, $\left(-\alpha,\frac{3\sqrt{15}}{8}\right)$.
H.



38. $f(x)=\sin x-x$ **A.** $D=R$ **B.** $x-$ intercept $=0=y-$ intercept **C.** $f(-x)=\sin(-x)-(-x)=-(\sin x-x)=-f(x)$, so f is odd. **D.** No asymptote **E.** $f'(x)=\cos x-1\leq 0$ for all x, so f is decreasing on $(-\infty,\infty)$. **F.** No extreme values **G.** $f''(x)=-\sin x\Rightarrow f''(x)>0\Leftrightarrow\sin x<0\Leftrightarrow(2n-1)\pi<x<2n\pi$, so f is CU on $((2n-1)\pi,2n\pi)$ and CD on $(2n\pi,(2n+1)\pi)$, n an integer. Points of inflection occur when $x=n\pi$.
H.



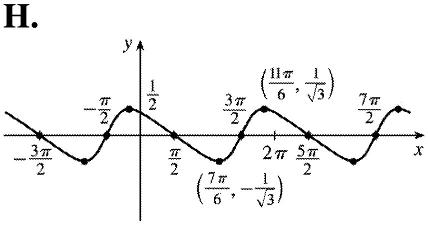
39. $y=f(x)=\dfrac{\sin x}{1+\cos x}\left[\overset{\text{when } \cos x\neq 1}{=}\dfrac{\sin x}{1+\cos x}\cdot\dfrac{1-\cos x}{1-\cos x}=\dfrac{\sin x(1-\cos x)}{\sin^2 x}=\dfrac{1-\cos x}{\sin x}=\csc x-\cot x\right]$ **A.** The domain of f is the set of all real numbers except odd integer multiples of π. **B.** y –intercept: $f(0)=0$; x –intercepts: $x=n\pi$, n an even integer. **C.** $f(-x)=-f(x)$, so f is an odd function; the graph is symmetric about the origin and has period 2π. **D.** When n is an odd integer, $\lim_{x\to(n\pi)^-}f(x)=\infty$ and $\lim_{x\to(n\pi)^+}f(x)=-\infty$, so $x=n\pi$ is a VA for each odd integer n. No HA. **E.**
$f'(x)=\dfrac{(1+\cos x)\cdot\cos x-\sin x(-\sin x)}{(1+\cos x)^2}=\dfrac{1+\cos x}{(1+\cos x)^2}=\dfrac{1}{1+\cos x}$. $f'(x)>0$ for all x except odd multiples of π, so f is increasing on $((2k-1)\pi,(2k+1)\pi)$ for each integer k. **F.** No extreme values **G.**
$f''(x)=\dfrac{\sin x}{(1+\cos x)^2}>0\Rightarrow\sin x>0\Rightarrow x\in(2k\pi,(2k+1)\pi)$ and $f''(x)<0$ on $((2k-1)\pi,2k\pi)$ for each integer k. f is CU on $(2k\pi,(2k+1)\pi)$ and CD on $((2k-1)\pi,2k\pi)$ for each integer k. f has IPs at $(2k\pi,0)$ for each integer k.
H



40. $y=f(x)=\cos x/(2+\sin x)$ **A.** $D=R$ *Note:* f is periodic with period 2π , so we determine B—G on $[0,2\pi]$. **B.** x –intercepts $\frac{\pi}{2},\frac{3\pi}{2}$, y –intercept $=f(0)=\frac{1}{2}$ **C.** No symmetry other than periodicity **D.** No asymptote **E.** $f'(x)=\frac{(2+\sin x)(-\sin x)-\cos x(\cos x)}{(2+\sin x)^2}=-\frac{2\sin x+1}{(2+\sin x)^2}$. $f'(x)>0\Leftrightarrow 2\sin x+1<0\Leftrightarrow$ $\sin x<-\frac{1}{2}\Leftrightarrow\frac{7\pi}{6}<x<\frac{11\pi}{6}$, so f is increasing on $\left(\frac{7\pi}{6},\frac{11\pi}{6}\right)$ and decreasing on $\left(0,\frac{7\pi}{6}\right)$, $\left(\frac{11\pi}{6},2\pi\right)$. **F.** Local minimum value $f\left(\frac{7\pi}{6}\right)=-\frac{1}{\sqrt{3}}$, local maximum value $f\left(\frac{11\pi}{6}\right)=\frac{1}{\sqrt{3}}$

G. $f''(x)=-\frac{(2+\sin x)^2(2\cos x)-(2\sin x+1)2(2+\sin x)\cos x}{(2+\sin x)^4}=-\frac{2\cos x(1-\sin x)}{(2+\sin x)^3}>0\Leftrightarrow\cos x<0\Leftrightarrow$ $\frac{\pi}{2}<x<\frac{3\pi}{2}$, so f is CU on $\left(\frac{\pi}{2},\frac{3\pi}{2}\right)$ and CD on $\left(0,\frac{\pi}{2}\right)$ and $\left(\frac{3\pi}{2},2\pi\right)$. IP at $\left(\frac{\pi}{2},0\right)$, $\left(\frac{3\pi}{2},0\right)$

H.



41. $y=1/\left(1+e^{-x}\right)$ **A.** $D=R$ **B.** No x –intercept; y –intercept $=f(0)=\frac{1}{2}$. **C.** No symmetry **D.** $\lim_{x\to\infty}1/\left(1+e^{-x}\right)=\frac{1}{1+0}=1$ and $\lim_{x\to-\infty}1/\left(1+e^{-x}\right)=0$ (since $\lim_{x\to-\infty}e^{-x}=\infty$), so f has horizontal asymptotes $y=0$ and $y=1$. **E.** $f'(x)=-\left(1+e^{-x}\right)^{-2}\left(-e^{-x}\right)=e^{-x}/\left(1+e^{-x}\right)^2$. This is positive for all x , so f is increasing on R . **F.** No extreme values

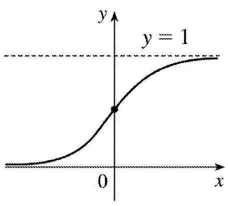
G.

$$f''(x)=\frac{\left(1+e^{-x}\right)^2\left(-e^{-x}\right)-e^{-x}(2)\left(1+e^{-x}\right)\left(-e^{-x}\right)}{\left(1+e^{-x}\right)^4}$$

$$=\frac{e^{-x}\left(e^{-x}-1\right)}{\left(1+e^{-x}\right)^{3}}$$

The second factor in the numerator is negative for $x>0$ and positive for $x<0$, and the other factors are always positive, so f is CU on $(-\infty,0)$ and CD on $(0,\infty)$. f has an inflection point at $\left(0,\frac{1}{2}\right)$.

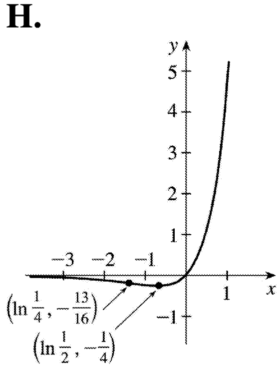
H.



42. $y=f(x)=e^{2x}-e^{x}$ **A.** $D=R$ **B.** $y-$ intercept: $f(0)=0$; x –intercepts: $f(x)=0\Rightarrow e^{2x}=e^{x}\Rightarrow e^{x}=1\Rightarrow x=0$. **C.** No symmetry **D.** $\lim\limits_{x\to-\infty} e^{2x}-e^{x}=0$, so $y=0$ is a HA. No VA. **E.** $f'(x)=2e^{2x}-e^{x}=e^{x}\left(2e^{x}-1\right)$, so $f'(x)>0$ $\Leftrightarrow e^{x}>\frac{1}{2}\Leftrightarrow x>\ln\frac{1}{2}=-\ln 2$ and $f'(x)<0\Leftrightarrow e^{x}<\frac{1}{2}\Leftrightarrow x<\ln\frac{1}{2}$, so f is decreasing on $\left(-\infty,\ln\frac{1}{2}\right)$ and increasing on $\left(\ln\frac{1}{2},\infty\right)$. **F.** Local minimum value $f\left(\ln\frac{1}{2}\right)=e^{2\ln(1/2)}-e^{\ln(1/2)}=\left(\frac{1}{2}\right)^{2}-\frac{1}{2}=-\frac{1}{4}$

G. $f''(x)=4e^{2x}-e^{x}=e^{x}\left(4e^{x}-1\right)$, so $f''(x)>0\Leftrightarrow$
$e^{x}>\frac{1}{4}\Leftrightarrow x>\ln\frac{1}{4}$ and $f''(x)<0\Leftrightarrow x<\ln\frac{1}{4}$.

H.



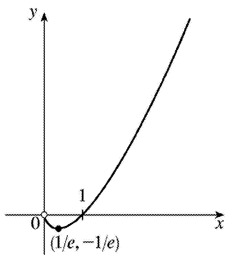
Thus, f is CD on $\left(-\infty,\ln\frac{1}{4}\right)$ and CU on $\left(\ln\frac{1}{4},\infty\right)$. f has an IP at $\left(\ln\frac{1}{4},\left(\frac{1}{4}\right)^{2}-\frac{1}{4}\right)=\left(\ln\frac{1}{4},-\frac{3}{16}\right)$.

43. $y=f(x)=x\ln x$ **A.** $D=(0,\infty)$ **B.** $x-$ intercept when $\ln x=0\Leftrightarrow x=1$, no $y-$ intercept

C. No symmetry **D.** $\lim_{x\to\infty} x\ln x=\infty$,

$\lim_{x\to 0^+} x\ln x=\lim_{x\to 0^+} \frac{\ln x}{1/x}=\lim_{x\to 0^+} \frac{1/x}{-1/x^2}=\lim_{x\to 0^+} (-x)=0$, no asymptote. **E.** $f'(x)=\ln x+1=0$ when $\ln x=-1 \Leftrightarrow x=e^{-1}$. $f'(x)>0 \Leftrightarrow \ln x>-1 \Leftrightarrow x>e^{-1}$, so f is increasing on $(1/e,\infty)$ and decreasing on $(0,1/e)$. **F.** $f(1/e)=-1/e$ is an absolute and local minimum value. **G.** $f''(x)=1/x>0$, so f is CU on $(0,\infty)$. No IP

H.



44. $y=f(x)=e^x/x$ **A.** $D=\{x|x\neq 0\}$ **B.** No intercept **C.** No symmetry **D.** $\lim_{x\to\infty} \frac{e^x}{x}=\lim_{x\to\infty} \frac{e^x}{1}=\infty$,

$\lim_{x\to -\infty} \frac{e^x}{x}=0$, so $y=0$ is a HA. \ $\lim_{x\to 0^+} \frac{e^x}{x}=\infty$, $\lim_{x\to 0^-} \frac{e^x}{x}=-\infty$, so $x=0$ is a VA.

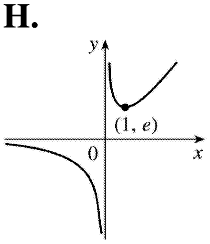
E. $f'(x)=\frac{xe^x-e^x}{x^2}>0 \Leftrightarrow (x-1)e^x>0 \Leftrightarrow x>1$,

so f is increasing on $(1,\infty)$, and decreasing on $(-\infty,0)$ and $(0,1)$. **F.** $f(1)=e$ is a local minimum value.

G. $f''(x)=\frac{x^2\left(xe^x\right)-2x\left(xe^x-e^x\right)}{x^4}=\frac{e^x\left(x^2-2x+2\right)}{x^3}>0$

$\Leftrightarrow x>0$ since $x^2-2x+2>0$ for all x . So f is CU on $(0,\infty)$ and CD on $(-\infty,0)$. No IP

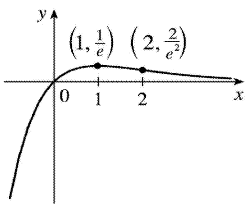
H.



45. $y=f(x)=xe^{-x}$ **A.** $D=R$ **B.** Intercepts are 0 **C.** No symmetry **D.** $\lim_{x\to\infty} xe^{-x}=\lim_{x\to\infty} \frac{x}{e^x}=\lim_{x\to\infty} \frac{1}{e^x}=0$, so $y=0$ is a HA.

$\lim_{x\to-\infty} xe^{-x}=-\infty$ **E.** $f'(x)=e^{-x}-xe^{-x}=e^{-x}(1-x)>0\Leftrightarrow x<1$, so f is increasing on $(-\infty,1)$ and decreasing on $(1,\infty)$. **F.** Absolute and local maximum value $f(1)=1/e$.

H.



G. $f''(x)=e^{-x}(x-2)>0\Leftrightarrow x>2$, so f is CU on $(2,\infty)$ and CD on $(-\infty,2)$. IP at $\left(2,2/e^2\right)$

46. $y=f(x)=\ln\left(x^2-3x+2\right)=\ln[(x-1)(x-2)]$

A. $D=\left\{x \text{in} R{:}x^2-3x+2>0\right\}=(-\infty,1)\cup(2,\infty)$.

B. y –intercept: $f(0)=\ln 2$; x –intercepts: $f(x)=0\Leftrightarrow x^2-3x+2=e^0\Leftrightarrow x^2-3x+1=0\Leftrightarrow x=\frac{3\pm\sqrt{5}}{2}$ $\Rightarrow x\approx 0.38$, 2.62 **C.** No symmetry **D.** $\lim_{x\to1^-} f(x)=\lim_{x\to2^+} f(x)=-\infty$, so $x=1$ and $x=2$ are VAs. No HA. **E.**

$f'(x)=\frac{2x-3}{x^2-3x+2}=\frac{2(x-3/2)}{(x-1)(x-2)}$, so $f'(x)<0$ for $x<1$ and $f'(x)>0$ for $x>2$. Thus, f is decreasing on $(-\infty,1)$ and increasing on $(2,\infty)$. **F.** No extreme values

G.

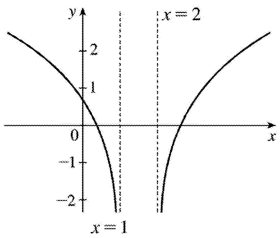
$$f''(x)=\frac{\left(x^2-3x+2\right)\cdot 2-(2x-3)^2}{\left(x^2-3x+2\right)^2}$$

$$=\frac{2x^2-6x+4-4x^2+12x-9}{\left(x^2-3x+2\right)^2}$$

$$=\frac{-2x^2+6x-5}{\left(x^2-3x+2\right)^2}$$

The numerator is negative for all x and the denominator is positive, so $f''(x)<0$ for all x in the domain of f . Thus, f is CD on $(-\infty,1)$ and $(2,\infty)$. No IP

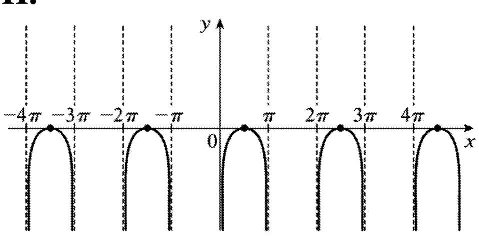
H.



47. $y=f(x)=\ln(\sin x)$
A.

$$D=\{x \text{ in } R \mid \sin x>0\} = \bigcup_{n=-\infty}^{\infty} (2n\pi,(2n+1)\pi)$$
$$=\cdots\cup(-4\pi,-3\pi)\cup(-2\pi,-\pi)\cup(0,\pi)\cup(2\pi,3\pi)\cup\cdots$$

B. No y –intercept; x –intercepts: $f(x)=0 \Leftrightarrow \ln(\sin x)=0 \Leftrightarrow \sin x=e^0=1 \Leftrightarrow x=2n\pi+\frac{\pi}{2}$ for each integer n .
C. f is periodic with period 2π . **D.** $\lim_{x\to(2n\pi)^+} f(x)=-\infty$ and $\lim_{x\to[(2n+1)\pi]^-} f(x)=-\infty$, so the lines $x=n\pi$ are VAs for all integers n . **E.** $f'(x)=\frac{\cos x}{\sin x}=\cot x$, so $f'(x)>0$ when $2n\pi<x<2n\pi+\frac{\pi}{2}$ for each integer n , and $f'(x)<0$ when $2n\pi+\frac{\pi}{2}<x<(2n+1)\pi$. Thus, f is increasing on $\left(2n\pi,2n\pi+\frac{\pi}{2}\right)$ and decreasing on $\left(2n\pi+\frac{\pi}{2},(2n+1)\pi\right)$ for each integer n . **F.** Local maximum values $f\left(2n\pi+\frac{\pi}{2}\right)=0$, no local minimum. **G.** $f''(x)=-\csc^2 x<0$, so f is CD on $(2n\pi,(2n+1)\pi)$ for each integer n. No IP
H.

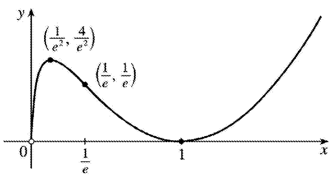


48. $y=f(x)=x(\ln x)^2$ **A.** $D=(0,\infty)$ **B.** x –intercept $=1$, no y –intercept **C.** No symmetry
D. $\lim_{x\to\infty} x(\ln x)^2=\infty$, $\lim_{x\to 0^+} x(\ln x)^2=\lim_{x\to 0^+}\frac{(\ln x)^2}{1/x}=\lim_{x\to 0^+}\frac{2(\ln x)(1/x)}{-1/x^2}=\lim_{x\to 0^+}\frac{2\ln x}{-1/x}=\lim_{x\to 0^+}\frac{2/x}{1/x^2}=\lim_{x\to 0^+} 2x=0$, no asymptote **E.** $f'(x)=(\ln x)^2+2\ln x=(\ln x)(\ln x+2)=0$ when $\ln x=0 \Leftrightarrow x=1$ and when $\ln x=-2 \Leftrightarrow x=e^{-2}$.
$f'(x)>0$ when $0<x<e^{-2}$ and when $x>1$, so
f is increasing on $\left(0,e^{-2}\right)$ and $(1,\infty)$ and decreasing on $\left(e^{-2},1\right)$.
F. Local maximum value $f\left(e^{-2}\right)=4e^{-2}$,

local minimum value $f(1)=0$

G. $f''(x)=2(\ln x)(1/x)+2/x=(2/x)(\ln x+1)=0$ when $\ln x=-1 \Leftrightarrow x=e^{-1}$. $f''(x)>0 \Leftrightarrow x>1/e$, so f is CU on $(1/e,\infty)$, CD on $(0,1/e)$.
IP at $(1/e,1/e)$

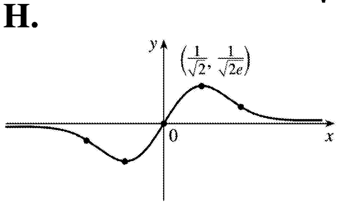
H.



49. $y=f(x)=xe^{-x^2}$ **A.** $D=R$ **B.** Intercepts are 0 **C.** $f(-x)=-f(x)$, so the curve is symmetric about the origin. **D.** $\lim_{x\to\pm\infty} xe^{-x^2} = \lim_{x\to\pm\infty} \frac{x}{e^{x^2}} = \lim_{x\to\pm\infty} \frac{1}{2xe^{x^2}} =0$, so $y=0$ is a HA. **E.**

$f'(x)=e^{-x^2}-2x^2e^{-x^2}=e^{-x^2}\left(1-2x^2\right)>0 \Leftrightarrow x^2<\frac{1}{2} \Leftrightarrow |x|<\frac{1}{\sqrt{2}}$, so f is increasing on $\left(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)$ and decreasing on $\left(-\infty,-\frac{1}{\sqrt{2}}\right)$ and $\left(\frac{1}{\sqrt{2}},\infty\right)$. **F.** Local maximum value $f\left(\frac{1}{\sqrt{2}}\right)=1/\sqrt{2e}$, local minimum value $f\left(-\frac{1}{\sqrt{2}}\right)=-1/\sqrt{2e}$ **G.** $f''(x)=-2xe^{-x^2}\left(1-2x^2\right)-4xe^{-x^2}=2xe^{-x^2}\left(2x^2-3\right)>0$ $\Leftrightarrow x>\sqrt{\frac{3}{2}}$ or $-\sqrt{\frac{3}{2}}<x<0$, so f is CU on $\left(\sqrt{\frac{3}{2}},\infty\right)$ and $\left(-\sqrt{\frac{3}{2}},0\right)$ and CD on $\left(-\infty,-\sqrt{\frac{3}{2}}\right)$ and $\left(0,\sqrt{\frac{3}{2}}\right)$.
IP are $(0,0)$ and $\left(\pm\sqrt{\frac{3}{2}},\pm\sqrt{\frac{3}{2}}\,e^{-3/2}\right)$.

H.



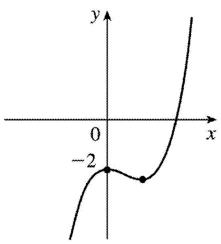
50. $y=f(x)=e^x-3e^{-x}-4x$ **A.** $D=R$ **B.** $y-$ intercept $=-2$; x –intercept ≈ 2.22 **C.** No symmetry
D. $\lim_{x\to\infty}\left(e^x-3e^{-x}-4x\right)=\lim_{x\to\infty} x\left(\frac{e^x}{x}-3\frac{e^{-x}}{x}-4\right)=\infty$, since $\lim_{x\to\infty}\frac{e^x}{x}=\lim_{x\to\infty}\frac{e^x}{1}=\infty$.
Similarly,

$\lim_{x\to -\infty}\left(e^{x}-3e^{-x}-4x\right)=-\infty$. No HA; no VA

E. $f'(x)=e^{x}+3e^{-x}-4=e^{-x}\left(e^{2x}-4e^{x}+3\right)=e^{-x}\left(e^{x}-3\right)\left(e^{x}-1\right)>0\Leftrightarrow e^{x}>3$ or $e^{x}<1\Leftrightarrow$ $x>\ln 3$ or $x<0$. So f is increasing on $(-\infty,0)$ and $(\ln 3,\infty)$ and decreasing on $(0,\ln 3)$. **F.** Local maximum value $f(0)=-2$,

local minimum value $f(\ln 3)=2-4\ln 3$ **G.** $f''(x)=e^{x}-3e^{-x}=e^{-x}\left(e^{2x}-3\right)>0\Leftrightarrow e^{2x}>3\Leftrightarrow x>\frac{1}{2}\ln 3$, so f is CU on $\left(\frac{1}{2}\ln 3,\infty\right)$ and CD on $\left(-\infty,\frac{1}{2}\ln 3\right)$. IP at $\left(\frac{1}{2}\ln 3,-2\ln 3\right)$.

H.



51. $y=f(x)=e^{3x}+e^{-2x}$ **A.** $D=R$ **B.** $y-$ intercept $=f(0)=2$; no x –intercept **C.** No symmetry **D.** No asymptotes

E. $f'(x)=3e^{3x}-2e^{-2x}$, so $f'(x)>0\Leftrightarrow 3e^{3x}>2e^{-2x}$ [multiply by e^{2x}] $\Leftrightarrow e^{5x}>\frac{2}{3}\Leftrightarrow 5x>\ln\frac{2}{3}\Leftrightarrow$

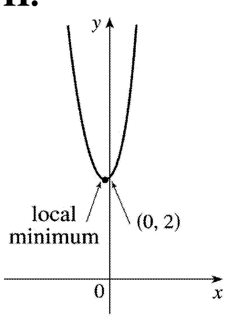
$x>\frac{1}{5}\ln\frac{2}{3}\approx -0.081$. Similarly, $f'(x)<0\Leftrightarrow x<\frac{1}{5}\ln\frac{2}{3}$.

f is decreasing on $\left(-\infty,\frac{1}{5}\ln\frac{2}{3}\right)$ and increasing on $\left(\frac{1}{5}\ln\frac{2}{3},\infty\right)$.

F. Local minimum value $f\left(\frac{1}{5}\ln\frac{2}{3}\right)=\left(\frac{2}{3}\right)^{3/5}+\left(\frac{2}{3}\right)^{-2/5}\approx 1.96$; no local maximum.

G. $f''(x)=9e^{3x}+4e^{-2x}$, so $f''(x)>0$ for all x , and f is CU on $(-\infty,\infty)$. No IP

H.



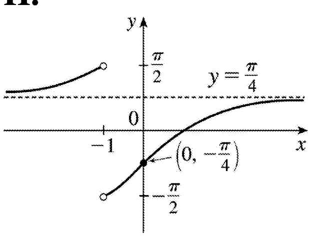
52. $y=f(x)=\tan^{-1}\left(\frac{x-1}{x+1}\right)$ **A.** $D=\{x\,|\,x\neq -1\}$

B. x –intercept $=1$, y –intercept

$=f(0)=\tan^{-1}(-1)=-\frac{\pi}{4}$ **C.** No symmetry **D.**
$\lim_{x\to\pm\infty}\tan^{-1}\left(\frac{x-1}{x+1}\right)=\lim_{x\to\pm\infty}\tan^{-1}\left(\frac{1-1/x}{1+1/x}\right)=\tan^{-1}1=\frac{\pi}{4}$, so $y=\frac{\pi}{4}$ is a HA. Also
$\lim_{x\to-1^+}\tan^{-1}\left(\frac{x-1}{x+1}\right)=-\frac{\pi}{2}$ and $\lim_{x\to-1^-}\tan^{-1}\left(\frac{x-1}{x+1}\right)=\frac{\pi}{2}$. **E.**

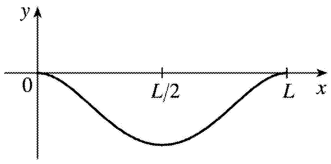
$$f'(x)=\frac{1}{1+[(x-1)/(x+1)]^2}\;\frac{(x+1)-(x-1)}{(x+1)^2}$$
$$=\frac{2}{(x+1)^2+(x-1)^2}=\frac{1}{x^2+1}>0$$

so f is increasing on $(-\infty,-1)$ and $(-1,\infty)$. **F.** No extreme values
H.

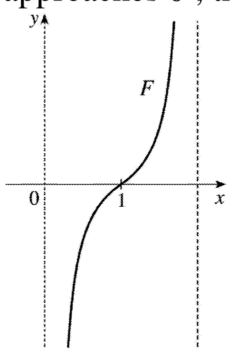


G. $f''(x)=-2x/\left(x^2+1\right)^2>0\Leftrightarrow x<0$, so f is CU on $(-\infty,-1)$ and $(-1,0)$, and CD on $(0,\infty)$.
IP at $\left(0,-\frac{\pi}{4}\right)$

53. $y=-\frac{W}{24EI}x^4+\frac{WL}{12EI}x^3-\frac{WL^2}{24EI}x^2=-\frac{W}{24EI}x^2\left(x^2-2Lx+L^2\right)=\frac{-W}{24EI}x^2(x-L)^2=cx^2(x-L)^2$ where $c=-\frac{W}{24EI}$ is a negative constant and $0\le x\le L$. We sketch $f(x)=cx^2(x-L)^2$ for $c=-1$. $f(0)=f(L)=0$
$f'(x)=cx^2[2(x-L)]+(x-L)^2(2cx)=2cx(x-L)[x+(x-L)]=2cx(x-L)(2x-L)$. So for $0<x<L$, $f'(x)>0\Leftrightarrow x(x-L)(2x-L)<0$ (since $c<0$) $\Leftrightarrow L/2<x<L$ and $f'(x)<0\Leftrightarrow 0<x<L/2$. So f is increasing on $(L/2,\mathrm{L})$ and decreasing on $(0,\mathrm{L}/2)$, and there is a local and absolute minimum at $(L/2,\mathrm{f}(L/2))=\left(L/2,\mathrm{cL}^4/16\right)$.
$f'(x)=2c[x(x-L)(2x-L)]\Rightarrow f''(x)=2c[1(x-L)(2x-L)+x(1)(2x-L)+x(x-L)(2)]=2c\left(6x^2-6Lx+L^2\right)=0\Leftrightarrow$
$x=\frac{6L\pm\sqrt{12L^2}}{12}=\frac{1}{2}L\pm\frac{\sqrt{3}}{6}L$, and these are the x –coordinates of the two inflection points.



54. $F(x)=-\frac{k}{x^2}+\frac{k}{(x-2)^2}$, where $k>0$ and $0<x<2$. For $0<x<2$, $x-2<0$, so $F'(x)=\frac{2k}{x^3}-\frac{2k}{(x-2)^3}>0$ and F is increasing. $\lim_{x\to 0^+} F(x)=-\infty$ and $\lim_{x\to 2^-} F(x)=\infty$, so $x=0$ and $x=2$ are vertical asymptotes. Notice that when the middle particle is at $x=1$, the net force acting on it is 0 . When $x>1$, the net force is positive, meaning that it acts to the right. And if the particle approaches $x=2$, the force on it rapidly becomes very large. When $x<1$, the net force is negative, so it acts to the left. If the particle approaches 0 , the force becomes very large to the left.



55.

$y=\frac{x^2+1}{x+1}$. Long division gives us:

$$\begin{array}{r|l} & \quad\quad x-1 \\ \hline x^2+2x & x^2 \quad\quad + 1 \\ & x^2+x \\ \hline & \quad -x+1 \\ & \quad -x+1 \\ \hline & \quad\quad\quad 2 \end{array}$$

Thus, $y=f(x)=\frac{x^2+1}{x+1}=x-1+\frac{2}{x+1}$ and $f(x)-(x-1)=\frac{2}{x+1}=\frac{\frac{2}{x}}{1+\frac{1}{x}}\to 0$ as $x\to\pm\infty$. So the line $y=x-1$ is a slant asymptote (SA).

56.

$y=\dfrac{2x^3+x^2+x+3}{x^2+2x}$. Long division gives us:

$$\begin{array}{r|r}
 & 2x-3 \\ \hline
x^2+2x & 2x^3+x^2+x+3 \\
 & 2x^3+4x^2 \\ \hline
 & -3x^2+x \\
 & -3x^2-6x \\ \hline
 & 7x+3
\end{array}$$

Thus, $y=f(x)=\dfrac{2x^3+x^2+x+3}{x^2+2x}=2x-3+\dfrac{7x+3}{x^2+2x}$ and $f(x)-(2x-3)=\dfrac{7x+3}{x^2+2x}=\dfrac{\frac{7}{x}+\frac{3}{x^2}}{1+\frac{2}{x}}\to 0$ as $x\to\pm\infty$. So the line $y=2x-3$ is a SA.

57.

$y=\dfrac{4x^3-2x^2+5}{2x^2+x-3}$. Long division gives us:

$$\begin{array}{r|r}
 & 2x-2 \\ \hline
2x^2+x-3 & 4x^3-2x^2 \qquad +3 \\
 & 4x^3+2x^2-6x \\ \hline
 & -4x^2+6x+5 \\
 & -4x^2-2x+6 \\ \hline
 & 8x-3
\end{array}$$

Thus, $y=f(x)=\dfrac{4x^3-2x^2+5}{2x^2+x-3}=2x-2+\dfrac{8x-1}{2x^2+x-3}$ and

$f(x)-(2x-2)=\dfrac{8x-1}{2x^2+x-3}=\dfrac{\frac{8}{x}-\frac{1}{x^2}}{2+\frac{1}{x}-\frac{3}{x^2}}\to 0$ as $x\to\pm\infty$. So the line $y=2x-2$ is a SA.

58.

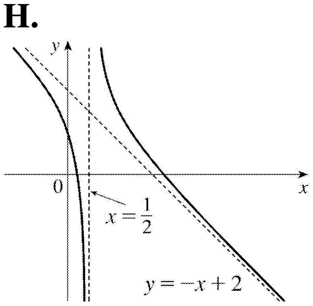
$y=\dfrac{5x^4+x^2+x}{x^3-x^2+2}$. Long division gives us:

$$\begin{array}{r|l} & 5x+2 \\ x^3-x^2+2 & 5x^4 \qquad +x^2 \quad +x \\ & 5x^4-5x^3 \qquad +10x \\ \hline & -5x^3+x^2-9x \\ & 5x^3-5x^2 \qquad +10 \\ \hline & 6x^2-9x-10 \end{array}$$

Thus, $y=f(x)=\dfrac{5x^4+x^2+x}{x^3-x^2+2}=5x+5+\dfrac{6x^2-9x-10}{x^3-x^2+2}$ and $f(x)-(5x+5)=\dfrac{6x^2-9x-10}{x^3-x^2+2}=\dfrac{\dfrac{6}{x}-\dfrac{9}{x^2}-\dfrac{10}{x^3}}{1-\dfrac{1}{x}+\dfrac{2}{x^3}}\to 0$ as $x\to\pm\infty$. So the line $y=5x+5$ is a SA.

59. $y=f(x)=\dfrac{-2x^2+5x-1}{2x-1}=-x+2+\dfrac{1}{2x-1}$ **A.** $D=\left\{x\in R \mid x\neq\dfrac{1}{2}\right\}=\left(-\infty,\dfrac{1}{2}\right)\cup\left(\dfrac{1}{2},\infty\right)$ **B.** y –intercept: $f(0)=1$; x –intercepts: $f(x)=0\Rightarrow -2x^2+5x-1=0\Rightarrow x=\dfrac{-5\pm\sqrt{17}}{-4}\Rightarrow x\approx 0.22$, 2.28 . **C.** No symmetry **D.** $\lim\limits_{x\to(1/2)^-} f(x)=-\infty$ and $\lim\limits_{x\to(1/2)^+} f(x)=\infty$, so $x=\dfrac{1}{2}$ is a VA. $\lim\limits_{x\to\pm\infty}\left[f(x)-(-x+2)\right]=\lim\limits_{x\to\pm\infty}\dfrac{1}{2x-1}=0$, so the line $y=-x+2$ is a SA. **E.** $f'(x)=-1-\dfrac{2}{(2x-1)^2}<0$ for $x\neq\dfrac{1}{2}$, so f is decreasing on $\left(-\infty,\dfrac{1}{2}\right)$ and $\left(\dfrac{1}{2},\infty\right)$. **F.** No extreme values **G.** $f'(x)=-1-2(2x-1)^{-2}\Rightarrow f''(x)=-2(-2)(2x-1)^{-3}(2)=\dfrac{8}{(2x-1)^3}$, so $f''(x)>0$ when $x>\dfrac{1}{2}$ and $f''(x)<0$ when $x<\dfrac{1}{2}$. Thus, f is CU on $\left(\dfrac{1}{2},\infty\right)$ and CD on $\left(-\infty,\dfrac{1}{2}\right)$. No IP

H.

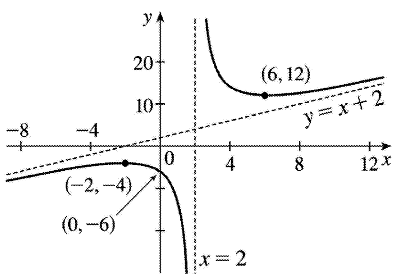


60. $y=f(x)=\dfrac{x^2+12}{x-2}=x+2+\dfrac{16}{x-2}$ **A.** $D=\{x\in R\mid x\neq 2\}=(-\infty,2)\cup(2,\infty)$ **B.** y –intercept: $f(0)=-6$; no x –intercepts. **C.** No symmetry **D.** $\lim_{x\to 2^-} f(x)=-\infty$ and $\lim_{x\to 2^+} f(x)=\infty$, so $x=2$ is a VA.

$\lim_{x\to\pm\infty}[f(x)-(x+2)]=\lim_{x\to\pm\infty}\dfrac{16}{x-2}=0$, so the line $y=x+2$ is a slant asymptote. **E.**

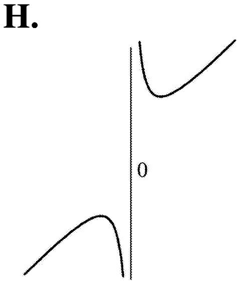
$f'(x)=1-\dfrac{16}{(x-2)^2}=\dfrac{x^2-4x-12}{(x-2)^2}=\dfrac{(x-6)(x+2)}{(x-2)^2}$, so $f'(x)>0$ when $x<-2$ or $x>6$ and $f'(x)<0$ when $-2<x<2$ or $2<x<6$. Thus, f is increasing on $(-\infty,-2)$ and $(6,\infty)$ and decreasing on $(-2,2)$ and $(2,6)$. **F.** Local maximum value $f(-2)=-4$, local minimum value $f(6)=12$ **G.** $f''(x)=16(-2)(x-2)^{-3}=\dfrac{32}{(x-2)^3}$, so $f''(x)>0$ for $x>2$ and $f''(x)<0$ for $x<2$. f is CU on $(2,\infty)$ and CD on $(-\infty,2)$. No IP

H.



61. $y=f(x)=\left(x^2+4\right)/x=x+4/x$ **A.** $D=\{x\mid x\neq 0\}=(-\infty,0)\cup(0,\infty)$ **B.** No intercept **C.** $f(-x)=-f(x)\Rightarrow$ symmetry about the origin **D.** $\lim_{x\to\infty}(x+4/x)=\infty$ but $f(x)-x=4/x\to 0$ as $x\to\pm\infty$, so $y=x$ is a slant asymptote. $\lim_{x\to 0^+}(x+4/x)=\infty$ and $\lim_{x\to 0^-}(x+4/x)=-\infty$, so $x=0$ is a VA. **E.** $f'(x)=1-4/x^2>0\Leftrightarrow x^2>4\Leftrightarrow x>2$ or $x<-2$, so f is increasing on $(-\infty,-2)$ and $(2,\infty)$ and decreasing on $(-2,0)$ and $(0,2)$. **F.** Local maximum value $f(-2)=-4$, local minimum value $f(2)=4$ **G.** $f''(x)=8/x^3>0\Leftrightarrow x>0$ so f is CU on $(0,\infty)$ and CD on $(-\infty,0)$. No IP

H.



62.

$y=f(x)=e^{x}-x$ **A.** $D=R$ **B.** No $x-$ intercept; y –intercept $=1$ **C.** No symmetry **D.** $\lim_{x\to -\infty}\left(e^{x}-x\right)=\infty$,

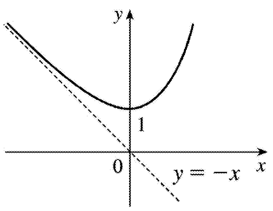
$\lim_{x\to \infty}\left(e^{x}-x\right)=\lim_{x\to \infty} x\left(\frac{e^{x}}{x}-1\right)=\infty$ since $\lim_{x\to \infty}\frac{e^{x}}{x}=\lim_{x\to \infty}\frac{e^{x}}{1}=\infty$.

$y=-x$ is a slant asymptote since $\left(e^{x}-x\right)-(-x)=e^{x}\to 0$ as $x\to -\infty$. **E.** $f'(x)=e^{x}-1>0\Leftrightarrow e^{x}>1\Leftrightarrow x>0$, so f is increasing on $(0,\infty)$ and decreasing on $(-\infty,0)$.

F. $f(0)=1$ is a local and absolute minimum value.

G. $f''(x)=e^{x}>0$ for all x , so f is CU on R . No IP

H.



63. $y=f(x)=\frac{2x^{3}+x^{2}+1}{x^{2}+1}=2x+1+\frac{-2x}{x^{2}+1}$ **A.** $D=R$ **B.** y –intercept: $f(0)=1$; x –intercept: $f(x)=0\Rightarrow$

$0=2x^{3}+x^{2}+1=(x+1)(2x^{2}-x+1)\Rightarrow x=-1$ **C.** No symmetry **D.** No VA

$\lim_{x\to \pm\infty}\left[f(x)-(2x+1)\right]=\lim_{x\to \pm\infty}\frac{-2x}{x^{2}+1}=\lim_{x\to \pm\infty}\frac{-2/x}{1+1/x^{2}}=0$, so the line $y=2x+1$ is a slant asymptote. **E.**

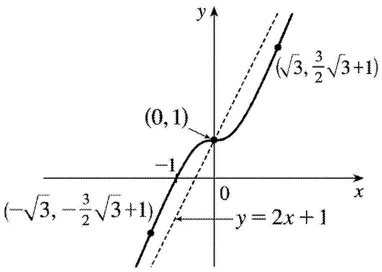
$'(x)=2+\frac{\left(x^{2}+1\right)(-2)-(-2x)(2x)}{\left(x^{2}+1\right)^{2}}=\frac{2\left(x^{4}+2x^{2}+1\right)-2x^{2}-2+4x^{2}}{\left(x^{2}+1\right)^{2}}=\frac{2x^{4}+6x^{2}}{(x^{2}+1)^{2}}=\frac{2x^{2}(x^{2}+3)}{(x^{2}+1)^{2}}$ so $f'(x)>0$ if $x\neq 0$. Thus, f is increasing on $(-\infty,0)$ and $(0,\infty)$. Since f is continuous at 0 , f is increasing on R .

F. No extreme values $f''(x)=\frac{\left(x^{2}+1\right)^{2}\cdot\left(8x^{3}+12x\right)-\left(2x^{4}+6x^{2}\right)\cdot 2\left(x^{2}+1\right)(2x)}{\left[\left(x^{2}+1\right)^{2}\right]^{2}}$

$=\frac{4x\left(x^{2}+1\right)\left(x^{2}+1\right)\left(2x^{2}+3\right)-2x^{4}-6x^{2}]}{\left(x^{2}+1\right)^{4}}=\frac{4x\left(-x^{2}+3\right)}{\left(x^{2}+1\right)^{3}}$ so $f''(x)>0$ for $x<-\sqrt{3}$ and $0<x<\sqrt{3}$, and

$f''(x)<0$ for $-\sqrt{3}<x<0$ and $x>\sqrt{3}$. f is CU on $\left(-\infty,-\sqrt{3}\right)$ and $\left(0,\sqrt{3}\right)$, and CD on $\left(-\sqrt{3},0\right)$ and $\left(\sqrt{3},\infty\right)$. There are three IPs: $(0,1)$, $\left(-\sqrt{3},-\frac{3}{2}\sqrt{3}+1\right)\approx(-1.73,-1.60)$, and $\left(\sqrt{3},\frac{3}{2}\sqrt{3}+1\right)\approx(1.73,3.60)$.

H.



64. $y=f(x)=\dfrac{(x+1)^3}{(x-1)^2}=\dfrac{x^3+3x^2+3x+1}{x^2-2x+1}=x+5+\dfrac{12x-4}{(x-1)^2}$ **A.** $D=\{x\in R\mid x\neq 1\}=(-\infty,1)\cup(1,\infty)$ **B.** y –intercept: $f(0)=1$; x –intercept: $f(x)=0\Rightarrow x=-1$ **C.** No symmetry **D.** $\lim_{x\to 1} f(x)=\infty$, so $x=1$ is a VA.

$\lim_{x\to\pm\infty}[f(x)-(x+5)]=\lim_{x\to\pm\infty}\dfrac{12x-4}{x^2-2x+1}=\lim_{x\to\pm\infty}\dfrac{\frac{12}{x}-\frac{4}{x^2}}{1-\frac{2}{x}^2+\frac{1}{x^2}}=0$, so the line $y=x+5$ is a SA. $f'(x)$

$=\dfrac{(x-1)^2\cdot 3(x+1)^2-(x+1)^3\cdot 2(x-1)}{\left[(x-1)^2\right]^2}=\dfrac{(x-1)(x+1)^2[3(x-1)-2(x+1)]}{(x-1)^4}=\dfrac{(x+1)^2(x-5)}{(x-1)^3}$ so $f'(x)>0$ when $x<-1$, $-1<x<1$, or $x>5$, and $f'(x)<0$ when $1<x<5$. f is increasing on $(-\infty,1)$ and $(5,\infty)$ and decreasing on $(1,5)$.

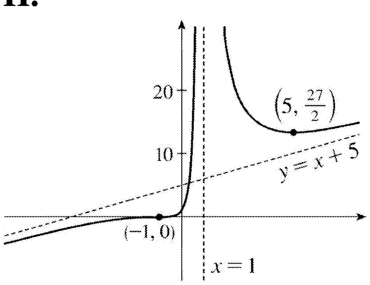
F. Local minimum value $f(5)=\dfrac{216}{16}=\dfrac{27}{2}$, no local maximum **G.** $f''(x)$

$=\dfrac{(x-1)^3\left[(x-1)^2+(x-5)\cdot 2(x+1)\right]-(x+1)^2(x-5)\cdot 3(x-1)^2}{\left[(x-1)^3\right]^2}$

$=\dfrac{(x-1)^2(x+1)\{(x-1)[(x+1)+2(x-5)]-3(x+1)(x-5)\}}{(x-1)^6}=\dfrac{(x+1)\left\{(x-1)[3x-9]-3\left(x^2-4x-5\right)\right\}}{(x-1)^4}=\dfrac{(x+1)(24)}{(x-1)^4}$

</ so $f''(x)>0$ if $-1<x<1$ or $x>1$, and $f''(x)<0$ if $x<-1$. Thus, f is CU on $(-1,1)$ and $(1,\infty)$ and CD on $(-\infty,-1)$. IP at $(-1,0)$

H.



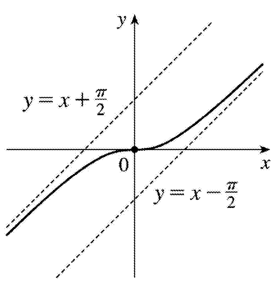
65. $y=f(x)=x-\tan^{-1}x$, $f'(x)=1-\frac{1}{1+x^2}=\frac{1+x^2-1}{1+x^2}=\frac{x^2}{1+x^2}$,

$f''(x)=\frac{\left(1+x^2\right)(2x)-x^2(2x)}{\left(1+x^2\right)^2}=\frac{2x\left(1+x^2-x^2\right)}{\left(1+x^2\right)^2}=\frac{2x}{\left(1+x^2\right)^2}$.

$\lim_{x\to\infty}\left[f(x)-\left(x-\frac{\pi}{2}\right)\right]=\lim_{x\to\infty}\left(\frac{\pi}{2}-\tan^{-1}x\right)=\frac{\pi}{2}-\frac{\pi}{2}=0$, so $y=x-\frac{\pi}{2}$ is a SA. Also,

$$\lim_{x\to-\infty}\left[f(x)-\left(x+\frac{\pi}{2}\right)\right]=\lim_{x\to-\infty}\left(-\frac{\pi}{2}-\tan^{-1}x\right)$$
$$=-\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)=0$$

so $y=x+\frac{\pi}{2}$ is also a SA. $f'(x)\geq 0$ for all x , with equality $\Leftrightarrow x=0$, so f is increasing on R . $f''(x)$ has the same sign as x , so f is CD on $(-\infty,0)$ and CU on $(0,\infty)$. $f(-x)=-f(x)$, so f is an odd function; its graph is symmetric about the origin. f has no local extreme values. Its only IP is at $(0,0)$.

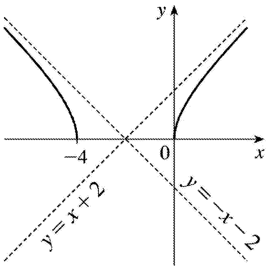


66. $y=f(x)=\sqrt{x^2+4x}=\sqrt{x(x+4)}$. $x(x+4)\geq 0\Leftrightarrow x\leq -4$ or $x\geq 0$, so $D=(-\infty,-4]\cup[0,\infty)$. y -intercept: $f(0)=0$; x -intercepts: $f(x)=0\Rightarrow x=-4$, 0 . $\sqrt{x^2+4x}\mp(x+2)$

$=\frac{\sqrt{x^2+4x}\mp(x+2)}{1}\cdot\frac{\sqrt{x^2+4x}\pm(x+2)}{\sqrt{x^2+4x}\pm(x+2)}=\frac{(x^2+4x)-(x^2+4x+4)}{\sqrt{x^2+4x}\pm(x+2)}=\frac{-4}{\sqrt{x^2+4x}\pm(x+2)}$ so

$\lim_{x\to\pm\infty}[f(x)\mp(x+2)]=0$. Thus, the graph of f approaches the slant asymptote $y=x+2$ as $x\to\infty$ and it approaches the slant asymptote $y=-(x+2)$ as $x\to-\infty$. $f'(x)=\frac{x+2}{\sqrt{x^2+4x}}$, so $f'(x)<0$ for $x<-4$ and $f'(x)>0$ for $x>0$; that is, f is decreasing on $(-\infty,-4)$ and increasing on $(0,\infty)$. There are no local extreme values. $f'(x)=(x+2)(x^2+4x)^{-1/2}\Rightarrow$

$f''(x)=(x+2)\cdot\left(-\frac{1}{2}\right)(x^2+4x)^{-3/2}\cdot(2x+4)+(x^2+4x)^{-1/2}=(x^2+4x)^{-3/2}\left[-(x+2)^2+(x^2+4x)\right]=-4\left(x^2+4x\right)^{-3/2}<0$ on D so f is CD on $(-\infty,-4)$ and $(0,\infty)$. No IP



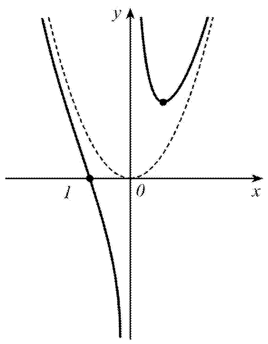
67. $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1\Rightarrow y=\pm\frac{b}{a}\sqrt{x^2-a^2}$. Now

$$\lim_{x\to\infty}\left[\frac{b}{a}\sqrt{x^2-a^2}-\frac{b}{a}x\right]=\frac{b}{a}\cdot\lim_{x\to\infty}\left(\sqrt{x^2-a^2}-x\right)\frac{\sqrt{x^2-a^2}+x}{\sqrt{x^2-a^2}+x}=\frac{b}{a}\cdot\lim_{x\to\infty}\frac{-a^2}{\sqrt{x^2-a^2}+x}=0,$$

which shows that $y=\frac{b}{a}x$ is a slant asymptote. Similarly,

$\lim_{x\to\infty}\left[-\frac{b}{a}\sqrt{x^2-a^2}-\left(-\frac{b}{a}x\right)\right]=-\frac{b}{a}\cdot\lim_{x\to\infty}\frac{-a^2}{\sqrt{x^2-a^2}+x}=0$, so $y=-\frac{b}{a}x$ is a slant asymptote.

68. $f(x)-x^2=\frac{x^3+1}{x}-x^2=\frac{x^3+1-x^3}{x}=\frac{1}{x}$, and $\lim_{x\to\pm\infty}\frac{1}{x}=0$. Therefore, $\lim_{x\to\pm\infty}\left[f(x)-x^2\right]=0$, and so the graph of f is asymptotic to that of $y=x^2$. For purposes of differentiation, we will use $f(x)=x^2+1/x$.
A. $D=\{x|x\neq 0\}$ **B.** No $y-$ intercept; to find the x –intercept, we set $y=0\Leftrightarrow x=-1$.
C. No symmetry **D.** $\lim_{x\to 0^+}\frac{x^3+1}{x}=\infty$ and $\lim_{x\to 0^-}\frac{x^3+1}{x}=-\infty$, so $x=0$ is a vertical asymptote. Also, the graph is asymptotic to the parabola $y=x^2$, as shown above. **E.** $f'(x)=2x-1/x^2>0\Leftrightarrow x>\frac{1}{\sqrt[3]{2}}$, so f is increasing on $\left(\frac{1}{\sqrt[3]{2}},\infty\right)$ and decreasing on $(-\infty,0)$ and $\left(0,\frac{1}{\sqrt[3]{2}}\right)$. **F.** Local minimum value $f\left(\frac{1}{\sqrt[3]{2}}\right)=\frac{3\sqrt[3]{3}}{2}$, no local maximum **G.** $f''(x)=2+2/x^3>0\Leftrightarrow x<-1$ or $x>0$, so f is CU on $(-\infty,-1)$ and $(0,\infty)$, and CD on $(-1,0)$. IP at $(-1,0)$
H.

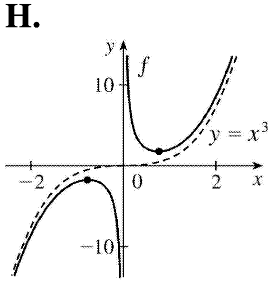


69. $\lim_{x\to\pm\infty}\left[f(x)-x^3\right]=\lim_{x\to\pm\infty}\frac{x^4+1}{x}-\frac{x^4}{x}=\lim_{x\to\pm\infty}\frac{1}{x}=0$, so the graph of f is asymptotic to that of $y=x^3$.

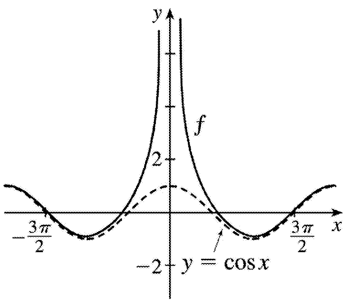
A. $D=\{x\,|\,x\neq 0\}$ **B.** No intercept **C.** f is symmetric about the origin. **D.** $\lim_{x\to 0^-}\left(x^3+\frac{1}{x}\right)=-\infty$ and $\lim_{x\to 0^+}\left(x^3+\frac{1}{x}\right)=\infty$, so $x=0$ is a vertical asymptote, and as shown above, the graph of f is asymptotic to that of $y=x^3$. **E.** $f'(x)=3x^2-1/x^2>0\Leftrightarrow x^4>\frac{1}{3}\Leftrightarrow |x|>\frac{1}{\sqrt[4]{3}}$, so f is increasing on $\left(-\infty,-\frac{1}{\sqrt[4]{3}}\right)$ and $\left(\frac{1}{\sqrt[4]{3}},\infty\right)$ and decreasing on $\left(-\frac{1}{\sqrt[4]{3}},0\right)$ and $\left(0,\frac{1}{\sqrt[4]{3}}\right)$. **F.** Local maximum value $f\left(-\frac{1}{\sqrt[4]{3}}\right)=-4\cdot 3^{-5/4}$, local minimum value $f\left(\frac{1}{\sqrt[4]{3}}\right)=4\cdot 3^{-5/4}$

G. $f''(x)=6x+2/x^3>0\Leftrightarrow x>0$, so f is CU on $(0,\infty)$ and CD on $(-\infty,0)$. No IP

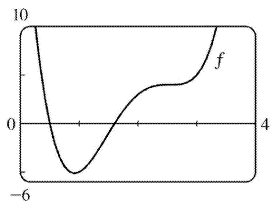
H.

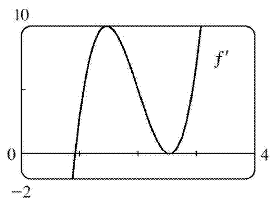


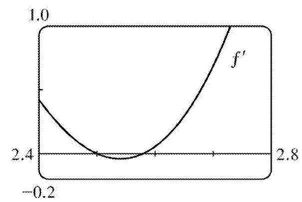
70. $\lim_{x\to\pm\infty}[f(x)-\cos\ x]=\lim_{x\to\pm\infty}1/x^2=0$, so the graph of f is asymptotic to that of $\cos\ x$. The intercepts can only be found approximately. $f(x)=f(-x)$, so f is even. $\lim_{x\to 0}\left(\cos\ x+\frac{1}{x^2}\right)=\infty$, so $x=0$ is a vertical asymptote. We don't need to calculate the derivatives, since we know the asymptotic behavior of the curve.

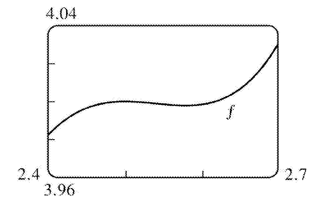

y
f
2
$-\frac{3\pi}{2}$
$\frac{3\pi}{2}$
x
−2
$y = \cos x$

1. $f(x)=4x^4-32x^3+89x^2-95x+29 \Rightarrow f'(x)=16x^3-96x^2+178x-95 \Rightarrow$
$f''(x)=48x^2-192x+178$. $f(x)=0 \Leftrightarrow x\approx 0.5$, 1.60 ; $f'(x)=0 \Leftrightarrow x\approx 0.92$, 2.5 , 2.58 and
$f''(x)=0 \Leftrightarrow x\approx 1.46$, 2.54 .

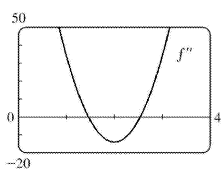




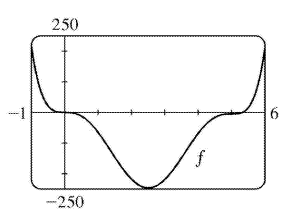


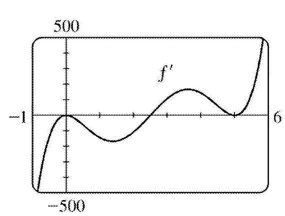


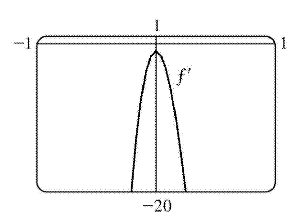
From the graphs of f' , we estimate that $f'<0$ and that f is decreasing on $(-\infty,0.92)$ and $(2.5,2.58)$, and that $f'>0$ and f is increasing on $(0.92,2.5)$ and $(2.58,\infty)$ with local minimum values $f(0.92)\approx -5.12$ and $f(2.58)\approx 3.998$ and local maximum value $f(2.5)=4$. The graphs of f' make it clear that f has a maximum and a minimum near $x=2.5$, shown more clearly in the fourth graph. From the graph of f'' , we estimate that $f''>0$ and that f is CU on $(-\infty,1.46)$ and $(2.54,\infty)$, and that $f''<0$ and f is CD on $(1.46,2.54)$. There are inflection points at about $(1.46,-1.40)$ and $(2.54,3.999)$.

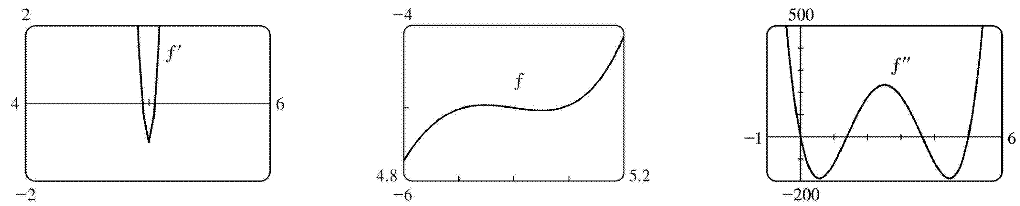


2. $f(x)=x^6-15x^5+75x^4-125x^3-x \Rightarrow f'(x)=6x^5-75x^4+300x^3-375x^2-1 \Rightarrow$
$f''(x)=30x^4-300x^3+900x^2-750x$. $f(x)=0 \Leftrightarrow x=0$ or $x\approx 5.33$; $f'(x)=0 \Leftrightarrow x\approx 2.50$, 4.95 , or 5.05;
$f''(x)=0 \Leftrightarrow x=0$, 5 or $x\approx 1.38$, 3.62 .



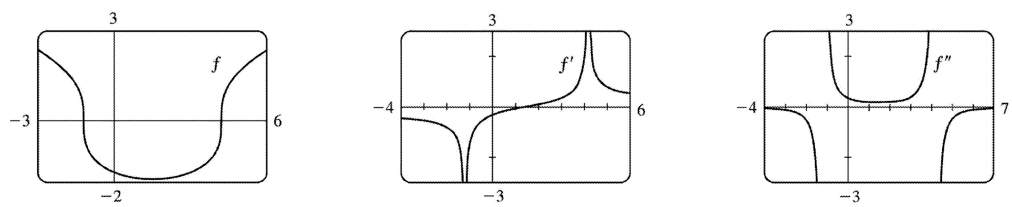






From the graphs of $f^{\prime}$, we estimate that f is decreasing on $(-\infty,2.50)$, increasing on $(2.50,4.95)$, decreasing on $(4.95,5.05)$, and increasing on $(5.05,\infty)$, with local minimum values $f(2.50)\approx-246.6$ and $f(5.05)\approx-5.03$, and local maximum value $f(4.95)\approx-4.965$ (notice the second graph of f). From the graph of $f^{\prime\prime}$, we estimate that f is CU on $(-\infty,0)$, CD on $(0,1.38)$, CU on $(1.38,3.62)$, CD on $(3.62,5)$, and CU on $(5,\infty)$. There are inflection points at $(0,0)$ and $(5,-5)$, and at about $(1.38,-126.38)$ and $(3.62,-128.62)$.

3. $f(x)=\sqrt[3]{x^2-3x-5}\Rightarrow f^{\prime}(x)=\frac{1}{3}\,\frac{2x-3}{\left(x^2-3x-5\right)^{2/3}}\Rightarrow f^{\prime\prime}(x)=-\frac{2}{9}\,\frac{x^2-3x+24}{\left(x^2-3x-5\right)^{5/3}}$

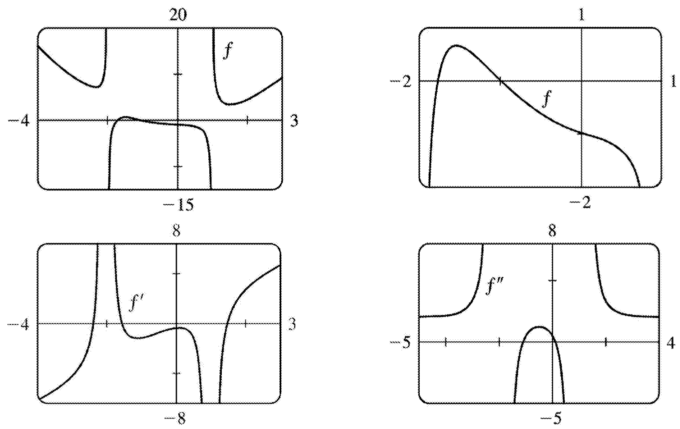


Note: With some CAS's, including Maple, it is necessary to define $f(x)=\frac{x^2-3x-5}{\left|x^2-3x-5\right|}\left|x^2-3x-5\right|^{1/3}$, since the CAS does not compute real cube roots of negative numbers. We estimate from the graph of $f^{\prime}$ that f is increasing on $(1.5,\infty)$, and decreasing on $(-\infty,1.5)$. f has no maximum. Minimum value: $f(1.5)\approx-1.9$.

From the graph of $f^{\prime\prime}$, we estimate that f is CU on $(-1.2,4.2)$ and CD on $(-\infty,-1.2)$ and $(4.2,\infty)$. IP at $(-1.2,0)$ and $(4.2,0)$.

4. $f(x)=\frac{x^4+x^3-2x^2+2}{x^2+x-2}\Rightarrow f^{\prime}(x)=2\,\frac{x^5+2x^4-3x^3-4x^2+2x-1}{\left(x^2+x-2\right)^2}\Rightarrow$

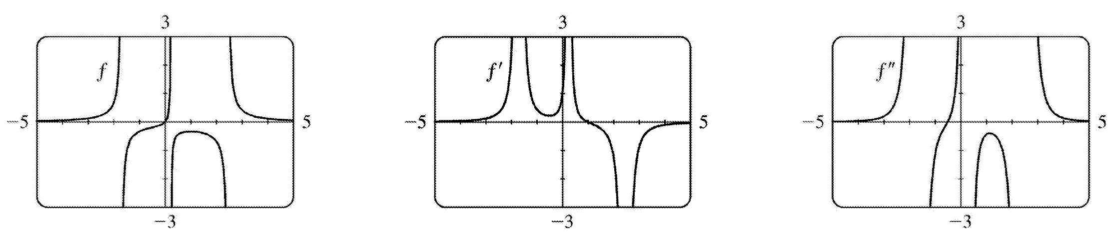
$$f''(x)=2\,\frac{x^6+3x^5-3x^4-11x^3+12x^2+18x-2}{\left(x^2+x-2\right)^3}$$



We estimate from the graph of f' that f is increasing on $(-2.4,-2)$, $(-2,-1.5)$ and $(1.5,\infty)$ and decreasing on $(-\infty,-2.4)$, $(-1.5,1)$ and $(1,1.5)$. Local maximum value: $f(-1.5)\approx 0.7$.

Local minimum values: $f(-2.4)\approx 7.2$, $f(1.5)\approx 3.4$. From the graph of f'' , we estimate that f is CU on $(-\infty,-2)$, $(-1.1,0.1)$ and $(1,\infty)$ and CD on $(-2,-1.1)$ and $(0.1,1)$.
f has IP at $(-1.1,0.2)$ and $(0.1,-1.1)$.

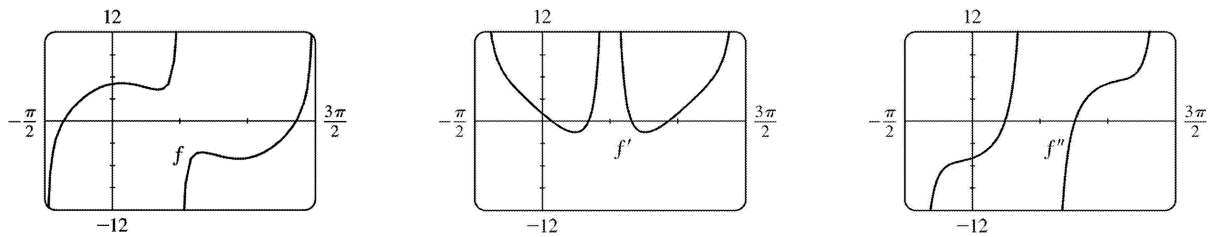
5. $f(x)=\dfrac{x}{x^3-x^2-4x+1}\Rightarrow f'(x)=\dfrac{-2x^3+x^2+1}{\left(x^3-x^2-4x+1\right)^2}\Rightarrow f''(x)=\dfrac{2\left(3x^5-3x^4+5x^3-6x^2+3x+4\right)}{\left(x^3-x^2-4x+1\right)^3}$



We estimate from the graph of f that $y=0$ is a horizontal asymptote, and that there are vertical asymptotes at $x=-1.7$, $x=0.24$, and $x=2.46$. From the graph of f' , we estimate that f is increasing on $(-\infty,-1.7)$, $(-1.7,0.24)$, and $(0.24,1)$, and that f is decreasing on $(1,2.46)$ and $(2.46,\infty)$. There is a local maximum value at $f(1)=-\dfrac{1}{3}$. From the graph of f'' , we estimate that f is CU on $(-\infty,-1.7)$, $(-0.506,0.24)$, and $(2.46,\infty)$, and that f is CD on $(-1.7,-0.506)$ and $(0.24,2.46)$.

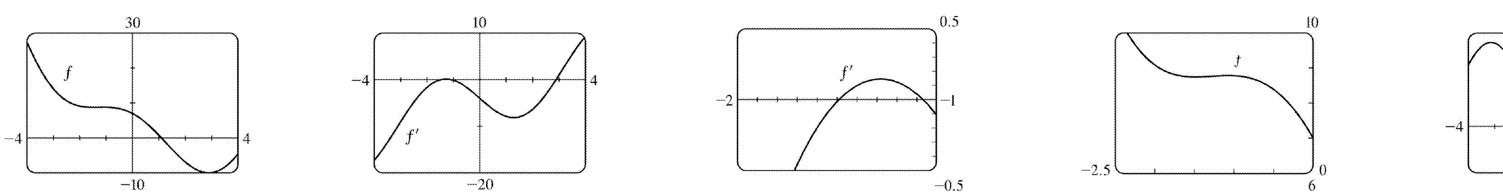
There is an inflection point at $(-0.506,-0.192)$.

6. $f(x)=\tan x+5\cos x \Rightarrow f'(x)=\sec^2 x-5\sin x \Rightarrow f''(x)=2\sec^2 x\tan x-5\cos x$. Since f is periodic with period 2π , and defined for all x except odd multiples of $\frac{\pi}{2}$, we graph f and its derivatives on $\left[-\frac{\pi}{2},\frac{3\pi}{2}\right]$.



We estimate from the graph of f' that f is increasing on $\left(-\frac{\pi}{2},0.21\right)$, $\left(1.07,\frac{\pi}{2}\right)$, $\left(\frac{\pi}{2},2.07\right)$, and $\left(2.93,\frac{3\pi}{2}\right)$, and decreasing on $(0.21,1.07)$ and $(2.07,2.93)$. Local minimum values: $f(1.07)\approx 4.23$, $f(2.93)\approx -5.10$. Local maximum values: $f(0.21)\approx 5.10$, $f(2.07)\approx -4.23$. From the graph of f'' , we estimate that f is CU on $\left(0.76,\frac{\pi}{2}\right)$ and $\left(2.38,\frac{3\pi}{2}\right)$, and CD on $\left(-\frac{\pi}{2},0.76\right)$ and $\left(\frac{\pi}{2},2.38\right)$. f has IP at $(0.76,4.57)$ and $(2.38,-4.57)$.

7. $f(x)=x^2-4x+7\cos x$, $-4\leq x\leq 4$. $f'(x)=2x-4-7\sin x \Rightarrow f''(x)=2-7\cos x$. $f(x)=0 \Leftrightarrow x\approx 1.10$; $f'(x)=0 \Leftrightarrow x\approx -1.49$, -1.07 , or 2.89 ; $f''(x)=0 \Leftrightarrow x=\pm\cos^{-1}\left(\frac{2}{7}\right)\approx \pm 1.28$



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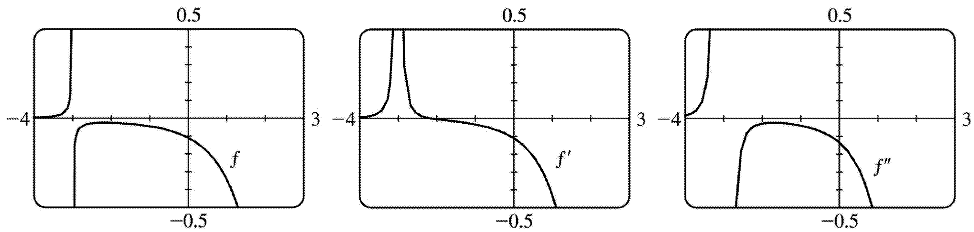
From the graphs of f' , we estimate that f is decreasing ($f'<0$) on $(-4,-1.49)$, increasing on $(-1.49,-1.07)$, decreasing on $(-1.07,2.89)$, and increasing on $(2.89,4)$, with local minimum values $f(-1.49)\approx 8.75$ and $f(2.89)\approx -9.99$ and local maximum value $f(-1.07)\approx 8.79$ (notice the second graph of f). From the graph of f'' , we estimate that f is CU ($f''>0$) on $(-4,-1.28)$, CD on $(-1.28,1.28)$, and CU on $(1.28,4)$. There are inflection points at about $(-1.28,8.77)$ and

$(1.28,-1.48)$.

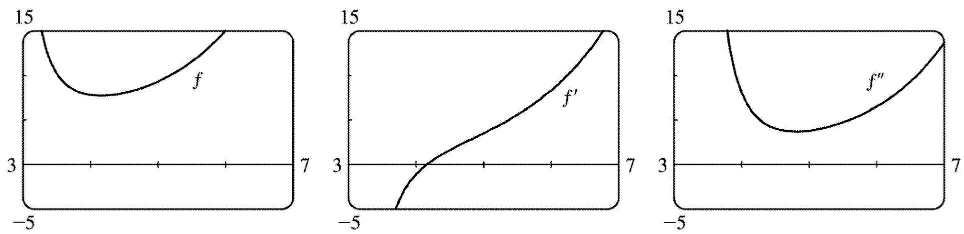
8. $f(x)=\dfrac{e^x}{x^2-9} \Rightarrow f'(x)=\dfrac{e^x(x^2-2x-9)}{\left(x^2-9\right)^2} \Rightarrow f''(x)=\dfrac{e^x(x^4-4x^3-12x^2+36x+99)}{\left(x^2-9\right)^3}$

There are vertical asymptotes at $x=\pm 3$. It is difficult to show all the pertinent features in one viewing rectangle, so we'll show f , f' , and f'' for $x<3$ and also for $x>3$.

For $x<3$

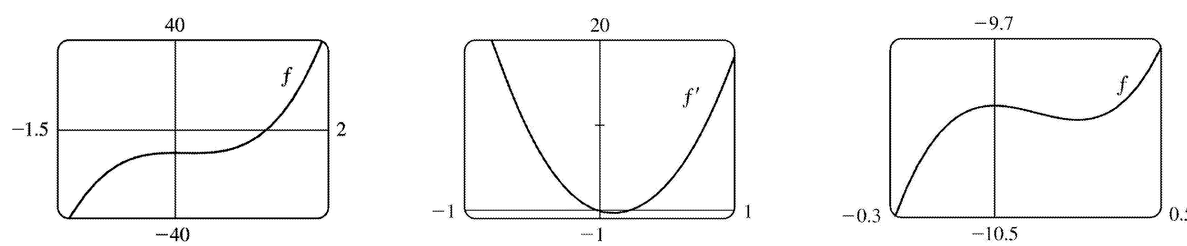


For $x>3$



We estimate from the graphs of f' and f that f is increasing on $(-\infty,-3)$, $(-3,-2.16)$, and $(4.16,\infty)$ and decreasing on $(-2.16,3)$ and $(3,4.16)$. There is a local maximum value of $f(-2.16)\approx-0.03$ and a local minimum value of $f(4.16)\approx 7.71$. From the graphs of f'' , we see that f is CU on $(-\infty,-3)$ and $(3,\infty)$ and CD on $(-3,3)$. There is no inflection point.

9. $f(x)=8x^3-3x^2-10 \Rightarrow f'(x)=24x^2-6x \Rightarrow f''(x)=48x-6$

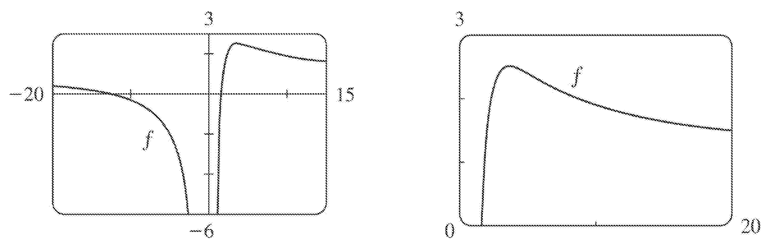


From the graphs, it appears that $f(x)=8x^3-3x^2-10$ increases on $(-\infty,0)$ and $(0.25,\infty)$ and decreases on $(0,0.25)$; that f has a local maximum value of $f(0)=-10.0$ and a local minimum value of $f(0.25)\approx-10.1$; that f is CU on $(0.1,\infty)$ and CD on $(-\infty,0.1)$; and that f has an IP at $(0.1,-10)$.

To find the exact values, note that $f'(x)=24x^2-6x=6x(4x-1)$, which is positive (f is increasing) for

$(-\infty,0)$ and $\left(\frac{1}{4},\infty\right)$, and negative (f is decreasing) on $\left(0,\frac{1}{4}\right)$. By the FDT, f has a local maximum at $x=0$: $f(0)=-10$; and f has a local minimum at $\frac{1}{4}$: $f\left(\frac{1}{4}\right)=\frac{1}{8}-\frac{3}{16}-10=-\frac{161}{16}$. $f''(x)=48x-6=6(8x-1)$, which is positive (f is CU) on $\left(\frac{1}{8},\infty\right)$ and negative (f is CD) on $\left(-\infty,\frac{1}{8}\right)$. f has an IP at $\left(\frac{1}{8},f\left(\frac{1}{8}\right)\right)=\left(\frac{1}{8},-\frac{321}{32}\right)$.

10.



From the graphs, it appears that f increases on $(0,3.6)$ and decreases on $(-\infty,0)$ and $(3.6,\infty)$; that f has a local maximum of $f(3.6)\approx 2.5$ and no local minima; that f is CU on $(5.5,\infty)$ and CD on $(-\infty,0)$ and $(0,5.5)$; and that f has an IP at $(5.5,2.3)$. $f(x)=\frac{x^2+11x-20}{x^2}=1+\frac{11}{x}-\frac{20}{x^2}\Rightarrow$ $f'(x)=-11x^{-2}+40x^{-3}=-x^{-3}(11x-40)$, which is positive (f is increasing) on $\left(0,\frac{40}{11}\right)$, and negative (f is decreasing) on $(-\infty,0)$ and on $\left(\frac{40}{11},\infty\right)$. By the FDT, f has a local maximum at $x=\frac{40}{11}$:

$$f\left(\frac{40}{11}\right)=\frac{\left(\frac{40}{11}\right)^2+11\left(\frac{40}{11}\right)-20}{\left(\frac{40}{11}\right)^2}=\frac{1600+11\cdot 11\cdot 40-20\cdot 121}{1600}=\frac{201}{80}$$

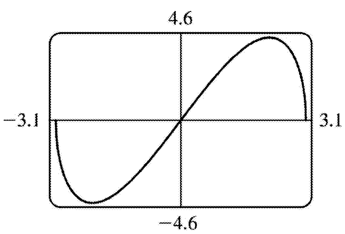
; and f has no local minimum. $f'(x)=-11x^{-2}+40x^{-3}\Rightarrow f''(x)=22x^{-3}-120x^{-4}=2x^{-4}(11x-60)$, which is positive (f is CU) on $\left(\frac{60}{11},\infty\right)$, and negative (f is CD) on $(-\infty,0)$ and $\left(0,\frac{60}{11}\right)$. f has an IP at $\left(\frac{60}{11},f\left(\frac{60}{11}\right)\right)=\left(\frac{60}{11},\frac{211}{90}\right)$.

11. From the graph, it appears that f increases on $(-2.1,2.1)$ and decreases on $(-3,-2.1)$ and $(2.1,3)$; that f has a local maximum of $f(2.1)\approx 4.5$ and a local minimum of $f(-2.1)\approx -4.5$; that f is CU on $(-3.0,0)$ and CD on $(0,3.0)$, and that f has an IP at $(0,0)$. $f(x)=x\sqrt{9-x^2}\Rightarrow$

$f'(x)=\frac{-x^2}{\sqrt{9-x^2}}+\sqrt{9-x^2}=\frac{9-2x^2}{\sqrt{9-x^2}}$, which is positive (f is increasing) on $\left(\frac{-3\sqrt{2}}{2},\frac{3\sqrt{2}}{2}\right)$ and negative (f is decreasing) on $\left(-3,\frac{-3\sqrt{2}}{2}\right)$ and $\left(\frac{3\sqrt{2}}{2},3\right)$. By the FDT, f has a local maximum value of $f\left(\frac{3\sqrt{2}}{2}\right)=\frac{3\sqrt{2}}{2}\sqrt{9-\left(\frac{3\sqrt{2}}{2}\right)^2}=\frac{9}{2}$; and f has a local minimum value of $f\left(\frac{-3\sqrt{2}}{2}\right)=-\frac{9}{2}$ (since f is an odd function). $f'(x)=\frac{-x^2}{\sqrt{9-x^2}}+\sqrt{9-x^2}\Rightarrow f''(x)$

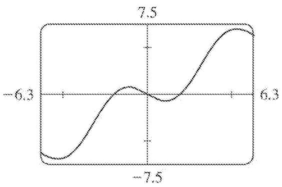
$$=\frac{\sqrt{9-x^2}(-2x)+x^2\left(\frac{1}{2}\right)\left(9-x^2\right)^{-1/2}(-2x)}{9-x^2}-x\left(9-x^2\right)^{-1/2}=\frac{-2x-x^3\left(9-x^2\right)^{-1}-x}{\sqrt{9-x^2}}$$

$=\frac{-3x}{\sqrt{9-x^2}}-\frac{x^3}{\left(9-x^2\right)^{3/2}}=\frac{x\left(2x^2-27\right)}{\left(9-x^2\right)^{3/2}}$ which is positive (f is CU) on $(-3,0)$ and negative (f is CD) on $(0,3)$. f has an IP at $(0,0)$.

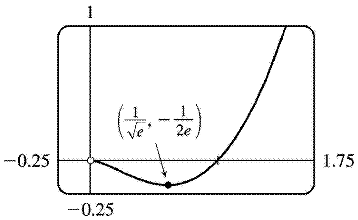


12. From the graph, it appears that f increases on $(-5.2,-1.0)$ and $(1.0,5.2)$ and decreases on $(-2\pi,-5.2)$, $(-1.0,1.0)$, and $(5.2,2\pi)$; that f has local maximum values of $f(-1.0)\approx 0.7$ and $f(5.2)\approx 7.0$ and local minimum values of $f(-5.2)\approx -7.0$ and $f(1.0)\approx -0.7$; that f is CU on $(-2\pi,-3.1)$ and $(0,3.1)$ and CD on $(-3.1,0)$ and $(3.1,2\pi)$, and that f has IP at $(0,0)$, $(-3.1,-3.1)$ and $(3.1,3.1)$. $f(x)=x-2\sin x\Rightarrow f'(x)=1-2\cos x$, which is positive (f is increasing) when $\cos x<\frac{1}{2}$, that is, on $\left(-\frac{5\pi}{3},-\frac{\pi}{3}\right)$ and $\left(\frac{\pi}{3},\frac{5\pi}{3}\right)$, and negative (f is decreasing) on $\left(-2\pi,-\frac{5\pi}{3}\right)$, $\left(-\frac{\pi}{3},\frac{\pi}{3}\right)$, and $\left(\frac{5\pi}{3},2\pi\right)$. By the FDT, f has local maximum values of $f\left(-\frac{\pi}{3}\right)=\frac{\pi}{3}+\sqrt{3}$ and $f\left(\frac{5\pi}{3}\right)=\frac{5\pi}{3}+\sqrt{3}$, and local minimum values of $f\left(-\frac{5\pi}{3}\right)=-\frac{5\pi}{3}-\sqrt{3}$ and $f\left(\frac{\pi}{3}\right)=-\frac{\pi}{3}-\sqrt{3}$. $f'(x)=1-2\cos x\Rightarrow f''(x)=2\sin x$, which is positive (f is CU) on $(-2\pi,-\pi)$

and $(0,\pi)$ and negative (f is CD) on $(-\pi,0)$ and $(\pi,2\pi)$. f has IP at $(0,0)$, $(-\pi,-\pi)$ and (π,π) .



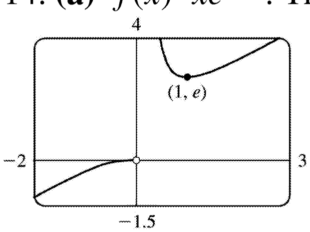
13. **(a)** $f(x)=x^2\ln x$. The domain of f is $(0,\infty)$.



(b) $\lim_{x\to0^+} x^2\ln x=\lim_{x\to0^+}\frac{\ln x}{1/x^2}=\lim_{x\to0^+}\frac{1/x}{-2/x^3}=\lim_{x\to0^+}\left(-\frac{x^2}{2}\right)=0$. There is a hole at $(0,0)$.

(c) It appears that there is an IP at about $(0.2,-0.06)$ and a local minimum at $(0.6,-0.18)$. $f(x)=x^2\ln x$ $\Rightarrow f'(x)=x^2(1/x)+(\ln x)(2x)=x(2\ln x+1)>0\Leftrightarrow\ln x>-\frac{1}{2}\Leftrightarrow x>e^{-1/2}$, so f is increasing on $\left(1/\sqrt{e},\infty\right)$, decreasing on $\left(0,1/\sqrt{e}\right)$. By the FDT, $f\left(1/\sqrt{e}\right)=-1/(2e)$ is a local minimum value. This point is approximately $(0.6065,-0.1839)$, which agrees with our estimate.
$f''(x)=x(2/x)+(2\ln x+1)=2\ln x+3>0\Leftrightarrow\ln x>-\frac{3}{2}\Leftrightarrow x>e^{-3/2}$, so f is CU on $\left(e^{-3/2},\infty\right)$ and CD on $\left(0,e^{-3/2}\right)$. IP is $\left(e^{-3/2},-3/\left(2e^3\right)\right)\approx(0.2231,-0.0747)$.

14. **(a)** $f(x)=xe^{1/x}$. The domain of f is $(-\infty,0)\cup(0,\infty)$.



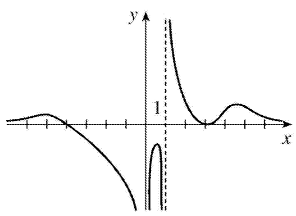
(b) $\lim_{x\to0^+} xe^{1/x}=\lim_{x\to0^+}\frac{e^{1/x}}{1/x}=\lim_{x\to0^+}\frac{e^{1/x}\left(-1/x^2\right)}{-1/x^2}=\lim_{x\to0^+} e^{1/x}=\infty$, so $x=0$ is a VA.
Also

$\lim_{x\to 0^-} xe^{1/x}=0$ since $1/x\to -\infty \Rightarrow e^{1/x}\to 0$.

(c) It appears that there is a local minimum at $(1,2.7)$. There are no IP and f is CD on $(-\infty,0)$ and CU on $(0,\infty)$.

$f(x)=xe^{1/x} \Rightarrow f'(x)=xe^{1/x}\left(-\frac{1}{x^2}\right)+e^{1/x}=e^{1/x}\left(1-\frac{1}{x}\right)>0 \Leftrightarrow \frac{1}{x}<1 \Leftrightarrow x<0$ or $x>1$, so f is increasing on $(-\infty,0)$ and $(1,\infty)$, and decreasing on $(0,1)$. By the FDT, $f(1)=e$ is a local minimum value, which agrees with our estimate.

$f''(x)=e^{1/x}\left(1/x^2\right)+(1-1/x)e^{1/x}\left(-1/x^2\right)=\left(e^{1/x}/x^2\right)(1-1+1/x)=e^{1/x}/x^3>0 \Leftrightarrow x>0$, so f is CU on $(0,\infty)$ and CD on $(-\infty,0)$. No IP.

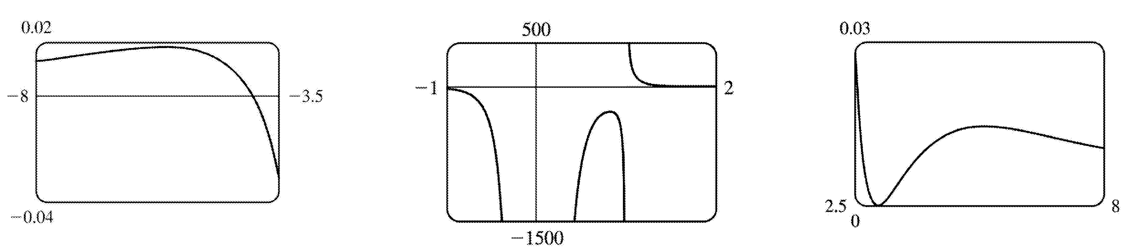


15.

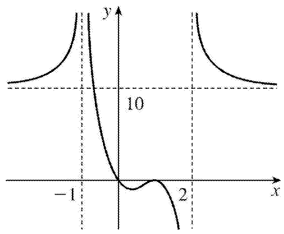
$f(x)=\frac{(x+4)(x-3)^2}{x^4(x-1)}$ has VA at $x=0$ and at $x=1$ since $\lim_{x\to 0} f(x)=-\infty$, $\lim_{x\to 1^-} f(x)=-\infty$ and $\lim_{x\to 1^+} f(x)=\infty$.

$$f(x) = \frac{\frac{x+4}{x}\cdot\frac{(x-3)^2}{x^2}}{\frac{x^4}{x^3}\cdot(x-1)} \qquad \text{[dividing numerator and denominator by } x^3 \text{]}$$

$=\frac{(1+4/x)(1-3/x)^2}{x(x-1)}\to 0$ as $x\to \pm\infty$, so f is asymptotic to the x –axis. Since f is undefined at $x=0$, it has no y – intercept. $f(x)=0 \Rightarrow (x+4)(x-3)^2=0 \Rightarrow x=-4$ or $x=3$, so f has x –intercepts -4 and 3 . Note, however, that the graph of f is only tangent to the x– axis and does not cross it at $x=3$, since f is positive as $x\to 3^-$ and as $x\to 3^+$.

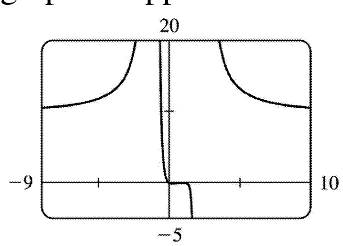


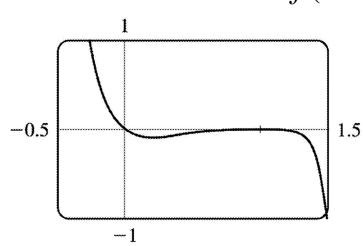
From these graphs, it appears that f has three maximum values and one minimum value. The maximum values are approximately $f(-5.6)=0.0182$, $f(0.82)=-281.5$ and $f(5.2)=0.0145$ and we know (since the graph is tangent to the $x-$ axis at $x=3$) that the minimum value is $f(3)=0$.

16. 

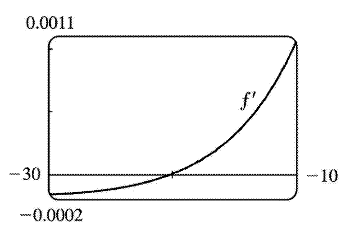
$f(x)=\dfrac{10x(x-1)^4}{(x-2)^3(x+1)^2}$ has VA at $x=-1$ and at $x=2$ since $\lim\limits_{x\to -1} f(x)=\infty$, $\lim\limits_{x\to 2^-} f(x)=-\infty$ and $\lim\limits_{x\to 2^+} f(x)=\infty$.

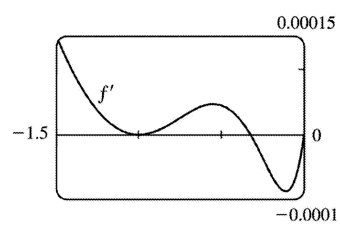
$f(x)=\dfrac{10(1-1/x)^4}{(1-2/x)^3(1+1/x)^2}\to 10$ as $x\to\pm\infty$, so f is asymptotic to the line $y=10$. $f(0)=0$, so f has a $y-$ intercept at 0 . $f(x)=0\Rightarrow 10x(x-1)^4=0\Rightarrow x=0$ or $x=1$. So f has x –intercepts 0 and 1 . Note, however, that f does not change sign at $x=1$, so the graph is tangent to the $x-$ axis and does not cross it. We know (since the graph is tangent to the $x-$ axis at $x=1$) that the maximum value is $f(1)=0$. From the graphs it appears that the minimum value is about $f(0.2)=-0.1$.

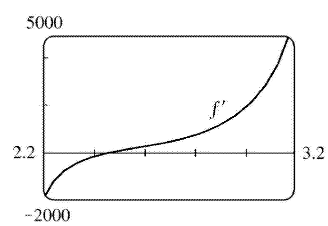




17. $f(x)=\dfrac{x^2(x+1)^3}{(x-2)^2(x-4)^4}\Rightarrow f'(x)=-\dfrac{x(x+1)^2\left(x^3+18x^2-44x-16\right)}{(x-2)^3(x-4)^5}$ (from CAS).



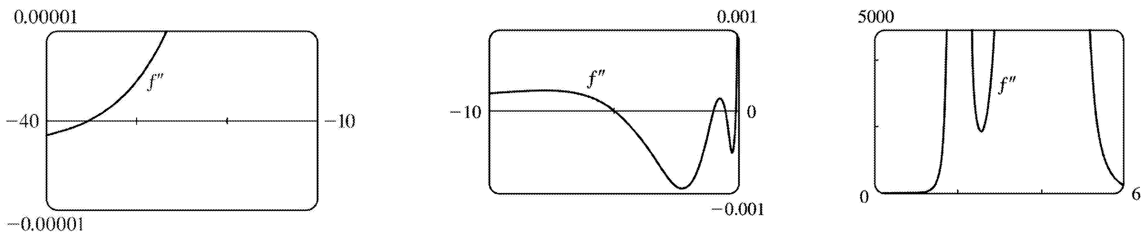




From the graphs of f' , it seems that the critical points which indicate extrema occur at $x\approx -20$, -0.3

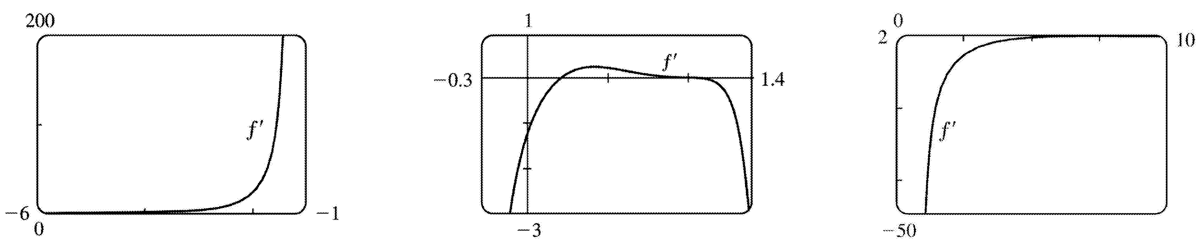
, and 2.5 , as estimated in Example 3. (There is another critical point at $x=-1$, but the sign of $f\,'$ does not change there.) We differentiate again, obtaining

$$f\,''(x)=2\,\frac{(x+1)\left(x^6+36x^5+6x^4-628x^3+684x^2+672x+64\right)}{(x-2)^4(x-4)^6}\ .$$

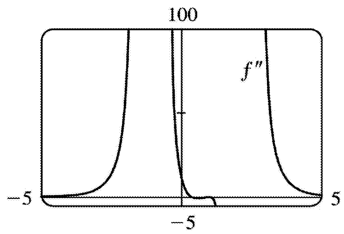


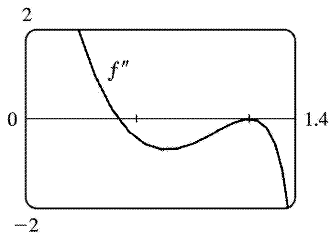
From the graphs of $f\,''$, it appears that f is CU on $(-35.3,-5.0)$, $(-1,-0.5)$, $(-0.1,2)$, $(2,4)$ and $(4,\infty)$ and CD on $(-\infty,-35.3)$, $(-5.0,-1)$ and $(-0.5,-0.1)$. We check back on the graphs of f to find the $y-$ coordinates of the inflection points, and find that these points are approximately $(-35.3,-0.015)$, $(-5.0,-0.005)$, $(-1,0)$, $(-0.5,0.00001)$, and $(-0.1,0.0000066)$.

18. $f(x)=\dfrac{10x(x-1)^4}{(x-2)^3(x+1)^2}\Rightarrow f\,'(x)=-20\,\dfrac{(x-1)^3(5x-1)}{(x-2)^4(x+1)^3}$ (from CAS).



From the graphs of $f\,'$, we estimate that f is increasing on $(-\infty,-1)$ and $(0.2,1)$ and decreasing on $(-1,0.2)$, $(1,2)$ and $(2,\infty)$. Differentiating $f\,'(x)$, we get $f\,''(x)=60\,\dfrac{(x-1)^2\left(5x^3-8x^2+17x-6\right)}{(x-2)^5(x+1)^4}$.

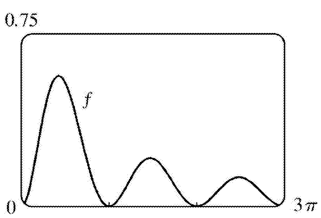


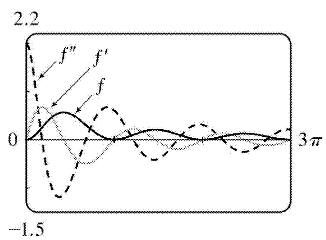


From the graphs of f'', it seems that f is CU on $(-\infty,-1.0)$, $(-1.0,0.4)$ and $(2.0,\infty)$, and CD on $(0.4,2)$. There is an inflection point at about $(0.4,-0.06)$.

19. $y=f(x)=\dfrac{\sin^2 x}{\sqrt{x^2+1}}$ with $0\le x\le 3\pi$. From a CAS, $y'=\dfrac{\sin x\left[2\left(x^2+1\right)\cos x-x\sin x\right]}{\left(x^2+1\right)^{3/2}}$ and

$$y''=\frac{\left(4x^4+6x^2+5\right)\cos^2 x-4x\left(x^2+1\right)\sin x\cos x-2x^4-2x^2-3}{\left(x^2+1\right)^{5/2}}.$$



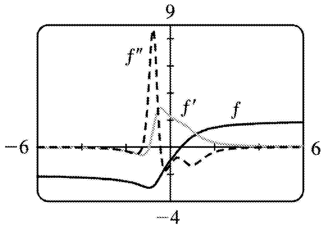


From the graph of f' and the formula for y', we determine that $y'=0$ when $x=\pi$, 2π, 3π, or $x\approx 1.3$, 4.6, or 7.8. So f is increasing on $(0,1.3)$, $(\pi,4.6)$, and $(2\pi,7.8)$. f is decreasing on $(1.3,\pi)$, $(4.6,2\pi)$, and $(7.8,3\pi)$. Local maximum values: $f(1.3)\approx 0.6$, $f(4.6)\approx 0.21$, and $f(7.8)\approx 0.13$. Local minimum values: $f(\pi)=f(2\pi)=0$. From the graph of f'', we see that $y''=0$ $\Leftrightarrow x\approx 0.6$, 2.1, 3.8, 5.4, 7.0, or 8.6. So f is CU on $(0,0.6)$, $(2.1,3.8)$, $(5.4,7.0)$, and $(8.6,3\pi)$. f is CD on $(0.6,2.1)$, $(3.8,5.4)$, and $(7.0,8.6)$. There are IP at $(0.6,0.25)$, $(2.1,0.31)$, $(3.8,0.10)$, $(5.4,0.11)$, $(7.0,0.061)$, and $(8.6,0.065)$.

20. $f(x)=\dfrac{2x-1}{\sqrt[4]{x^4+x+1}}\Rightarrow$

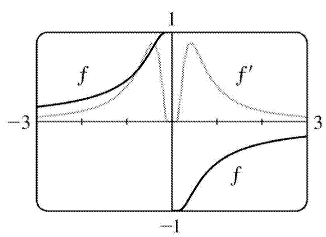
$f'(x)=\dfrac{4x^3+6x+9}{4\left(x^4+x+1\right)^{5/4}}\Rightarrow$

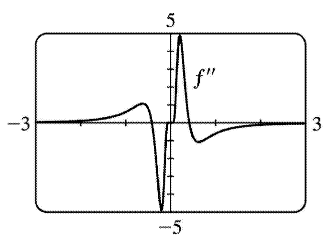
$$f''(x)=-\frac{32x^6+96x^4+152x^3-48x^2+6x+21}{16\left(x^4+x+1\right)^{9/4}}$$



From the graph of f', f appears to be decreasing on $(-\infty,-0.94)$ and increasing on $(-0.94,\infty)$. There is a local minimum value of $f(-0.94)\approx-3.01$. From the graph of f'', f appears to be CU on $(-1.25,-0.44)$ and CD on $(-\infty,-1.25)$ and $(-0.44,\infty)$. There are inflection points at $(-1.25,-2.87)$ and $(-0.44,-2.14)$.

21. $y=f(x)=\dfrac{1-e^{1/x}}{1+e^{1/x}}$. From a CAS, $y'=\dfrac{2e^{1/x}}{x^2\left(1+e^{1/x}\right)^2}$ and $y''=\dfrac{-2e^{1/x}\left(1-e^{1/x}+2x+2xe^{1/x}\right)}{x^4\left(1+e^{1/x}\right)^3}$.

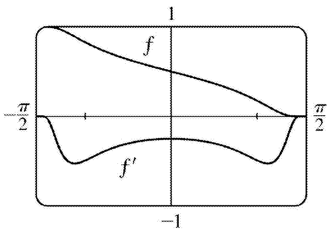


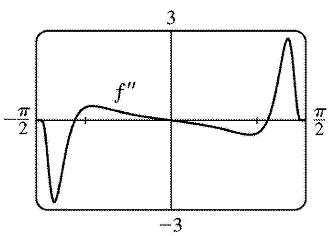


f is an odd function defined on $(-\infty,0)\cup(0,\infty)$. Its graph has no x– or y–intercepts. Since $\lim_{x\to\pm\infty} f(x)=0$, the x–axis is a HA. $f'(x)>0$ for $x\neq0$, so f is increasing on $(-\infty,0)$ and $(0,\infty)$. It has no local extreme values. $f''(x)=0$ for $x\approx\pm0.417$, so f is CU on $(-\infty,-0.417)$, CD on $(-0.417,0)$, CU on $(0,0.417)$, and CD on $(0.417,\infty)$. f has IPs at $(-0.417,0.834)$ and $(0.417,-0.834)$.

22. $y=f(x)=\dfrac{1}{1+e^{\tan x}}$. From a CAS,

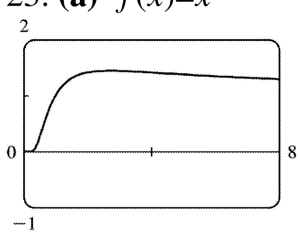
$y'=-\dfrac{e^{\tan x}}{\cos^2 x\left(1+e^{\tan x}\right)^2}$ and $y''=-\dfrac{e^{\tan x}\left[e^{\tan x}(2\sin x\cos x-1)+2\sin x\cos x+1\right]}{\cos^4 x\left(1+e^{\tan x}\right)^3}$. f is a periodic function with period π that has positive values throughout its domain, which consists of all real numbers except odd multiples of $\dfrac{\pi}{2}$ (that is, $\pm\dfrac{\pi}{2}$, $\pm\dfrac{3\pi}{2}$, $\pm\dfrac{5\pi}{2}$, and so on). f has y –intercept $\dfrac{1}{2}$, but no x –intercepts. We graph f , f' , and f'' on one period, $\left(-\dfrac{\pi}{2},\dfrac{\pi}{2}\right)$.





Since $f'(x)<0$ for all x in the domain of f , f is decreasing on the intervals between odd multiples of $\dfrac{\pi}{2}$. $f''(x)=0$ for $x=0+n\pi$ and for $x\approx\pm 1.124+n\pi$, so f is CD on $\left(-\dfrac{\pi}{2},-1.124\right)$, CU on $(-1.124,0)$, CD on $(0,1.124)$, and CU on $\left(1.124,\dfrac{\pi}{2}\right)$. Since f is periodic, this behavior repeats on every interval of length π . f has IPs at $(-1.124+n\pi,0.890)$, $\left(n\pi,\dfrac{1}{2}\right)$, and $(1.124+n\pi,0.110)$.

23. **(a)** $f(x)=x^{1/x}$

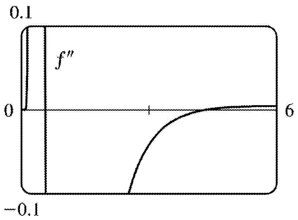


(b) Recall that $a^b=e^{b\ln a}$. $\lim_{x\to 0^+} x^{1/x}=\lim_{x\to 0^+} e^{(1/x)\ln x}$. As $x\to 0^+$, $\dfrac{\ln x}{x}\to-\infty$, so $x^{1/x}=e^{(1/x)\ln x}\to 0$. This indicates that there is a hole at $(0,0)$. As $x\to\infty$, we have the indeterminate form ∞^0 .

$\lim_{x\to\infty} x^{1/x}=\lim_{x\to\infty} e^{(1/x)\ln x}$, but $\lim_{x\to\infty} \frac{\ln x}{x}=\lim_{x\to\infty} \frac{1/x}{1}=0$, so $\lim_{x\to\infty} x^{1/x}=e^0=1$. This indicates that $y=1$ is a HA.

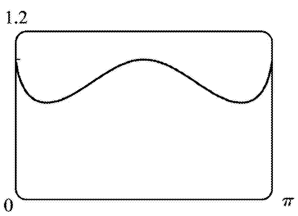
(c) Estimated maximum: $(2.72,1.45)$. No estimated minimum. We use logarithmic differentiation to find any critical numbers. $y=x^{1/x} \Rightarrow \ln y=\frac{1}{x}\ln x \Rightarrow \frac{y'}{y}=\frac{1}{x}\cdot\frac{1}{x}+(\ln x)\left(-\frac{1}{x^2}\right) \Rightarrow$

$y'=x^{1/x}\left(\frac{1-\ln x}{x^2}\right)=0 \Rightarrow \ln x=1 \Rightarrow x=e$. For $0<x<e$, $y'>0$ and for $x>e$, $y'<0$, so $f(e)=e^{1/e}$ is a local maximum value. This point is approximately $(2.7183,1.4447)$, which agrees with our estimate.

(d) 

From the graph, we see that $f''(x)=0$ at $x\approx 0.58$ and $x\approx 4.37$. Since f'' changes sign at these values, they are x-coordinates of inflection points.

24. **(a)** $f(x)=(\sin x)^{\sin x}$ is continuous where $\sin x>0$, that is, on intervals of the form $(2n\pi,(2n+1)\pi)$, so we have graphed f on $(0,\pi)$.

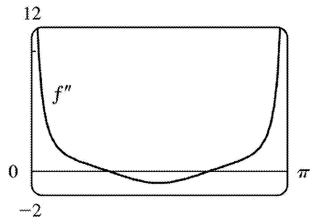


(b) $y=(\sin x)^{\sin x} \Rightarrow \ln y=\sin x\ln \sin x$, so

$\lim_{x\to 0^+} \ln y=\lim_{x\to 0^+} \sin x\ln \sin x=\lim_{x\to 0^+} \frac{\ln \sin x}{\csc x}=\lim_{x\to 0^+} \frac{\cot x}{-\csc x\cot x}=\lim_{x\to 0^+} (-\sin x)=0 \Rightarrow \lim_{x\to 0^+} y=e^0=1$.

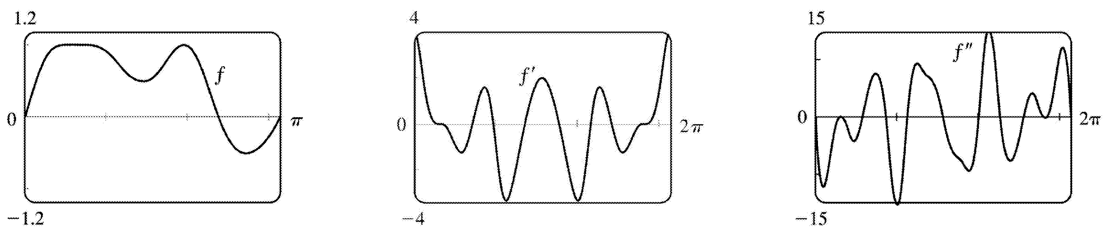
(c) It appears that we have a local maximum at $(1.57,1)$ and local minima at $(0.38,0.69)$ and $(2.76,0.69)$. $y=(\sin x)^{\sin x} \Rightarrow \ln y=\sin x\ln \sin x \Rightarrow$

$\frac{y'}{y}=(\sin x)\left(\frac{\cos x}{\sin x}\right)+(\ln \sin x)\cos x=\cos x(1+\ln \sin x)\Rightarrow y'=(\sin x)^{\sin x}(\cos x)(1+\ln \sin x)$. $y'=0\Rightarrow$ $\cos x=0$ or $\ln \sin x=-1\Rightarrow x_2=\frac{\pi}{2}$ or $\sin x=e^{-1}$. On $(0,\pi)$, $\sin x=e^{-1}\Rightarrow x_1=\sin^{-1}\left(e^{-1}\right)$ and $x_3=\pi-\sin^{-1}\left(e^{-1}\right)$. Approximating these points gives us $\left(x_1,f\left(x_1\right)\right)\approx(0.3767,0.6922)$, $\left(x_2,f\left(x_2\right)\right)\approx(1.5708,1)$, and $\left(x_3,f\left(x_3\right)\right)\approx(2.7649,0.6922)$. The approximations confirm our estimates.

(d) 

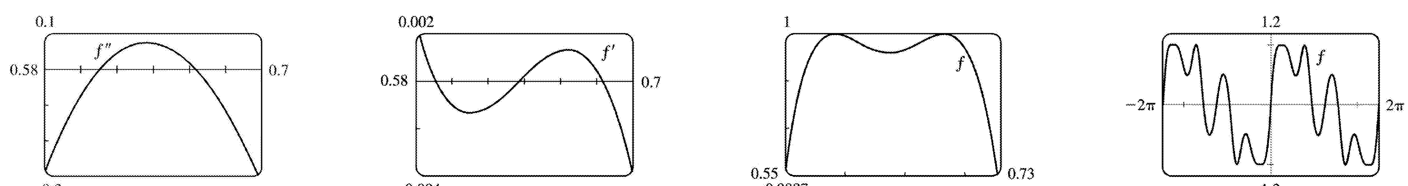
From the graph, we see that $f''(x)=0$ at $x\approx 0.94$ and $x\approx 2.20$. Since f'' changes sign at these values, they are x –coordinates of inflection points.

25.



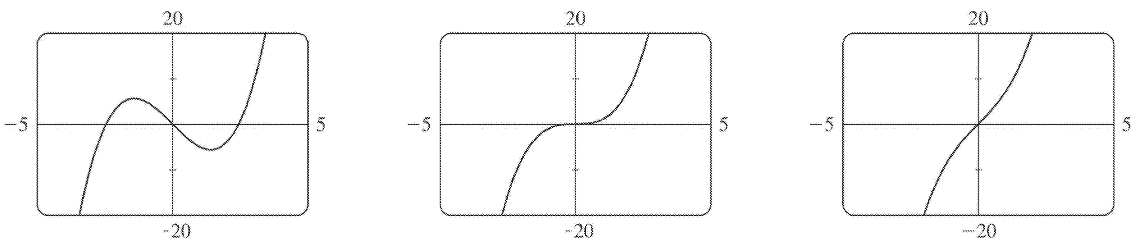
From the graph of $f(x)=\sin(x+\sin 3x)$ in the viewing rectangle $[0,\pi]$ by $[-1.2,1.2]$, it looks like f has two maxima and two minima. If we calculate and graph $f'(x)=[\cos(x+\sin 3x)](1+3\cos 3x)$ on $[0,2\pi]$,

we see that the graph of f' appears to be almost tangent to the x– axis at about $x=0.7$. The graph of $f''=-[\sin(x+\sin 3x)](1+3\cos 3x)^2+\cos(x+\sin 3x)(-9\sin 3x)$ is even more interesting near this x– value: it seems to just touch the x– axis.



If we zoom in on this place on the graph of f'', we see that f'' actually does cross the axis twice near $x=0.65$, indicating a change in concavity for a very short interval. If we look at the graph of f' on the same interval, we see that it changes sign three times near $x=0.65$, indicating that what we had thought was a broad extremum at about $x=0.7$ actually consists of three extrema (two maxima and a minimum). These maximum values are roughly $f(0.59)=1$ and $f(0.68)=1$, and the minimum value is roughly $f(0.64)=0.99996$. There are also a maximum value of about $f(1.96)=1$ and minimum values of about $f(1.46)=0.49$ and $f(2.73)=-0.51$. The points of inflection on $(0,\pi)$ are about $(0.61,0.99998)$, $(0.66,0.99998)$, $(1.17,0.72)$, $(1.75,0.77)$, and $(2.28,0.34)$. On $(\pi,2\pi)$, they are about $(4.01,-0.34)$, $(4.54,-0.77)$, $(5.11,-0.72)$, $(5.62,-0.99998)$, and $(5.67,-0.99998)$. There are also IP at $(0,0)$ and $(\pi,0)$. Note that the function is odd and periodic with period 2π, and it is also rotationally symmetric about all points of the form $((2n+1)\pi,0)$, n an integer.

26. $f(x)=x^3+cx=x\left(x^2+c\right) \Rightarrow f'(x)=3x^2+c \Rightarrow f''(x)=6x$



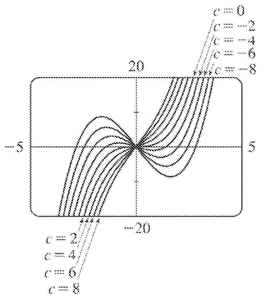
$c=-6$ $c=0$ $c=6$

x-intercepts: When $c\geq 0$, 0 is the only x-intercept. When $c<0$, the x-intercepts are 0 and $\pm\sqrt{-c}$. y-intercept $=f(0)=0$. f is odd, so the graph is symmetric with respect to the origin. $f''(x)<0$ for $x<0$ and $f''(x)>0$ for $x>0$, so f is CD on $(-\infty,0)$ and CU on $(0,\infty)$. The origin is the only inflection point.

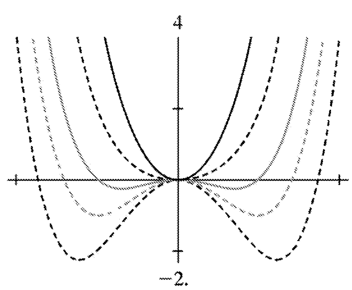
If $c>0$, then $f'(x)>0$ for all x, so f is increasing and has no local maximum or minimum.

If $c=0$, then $f'(x)\geq 0$ with equality at $x=0$, so again f is increasing and has no local maximum or minimum.

If $c<0$, then $f'(x)=3\left[x^2-(-c/3)\right]=3\left(x+\sqrt{-c/3}\right)\left(x-\sqrt{-c/3}\right)$, so $f'(x)>0$ on $\left(-\infty,-\sqrt{-c/3}\right)$ and $\left(\sqrt{-c/3},\infty\right)$; $f'(x)<0$ on $\left(-\sqrt{-c/3},\sqrt{-c/3}\right)$. It follows that $f\left(-\sqrt{-c/3}\right)=-\frac{2}{3}c\sqrt{-c/3}$ is a local maximum value and $f\left(\sqrt{-c/3}\right)=\frac{2}{3}c\sqrt{-c/3}$ is a local minimum value. As c decreases (toward more negative values), the local maximum and minimum move further apart. There is no absolute maximum or minimum value. The only transitional value of c corresponding to a change in character of the graph is $c=0$.



27. $f(x)=x^4+cx^2=x^2\left(x^2+c\right)$. Note that f is an even function. For $c\geq 0$, the only x –intercept is the point $(0,0)$. We calculate $f'(x)=4x^3+2cx=4x\left(x^2+\frac{1}{2}c\right)\Rightarrow f''(x)=12x^2+2c$. If $c\geq 0$, $x=0$ is the only critical point and there is no inflection point. As we can see from the examples, there is no change in the basic shape of the graph for $c\geq 0$; it merely becomes steeper as c increases. For $c=0$, the graph is the simple curve
$y=x^4$. For $c<0$, there are x– intercepts at 0 and at $\pm\sqrt{-c}$. Also, there is a maximum at $(0,0)$, and there are minima at $\left(\pm\sqrt{-\frac{1}{2}c},-\frac{1}{4}c^2\right)$. As $c\to-\infty$, the x– coordinates of these minima get larger in absolute value, and the minimum points move downward. There are inflection points at $\left(\pm\sqrt{-\frac{1}{6}c},-\frac{5}{36}c^2\right)$, which also move away from the origin as $c\to-\infty$.



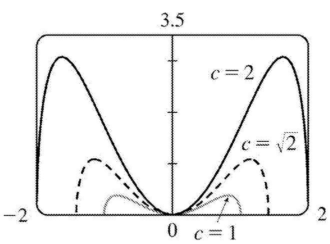
28. We need only consider the function $f(x)=x^2\sqrt{c^2-x^2}$ for $c\geq 0$, because if c is replaced by $-c$, the function is unchanged. For $c=0$, the graph consists of the single point $(0,0)$. The domain of f is $[-c,c]$, and the graph of f is symmetric about the y– axis.

$f'(x)=2x\sqrt{c^2-x^2}+x^2\dfrac{-2x}{2\sqrt{c^2-x^2}}=2x\sqrt{c^2-x^2}-\dfrac{x^3}{\sqrt{c^2-x^2}}=\dfrac{2x\left(c^2-x^2\right)-x^3}{\sqrt{c^2-x^2}}=-\dfrac{3x\left(x^2-\frac{2}{3}c^2\right)}{\sqrt{c^2-x^2}}$. So we see that all members of the family of curves have horizontal tangents at $x=0$, since $f'(0)=0$ for all $c>0$.
Also, the tangents to all the curves become very steep as $x\to\pm c$, since

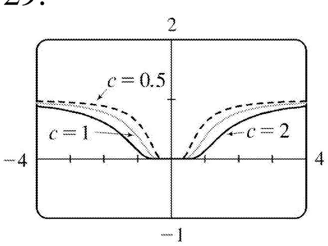
$\lim_{x\to -c^+} f'(x)=\infty$ and $\lim_{x\to c^-} f'(x)=-\infty$. We set $f'(x)=0 \Leftrightarrow x=0$ or $x^2-\frac{2}{3}c^2=0$, so the absolute maximum values are $f\left(\pm\sqrt{\frac{2}{3}}\,c\right)=\frac{2}{3\sqrt{3}}c^3$.

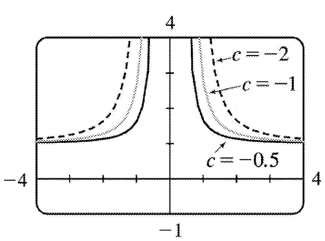
$$f''(x)=\frac{\left(-9x^2+2c^2\right)\sqrt{c^2-x^2}-\left(-3x^3+2c^2x\right)\left(-x/\sqrt{c^2-x^2}\right)}{c^2-x^2}=\frac{6x^4-9c^2x^2+2c^4}{\left(c^2-x^2\right)^{3/2}} .$$

Using the quadratic formula, we find that $f''(x)=0 \Leftrightarrow x^2=\frac{9c^2\pm c^2\sqrt{33}}{12}$. Since $-c<x<c$, we take $x^2=\frac{9-\sqrt{33}}{12}c^2$, so the inflection points are $\left(\pm\sqrt{\frac{9-\sqrt{33}}{12}}\,c, \frac{\left(9-\sqrt{33}\right)\left(\sqrt{33}-3\right)}{144}c^3\right)$.
From these calculations we can see that the maxima and the points of inflection get both horizontally and vertically further from the origin as c increases. Since all of the functions have two maxima and two inflection points, we see that the basic shape of the curve does not change as c changes.



29.

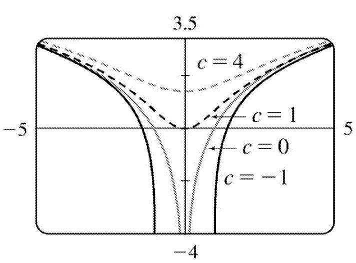




$c=0$ is a transitional value — we get the graph of $y=1$. For $c>0$, we see that there is a HA at $y=1$, and that the graph spreads out as c increases. At first glance there appears to be a minimum at $(0,0)$, but $f(0)$ is undefined, so there is no minimum or maximum. For $c<0$, we still have the HA at $y=1$,

but the range is $(1,\infty)$ rather than $(0,1)$. We also have a VA at $x=0$. $f(x)=e^{-c/x^2} \Rightarrow$ $f'(x)=e^{-c/x^2}\left(-2c/x^3\right) \Rightarrow f''(x)=\dfrac{2c\left(2c-3x^2\right)}{x^6 e^{c/x^2}}$. $f'(x)\neq 0$ and $f'(x)$ exists for all $x\neq 0$ (and 0 is not in the domain of f), so there are no maxima or minima. $f''(x)=0 \Rightarrow x=\pm\sqrt{2c/3}$, so if $c>0$, the inflection points spread out as c increases,
and if $c<0$, there are no IP. For $c>0$, there are IP at $\left(\pm\sqrt{2c/3}, e^{-3/2}\right)$. Note that the y –coordinate of the IP is constant.

30. We see that if $c\leq 0$, $f(x)=\ln\left(x^2+c\right)$ is only defined for $x^2>-c \Rightarrow |x|>\sqrt{-c}$, and $\lim\limits_{x\to\sqrt{-c}^{+}} f(x)=\lim\limits_{x\to-\sqrt{-c}^{-}} f(x)=-\infty$, since $\ln y\to-\infty$ as $y\to 0$. Thus, for $c<0$, there are vertical asymptotes at $x=\pm\sqrt{c}$, and as c decreases (that is, $|c|$ increases), the asymptotes get further apart. For $c=0$, $\lim\limits_{x\to 0} f(x)=-\infty$, so there is a vertical asymptote at $x=0$. If $c>0$, there are no asymptotes. To find the extrema
and inflection points, we differentiate: $f(x)=\ln\left(x^2+c\right) \Rightarrow f'(x)=\dfrac{1}{x^2+c}(2x)$, so by the First Derivative Test there is a local and absolute minimum at $x=0$. Differentiating again, we get $f''(x)=\dfrac{1}{x^2+c}(2)+2x\left[-\left(x^2+c\right)^{-2}(2x)\right]=\dfrac{2\left(c-x^2\right)}{\left(x^2+c\right)^2}$. Now if $c\leq 0$, f'' is always negative, so f is concave down on both of the intervals on which it is defined. If $c>0$, then f'' changes sign when $c=x^2 \Leftrightarrow x=\pm\sqrt{c}$. So for $c>0$ there are inflection points at $x=\pm\sqrt{c}$, and as c increases, the inflection points get further apart.



31. Note that $c=0$ is a transitional value at which the graph consists of the x– axis. Also, we can see that if we substitute $-c$ for c , the function $f(x)=\dfrac{cx}{1+c^2x^2}$ will be reflected in the x– axis, so we investigate only positive
values of c (except $c=-1$, as a demonstration of this reflective property). Also, f is an odd function.

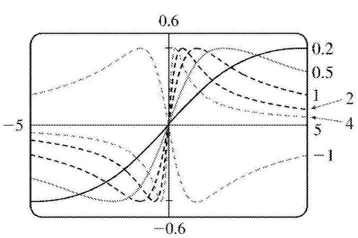
$\lim_{x\to\pm\infty} f(x)=0$, so $y=0$ is a horizontal asymptote for all c . We calculate

$f'(x)=\dfrac{\left(1+c^2x^2\right)c-cx\left(2c^2x\right)}{\left(1+c^2x^2\right)^2}=-\dfrac{c\left(c^2x^2-1\right)}{\left(1+c^2x^2\right)^2}$. $f'(x)=0\Leftrightarrow c^2x^2-1=0\Leftrightarrow x=\pm 1/c$. So there is an absolute maximum value of $f(1/c)=\dfrac{1}{2}$ and an absolute minimum value of $f(-1/c)=-\dfrac{1}{2}$. These extrema have the same value regardless of c , but the maximum points move closer to the $y-$ axis as c increases.

$$f''(x)=\frac{\left(-2c^3x\right)\left(1+c^2x^2\right)^2-\left(-c^3x^2+c\right)}{\left(1+c^2x^2\right)^4}$$

$$=\frac{\left(-2c^3x\right)\left(1+c^2x^2\right)+\left(c^3x^2-c\right)\left(4c^2x\right)}{\left(1+c^2x^2\right)^3}=\frac{2c^3x\left(c^2x^2-3\right)}{\left(1+c^2x^2\right)^3}$$

$f''(x)=0\Leftrightarrow x=0$ or $\pm\sqrt{3}/c$, so there are inflection points at $(0,0)$ and at $\left(\pm\sqrt{3}/\text{c},\pm\sqrt{3}/4\right)$.

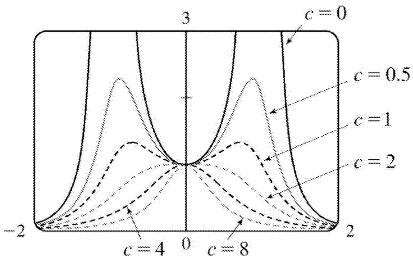


Again, the $y-$ coordinate of the inflection points does not depend on c , but as c increases, both inflection points approach the $y-$ axis.

32. Note that $f(x)=\dfrac{1}{\left(1-x^2\right)^2+cx^2}$ is an even function, and also that $\lim_{x\to\pm\infty} f(x)=0$ for any value of c , so $y=0$ is a horizontal asymptote. We calculate the derivatives:

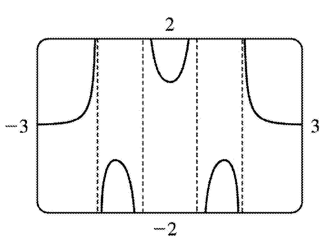
$f'(x)=\dfrac{-4\left(1-x^2\right)x+2cx}{\left[\left(1-x^2\right)^2+cx^2\right]^2}=\dfrac{4x\left[x^2+\left(\frac{1}{2}c-1\right)\right]}{\left[\left(1-x^2\right)^2+cx^2\right]^2}$, and

$f''(x)=2\dfrac{10x^6+(9c-18)x^4+\left(3c^2-12c+6\right)x^2+2-c}{\left[x^4+(c-2)x^2+1\right]^3}$.

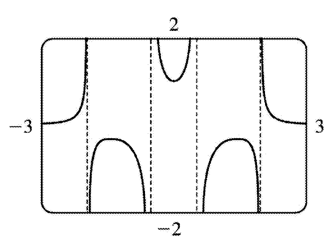


We first consider the case $c>0$. Then the denominator of $f^{\prime}$ is positive, that is, $\left(1-x^2\right)^2+cx^2>0$ for all x , so f has domain R and also $f>0$. If $\frac{1}{2}c-1\geq 0$; that is, $c\geq 2$, then the only critical point is $f(0)=1$, a maximum. Graphing a few examples for $c\geq 2$ shows that there are two IP which approach the y– axis as $c\rightarrow\infty$. $c=2$ and $c=0$ are transitional values of c at which the shape of the curve changes. For $0<c<2$, there are three critical points: $f(0)=1$, a minimum value, and $f\left(\pm\sqrt{1-\frac{1}{2}c}\right)=\frac{1}{c(1-c/4)}$, both maximum values. As c decreases from 2 to 0 , the maximum values get larger and larger, and the x– values at which they occur go from 0 to ± 1 . Graphs show that there are four inflection points for $0<c<2$, and that they get farther away from the origin, both vertically and horizontally, as $c\rightarrow 0^+$. For $c=0$, the function is simply asymptotic to the x– axis and to the lines $x=\pm 1$, approaching $+\infty$ from both sides of each. The y –intercept is 1 , and $(0,1)$ is a local minimum. There are no inflection points. Now if $c<0$, we can write

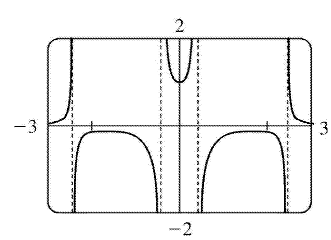
$f(x)=\frac{1}{\left(1-x^2\right)^2+cx^2}=\frac{1}{\left(1-x^2\right)^2-\left(\sqrt{-c}\,x\right)^2}=\frac{1}{\left(x^2-\sqrt{-c}\,x-1\right)\left(x^2+\sqrt{-c}\,x-1\right)}$. So f has vertical asymptotes where $x^2\pm\sqrt{-c}\,x-1=0\Leftrightarrow x=\left|\left(-\sqrt{-c}\pm\sqrt{4-c}\right)\right/2$ or $x=\left|\left(\sqrt{-c}\pm\sqrt{4-c}\right)\right/2$. As c decreases, the two exterior asymptotes move away from the origin, while the two interior ones move toward it. We graph a few examples to see the behavior of the graph near the asymptotes, and the nature of the critical points $x=0$ and $x=\pm\sqrt{1-\frac{1}{2}c}$:



$c=-1$



$c=-1$

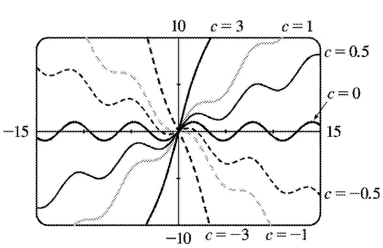


$c=-1$

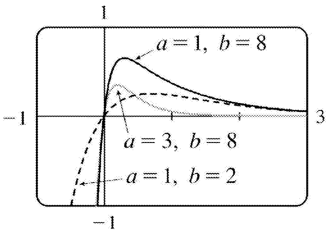
We see that there is one local minimum value, $f(0)=1$, and there are two local maximum values,

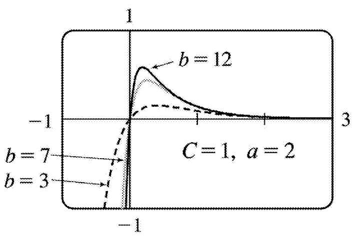
$f\left(\pm\sqrt{1-\frac{1}{2}c}\right)=\frac{1}{c(1-c/4)}$ as before. As c decreases, the $x-$ values at which these maxima occur get larger, and the maximum values themselves approach 0 , though they are always negative.

33. $f(x)=cx+\sin x \Rightarrow f'(x)=c+\cos x \Rightarrow f''(x)=-\sin x$
$f(-x)=-f(x)$, so f is an odd function and its graph is symmetric with respect to the origin.
$f(x)=0 \Leftrightarrow \sin x=-cx$, so 0 is always an x –intercept.
$f'(x)=0 \Leftrightarrow \cos x=-c$, so there is no critical number when $|c|>1$. If $|c|\leq 1$, then there are infinitely many critical numbers. If x_1 is the unique solution of $\cos x=-c$ in the interval $[0,\pi]$, then the critical numbers are $2n\pi\pm x_1$, where n ranges over the integers. (Special cases: When $c=1$, $x_1=0$; when $c=0$, $x=\frac{\pi}{2}$; and when $c=-1$, $x_1=\pi$.)
$f''(x)<0 \Leftrightarrow \sin x>0$, so f is CD on intervals of the form $(2n\pi,(2n+1)\pi)$. f is CU on intervals of the form $((2n-1)\pi,2n\pi)$. The inflection points of f are the points $(2n\pi,2n\pi c)$, where n is an integer.
If $c\geq 1$, then $f'(x)\geq 0$ for all x , so f is increasing and has no extremum. If $c\leq -1$, then $f'(x)\leq 0$ for all x , so f is decreasing and has no extremum. If $|c|<1$, then $f'(x)>0 \Leftrightarrow \cos x>-c \Leftrightarrow x$ is in an interval of the form $\left(2n\pi-x_1,2n\pi+x_1\right)$ for some integer n . These are the intervals on which f is increasing. Similarly, we
find that f is decreasing on the intervals of the form $\left(2n\pi+x_1,2(n+1)\pi-x_1\right)$. Thus, f has local maxima at the points $2n\pi+x_1$, where f has the values $c(2n\pi+x_1)+\sin x_1=c(2n\pi+x_1)+\sqrt{1-c^2}$, and f has local minima at the points $2n\pi-x_1$, where we have
$f(2n\pi-x_1)=c(2n\pi-x_1)-\sin x_1=c(2n\pi-x_1)-\sqrt{1-c^2}$ .The transitional values of c are -1 and 1 . The inflection points move vertically, but not horizontally, when c changes. When $|c|\geq 1$, there is no extremum. For $|c|<1$, the maxima are spaced 2π apart horizontally, as are the minima. The horizontal spacing between maxima and adjacent minima is regular (and equals π) when $c=0$, but the horizontal space between a local maximum and the nearest local minimum shrinks as $|c|$ approaches 1 .



34. For $f(t)=C\left(e^{-at}-e^{-bt}\right)$, C affects only vertical stretching, so we let $C=1$. From the first figure, we notice that the graphs all pass through the origin, approach the t –axis as t increases, and approach $-\infty$ as $t\to-\infty$. Next we let $a=2$ and produce the second figure.

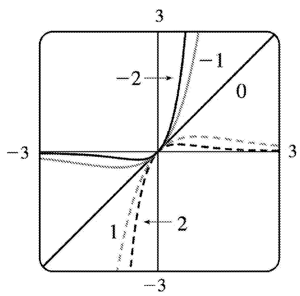




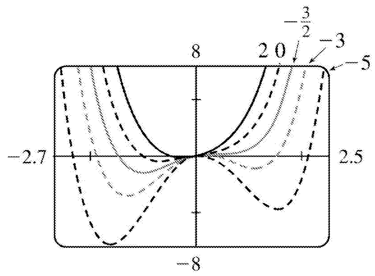
Here, as b increases, the slope of the tangent at the origin increases and the local maximum value increases. $f(t)=e^{-2t}-e^{-bt}\Rightarrow f'(t)=be^{-bt}-2e^{-2t}$. $f'(0)=b-2$, which increases as b increases.
$f'(t)=0\Rightarrow be^{-bt}=2e^{-2t}\Rightarrow \frac{b}{2}=e^{(b-2)t}\Rightarrow \ln\frac{b}{2}=(b-2)t\Rightarrow t=t_1=\frac{\ln b-\ln 2}{b-2}$, which decreases as b increases (the maximum is getting closer to the y –axis). $f(t_1)=\frac{(b-2)2^{2/(b-2)}}{b^{1+2/(b-2)}}$. We can show that this value increases as b increases by considering it to be a function of b and graphing its derivative with respect to b , which is always positive.

35. If $c<0$, then $\lim_{x\to-\infty} f(x)=\lim_{x\to-\infty}\frac{x}{e^{cx}}=\lim_{x\to-\infty}\frac{1}{ce^{cx}}=0$, and $\lim_{x\to\infty} f(x)=\infty$.
If $c>0$, then $\lim_{x\to-\infty} f(x)=-\infty$, and $\lim_{x\to\infty} f(x)=\lim_{x\to\infty}\frac{1}{ce^{cx}}=0$.
If $c=0$, then $f(x)=x$, so $\lim_{x\to\pm\infty} f(x)=\pm\infty$ respectively.
So we see that $c=0$ is a transitional value. We now exclude the case $c=0$, since we know how the function behaves in that case. To find the maxima and minima of f , we differentiate: $f(x)=xe^{-cx}\Rightarrow$ $f'(x)=x\left(-ce^{-cx}\right)+e^{-cx}=(1-cx)\,e^{-cx}$. This is 0 when $1-cx=0\Leftrightarrow x=1/c$. If $c<0$ then this represents a minimum value of $f(1/c)=1/(ce)$, since $f'(x)$ changes from negative to positive at $x=1/c$;

and if $c>0$, it represents a maximum value. As $|c|$ increases, the maximum or minimum point gets closer to the origin. To find the inflection points, we differentiate again: $f'(x)=e^{-cx}(1-cx) \Rightarrow$ $f''(x)=e^{-cx}(-c)+(1-cx)\left(-ce^{-cx}\right)=(cx-2)ce^{-cx}$. This changes sign when $cx-2=0 \Leftrightarrow x=2/c$. So as $|c|$ increases, the points of inflection get closer to the origin.

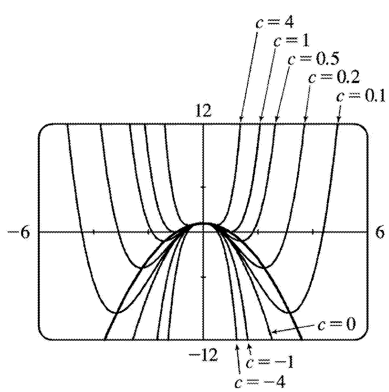


36. For $c=0$, there is no inflection point; the curve is CU everywhere. If c increases, the curve simply becomes steeper, and there are still no inflection points. If c starts at 0 and decreases, a slight upward bulge appears near $x=0$, so that there are two inflection points for any $c<0$. This can be seen algebraically by calculating the second derivative: $f(x)=x^4+cx^2+x \Rightarrow f'(x)=4x^3+2cx+1 \Rightarrow$ $f''(x)=12x^2+2c$. Thus, $f''(x)>0$ when $c>0$. For $c<0$, there are inflection points when $x=\pm\sqrt{-\frac{1}{6}c}$. For $c=0$, the graph has one critical number, at the absolute minimum somewhere around $x=-0.6$. As c increases, the number of critical points does not change. If c instead decreases from 0 , we see that the graph eventually sprouts another local minimum, to the right of the origin, somewhere between $x=1$ and $x=2$. Consequently, there is also a maximum near $x=0$.
After a bit of experimentation, we find that at $c=-1.5$, there appear to be two critical numbers: the absolute minimum at about $x=-1$, and a horizontal tangent with no extremum at about $x=0.5$. For any c smaller than this there will be 3 critical points, as shown in the graphs with $c=-3$ and with $c=-5$. To prove this algebraically, we calculate $f'(x)=4x^3+2cx+1$. Now if we substitute our value of $c=-1.5$, the formula for $f'(x)$ becomes $4x^3-3x+1=(x+1)(2x-1)^2$. This has a double root at $x=\frac{1}{2}$, indicating that the function has two critical points: $x=-1$ and $x=\frac{1}{2}$, just as we had guessed from the graph.



37. **(a)** $f(x)=cx^4-2x^2+1$. For $c=0$, $f(x)=-2x^2+1$, a parabola whose vertex, $(0,1)$, is the absolute maximum. For $c>0$, $f(x)=cx^4-2x^2+1$ opens upward with two minimum points. As $c\to 0$, the minimum points spread apart and move downward; they are below the x –axis for $0<c<1$ and above for $c>1$. For $c<0$, the graph opens downward, and has an absolute maximum at $x=0$ and no local minimum.

(b) $f'(x)=4cx^3-4x=4cx\left(x^2-1/c\right)$ ($c\neq 0$). If $c\leq 0$, 0 is the only critical number. $f''(x)=12cx^2-4$, so $f''(0)=-4$ and there is a local maximum at $(0,f(0))=(0,1)$, which lies on $y=1-x^2$. If $c>0$, the critical numbers are 0 and $\pm 1/\sqrt{c}$. As before, there is a local maximum at $(0,f(0))=(0,1)$, which lies on $y=1-x^2$. $f''(\pm 1/\sqrt{c})=12-4=8>0$, so there is a local minimum at $x=\pm 1/\sqrt{c}$. Here $f(\pm 1/\sqrt{c})=c\left(1/c^2\right)-2/c+1=-1/c+1$. But $\left(\pm 1/\sqrt{c},-1/c+1\right)$ lies on $y=1-x^2$ since $1-(\pm 1/\sqrt{c})^2=1-1/c$.



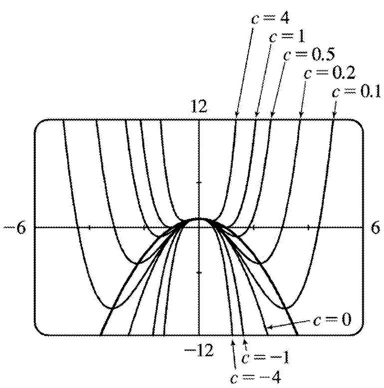
38. **(a)** $f(x)=2x^3+cx^2+2x\Rightarrow f'(x)=6x^2+2cx+2=2\left(3x^2+cx+1\right)$. $f'(x)=0\Leftrightarrow x=\dfrac{-c\pm\sqrt{c^2-12}}{6}$. So f has critical points $\Leftrightarrow c^2-12\geq 0\Leftrightarrow |c|\geq 2\sqrt{3}$. For $c=\pm 2\sqrt{3}$, $f'(x)\geq 0$ on $(-\infty,\infty)$, so f' does not change signs at $-c/6$, and there is no extremum. If $c^2-12>0$, then f' changes from positive to negative at $x=\dfrac{-c-\sqrt{c^2-12}}{6}$ and from negative to positive at $x=\dfrac{-c+\sqrt{c^2-12}}{6}$. So f has a local maximum at

$x=\dfrac{-c-\sqrt{c^2-12}}{6}$ and a local minimum at $x=\dfrac{-c+\sqrt{c^2-12}}{6}$.

(b) Let x_0 be a critical number for $f(x)$. Then $f'(x_0)=0 \Rightarrow 3x_0^2+cx_0+1=0 \Leftrightarrow c=\dfrac{-1-3x_0^2}{x_0}$. Now

$$f(x_0) = 2x_0^3+cx_0^2+2x_0=2x_0^3+x_0^2\left(\frac{-1-3x_0^2}{x_0}\right)+2x_0 = 2x_0^3-x_0-3x_0^3+2x_0=x_0-x_0^3$$

So the point is $\left(x_0,y_0\right)=\left(x_0,x_0-x_0^3\right)$; that is, the point lies on the curve $y=x-x^3$.



1. **(a)**

First Number	Second Number	Product
1	22	22
2	21	42
3	20	60
4	19	76
5	18	90
6	17	102
7	16	112
8	15	120
9	14	126
10	13	130
11	12	132

We needn't consider pairs where the first number is larger than the second, since we can just interchange the numbers in such cases. The answer appears to be 11 and 12 , but we have considered only integers in the table.

(b) Call the two numbers x and y . Then $x+y=23$, so $y=23-x$. Call the product P . Then $P=xy=x(23-x)=23x-x^2$, so we wish to maximize the function $P(x)=23x-x^2$. Since $P'(x)=23-2x$, we see that $P'(x)=0\Leftrightarrow x=\frac{23}{2}=11.5$. Thus, the maximum value of P is $P(11.5)=(11.5)^2=132.25$ and it occurs when $x=y=11.5$.

Or: Note that $P''(x)=-2<0$ for all x , so P is everywhere concave downward and the local maximum at $x=11.5$ must be an absolute maximum.

2. The two numbers are $x+100$ and x . Minimize $f(x)=(x+100)x=x^2+100x$. $f'(x)=2x+100=0\Rightarrow x=-50$. Since $f''(x)=2>0$, there is an absolute minimum at $x=-50$. The two numbers are 50 and -50 .

3. The two numbers are x and $\frac{100}{x}$, where $x>0$. Minimize $f(x)=x+\frac{100}{x}$. $f'(x)=1-\frac{100}{x^2}=\frac{x^2-100}{x^2}$.

The critical number is $x=10$. Since $f'(x)<0$ for $0<x<10$ and $f'(x)>0$ for $x>10$, there is an absolute minimum at $x=10$. The numbers are 10 and 10 .

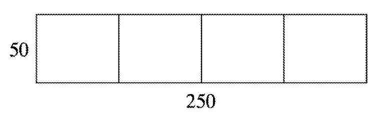
4. Let $x>0$ and let $f(x)=x+1/x$. We wish to minimize $f(x)$. Now $f'(x)=1-\frac{1}{x^2}=\frac{1}{x^2}\left(x^2-1\right)=\frac{1}{x^2}(x+1)(x-1)$, so the only critical number in $(0,\infty)$ is 1 .

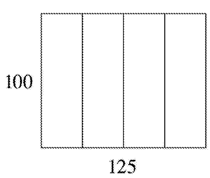
$f'(x)<0$ for $0<x<1$ and $f'(x)>0$ for $x>1$, so f has an absolute minimum at $x=1$, and $f(1)=2$.
Or: $f''(x)=2/x^3>0$ for all $x>0$, so f is concave upward everywhere and the critical point $(1,2)$ must correspond to a local minimum for f .

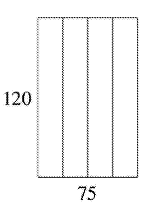
5. If the rectangle has dimensions x and y , then its perimeter is $2x+2y=100$ m, so $y=50-x$. Thus, the area is $A=xy=x(50-x)$. We wish to maximize the function $A(x)=x(50-x)=50x-x^2$, where $0<x<50$. Since $A'(x)=50-2x=-2(x-25)$, $A'(x)>0$ for $0<x<25$ and $A'(x)<0$ for $25<x<50$. Thus, A has an absolute maximum at $x=25$, and $A(25)=25^2=625$ m^2 . The dimensions of the rectangle that maximize its area are $x=y=25$ m. (The rectangle is a square.)

6. If the rectangle has dimensions x and y , then its area is $xy=1000$ m^2 , so $y=1000/x$. The perimeter $P=2x+2y=2x+2000/x$. We wish to minimize the function $P(x)=2x+2000/x$ for $x>0$.
$P'(x)=2-2000/x^2=\left(2/x^2\right)\left(x^2-1000\right)$, so the only critical number in the domain of P is $x=\sqrt{1000}$.
$P''(x)=4000/x^3>0$, so P is concave upward throughout its domain and $P\left(\sqrt{1000}\right)=4\sqrt{1000}$ is an absolute minimum value. The dimensions of the rectangle with minimal perimeter are
$x=y=\sqrt{1000}=10\sqrt{10}$ m.
(The rectangle is a square.)

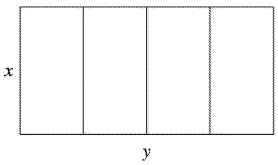
7. **(a)**







The areas of the three figures are 12 , 500 , 12 , 500 , and 9000 ft^2 . There appears to be a maximum area of at least 12 , 500 ft^2 .
(b) Let x denote the length of each of two sides and three dividers.
Let y denote the length of the other two sides.

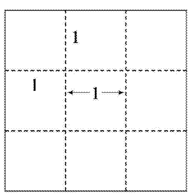


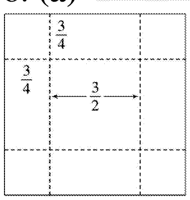
(c) Area A= length × width $=y\cdot x$

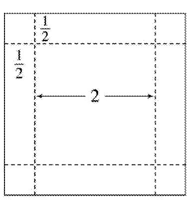
(d) Length of fencing $=750 \Rightarrow 5x+2y=750$

(e) $5x+2y=750 \Rightarrow y=375-\frac{5}{2}x \Rightarrow A(x)=\left(375-\frac{5}{2}x\right)x=375x-\frac{5}{2}x^2$

(f) $A'(x)=375-5x=0 \Rightarrow x=75$. Since $A''(x)=-5<0$ there is an absolute maximum when $x=75$. Then $y=\frac{375}{2}=187.5$. The largest area is $75\left(\frac{375}{2}\right)=14,062.5$ ft^2 . These values of x and y are between the values in the first and second figures in part (a). Our original estimate was low.

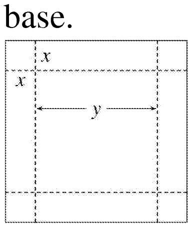
8. **(a)** 





The volumes of the resulting boxes are 1 , 1.6875 , and 2 ft^3 . There appears to be a maximum volume of at least 2 ft^3 .

(b) Let x denote the length of the side of the square being cut out. Let y denote the length of the base.



(c) Volume V= length × width × height $\Rightarrow V=y\cdot y\cdot x=xy^2$

(d) Length of cardboard $=3 \Rightarrow x+y+x=3 \Rightarrow y+2x=3$

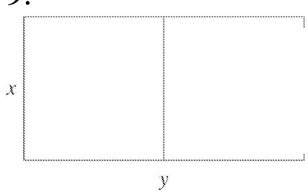
(e) $y+2x=3 \Rightarrow y=3-2x \Rightarrow V(x)=x(3-2x)^2$

(f) $V(x)=x(3-2x)^2 \Rightarrow$

$V'(x)=x\cdot 2(3-2x)(-2)+(3-2x)^2\cdot 1=(3-2x)[-4x+(3-2x)]=(3-2x)(-6x+3)$,

so the critical numbers are $x=\frac{3}{2}$ and $x=\frac{1}{2}$. Now $0\le x\le \frac{3}{2}$ and $V(0)=V\left(\frac{3}{2}\right)=0$, so the maximum is $V\left(\frac{1}{2}\right)=\left(\frac{1}{2}\right)(2)^2=2\text{ ft}^3$, which is the value found from our third figure in part (a).

9.

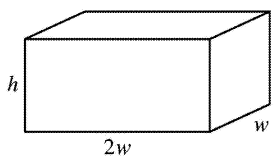


$xy=1.5\times 10^6$, so $y=1.5\times 10^6/x$. Minimize the amount of fencing, which is $3x+2y=3x+2\left(1.5\times 10^6/x\right)=3x+3\times 10^6/x=F(x)$. $F'(x)=3-3\times 10^6/x^2=3\left(x^2-10^6\right)/x^2$. The critical number is $x=10^3$ and $F'(x)<0$ for $0<x<10^3$ and $F'(x)>0$ if $x>10^3$, so the absolute minimum occurs when $x=10^3$ and $y=1.5\times 10^3$. The field should be 1000 feet by 1500 feet with the middle fence parallel to the short side of the field.

10. Let b be the length of the base of the box and h the height. The volume is $32,000=b^2h \Rightarrow h=32,000/b^2$. The surface area of the open box is $S=b^2+4hb=b^2+4(32,000/b^2)b=b^2+4\,(32{,}000)\,/b$. So $S'(b)=2b-4(32,000)/b^2=2\left(b^3-64{,}000\right)/b^2=0 \Leftrightarrow b=\sqrt[3]{64{,}000}=40$. This gives an absolute minimum since $S'(b)<0$ if $0<b<40$ and $S'(b)>0$ if $b>40$. The box should be $40\times 40\times 20$.

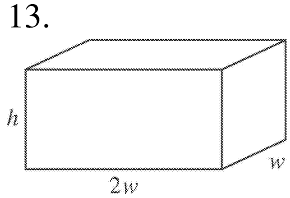
11. Let b be the length of the base of the box and h the height. The surface area is $1200=b^2+4hb \Rightarrow h=\left(1200-b^2\right)/(4b)$. The volume is $V=b^2h=b^2\left(1200-b^2\right)/4b=300b-b^3/4 \Rightarrow V'(b)=300-\frac{3}{4}b^2$. $V'(b)=0 \Rightarrow 300=\frac{3}{4}b^2 \Rightarrow b^2=400 \Rightarrow b=\sqrt{400}=20$. Since $V'(b)>0$ for $0<b<20$ and $V'(b)<0$ for $b>20$, there is an absolute maximum when $b=20$ by the First Derivative Test for Absolute Extreme Values (see page 280). If $b=20$, then $h=\left(1200-20^2\right)/(4\cdot 20)=10$, so the largest possible volume is $b^2h=(20)^2(10)=4000\text{ cm}^3$.

12.



$V=lwh \Rightarrow 10=(2w)(w)h=2w^2h$, so $h=5/w^2$. The cost is $10\left(2w^2\right)+6\left[2(2wh)+2(hw)\right]=20w^2+36wh$, so $C(w)=20w^2+36w\left(5/w^2\right)=20w^2+180/w$. $C'(w)=40w-180/w^2=40\left(w^3-\frac{9}{2}\right)/w^2 \Rightarrow w=\sqrt[3]{\frac{9}{2}}$ is the critical number. There is an absolute minimum for C when $w=\sqrt[3]{\frac{9}{2}}$ since $C'(w)<0$ for $0<w<\sqrt[3]{\frac{9}{2}}$ and $C'(w)>0$ for $w>\sqrt[3]{\frac{9}{2}}$. $C\left(\sqrt[3]{\frac{9}{2}}\right)=20\left(\sqrt[3]{\frac{9}{2}}\right)^2+\frac{180}{\sqrt[3]{9/2}}\approx\$163.54$.

13.



$10=(2w)(w)h=2w^2h$, so $h=5/w^2$. The cost is $C(w)=10\left(2w^2\right)+6\left[2(2wh)+2hw\right]+6\left(2w^2\right)=$ $32w^2+36wh=32w^2+180/w$ $C'(w)=64w-180/w^2=4\left(16w^3-45\right)/w^2 \Rightarrow w=\sqrt[3]{\frac{45}{16}}$ is the critical number. $C'(w)<0$ for $0<w<\sqrt[3]{\frac{45}{16}}$ and $C'(w)>0$ for $w>\sqrt[3]{\frac{45}{16}}$. The minimum cost is $C\left(\sqrt[3]{\frac{45}{16}}\right)=32(2.8125)^{2/3}+180/\sqrt{2.8125}\approx\$191.28$.

14. **(a)** Let the rectangle have sides x and y and area A , so $A=xy$ or $y=A/x$. The problem is to minimize the perimeter $=2x+2y=2x+2A/x=P(x)$. Now $P'(x)=2-2A/x^2=2\left(x^2-A\right)/x^2$. So the critical number is $x=\sqrt{A}$. Since $P'(x)<0$ for $0<x<\sqrt{A}$ and $P'(x)>0$ for $x>\sqrt{A}$, there is an absolute minimum at $x=\sqrt{A}$. The sides of the rectangle are $\sqrt{A}$ and $A/\sqrt{A}=\sqrt{A}$, so the rectangle is a square.

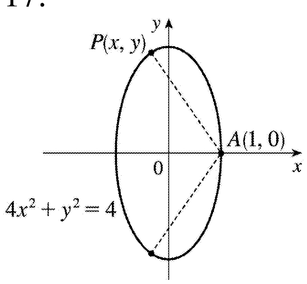
(b) Let p be the perimeter and x and y the lengths of the sides, so $p=2x+2y\Rightarrow 2y=p-2x\Rightarrow y=\frac{1}{2}p-x$. The area is $A(x)=x\left(\frac{1}{2}p-x\right)=\frac{1}{2}px-x^2$. Now $A'(x)=0\Rightarrow \frac{1}{2}p-2x=0\Rightarrow 2x=\frac{1}{2}p\Rightarrow x=\frac{1}{4}p$. Since $A''(x)=-2<0$, there is an absolute maximum for A when $x=\frac{1}{4}p$ by the Second Derivative Test. The sides of the rectangle are

$\frac{1}{4}p$ and $\frac{1}{2}p-\frac{1}{4}p=\frac{1}{4}p$, so the rectangle is a square.

15. The distance from a point (x,y) on the line $y=4x+7$ to the origin is $\sqrt{(x-0)^2+(y-0)^2}=\sqrt{x^2+y^2}$. However, it is easier to work with the *square* of the distance; that is, $D(x)=\left(\sqrt{x^2+y^2}\right)^2=x^2+y^2=x^2+(4x+7)^2$. Because the distance is positive, its minimum value will occur at the same point as the minimum value of D .
$D'(x)=2x+2(4x+7)(4)=34x+56$, so $D'(x)=0\Leftrightarrow x=-\frac{28}{17}$.
$D''(x)=34>0$, so D is concave upward for all x . Thus, D has an absolute minimum at $x=-\frac{28}{17}$. The point closest to the origin is $(x,y)=\left(-\frac{28}{17},4\left(-\frac{28}{17}\right)+7\right)=\left(-\frac{28}{17},\frac{7}{17}\right)$.

16. The square of the distance from a point (x,y) on the line $y=-6x+9$ to the point $(-3,1)$ is $D(x)=(x+3)^2+(y-1)^2=(x+3)^2+(-6x+8)^2=37x^2-90x+73$. $D'(x)=74x-90$, so $D'(x)=0\Leftrightarrow x=\frac{45}{37}$.
$D''(x)=74>0$, so D is concave upward for all x . Thus, D has an absolute minimum at $x=\frac{45}{37}$. The point on the line closest to $(-3,1)$ is $\left(\frac{45}{37},\frac{63}{37}\right)$.

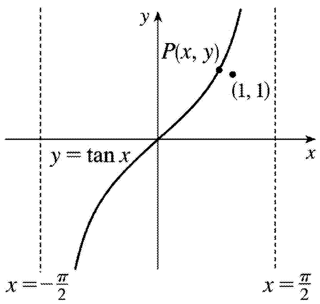
17.



From the figure, we see that there are two points that are farthest away from $A(1,0)$. The distance d from A to an arbitrary point $P(x,y)$ on the ellipse is $d=\sqrt{(x-1)^2+(y-0)^2}$ and the square of the distance is $S=d^2=x^2-2x+1+y^2=x^2-2x+1+\left(4-4x^2\right)=-3x^2-2x+5$. $S'=-6x-2$ and $S'=0\Rightarrow x=-\frac{1}{3}$. Now $S''=-6<0$, so we know that S has a maximum at $x=-\frac{1}{3}$. Since $-1\le x\le 1$, $S(-1)=4$, $S\left(-\frac{1}{3}\right)=\frac{16}{3}$, and $S(1)=0$, we see that the maximum distance is $\sqrt{\frac{16}{3}}$. The corresponding y –values are

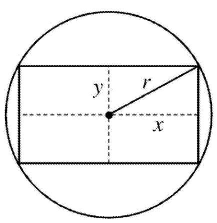
$y=\pm\sqrt{4-4\left(-\frac{1}{3}\right)^2}=\pm\sqrt{\frac{32}{9}}=\pm\frac{4}{3}\sqrt{2}\approx\pm1.89$. The points are $\left(-\frac{1}{3},\pm\frac{4}{3}\sqrt{2}\right)$.

18.



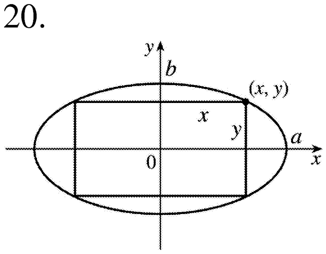
The distance d from $(1,1)$ to an arbitrary point P(x,y) on the curve $y=\tan x$ is $d=\sqrt{(x-1)^2+(y-1)^2}$ and the square of the distance is $S=d^2=(x-1)^2+(\tan x-1)^2$. $S'=2(x-1)+2(\tan x-1)\sec^2 x$. Graphing S' on $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ gives us a zero at $x\approx0.82$, and so $\tan x\approx1.08$. The point on $y=\tan x$ that is closest to $(1,1)$ is approximately $(0.82,1.08)$.

19.



The area of the rectangle is $(2x)(2y)=4xy$. Also $r^2=x^2+y^2$ so $y=\sqrt{r^2-x^2}$, so the area is $A(x)=4x\sqrt{r^2-x^2}$. Now $A'(x)=4\left(\sqrt{r^2-x^2}-\frac{x^2}{\sqrt{r^2-x^2}}\right)=4\frac{r^2-2x^2}{\sqrt{r^2-x^2}}$. The critical number is $x=\frac{1}{\sqrt{2}}r$. Clearly this gives a maximum. $y=\sqrt{r^2-\left(\frac{1}{\sqrt{2}}r\right)^2}=\sqrt{\frac{1}{2}r^2}=\frac{1}{\sqrt{2}}r=x$, which tells us that the rectangle is a square. The dimensions are $2x=\sqrt{2}r$ and $2y=\sqrt{2}r$.

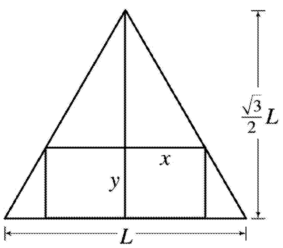
20.



The area of the rectangle is $(2x)(2y)=4xy$. Now

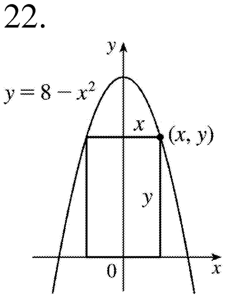
$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ gives $y=\frac{b}{a}\sqrt{a^2-x^2}$, so we maximize $A(x)=4\frac{b}{a}x\sqrt{a^2-x^2}$ $A'(x)=$
$\frac{4b}{a}\left[x\cdot\frac{1}{2}\left(a^2-x^2\right)^{-1/2}(-2x)+\left(a^2-x^2\right)^{1/2}\cdot 1\right]=\frac{4b}{a}\left(a^2-x^2\right)^{-1/2}\left[-x^2+a^2-x^2\right]=\frac{4b}{a\sqrt{a^2-x^2}}\left[a^2-2x^2\right]$ So the critical number is $x=\frac{1}{\sqrt{2}}a$, and this clearly gives a maximum. Then $y=\frac{1}{\sqrt{2}}b$, so the maximum area is $4\left(\frac{1}{\sqrt{2}}a\right)\left(\frac{1}{\sqrt{2}}b\right)=2ab$.

21.



The height h of the equilateral triangle with sides of length L is $\frac{\sqrt{3}}{2}L$, since $h^2+(L/2)^2=L^2\Rightarrow$ $h^2=L^2-\frac{1}{4}L^2=\frac{3}{4}L^2\Rightarrow h=\frac{\sqrt{3}}{2}L$. Using similar triangles, $\frac{\frac{\sqrt{3}}{2}L-y}{x}=\frac{\frac{\sqrt{3}}{2}L}{L/2}=\sqrt{3}\Rightarrow\sqrt{3}x=\frac{\sqrt{3}}{2}L-y\Rightarrow$ $y=\frac{\sqrt{3}}{2}L-\sqrt{3}x\Rightarrow y=\frac{\sqrt{3}}{2}(L-2x)$. The area of the inscribed rectangle is $A(x)=(2x)y=\sqrt{3}x(L-2x)=\sqrt{3}Lx-2\sqrt{3}x^2$, where $0\le x\le L/2$. Now $0=A'(x)=\sqrt{3}L-4\sqrt{3}x\Rightarrow$ $x=\sqrt{3}L\big/\left(4\sqrt{3}\right)=L/4$. Since $A(0)=A(L/2)=0$, the maximum occurs when $x=L/4$, and $y=\frac{\sqrt{3}}{2}L-\frac{\sqrt{3}}{4}L=\frac{\sqrt{3}}{4}L$, so the dimensions are $L/2$ and $\frac{\sqrt{3}}{4}L$.

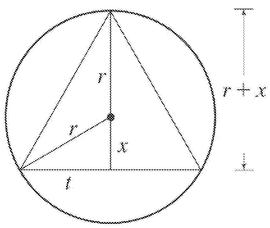
22.



The rectangle has area $A(x)=2xy=2x\left(8-x^2\right)=16x-2x^3$, where $0\le x\le 2\sqrt{2}$. Now $A'(x)=16-6x^2=0\Rightarrow$

$x=2\sqrt{\frac{2}{3}}$. Since $A(0)=A(2\sqrt{2})=0$, there is a maximum when $x=2\sqrt{\frac{2}{3}}$. Then $y=\frac{16}{3}$, so the rectangle has dimensions $4\sqrt{\frac{2}{3}}$ and $\frac{16}{3}$.

23.

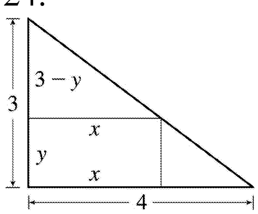


The area of the triangle is $A(x)=\frac{1}{2}(2t)(r+x)=t(r+x)=\sqrt{r^2-x^2}(r+x)$. 0=

$A'(x)=r\frac{-2x}{2\sqrt{r^2-x^2}}+\sqrt{r^2-x^2}+x\frac{-2x}{2\sqrt{r^2-x^2}}=-\frac{x^2+rx}{\sqrt{r^2-x^2}}+\sqrt{r^2-x^2}\Rightarrow\frac{x^2+rx}{\sqrt{r^2-x^2}}=\sqrt{r^2-x^2}\Rightarrow x^2+rx=r^2-x^2$

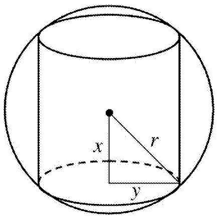
$\Rightarrow 0=2x^2+rx-r^2=(2x-r)(x+r)\Rightarrow x=\frac{1}{2}r$ or $x=-r$. Now $A(r)=0=A(-r)\Rightarrow$ the maximum occurs where $x=\frac{1}{2}r$, so the triangle has height $r+\frac{1}{2}r=\frac{3}{2}r$ and base $2\sqrt{r^2-\left(\frac{1}{2}r\right)^2}=2\sqrt{\frac{3}{4}r^2}=\sqrt{3}r$.

24.



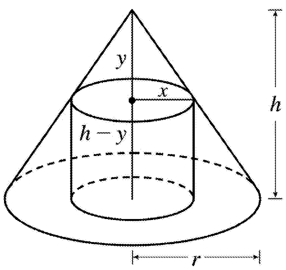
The rectangle has area xy . By similar triangles $\frac{3-y}{x}=\frac{3}{4}\Rightarrow -4y+12=3x$ or $y=-\frac{3}{4}x+3$. So the area is $A(x)=x\left(-\frac{3}{4}x+3\right)=-\frac{3}{4}x^2+3x$ where $0\leq x\leq 4$. Now $0=A'(x)=-\frac{3}{2}x+3\Rightarrow x=2$ and $y=\frac{3}{2}$. Since $A(0)=A(4)=0$, the maximum area is $A(2)=2\left(\frac{3}{2}\right)=3\text{ cm}^2$.

25.



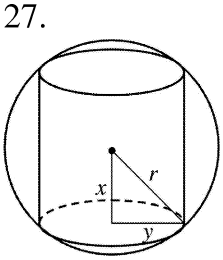
The cylinder has volume $V=\pi y^2(2x)$. Also $x^2+y^2=r^2 \Rightarrow y^2=r^2-x^2$, so $V(x)=\pi\left(r^2-x^2\right)(2x)=2\pi\left(r^2x-x^3\right)$, where $0\le x\le r$. $V'(x)=2\pi\left(r^2-3x^2\right)=0\Rightarrow x=r/\sqrt{3}$. Now $V(0)=V(r)=0$, so there is a maximum when $x=r/\sqrt{3}$ and $V\left(r/\sqrt{3}\right)=\pi\left(r^2-r^2/3\right)\left(2r/\sqrt{3}\right)=4\pi r^3/\left(3\sqrt{3}\right)$.

26.



By similar triangles, $y/x=h/r$, so $y=hx/r$. The volume of the cylinder is $\pi x^2(h-y)=\pi hx^2-(\pi h/r)x^3=V(x)$. Now $V'(x)=2\pi hx-(3\pi h/r)x^2=\pi hx(2-3x/r)$. So $V'(x)=0\Rightarrow x=0$ or $x=\frac{2}{3}r$. The maximum clearly occurs when $x=\frac{2}{3}r$ and then the volume is $\pi hx^2-(\pi h/r)x^3=\pi hx^2(1-x/r)=\pi\left(\frac{2}{3}r\right)^2h\left(1-\frac{2}{3}\right)=\frac{4}{27}\pi r^2h$.

27.



The cylinder has surface area 2(area of the base)+(lateral surface area)$=2\pi$(radius)$^2+2\pi$(radius)(height)$=2\pi y^2+2\pi y(2x)$. Now $x^2+y^2=r^2\Rightarrow y^2=r^2-x^2\Rightarrow y=\sqrt{r^2-x^2}$, so the surface area is $S(x)=2\pi\left(r^2-x^2\right)+4\pi x\sqrt{r^2-x^2},\ 0\le x\le r=2\pi r^2-2\pi x^2+4\pi\left(x\sqrt{r^2-x^2}\right)$ Thus, $S'(x)$ $=0-4\pi x+4\pi\left[x\cdot\frac{1}{2}\left(r^2-x^2\right)^{-1/2}(-2x)+\left(r^2-x^2\right)^{1/2}\cdot 1\right]$

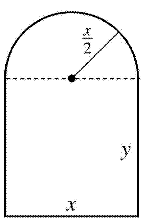
$$=4\pi\left[-x-\frac{x^2}{\sqrt{r^2-x^2}}+\sqrt{r^2-x^2}\right]=4\pi\cdot\frac{-x\sqrt{r^2-x^2}-x^2+r^2-x^2}{\sqrt{r^2-x^2}}$$

$S'(x)=0\Rightarrow x\sqrt{r^2-x^2}=r^2-2x^2\ (\,*\,)\Rightarrow\left(x\sqrt{r^2-x^2}\right)^2=\left(r^2-2x^2\right)^2\Rightarrow x^2\left(r^2-x^2\right)=r^4-4r^2x^2+4x^4\Rightarrow$ $r^2x^2-x^4=r^4-4r^2x^2+4x^4\Rightarrow 5x^4-5r^2x^2+r^4=0$. This is a quadratic equation in x^2 . By the quadratic formula, $x^2=\frac{5\pm\sqrt{5}}{10}r^2$, but we reject the root with the + sign since it doesn't satisfy (*). So $x=\sqrt{\frac{5-\sqrt{5}}{10}}\,r$. Since $S(0)=S(r)=0$, the maximum surface area occurs at the critical number and $x^2=\frac{5-\sqrt{5}}{10}r^2\Rightarrow y^2=r^2-\frac{5-\sqrt{5}}{10}r^2=\frac{5+\sqrt{5}}{10}r^2\Rightarrow$ the surface area is

$$2\pi\left(\frac{5+\sqrt{5}}{10}\right)r^2+4\pi\sqrt{\frac{5-\sqrt{5}}{10}}\sqrt{\frac{5+\sqrt{5}}{10}}\,r^2=\pi r^2\left[2\cdot\frac{5+\sqrt{5}}{10}+4\frac{\sqrt{(5-\sqrt{5})(5+\sqrt{5})}}{10}\right]$$

$$=\pi r^2\left[\frac{5+\sqrt{5}}{5}+\frac{2\sqrt{20}}{5}\right]=\pi r^2\left[\frac{5+\sqrt{5}+2\cdot 2\sqrt{5}}{5}\right]=\pi r^2\left[\frac{5+5\sqrt{5}}{5}\right]=\pi r^2\left(1+\sqrt{5}\right).$$

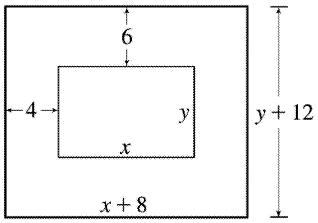
28.



Perimeter=30$\Rightarrow 2y+x+\pi\left(\frac{x}{2}\right)=30\Rightarrow y=\frac{1}{2}\left(30-x-\frac{\pi x}{2}\right)=15-\frac{x}{2}-\frac{\pi x}{4}$. The area is the area of the rectangle plus the area of the semicircle, or $xy+\frac{1}{2}\pi\left(\frac{x}{2}\right)^2$, so

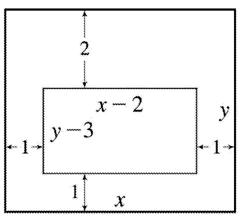
$A(x)=x\left(15-\frac{x}{2}-\frac{\pi x}{4}\right)+\frac{1}{8}\pi x^2=15x-\frac{1}{2}x^2-\frac{\pi}{8}x^2$. $A'(x)=15-\left(1+\frac{\pi}{4}\right)x=0\Rightarrow x=\frac{15}{1+\pi/4}=\frac{60}{4+\pi}$. $A''(x)=-\left(1+\frac{\pi}{4}\right)<0$, so this gives a maximum. The dimensions are $x=\frac{60}{4+\pi}$ ft and $y=15-\frac{30}{4+\pi}-\frac{15\pi}{4+\pi}=\frac{60+15\pi-30-15\pi}{4+\pi}=\frac{30}{4+\pi}$ ft, so the height of the rectangle is half the base.

29.



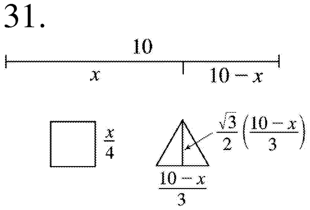
$xy=384 \Rightarrow y=384/x$. Total area is $A(x)=(8+x)(12+384/x)=12(40+x+256/x)$, so $A'(x)=12\left(1-256/x^2\right)=0 \Rightarrow x=16$. There is an absolute minimum when $x=16$ since $A'(x)<0$ for $0<x<16$ and $A'(x)>0$ for $x>16$. When $x=16$, $y=384/16=24$, so the dimensions are 24 cm and 36 cm.

30.



$xy=180$, so $y=180/x$. The printed area is $(x-2)(y-3)=(x-2)(180/x-3)=186-3x-360/x=A(x)$. $A'(x)=-3+360/x^2=0$ when $x^2=120 \Rightarrow x=2\sqrt{30}$. This gives an absolute maximum since $A'(x)>0$ for $0<x<2\sqrt{30}$ and $A'(x)<0$ for $x>2\sqrt{30}$. When $x=2\sqrt{30}$, $y=180/(2\sqrt{30})$, so the dimensions are $2\sqrt{30}$ in. and $90/\sqrt{30}$ in.

31.



Let x be the length of the wire used for the square. The total area is

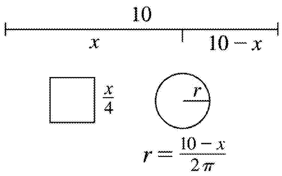
$$A(x)=\left(\frac{x}{4}\right)^2+\frac{1}{2}\left(\frac{10-x}{3}\right)\frac{\sqrt{3}}{2}\left(\frac{10-x}{3}\right)=\frac{1}{16}x^2+\frac{\sqrt{3}}{36}(10-x)^2,\ 0\le x\le 10$$

$A'(x)=\frac{1}{8}x-\frac{\sqrt{3}}{18}(10-x)=0 \Leftrightarrow \frac{9}{72}x+\frac{4\sqrt{3}}{72}x-\frac{40\sqrt{3}}{72}=0 \Leftrightarrow x=\frac{40\sqrt{3}}{9+4\sqrt{3}}$. Now $A(0)=\left(\frac{\sqrt{3}}{36}\right)100\approx 4.81$, $A(10)=\frac{100}{16}=6.25$ and $A\left(\frac{40\sqrt{3}}{9+4\sqrt{3}}\right)\approx 2.72$, so

(a) The maximum area occurs when $x=10$ m, and all the wire is used for the square.

(b) The minimum area occurs when $x=\frac{40\sqrt{3}}{9+4\sqrt{3}}\approx 4.35$ m.

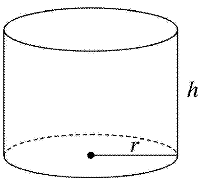
32.



Total area is $A(x)=\left(\dfrac{x}{4}\right)^2+\pi\left(\dfrac{10-x}{2\pi}\right)^2=\dfrac{x^2}{16}+\dfrac{(10-x)^2}{4\pi}$, $0\le x\le 10$.

$A'(x)=\dfrac{x}{8}-\dfrac{10-x}{2\pi}=\left(\dfrac{1}{2\pi}+\dfrac{1}{8}\right)x-\dfrac{5}{\pi}=0\Rightarrow x=40/(4+\pi)$. $A(0)=25/\pi\approx 7.96$, $A(10)=6.25$, and $A(40/(4+\pi))\approx 3.5$, so the maximum occurs when $x=0$ m and the minimum occurs when $x=40/(4+\pi)$ m.

33.

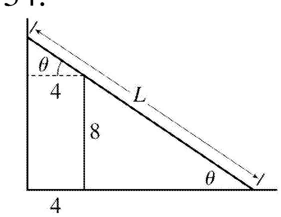


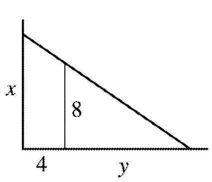
The volume is $V=\pi r^2 h$ and the surface area is $S(r)=\pi r^2+2\pi rh=\pi r^2+2\pi r\left(\dfrac{V}{\pi r^2}\right)=\pi r^2+\dfrac{2V}{r}$.

$S'(r)=2\pi r-\dfrac{2V}{r^2}=0\Rightarrow 2\pi r^3=2V\Rightarrow r=\sqrt[3]{\dfrac{V}{\pi}}$ cm.

This gives an absolute minimum since $S'(r)<0$ for $0<r<\sqrt[3]{\dfrac{V}{\pi}}$ and $S'(r)>0$ for $r>\sqrt[3]{\dfrac{V}{\pi}}$. When $r=\sqrt[3]{\dfrac{V}{\pi}}$, $h=\dfrac{V}{\pi r^2}=\dfrac{V}{\pi(V/\pi)^{2/3}}=\sqrt[3]{\dfrac{V}{\pi}}$ cm.

34.



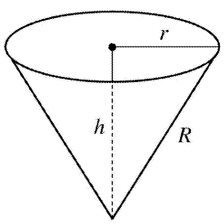


$L = 8\theta + 4\sec\theta$, $0 < \theta < \dfrac{\pi}{2}$, $\dfrac{dL}{d\theta} = -8\theta\ \cot\theta + 4\sec\theta\ \tan\theta = 0$ when $\sec\theta\ \tan\theta = 2\theta\ \cot\theta \Leftrightarrow \tan^3\theta = 2 \Leftrightarrow$ $\tan\theta = \sqrt[3]{2} \Leftrightarrow \theta = \tan^{-1}\sqrt[3]{2}$.

$dL/d\theta < 0$ when $0 < \theta < \tan^{-1}\sqrt[3]{2}$, $dL/d\theta > 0$ when $\tan^{-1}\sqrt[3]{2} < \theta < \dfrac{\pi}{2}$, so L has an absolute minimum when $\theta = \tan^{-1}\sqrt[3]{2}$, and the shortest ladder has length $L = 8\dfrac{\sqrt{1+2^{2/3}}}{2^{1/3}} + 4\sqrt{1+2^{2/3}} \approx 16.65$ ft.

Another method: Minimize $L^2 = x^2 + (4+y)^2$, where $\dfrac{x}{4+y} = \dfrac{8}{y}$.

35.

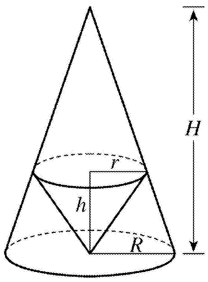


$h^2 + r^2 = R^2 \Rightarrow V = \dfrac{\pi}{3} r^2 h = \dfrac{\pi}{3}\left(R^2 - h^2\right) h = \dfrac{\pi}{3}\left(R^2 h - h^3\right)$. $V'(h) = \dfrac{\pi}{3}\left(R^2 - 3h^2\right) = 0$ when $h = \dfrac{1}{\sqrt{3}} R$. This gives an absolute maximum, since $V'(h) > 0$ for $0 < h < \dfrac{1}{\sqrt{3}} R$ and $V'(h) < 0$ for $h > \dfrac{1}{\sqrt{3}} R$. The maximum volume is $V\left(\dfrac{1}{\sqrt{3}} R\right) = \dfrac{\pi}{3}\left(\dfrac{1}{\sqrt{3}} R^3 - \dfrac{1}{3\sqrt{3}} R^3\right) = \dfrac{2}{9\sqrt{3}} \pi R^3$.

36. The volume and surface area of a cone with radius r and height h are given by $V = \dfrac{1}{3}\pi r^2 h$ and $S = \pi r\sqrt{r^2 + h^2}$. We'll minimize $A = S^2$ subject to $V = 27$. $V = 27 \Rightarrow \dfrac{1}{3}\pi r^2 h = 27 \Rightarrow r^2 = \dfrac{81}{\pi h}$ **(1)** .

$A = \pi^2 r^2 (r^2 + h^2) = \pi^2\left(\dfrac{81}{\pi h}\right)\left(\dfrac{81}{\pi h} + h^2\right) = \dfrac{81^2}{h^2} + 81\pi h$, so $A' = 0 \Rightarrow \dfrac{-2\cdot 81^2}{h^3} + 81\pi = 0 \Rightarrow 81\pi = \dfrac{2\cdot 81^2}{h^3} \Rightarrow$ $h^3 = \dfrac{162}{\pi} \Rightarrow h = \sqrt[3]{\dfrac{162}{\pi}} = 3\sqrt[3]{\dfrac{6}{\pi}} \approx 3.722$. From **(1)** , $r^2 = \dfrac{81}{\pi h} = \dfrac{81}{\pi\cdot 3\sqrt[3]{6/\pi}} = \dfrac{27}{\sqrt[3]{6\pi^2}} \Rightarrow$ $r = \dfrac{3\sqrt{3}}{\sqrt[6]{6\pi^2}} \approx 2.632$. $A'' = 6\cdot 81^2/h^4 > 0$, so A and hence S has an absolute minimum at these values of r and h .

37.



By similar triangles, $\frac{H}{R}=\frac{H-h}{r}$ **(1)** . The volume of the inner cone is $V=\frac{1}{3}\pi r^2 h$, so we'll solve **(1)** for h . $\frac{Hr}{R}=H-h \Rightarrow h=H-\frac{Hr}{R}=\frac{HR-Hr}{R}=\frac{H}{R}(R-r)$ **(2)** .

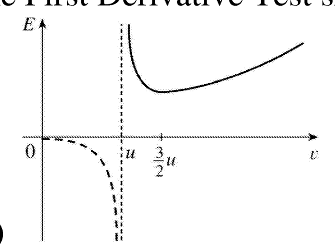
Thus, $V(r)=\frac{\pi}{3}r^2\cdot\frac{H}{R}(R-r)=\frac{\pi H}{3R}(Rr^2-r^3)\Rightarrow V'(r)=\frac{\pi H}{3R}(2Rr-3r^2)=\frac{\pi H}{3R}r(2R-3r)$.

$V'(r)=0\Rightarrow r=0$ or $2R=3r\Rightarrow r=\frac{2}{3}R$ and from **(2)** , $h=\frac{H}{R}\left(R-\frac{2}{3}R\right)=\frac{H}{R}\left(\frac{1}{3}R\right)=\frac{1}{3}H$.

$V'(r)$ changes from positive to negative at $r=\frac{2}{3}R$, so the inner cone has a maximum volume of $V=\frac{1}{3}\pi r^2 h=\frac{1}{3}\pi\left(\frac{2}{3}R\right)^2\left(\frac{1}{3}H\right)=\frac{4}{27}\cdot\frac{1}{3}\pi R^2 H$, which is approximately 15% of the volume of the larger cone.

38. **(a)** $E(v)=\frac{aLv^3}{v-u}\Rightarrow E'(v)=aL\frac{(v-u)3v^2-v^3}{(v-u)^2}=0$ when $2v^3=3uv^2\Rightarrow 2v=3u\Rightarrow v=\frac{3}{2}u$.

The First Derivative Test shows that this value of v gives the minimum value of E .

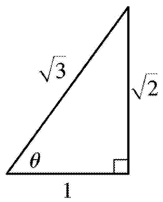
(b) 

39. $S=6sh-\frac{3}{2}s^2\cot\theta+3s^2\frac{\sqrt{3}}{2}\ \theta$

(a) $\frac{dS}{d\theta}=\frac{3}{2}s^2\ {}^2\theta-3s^2\frac{\sqrt{3}}{2}\ \theta\cot\theta$ or $\frac{3}{2}s^2\ \theta\left(\ \theta-\sqrt{3}\cot\theta\right)$.

(b) $\frac{dS}{d\theta}=0$ when $\theta-\sqrt{3}\cot\theta=0\Rightarrow\frac{1}{\sin\theta}-\sqrt{3}\frac{\cos\theta}{\sin\theta}=0\Rightarrow\cos\theta=\frac{1}{\sqrt{3}}$. The First Derivative Test shows that the minimum surface area occurs when $\theta=\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)\approx 55^\circ$.

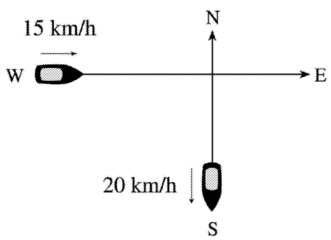
(c)



If $\cos\theta = \frac{1}{\sqrt{3}}$, then $\cot\theta = \frac{1}{\sqrt{2}}$ and $\theta = \frac{\sqrt{3}}{\sqrt{2}}$, so the surface area is $S <=$

$$6sh - \frac{3}{2} s^2 \frac{1}{\sqrt{2}} + 3s^2 \frac{\sqrt{3}}{2} \frac{\sqrt{3}}{\sqrt{2}} = 6sh - \frac{3}{2\sqrt{2}} s^2 + \frac{9}{2\sqrt{2}} s^2 = 6sh + \frac{6}{2\sqrt{2}} s^2 = 6s\left(h + \frac{1}{2\sqrt{2}} s\right)$$

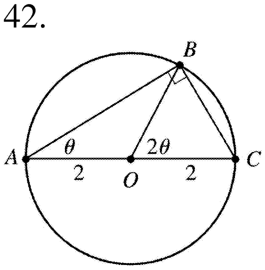
40.



Let t be the time, in hours, after 2:00 P.M. The position of the boat heading south at time t is $(0,-20t)$. The position of the boat heading east at time t is $(-15+15t,0)$. If $D(t)$ is the distance between the boats at time t , we minimize $f(t)=[D(t)]^2=20^2t^2+15^2(t-1)^2$. $f'(t)=800t+450(t-1)=1250t-450=0$ when $t=\frac{450}{1250}=0.36$ h. 0.36 h $\times \frac{60\text{min}}{\text{h}}=21.6$ min $=21$ min 36 s. Since $f''(t)>0$, this gives a minimum, so the boats are closest together at 2:21:36 P.M.

41. Here $T(x)=\frac{\sqrt{x^2+25}}{6}+\frac{5-x}{8}$, $0\le x\le 5 \Rightarrow T'(x)=\frac{x}{6\sqrt{x^2+25}}-\frac{1}{8}=0 \Leftrightarrow 8x=6\sqrt{x^2+25} \Leftrightarrow$ $16x^2=9\left(x^2+25\right) \Leftrightarrow x=\frac{15}{\sqrt{7}}$. But $\frac{15}{\sqrt{7}}>5$, so T has no critical number. Since $T(0)\approx 1.46$ and $T(5)\approx 1.18$, he should row directly to B .

42.

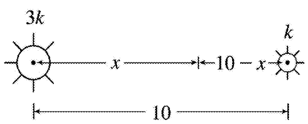


In isosceles triangle AOB , $\angle O=180^\circ-\theta-\theta$, so $\angle BOC=2\theta$. The distance rowed is $4\cos\theta$ while the distance walked is the length of arc $BC=2(2\theta)=4\theta$. The time taken is given by

$T(\theta)=\dfrac{4\cos\theta}{2}+\dfrac{4\theta}{4}=2\cos\theta+\theta$, $0\leq\theta\leq\dfrac{\pi}{2}$. $T'(\theta)=-2\sin\theta+1=0\Leftrightarrow\sin\theta=\dfrac{1}{2}\Rightarrow\theta=\dfrac{\pi}{6}$.

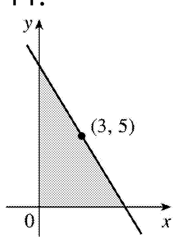
Check the value of T at $\theta=\dfrac{\pi}{6}$ and at the endpoints of the domain of T ; that is, $\theta=0$ and $\theta=\dfrac{\pi}{2}$. $T(0)=2$, $T\left(\dfrac{\pi}{6}\right)=\sqrt{3}+\dfrac{\pi}{6}\approx 2.26$, and $T\left(\dfrac{\pi}{2}\right)=\dfrac{\pi}{2}\approx 1.57$. Therefore, the minimum value of T is $\dfrac{\pi}{2}$ when $\theta=\dfrac{\pi}{2}$; that is, the woman should walk all the way. Note that $T''(\theta)=-2\cos\theta<0$ for $0\leq\theta<\dfrac{\pi}{2}$, so $\theta=\dfrac{\pi}{6}$ gives a maximum time.

43.



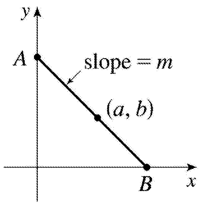
The total illumination is $I(x)=\dfrac{3k}{x^2}+\dfrac{k}{(10-x)^2}$, $0<x<10$. Then $I'(x)=\dfrac{-6k}{x^3}+\dfrac{2k}{(10-x)^3}=0\Rightarrow$ $6k(10-x)^3=2kx^3\Rightarrow 3(10-x)^3=x^3\Rightarrow\sqrt[3]{3}(10-x)=x\Rightarrow 10\sqrt[3]{3}-\sqrt[3]{3}\,x=x\Rightarrow 10\sqrt[3]{3}=x+\sqrt[3]{3}\,x\Rightarrow$ $10\sqrt[3]{3}=\left(1+\sqrt[3]{3}\right)x\Rightarrow x=\dfrac{10\sqrt[3]{3}}{1+\sqrt[3]{3}}\approx 5.9$ ft. This gives a minimum since $I''(x)>0$ for $0<x<10$.

44.



The line with slope m (where $m<0$) through $(3,5)$ has equation $y-5=m(x-3)$ or $y=mx+(5-3m)$. The $y-$ intercept is $5-3m$ and the $x-$ intercept is $-5/m+3$. So the triangle has area $A(m)=\dfrac{1}{2}(5-3m)(-5/m+3)=15-25/(2m)-\dfrac{9}{2}m$. Now $A'(m)=\dfrac{25}{2m^2}-\dfrac{9}{2}=0\Leftrightarrow m^2=\dfrac{25}{9}\Rightarrow m=-\dfrac{5}{3}$ (since $m<0$). $A''(m)=-\dfrac{25}{m^3}>0$, so there is an absolute minimum when $m=-\dfrac{5}{3}$. Thus, an equation of the line is $y-5=-\dfrac{5}{3}(x-3)$ or $y=-\dfrac{5}{3}x+10$.

45.



Every line segment in the first quadrant passing through (a,b) with endpoints on the x – and y –axes satisfies an equation of the form $y-b=m(x-a)$, where $m<0$. By setting $x=0$ and then $y=0$, we find its endpoints, $A(0,b-am)$ and $B\left(a-\frac{b}{m},0\right)$. The distance d from A to B is given by

$$d=\sqrt{\left[\left(a-\frac{b}{m}\right)-0\right]^2+[0-(b-am)]^2} .$$

It follows that the square of the length of the line segment, as a function of m , is given by

$$S(m)=\left(a-\frac{b}{m}\right)^2+(am-b)^2=a^2-\frac{2ab}{m}+\frac{b^2}{m^2}+a^2m^2-2abm+b^2 . \text{ Thus,}$$

$$S'(m)=\frac{2ab}{m^2}-\frac{2b^2}{m^3}+2a^2m-2ab=\frac{2}{m^3}\left(abm-b^2+a^2m^4-abm^3\right)$$

$$=\frac{2}{m^3}\left[b(am-b)+am^3(am-b)\right]=\frac{2}{m^3}(am-b)\left(b+am^3\right)$$

Thus, $S'(m)=0\Leftrightarrow m=b/a$ or $m=-\sqrt[3]{\frac{b}{a}}$. Since $b/a>0$ and $m<0$, m must equal $-\sqrt[3]{\frac{b}{a}}$. Since $\frac{2}{m^3}<0$, we see that $S'(m)<0$ for $m<-\sqrt[3]{\frac{b}{a}}$ and $S'(m)>0$ for $m>-\sqrt[3]{\frac{b}{a}}$. Thus, S has its absolute minimum value when $m=-\sqrt[3]{\frac{b}{a}}$. That value is

$$S\left(-\sqrt[3]{\frac{b}{a}}\right)=\left(a+b\sqrt[3]{\frac{a}{b}}\right)^2+\left(-a\sqrt[3]{\frac{b}{a}}-b\right)^2=\left(a+\sqrt[3]{ab^2}\right)^2+\left(\sqrt[3]{a^2b}+b\right)^2$$

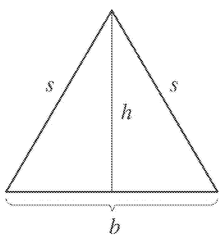
$$=a^2+2a^{4/3}b^{2/3}+a^{2/3}b^{4/3}+a^{4/3}b^{2/3}+2a^{2/3}b^{4/3}+b^2=a^2+3a^{4/3}b^{2/3}+3a^{2/3}b^{4/3}+b^2$$

The last expression is of the form $x^3+3x^2y+3xy^2+y^3\left[=(x+y)^3\right]$ with $x=a^{2/3}$ and $y=b^{2/3}$, so we can write it as $(a^{2/3}+b^{2/3})^3$ and the shortest such line segment has length $\sqrt{S}=(a^{2/3}+b^{2/3})^{3/2}$.

46. $y=1+40x^3-3x^5\Rightarrow y'=120x^2-15x^4$, so the tangent line to the curve at $x=a$ has slope $m(a)=120a^2-15a^4$. Now $m'(a)=240a-60a^3=-60a(a^2-4)=-60a(a+2)(a-2)$, so $m'(a)>0$ for $a<-2$,

and $0<a<2$, and $m'(a)<0$ for $-2<a<0$ and $a>2$. Thus, m is increasing on $(-\infty,-2)$, decreasing on $(-2,0)$, increasing on $(0,2)$, and decreasing on $(2,\infty)$. Clearly, $m(a)\to-\infty$ as $a\to\pm\infty$, so the maximum value of $m(a)$ must be one of the two local maxima, $m(-2)$ or $m(2)$. But both $m(-2)$ and $m(2)$ equal $120\cdot 2^2-15\cdot 2^4=480-240=240$. So 240 is the largest slope, and it occurs at the points $(-2,-223)$ and $(2,225)$. *Note:* $a=0$ corresponds to a local *minimum* of m .

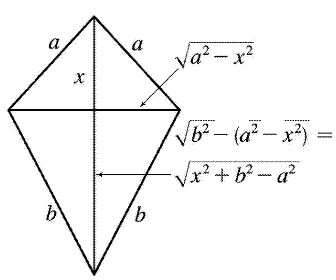
47.



Here $s^2=h^2+b^2/4$, so $h^2=s^2-b^2/4$. The area is $A=\frac{1}{2}b\sqrt{s^2-b^2/4}$. Let the perimeter be p , so $2s+b=p$ or $s=(p-b)/2\Rightarrow A(b)=\frac{1}{2}b\sqrt{(p-b)^2/4-b^2/4}=b\sqrt{p^2-2pb}/4$. Now

$$A'(b)=\frac{\sqrt{p^2-2pb}}{4}-\frac{bp/4}{\sqrt{p^2-2pb}}=\frac{-3pb+p^2}{4\sqrt{p^2-2pb}}$$

. Therefore, $A'(b)=0\Rightarrow -3pb+p^2=0\Rightarrow b=p/3$. Since $A'(b)>0$ for $b<p/3$ and $A'(b)<0$ for $b>p/3$, there is an absolute maximum when $b=p/3$. But then $2s+p/3=p$, so $s=p/3\Rightarrow s=b\Rightarrow$ the triangle is equilateral.



48.

See the figure. The area is given by

$$A(x)=\frac{1}{2}\left(2\sqrt{a^2-x^2}\right)x+\frac{1}{2}\left(2\sqrt{a^2-x^2}\right)\left(\sqrt{x^2+b^2-a^2}\right)=\sqrt{a^2-x^2}\left(x+\sqrt{x^2+b^2-a^2}\right)$$

for $0\le x\le a$. Now

$$A'(x)=\sqrt{a^2-x^2}\left(1+\frac{x}{\sqrt{x^2+b^2-a^2}}\right)+\left(x+\sqrt{x^2+b^2-a^2}\right)\frac{-x}{\sqrt{a^2-x^2}}=0\Leftrightarrow$$

$$\frac{x}{\sqrt{a^2-x^2}}\left(x+\sqrt{x^2+b^2-a^2}\right)=\sqrt{a^2-x^2}\left(\frac{x+\sqrt{x^2+b^2-a^2}}{\sqrt{x^2+b^2-a^2}}\right).$$

Except for the trivial case where $x=0$, $a=b$ and $A(x)=0$, we have

$x+\sqrt{x^2+b^2-a^2}>0$. Hence, cancelling this factor gives $\dfrac{x}{\sqrt{a^2-x^2}}=\dfrac{\sqrt{a^2-x^2}}{\sqrt{x^2+b^2-a^2}}\Rightarrow x\sqrt{x^2+b^2-a^2}=a^2-x^2$

$\Rightarrow x^2\left(x^2+b^2-a^2\right)=a^4-2a^2x^2+x^4\Rightarrow x^2\left(b^2-a^2\right)=a^4-2a^2x^2\Rightarrow x^2\left(b^2+a^2\right)=a^4\Rightarrow x=\dfrac{a^2}{\sqrt{a^2+b^2}}$.

Now we must check the value of A at this point as well as at the endpoints of the domain to see which gives the maximum value. $A(0)=a\sqrt{b^2-a^2}$, $A(a)=0$ and

$$A\left(\frac{a^2}{\sqrt{a^2+b^2}}\right)=\sqrt{a^2-\left(\frac{a^2}{\sqrt{a^2+b^2}}\right)^2}\left[\frac{a^2}{\sqrt{a^2+b^2}}+\sqrt{\left(\frac{a^2}{\sqrt{a^2+b^2}}\right)^2+b^2-a^2}\right]=$$

$$\frac{ab}{\sqrt{a^2+b^2}}\left[\frac{a^2}{\sqrt{a^2+b^2}}+\frac{b^2}{\sqrt{a^2+b^2}}\right]=\frac{ab\left(a^2+b^2\right)}{a^2+b^2}=ab=0$$

Since $b\geq\sqrt{b^2-a^2}$, $A\left(a^2/\sqrt{a^2+b^2}\right)\geq A(0)$. So there is an absolute maximum when $x=\dfrac{a^2}{\sqrt{a^2+b^2}}$.

In this case the horizontal piece should be $\dfrac{2ab}{\sqrt{a^2+b^2}}$ and the vertical piece should be

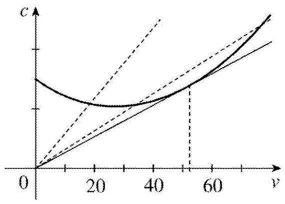
$\dfrac{a^2+b^2}{\sqrt{a^2+b^2}}=\sqrt{a^2+b^2}$.

49. Note that $|AD|=|AP|+|PD|\Rightarrow 5=x+|PD|\Rightarrow |PD|=5-x$. Using the Pythagorean Theorem for ΔPDB and ΔPDC gives us

$$\begin{aligned} L(x) &= |AP|+|BP|+|CP|=x+\sqrt{(5-x)^2+2^2}+\sqrt{(5-x)^2+3^2} \\ &= x+\sqrt{x^2-10x+29}+\sqrt{x^2-10x+34} \end{aligned}$$

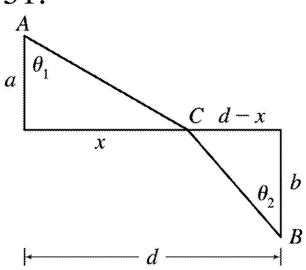
$\Rightarrow L'(x)=1+\dfrac{x-5}{\sqrt{x^2-10x+29}}+\dfrac{x-5}{\sqrt{x^2-10x+34}}$ From the graphs of L and L' , it seems that the minimum value of L is about $L(3.59)=9.35$ m.

50.



We note that since c is the consumption in gallons per hour, and v is the velocity in miles per hour, then $\frac{c}{v} = \frac{\text{gallons/hour}}{\text{miles/hour}} = \frac{\text{gallons}}{\text{mile}}$ gives us the consumption in gallons per mile, that is, the quantity G. To find the minimum, we calculate $\frac{dG}{dv} = \frac{d}{dv}\left(\frac{c}{v}\right) = \frac{v\frac{dc}{dv} - c\frac{dv}{dv}}{v^2} = \frac{v\frac{dc}{dv} - c}{v^2}$. This is 0 when $v\frac{dc}{dv} - c = 0 \Leftrightarrow \frac{dc}{dv} = \frac{c}{v}$. This implies that the tangent line of $c(v)$ passes through the origin, and this occurs when $v \approx 53$ mi / h. Note that the slope of the secant line through the origin and a point $(v, c(v))$ on the graph is equal to $G(v)$, and it is intuitively clear that G is minimized in the case where the secant is in fact a tangent.

51.



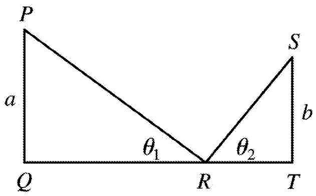
The total time is

$$T(x) \lt/ = (\text{time from } A \text{ to } C) + (\text{time from } C \text{ to } B) = \frac{\sqrt{a^2+x^2}}{v_1} + \frac{\sqrt{b^2+(d-x)^2}}{v_2}, \; 0<x<d$$

$$T'(x) = \frac{x}{v_1\sqrt{a^2+x^2}} - \frac{d-x}{v_2\sqrt{b^2+(d-x)^2}} = \frac{\sin\theta_1}{v_1} - \frac{\sin\theta_2}{v_2}$$

The minimum occurs when $T'(x) = 0 \Rightarrow \frac{\sin\theta_1}{v_1} = \frac{\sin\theta_2}{v_2}$.

52.



If $d=|QT|$, we minimize $f\left(\theta_1\right)=|PR|+|RS|=a\ \theta_1+b\ \theta_2$. Differentiating with respect to θ_1 , and setting $\frac{df}{d\theta_1}$ equal to 0 , we get $\frac{df}{d\theta_1}=0=-a\ \theta_1\cot\theta_1-b\ \theta_2\cot\theta_2\frac{d\theta_2}{d\theta_1}$.

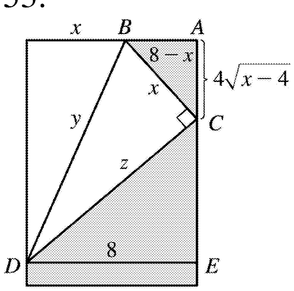
So we need to find an expression for $\frac{d\theta_2}{d\theta_1}$. We can do this by observing that $|QT|$= constant $=a\cot\theta_1+b\cot\theta_2$.

Differentiating this equation implicitly with respect to θ_1 , we get $-a\ ^2\theta_1-b\ ^2\theta_2\frac{d\theta_2}{d\theta_1}=0\Rightarrow$ $\frac{d\theta_2}{d\theta_1}=-\frac{a\ ^2\theta_1}{b\ ^2\theta_2}$. We substitute this into the expression for $\frac{df}{d\theta_1}$ to get

$-a\ \theta_1\cot\theta_1-b\ \theta_2\cot\theta_2\left(-\frac{a\ ^2\theta_1}{b\ ^2\theta_2}\right)=0\Leftrightarrow-a\csc\theta_1\cot\theta_1+a\frac{\csc^2\theta_1\cot\theta_2}{\csc\theta_2}=0\Leftrightarrow$

$\cot\theta_1\csc\theta_2=\csc\theta_1\cot\theta_2\Leftrightarrow\frac{\cot\theta_1}{\csc\theta_1}=\frac{\cot\theta_2}{\csc\theta_2}\Leftrightarrow\cos\theta_1=\cos\theta_2$. Since θ_1 and θ_2 are both acute, we have $\theta_1=\theta_2$.

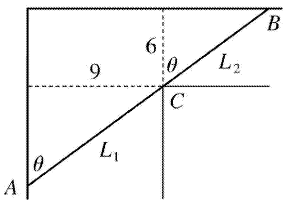
53.



$y^2=x^2+z^2$, but triangles *CDE* and *BCA* are similar, so $z/8=x\,/\left(4\sqrt{x-4}\right)\ \Rightarrow z=2x\,/\sqrt{x-4}$. Thus, we minimize $f(x)=y^2=x^2+4x^2/(x-4)=x^3/(x-4)$, $4<x\leq 8$.

$f'(x) = \dfrac{(x-4)\left(3x^2\right) - x^3}{(x-4)^2} = \dfrac{x^2[3(x-4)-x]}{(x-4)^2} = \dfrac{2x^2(x-6)}{(x-4)^2} = 0$ when $x = 6$. $f'(x) < 0$ when $x < 6$, $f'(x) > 0$ when $x > 6$, so the minimum occurs when $x = 6$ in.

54.



Paradoxically, we solve this maximum problem by solving a minimum problem. Let L be the length of the line ACB going from wall to wall touching the inner corner C . As $\theta \to 0$ or $\theta \to \dfrac{\pi}{2}$, we have $L \to \infty$ and there will be an angle that makes L a minimum. A pipe of this length will just fit around the corner.

From the diagram, $L = L_1 + L_2 = 9\csc\theta + 6\sec\theta \Rightarrow dL/d\theta = -9\csc\theta\ \cot\theta + 6\sec\theta\ \tan\theta = 0$ when

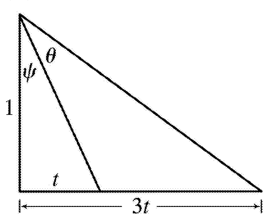
$6\sec\theta\ \tan\theta = 9\csc\theta\ \cot\theta \Leftrightarrow \tan^3\theta = \dfrac{9}{6} = 1.5 \Leftrightarrow \tan\theta = \sqrt{[3]1.5}$. Then $\sec^2\theta = 1 + \left(\dfrac{3}{2}\right)^{2/3}$ and

$\csc^2\theta = 1 + \left(\dfrac{3}{2}\right)^{-2/3}$, so the longest pipe has length

$L = 9\left[1 + \left(\dfrac{3}{2}\right)^{-2/3}\right]^{1/2} + 6\left[1 + \left(\dfrac{3}{2}\right)^{2/3}\right]^{1/2} \approx 21.07$ ft.

Or, use $\theta = \tan^{-1}\left(\sqrt[3]{1.5}\right) \approx 0.852 \Rightarrow L = 9\ \theta + 6\sec\theta \approx 21.07$ ft.

55.



It suffices to maximize $\tan\theta$. Now $\dfrac{3t}{1} = \tan\ (\psi + \theta) = \dfrac{\tan\psi + \tan\theta}{1 - \tan\psi\tan\theta} = \dfrac{t + \tan\theta}{1 - t\tan\theta}$. So

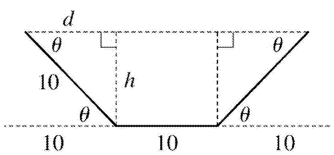
$3t\ (1 - t\tan\theta) = t + \tan\theta \Rightarrow 2t = \left(1 + 3t^2\right)\tan\theta \Rightarrow \tan\theta = \dfrac{2t}{1+3t^2}$. Let $f(t) = \tan\theta = \dfrac{2t}{1+3t^2} \Rightarrow$

$f'(t) = \dfrac{2\left(1+3t^2\right) - 2t(6t)}{\left(1+3t^2\right)^2} = \dfrac{2\left(1-3t^2\right)}{\left(1+3t^2\right)^2} = 0 \Leftrightarrow 1 - 3t^2 = 0 \Leftrightarrow t = \dfrac{1}{\sqrt{3}}$ since $t \geq 0$.

Now

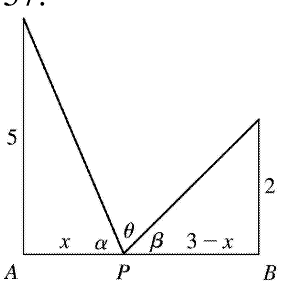
$f'(t)>0$ for $0\leq t<\frac{1}{\sqrt{3}}$ and $f'(t)<0$ for $t>\frac{1}{\sqrt{3}}$, so f has an absolute maximum when $t=\frac{1}{\sqrt{3}}$ and $\tan\theta=\frac{2(1/\sqrt{3})}{1+3(1/\sqrt{3})^2}=\frac{1}{\sqrt{3}}\Rightarrow\theta=\frac{\pi}{6}$. Substituting for t and θ in $3t=\tan(\psi+\theta)$ gives us $\sqrt{3}=\tan\left(\psi+\frac{\pi}{6}\right)\Rightarrow\psi=\frac{\pi}{6}$.

56.



We maximize the cross–sectional area $A(\theta)=$
$10h+2\left(\frac{1}{2}dh\right)=10h+dh=10(10\sin\theta)+(10\cos\theta)(10\sin\theta)>=100(\sin\theta+\sin\theta\cos\theta),\ 0\leq\theta\leq\frac{\pi}{2}$
$A'(\theta)=100\left(\cos\theta+\cos^2\theta-\sin^2\theta\right)=100\left(\cos\theta+2\cos^2\theta-1\right)=100(2\cos\theta-1)(\cos\theta+1)=0$ when $\cos\theta=\frac{1}{2}\Leftrightarrow\theta=\frac{\pi}{3}$. ($\cos\theta\neq-1$ since $0\leq\theta\leq\frac{\pi}{2}$.)
Now $A(0)=0$, $A\left(\frac{\pi}{2}\right)=100$ and $A\left(\frac{\pi}{3}\right)=75\sqrt{3}\approx129.9$, so the maximum occurs when $\theta=\frac{\pi}{3}$.

57.



From the figure, $\tan\alpha=\frac{5}{x}$ and $\tan\beta=\frac{2}{3-x}$. Since $\alpha+\beta+\theta=180^\circ=\pi$,
$\theta=\pi-\tan^{-1}\left(\frac{5}{x}\right)-\tan^{-1}\left(\frac{2}{3-x}\right)\Rightarrow$

$$\frac{d\theta}{dx}=-\frac{1}{1+\left(\frac{5}{x}\right)^2}\left(-\frac{5}{x^2}\right)-\frac{1}{1+\left(\frac{2}{3-x}\right)^2}\left[\frac{2}{(3-x)^2}\right]$$

$$=\frac{x^2}{x^2+25}\cdot\frac{5}{x^2}-\frac{(3-x)^2}{(3-x)^2+4}\cdot\frac{2}{(3-x)^2}.$$

Now $\frac{d\theta}{dx}=0 \Rightarrow \frac{5}{x^2+25}=\frac{2}{x^2-6x+13} \Rightarrow 2x^2+50=5x^2-30x+65 \Rightarrow$
$3x^2-30x+15=0 \Rightarrow x^2-10x+5=0 \Rightarrow x=5\pm 2\sqrt{5}$. We reject the root with the + sign, since it is larger than 3 . $d\theta/dx>0$ for $x<5-2\sqrt{5}$ and $d\theta/dx<0$ for $x>5-2\sqrt{5}$, so θ is maximized when $|AP|=x=5-2\sqrt{5}\approx 0.53$.

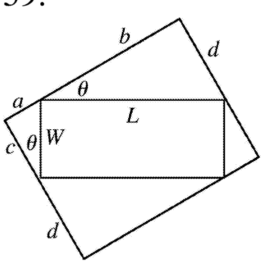
58. Let x be the distance from the observer to the wall. Then, from the given figure, <
$\theta=\tan^{-1}\left(\frac{h+d}{x}\right)-\tan^{-1}\left(\frac{d}{x}\right)$, $x>0 \Rightarrow$

$$\frac{d\theta}{dx} = \frac{1}{1+[(h+d)/x]^2}\left[-\frac{h+d}{x^2}\right]-\frac{1}{1+(d/x)^2}\left[-\frac{d}{x^2}\right]=-\frac{h+d}{x^2+(h+d)^2}+\frac{d}{x^2+d^2}$$

$$= \frac{d\left[x^2+(h+d)^2\right]-(h+d)\left(x^2+d^2\right)}{\left[x^2+(h+d)^2\right]\left(x^2+d^2\right)}=\frac{h^2d+hd^2-hx^2}{\left[x^2+(h+d)^2\right]\left(x^2+d^2\right)}=0 \Leftrightarrow$$

$hx^2=h^2d+hd^2 \Leftrightarrow x^2=hd+d^2 \Leftrightarrow x=\sqrt{d(h+d)}$. Since $d\theta/dx>0$ for all $x<\sqrt{d(h+d)}$ and $d\theta/dx<0$ for all $x>\sqrt{d(h+d)}$, the absolute maximum occurs when $x=\sqrt{d(h+d)}$.

59.



In the small triangle with sides a and c and hypotenuse W , $\sin\theta=\frac{a}{W}$ and $\cos\theta=\frac{c}{W}$. In the triangle with sides b and d and hypotenuse L , $\sin\theta=\frac{d}{L}$ and $\cos\theta=\frac{b}{L}$. Thus, $a=W\sin\theta$, $c=W\cos\theta$, $d=L\sin\theta$, and $b=L\cos\theta$, so the area of the circumscribed rectangle is

$$\begin{aligned} A(\theta) &= (a+b)(c+d)=(W\sin\theta+L\cos\theta)(W\cos\theta+L\sin\theta)=1-12\text{pt} \\ &= W^2\sin\theta\cos\theta+WL\sin^2\theta+LW\cos^2\theta+L^2\sin\theta\cos\theta=1-12\text{pt} \\ &= LW\sin^2\theta+LW\cos^2\theta+\left(L^2+W^2\right)\sin\theta\cos\theta=1-12\text{pt} \\ &= LW\left(\sin^2\theta+\cos^2\theta\right)+\left(L^2+W^2\right)\cdot\frac{1}{2}\cdot 2\sin\theta\cos\theta=1-12\text{pt} \\ &= LW+\frac{1}{2}\left(L^2+W^2\right)\sin 2\theta,\ 0\le\theta\le\frac{\pi}{2} \end{aligned}$$

This expression shows, without calculus, that the maximum value of $A(\theta)$ occurs when $\sin 2\theta = 1 \Leftrightarrow$
$2\theta = \dfrac{\pi}{2}$
$\Rightarrow \theta = \dfrac{\pi}{4}$. So the maximum area is $A\left(\dfrac{\pi}{4}\right) = LW + \dfrac{1}{2}\left(L^2 + W^2\right) = \dfrac{1}{2}\left(L^2 + 2LW + W^2\right) = \dfrac{1}{2}\left(L+W\right)^2$.

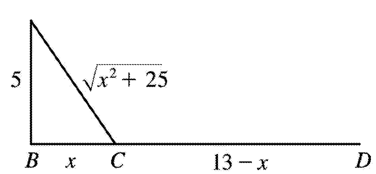
60. **(a)** Let D be the point such that $a = |AD|$. From the figure, $\sin\theta = \dfrac{b}{|BC|} \Rightarrow |BC| = b\ \theta$ and
$\cos\theta = \dfrac{|BD|}{|BC|} = \dfrac{a - |AB|}{|BC|} \Rightarrow |BC| = (a - |AB|)\sec\theta$. Eliminating $|BC|$ gives $(a - |AB|)\sec\theta = b\ \theta \Rightarrow$
$b\cot\theta = a - |AB| \Rightarrow |AB| = a - b\cot\theta$. The total resistance is
$R(\theta) = C\dfrac{|AB|}{r_1^4} + C\dfrac{|BC|}{r_2^4} = C\left(\dfrac{a - b\cot\theta}{r_1^4} + \dfrac{b\csc\theta}{r_2^4}\right)$.

(b) $R'(\theta) = C\left(\dfrac{b\csc^2\theta}{r_1^4} - \dfrac{b\csc\theta\cot\theta}{r_2^4}\right) = bC\csc\theta\left(\dfrac{\csc\theta}{r_1^4} - \dfrac{\cot\theta}{r_2^4}\right)$.

$R'(\theta) = 0 \Leftrightarrow \dfrac{\csc\theta}{r_1^4} = \dfrac{\cot\theta}{r_2^4} \Leftrightarrow \dfrac{r_2^4}{r_1^4} = \dfrac{\cot\theta}{\csc\theta} = \cos\theta$.

$R'(\theta) > 0 \Leftrightarrow \dfrac{\csc\theta}{r_1^4} > \dfrac{\cot\theta}{r_2^4} \Rightarrow \cos\theta < \dfrac{r_2^4}{r_1^4}$ and $R'(\theta) < 0$ when $\cos\theta > \dfrac{r_2^4}{r_1^4}$, so there is an absolute minimum when $\cos\theta = r_2^4 / r_1^4$.

(c) When $r_2 = \dfrac{2}{3} r_1$, we have $\cos\theta = \left(\dfrac{2}{3}\right)^4$, so $\theta = \cos^{-1}\left(\dfrac{2}{3}\right)^4 \approx 79^\circ$.

61. **(a)**


If k = energy / km over land, then energy / km over water $= 1.4k$. So the total energy is
$E = 1.4k\sqrt{25 + x^2} + k(13 - x)$, $0 \le x \le 13$, and so $\dfrac{dE}{dx} = \dfrac{1.4kx}{\left(25 + x^2\right)^{1/2}} - k$.

Set $\dfrac{dE}{dx} = 0$: $1.4kx = k\left(25 + x^2\right)^{1/2} \Rightarrow 1.96x^2 = x^2 + 25 \Rightarrow 0.96x^2 = 25 \Rightarrow x = \dfrac{5}{\sqrt{0.96}} \approx 5.1$. Testing against the value of E at the endpoints: $E(0)=1.4k(5)+13k=20k$, $E(5.1)\approx 17.9k$, $E(13)\approx 19.5k$. Thus, to minimize energy, the bird should fly to a point about 5.1 km from B .

(b) If W / L is large, the bird would fly to a point C that is closer to B than to D to minimize the energy used flying over water. If W / L is small, the bird would fly to a point C that is closer to D than to B to minimize the distance of the flight. $E=W\sqrt{25+x^2}+L(13-x)\Rightarrow \dfrac{dE}{dx}=\dfrac{Wx}{\sqrt{25+x^2}}-L=0$ when

$\dfrac{W}{L}=\dfrac{\sqrt{25+x^2}}{x}$. By the same sort of argument as in part (a), this ratio will give the minimal expenditure of energy if the bird heads for the point x km from B .

(c) For flight direct to D , $x=13$, so from part (b), $W / L=\dfrac{\sqrt{25+13^2}}{13}\approx 1.07$. There is no value of W / L for which the bird should fly directly to B . But note that $\lim_{x\to 0^+}(W / L)=\infty$, so if the point at which E is a minimum is close to B , then W / L is large.

(d) Assuming that the birds instinctively choose the path that minimizes the energy expenditure, we can use the equation for $dE/dx=0$ from part (a) with $1.4k=c$, $x=4$, and $k=1$: $(c)(4)=1\cdot\left(25+4^2\right)^{1/2}\Rightarrow$ $c=\sqrt{41}/4\approx 1.6$.

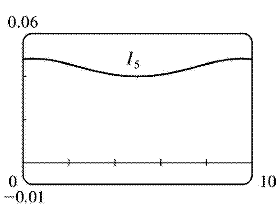
62. **(a)** $I(x)\propto\dfrac{\text{strength of source}}{(\text{distance from source})^2}$. Adding the intensities from the left and right lightbulbs,

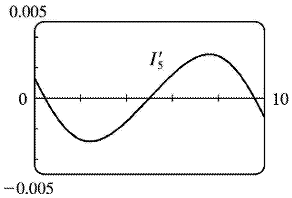
$$I(x)=\frac{k}{x^2+d^2}+\frac{k}{(10-x)^2+d^2}=\frac{k}{x^2+d^2}+\frac{k}{x^2-20x+100+d^2}.$$

(b) The magnitude of the constant k won't affect the location of the point of maximum intensity, so for convenience we take $k=1$. $I'(x)=-\dfrac{2x}{\left(x^2+d^2\right)^2}-\dfrac{2(x-10)}{\left(x^2-20x+100+d^2\right)^2}$.

Substituting $d=5$ into the equations for $I(x)$ and $I'(x)$, we get

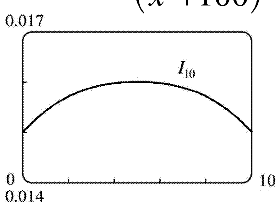
$$I_5(x)=\frac{1}{x^2+25}+\frac{1}{x^2-20x+125}\quad\text{and}\quad I_5'(x)=-\frac{2x}{\left(x^2+25\right)^2}-\frac{2(x-10)}{\left(x^2-20x+125\right)^2}$$

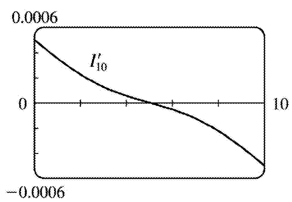




From the graphs, it appears that $I_5(x)$ has a minimum at $x=5$ m.

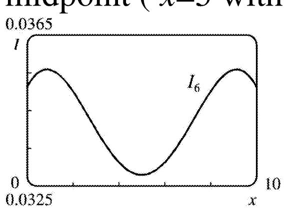
(c) Substituting $d=10$ into the equations for $I(x)$ and $I'(x)$ gives $I_{10}(x)=\dfrac{1}{x^2+100}+\dfrac{1}{x^2-20x+200}$ and $I_{10}'(x)=-\dfrac{2x}{\left(x^2+100\right)^2}-\dfrac{2(x-10)}{\left(x^2-20x+200\right)^2}$.

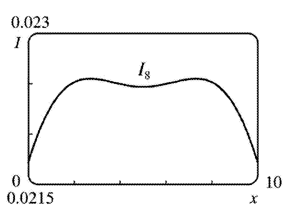


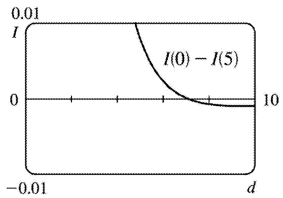


From the graphs, it seems that for $d=10$, the intensity is minimized at the endpoints, that is, $x=0$ and $x=10$. The midpoint is now the most brightly lit point!

(d) From the first figures in parts (b) and (c), we see that the minimal illumination changes from the midpoint ($x=5$ with $d=5$) to the endpoints ($x=0$ and $x=10$ with $d=10$).





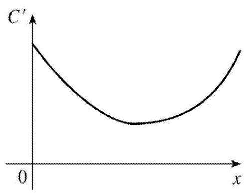


So we try $d=6$ (see the first figure) and we see that the minimum value still occurs at $x=5$. Next, we let $d=8$ (see the second figure) and we see that the minimum value occurs at the endpoints. It appears that for some value of d between 6 and 8 , we must have minima at both the midpoint and the endpoints, that is, $I(5)$ must equal $I(0)$. To find this value of d , we solve $I(0)=I(5)$ (with $k=1$):
$\frac{1}{d^2}+\frac{1}{100+d^2}=\frac{1}{25+d^2}+\frac{1}{25+d^2}=\frac{2}{25+d^2} \Rightarrow \left(25+d^2\right)\left(100+d^2\right)+d^2\left(25+d^2\right)=2d^2\left(100+d^2\right) \Rightarrow$
$2500+125d^2+d^4+25d^2+d^4=200d^2+2d^4 \Rightarrow 2500=50d^2 \Rightarrow d^2=50 \Rightarrow d=5\sqrt{2}\approx 7.071$ (for $0\leq d\leq 10$). The third figure, a graph of $I(0)–I(5)$ with d independent, confirms that $I(0)–I(5)=0$, that is, $I(0)=I(5)$, when $d=5\sqrt{2}$. Thus, the point of minimal illumination changes abruptly from the midpoint to the endpoints when $d=5\sqrt{2}$.

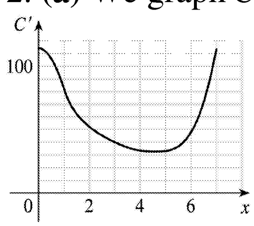
1. **(a)** $C(0)$ represents the fixed costs of production, such as rent, utilities, machinery etc., which are incurred even when nothing is produced.

(b) The inflection point is the point at which $C''(x)$ changes from negative to positive; that is, the marginal cost $C'(x)$ changes from decreasing to increasing. Thus, the marginal cost is minimized at the inflection point.

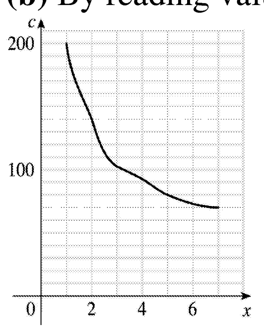
(c) The marginal cost function is $C'(x)$. We graph it as in Example 1 in Section.



2. **(a)** We graph C' as in Example 1 in Section.



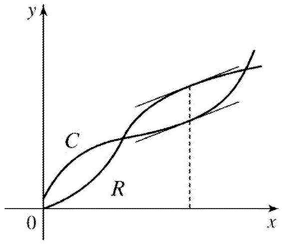
(b) By reading values of $C(x)$ from its graph, we can plot $c(x)=C(x)/x$.



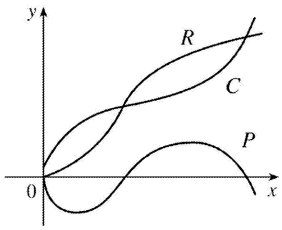
(c) Since the graph in part (b) is decreasing, we estimate that the minimum value of $c(x)$ occurs at $x=7$. The average cost and the marginal cost are equal at that value. See the box preceding Example 1.

3. $c(x)=21.4-0.002x$ and $c(x)=C(x)/x \Rightarrow C(x)=21.4x-0.002x^2$. $C'(x)=21.4-0.004x$ and $C'(1000)=17.4$. This means that the cost of producing the 1001 st unit is about \$17.40 .

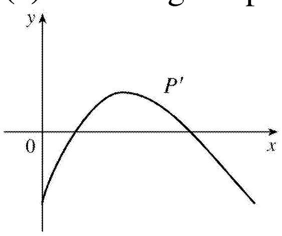
4. **(a)** Profit is maximized when the marginal revenue is equal to the marginal cost; that is, when R and C have equal slopes. See the box preceding Example 2.



(b) $P(x)=R(x)-C(x)$ is sketched.



(c) The marginal profit function is defined as $P'(x)$.



5. **(a)** The cost function is $C(x)=40,000+300x+x^2$, so the cost at a production level of 1000 is $C(1000)=\$1,340,000$. The average cost function is $c(x)=\frac{C(x)}{x}=\frac{40,000}{x}+300+x$ and $c(1000)=\$1340/$ unit. The marginal cost function is $C'(x)=300+2x$ and $C'(1000)=\$2300/$ unit.

(b) See the box preceding Example 1. We must have $C'(x)=c(x)\Leftrightarrow 300+2x=\frac{40,000}{x}+300+x\Leftrightarrow x=\frac{40,000}{x}\Rightarrow x^2=40,000\Rightarrow x=\sqrt{40,000}=200$. This gives a minimum value of the average cost function $c(x)$ since $c''(x)=\frac{80,000}{x^3}>0$.

(c) The minimum average cost is $c(200)=\$700/$ unit.

6. **(a)** $C(x)=25,000+120x+0.1x^2$, $C(1000)=\$245,000$. $c(x)=\frac{C(x)}{x}=\frac{25,000}{x}+120+0.1x$, $c(1000)=\$245/$ unit. $C'(x)=120+0.2x$, $C'(1000)=\$320/$ unit.

(b) We must have $C'(x)=c(x)\Leftrightarrow 120+0.2x=\frac{25,000}{x}+120+0.1x\Leftrightarrow 0.1x=\frac{25,000}{x}\Rightarrow 0.1x^2=25,000\Rightarrow$

$x=\sqrt{250{,}000}=500$. This gives a minimum since $c''(x)=\dfrac{50{,}000}{x^3}>0$.

(c) The minimum average cost is $c(500)=\$220.00$/ unit.

7. **(a)** $C(x)=16{,}000+200x+4x^{3/2}$, $C(1000)=16{,}000+200{,}000+40{,}000\sqrt{10}\approx 216{,}000+126{,}491$, so $C(1000)\approx \$342{,}491$. $c(x)=C(x)/x=\dfrac{16{,}000}{x}+200+4x^{1/2}$, $c(1000)\approx \$342.49$/unit .

$C'(x)=200+6x^{1/2}$, $C'(1000)=200+60\sqrt{10}\approx \$389.74$/unit .

(b) We must have $C'(x)=c(x)\Leftrightarrow 200+6x^{1/2}=\dfrac{16{,}000}{x}+200+4x^{1/2}\Leftrightarrow 2x^{3/2}=16{,}000\Leftrightarrow x=(8{,}000)^{2/3}=400$ units. To check that this is a minimum, we calculate $c'(x)=\dfrac{-16{,}000}{x^2}+\dfrac{2}{\sqrt{x}}=\dfrac{2}{x^2}(x^{3/2}-8000)$. This is negative for $x<(8000)^{2/3}=400$, zero at $x=400$, and positive for $x>400$, so c is decreasing on $(0,400)$ and increasing on $(400,\infty)$. Thus, c has an absolute minimum at $x=400$.

(c) The minimum average cost is $c(400)=40+200+80=\$320$/ unit.

8. **(a)** $C(x)=10{,}000+340x-0.3x^2+0.0001x^3$, $C(1000)=\$150{,}000$.

$c(x)=C(x)/x=\dfrac{10{,}000}{x}+340-0.3x+0.0001x^2$, $c(1000)=\$150$/unit . $C'(x)=340-0.6x+0.0003x^2$, $C'(1000)=\$40$/unit .

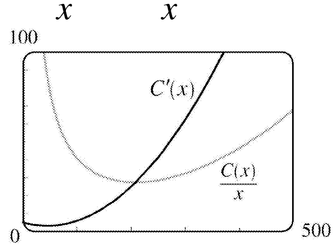
(b) We must have $C'(x)=c(x)\Leftrightarrow 340-0.6x+0.0003x^2=\dfrac{10{,}000}{x}+340-0.3x+0.0001x^2\Leftrightarrow$ $0.0002x^2=\dfrac{10{,}000}{x}+0.3x\Leftrightarrow 0.0002x^3-0.3x^2-10{,}000=0\Leftrightarrow x^3-1500x^2-50{,}000{,}000=0\Rightarrow$ $x\approx 1521.60\approx 1522$ units. This gives a minimum since $c''(x)=\dfrac{20{,}000}{x^3}+0.0002>0$.

(c) The minimum average cost is about $c(1521.60)\approx \$121.62$/ unit.

9. **(a)** $C(x)=3700+5x-0.04x^2+0.0003x^3\Rightarrow C'(x)=5-0.08x+0.0009x^2$ (marginal cost).

$c(x)=\dfrac{C(x)}{x}=\dfrac{3700}{x}+5-0.04x+0.0003x^2$ (average cost).

(b)



The graphs intersect at $(208.51,27.45)$, so the production level that minimizes average cost is about

209 units.

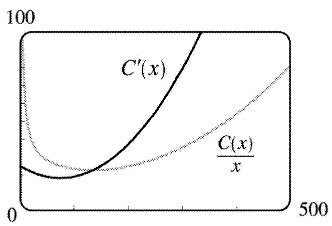
(c) $c'(x)=-\frac{3700}{x^2}-0.04+0.0006x=0 \Rightarrow 3700+0.04x^2-0.0006x^3=0 \Rightarrow x_1 \approx 208.51$. $c\left(x_1\right) \approx \$27.45$/ unit.

(d) The marginal cost is given by $C'(x)$, so to find its minimum value we'll find the derivative of C' ; that is, C'' . $C''(x)=-0.08+0.0018x=0 \Rightarrow x_1=\frac{800}{18}=44.4\overline{4}$. $C'\left(x_1\right)=\$3.22$/ unit.

$C'''(x)=0.0018>0$ for all x , so this is the minimum marginal cost. C''' is the second derivative of C' . cost is given by $C'(x)$.

10. **(a)** $C(x)=339+25x-0.09x^2+0.0004x^3 \Rightarrow C'(x)=25-0.18x+0.0012x^2$ (marginal cost).

$c(x)=\frac{C(x)}{x}=\frac{339}{x}+25-0.09x+0.0004x^2$ (average cost).



(b) The graphs intersect at $(135.56,22.65)$, so the production level that minimizes average cost is about 136 units.

(c) $c'(x)=-\frac{339}{x^2}-0.09+0.0008x=0 \Rightarrow x_1 \approx 135.56$. $c\left(x_1\right) \approx \$22.65$/ unit.

(d) $C''(x)=-0.18+0.0024x=0 \Rightarrow x=\frac{1800}{24}=75$. $C'(75)=\$18.25$/ unit.

$C'''(x)=0.0024>0$ for all x , so this is the minimum marginal cost.

11. $C(x)=680+4x+0.01x^2$, $p(x)=12 \Rightarrow R(x)=xp(x)=12x$. If the profit is maximum, then $R'(x)=C'(x)$ $\Rightarrow 12=4+0.02x \Rightarrow 0.02x=8 \Rightarrow x=400$. The profit is maximized if $P''(x)<0$, but since $P''(x)=R''(x)-C''(x)$, we can just check the condition $R''(x)<C''(x)$. Now $R''(x)=0<0.02=C''(x)$, so $x=400$ gives a maximum.

12. $C(x)=680+4x+0.01x^2$, $p(x)=12-x/500$. Then $R(x)=xp(x)=12x-x^2/500$. If the profit is maximum, then $R'(x)=C'(x) \Leftrightarrow 12-x/250=4+0.02x \Leftrightarrow 8=0.024x \Leftrightarrow x=8/0.024=\frac{1000}{3}$. The profit is maximized if $P''(x)<0$, but since $P''(x)=R''(x)-C''(x)$, we can just check the condition $R''(x)<C''(x)$.

Now $R''(x)=-\frac{1}{250}<0.02=C''(x)$, so $x=\frac{1000}{3}$ gives a maximum.

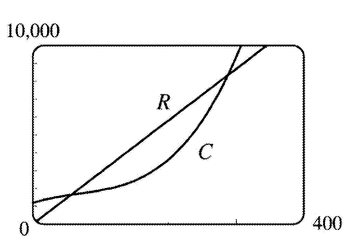
13. $C(x)=1450+36x-x^2+0.001x^3$, $p(x)=60-0.01x$. Then $R(x)=xp(x)=60x-0.01x^2$. If the profit is maximum, then $R'(x)=C'(x)\Leftrightarrow 60-0.02x=36-2x+0.003x^2 \Rightarrow 0.003x^2-1.98x-24=0$. By the quadratic formula, $x=\frac{1.98\pm\sqrt{(-1.98)^2+4(0.003)(24)}}{2(0.003)}=\frac{1.98\pm\sqrt{4.2084}}{0.006}$. Since $x>0$, $x\approx(1.98+2.05)/0.006\approx 672$. Now $R''(x)=-0.02$ and $C''(x)=-2+0.006x\Rightarrow C''(672)=2.032\Rightarrow$ $R''(672)<C''(672)\Rightarrow$ there is a maximum at $x=672$.

14. $C(x)=16{,}000+500x-1.6x^2+0.004x^3$, $p(x)=1700-7x$. Then $R(x)=xp(x)=1700x-7x^2$. If the profit is maximum, then $R'(x)=C'(x)\Leftrightarrow 1700-14x=500-3.2x+0.012x^2 \Leftrightarrow 0.012x^2+10.8x-1200=0\Leftrightarrow$ $x^2+900x-100{,}000=0\Leftrightarrow(x+1000)(x-100)=0\Leftrightarrow x=100$ (since $x>0$). The profit is maximized if $P''(x)<0$, but since $P''(x)=R''(x)-C''(x)$, we can just check the condition $R''(x)<C''(x)$. Now $R''(x)=-14<-3.2+0.024x=C''(x)$ for $x>0$, so there is a maximum at $x=100$.

15. $C(x)=0.001x^3-0.3x^2+6x+900$. The marginal cost is $C'(x)=0.003x^2-0.6x+6$. $C'(x)$ is increasing when $C''(x)>0\Leftrightarrow 0.006x-0.6>0\Leftrightarrow x>0.6/0.006=100$. So $C'(x)$ starts to increase when $x=100$.

16. $C(x)=0.0002x^3-0.25x^2+4x+1500$. The marginal cost is $C'(x)=0.0006x^2-0.50x+4$. $C'(x)$ is increasing when $C''(x)>0\Leftrightarrow 0.0012x-0.5>0\Leftrightarrow x>0.5/0.0012\approx 417$. So $C'(x)$ starts to increase when $x=417$.

17. **(a)** $C(x)=1200+12x-0.1x^2+0.0005x^3$. $R(x)=xp(x)=29x-0.00021x^2$. Since the profit is maximized when $R'(x)=C'(x)$, we examine the curves R and C in the figure, looking for $x-$ values at which the slopes of the tangent lines are equal. It appears that $x=200$ is a good estimate.



(b) $R'(x)=C'(x)\Rightarrow 29-0.00042x=12-0.2x+0.0015x^2\Rightarrow 0.0015x^2-0.19958x-17=0\Rightarrow x\approx 192.06$ (for $x>0$). As in Exercise 11, $R''(x)<C''(x)\Rightarrow -0.00042<-0.2+0.003x\Leftrightarrow 0.003x>0.19958\Leftrightarrow x>66.5$. Our value of 192 is in this range, so we have a maximum profit when we produce 192 yards of fabric.

18. **(a)** Cost=setupcost+manufacturingcost$\Rightarrow C(x)=500+m(x)=500+20x-5x^{3/4}+0.01x^2$. We can solve $x(p)=320-7.7p$ for p in terms of x to find the demand (or price) function. $x=320-7.7p\Rightarrow 7.7p=320-x$ $\Rightarrow p(x)=\frac{320-x}{7.7}$. $R(x)=xp(x)=\frac{320x-x^2}{7.7}$.
(b) $C'(x)=R'(x)\Rightarrow 20-\frac{15}{4}x^{-1/4}+0.02x=\frac{320-2x}{7.7}\Rightarrow x\approx 81.53$ planes, and $p(x)=\$30.97$ million. The maximum profit associated with these values is about \$463.59 million.

19. **(a)** We are given that the demand function p is linear and $p(27,000)=10$, $p(33,000)=8$, so the slope is $\frac{10-8}{27,000-33,000}=-\frac{1}{3000}$ and an equation of the line is $y-10=\left(-\frac{1}{3000}\right)(x-27,000)\Rightarrow$ $y=p(x)=-\frac{1}{3000}x+19=19-(x/3000)$.
(b) The revenue is $R(x)=xp(x)=19x-(x^2/3000)\Rightarrow R'(x)=19-(x/1500)=0$ when $x=28,500$. Since $R''(x)=-1/1500<0$, the maximum revenue occurs when $x=28,500\Rightarrow$ the price is $p(28,500)=\$9.50$.

20. **(a)** Let $p(x)$ be the demand function. Then $p(x)$ is linear and $y=p(x)$ passes through $(20,10)$ and $(18,11)$, so the slope is $-\frac{1}{2}$ and an equation of the line is $y-10=-\frac{1}{2}(x-20)\Leftrightarrow y=-\frac{1}{2}x+20$. Thus, the demand is $p(x)=-\frac{1}{2}x+20$ and the revenue is $R(x)=xp(x)=-\frac{1}{2}x^2+20x$.
(b) The cost is $C(x)=6x$, so the profit is $P(x)=R(x)-C(x)=-\frac{1}{2}x^2+14x$. Then $0=P'(x)=-x+14\Rightarrow x=14$. Since $P''(x)=-1<0$, the selling price for maximum profit is $p(14)=-\frac{1}{2}(14)+20=\$13$.

21. **(a)** As in Example 3, we see that the demand function p is linear. We are given that $p(1000)=450$ and deduce that $p(1100)=440$, since a \$10 reduction in price increases sales by 100 per week. The slope for p is $\frac{440-450}{1100-1000}=-\frac{1}{10}$, so an equation is $p-450=-\frac{1}{10}(x-1000)$ or $p(x)=-\frac{1}{10}x+550$.
(b) $R(x)=xp(x)=-\frac{1}{10}x^2+550x$. $R'(x)=-\frac{1}{5}x+550=0$ when $x=5(550)=2750$. $p(2750)=275$, so the rebate should be $450-275=\$175$.
(c) $C(x)=68,000+150x\Rightarrow$

$P(x)=R(x)-C(x)=-\frac{1}{10}x^2+550x-68,000-150x=-\frac{1}{10}x^2+400x-68,000$, $P'(x)=-\frac{1}{5}x+400=0$ when $x=2000$. $p(2000)=350$. Therefore, the rebate to maximize profits should be $450-350=\$100$.

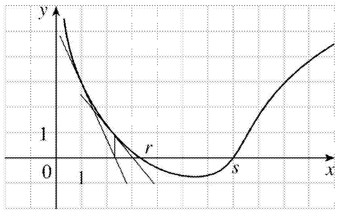
22. Let x denote the number of \$10 increases in rent. Then the price is $p(x)=800+10x$, and the number of units occupied is $100-x$. Now the revenue is

$$\begin{aligned} R(x) &= (\text{rental price per unit})\times(\text{number of unit srented}) \\ &= (800+10x)(100-x)=-10x^2+200x+80{,}000 \text{ for } 0\le x\le 100 \Rightarrow \end{aligned}$$

$R'(x)=-20x+200=0\Leftrightarrow x=10$. This is a maximum since $R''(x)=-20<0$ for all x . Now we must check the value of $R(x)=(800+10x)(100-x)$ at $x=10$ and at the endpoints of the domain to see which value of x gives the maximum value of R . $R(0)=80,000$, $R(10)=(900)(90)=81,000$, and $R(100)=(1800)(0)=0$. Thus, the maximum revenue of \$81 , 000/ week occurs when 90 units are occupied at a rent of \$900/ week.

23. If the reorder quantity is x , then the manager places $\frac{800}{x}$ orders per year. Storage costs for the year are $\frac{1}{2}x\cdot 4=2x$ dollars. Handling costs are \$100 per delivery, for a total of $\frac{800}{x}\cdot 100=\frac{80{,}000}{x}$ dollars. The total costs for the year are $C(x)=2x+\frac{80{,}000}{x}$. To minimize $C(x)$, we calculate $C'(x)=2-\frac{80{,}000}{x^2}=\frac{2}{x^2}(x^2-40,000)$. This is negative when $x<200$, zero when $x=200$, and positive when $x>200$, so C is decreasing on $(0,200)$ and increasing on $(200,\infty)$, reaching its absolute minimum when $x=200$. Thus, the optimal reorder quantity is 200 cases. The manager will place 4 orders per year for a total cost of $C(200)=\$800$.

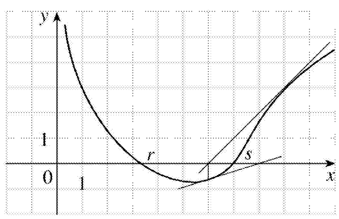
24. She will have A/n dollars after each withdrawal and 0 dollars just before the next withdrawal, so her average cash balance at any given time is $\frac{1}{2}(A/n+0)=A/(2n)$. The transaction costs for n withdrawals are nT . The lost interest cost on the average cash balance is $[A/(2n)]\cdot R$. Thus, the total cost for n transactions is $C(n)=nT+\frac{AR}{2n}$. Now $C'(n)=T-\frac{AR}{2n^2}$ and $C'(n)=0\Rightarrow \frac{AR}{2n^2}=T\Rightarrow n^2=\frac{AR}{2T}\Rightarrow$ $n=\sqrt{\frac{AR}{2T}}$, the value of n that minimizes total costs since $C''(n)=-\frac{AR}{n^3}<0$. Thus, the optimal average cash balance is $\frac{A}{2n}=\frac{A\sqrt{2T}}{2\sqrt{AR}}=\frac{\sqrt{AT}}{\sqrt{2R}}=\sqrt{\frac{AT}{2R}}$.

1. **(a)** 

The tangent line at $x=1$ intersects the x –axis at $x\approx 2.3$, so $x_2\approx 2.3$. The tangent line at $x=2.3$ intersects the x –axis at $x\approx 3$, so $x_3\approx 3.0$.

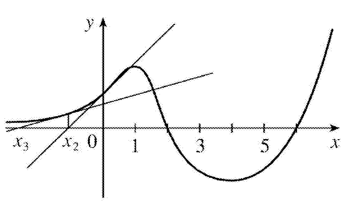
(b) $x_1=5$ would *not* be a better first approximation than $x_1=1$ since the tangent line is nearly horizontal. In fact, the second approximation for $x_1=5$ appears to be to the left of $x=1$.

2.

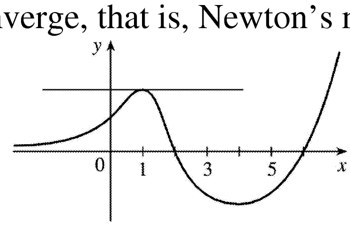


The tangent line at $x=9$ intersects the x –axis at $x\approx 6.0$, so $x_2\approx 6.0$. The tangent line at $x=6.0$ intersects the x –axis at $x\approx 8.0$, so $x_3\approx 8.0$.

3. Since $x_1=3$ and $y=5x-4$ is tangent to $y=f(x)$ at $x=3$, we simply need to find where the tangent line intersects the x– axis. $y=0\Rightarrow 5x_2-4=0\Rightarrow x_2=\frac{4}{5}$.

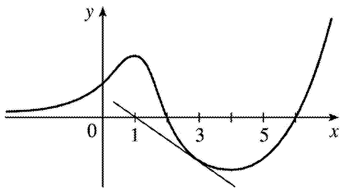


4. **(a)**
If $x_1=0$, then x_2 is negative, and x_3 is even more negative. The sequence of approximations does not converge, that is, Newton's method fails.



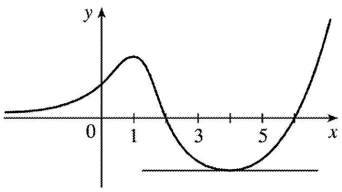
(b)

If $x_1=1$, the tangent line is horizontal and Newton's method fails.



(c)

If $x_1=3$, then $x_2=1$ and we have the same situation as in part (b). Newton's method fails again.



(d)

If $x_1=4$, the tangent line is horizontal and Newton's method fails.

5. $f(x)=x^3+2x-4 \Rightarrow f'(x)=3x^2+2$, so $x_{n+1}=x_n-\dfrac{x_n^3+2x_n-4}{3x_n^2+2}$. Now $x_1=1 \Rightarrow x_2=1-\dfrac{1+2-4}{3\cdot 1^2+2}=1-\dfrac{-1}{5}=1.2 \Rightarrow$

$x_3=1.2-\dfrac{(1.2)^3+2(1.2)-4}{3(1.2)^2+2}\approx 1.1797$.

6. $f(x)=x^3-x^2-1 \Rightarrow f'(x)=3x^2-2x$, so $x_{n+1}=x_n-\dfrac{f\left(x_n\right)}{f'\left(x_n\right)}=x_n-\dfrac{x_n^3-x_n^2-1}{3x_n^2-2x_n}$.

Now $x_1=1 \Rightarrow x_2=1-\dfrac{1-1-1}{3-2}=2 \Rightarrow x_3=2-\dfrac{2^3-2^2-1}{3\cdot 2^2-2\cdot 2}=1.625$.

7. $f(x)=x^4-20 \Rightarrow f'(x)=4x^3$, so $x_{n+1}=x_n-\dfrac{f\left(x_n\right)}{f'\left(x_n\right)}=x_n-\dfrac{x_n^4-20}{4x_n^3}$.

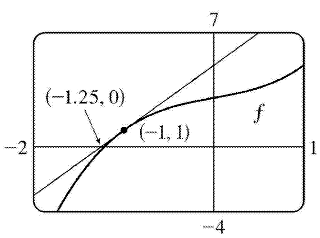
Now $x_1=2 \Rightarrow x_2=2-\dfrac{2^4-20}{4(2)^3}=2.125 \Rightarrow x_3=2.125-\dfrac{(2.125)^4-20}{4(2.125)^3}\approx 2.1148$.

8. $f(x)=x^5+2 \Rightarrow f'(x)=5x^4$, so

$x_{n+1}=x_n-\dfrac{x_n^5+2}{5x_n^4}$. Now $x_1=-1\Rightarrow x_2=-1-\dfrac{(-1)^5+2}{5\cdot(-1)^4}=-1-\dfrac{1}{5}=-1.2\Rightarrow x_3=-1.2-\dfrac{(-1.2)^5+2}{5(-1.2)^4}\approx-1.1529$.

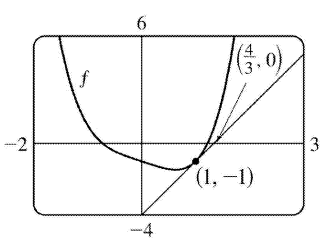
9. $f(x)=x^3+x+3\Rightarrow f'(x)=3x^2+1$, so $x_{n+1}=x_n-\dfrac{x_n^3+x_n+3}{3x_n^2+1}$. Now $x_1=-1\Rightarrow$

$x_2=-1-\dfrac{(-1)^3+(-1)+3}{3(-1)^2+1}=-1-\dfrac{-1-1+3}{3+1}=-1-\dfrac{1}{4}=-1.25$. Newton's method follows the tangent line at $(-1,1)$ down to its intersection with the x –axis at $(-1.25,0)$, giving the second approximation $x_2=-1.25$.



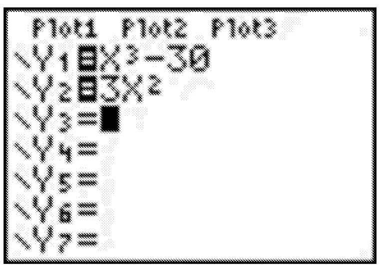
10. $f(x)=x^4-x-1\Rightarrow f'(x)=4x^3-1$, so $x_{n+1}=x_n-\dfrac{x_n^4-x_n-1}{4x_n^3-1}$. Now $x_1=1\Rightarrow x_2=1-\dfrac{1^4-1-1}{4\cdot 1^3-1}=1-\dfrac{-1}{3}=\dfrac{4}{3}$.

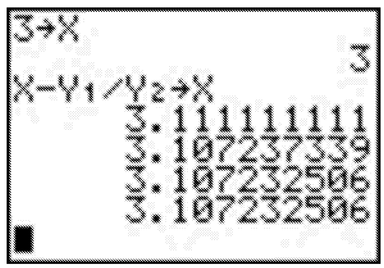
Newton's method follows the tangent line at $(1,-1)$ up to its intersection with the x –axis at $\left(\dfrac{4}{3},0\right)$, giving the second approximation $x_2=\dfrac{4}{3}$.



11. To approximate $x=\sqrt[3]{30}$ (so that $x^3=30$), we can take $f(x)=x^3-30$. So $f'(x)=3x^2$, and thus,

$x_{n+1}=x_n-\dfrac{x_n^3-30}{3x_n^2}$. Since $\sqrt[3]{27}=3$ and 27 is close to 30 , we'll use $x_1=3$. We need to find approximations until they agree to eight decimal places. $x_1=3\Rightarrow x_2\approx 3.11111111$, $x_3\approx 3.10723734$, $x_4\approx 3.10723251\approx x_5$. So $\sqrt[3]{30}\approx 3.10723251$, to eight decimal places. Here is a quick and easy method for finding the iterations for Newton's method on a programmable calculator. (The screens shown are from the TI–83 Plus, but the method is similar on other calculators.) Assign $f(x)=x^3-30$ to Y_1 , and $f'(x)=3x^2$ to Y_2 . Now store $x_1=3$ in X and then enter $X-Y_1/Y_2\to X$ to get $x_2=3.\overline{1}$. By successively pressing the ENTER key, you get the approximations x_3 , x_4 , … .





In Derive, load the utility file NEWTON$(x\text{^}3-30,x,3)$ and then APPROXIMATE to get . You can request a specific iteration by adding a fourth argument. For example, NEWTON$(x\text{^}3-30,x,3,\ 2)$ gives [3,3.11111111,3.10723733] .
In Maple, make the assignments f:=$x\to x$^3–30; , g:=$x\to x-f(x)/D(f)(x)$; , and x:=3.; . Repeatedly execute the command x:=$g(x)$; to generate successive approximations.

In Mathematica, make the assignments $f[x]$:=x^3–30 , $g[x]$:=$x-f[x]/f'[x]$, and x=3. Repeatedly execute the command $x=g[x]$ to generate successive approximations.

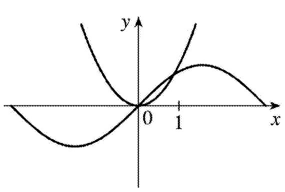
12. $f(x)=x^7-1000\Rightarrow f'(x)=7x^6$, so $x_{n+1}=x_n-\dfrac{x_n^7-1000}{7x_n^6}$. We need to find approximations until they agree to eight decimal places. $x_1=3\Rightarrow x_2\approx 2.76739173$, $x_3\approx 2.69008741$, $x_4\approx 2.68275645$, $x_5\approx 2.68269580\approx x_6$. So $\sqrt[7]{1000}\approx 2.68269580$, to eight decimal places.

13. $f(x)=2x^3-6x^2+3x+1\Rightarrow f'(x)=6x^2-12x+3\Rightarrow$

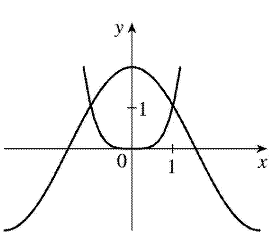
$x_{n+1}=x_n-\dfrac{2x_n^3-6x_n^2+3x_n+1}{6x_n^2-12x_n+3}$. We need to find approximations until they agree to six decimal places. $x_1=2.5\Rightarrow x_2\approx 2.285714$, $x_3\approx 2.228824$, $x_4\approx 2.224765$, $x_5\approx 2.224745\approx x_6$. So the root is 2.224745 , to six decimal places.

14. $f(x)=x^4+x-4\Rightarrow f'(x)=4x^3+1\Rightarrow x_{n+1}=x_n-\dfrac{x_n^4+x_n-4}{4x_n^3+1}$. $x_1=1.5\Rightarrow x_2\approx 1.323276$, $x_3\approx 1.285346$, $x_4\approx 1.283784$, $x_5\approx 1.283782\approx x_6$. So the root is 1.283782 , to six decimal places.

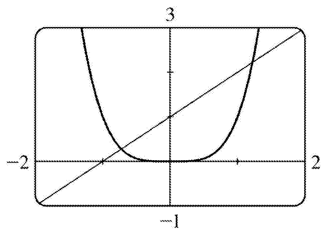
15. $\sin x=x^2$, so $f(x)=\sin x-x^2\Rightarrow f'(x)=\cos x-2x\Rightarrow x_{n+1}=x_n-\dfrac{\sin x_n-x_n^2}{\cos x_n-2x_n}$. From the figure, the positive root of $\sin x=x^2$ is near 1 . $x_1=1\Rightarrow x_2\approx 0.891396$, $x_3\approx 0.876985$, $x_4\approx 0.876726\approx x_5$. So the positive root is 0.876726 , to six decimal places.



16. $2\cos x=x^4$, so $f(x)=2\cos x-x^4\Rightarrow f'(x)=-2\sin x-4x^3\Rightarrow x_{n+1}=x_n-\dfrac{2\cos x_n-x_n^4}{-2\sin x_n-4x_n^3}$. From the figure, the positive root of $2\cos x=x^4$ is near 1.$x_1=1\Rightarrow x_2\approx 1.014184$, $x_3\approx 1.013958\approx x_4$. So the positive root is 1.013958 , to six decimal places.



17.

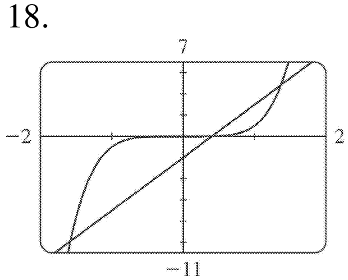


From the graph, we see that there appear to be points of intersection near $x=-0.7$ and $x=1.2$. Solving $x^4=1+x$ is the same as solving $f(x)=x^4-x-1=0$. $f(x)=x^4-x-1\Rightarrow f'(x)=4x^3-1$, so $x_{n+1}=x_n-\dfrac{x_n^4-x_n-1}{4x_n^3-1}$.

$x_1=-0.7$	$x_1=1.2$
$x_2\approx-0.725253$	$x_2\approx1.221380$
$x_3\approx-0.724493$	$x_3\approx1.220745$
$x_4\approx-0.724492\approx x_5$	$x_4\approx1.220744\approx x_5$

To six decimal places, the roots of the equation are -0.724492 and 1.220744 .

18.



From the graph, we see that reasonable first approximations are $x=0.5$ and $x=\pm 1.5$. $f(x)=x^5-5x+2\Rightarrow f'(x)=5x^4-5$, so $x_{n+1}=x_n-\dfrac{x_n^5-5x_n+2}{5x_n^4-5}$.

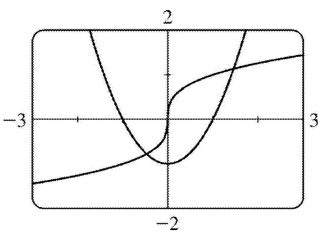
$x_1=-1.5$
$x_2\approx-1.593846$
$x_3\approx-1.582241$
$x_4\approx-1.582036\approx x_5$

$x_1=0.5$
$x_2=0.4$
$x_3\approx0.402102\approx x_4$

$x_1=1.5$
$x_2\approx1.396923$
$x_3\approx1.373078$
$x_4\approx1.371885$
$x_5\approx1.371882\approx x_6$

To six decimal places, the roots are -1.582036 , 0.402102 , and 1.371882 .

19.



From the graph, we see that there appear to be points of intersection near $x=-0.5$ and $x=1.5$. Solving $\sqrt[3]{x}=x^2-1$ is the same as solving $f(x)=\sqrt[3]{x}-x^2+1=0$. $f(x)=\sqrt[3]{x}-x^2+1\Rightarrow f'(x)=\frac{1}{3}x^{-2/3}-2x$, so

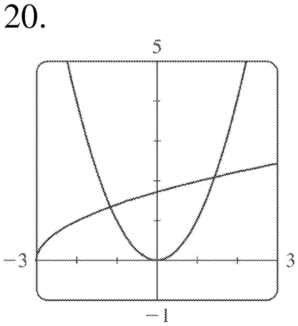
$$x_{n+1}=x_n-\frac{\sqrt[3]{x_n}-x_n^2+1}{\frac{1}{3}x_n^{-2/3}-2x_n} .$$

$x_1=-0.5$
$x_2\approx-0.471421$
$x_3\approx-0.471074\approx x_4$

$x_1=1.5$
$x_2\approx1.461653$
$x_3\approx1.461070\approx x_4$

To six decimal places, the roots are -0.471074 and 1.461070 .

20.



From the graph, we see that there appear to be points of intersection near $x=-1.2$ and $x=1.5$. Solving

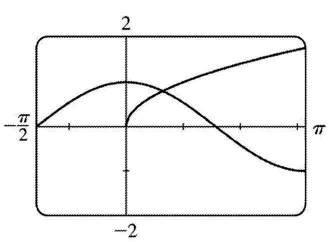
$\sqrt{x+3}=x^2$ is the same as solving $f(x)=x^2-\sqrt{x+3}=0$. $f(x)=x^2-\sqrt{x+3} \Rightarrow f'(x)=2x-\dfrac{1}{2\sqrt{x+3}}$, so

$$x_{n+1}=x_n-\frac{x_n^2-\sqrt{x_n+3}}{2x_n-1\big/\left(2\sqrt{x_n+3}\right)} .$$

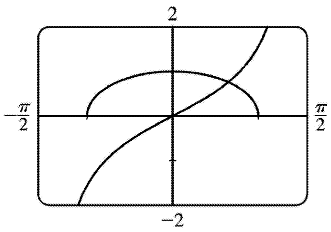
$x_1=-1.2$	$x_1=1.5$
$x_2\approx-1.164526$	$x_2\approx1.453449$
$x_3\approx-1.164035\approx x_4$	$x_3\approx1.452627\approx x_4$

To six decimal places, the roots of the equation are -1.164035 and 1.452627 .

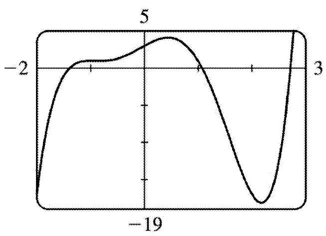
21. From the graph, there appears to be a point of intersection near $x=0.6$. Solving $\cos x=\sqrt{x}$ is the same as solving $f(x)=\cos x-\sqrt{x}=0$. $f(x)=\cos x-\sqrt{x} \Rightarrow f'(x)=-\sin x-1/\left(2\sqrt{x}\right)$, so $x_{n+1}=x_n-\dfrac{\cos x_n-\sqrt{x_n}}{-\sin x_n-1/\left(2\sqrt{x}\right)}$. Now $x_1=0.6\Rightarrow$ $x_2\approx0.641928$, $x_3\approx0.641714\approx x_4$. To six decimal places, the root of the equation is 0.641714 .



22. From the graph, there appears to be a point of intersection near $x=0.7$. Solving $\tan x=\sqrt{1-x^2}$ is the same as solving $f(x)=\tan x-\sqrt{1-x^2}=0$. $f(x)=\tan x-\sqrt{1-x^2} \Rightarrow f'(x)=\sec^2 x+x/\sqrt{1-x^2}$, so $x_{n+1}=x_n-\dfrac{\tan x_n-\sqrt{1-x_n^2}}{\sec^2 x_n+x_n/\sqrt{1-x_n^2}}$. $x_1=0.7\Rightarrow x_2\approx0.652356$, $x_3\approx0.649895$, $x_4\approx0.649889\approx x_5$. To six decimal places, the root of the equation is 0.649889 .



23.

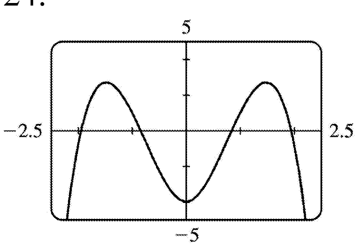


$f(x)=x^5-x^4-5x^3-x^2+4x+3 \Rightarrow f'(x)=5x^4-4x^3-15x^2-2x+4 \Rightarrow x_{n+1}=x_n-\dfrac{x_n^5-x_n^4-5x_n^3-x_n^2+4x_n+3}{5x_n^4-4x_n^3-15x_n^2-2x_n+4}$. From the graph of f , there appear to be roots near -1.4 , 1.1 , and 2.7 .

$x_1=-1.4$	$x_1=1.1$	$x_1=2.7$
$x_2\approx-1.39210970$	$x_2\approx1.07780402$	$x_2\approx2.72046250$
$x_3\approx-1.39194698$	$x_3\approx1.07739442$	$x_3\approx2.71987870$
$x_4\approx-1.39194691\approx x_5$	$x_4\approx1.07739428\approx x_5$	$x_4\approx2.71987822\approx x_5$

To eight decimal places, the roots of the equation are -1.39194691 , 1.07739428 , and 2.71987822 .

24.



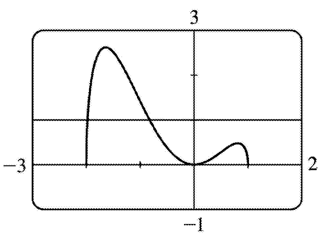
Solving $x^2\left(4-x^2\right)=\dfrac{4}{x^2+1}$ is the same as solving $f(x)=4x^2-x^4-\dfrac{4}{x^2+1}=0$. $f'(x)=8x-4x^3+\dfrac{8x}{\left(x^2+1\right)^2} \Rightarrow$ $x_{n+1}=x_n-\dfrac{4x_n^2-x_n^4-4/\left(x_n^2+1\right)}{8x_n-4x_n^3+8x_n/\left(x_n^2+1\right)^2}$. From the graph of $f(x)$, there appear to be roots near $x=\pm1.9$ and

$x=\pm 0.8$. Since f is even, we only need to find the positive roots.

$$\begin{array}{ll} x_1=0.8 & x_1=1.9 \\ x_2\approx 0.84287645 & x_2\approx 1.94689103 \\ x_3\approx 0.84310820 & x_3\approx 1.94383891 \\ x_4\approx 0.84310821\approx x_5 & x_4\approx 1.94382538\approx x_5 \end{array}$$

To eight decimal places, the roots of the equation are ± 0.84310821 and ± 1.94382538 .

25.



From the graph, $y=x^2\sqrt{2-x-x^2}$ and $y=1$ intersect twice, at $x\approx -2$ and at $x\approx -1$. $f(x)=x^2\sqrt{2-x-x^2}-1$ $\Rightarrow$

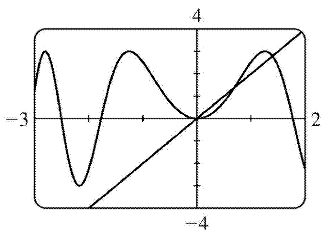
$$\begin{aligned} f'(x) &= x^2\cdot\frac{1}{2}\left(2-x-x^2\right)^{-1/2}(-1-2x)+\left(2-x-x^2\right)^{1/2}\cdot 2x \\ &= \frac{1}{2}x\left(2-x-x^2\right)^{-1/2}\left[x(-1-2x)+4\left(2-x-x^2\right)\right] \\ &= \frac{x\left(8-5x-6x^2\right)}{2\sqrt{(2+x)(1-x)}}\ , \end{aligned}$$

so $x_{n+1}=x_n-\dfrac{x^2\sqrt{2-x_n-x_n^2}-1}{\dfrac{x_n\left(8-5x_n-6x_n^2\right)}{2\sqrt{2(2+x_n)(1-x_n)}}}$. Trying $x_1=-2$ won't work because $f'(-2)$ is undefined, so we'll try $x_1=-1.95$.

$x_1=-1.95$	$x_1=-0.8$
$x_2\approx-1.98580357$	$x_2\approx-0.82674444$
$x_3\approx-1.97899778$	$x_3\approx-0.82646236$
$x_4\approx-1.97807848$	$x_4\approx-0.82646233\approx x_5$
$x_5\approx-1.97806682$	
$x_6\approx-1.97806681\approx x_7$	

To eight decimal places, the roots of the equation are -1.97806681 and -0.82646233 .

26.



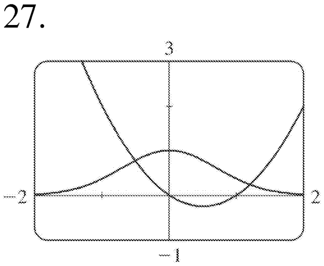
From the equations $y=3\sin(x^2)$ and $y=2x$ and the graph, we deduce that one root of the equation $3\sin(x^2)=2x$ is $x=0$. We also see that the graphs intersect at approximately $x=0.7$ and $x=1.4$.

$f(x)=3\sin(x^2)-2x \Rightarrow f'(x)=3\cos(x^2)\cdot 2x-2$, so $x_{n+1}=x_n-\dfrac{3\sin(x_n^2)-2x_n}{6x_n\cos(x_n^2)-2}$.

$x_1=0.7$	$x_1=1.4$
$x_2\approx0.69303689$	$x_2\approx1.39530295$
$x_3\approx0.69299996\approx x_4$	$x_3\approx1.39525078$
	$x_4\approx1.39525077\approx x_5$

To eight decimal places, the roots of the equation are 0.69299996 and 1.39525077 .

27.

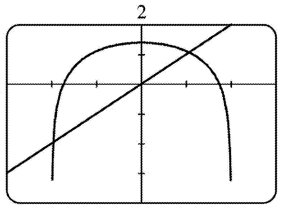


From the graph, we see that $y=e^{-x^2}$ and $y=x^2-x$ intersect twice. Good first approximations are $x=-0.5$ and $x=1.1$. $f(x)=e^{-x^2}-x^2+x \Rightarrow f'(x)=-2xe^{-x^2}-2x+1$, so $x_{n+1}=x_n-\dfrac{e^{-x_n^2}-x_n^2+x_n}{-2x_n e^{-x_n^2}-2x_n+1}$.

$x_1=-0.5$	$x_1=1.1$
$x_2\approx-0.51036446$	$x_2\approx1.20139754$
$x_3\approx-0.51031156\approx x_4$	$x_3\approx1.19844118$
	$x_4\approx1.19843871\approx x_5$

To eight decimal places, the roots of the equation are -0.51031156 and 1.19843871 .

28.



From the graph, $y=\ln\left(4-x^2\right)$ and $y=x$ intersect twice, at $x\approx-2$ and at $x\approx1$. $f(x)=\ln\left(4-x^2\right)-x \Rightarrow f'(x)=\dfrac{-2x}{4-x^2}-1$, so $x_{n+1}=x_n-\dfrac{\ln\left(4-x_n^2\right)-x_n}{\left[-2x_n/\left(4-x_n^2\right)\right]-1}$. Trying $x_1=-2$ won't work because it's not in the domain of $y=\ln\left(4-x^2\right)$. Trying $x_1=-1.9$ also fails after one iteration because the approximation x_2 is less than -2 . We try $x_1=-1.99$.

$x_1=-1.99$	$x_1=1.1$
$x_2\approx-1.97753026$	$x_2\approx1.05864851$
$x_3\approx-1.96741777$	$x_3\approx1.05800655$
$x_4\approx-1.96475281$	$x_4\approx1.05800640\approx x_5$
$x_5\approx-1.96463580$	
$x_6\approx-1.96463560\approx x_7$	

To eight decimal places, the roots of the equation are -1.96463560 and 1.05800640 .

29. **(a)** $f(x)=x^2-a \Rightarrow f'(x)=2x$, so Newton's method gives

$$x_{n+1}=x_n-\frac{x_n^2-a}{2x_n}=x_n-\frac{1}{2}x_n+\frac{a}{2x_n}=\frac{1}{2}x_n+\frac{a}{2x_n}=\frac{1}{2}\left(x_n+\frac{a}{x_n}\right).$$

(b) Using (a) with $a=1000$ and $x_1=\sqrt{900}=30$, we get $x_2\approx 31.666667$, $x_3\approx 31.622807$, and $x_4\approx 31.622777\approx x_5$. So $\sqrt{1000}\approx 31.622777$.

30. **(a)** $f(x)=\frac{1}{x}-a \Rightarrow f'(x)=-\frac{1}{x^2}$, so $x_{n+1}=x_n-\frac{1/x_n-a}{-1/x_n^2}=x_n+x_n-ax_n^2=2x_n-ax_n^2$.

(b) Using (a) with $a=1.6894$ and $x_1=\frac{1}{2}=0.5$, we get $x_2=0.5754$, $x_3\approx 0.588485$, and $x_4\approx 0.588789\approx x_5$. So $1/1.6984\approx 0.588789$.

31. $f(x)=x^3-3x+6 \Rightarrow f'(x)=3x^2-3$. If $x_1=1$, then $f'\left(x_1\right)=0$ and the tangent line used for approximating x_2 is horizontal. Attempting to find x_2 results in trying to divide by zero.

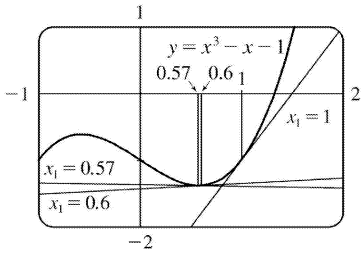
32. $x^3-x=1 \Leftrightarrow x^3-x-1=0$. $f(x)=x^3-x-1 \Rightarrow f'(x)=3x^2-1$, so $x_{n+1}=x_n-\frac{x_n^3-x_n-1}{3x_n^2-1}$.

(a) $x_1=1$, $x_2=1.5$, $x_3\approx 1.347826$, $x_4\approx 1.325200$, $x_5\approx 1.324718\approx x_6$

(b) $x_1=0.6$, $x_2=17.9$, $x_3\approx 11.946802$, $x_4\approx 7.985520$, $x_5\approx 5.356909$, $x_6\approx 3.624996$, $x_7\approx 2.505589$, $x_8\approx 1.820129$, $x_9\approx 1.461044$, $x_{10}\approx 1.339323$, $x_{11}\approx 1.324913$, $x_{12}\approx 1.324718\approx x_{13}$

(c) $x_1=0.57$, $x_2\approx -54.165455$, $x_3\approx -36.114293$, $x_4\approx -24.082094$, $x_5\approx -16.063387$, $x_6\approx -10.721483$, $x_7\approx -7.165534$, $x_8\approx -4.801704$, $x_9\approx -3.233425$, $x_{10}\approx -2.193674$, $x_{11}\approx -1.496867$, $x_{12}\approx -0.997546$, $x_{13}\approx -0.496305$, $x_{14}\approx -2.894162$, $x_{15}\approx -1.967962$, $x_{16}\approx -1.341355$, $x_{17}\approx -0.870187$, $x_{18}\approx -0.249949$, $x_{19}\approx -1.192219$, $x_{20}\approx -0.731952$, $x_{21}\approx 0.355213$, $x_{22}\approx -1.753322$, $x_{23}\approx -1.189420$, $x_{24}\approx -0.729123$, $x_{25}\approx 0.377844$, $x_{26}\approx -1.937872$, $x_{27}\approx -1.320350$, $x_{28}\approx -0.851919$, $x_{29}\approx -0.200959$, $x_{30}\approx -1.119386$, $x_{31}\approx -0.654291$,

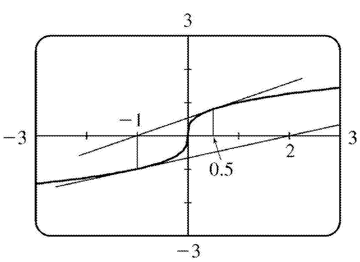
$x_{32} \approx 1.547010$, $x_{33} \approx 1.360051$, $x_{34} \approx 1.325828$, $x_{35} \approx 1.324719$, $x_{36} \approx 1.324718 \approx x_{37}$.



(d)

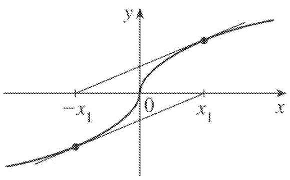
From the figure, we see that the tangent line corresponding to $x_1=1$ results in a sequence of approximations that converges quite quickly ($x_5 \approx x_6$). The tangent line corresponding to $x_1=0.6$ is close to being horizontal, so x_2 is quite far from the root. But the sequence still converges — just a little more slowly ($x_{12} \approx x_{13}$). Lastly, the tangent line corresponding to $x_1=0.57$ is very nearly horizontal, x_2 is farther away from the root, and the sequence takes more iterations to converge ($x_{36} \approx x_{37}$) .

33. For $f(x)=x^{1/3}$, $f'(x)=\frac{1}{3}x^{-2/3}$ and $x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}=x_n-\frac{x_n^{1/3}}{\frac{1}{3}x_n^{-2/3}}=x_n-3x_n=-2x_n$. Therefore, each successive approximation becomes twice as large as the previous one in absolute value, so the sequence of approximations fails to converge to the root, which is 0 . In the figure, we have $x_1=0.5$, $x_2=-2(0.5)=-1$, and $x_3=-2(-1)=2$.



34. According to Newton's Method, for $x_n>0$, $x_{n+1}=x_n-\frac{\sqrt{x_n}}{1/\left(2\sqrt{x_n}\right)}=x_n-2x_n=-x_n$ and for $x_n<0$,

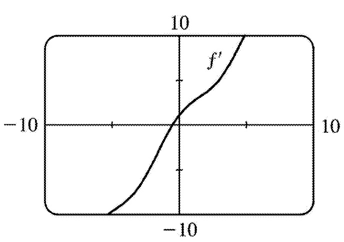
$x_{n+1}=x_n-\frac{-\sqrt{-x_n}}{1/\left(2\sqrt{-x_n}\right)}=x_n-\left[-2\left(-x_n\right)\right]=-x_n$. So we can see that after choosing any value x_1 the subsequent values will alternate between $-x_1$ and x_1 and never approach the root.



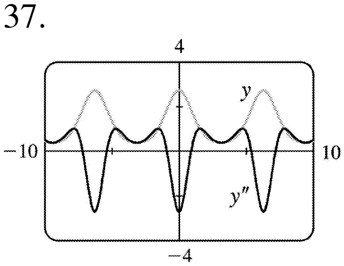
35. **(a)** $f(x)=3x^4-28x^3+6x^2+24x \Rightarrow f'(x)=12x^3-84x^2+12x+24 \Rightarrow f''(x)=36x^2-168x+12$. Now to solve $f'(x)=0$, try $x_1=\frac{1}{2} \Rightarrow x_2=x_1-\dfrac{f'(x_1)}{f''(x_1)}=\frac{2}{3} \Rightarrow x_3\approx 0.6455 \Rightarrow x_4\approx 0.6452 \Rightarrow x_5\approx 0.6452$. Now try $x_1=6 \Rightarrow x_2=7.12 \Rightarrow$
$x_3\approx 6.8353 \Rightarrow x_4\approx 6.8102 \Rightarrow x_5\approx 6.8100$. Finally try $x_1=-0.5 \Rightarrow x_2\approx -0.4571 \Rightarrow x_3\approx -0.4552 \Rightarrow$
$x_4\approx -0.4552$. Therefore, $x=-0.455$, 6.810 and 0.645 are all critical numbers correct to three decimal places.

(b) $f(-1)=13$, $f(7)=-1939$, $f(6.810)\approx -1949.07$, $f(-0.455)\approx -6.912$, $f(0.645)\approx 10.982$. Therefore, $f(6.810)\approx -1949.07$ is the absolute minimum correct to two decimal places.

36. $f(x)=x^2+\sin x \Rightarrow f'(x)=2x+\cos x$. $f'(x)$ exists for all x , so to find the minimum of f , we can examine the zeros of f' . From the graph of f' , we see that a good choice for x_1 is $x_1=-0.5$. Use $g(x)=2x+\cos x$ and $g'(x)=2-\sin x$ to obtain $x_2\approx -0.450627$, $x_3\approx -0.450184\approx x_4$. Since $f''(x)=2-\sin x>0$ for all x , $f(-0.450184)\approx -0.232466$ is the absolute minimum.

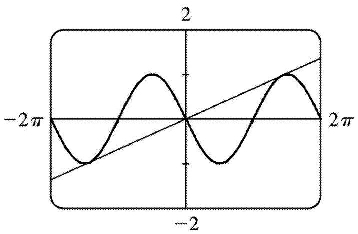


37.



From the figure, we see that $y=f(x)=e^{\cos x}$ is periodic with period 2π . To find the x –coordinates of the IP, we only need to approximate the zeros of y'' on $[0,\pi]$. $f'(x)=-e^{\cos x}\sin x\Rightarrow$ $f''(x)=e^{\cos x}\left(\sin^2 x-\cos x\right)$. Since $e^{\cos x}\neq 0$, we will use Newton's method with $g(x)=\sin^2 x-\cos x$, $g'(x)=2\sin x\cos x+\sin x$, and $x_1=1$. $x_2\approx 0.904173$, $x_3\approx 0.904557\approx x_4$. Thus, $(0.904557,1.855277)$ is the IP.

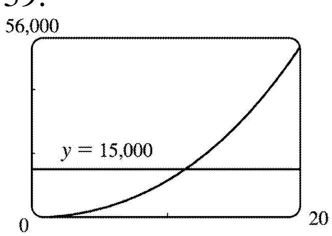
38.



$f(x)=-\sin x\Rightarrow f'(x)=-\cos x$. At $x=a$, the slope of the tangent line is $f'(a)=-\cos a$. The line through the origin and $(a,f(a))$ is $y=\dfrac{-\sin a-0}{a-0}x$. If this line is to be tangent to f at $x=a$, then its slope must equal $f'(a)$. Thus, $\dfrac{-\sin a}{a}=-\cos a\Rightarrow \tan a=a$. To solve this equation using Newton's method, let $g(x)=\tan x-x$,

$g'(x)=\sec^2 x-1$, and $x_{n+1}=x_n-\dfrac{\tan x_n-x_n}{\sec^2 x_n-1}$ with $x_1=4.5$ (estimated from the figure). $x_2\approx 4.493614$, $x_3\approx 4.493410$, $x_4\approx 4.493409\approx x_5$. Thus, the slope of the line that has the largest slope is $f'\left(x_5\right)\approx 0.217234$.

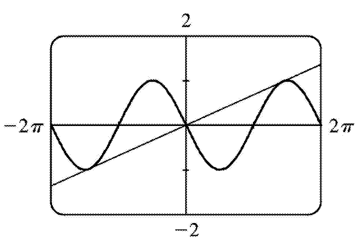
39.



The volume of the silo, in terms of its radius, is $V(r)=\pi r^2(30)+\frac{1}{2}\left(\frac{4}{3}\pi r^3\right)=30\pi r^2+\frac{2}{3}\pi r^3$. From a graph of V , we see that $V(r)=15,000$ at $r\approx 11$ ft. Now we use Newton's method to solve the equation $V(r)-15,000=0$. $\dfrac{dV}{dr}=60\pi r+2\pi r^2$, so

$r_{n+1}=r_n-\dfrac{30\pi r_n^2+\frac{2}{3}\pi r_n^3-15{,}000}{60\pi r_n+2\pi r_n^2}$. Taking $r_1=11$, we get $r_2\approx 11.2853$, $r_3\approx 11.2807\approx r_4$. So in order for the silo to hold 15 , 000 ft ^{3} of grain, its radius must be about 11.2807 ft.

40. Let the radius of the circle be r . Using $s=r\theta$, we have $5=r\theta$ and so $r=5/\theta$. From the Law of Cosines we get $4^2=r^2+r^2-2\cdot r\cdot r\cdot\cos\theta\Leftrightarrow 16=2r^2(1-\cos\theta)=2(5/\theta)^2(1-\cos\theta)$. Multiplying by θ^2 gives $16\theta^2=50(1-\cos\theta)$, so we take $f(\theta)=16\theta^2+50\cos\theta-50$ and $f'(\theta)=32\theta-50\sin\theta$. The formula for Newton's method is $\theta_{n+1}=\theta_n-\dfrac{16\theta_n^2+50\cos\theta_n-50}{32\theta_n-50\sin\theta_n}$. From the graph of f , we can use $\theta_1=2.2$, giving us $\theta_2\approx 2.2662$, $\theta_3\approx 2.2622\approx\theta_4$. So correct to four decimal places, the angle is 2.2622 radians $\approx 130^\circ$.



41. In this case, $A=18{,}000$, $R=375$, and $n=5(12)=60$. So the formula $A=\dfrac{R}{i}\left[1-(1+i)^{-n}\right]$ becomes $18{,}000=\dfrac{375}{x}\left[1-(1+x)^{-60}\right]\Leftrightarrow 48x=1-(1+x)^{-60}$ [multiply each term by $(1+x)^{60}$] $\Leftrightarrow$ $48x(1+x)^{60}-(1+x)^{60}+1=0$. Let the LHS be called $f(x)$, so that

$$\begin{aligned} f'(x) &= 48x(60)(1+x)^{59}+48(1+x)^{60}-60(1+x)^{59} \\ &= 12(1+x)^{59}\left[4x(60)+4(1+x)-5\right]=12(1+x)^{59}(244x-1) \end{aligned}$$

$x_{n+1}=x_n-\dfrac{48x_n\left(1+x_n\right)^{60}-\left(1+x_n\right)^{60}+1}{12\left(1+x_n\right)^{59}\left(244x_n-1\right)}$. An interest rate of 1% per month seems like a reasonable estimate for $x=i$. So let $x_1=1\%=0.01$, and we get $x_2\approx 0.0082202$, $x_3\approx 0.0076802$, $x_4\approx 0.0076291$, $x_5\approx 0.0076286\approx x_6$. Thus, the dealer is charging a monthly interest rate of 0.76286% (or 9.55% per year, compounded monthly).

42. **(a)** $p(x)=x^5-(2+r)x^4+(1+2r)x^3-(1-r)x^2+2(1-r)x+r-1 \Rightarrow$
$p'(x)=5x^4-4(2+r)x^3+3(1+2r)x^2-2(1-r)x+2(1-r)$. So we use

$$x_{n+1}=x_n-\frac{x_n^5-(2+r)x_n^4+(1+2r)x_n^3-(1-r)x_n^2+2(1-r)x_n+r-1}{5x_n^4-4(2+r)x_n^3+3(1+2r)x_n^2-2(1-r)x_n+2(1-r)}$$

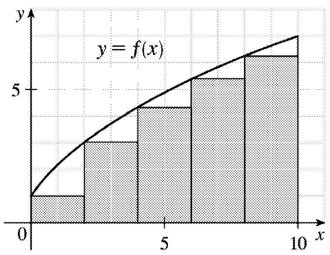
. We substitute in the value $r\approx 3.04042\times 10^{-6}$ in order to evaluate the approximations numerically. The libration point L_1 is slightly less than 1 AU from the Sun, so we take $x_1=0.95$ as our first approximation, and get $x_2\approx 0.96682$, $x_3\approx 0.97770$, $x_4\approx 0.98451$, $x_5\approx 0.98830$, $x_6\approx 0.98976$, $x_7\approx 0.98998$, $x_8\approx 0.98999\approx x_9$.
So, to five decimal places, L_1 is located 0.98999 AU from the Sun (or 0.01001 AU from Earth).
(b) In this case we use Newton's method with the function
$p(x)-2rx^2=x^5-(2+r)x^4+(1+2r)x^3-(1+r)x^2+2(1-r)x+r-1 \Rightarrow$
$\left[p(x)-2rx^2\right]'=5x^4-4(2+r)x^3+3(1+2r)x^2-2(1+r)x+2(1-r)$. So

$$x_{n+1}=x_n-\frac{x_n^5-(2+r)x_n^4+(1+2r)x_n^3-(1+r)x_n^2+2(1-r)x_n+r-1}{5x_n^4-4(2+r)x_n^3+3(1+2r)x_n^2-2(1+r)x_n+2(1-r)}$$

. Again, we substitute $r\approx 3.04042\times 10^{-6}$. L_2 is slightly more than 1 AU from the Sun and, judging from the result of part (a), probably less than 0.02 AU from Earth. So we take $x_1=1.02$ and get $x_2\approx 1.01422$, $x_3\approx 1.01118$, $x_4\approx 1.01018$, $x_5\approx 1.01008\approx x_6$. So, to five decimal places, L_2 is located 1.01008 AU from the Sun (or 0.01008 AU from Earth).

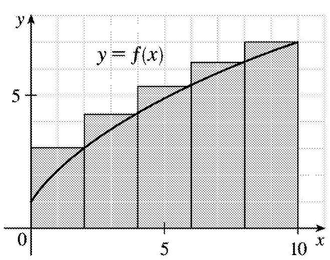
1. **(a)** Since f is *increasing* , we can obtain a *lower* estimate by using *left* endpoints. We are instructed to use five rectangles, so $n=5$.

$$L_5 = \sum_{i=1}^{5} f\left(x_{i-1}\right) \Delta x$$
$$= f\left(x_0\right)\cdot 2 + f\left(x_1\right)\cdot 2 + f\left(x_2\right)\cdot 2 + f\left(x_3\right)\cdot 2 + f\left(x_4\right)\cdot 2$$
$$= 2[f(0)+f(2)+f(4)+f(6)+f(8)]$$
$$\approx 2(1+3+4.3+5.4+6.3)=2(20)=40$$



Since f is *increasing* , we can obtain an *upper* estimate by using *right* endpoints.

$$R_5 = \sum_{i=1}^{5} f\left(x_i\right) \Delta x$$
$$= 2\left[f\left(x_1\right)+f\left(x_2\right)+f\left(x_3\right)+f\left(x_4\right)+f\left(x_5\right)\right]$$
$$= 2[f(2)+f(4)+f(6)+f(8)+f(10)]$$
$$\approx 2(3+4.3+5.4+6.3+7)=2(26)=52$$



Comparing R_5 to L_5 , we see that we have added the area of the rightmost upper rectangle, $f(10)\cdot 2$, to the sum and subtracted the area of the leftmost lower rectangle, $f(0)\cdot 2$, from the sum. aaaaa

(b)

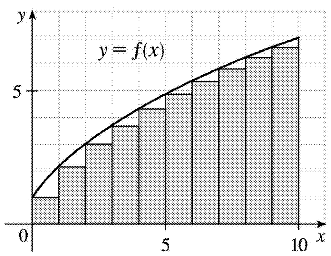
$$L_{10} = \sum_{i=1}^{10} f\left(x_{i-1}\right) \Delta x$$

$=1\left[f\left(x_0\right)+f\left(x_1\right)+\cdots+f\left(x_9\right)\right]$

$=f(0)+f(1)+\cdots+f(9)$

$\approx 1+2.1+3+3.7+4.3+4.9+5.4+5.8+6.3+6.7$

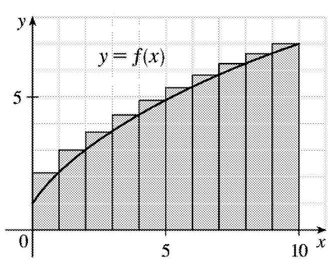
$=43.2$



$R_{10}=\sum_{i=1}^{10} f\left(x_i\right)\Delta x=f(1)+f(2)+\cdots+f(10)$

$=L_{10}+1\cdot f(10)-1\cdot f(0)$ [add rightmost upper rectangle, subtract leftmost lower rectangle]

$=43.2+7-1=49.2$



2. **(a)**

(i)

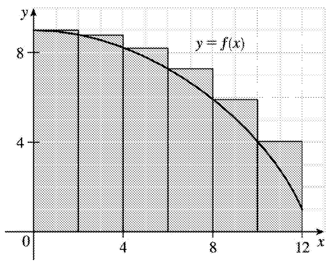
$L_6=\sum_{i=1}^{6} f\left(x_{i-1}\right)\Delta x$

$=2\left[f\left(x_0\right)+f\left(x_1\right)+f\left(x_2\right)+f\left(x_3\right)+f\left(x_4\right)+f\left(x_5\right)\right]$

$=2[f(0)+f(2)+f(4)+f(6)+f(8)+f(10)]$

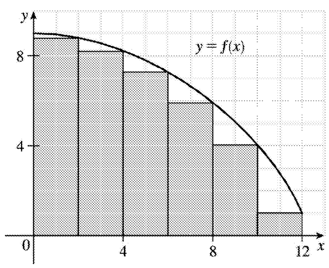
$\approx 2(9+8.8+8.2+7.3+5.9+4.1)$

$=2(43.3)=86.6$



(ii) $R_6 = L_6 + 2\cdot f(12) - 2\cdot f(0)$

$\approx 86.6 + 2(1) - 2(9) = 70.6$

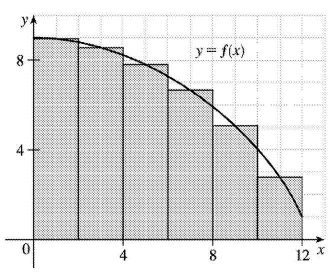


(iii) $M_6 = \sum_{i=1}^{6} f\left(x_i^*\right)\Delta x$

$= 2[f(1) + f(3) + f(5) + f(7) + f(9) + f(11)]$

$\approx 2(8.9 + 8.5 + 7.8 + 6.6 + 5.1 + 2.8)$

$= 2(39.7) = 79.4$



(b) Since f is *decreasing* , we obtain an *overestimate* by using *left* endpoints; that is, L_6 .

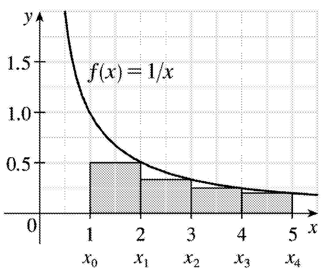
(c) Since f is *decreasing* , we obtain an *underestimate* by using *right* endpoints; that is, R_6 .

(d) M_6 gives the best estimate, since the area of each rectangle appears to be closer to the true area than the overestimates and underestimates in L_6 and R_6 .

3. **(a)**

$$
\begin{aligned}
R_4 &= \sum_{i=1}^{4} f\left(x_i\right)\Delta x \\
&= f\left(x_1\right)\cdot 1 + f\left(x_2\right)\cdot 1 + f\left(x_3\right)\cdot 1 + f\left(x_4\right)\cdot 1 \\
&= f(2)+f(3)+f(4)+f(5) \\
&= \frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}=\frac{77}{60}=1.28\overline{3}
\end{aligned}
$$

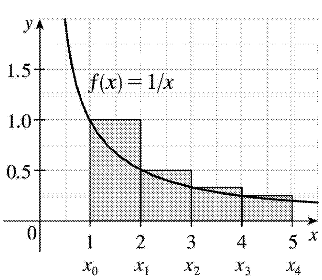
Since f is *decreasing* on $[1,5]$, an *underestimate* is obtained by using the *right* endpoint approximation, R_4 .



(b)

$$
\begin{aligned}
L_4 &= \sum_{i=1}^{4} f\left(x_{i-1}\right)\Delta x \\
&= f(1)+f(2)+f(3)+f(4) \\
&= 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}=\frac{25}{12}=2.08\overline{3}
\end{aligned}
$$

L_4 is an overestimate. Alternatively, we could just add the area of the leftmost upper rectangle and subtract the area of the rightmost lower rectangle; that is, $L_4=R_4+f(1)\cdot 1-f(5)\cdot 1$.

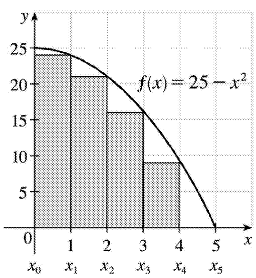


4. **(a)**

$$
R_5 = \sum_{i=1}^{5} f\left(x_i\right)\Delta x
$$

$$=f\left(x_1\right)\cdot 1+f\left(x_2\right)\cdot 1+f\left(x_3\right)\cdot 1+f\left(x_4\right)\cdot 1+f\left(x_5\right)\cdot 1$$
$$=f(1)+f(2)+f(3)+f(4)+f(5)$$
$$=24+21+16+9+0=70$$

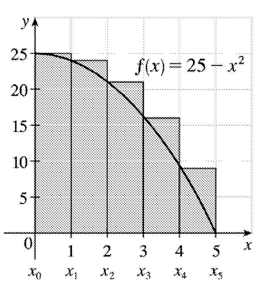
Since f is decreasing on $[0,5]$, R_5 is an underestimate.



(b)

$$L_5=\sum_{i=1}^{5} f\left(x_{i-1}\right)\Delta x$$
$$=f(0)+f(1)+f(2)+f(3)+f(4)$$
$$=25+24+21+16+9=95$$

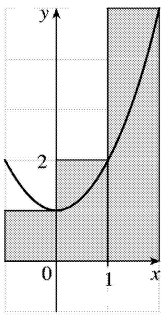
L_5 is an overestimate.

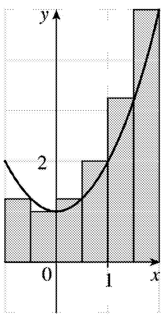


5. **(a)** $f(x)=1+x^2$ and $\Delta x=\dfrac{2-(-1)}{3}=1\Rightarrow R_3=1\cdot f(0)+1\cdot f(1)+1\cdot f(2)=1\cdot 1+1\cdot 2+1\cdot 5=8$.

$\Delta x=\dfrac{2-(-1)}{6}=0.5\Rightarrow$

$$R_6=0.5[f(-0.5)+f(0)+f(0.5)+f(1)+f(1.5)+f(2)]$$
$$=0.5(1.25+1+1.25+2+3.25+5)$$
$$=0.5(13.75)=6.875$$

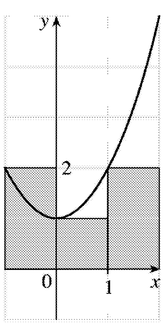


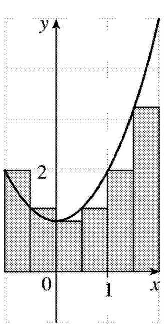


(b)

$L_3 = 1\cdot f(-1) + 1\cdot f(0) + 1\cdot f(1) = 1\cdot 2 + 1\cdot 1 + 1\cdot 2 = 5$

$$\begin{aligned} L_6 &= 0.5[f(-1)+f(-0.5)+f(0)+f(0.5)+f(1)+f(1.5)] \\ &= 0.5(2+1.25+1+1.25+2+3.25) \\ &= 0.5(10.75) = 5.375 \end{aligned}$$

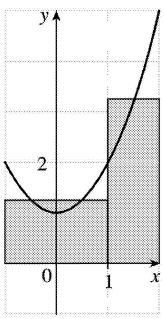


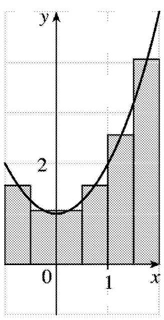


(c)

$$M_3 = 1\cdot f(-0.5) + 1\cdot f(0.5) + 1\cdot f(1.5)$$
$$= 1\cdot 1.25 + 1\cdot 1.25 + 1\cdot 3.25 = 5.75$$

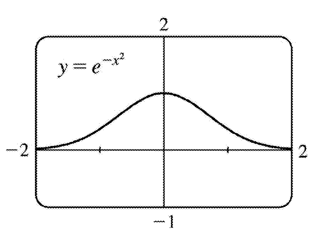
$$M_6 = 0.5[f(-0.75) + f(-0.25) + f(0.25)]$$
$$+ f(0.75) + f(1.25) + f(1.75)]$$
$$= 0.5(1.5625 + 1.0625 + 1.0625 + 1.5625 + 2.5625 + 4.0625)$$
$$= 0.5(11.875) = 5.9375$$





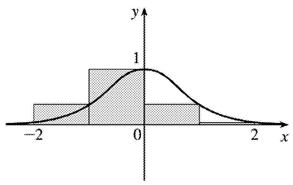
(d) M_6 appears to be the best estimate.

6. **(a)**

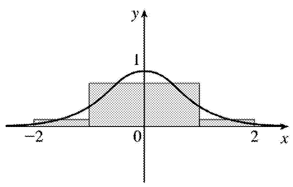


(b) $f(x) = e^{-x^2}$ and $\Delta x = \dfrac{2-(-2)}{4} = 1 \Rightarrow$

(i) $R_4 = 1\cdot f(-1) + 1\cdot f(0) + 1\cdot f(1) + 1\cdot f(2) = e^{-1} + 1 + e^{-1} + e^{-4} \approx 1.754$

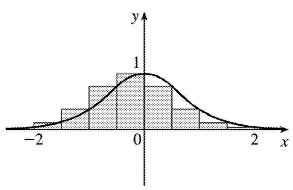


(ii) $M_4 = 1 \cdot f(-1.5) + 1 \cdot f(-0.5) + 1 \cdot f(0.5) + 1 \cdot f(1.5) = e^{-2.25} + e^{-0.25} + e^{-0.25} + e^{-2.25} \approx 1.768$

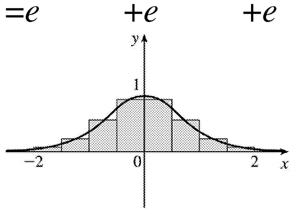


(c)

(i) $R_8 = 0.5\,[f(-1.5) + f(-1) + f(-0.5) + f(0) \,| + f(0.5) + f(1) + f(1.5) + f(2)] = e^{-2.25} + e^{-1} + e^{-0.25} + 1$
$+ e^{-0.25} + e^{-1} + e^{-2.25} + e^{-4} \approx 1.761$

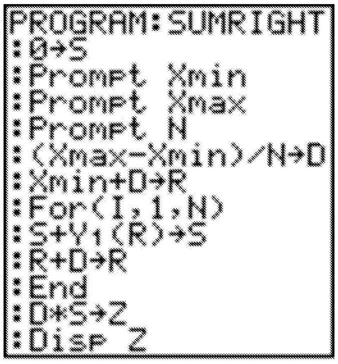


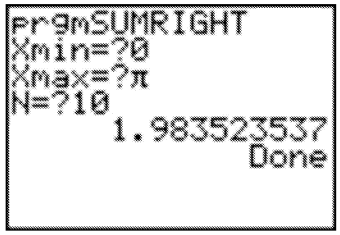
(ii) Due to the symmetry of the figure, we see that $M_8 = (0.5)(2)[f(0.25) + f(0.75) + f(1.25) + f(1.75)]$
$= e^{-0.0625} + e^{-0.5625} + e^{-1.5625} + e^{3.0625} \approx 1.766$



7. Here is one possible algorithm (ordered sequence of operations) for calculating the sums:
(a) Let SUM =0 , X_MIN =0 , X_MAX $=\pi$, N =10 (or 30 or 50 , depending on which sum we are calculating), DELTA_X =(XMAX–XMIN) / N, and RIGHT_ENDPOINT = X_MIN + DELTA_X.
(b) Repeat steps 2a, 2b in sequence until RIGHT_ENDPOINT > X_MAX.
(c) Add sin (RIGHTENDPOINT) to SUM.
(d) Add DELTA_X to RIGHT_ENDPOINT.
At the end of this procedure, (DELTAX)$\cdot$(SUM) is equal to the answer we are looking for. We find that $R_{10} = \frac{\pi}{10} \sum_{i=1}^{10} \sin\left(\frac{i\pi}{10}\right) \approx 1.9835$, $R_{30} = \frac{\pi}{30} \sum_{i=1}^{30} \sin\left(\frac{i\pi}{30}\right) \approx 1.9982$, and $R_{50} = \frac{\pi}{50} \sum_{i=1}^{50} \sin\left(\frac{i\pi}{50}\right) \approx 1.9993$. It appears that the exact area is 2 .
Shown below is program SUMRIGHT and its output from a TI–83 Plus calculator. To generalize the

program, we have input (rather than assigned) values for Xmin, Xmax, and N. Also, the function, $\sin x$, is assigned to Y_1, enabling us to evaluate any right sum merely by changing Y_1 and running the program.



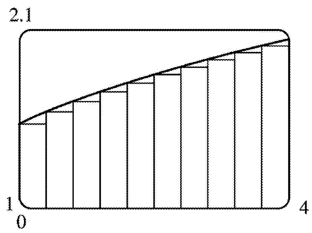


8. We can use the algorithm from Exercise 7 with X_MIN =1 , X_MAX =2 , and $1/(\text{RIGHTENDPOINT})^2$ instead of $\sin(\text{RIGHTENDPOINT})$ in step 2a. We find that $R_{10}=\frac{1}{10}\sum_{i=1}^{10}\frac{1}{(1+i/10)^2}\approx 0.4640$, $R_{30}=\frac{1}{30}\sum_{i=1}^{30}\frac{1}{(1+i/30)^2}\approx 0.4877$, and $R_{50}=\frac{1}{50}\sum_{i=1}^{50}\frac{1}{(1+i/50)^2}\approx 0.4926$. It appears that the exact area is $\frac{1}{2}$.

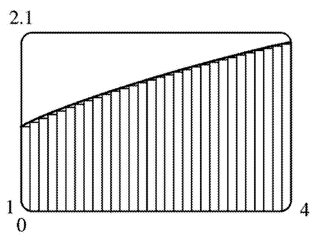
9. In Maple, we have to perform a number of steps before getting a numerical answer. After loading the student package we use the command sum: =leftsum(x^(1 / 2) , x=1..4,10 [or 30, or 50]) ; which gives us the expression in summation notation. To get a numerical approximation to the sum, we use evalf(left_sum);. Mathematica does not have a special command for these sums, so we must type them in manually. For example, the first left sum is given by (3 / 10)*Sum[Sqrt[1 + 3 (i − 1) / 10] , {i, 1, 10}] ,], and we use the N command on the resulting output to get a numerical approximation.
In Derive, we use the LEFT_RIEMANN command to get the left sums, but must define the right sums ourselves. (We can define a new function using LEFT_RIEMANN with k ranging from 1 to n instead of from 0 to $n-1$.)

(a) With $f(x)=\sqrt{x}$, $1\le x\le 4$, the left sums are of the form $L_n=\frac{3}{n}\sum_{i=1}^{n}\sqrt{1+\frac{3(i-1)}{n}}$. Specifically, $L_{10}\approx 4.5148$, $L_{30}\approx 4.6165$, and $L_{50}\approx 4.6366$. The right sums are of the form $R_n=\frac{3}{n}\sum_{i=1}^{n}\sqrt{1+\frac{3i}{n}}$. Specifically, $R_{10}\approx 4.8148$, $R_{30}\approx 4.7165$, and $R_{50}\approx 4.6966$.

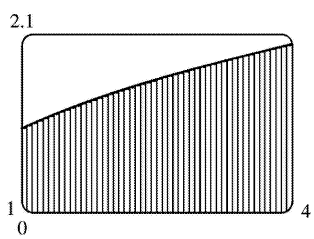
(b) In Maple, we use the leftbox and rightbox commands (with the same arguments as leftsum and rightsum above) to generate the graphs.



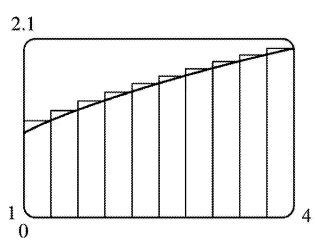
left endpoints, n=10



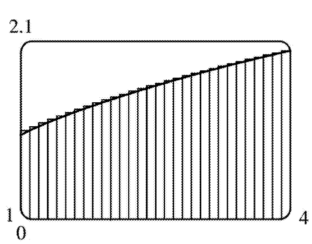
left endpoints, n=30



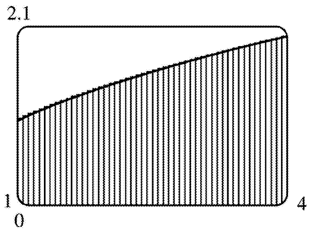
left endpoints, n=50



right endpoints, n=10



right endpoints, n=30



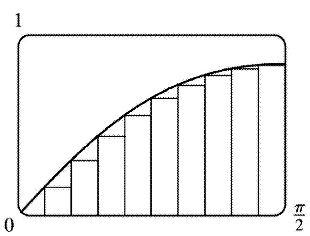
right endpoints, n=50

(c) We know that since $\sqrt{x}$ is an increasing function on $(1,4)$, all of the left sums are smaller than the actual area, and all of the right sums are larger than the actual area. Since the left sum with n=50 is about 4.637>4.6 and the right sum with n=50 is about 4.697<4.7 , we conclude that $4.6<L_{50}<$ exact area $<R_{50}<4.7$, so the exact area is between 4.6 and 4.7 .

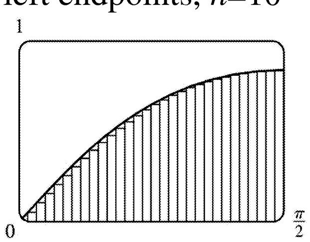
10.

(a) With $f(x)=\sin(\sin x)$, $0\le x\le \frac{\pi}{2}$, the left sums are of the form $L_n=\frac{\pi}{2n}\sum_{i=1}^{n}\sin\left(\sin\frac{\pi(i-1)}{2n}\right)$. In particular, $L_{10}\approx 0.8251$, $L_{30}\approx 0.8710$, and $L_{50}\approx 0.8799$. The right sums are of the form $R_n=\frac{\pi}{2n}\sum_{i=1}^{n}\sin\left(\sin\frac{\pi i}{2n}\right)$. In particular, $R_{10}\approx 0.9573$, $R_{30}\approx 0.9150$, and $R_{50}\approx 0.9064$.

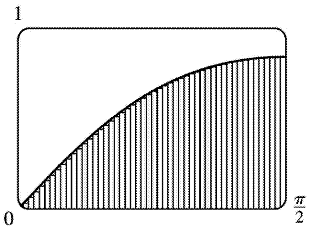
(b) In Maple, we use the leftbox and rightbox commands (with the same arguments as leftsum and rightsum above) to generate the graphs.

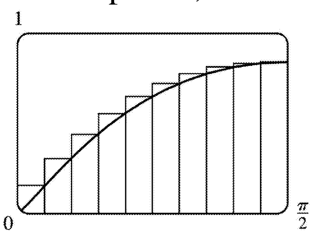


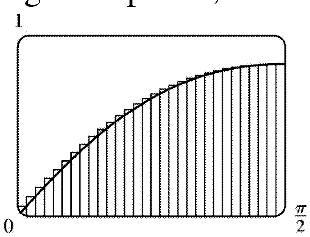
left endpoints, n=10

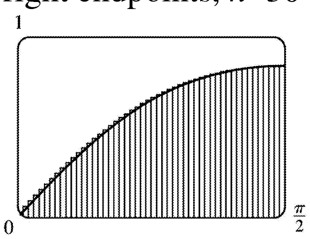


left endpoints, n=30


left endpoints, n=50


right endpoints, n=10


right endpoints, n=30


righnt endpoints, n=10

(c) We know that since sin (sin x) is an increasing function on $\left(0, \frac{\pi}{2}\right)$, all of the left sums are smaller than the actual area, and all of the right sums are larger than the actual area. Since the left sum with n=50 is about $0.8799>0.87$ and the right sum with n=50 is about $0.9064<0.91$, we conclude that $0.87<L_{50}<$ exact area $<R_{50}<0.91$, so the exact area is between 0.87 and 0.91 .

11. Since v is an increasing function, L_6 will give us a lower estimate and R_6 will give us an upper estimate.

$$L_6 = (0\text{ft/s})\,(0.5\text{s})+(6.2)\,(0.5)+(10.8)\,(0.5)+(14.9)\,(0.5)+(18.1)\,(0.5)+(19.4)\,(0.5)$$
$$=0.5\,(69.4)=34.7\text{ ft}$$

$$R_6=0.5\,(6.2+10.8+14.9+18.1+19.4+20.2)=0.5\,(89.6)=44.8\text{ ft}$$

12.
(a)

$$d \approx L_5 = (30 \text{ ft / s })(12 \text{ s }) + 28 \cdot 12 + 25 \cdot 12 + 22 \cdot 12 + 24 \cdot 12$$
$$= (30+28+25+22+24) \cdot 12 = 129 \cdot 12 = 1548 \text{ ft}$$

(b) $d \approx R_5 = (28+25+22+24+27) \cdot 12 = 126 \cdot 12 = 1512 \text{ ft}$

(c) The estimates are neither lower nor upper estimates since v is neither an increasing nor a decreasing function of t .

13. Lower estimate for oil leakage: $R_5 = (7.6+6.8+6.2+5.7+5.3)(2) = (31.6)(2) = 63.2$ L.
Upper estimate for oil leakage: $L_5 = (8.7+7.6+6.8+6.2+5.7)(2) = (35)(2) = 70$ L.

14. We can find an upper estimate by using the final velocity for each time interval. Thus, the distance d traveled after 62 seconds can be approximated by

$$d = \sum_{i=1}^{6} v\left(t_i\right) \Delta t_i = (185\text{ft/s})\,(10\text{s}) + 319 \cdot 5 + 447 \cdot 5 + 742 \cdot 12 + 1325 \cdot 27 + 1445 \cdot 3 = 54\text{ , }694 \text{ ft}$$

15. For a decreasing function, using left endpoints gives us an overestimate and using right endpoints results in an underestimate. We will use M_6 to get an estimate. $\Delta t = 1$, so

$$M_6 = 1[v(0.5)+v(1.5)+v(2.5)+v(3.5)+v(4.5)+v(5.5)]$$
$$\approx 55+40+28+18+10+4 = 155\text{ft}$$

For a very rough check on the above calculation, we can draw a line from $(0{,}70)$ to $(6{,}0)$ and calculate the area of the triangle: $\frac{1}{2}(70)(6) = 210$. This is clearly an overestimate, so our midpoint estimate of 155 is reasonable.

16. For an increasing function, using left endpoints gives us an underestimate and using right endpoints results in an overestimate. We will use M_6 to get an estimate. $\Delta t = \frac{30-0}{6} = 5 \text{ s} = \frac{5}{3600}$ h $= \frac{1}{720}$ h.

$$M_6 = \frac{1}{720}[v(2.5)+v(7.5)+v(12.5)+v(17.5)+v(22.5)+v(27.5)]$$
$$= \frac{1}{720}(31.25+66+88+103.5+113.75+119.25) = \frac{1}{720}(521.75) \approx 0.725\text{km}$$

For a very rough check on the above calculation, we can draw a line from $(0{,}0)$ to $(30{,}120)$ and calculate the area
of the triangle:

$\frac{1}{2}(30)(120)=1800$. Divide by 3600 to get 0.5 , which is clearly an underestimate, making our midpoint estimate of 0.725 seem reasonable. Of course, answers will vary due to different readings of the graph.

17. $f(x)=\sqrt[4]{x}$, $1\le x\le 16$. $\Delta x=(16-1)/n=15/n$ and $x_i=1+i\,\Delta x=1+15i/n$.

$A=\lim_{n\to\infty} R_n=\lim_{n\to\infty}\sum_{i=1}^{n} f\left(x_i\right)\Delta x=\lim_{n\to\infty}\sum_{i=1}^{n}\sqrt[4]{1+\frac{15i}{n}}\cdot\frac{15}{n}$.

18. $f(x)=\frac{\ln x}{x}$, $3\le x\le 10$. $\Delta x=(10-3)/n=7/n$ and $x_i=3+i\,\Delta x=3+7i/n$.

$A=\lim_{n\to\infty} R_n=\lim_{n\to\infty}\sum_{i=1}^{n} f\left(x_i\right)\Delta x=\lim_{n\to\infty}\sum_{i=1}^{n}\frac{\ln(3+7i/n)}{3+7i/n}\cdot\frac{7}{n}$.

19. $f(x)=x\cos x$, $0\le x\le\frac{\pi}{2}$. $\Delta x=(\frac{\pi}{2}-0)/n=\frac{\pi}{2}/n$ and $x_i=0+i\,\Delta x=\frac{\pi}{2}\,i/n$.

$A=\lim_{n\to\infty} R_n=\lim_{n\to\infty}\sum_{i=1}^{n} f\left(x_i\right)\Delta x=\lim_{n\to\infty}\sum_{i=1}^{n}\frac{i\pi}{2n}\cos\left(\frac{i\pi}{2n}\right)\cdot\frac{\pi}{2n}$.

20. $\lim_{n\to\infty}\sum_{i=1}^{n}\frac{2}{n}\left(5+\frac{2i}{n}\right)^{10}$ can be interpreted as the area of the region lying under the graph of $y=(5+x)^{10}$ on the interval $[0,2]$, since for $y=(5+x)^{10}$ on $[0,2]$ with $\Delta x=\frac{2-0}{n}=\frac{2}{n}$, $x_i=0+i\,\Delta x=\frac{2i}{n}$, and $x_i^*=x_i$, the expression for the area is $A=\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x=\lim_{n\to\infty}\sum_{i=1}^{n}\left(5+\frac{2i}{n}\right)^{10}\frac{2}{n}$. Note that the answer is not unique. We could use $y=x^{10}$ on $[5,7]$ or, in general, $y=((5-n)+x)^{10}$ on $[n,n+2]$.

21. $\lim_{n\to\infty}\sum_{i=1}^{n}\frac{\pi}{4n}\tan\frac{i\pi}{4n}$ can be interpreted as the area of the region lying under the graph of $y=\tan x$ on the interval $\left[0,\frac{\pi}{4}\right]$, since for $y=\tan x$ on $\left[0,\frac{\pi}{4}\right]$ with $\Delta x=\frac{\pi/4-0}{n}=\frac{\pi}{4n}$, $x_i=0+i\,\Delta x=\frac{i\pi}{4n}$, and $x_i^*=x_i$, the expression for the area is $A=\lim_{n\to\infty}\sum_{i=1}^{n} f\left(x_i^*\right)\Delta x=\lim_{n\to\infty}\sum_{i=1}^{n}\tan\left(\frac{i\pi}{4n}\right)\frac{\pi}{4n}$. Note that this answer is not unique, since the expression for the area is the same for the function $y=\tan(x-k\pi)$ on the interval $\left[k\pi,k\pi+\frac{\pi}{4}\right]$, where k is any integer.

22. **(a)**

$\Delta x=\frac{1-0}{n}=\frac{1}{n}$ and $x_i=0+i\,\Delta x=\frac{i}{n}$. $A=\lim_{n\to\infty} R_n=\lim_{n\to\infty} \sum_{i=1}^{n} f\left(x_i\right)\Delta x=\lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{i}{n}\right)^3\cdot\frac{1}{n}$.

(b) $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{i^3}{n^3}\cdot\frac{1}{n}=\lim_{n\to\infty} \frac{1}{n^4}\sum_{i=1}^{n} i^3=\lim_{n\to\infty} \frac{1}{n^4}\left[\frac{n(n+1)}{2}\right]^2=\lim_{n\to\infty} \frac{(n+1)^2}{4n^2}=\frac{1}{4}\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^2=\frac{1}{4}$

23. **(a)** $y=f(x)=x^5$. $\Delta x=\frac{2-0}{n}=\frac{2}{n}$ and $x_i=0+i\,\Delta x=\frac{2i}{n}$.

$A=\lim_{n\to\infty} R_n=\lim_{n\to\infty} \sum_{i=1}^{n} f\left(x_i\right)\Delta x=\lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{2i}{n}\right)^5\cdot\frac{2}{n}=\lim_{n\to\infty} \sum_{i=1}^{n} \frac{32i^5}{n^5}\cdot\frac{2}{n}=\lim_{n\to\infty} \frac{64}{n^6}\sum_{i=1}^{n} i^5$.

(b) $\sum_{i=1}^{n} i^5=\frac{n^2(n+1)^2\left(2n^2+2n-1\right)}{12}$

(c) $\lim_{n\to\infty} \frac{64}{n^6}\cdot\frac{n^2(n+1)^2\left(2n^2+2n-1\right)}{12}=\frac{64}{12}\lim_{n\to\infty} \frac{\left(n^2+2n+1\right)\left(2n^2+2n-1\right)}{n^2\cdot n^2}$

$=\frac{16}{3}\lim_{n\to\infty}\left(1+\frac{2}{n}+\frac{1}{n^2}\right)\left(2+\frac{2}{n}-\frac{1}{n^2}\right)=\frac{16}{3}\cdot 1\cdot 2=\frac{32}{3}$

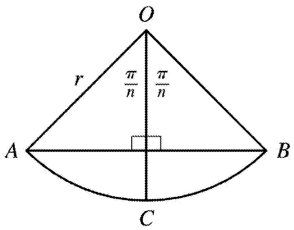
24. From Example 3(a), we have $A=\lim_{n\to\infty} \frac{2}{n}\sum_{i=1}^{n} e^{-2i/n}$. Using a CAS, $\sum_{i=1}^{n} e^{-2i/n}=\frac{e^{-2}\left(e^2-1\right)}{e^{2/n}-1}$ and $\lim_{n\to\infty} \frac{2}{n}\cdot\frac{e^{-2}\left(e^2-1\right)}{e^{2/n}-1}=e^{-2}\left(e^2-1\right)\approx 0.8647$, whereas the estimate from Example 3(b) using M_{10} was 0.8632 .

25. $y=f(x)=\cos x$. $\Delta x=\frac{b-0}{n}=\frac{b}{n}$ and $x_i=0+i\,\Delta x=\frac{bi}{n}$.

$$A=\lim_{n\to\infty} R_n=\lim_{n\to\infty} \sum_{i=1}^{n} f\left(x_i\right)\Delta x=\lim_{n\to\infty} \sum_{i=1}^{n} \cos\left(\frac{bi}{n}\right)\cdot\frac{b}{n}=\lim_{n\to\infty}\left[\frac{b\sin\left(b\left(\frac{1}{2n}+1\right)\right)}{2n\sin\left(\frac{b}{2n}\right)}-\frac{b}{2n}\right]=\sin b$$

If $b=\frac{\pi}{2}$, then $A=\sin\frac{\pi}{2}=1$.

26. **(a)**



The diagram shows one of the n congruent triangles, $\triangle AOB$, with central angle $2\pi/n$. O is the center of the circle and AB is one of the sides of the polygon. Radius OC is drawn so as to bisect $\angle AOB$. It follows that OC intersects AB at right angles and bisects AB. Thus, $\triangle AOB$ is divided into two right triangles with legs of length $\frac{1}{2}(AB) = r\sin(\pi/n)$ and $r\cos(\pi/n)$.

$\triangle AOB$ has area $2\cdot\frac{1}{2}[r\sin(\pi/n)][r\cos(\pi/n)] = r^2\sin(\pi/n)\cos(\pi/n) = \frac{1}{2}r^2\sin(2\pi/n)$, so $A_n = n\cdot$ area $(\triangle AOB) = \frac{1}{2}nr^2\sin(2\pi/n)$.

(b) To use Equation 3.4.2, $\lim\limits_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$, we need to have the same expression in the denominator as we have in the argument of the sine function — in this case, $2\pi/n$.

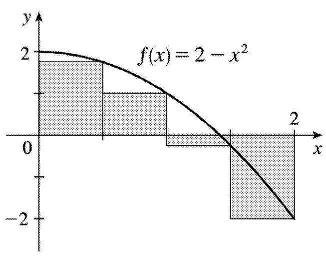
$\lim\limits_{n\to\infty} A_n = \lim\limits_{n\to\infty}\frac{1}{2}nr^2\sin(2\pi/n) = \lim\limits_{n\to\infty}\frac{1}{2}nr^2\frac{\sin(2\pi/n)}{2\pi/n}\cdot\frac{2\pi}{n} = \lim\limits_{n\to\infty}\frac{\sin(2\pi/n)}{2\pi/n}\pi r^2$. Let $\theta = \frac{2\pi}{n}$.

Then as $n\to\infty$, $\theta\to 0$, so $\lim\limits_{n\to\infty}\frac{\sin(2\pi/n)}{2\pi/n}\pi r^2 = \lim\limits_{\theta\to 0}\frac{\sin\theta}{\theta}\pi r^2 = (1)\pi r^2 = \pi r^2$.

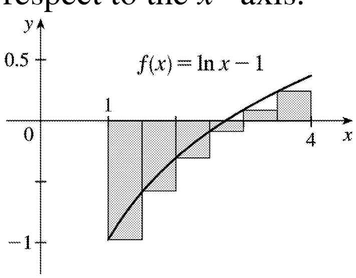
1.

$$R_4 = \sum_{i=1}^{4} f\left(x_i\right) \Delta x$$
$$= 0.5[f(0.5)+f(1)+f(1.5)+f(2)]$$
$$= 0.5[1.75+1+(-0.25)+(-2)]$$
$$= 0.5(0.5) = 0.25$$

The Riemann sum represents the sum of the areas of the two rectangles above the x –axis minus the sum of the areas of the two rectangles below the x –axis; that is, the *net area* of the rectangles with respect to the x –axis.



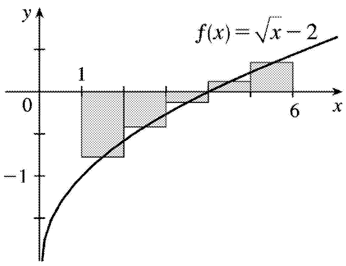
2. $L_6 = \sum_{i=1}^{6} f\left(x_{i-1}\right) \Delta x = 0.5[f(1)+f(1.5)+f(2)+f(2.5)+f(3)+f(3.5)]$
$\approx 0.5(-1-0.5945349-0.3068528-0.0837093+0.0986123+0.2527630) = 0.5(-1.6337217) \approx -0.816861$
The Riemann sum represents the sum of the areas of the two rectangles above the x –axis minus the sum of the areas of the four rectangles below the x –axis; that is, the *net area* of the rectangles with respect to the x –axis.



3.

$$M_5 = \sum_{i=1}^{5} f\left(\bar{x}_i\right) \Delta x$$
$$= 1[f(1.5)+f(2.5)+f(3.5)$$
$$+f(4.5)+f(5.5)]$$
$$\approx -0.856759$$

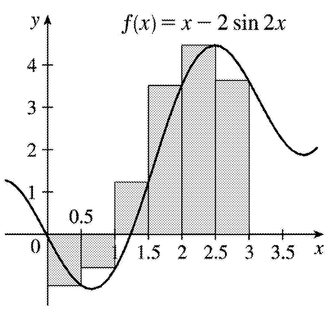
The Riemann sum represents the sum of the areas of the two rectangles above the x –axis minus the sum of the areas of the three rectangles below the x –axis.



4. **(a)**

$$R_6 = \sum_{i=1}^{6} f\left(x_i\right) \Delta x$$
$$= 0.5[f(0.5) + f(1) + f(1.5) + f(2)$$
$$+ f(2.5) + f(3)]$$
$$\approx 5.353254$$

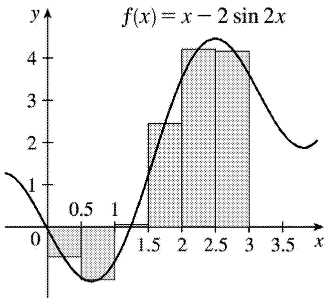
The Riemann sum represents the sum of the areas of the four rectangles above the x –axis minus the sum of the areas of the two rectangles below the x –axis.



(b)

$$M_6 = \sum_{i=1}^{6} f\left(\bar{x}_i\right) \Delta x$$
$$= 0.5[f(0.25) + f(0.75) + f(1.25) + f(1.75)$$
$$+ f(2.25) + f(2.75)]$$
$$\approx 4.458461$$

The Riemann sum represents the sum of the areas of the four rectangles above the x –axis minus the sum of the areas of the two rectangles below the x –axis.



5. $\Delta x = (b-a)/n = (8-0)/4 = 8/4 = 2$.

(a) Using the right endpoints to approximate $\int_0^8 f(x)\,dx$, we have

$\sum_{i=1}^{4} f\left(x_i\right)\Delta x = 2[f(2)+f(4)+f(6)+f(8)] \approx 2[1+2+(-2)+1] = 4$.

(b) Using the left endpoints to approximate $\int_0^8 f(x)\,dx$, we have

$\sum_{i=1}^{4} f\left(x_{i-1}\right)\Delta x = 2[f(0)+f(2)+f(4)+f(6)] \approx 2[2+1+2+(-2)] = 6$.

(c) Using the midpoint of each subinterval to approximate $\int_0^8 f(x)\,dx$, we have

$\sum_{i=1}^{4} f\left(\bar{x}_i\right)\Delta x = 2[f(1)+f(3)+f(5)+f(7)] \approx 2[3+2+1+(-1)] = 10$.

6. **(a)** Using the right endpoints to approximate $\int_{-3}^{3} g(x)\,dx$, we have

$$\begin{aligned}\sum_{i=1}^{6} g\left(x_i\right)\Delta x &= 1[g(-2)+g(-1)+g(0)+g(1)+g(2)+g(3)] \\ &\approx 1-0.5-1.5-1.5-0.5+2.5 = -0.5\end{aligned}$$

(b) Using the left endpoints to approximate $\int_{-3}^{3} g(x)\,dx$, we have

$$\begin{aligned}\sum_{i=1}^{6} g\left(x_{i-1}\right)\Delta x &= 1[g(-3)+g(-2)+g(-1)+g(0)+g(1)+g(2)] \\ &\approx 2+1-0.5-1.5-1.5-0.5 = -1\end{aligned}$$

(c) Using the midpoint of each subinterval to approximate $\int_{-3}^{3} g(x)\,dx$, we have

$$\begin{aligned}\sum_{i=1}^{6} g\left(\bar{x}_i\right)\Delta x &= 1[g(-2.5)+g(-1.5)+g(-0.5)+g(0.5)+g(1.5)+g(2.5)] \\ &\approx 1.5+0-1-1.75-1+0.5 = -1.75\end{aligned}$$

7. Since f is increasing, $L_5 \le \int_0^{25} f(x)\,dx \le R_5$.

Lower estimate $= L_5=\sum_{i=1}^{5} f\left(x_{i-1}\right)\Delta x=5[f(0)+f(5)+f(10)+f(15)+f(20)]$

$= 5(-42-37-25-6+15)=5(-95)=-475$

Upper estimate $= R_5=\sum_{i=1}^{5} f\left(x_i\right)\Delta x=5[f(5)+f(10)+f(15)+f(20)+f(25)]$

$= 5(-37-25-6+15+36)=5(-17)=-85$

8. **(a)** Using the right endpoints to approximate $\int_0^6 f(x)\,dx$, we have

$\sum_{i=1}^{3} f\left(x_i\right)\Delta x=2[f(2)+f(4)+f(6)]=2(8.3+2.3-10.5)=0.2$

(b) Using the left endpoints to approximate $\int_0^6 f(x)\,dx$, we have

$\sum_{i=1}^{3} f\left(x_{i-1}\right)\Delta x=2[f(0)+f(2)+f(4)]=2(9.3+8.3+2.3)=39.8$

(c) Using the midpoint of each interval to approximate $\int_0^6 f(x)\,dx$, we have

$\sum_{i=1}^{3} f\left(\bar{x}_i\right)\Delta x=2\,[f(1)+f(3)+f(5)]=2(9.0+6.5-7.6)=15.8$.

The estimate using the right endpoints must be less than $\int_0^6 f(x)\,dx$, since if we take x_i^* to be the right endpoint x_i of each interval, then $f\left(x_i\right)\leq f(x)$ for all x on $\left[x_{i-1},x_i\right]$, which implies that $f\left(x_i\right)\Delta x\leq\int_{x_{i-1}}^{x_i} f(x)\,dx$, and so the sum $\sum_{i=1}^{3}\left[f\left(x_i\right)\Delta x\right]\leq\sum_{i=1}^{3}\left[\int_{x_{i-1}}^{x_i} f(x)\,dx\right]=\int_0^6 f(x)\,dx$.

Similarly, if we take x_i^* to be the left endpoint x_{i-1} of each interval, then $f\left(x_{i-1}\right)\geq f(x)$ for all x on $\left[x_{i-1},x_i\right]$, and so $\sum_{i=1}^{3}\left[f\left(x_{i-1}\right)\Delta x\right]\geq\int_0^6 f(x)\,dx$. We cannot say anything about the midpoint estimate.

9. $\Delta x=(10-2)/4=2$, so the endpoints are 2 , 4 , 6 , 8 , and 10 , and the midpoints are 3 , 5 , 7 , and 9 . The Midpoint Rule gives

$\int_2^{10}\sqrt{x^3+1}\,dx\approx\sum_{i=1}^{4} f\left(\bar{x}_i\right)\Delta x=2\left(\sqrt{3^3+1}+\sqrt{5^3+1}+\sqrt{7^3+1}+\sqrt{9^3+1}\right)\approx 124.1644$.

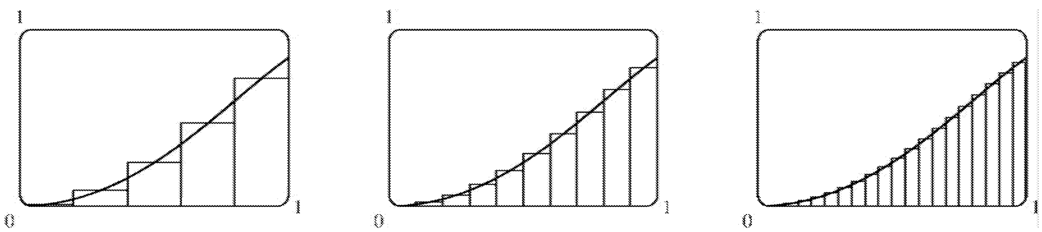
10. $\Delta x=(\pi-0)/6=\frac{\pi}{6}$, so the endpoints are 0 , $\frac{\pi}{6}$, $\frac{2\pi}{6}$, $\frac{3\pi}{6}$, $\frac{4\pi}{6}$, $\frac{5\pi}{6}$, and $\frac{6\pi}{6}$, and the midpoints are $\frac{\pi}{12}$, $\frac{3\pi}{12}$, $\frac{5\pi}{12}$, $\frac{7\pi}{12}$, $\frac{9\pi}{12}$, and $\frac{11\pi}{12}$. The Midpoint Rule gives

$\int_0^{\pi}\sec\,(x/3)\,dx\approx\sum_{i=1}^{6} f\left(\bar{x}_i\right)\Delta x=\frac{\pi}{6}\left(\sec\,\frac{\pi}{36}+\sec\,\frac{3\pi}{36}+\sec\,\frac{5\pi}{36}+\sec\,\frac{7\pi}{36}+\sec\,\frac{9\pi}{36}+\sec\,\frac{11\pi}{36}\right)\approx 3.9379$

.

11. $\Delta x=(1-0)/5=0.2$, so the endpoints are 0, 0.2, 0.4, 0.6, 0.8, and 1, and the midpoints are 0.1, 0.3, 0.5, 0.7, and 0.9. The Midpoint Rule gives
$\int_0^1 \sin (x^2) dx \approx \sum_{i=1}^{5} f\left(\bar{x}_i\right) \Delta x = 0.2\left[\sin (0.1)^2 + \sin (0.3)^2 + \sin (0.5)^2 + \sin (0.7)^2 + \sin (0.9)^2\right] \approx 0.3084$.

12. $\Delta x=(5-1)/4=1$, so the endpoints are 1, 2, 3, 4, and 5, and the midpoints are 1.5, 2.5, 3.5, and 4.5. The Midpoint Rule gives
$\int_1^5 x^2 e^{-x} dx \approx \sum_{n=1}^{4} f\left(\bar{x}_i\right) \Delta x = 1\left[(1.5)^2 e^{-1.5} + (2.5)^2 e^{-2.5} + (3.5)^2 e^{-3.5} + (4.5)^2 e^{-4.5}\right] \approx 1.6099$.
13. In Maple, we use the command with(student) to load the sum and box commands, then m:=middlesum(sin(x^),x=0..1,5); which gives us the sum in summation notation, then M:=evalf(m); which gives $M_5 \approx 0.30843908$, confirming the result of Exercise 11. The command middlebox(sin(x^),x=0..1,5) generates the graph. Repeating for $n=10$ and $n=20$ gives $M_{10} \approx 0.30981629$ and $M_{20} \approx 0.31015563$.



14. See the solution to Exercise 5.1.7 for a possible algorithm to calculate the sums. With $\Delta x=(1-0)/100=0.01$ and subinterval endpoints 1, 1.01, 1.02, … , 1.99, 2, we calculate that the left Riemann sum is $L_{100} = \sum_{i=1}^{100} \sin (x_{i-1}) \Delta x \approx 0.30607$, and the right Riemann sum is
$R_{100} = \sum_{i=1}^{100} \sin \left(x_i^2\right) \Delta x \approx 0.31448$.
Since $f(x) = \sin \left(x^2\right)$ is an increasing function, we must have $L_{100} \le \int_0^1 \sin \left(x^2\right) dx \le R_{100}$, so $0.306 < L_{100} \le \int_0^1 \sin \left(x^2\right) dx \le R_{100} < 0.315$. Therefore, the approximate value $0.3084 \approx 0.31$ in Exercise 11 must be accurate to two decimal places.

15. We'll create the table of values to approximate $\int_0^{\pi} \sin x dx$ by using the program in the solution to Exercise 5.1.7 with $Y_1 = \sin x$, Xmin=0 , $X = \pi$, and $n=5$, 10 , 50 , and 100 .
The values of R_n appear to be approaching 2 .

n	R_n

5	1.933766
10	1.983524
50	1.999342
100	1.999836

16. $\int_0^2 e^{-x^2}\,dx$ with $n=5$, 10 , 50 , and 100 .

n	L_n	R_n
5	1.077467	0.684794
10	0.980007	0.783670
50	0.901705	0.862438
100	0.891896	0.872262

The value of the integral lies between 0.872 and 0.892 . Note that $f(x)=e^{-x^2}$ is decreasing on $(0,2)$. We cannot make a similar statement for $\int_{-1}^2 e^{-x^2}\,dx$ since f is increasing on $(-1,0)$.

17. On $[0,\pi]$, $\lim_{n\to\infty}\sum_{i=1}^n x_i \sin x_i\,\Delta x=\int_0^\pi x\sin x\,dx$.

18. On $[1,5]$, $\lim_{n\to\infty}\sum_{i=1}^n \frac{e^{x_i}}{1+x_i}\,\Delta x=\int_1^5 \frac{e^x}{1+x}\,dx$.

19. On $[1,8]$, $\lim_{n\to\infty}\sum_{i=1}^n \sqrt{2x_i^*+\left(x_i^*\right)^2}\,\Delta x=\int_1^8\sqrt{2x+x^2}\,dx$.

20. On [0,2] , $\lim_{n\to\infty}\sum_{i=1}^n \,\Delta x=\int_0^2(4-3x^2+6x^5)dx$.

21. Note that $\Delta x=\frac{5-(-1)}{n}=\frac{6}{n}$ and $x_i=-1+i\,\Delta x=-1+\frac{6i}{n}$.

$$\int_{-1}^5(1+3x)dx = \lim_{n\to\infty}\sum_{i=1}^n f\left(x_i\right)\Delta x=\lim_{n\to\infty}\sum_{i=1}^n\left[1+3\left(-1+\frac{6i}{n}\right)\right]\frac{6}{n}$$

$$= \lim_{n\to\infty}\frac{6}{n}\sum_{i=1}^n\left[-2+\frac{18i}{n}\right]=\lim_{n\to\infty}\frac{6}{n}\left[\sum_{i=1}^n(-2)+\sum_{i=1}^n\frac{18i}{n}\right]$$

$$=\lim_{n\to\infty}\frac{6}{n}\left[-2n+\frac{18}{n}\sum_{i=1}^{n} i\right]=\lim_{n\to\infty}\frac{6}{n}\left[-2n+\frac{18}{n}\cdot\frac{n(n+1)}{2}\right]$$

$$=\lim_{n\to\infty}\left[-12+\frac{108}{n^2}\cdot\frac{n(n+1)}{2}\right]=\lim_{n\to\infty}\left[-12+54\frac{n+1}{n}\right]$$

$$=\lim_{n\to\infty}\left[-12+54\left(1+\frac{1}{n}\right)\right]=-12+54\cdot 1=42$$

22.

$$\int_1^4 (x^2+2x-5)\,dx=\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x\ [\ \Delta x=3/n \text{ and } x_i=1+3i/n\ ]$$

$$=\lim_{n\to\infty}\sum_{i=1}^{n}\left[\left(1+\frac{3i}{n}\right)^2+2\left(1+\frac{3i}{n}\right)-5\right]\left(\frac{3}{n}\right)$$

$$=\lim_{n\to\infty}\frac{3}{n}\left[\sum_{i=1}^{n}\left(1+\frac{6i}{n}+\frac{9i^2}{n^2}+2+\frac{6i}{n}-5\right)\right]$$

$$=\lim_{n\to\infty}\frac{3}{n}\left[\sum_{i=1}^{n}\left(\frac{9}{n^2}\cdot i^2+\frac{12}{n}\cdot i-2\right)\right]$$

$$=\lim_{n\to\infty}\frac{3}{n}\left[\frac{9}{n^2}\sum_{i=1}^{n} i^2+\frac{12}{n}\sum_{i=1}^{n} i-\sum_{i=1}^{n} 2\right]$$

$$=\lim_{n\to\infty}\left(\frac{27}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}+\frac{36}{n^2}\cdot\frac{n(n+1)}{2}-\frac{6}{n}\cdot n\right)$$

$$=\lim_{n\to\infty}\left(\frac{9}{2}\cdot\frac{n+1}{n}\cdot\frac{2n+1}{n}+18\cdot\frac{n+1}{n}-6\right)$$

$$=\lim_{n\to\infty}\left[\frac{9}{2}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)+18\left(1+\frac{1}{n}\right)-6\right]=\frac{9}{2}\cdot 1\cdot 2+18\cdot 1-6=21$$

23. Note that $\Delta x=\frac{2-0}{n}=\frac{2}{n}$ and $x_i=0+i\Delta x=\frac{2i}{n}$.

$$\int_0^2\left(2-x^2\right)dx=\lim_{n\to\infty}\sum_{i=1}^{n} f\left(x_i\right)\Delta x=\lim_{n\to\infty}\sum_{i=1}^{n}\left(2-\frac{4i^2}{n^2}\right)\left(\frac{2}{n}\right)$$

$$=\lim_{n\to\infty}\frac{2}{n}\left[\sum_{i=1}^{n} 2-\frac{4}{n^2}\sum_{i=1}^{n} i^2\right]=\lim_{n\to\infty}\frac{2}{n}\left(2n-\frac{4}{n^2}\sum_{i=1}^{n} i^2\right)$$

$$=\lim_{n\to\infty}\left[4-\frac{8}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}\right]=\lim_{n\to\infty}\left(4-\frac{4}{3}\cdot\frac{n+1}{n}\cdot\frac{2n+1}{n}\right)$$

$$=\lim_{n\to\infty}\left[4-\frac{4}{3}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)\right]=4-\frac{4}{3}\cdot 1\cdot 2=\frac{4}{3}$$

24.

$$\int_0^5\left(1+2x^3\right)dx=\lim_{n\to\infty}\sum_{i=1}^{n}f\left(x_i\right)\Delta x$$

$$=\lim_{n\to\infty}\sum_{i=1}^{n}\left(1+2\cdot\frac{125i^3}{n^3}\right)\left(\frac{5}{n}\right)=\lim_{n\to\infty}\frac{5}{n}\left[\sum_{i=1}^{n}1+\frac{250}{n^3}\sum_{i=1}^{n}i^3\right]$$

$$=\lim_{n\to\infty}\frac{5}{n}\left(1\cdot n+\frac{250}{n^3}\sum_{i=1}^{n}i^3\right)=\lim_{n\to\infty}\left[5+\frac{1250}{n^4}\cdot\frac{n^2(n+1)^2}{4}\right]$$

$$=\lim_{n\to\infty}\left[5+312.5\cdot\frac{(n+1)^2}{n^2}\right]=\lim_{n\to\infty}\left[5+312.5\left(1+\frac{1}{n}\right)^2\right]$$

$$=5+312.5=317.5$$

25. Note that $\Delta x=\frac{2-1}{n}=\frac{1}{n}$ and $x_i=1+i\Delta x=1+i\,(1/n)=1+i/n$.

$$\int_1^2 x^3\,dx=\lim_{n\to\infty}\sum_{i=1}^{n}f\left(x_i\right)\Delta x=\lim_{n\to\infty}\sum_{i=1}^{n}\left(1+\frac{i}{n}\right)^3\left(\frac{1}{n}\right)=\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\left(\frac{n+i}{n}\right)^3$$

$$=\lim_{n\to\infty}\frac{1}{n^4}\sum_{i=1}^{n}\left(n^3+3n^2i+3ni^2+i^3\right)=\lim_{n\to\infty}\frac{1}{n^4}\left[\sum_{i=1}^{n}n^3+\sum_{i=1}^{n}3n^2i+\sum_{i=1}^{n}3ni^2+\sum_{i=1}^{n}i^3\right]$$

$$=\lim_{n\to\infty}\frac{1}{n^4}\left[n\cdot n^3+3n^2\sum_{i=1}^{n}i+3n\sum_{i=1}^{n}i^2+\sum_{i=1}^{n}i^3\right]$$

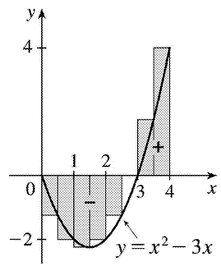
$$=\lim_{n\to\infty}\left[1+\frac{3}{n^2}\cdot\frac{n(n+1)}{2}+\frac{3}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}+\frac{1}{n^4}\cdot\frac{n^2(n+1)^2}{4}\right]$$

$$=\lim_{n\to\infty}\left[1+\frac{3}{2}\cdot\frac{n+1}{n}+\frac{1}{2}\cdot\frac{n+1}{n}\cdot\frac{2n+1}{n}+\frac{1}{4}\cdot\frac{(n+1)^2}{n^2}\right]$$

$$=\lim_{n\to\infty}\left[1+\frac{3}{2}\left(1+\frac{1}{n}\right)+\frac{1}{2}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)+\frac{1}{4}\left(1+\frac{1}{n}\right)^2\right]=1+\frac{3}{2}+\frac{1}{2}\cdot 2+\frac{1}{4}=3.75$$

26. **(a)** $\Delta x=(4-0)/8=0.5$ and $x_i^*=x_i=0.5i$.

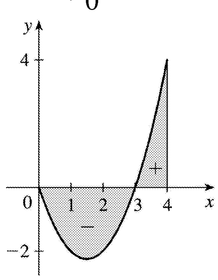
$$\int_0^4 \left(x^2-3x\right)dx\approx \sum_{i=1}^{8} f\left(x_i^*\right)\Delta x$$
$$=0.5\left\{\left[0.5^2-3(0.5)\right]+\left[1.0^2-3(1.0)\right]+\cdots\right.$$
$$\left.+\left[3.5^2-3(3.5)\right]+\left[4.0^2-3(4.0)\right]\right\}$$
$$=\frac{1}{2}\left(-\frac{5}{4}-2-\frac{9}{4}-2-\frac{5}{4}+0+\frac{7}{4}+4\right)=-1.5$$

(b) 

(c) $\int_0^4 \left(x^2-3x\right)dx=\lim_{n\to\infty}\sum_{i=1}^{n}\left[\left(\frac{4i}{n}\right)^2-3\left(\frac{4i}{n}\right)\right]\left(\frac{4}{n}\right)$

$$=\lim_{n\to\infty}\frac{4}{n}\left[\frac{16}{n^2}\sum_{i=1}^{n} i^2-\frac{12}{n}\sum_{i=1}^{n} i\right]$$
$$=\lim_{n\to\infty}\left[\frac{64}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}-\frac{48}{n^2}\cdot\frac{n(n+1)}{2}\right]$$
$$=\lim_{n\to\infty}\left[\frac{32}{3}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)-24\left(1+\frac{1}{n}\right)\right]$$
$$=\frac{32}{3}\cdot 2-24=-\frac{8}{3}$$

(d) $\int_0^4 \left(x^2-3x\right)dx=A_1-A_2$, where A_1 is the area marked + and A_2 is the area marked − .



27.

$$\int_a^b x\,dx=\lim_{n\to\infty}\frac{b-a}{n}\sum_{i=1}^{n}\left[a+\frac{b-a}{n}i\right]=\lim_{n\to\infty}\left[\frac{a(b-a)}{n}\sum_{i=1}^{n}1+\frac{(b-a)^2}{n^2}\sum_{i=1}^{n}i\right]$$

$$=\lim_{n\to\infty}\left[\frac{a(b-a)}{n}n+\frac{(b-a)^2}{n^2}\cdot\frac{n(n+1)}{2}\right]=a(b-a)+\lim_{n\to\infty}\frac{(b-a)^2}{2}\left(1+\frac{1}{n}\right)$$

$$=a(b-a)+\frac{1}{2}(b-a)^2=(b-a)\left(a+\frac{1}{2}b-\frac{1}{2}a\right)=(b-a)\frac{1}{2}(b+a)=\frac{1}{2}\left(b^2-a^2\right)$$

28.

$$\int_a^b x^2\,dx=\lim_{n\to\infty}\frac{b-a}{n}\sum_{i=1}^{n}\left[a+\frac{b-a}{n}i\right]^2=\lim_{n\to\infty}\frac{b-a}{n}\sum_{i=1}^{n}\left[a^2+2a\frac{b-a}{n}i+\frac{(b-a)^2}{n^2}i^2\right]$$

$$=\lim_{n\to\infty}\left[\frac{(b-a)^3}{n^3}\sum_{i=1}^{n}i^2+\frac{2a(b-a)^2}{n^2}\sum_{i=1}^{n}i+\frac{a^2(b-a)}{n}\sum_{i=1}^{n}1\right]$$

$$=\lim_{n\to\infty}\left[\frac{(b-a)^3}{n^3}\frac{n(n+1)(2n+1)}{6}+\frac{2a(b-a)^2}{n^2}\frac{n(n+1)}{2}+\frac{a^2(b-a)}{n}n\right]$$

$$=\lim_{n\to\infty}\left[\frac{(b-a)^3}{6}\cdot1\cdot\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)+a(b-a)^2\cdot1\cdot\left(1+\frac{1}{n}\right)+a^2(b-a)\right]$$

$$=\frac{(b-a)^3}{3}+a(b-a)^2+a^2(b-a)=\frac{b^3-3ab^2+3a^2b-a^3}{3}+ab^2-2a^2b+a^3+a^2b-a^3$$

$$=\frac{b^3}{3}-\frac{a^3}{3}-ab^2+a^2b+ab^2-a^2b=\frac{b^3-a^3}{3}$$

29. $f(x)=\dfrac{x}{1+x^5}$, $a=2$, $b=6$, and $\Delta x=\dfrac{6-2}{n}=\dfrac{4}{n}$. Using Equation 3, we get $x_i^*=x_i=2+i\,\Delta x=2+\dfrac{4i}{n}$, so

$$\int_2^6\frac{x}{1+x^5}\,dx=\lim_{n\to\infty}R_n=\lim_{n\to\infty}\sum_{i=1}^{n}\frac{2+\frac{4i}{n}}{1+\left(2+\frac{4i}{n}\right)^5}\cdot\frac{4}{n}\ .$$

30. $\Delta x=\dfrac{10-1}{n}=\dfrac{9}{n}$ and $x_i=1+i\,\Delta x=1+\dfrac{9i}{n}$, so

$\int_1^{10}(x-4\ln x)dx=\lim_{n\to\infty} R_n=\lim_{n\to\infty} \sum_{i=1}^{n}\left[\left(1+\frac{9i}{n}\right)-4\ln\left(1+\frac{9i}{n}\right)\right]\cdot\frac{9}{n}$.

31. $\Delta x=(\pi-0)/n=\pi/n$ and $x_i^*=x_i=\pi i/n$.

$\int_0^{\pi}\sin 5x dx=\lim_{n\to\infty}\sum_{i=1}^{n}\left(\sin 5x_i\right)\left(\frac{\pi}{n}\right)=\lim_{n\to\infty}\sum_{i=1}^{n}\left(\sin\frac{5\pi i}{n}\right)\frac{\pi}{n}=\pi\lim_{n\to\infty}\frac{1}{n}\cot\left(\frac{5\pi}{2n}\right)=\pi\left(\frac{2}{5\pi}\right)=\frac{2}{5}$

32. $\Delta x=(10-2)/n=8/n$ and $x_i^*=x_i=2+8i/n$.

$$\int_2^{10}x^6 dx=\lim_{n\to\infty}\sum_{i=1}^{n}\left(2+\frac{8i}{n}\right)^6\left(\frac{8}{n}\right)=8\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\left(2+\frac{8i}{n}\right)^6$$

$$=8\lim_{n\to\infty}\frac{1}{n}\cdot\frac{64\left(58{,}593n^6+164{,}052n^5+131{,}208n^4-27{,}776n^2+2048\right)}{21n^5}$$

$$=8\left(\frac{1{,}249{,}984}{7}\right)=\frac{9{,}999{,}872}{7}\approx 1{,}428{,}553.1$$

33. **(a)** Think of $\int_0^2 f(x)dx$ as the area of a trapezoid with bases 1 and 3 and height 2 . The area of a trapezoid is $A=\frac{1}{2}(b+B)h$, so $\int_0^2 f(x)dx=\frac{1}{2}(1+3)2=4$.

(b)

$$\int_0^5 f(x)dx=\underbrace{\int_0^2 f(x)dx}_{\text{trapezoid}}+\underbrace{\int_2^3 f(x)dx}_{\text{rectangle}}+\underbrace{\int_3^5 f(x)dx}_{\text{triangle}}$$

$$=\frac{1}{2}(1+3)2+3\cdot 1+\frac{1}{2}\cdot 2\cdot 3=4+3+3=10$$

(c) $\int_5^7 f(x)dx$ is the negative of the area of the triangle with base 2 and height 3 .

$\int_5^7 f(x)dx=-\frac{1}{2}\cdot 2\cdot 3=-3$.

(d) $\int_7^9 f(x)dx$ is the negative of the area of a trapezoid with bases 3 and 2 and height 2 , so it equals $-\frac{1}{2}(B+b)h=-\frac{1}{2}(3+2)2=-5$. Thus, $\int_0^9 f(x)dx=\int_0^5 f(x)dx+\int_5^7 f(x)dx+\int_7^9 f(x)dx=10+(-3)+(-5)=2$.

34. **(a)**

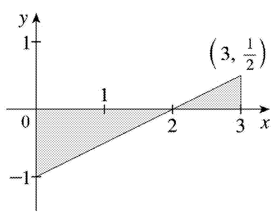
$\int_0^2 g(x)dx = \frac{1}{2} \cdot 4 \cdot 2 = 4$ (area of a triangle)

(b) $\int_2^6 g(x)dx = -\frac{1}{2}\pi(2)^2 = -2\pi$ (negative of the area of a semicircle)

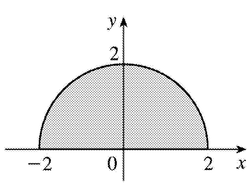
(c) $\int_6^7 g(x)dx = \frac{1}{2} \cdot 1 \cdot 1 = \frac{1}{2}$ (area of a triangle)

$\int_0^7 g(x)dx = \int_0^2 g(x)dx + \int_2^6 g(x)dx + \int_6^7 g(x)dx = 4 - 2\pi + \frac{1}{2} = 4.5 - 2\pi$

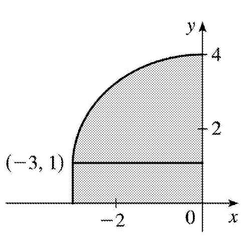
35. $\int_0^3 \left(\frac{1}{2}x - 1\right) dx$ can be interpreted as the area of the triangle above the x-axis minus the area of the triangle below the x-axis; that is, $\frac{1}{2}(1)\left(\frac{1}{2}\right) - \frac{1}{2}(2)(1) = \frac{1}{4} - 1 = -\frac{3}{4}$.



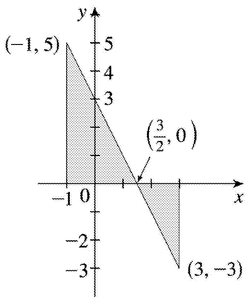
36. $\int_{-2}^2 \sqrt{4 - x^2}\, dx$ can be interpreted as the area under the graph of $f(x) = \sqrt{4 - x^2}$ between $x = -2$ and $x = 2$. This is equal to half the area of the circle with radius 2 , so $\int_{-2}^2 \sqrt{4 - x^2}\, dx = \frac{1}{2}\pi \cdot 2^2 = 2\pi$.



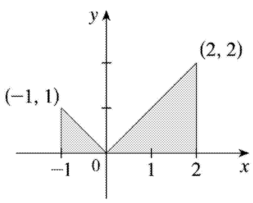
37. $\int_{-3}^0 \left(1 + \sqrt{9 - x^2}\right) dx$ can be interpreted as the area under the graph of $f(x) = 1 + \sqrt{9 - x^2}$ between $x = -3$ and $x = 0$. This is equal to one–quarter the area of the circle with radius 3 , plus the area of the rectangle, so $\int_{-3}^0 \left(1 + \sqrt{9 - x^2}\right) dx = \frac{1}{4}\pi \cdot 3^2 + 1 \cdot 3 = 3 + \frac{9}{4}\pi$.



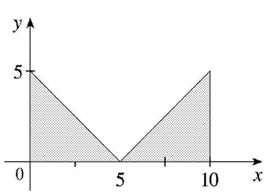
38. $\int_{-1}^{3}(3-2x)\,dx$ can be interpreted as the area of the triangle above the x –axis minus the area of the triangle below the x –axis; that is, $\frac{1}{2}\left(\frac{5}{2}\right)(5)-\frac{1}{2}\left(\frac{3}{2}\right)(3)=\frac{25}{4}-\frac{9}{4}=4$.



39. $\int_{-1}^{2}|x|\,dx$ can be interpreted as the sum of the areas of the two shaded triangles; that is, $\frac{1}{2}(1)(1)+\frac{1}{2}(2)(2)=\frac{1}{2}+\frac{4}{2}=\frac{5}{2}$.



40. $\int_{0}^{10}|x-5|\,dx$ can be interpreted as the sum of the areas of the two shaded triangles; that is, $2\left(\frac{1}{2}\right)(5)(5)=25$.



41.

$$\begin{aligned}\int_{9}^{4}\sqrt{t}\,dt &= -\int_{4}^{9}\sqrt{t}\,dt\\ &= -\int_{4}^{9}\sqrt{x}\,dx\\ &= -\frac{38}{3}\end{aligned}$$

42. $\int_1^1 x^2 \cos x\,dx = 0$ since the limits of integration are equal.

43. $\int_0^1 \left(5-6x^2\right)dx = \int_0^1 5\,dx - 6\int_0^1 x^2\,dx = 5(1-0) - 6\left(\frac{1}{3}\right) = 5-2 = 3$

44. $\int_1^3 \left(2e^x - 1\right)dx = 2\int_1^3 e^x\,dx - \int_1^3 1\,dx = 2\left(e^3 - e\right) - 1(3-1) = 2e^3 - 2e - 2$

45. $\int_1^3 e^{x+2}\,dx = \int_1^3 e^x \cdot e^2\,dx = e^2\int_1^3 e^x\,dx = e^2\left(e^3 - e\right) = e^5 - e^3$

46.

$$\int_0^{\pi/2}(2\cos x - 5x)\,dx = \int_0^{\pi/2} 2\cos x\,dx - \int_0^{\pi/2} 5x\,dx = 2\int_0^{\pi/2}\cos x\,dx - 5\int_0^{\pi/2} x\,dx$$

$$= 2(1) - 5\,\frac{(\pi/2)^2 - 0^2}{2} = 2 - \frac{5\pi^2}{8}$$

47.

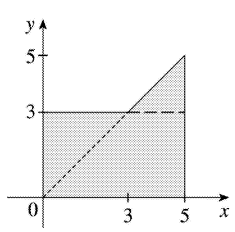
$$\int_{-2}^{2} f(x)\,dx + \int_2^5 f(x)\,dx - \int_{-2}^{-1} f(x)\,dx = \int_{-2}^{5} f(x)\,dx + \int_{-1}^{-2} f(x)\,dx$$

$$= \int_{-1}^{5} f(x)\,dx$$

48. $\int_1^4 f(x)\,dx = \int_1^5 f(x)\,dx - \int_4^5 f(x)\,dx = 12 - 3.6 = 8.4$

49. $\int_0^9 [2f(x) + 3g(x)]\,dx = 2\int_0^9 f(x)\,dx + 3\int_0^9 g(x)\,dx = 2(37) + 3(16) = 122$

50. If $f(x) = \begin{cases} 3 & \text{for } x<3 \\ x & \text{for } x \geq 3 \end{cases}$, then $\int_0^5 f(x)\,dx$ can be interpreted as the area of the shaded region, which consists of a 5–by–3 rectangle surmounted by an isosceles right triangle whose legs have length 2. Thus, $\int_0^5 f(x)\,dx = 5(3) + \frac{1}{2}(2)(2) = 17$.



51. $0 \le \sin x < 1$ on $\left[0, \frac{\pi}{4}\right]$, so $\sin^3 x \le \sin^2 x$ on $\left[0, \frac{\pi}{4}\right]$. Hence, $\int_0^{\pi/4} \sin^3 x\,dx \le \int_0^{\pi/4} \sin^2 x\,dx$ (Property 7).

52. $5-x \ge 3 \ge x+1$ on $[1,2]$, so $\sqrt{5-x} \ge \sqrt{x+1}$ and $\int_1^2 \sqrt{5-x}\,dx \ge \int_1^2 \sqrt{x+1}\,dx$.

53. If $-1 \le x \le 1$, then $0 \le x^2 \le 1$ and $1 \le 1+x^2 \le 2$, so $1 \le \sqrt{1+x^2} \le \sqrt{2}$ and $1[1-(-1)] \le \int_{-1}^1 \sqrt{1+x^2}\,dx \le \sqrt{2}[1-(-1)]$; that is, $2 \le \int_{-1}^1 \sqrt{1+x^2}\,dx \le 2\sqrt{2}$.

54. $\frac{1}{2} \le \sin x \le 1$ for $\frac{\pi}{6} \le x \le \frac{\pi}{2}$, so $\frac{1}{2}\left(\frac{\pi}{2} - \frac{\pi}{6}\right) \le \int_{\pi/6}^{\pi/2} \sin x\,dx \le 1\left(\frac{\pi}{2} - \frac{\pi}{6}\right)$; that is, $\frac{\pi}{6} \le \int_{\pi/6}^{\pi/2} \sin x\,dx \le \frac{\pi}{3}$.

55. If $1 \le x \le 2$, then $\frac{1}{2} \le \frac{1}{x} \le 1$, so $\frac{1}{2}(2-1) \le \int_1^2 \frac{1}{x}\,dx \le 1(2-1)$ or $\frac{1}{2} \le \int_1^2 \frac{1}{x}\,dx \le 1$.

56. If $0 \le x \le 2$, then $0 \le x^3 \le 8$, so $1 \le x^3+1 \le 9$ and $1 \le \sqrt{x^3+1} \le 3$. Thus, $1(2-0) \le \int_0^2 \sqrt{x^3+1}\,dx \le 3(2-0)$ that is, $2 \le \int_0^2 \sqrt{x^3+1}\,dx \le 6$.

57. If $\frac{\pi}{4} \le x \le \frac{\pi}{3}$, then $1 \le \tan x \le \sqrt{3}$, so $1\left(\frac{\pi}{3} - \frac{\pi}{4}\right) \le \int_{\pi/4}^{\pi/3} \tan x\,dx \le \sqrt{3}\left(\frac{\pi}{3} - \frac{\pi}{4}\right)$ or $\frac{\pi}{12} \le \int_{\pi/4}^{\pi/3} \tan x\,dx \le \frac{\pi}{12}\sqrt{3}$.

58. Let $f(x)=x^3-3x+3$ for $0 \le x \le 2$. Then $f'(x)=3x^2-3=3(x+1)(x-1)$, so f is decreasing on $(0,1)$ and increasing on $(1,2)$. f has the absolute minimum value $f(1)=1$. Since $f(0)=3$ and $f(2)=5$, the absolute maximum value of f is $f(2)=5$. Thus, $1 \le x^3-3x+3 \le 5$ for x in $[0,2]$. It follows from Property 8 that $1\cdot(2-0) \le \int_0^2 \left(x^3-3x+3\right)dx \le 5\cdot(2-0)$; that is, $2 \le \int_0^2 \left(x^3-3x+3\right)dx \le 10$.

59. The only critical number of $f(x)=xe^{-x}$ on $[0,2]$ is $x=1$. Since $f(0)=0$, $f(1)=e^{-1} \approx 0.368$, and $f(2)=2e^{-2} \approx 0.271$, we know that the absolute minimum value of f on $[0,2]$ is 0, and the absolute maximum is e^{-1}. By Property 8 ,$0 \le xe^{-x} \le e^{-1}$ for $0 \le x \le 2 \Rightarrow 0(2-0) \le \int_0^2 xe^{-x}\,dx \le e^{-1}(2-0) \Rightarrow$

$0 \le \int_0^2 xe^{-x}\,dx \le 2/e$.

60. If $\frac{1}{4}\pi \le x \le \frac{3}{4}\pi$, then $\frac{\sqrt{2}}{2} \le \sin x \le 1$ and $\frac{1}{2} \le \sin^2 x \le 1$, so $\frac{1}{2}\left(\frac{3}{4}\pi - \frac{1}{4}\pi\right) \le \int_{\pi/4}^{3\pi/4} \sin^2 x\,dx \le 1\left(\frac{3}{4}\pi - \frac{1}{4}\pi\right)$; that is, $\frac{1}{4}\pi \le \int_{\pi/4}^{3\pi/4} \sin^2 x\,dx \le \frac{1}{2}\pi$.

61. $\sqrt{x^4+1} \ge \sqrt{x^4} = x^2$, so $\int_1^3 \sqrt{x^4+1}\,dx \ge \int_1^3 x^2\,dx = \frac{1}{3}\left(3^3 - 1^3\right) = \frac{26}{3}$.

62. $0 \le \sin x \le 1$ for $0 \le x \le \frac{\pi}{2}$, so $x\sin x \le x \Rightarrow \int_0^{\pi/2} x\sin x\,dx \le \int_0^{\pi/2} x\,dx = \frac{1}{2}\left[\left(\frac{\pi}{2}\right)^2 - 0^2\right] = \frac{\pi^2}{8}$.

63. Using a regular partition and right endpoints as in the proof of Property 2, we calculate $\int_a^b cf(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^n cf\left(x_i\right)\Delta x_i = \lim_{n\to\infty} c\sum_{i=1}^n f\left(x_i\right)\Delta x_i = c\lim_{n\to\infty} \sum_{i=1}^n f\left(x_i\right)\Delta x_i = c\int_a^b f(x)\,dx$.

64. As in the proof of Property 2, we write $\int_a^b f(x)\,dx = {}_{n\to\infty}\sum_{i=1}^n f\left(x_i\right)\Delta x$. Now $f\left(x_i\right) \ge 0$ and $\Delta x \ge 0$, so $f\left(x_i\right)\Delta x \ge 0$ and therefore $\sum_{i=1}^n f\left(x_i\right)\Delta x \ge 0$. But the limit of nonnegative quantities is nonnegative by Theorem 2.3.2, so $\int_a^b f(x)\,dx \ge 0$.

65. Since $-|f(x)| \le f(x) \le |f(x)|$, it follows from Property 7 that

$$-\int_a^b |f(x)|\,dx \le \int_a^b f(x)\,dx \le \int_a^b |f(x)|\,dx \Rightarrow \left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx$$

Note that the definite integral is a real number, and so the following property applies: $-a \le b \le a \Rightarrow |b| \le a$ for all real numbers b and nonnegative numbers a .

66. $\left|\int_0^{2\pi} f(x)\sin 2x\,dx\right| \le \int_0^{2\pi} |f(x)\sin 2x|\,dx = \int_0^{2\pi} |f(x)|\,|\sin 2x|\,dx \le \int_0^{2\pi} |f(x)|\,dx$ by Property 7 , since $|\sin 2x| \le 1 \Rightarrow |f(x)|\,|\sin 2x| \le |f(x)|$.

67. $\lim_{n\to\infty} \sum_{i=1}^n \frac{i^4}{n^5} = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^n \left(\frac{i}{n}\right)^4 = \int_0^1 x^4\,dx$

68.

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\frac{1}{1+(i/n)^2}=\int_0^1\frac{dx}{1+x^2}$$

69. Choose $x_i=1+\frac{i}{n}$ and $x_i^*=\sqrt{x_{i-1}x_i}=\sqrt{\left(1+\frac{i-1}{n}\right)\left(1+\frac{i}{n}\right)}$. Then

$$\int_1^2 x^{-2}dx = \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\frac{1}{\left(1+\frac{i-1}{n}\right)\left(1+\frac{i}{n}\right)}$$

$$= \lim_{n\to\infty} n\sum_{i=1}^{n}\frac{1}{(n+i-1)(n+i)}$$

$$= \lim_{n\to\infty} n\sum_{i=1}^{n}\left(\frac{1}{n+i-1}-\frac{1}{n+i}\right) \quad \text{[by the hint]}$$

$$= \lim_{n\to\infty} n\left(\sum_{i=0}^{n-1}\frac{1}{n+i}-\sum_{i=1}^{n}\frac{1}{n+i}\right)$$

$$= \lim_{n\to\infty} n\left(\left[\frac{1}{n}+\frac{1}{n+1}+\cdots+\frac{1}{2n-1}\right]-\left[\frac{1}{n+1}+\cdots+\frac{1}{2n-1}+\frac{1}{2n}\right]\right)$$

$$= \lim_{n\to\infty} n\left(\frac{1}{n}-\frac{1}{2n}\right)=\lim_{n\to\infty}\left(1-\frac{1}{2}\right)=\frac{1}{2}$$

1. The precise version of this statement is given by the Fundamental Theorem of Calculus. See the statement of this theorem and the paragraph that follows it at the end of Section 5.3.

2. **(a)**

$g(x) = \int_0^x f(t)\,dt$, so $g(0) = \int_0^0 f(t)\,dt = 0$.

$g(1) = \int_0^1 f(t)\,dt = \frac{1}{2}\cdot 1\cdot 1$ [area of triangle] $= \frac{1}{2}$.

$g(2) = \int_0^2 f(t)\,dt = \int_0^1 f(t)\,dt + \int_1^2 f(t)\,dt$ [below the x –axis]

$= \frac{1}{2} - \frac{1}{2}\cdot 1\cdot 1 = 0.$

$g(3) = g(2) + \int_2^3 f(t)\,dt = 0 - \frac{1}{2}\cdot 1\cdot 1 = -\frac{1}{2}$.

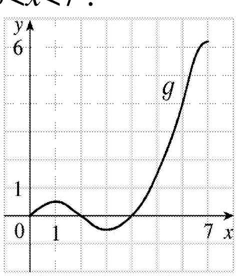
$g(4) = g(3) + \int_3^4 f(t)\,dt = -\frac{1}{2} + \frac{1}{2}\cdot 1\cdot 1 = 0.$

$g(5) = g(4) + \int_4^5 f(t)\,dt = 0 + 1.5 = 1.5.$

$g(6) = g(5) + \int_5^6 f(t)\,dt = 1.5 + 2.5 = 4.$

(b) $g(7) = g(6) + \int_6^7 f(t)\,dt \approx 4 + 2.2$ [estimate from the graph] $= 6.2$.

(c) The answers from part (a) and part (b) indicate that g has a minimum at $x = 3$ and a maximum at $x = 7$. This makes sense from the graph of f since we are subtracting area on $1 < x < 3$ and adding area on $3 < x < 7$.

(d) 

3. **(a)**

$g(x) = \int_0^x f(t)\,dt$.

$g(0) = \int_0^0 f(t)\,dt = 0$

$g(1) = \int_0^1 f(t)\,dt = 1 \cdot 2 = 2$,

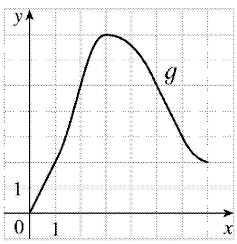
$g(2) = \int_0^2 f(t)\,dt = \int_0^1 f(t)\,dt + \int_1^2 f(t)\,dt = g(1) + \int_1^2 f(t)\,dt$

$= 2 + 1 \cdot 2 + \frac{1}{2} \cdot 1 \cdot 2 = 5$,

$g(3) = \int_0^3 f(t)\,dt = g(2) + \int_2^3 f(t)\,dt = 5 + \frac{1}{2} \cdot 1 \cdot 4 = 7$,

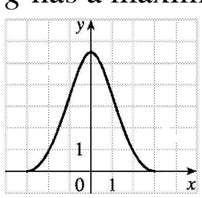
$g(6) = g(3) + \int_3^6 f(t)\,dt$

$= 7 + \left[-\left(\frac{1}{2} \cdot 2 \cdot 2 + 1 \cdot 2 \right) \right] = 7 - 4 = 3$

(b) 

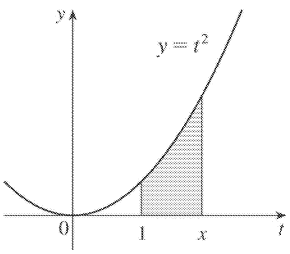
(c) g is increasing on $(0,3)$ because as x increases from 0 to 3 , we keep adding more area.
(d) g has a maximum value when we start subtracting area; that is, at $x=3$.

4. **(a)** $g(-3) = \int_{-3}^{-3} f(t)\,dt = 0$, $g(3) = \int_{-3}^{3} f(t)\,dt = \int_{-3}^{0} f(t)\,dt + \int_0^3 f(t)\,dt = 0$ by symmetry, since the area above the $x-$ axis is the same as the area below the axis.
(b) From the graph, it appears that to the nearest $\frac{1}{2}$, $g(-2) = \int_{-3}^{-2} f(t)\,dt \approx 1$, $g(-1) = \int_{-3}^{-1} f(t)\,dt \approx 3\frac{1}{2}$, and $g(0) = \int_{-3}^{0} f(t)\,dt \approx 5\frac{1}{2}$.
(c) g is increasing on $(-3,0)$ because as x increases from -3 to 0 , we keep adding more area.
(d) g has a maximum value when we start subtracting area; that is, at $x=0$.

(e) 

(f) The graph of $g'(x)$ is the same as that of $f(x)$, as indicated by FTC1.

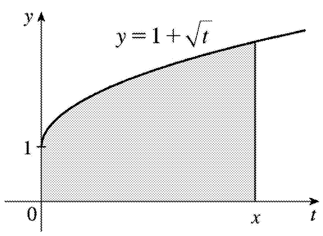
5.



(a) By FTC1 with $f(t)=t^2$ and $a=1$, $g(x)=\int_1^x t^2\,dt \Rightarrow g'(x)=f(x)=x^2$.

(b) Using FTC2, $g(x)=\int_1^x t^2\,dt=\left[\frac{1}{3}t^3\right]_1^x=\frac{1}{3}x^3-\frac{1}{3} \Rightarrow g'(x)=x^2$.

6.



(a) By FTC1 with $f(t)=1+\sqrt{t}$ and $a=0$, $g(x)=\int_0^x\left(1+\sqrt{t}\right)dt \Rightarrow g'(x)=f(x)=1+\sqrt{x}$.

(b) Using FTC2, $g(x)=\int_0^x\left(1+\sqrt{t}\right)dt=\left[t+\frac{2}{3}t^{3/2}\right]_0^x=x+\frac{2}{3}x^{3/2} \Rightarrow g'(x)=1+x^{1/2}=1+\sqrt{x}$.

7. $f(t)=\sqrt{1+2t}$ and $g(x)=\int_0^x\sqrt{1+2t}\,dt$, so by FTC1, $g'(x)=f(x)=\sqrt{1+2x}$.

8. $f(t)=\ln t$ and $g(x)=\int_1^x \ln t\,dt$, so by FTC1, $g'(x)=f(x)=\ln x$.

9. $f(t)=t^2\sin t$ and $g(y)=\int_2^y t^2\sin t\,dt$, so by FTC1, $g'(y)=f(y)=y^2\sin y$.

10. $f(x)=\dfrac{1}{x+x^2}$ and $g(u)=\int_3^u \dfrac{1}{x+x^2}\,dx$, so $g'(u)=f(u)=\dfrac{1}{u+u^2}$.

11. $F(x)=\int_x^2\cos\left(t^2\right)dt=-\int_2^x\cos\left(t^2\right)dt \Rightarrow F'(x)=-\cos\left(x^2\right)$

12. $f(\theta)=\tan\theta$ and $F(x)=\int_x^{10}\tan\theta\,d\theta=-\int_{10}^x\tan\theta\,d\theta$, so by FTC1, $F'(x)=-f(x)=-\tan x$.

13. Let $u=\frac{1}{x}$. Then $\frac{du}{dx}=-\frac{1}{x^2}$. Also, $\frac{dh}{dx}=\frac{dh}{du}\frac{du}{dx}$, so

$h'(x)=\frac{d}{dx}\int_2^{1/x}\arctan t\,dt=\frac{d}{du}\int_2^{u}\arctan t\,dt\cdot\frac{du}{dx}=\arctan u\frac{du}{dx}=-\frac{\arctan(1/x)}{x^2}$.

14. Let $u=x^2$. Then $\frac{du}{dx}=2x$. Also, $\frac{dh}{dx}=\frac{dh}{du}\frac{du}{dx}$, so

$h'(x)=\frac{d}{dx}\int_0^{x^2}\sqrt{1+r^3}\,dr=\frac{d}{du}\int_0^{u}\sqrt{1+r^3}\,dr\cdot\frac{du}{dx}=\sqrt{1+u^3}(2x)=2x\sqrt{1+(x^2)^3}=2x\sqrt{1+x^6}$.

15. Let $u=\sqrt{x}$. Then $\frac{du}{dx}=\frac{1}{2\sqrt{x}}$. Also, $\frac{dy}{dx}=\frac{dy}{du}\frac{du}{dx}$, so

$y'=\frac{d}{dx}\int_3^{\sqrt{x}}\frac{\cos t}{t}\,dt=\frac{d}{du}\int_3^{u}\frac{\cos t}{t}\,dt\cdot\frac{du}{dx}=\frac{\cos u}{u}\cdot\frac{1}{2\sqrt{x}}=\frac{\cos\sqrt{x}}{\sqrt{x}}\cdot\frac{1}{2\sqrt{x}}=\frac{\cos\sqrt{x}}{2x}$.

16. Let $u=\cos x$. Then $\frac{du}{dx}=-\sin x$. Also, $\frac{dy}{dx}=\frac{dy}{du}\frac{du}{dx}$, so

$$y'=\frac{d}{dx}\int_1^{\cos x}(t+\sin t)\,dt=\frac{d}{du}\int_1^{u}(t+\sin t)\,dt\cdot\frac{du}{dx}$$
$$=(u+\sin u)\cdot(-\sin x)=-\sin x[\cos x+\sin(\cos x)]$$

17. Let $w=1-3x$. Then $\frac{dw}{dx}=-3$. Also, $\frac{dy}{dx}=\frac{dy}{dw}\frac{dw}{dx}$, so

$$y'=\frac{d}{dx}\int_{1-3x}^{1}\frac{u^3}{1+u^2}\,du=\frac{d}{dw}\int_{w}^{1}\frac{u^3}{1+u^2}\,du\cdot\frac{dw}{dx}$$
$$=-\frac{d}{dw}\int_1^{w}\frac{u^3}{1+u^2}\,du\cdot\frac{dw}{dx}=-\frac{w^3}{1+w^2}(-3)=\frac{3(1-3x)^3}{1+(1-3x)^2}$$

18. Let $u=e^x$. Then $\frac{du}{dx}=e^x$. Also, $\frac{dy}{dx}=\frac{dy}{du}\frac{du}{dx}$, so

$y'=\frac{d}{dx}\int_{e^x}^{0}\sin^3 t\,dt=\frac{d}{du}\int_{u}^{0}\sin^3 t\,dt\cdot\frac{du}{dx}=-\frac{d}{du}\int_0^{u}\sin^3 t\,dt\cdot\frac{du}{dx}=-\sin^3 u\cdot e^x=-e^x\sin^3\left(e^x\right)$.

19. $\int_{-1}^{3}x^5\,dx=\left[\frac{x^6}{6}\right]_{-1}^{3}=\frac{3^6}{6}-\frac{(-1)^6}{6}=\frac{729-1}{6}=\frac{364}{3}$

20. $\int_{-2}^{5} 6\,dx = [6x]_{-2}^{5} = 6[5-(-2)] = 6(7) = 42$

21. $\int_{2}^{8} (4x+3)\,dx = \left[\frac{4}{2} x^2 + 3x \right]_{2}^{8} = \left(2 \cdot 8^2 + 3 \cdot 8\right) - \left(2 \cdot 2^2 + 3 \cdot 2\right) = 152 - 14 = 138$

22. $\int_{0}^{4} \left(1+3y-y^2\right) dy = \left[y + \frac{3}{2} y^2 - \frac{1}{3} y^3 \right]_{0}^{4} = \left(4 + \frac{3}{2} \cdot 16 - \frac{1}{3} \cdot 64 \right) - (0) = \frac{20}{3}$

23. $\int_{0}^{1} x^{4/5}\,dx = \left[\frac{5}{9} x^{9/5} \right]_{0}^{1} = \frac{5}{9} - 0 = \frac{5}{9}$

24. $\int_{1}^{8} \sqrt[3]{x}\,dx = \int_{1}^{8} x^{1/3}\,dx = \left[\frac{3}{4} x^{4/3} \right]_{1}^{8} = \frac{3}{4} (8^{4/3} - 1^{4/3}) = \frac{3}{4} (2^4 - 1) = \frac{3}{4} (16-1) = \frac{3}{4} (15) = \frac{45}{4}$

25. $\int_{1}^{2} \frac{3}{t^4}\,dt = 3\int_{1}^{2} t^{-4}\,dt = 3\left[\frac{t^{-3}}{-3} \right]_{1}^{2} = \frac{3}{-3} \left[\frac{1}{t^3} \right]_{1}^{2} = -1 \left(\frac{1}{8} - 1 \right) = \frac{7}{8}$

26. $\int_{-2}^{3} x^{-5}\,dx$ does not exist because the function $f(x) = x^{-5}$ has an infinite discontinuity at $x=0$; that is, f is discontinuous on the interval $[-2,3]$.

27. $\int_{-5}^{5} \frac{2}{x^3}\,dx$ does not exist because the function $f(x) = \frac{2}{x^3}$ has an infinite discontinuity at $x=0$; that is, f is discontinuous on the interval $[-5,5]$.

28. $\int_{\pi}^{2\pi} \cos\theta\,d\theta = [\sin\theta]_{\pi}^{2\pi} = \sin 2\pi - \sin\pi = 0 - 0 = 0$

29. $\int_{0}^{2} x(2+x^5)\,dx = \int_{0}^{2} (2x+x^6)\,dx = \left[x^2 + \frac{1}{7} x^7 \right]_{0}^{2} = \left(4 + \frac{128}{7} \right) - (0+0) = \frac{156}{7}$

30. $\int_{1}^{4} \frac{1}{\sqrt{x}}\,dx = \int_{1}^{4} x^{-1/2}\,dx = \left[\frac{x^{1/2}}{1/2} \right]_{1}^{4} = \left[2x^{1/2} \right]_{1}^{4} = 2\sqrt{4} - 2\sqrt{1} = 4 - 2 = 2$

31. $\int_{0}^{\pi/4} \sec^2 t\,dt = [\tan t]_{0}^{\pi/4} = \tan\frac{\pi}{4} - \tan 0 = 1 - 0 = 1$

32.

$\int_0^1 (3+x\sqrt{x})\,dx=\int_0^1 (3+x^{3/2})\,dx=\left[3x+\frac{2}{5}x^{5/2}\right]_0^1=\left[\left(3+\frac{2}{5}\right)-0\right]=\frac{17}{5}$

33. $\int_{\pi}^{2\pi} \csc^2\theta\, d\theta$ does not exist because the function $f(\theta)=\csc^2\theta$ has infinite discontinuities at $\theta=\pi$ and $\theta=2\pi$; that is, f is discontinuous on the interval $[\pi,2\pi]$.

34. $\int_0^{\pi/6} \csc\theta\cot\theta\, d\theta$ does not exist because the function $f(\theta)=\csc\theta\cot\theta$ has an infinite discontinuity at $\theta=0$;
that is, f is discontinuous on the interval $\left[0,\frac{\pi}{6}\right]$.

35. $\int_1^9 \frac{1}{2x}\,dx=\frac{1}{2}\int_1^9 \frac{1}{x}\,dx=\frac{1}{2}\,[\ln|x|]_1^9=\frac{1}{2}\,(\ln 9-\ln 1)=\frac{1}{2}\ln 9-0=\ln 9^{1/2}=\ln 3$

36. $\int_0^1 10^x\,dx=\left[\frac{10^x}{\ln 10}\right]_0^1=\frac{10}{\ln 10}-\frac{1}{\ln 10}=\frac{9}{\ln 10}$

37.

$$\int_{1/2}^{\sqrt{3}/2} \frac{6}{\sqrt{1-t^2}}\,dt=6\int_{1/2}^{\sqrt{3}/2}\frac{1}{\sqrt{1-t^2}}\,dt=6\left[\sin^{-1}t\right]_{1/2}^{\sqrt{3}/2}=6\left[\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)-\sin^{-1}\left(\frac{1}{2}\right)\right]$$
$$=6\left(\frac{\pi}{3}-\frac{\pi}{6}\right)=6\left(\frac{\pi}{6}\right)=\pi$$

38. $\int_0^1 \frac{4}{t^2+1}\,dt=4\int_0^1\frac{1}{1+t^2}\,dt=4\left[\tan^{-1}t\right]_0^1=4\left(\tan^{-1}1-\tan^{-1}0\right)=4\left(\frac{\pi}{4}-0\right)=\pi$

39. $\int_{-1}^1 e^{u+1}\,du=\left[e^{u+1}\right]_{-1}^1=e^2-e^0=e^2-1$ [or start with $e^{u+1}=e^u e^1$]

40.

$$\int_1^2 \frac{4+u^2}{u^3}\,du=\int_1^2\left(4u^{-3}+u^{-1}\right)du=\left[\frac{4}{-2}u^{-2}+\ln|u|\right]_1^2=\left[\frac{-2}{u^2}+\ln u\right]_1^2$$
$$=\left(-\frac{1}{2}+\ln 2\right)-(-2+\ln 1)=\frac{3}{2}+\ln 2$$

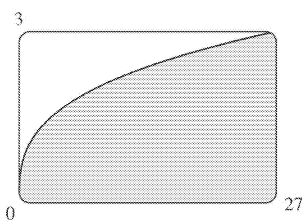
41. $\int_0^2 f(x)dx=\int_0^1 x^4\,dx+\int_1^2 x^5\,dx=\left[\frac{1}{5}x^5\right]_0^1+\left[\frac{1}{6}x^6\right]_1^2=\left(\frac{1}{5}-0\right)+\left(\frac{64}{6}-\frac{1}{6}\right)=10.7$

42.

$$\int_{-\pi}^{\pi} f(x)dx=\int_{-\pi}^{0} x\,dx+\int_0^{\pi}\sin x\,dx=\left[\frac{1}{2}x^2\right]_{-\pi}^{0}-[\cos x]_0^{\pi}=\left(0-\frac{\pi^2}{2}\right)-(\cos\pi-\cos 0)$$

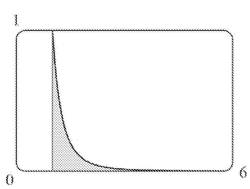
$$=-\frac{\pi^2}{2}-(-1-1)=2-\frac{\pi^2}{2}$$

43. From the graph, it appears that the area is about 60 . The actual area is $\int_0^{27} x^{1/3}\,dx=\left[\frac{3}{4}x^{4/3}\right]_0^{27}=\frac{3}{4}\cdot 81-0=\frac{243}{4}=60.75$. This is $\frac{3}{4}$ of the area of the viewing rectangle.

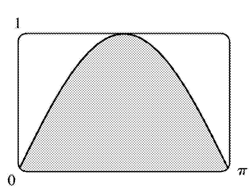


44. From the graph, it appears that the area is about $\frac{1}{3}$. The actual area is

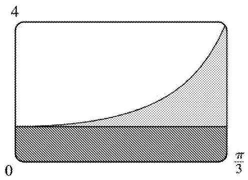
$$\int_1^6 x^{-4}\,dx=\left[\frac{x^{-3}}{-3}\right]_1^6=\left[\frac{-1}{3x^3}\right]_1^6=-\frac{1}{3\cdot 216}+\frac{1}{3}=\frac{215}{648}\approx 0.3318 .$$



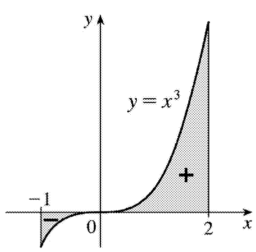
45. It appears that the area under the graph is about $\frac{2}{3}$ of the area of the viewing rectangle, or about $\frac{2}{3}\pi\approx 2.1$. The actual area is $\int_0^{\pi}\sin x\,dx=[-\cos x]_0^{\pi}=(-\cos\pi)-(-\cos 0)=-(-1)+1=2$.



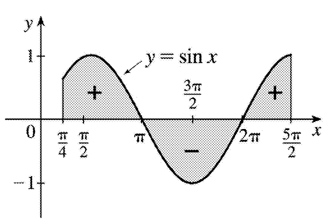
46. Splitting up the region as shown, we estimate that the area under the graph is $\frac{\pi}{3}+\frac{1}{4}\left(3\cdot\frac{\pi}{3}\right)\approx 1.8$. The actual area is $\int_0^{\pi/3}\sec^2 x\,dx=[\tan x]_0^{\pi/3}=\sqrt{3}-0=\sqrt{3}\approx 1.73$.



47. $\int_{-1}^{2} x^3\,dx=\left[\frac{1}{4}x^4\right]_{-1}^{2}=4-\frac{1}{4}=\frac{15}{4}=3.75$



48. $\int_{\pi/4}^{5\pi/2}\sin x\,dx=[-\cos x]_{\pi/4}^{5\pi/2}=0+\frac{\sqrt{2}}{2}=\frac{\sqrt{2}}{2}$



49. $g(x)=\int_{2x}^{3x}\frac{u^2-1}{u^2+1}\,du=\int_{2x}^{0}\frac{u^2-1}{u^2+1}\,du+\int_{0}^{3x}\frac{u^2-1}{u^2+1}\,du=-\int_{0}^{2x}\frac{u^2-1}{u^2+1}\,du+\int_{0}^{3x}\frac{u^2-1}{u^2+1}\,du\Rightarrow$

$g'(x)=-\frac{(2x)^2-1}{(2x)^2+1}\cdot\frac{d}{dx}(2x)+\frac{(3x)^2-1}{(3x)^2+1}\cdot\frac{d}{dx}(3x)=-2\cdot\frac{4x^2-1}{4x^2+1}+3\cdot\frac{9x^2-1}{9x^2+1}$

50. $g(x)=\int_{\tan x}^{x^2}\frac{1}{\sqrt{2+t^4}}\,dt=\int_{\tan x}^{1}\frac{dt}{\sqrt{2+t^4}}+\int_{1}^{x^2}\frac{dt}{\sqrt{2+t^4}}=-\int_{1}^{\tan x}\frac{dt}{\sqrt{2+t^4}}+\int_{1}^{x^2}\frac{dt}{\sqrt{2+t^4}}\Rightarrow$

$g'(x)=\frac{-1}{\sqrt{2+\tan^4 x}}\frac{d}{dx}(\tan x)+\frac{1}{\sqrt{2+x^8}}\frac{d}{dx}\left(x^2\right)=-\frac{\sec^2 x}{\sqrt{2+\tan^4 x}}+\frac{2x}{\sqrt{2+x^8}}$

51. $y=\int_{\sqrt{x}}^{x^3}\sqrt{t}\sin t\,dt=\int_{\sqrt{x}}^{1}\sqrt{t}\sin t\,dt+\int_{1}^{x^3}\sqrt{t}\sin t\,dt=-\int_{1}^{\sqrt{x}}\sqrt{t}\sin t\,dt+\int_{1}^{x^3}\sqrt{t}\sin t\,dt\Rightarrow$

$y'=-\sqrt[4]{x}\,(\sin\sqrt{x})\cdot\frac{d}{dx}(\sqrt{x})+x^{3/2}\sin\left(x^3\right)\cdot\frac{d}{dx}\left(x^3\right)=-\frac{\sqrt[4]{x}\sin\sqrt{x}}{2\sqrt{x}}+x^{3/2}\sin\left(x^3\right)\left(3x^2\right)$

$=3x^{7/2}\sin\left(x^3\right)-\frac{\sin\sqrt{x}}{2\sqrt[4]{x}}$

52. $y=\int_{\cos x}^{5x}\cos\left(u^2\right)du=\int_0^{5x}\cos\left(u^2\right)du-\int_0^{\cos x}\cos\left(u^2\right)du\Rightarrow$

$y'=\cos\left(25x^2\right)\cdot\frac{d}{dx}(5x)-\cos\left(\cos^2 x\right)\cdot\frac{d}{dx}(\cos x)=\cos\left(25x^2\right)\cdot 5-\cos\left(\cos^2 x\right)\cdot(-\sin x)$

$=5\cos\left(25x^2\right)+\sin x\cos\left(\cos^2 x\right)$

53. $F(x)=\int_1^x f(t)\,dt\Rightarrow F'(x)=f(x)=\int_1^{x^2}\frac{\sqrt{1+u^4}}{u}\,du\left[\text{ since } f(t)=\int_1^{t^2}\frac{\sqrt{1+u^4}}{u}\,du\right]\Rightarrow$

$F''(x)=f'(x)=\frac{\sqrt{1+(x^2)^4}}{x^2}\cdot\frac{d}{dx}\left(x^2\right)=\frac{\sqrt{1+x^8}}{x^2}\cdot 2x=\frac{2\sqrt{1+x^8}}{x}$. So $F''(2)=\sqrt{1+2^8}=\sqrt{257}$.

54. For the curve to be concave upward, we must have $y''>0$. $y=\int_0^x\frac{1}{1+t+t^2}\,dt\Rightarrow y'=\frac{1}{1+x+x^2}\Rightarrow$

$y''=\frac{-(1+2x)}{\left(1+x+x^2\right)^2}$. For this expression to be positive, we must have $(1+2x)<0$, since $\left(1+x+x^2\right)^2>0$ for all x. $(1+2x)<0\Leftrightarrow x<-\frac{1}{2}$. Thus, the curve is concave upward on $\left(-\infty,-\frac{1}{2}\right)$.

55. By FTC2, $\int_1^4 f'(x)\,dx=f(4)-f(1)$, so $17=f(4)-12\Rightarrow f(4)=17+12=29$.

56. **(a)** $\text{erf}(x)=\frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt\Rightarrow\int_0^x e^{-t^2}dt=\frac{\sqrt{\pi}}{2}\text{erf}(x)$. By Property 5 of definite integrals in Section 5.2, $\int_0^b e^{-t^2}dt=\int_0^a e^{-t^2}dt+\int_a^b e^{-t^2}dt$, so

$\int_a^b e^{-t^2}dt=\int_0^b e^{-t^2}dt-\int_0^a e^{-t^2}dt=\frac{\sqrt{\pi}}{2}\text{erf}(b)-\frac{\sqrt{\pi}}{2}\text{erf}(a)=\frac{1}{2}\sqrt{\pi}\,[\text{erf}(b)-\text{erf}(a)]$.

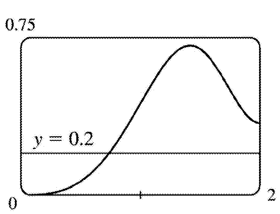
(b) $y=e^{x^2}\text{erf}(x)\Rightarrow$

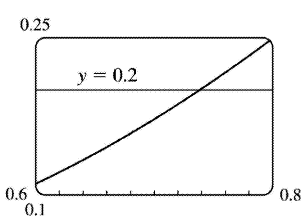
$y'=2xe^{x^2}\operatorname{erf}(x)+e^{x^2}\operatorname{erf}'(x)=2xy+e^{x^2}\cdot\frac{2}{\sqrt{\pi}}e^{-x^2}$ [by FTC1] $=2xy+2/\sqrt{\pi}$.

57. **(a)** The Fresnel function $S(x)=\int_0^x \sin\left(\frac{\pi}{2}t^2\right)dt$ has local maximum values where $0=S'(x)=\sin\left(\frac{\pi}{2}x^2\right)$ and S' changes from positive to negative. For $x>0$, this happens when $\frac{\pi}{2}x^2=(2n-1)\pi \Leftrightarrow x^2=2(2n-1) \Leftrightarrow x=\sqrt{4n-2}$, n any positive integer. For $x<0$, S' changes from positive to negative where $\frac{\pi}{2}x^2=2n\pi \Leftrightarrow x^2=4n \Leftrightarrow x=-2\sqrt{n}$. S' does not change sign at $x=0$.

(b) S is concave upward on those intervals where $S''(x)>0$. Differentiating our expression for $S'(x)$, we get $S''(x)=\cos\left(\frac{\pi}{2}x^2\right)\left(2\frac{\pi}{2}x\right)=\pi x\cos\left(\frac{\pi}{2}x^2\right)$. For $x>0$, $S''(x)>0$ where $\cos\left(\frac{\pi}{2}x^2\right)>0 \Leftrightarrow 0<\frac{\pi}{2}x^2<\frac{\pi}{2}$ or $\left(2n-\frac{1}{2}\right)\pi<\frac{\pi}{2}x^2<\left(2n+\frac{1}{2}\right)\pi$, n any integer $\Leftrightarrow 0<x<1$ or $\sqrt{4n-1}<x<\sqrt{4n+1}$, n any positive integer. For $x<0$, $S''(x)>0$ where $\cos\left(\frac{\pi}{2}x^2\right)<0 \Leftrightarrow \left(2n-\frac{3}{2}\right)\pi<\frac{\pi}{2}x^2<\left(2n-\frac{1}{2}\right)\pi$, n any integer $\Leftrightarrow 4n-3<x^2<4n-1 \Leftrightarrow \sqrt{4n-3}<|x|<\sqrt{4n-1} \Rightarrow \sqrt{4n-3}<-x<\sqrt{4n-1} \Rightarrow -\sqrt{4n-3}>x>-\sqrt{4n-1}$, so the intervals of upward concavity for $x<0$ are $\left(-\sqrt{4n-1},-\sqrt{4n-3}\right)$, n any positive integer. To summarize: S is concave upward on the intervals $(0,1)$, $\left(-\sqrt{3},-1\right)$, $\left(\sqrt{3},\sqrt{5}\right)$, $\left(-\sqrt{7},-\sqrt{5}\right)$, $\left(\sqrt{7},3\right)$, … .

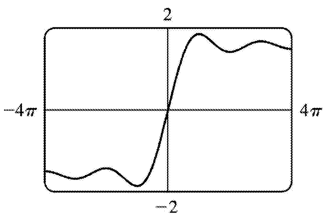
(c) In Maple, we use plot({int(sin(Pi*t^2/2),t=0..x),0.2},x=0..2); . Note that Maple recognizes the Fresnel function, calling it FresnelS(x) . In Mathematica, we use Plot[{Integrate[Sin[Pi*t^2/2],{t,0,x}],0.2},{x,0,2}] . In Derive, we load the utility file FRESNEL and plot FRESNEL_SIN(x) . From the graphs, we see that $\int_0^x \sin\left(\frac{\pi}{2}t^2\right)dt=0.2$ at $x\approx 0.74$.





58. **(a)** In Maple, we should start by setting si:=int(sin(t)/t,t=0..x); . In Mathematica, the command is

si=Integrate[Sin[t]/t,{t,0,x}] . Note that both systems recognize this function; Maple calls it Si(x) and Mathematica calls it SinIntegral[x] . In Maple, the command to generate the graph is plot(si,x=–4*Pi..4*Pi); . In Mathematica, it is plot[si,{x,–4*Pi.,4*Pi}] . In Derive, we load the utility file EXP_INT and plot SI(x) .



(b) SI(x) has local maximum values where $\mathrm{Si}'(x)$ changes from positive to negative, passing through 0 . From the Fundamental Theorem we know that $\mathrm{Si}'(x)=\frac{d}{dx}\int_0^x \frac{\sin t}{t}\,dt=\frac{\sin x}{x}$, so we must have $\sin x=0$ for a maximum, and for $x>0$ we must have $x=(2n-1)\pi$, n any positive integer, for Si' to be changing from positive to negative at x . For $x<0$, we must have $x=2n\pi$, n any positive integer, for a maximum, since the denominator of $\mathrm{Si}'(x)$ is negative for $x<0$. Thus, the local maxima occur at $x=\pi,-2\pi,3\pi,-4\pi,5\pi,-6\pi,\ldots$.

(c) To find the first inflection point, we solve $\mathrm{Si}''(x)=\frac{\cos x}{x}-\frac{\sin x}{x^2}=0$. We can see from the graph that the first inflection point lies somewhere between $x=3$ and $x=5$. Using a root finder gives the value $x\approx 4.4934$. To find the $y-$ coordinate of the inflection point, we evaluate $\mathrm{Si}(4.4934)\approx 1.6556$. So the coordinates of the first inflection point to the right of the origin are about $(4.4934,1.6556)$. Alternatively, we could graph $S''(x)$ and estimate the first positive $x-$ value at which it changes sign.

(d) It seems from the graph that the function has horizontal asymptotes at $y\approx 1.5$, with $\lim_{x\to\pm\infty}\mathrm{Si}(x)\approx\pm 1.5$ respectively. Using the limit command, we get $\lim_{x\to\infty}\mathrm{Si}(x)=\frac{\pi}{2}$. Since Si(x) is an odd function, $\lim_{x\to-\infty}\mathrm{Si}(x)=-\frac{\pi}{2}$. So Si(x) has the horizontal asymptotes $y=\pm\frac{\pi}{2}$.

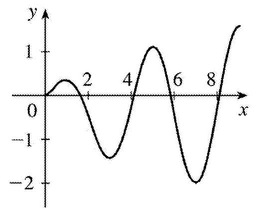
(e) We use the fsolve command in Maple (or FindRoot in Mathematica) to find that the solution is $x\approx 1.1$. Or, as in Exercise (c), we graph $y=\mathrm{Si}(x)$ and $y=1$ on the same screen to see where they intersect.

59. **(a)** By FTC1, $g'(x)=f(x)$. So $g'(x)=f(x)=0$ at $x=1,3,5,7$, and 9 . g has local maxima at $x=1$ and 5 (since $f=g'$ changes from positive to negative there) and local minima at $x=3$ and 7 . There is no local maximum or minimum at $x=9$, since f is not defined for $x>9$.

(b) We can see from the graph that $\left|\int_0^1 f\,dt\right|<\left|\int_1^3 f\,dt\right|<\left|\int_3^5 f\,dt\right|<\left|\int_5^7 f\,dt\right|<\left|\int_7^9 f\,dt\right|$. So

$g(1)=\left|\int_0^1 f\,dt\right|$, $g(5)=\int_0^5 f\,dt=g(1)-\left|\int_1^3 f\,dt\right|+\left|\int_3^5 f\,dt\right|$, and $g(9)=\int_0^9 f\,dt=g(5)-\left|\int_5^7 f\,dt\right|+\left|\int_7^9 f\,dt\right|$. Thus, $g(1)<g(5)<g(9)$, and so the absolute maximum of $g(x)$ occurs at $x=9$.

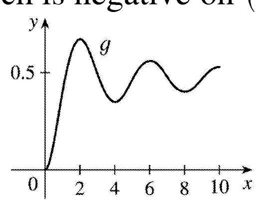
(c) g is concave downward on those intervals where $g''<0$. But $g'(x)=f(x)$, so $g''(x)=f'(x)$, which is negative on (approximately) $\left(\frac{1}{2},2\right)$, $(4,6)$ and $(8,9)$. So g is concave downward on these intervals.

(d) 

60. **(a)** By FTC1, $g'(x)=f(x)$. So $g'(x)=f(x)=0$ at $x=2$, 4 , 6 , 8 , and 10 . g has local maxima at $x=2$ and 6 (since $f=g'$ changes from positive to negative there) and local minima at $x=4$ and 8 . There is no local maximum or minimum at $x=10$, since f is not defined for $x>10$.

(b) We can see from the graph that $\left|\int_0^2 f\,dt\right|>\left|\int_2^4 f\,dt\right|>\left|\int_4^6 f\,dt\right|>\left|\int_6^8 f\,dt\right|>\left|\int_8^{10} f\,dt\right|$. So $g(2)=\left|\int_0^2 f\,dt\right|$, $g(6)=\int_0^6 f\,dt=g(2)-\left|\int_2^4 f\,dt\right|+\left|\int_4^6 f\,dt\right|$, and $g(10)=\int_0^{10} f\,dt=g(6)-\left|\int_6^8 f\,dt\right|+\left|\int_8^{10} f\,dt\right|$. Thus, $g(2)>g(6)>g(10)$, and so the absolute maximum of $g(x)$ occurs at $x=2$.

(c) g is concave downward on those intervals where $g''<0$. But $g'(x)=f(x)$, so $g''(x)=f'(x)$, which is negative on $(1,3)$, $(5,7)$ and $(9,10)$. So g is concave downward on these intervals.

(d) 

61. $\lim_{n\to\infty}\sum_{i=1}^{n}\frac{i^3}{n^4}=\lim_{n\to\infty}\frac{1-0}{n}\sum_{i=1}^{n}\left(\frac{i}{n}\right)^3=\int_0^1 x^3\,dx=\left[\frac{x^4}{4}\right]_0^1=\frac{1}{4}$

62. $\lim_{n\to\infty}\frac{1}{n}\left(\sqrt{\frac{1}{n}}+\sqrt{\frac{2}{n}}+\cdots+\sqrt{\frac{n}{n}}\right)=\lim_{n\to\infty}\frac{1-0}{n}\sum_{i=1}^{n}\sqrt{\frac{i}{n}}=\int_0^1\sqrt{x}\,dx=\left[\frac{2x^{3/2}}{3}\right]_0^1=\frac{2}{3}-0=\frac{2}{3}$

63. Suppose $h<0$. Since f is continuous on $[x+\mathrm{h},\mathrm{x}]$, the Extreme Value Theorem says that there are

numbers u and v in $[x+h,x]$ such that $f(u)=m$ and $f(v)=M$, where m and M are the absolute minimum and maximum values of f on $[x+h,x]$. By Property 8 of integrals, $m(-h)\leq\int_{x+h}^{x}f(t)\,dt\leq M(-h)$; that is, $f(u)(-h)\leq -\int_{x}^{x+h}f(t)\,dt\leq f(v)(-h)$. Since $-h>0$, we can divide this inequality by $-h$: $f(u)\leq \frac{1}{h}\int_{x}^{x+h}f(t)\,dt\leq f(v)$. By Equation 2, $\frac{g(x+h)-g(x)}{h}=\frac{1}{h}\int_{x}^{x+h}f(t)\,dt$ for $h\neq 0$, and hence $f(u)\leq \frac{g(x+h)-g(x)}{h}\leq f(v)$, which is Equation 3 in the case where $h<0$.

64.

$$\begin{aligned}\frac{d}{dx}\int_{g(x)}^{h(x)}f(t)\,dt &=\frac{d}{dx}\left[\int_{g(x)}^{a}f(t)\,dt+\int_{a}^{h(x)}f(t)\,dt\right] \text{ (where } a \text{ is in the domain of } f\text{)}\\ &=\frac{d}{dx}\left[-\int_{a}^{g(x)}f(t)\,dt\right]+\frac{d}{dx}\left[\int_{a}^{h(x)}f(t)\,dt\right]=-f(g(x))\,g'(x)+f(h(x))\,h'(x)\\ &=f(h(x))\,h'(x)-f(g(x))\,g'(x)\end{aligned}$$

65. **(a)** Let $f(x)=\sqrt{x}\Rightarrow f'(x)=1/(2\sqrt{x})>0$ for $x>0\Rightarrow f$ is increasing on $(0,\infty)$. If $x\geq 0$, then $x^3\geq 0$, so $1+x^3\geq 1$ and since f is increasing, this means that $f\left(1+x^3\right)\geq f(1)\Rightarrow \sqrt{1+x^3}\geq 1$ for $x\geq 0$. Next let $g(t)=t^2-t\Rightarrow g'(t)=2t-1\Rightarrow g'(t)>0$ when $t\geq 1$. Thus, g is increasing on $(1,\infty)$. And since $g(1)=0$, $g(t)\geq 0$ when $t\geq 1$. Now let $t=\sqrt{1+x^3}$, where $x\geq 0$. $\sqrt{1+x^3}\geq 1$ (from above) $\Rightarrow t\geq 1\Rightarrow g(t)\geq 0\Rightarrow$ $\left(1+x^3\right)-\sqrt{1+x^3}\geq 0$ for $x\geq 0$. Therefore, $1\leq\sqrt{1+x^3}\leq 1+x^3$ for $x\geq 0$.

(b) From part (a) and Property 7: $\int_0^1 1\,dx\leq\int_0^1\sqrt{1+x^3}\,dx\leq\int_0^1\left(1+x^3\right)dx\Leftrightarrow$ $[x]_0^1\leq\int_0^1\sqrt{1+x^3}\,dx\leq\left[x+\frac{1}{4}x^4\right]_0^1\Leftrightarrow 1\leq\int_0^1\sqrt{1+x^3}\,dx\leq 1+\frac{1}{4}=1.25$.

66. **(a)** If $x<0$, then $g(x)=\int_0^x f(t)\,dt=\int_0^x 0\,dt=0$.

If $0\leq x\leq 1$, then $g(x)=\int_0^x f(t)\,dt=\int_0^x t\,dt=\left[\frac{1}{2}t^2\right]_0^x=\frac{1}{2}x^2$.

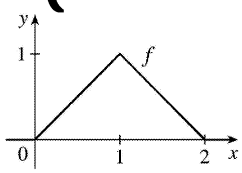
If $1<x\leq 2$, then

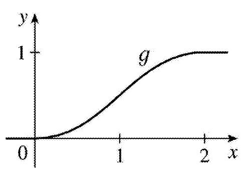
$$\begin{aligned}g(x) &=\int_0^x f(t)\,dt=\int_0^1 f(t)\,dt+\int_1^x f(t)\,dt\\ &=g(1)+\int_1^x(2-t)\,dt=\frac{1}{2}(1)^2+\left[2t-\frac{1}{2}t^2\right]_1^x\end{aligned}$$

$$=\frac{1}{2}+\left(2x-\frac{1}{2}x^2\right)-\left(2-\frac{1}{2}\right)=2x-\frac{1}{2}x^2-1\ .$$

If $x>2$, then $g(x)=\int_0^x f(t)\,dt=g(2)+\int_2^x 0\,dt=1+0=1$. So

$$g(x)=\begin{cases} 0 & \text{if } x<0 \\ \frac{1}{2}x^2 & \text{if } 0\le x\le 1 \\ 2x-\frac{1}{2}x^2-1 & \text{if } 1<x\le 2 \\ 1 & \text{if } x>2 \end{cases}$$

(b) 



(c) f is not differentiable at its corners at $x=0$, 1 , and 2 . f is differentiable on $(-\infty,0)$, $(0,1)$, $(1,2)$ and $(2,\infty)$. g is differentiable on $(-\infty,\infty)$.

67. Using FTC1, we differentiate both sides of $6+\int_a^x \frac{f(t)}{t^2}\,dt=2\sqrt{x}$ to get $\frac{f(x)}{x^2}=2\,\frac{1}{2\sqrt{x}}\Rightarrow f(x)=x^{3/2}$.

To find a , we substitute $x=a$ in the original equation to obtain $6+\int_a^a \frac{f(t)}{t^2}\,dt=2\sqrt{a}\Rightarrow 6+0=2\sqrt{a}\Rightarrow$

$3=\sqrt{a}\Rightarrow a=9$.

68. $B=3A\Rightarrow\int_0^b e^x\,dx=3\int_0^a e^x\,dx\Rightarrow\left[e^x\right]_0^b=3\left[e^x\right]_0^a\Rightarrow e^b-1=3\left(e^a-1\right)\Rightarrow$
$e^b=3e^a-2\Rightarrow b=\ln\left(3e^a-2\right)$

69. **(a)** Let $F(t)=\int_0^t f(s)\,ds$. Then, by FTC1, $F'(t)=f(t)=$ rate of depreciation, so $F(t)$ represents the loss in value over the interval $[0,t]$.

(b) $C(t)=\frac{1}{t}\left[A+\int_0^t f(s)\,ds\right]=\frac{A+F(t)}{t}$ represents the average expenditure per unit of t during the interval $[0,t]$, assuming that there has been only one overhaul during that time period. The company wants to minimize average expenditure.

(c) $C(t)=\frac{1}{t}\left[A+\int_0^t f(s)\,ds\right]$. Using FTC1, we have $C'(t)=-\frac{1}{t^2}\left[A+\int_0^t f(s)\,ds\right]+\frac{1}{t}f(t)$.

$C'(t)=0\Rightarrow t\,f(t)=A+\int_0^t f(s)\,ds\Rightarrow f(t)=\frac{1}{t}\left[A+\int_0^t f(s)\,ds\right]=C(t)$.

70. **(a)** $C(t)=\frac{1}{t}\int_0^t[f(s)+g(s)]\,ds$. Using FTC1 and the Product Rule, we have

$C'(t)=\frac{1}{t}[f(t)+g(t)]-\frac{1}{t^2}\int_0^t[f(s)+g(s)]\,ds$. Set $C'(t)=0$: $\frac{1}{t}[f(t)+g(t)]-\frac{1}{t^2}\int_0^t[f(s)+g(s)]\,ds=0\Rightarrow$

$[f(t)+g(t)]-\frac{1}{t}\int_0^t[f(s)+g(s)]\,ds=0\Rightarrow[f(t)+g(t)]-C(t)=0\Rightarrow C(t)=f(t)+g(t)$.

(b) For $0\le t\le 30$, we have $D(t)=\int_0^t\left(\frac{V}{15}-\frac{V}{450}s\right)ds=\left[\frac{V}{15}s-\frac{V}{900}s^2\right]_0^t=\frac{V}{15}t-\frac{V}{900}t^2$.

So $D(t)=V\Rightarrow\frac{V}{15}t-\frac{V}{900}t^2=V\Rightarrow 60t-t^2=900\Rightarrow t^2-60t+900=0\Rightarrow$

$(t-30)^2=0\Rightarrow t=30$. So the length of time T is 30 months.

(c)

$$C(t)=\frac{1}{t}\int_0^t\left(\frac{V}{15}-\frac{V}{450}s+\frac{V}{12{,}900}s^2\right)ds=\frac{1}{t}\left[\frac{V}{15}s-\frac{V}{900}s^2+\frac{V}{38{,}700}s^3\right]_0^t$$
$$=\frac{1}{t}\left(\frac{V}{15}t-\frac{V}{900}t^2+\frac{V}{38{,}700}t^3\right)=\frac{V}{15}-\frac{V}{900}t+\frac{V}{38{,}700}t^2\Rightarrow$$

$C'(t)=-\frac{V}{900}+\frac{V}{19{,}350}t=0$ when $\frac{1}{19{,}350}t=\frac{1}{900}\Rightarrow t=21.5$.

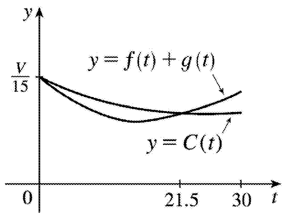
$C(21.5)=\frac{V}{15}-\frac{V}{900}(21.5)+\frac{V}{38{,}700}(21.5)^2\approx 0.05472V$, $C(0)=\frac{V}{15}\approx 0.06667V$, and

$C(30)=\frac{V}{15}-\frac{V}{900}(30)+\frac{V}{38{,}700}(30)^2\approx 0.05659V$, so the absolute minimum is $C(21.5)\approx 0.05472V$.

(d) As in part (c), we have $C(t)=\frac{V}{15}-\frac{V}{900}t+\frac{V}{38{,}700}t^2$, so $C(t)=f(t)+g(t)\Leftrightarrow$

$\frac{V}{15}-\frac{V}{900}t+\frac{V}{38{,}700}t^2=\frac{V}{15}-\frac{V}{450}t+\frac{V}{12{,}900}t^2\Leftrightarrow t^2\left(\frac{1}{12{,}900}-\frac{1}{38{,}700}\right)=t\left(\frac{1}{450}-\frac{1}{900}\right)\Leftrightarrow$

$t=\frac{1/900}{2/38{,}700}=\frac{43}{2}=21.5$. This is the value of t that we obtained as the critical number of C in part (c), so we have verified the result of (a) in this case.



1. $\frac{d}{dx}\left[\sqrt{x^2+1}+C\right]=\frac{d}{dx}\left[\left(x^2+1\right)^{1/2}+C\right]=\frac{1}{2}\left(x^2+1\right)^{-1/2}\cdot 2x=\frac{x}{\sqrt{x^2+1}}$

2. $\frac{d}{dx}\left[x\sin x+\cos x+C\right]=x\cos x+(\sin x)\cdot 1-\sin x=x\cos x$

3. $\frac{d}{dx}\left[\frac{x}{a^2\sqrt{a^2-x^2}}+C\right]=\frac{1}{a^2}\frac{\sqrt{a^2-x^2}-x(-x)\Big/\sqrt{a^2-x^2}}{a^2-x^2}=\frac{1}{a^2}\frac{\left(a^2-x^2\right)+x^2}{\left(a^2-x^2\right)^{3/2}}=\frac{1}{\sqrt{\left(a^2-x^2\right)^3}}$

4.

$$\frac{d}{dx}\left[-\frac{\sqrt{x^2+a^2}}{a^2x}+C\right]=-\frac{1}{a^2}\frac{d}{dx}\left[\frac{\sqrt{x^2+a^2}}{x}\right]=-\frac{x\left(x/\sqrt{x^2+a^2}\right)-\sqrt{x^2+a^2}\cdot 1}{a^2x^2}$$

$$=-\frac{x^2-\left(x^2+a^2\right)}{a^2x^2\sqrt{x^2+a^2}}=\frac{1}{x^2\sqrt{x^2+a^2}}$$

5. $\int x^{-3/4}dx=\frac{x^{-3/4+1}}{-3/4+1}+C=\frac{x^{1/4}}{1/4}+C=4x^{1/4}+C$

6. $\int\sqrt[3]{x}\,dx=\int x^{1/3}dx=\frac{x^{4/3}}{4/3}+C=\frac{3}{4}x^{4/3}+C$

7. $\int\left(x^3+6x+1\right)dx=\frac{x^4}{4}+6\frac{x^2}{2}+x+C=\frac{1}{4}x^4+3x^2+x+C$

8. $\int x\left(1+2x^4\right)dx=\int\left(x+2x^5\right)dx=\frac{x^2}{2}+2\frac{x^6}{6}+C=\frac{1}{2}x^2+\frac{1}{3}x^6+C$

9. $\int(1-t)\left(2+t^2\right)dt=\int\left(2-2t+t^2-t^3\right)dt=2t-2\frac{t^2}{2}+\frac{t^3}{3}-\frac{t^4}{4}+C=2t-t^2+\frac{1}{3}t^3-\frac{1}{4}t^4+C$

10. $\int\left(x^2+1+\frac{1}{x^2+1}\right)dx=\frac{x^3}{3}+x+\tan^{-1}x+C$

11. $\int(2-\sqrt{x})^2dx=\int(4-4\sqrt{x}+x)dx=4x-4\dfrac{x^{3/2}}{3/2}+\dfrac{x^2}{2}+C=4x-\dfrac{8}{3}x^{3/2}+\dfrac{1}{2}x^2+C$

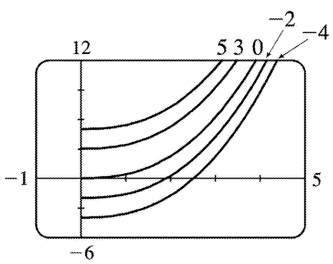
12. $\int\left(3e^u+\sec^2u\right)du=3e^u+\tan u+C$

13. $\int\dfrac{\sin x}{1-\sin^2x}dx=\int\dfrac{\sin x}{\cos^2x}dx=\int\dfrac{1}{\cos x}\cdot\dfrac{\sin x}{\cos x}dx=\int\sec x\tan x\,dx=\sec x+C$

14. $\int\dfrac{\sin 2x}{\sin x}dx=\int\dfrac{2\sin x\cos x}{\sin x}dx=\int 2\cos x\,dx=2\sin x+C$

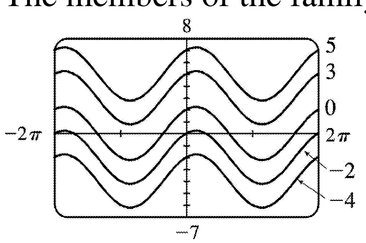
15. $\int x\sqrt{x}\,dx=\int x^{3/2}dx=\dfrac{2}{5}x^{5/2}+C$.

The members of the family in the figure correspond to $C=5$, 3 , 0 , -2 , and -4 .



16. $\int(\cos x-2\sin x)dx=\sin x+2\cos x+C$.

The members of the family in the figure correspond to $C=5$, 3 , 0 , -2 , and -4 .



17. $\int_0^2(6x^2-4x+5)dx=\left[6\cdot\dfrac{1}{3}x^3-4\cdot\dfrac{1}{2}x^2+5x\right]_0^2=\left[2x^3-2x^2+5x\right]_0^2=(16-8+10)-0=18$

18.

$$\int_1^3(1+2x-4x^3)dx=\left[x+2\cdot\frac{1}{2}x^2-4\cdot\frac{1}{4}x^4\right]_1^3=\left[x+x^2-x^4\right]_1^3$$
$$=(3+9-81)-(1+1-1)=-69-1=-70$$

19. $\int_{-1}^0\left(2x-e^x\right)dx=\left[x^2-e^x\right]_{-1}^0=(0-1)-\left(1-e^{-1}\right)=-2+1/e$

20. $\int_{-2}^{0}(u^5-u^3+u^2)\,du=\left[\frac{1}{6}u^6-\frac{1}{4}u^4+\frac{1}{3}u^3\right]_{-2}^{0}=0-\left(\frac{32}{3}-4-\frac{8}{3}\right)=-4$

21.

$$\int_{-2}^{2}(3u+1)^2\,du=\int_{-2}^{2}\left(9u^2+6u+1\right)du=\left[9\cdot\frac{1}{3}u^3+6\cdot\frac{1}{2}u^2+u\right]_{-2}^{2}=\left[3u^3+3u^2+u\right]_{-2}^{2}$$
$$=(24+12+2)-(-24+12-2)=38-(-14)=52$$

22.

$$\int_{0}^{4}(2v+5)(3v-1)\,dv=\int_{0}^{4}(6v^2+13v-5)\,dv=\left[6\cdot\frac{1}{3}v^3+13\cdot\frac{1}{2}v^2-5v\right]_{0}^{4}=\left[2v^3+\frac{13}{2}v^2-5v\right]_{0}^{4}$$
$$=(128+104-20)-0=212$$

23. $\int_{1}^{4}\sqrt{t}\,(1+t)\,dt=\int_{1}^{4}(t^{1/2}+t^{3/2})\,dt=\left[\frac{2}{3}t^{3/2}+\frac{2}{5}t^{5/2}\right]_{1}^{4}=\left(\frac{16}{3}+\frac{64}{5}\right)-\left(\frac{2}{3}+\frac{2}{5}\right)=\frac{14}{3}+\frac{62}{5}=\frac{256}{15}$

24. $\int_{0}^{9}\sqrt{2t}\,dt=\int_{0}^{9}\sqrt{2}\,t^{1/2}\,dt=\left[\sqrt{2}\cdot\frac{2}{3}t^{3/2}\right]_{0}^{9}=\sqrt{2}\cdot\frac{2}{3}\cdot 27-0=18\sqrt{2}$

25. $\int_{-2}^{-1}\left(4y^3+\frac{2}{y^3}\right)dy=\left[4\cdot\frac{1}{4}y^4+2\cdot\frac{1}{-2}y^{-2}\right]_{-2}^{-1}=\left[y^4-\frac{1}{y^2}\right]_{-2}^{-1}=(1-1)-\left(16-\frac{1}{4}\right)=-\frac{63}{4}$

26. $\int_{1}^{2}\frac{y+5y^7}{y^3}\,dy=\int_{1}^{2}(y^{-2}+5y^4)\,dy=\left[-y^{-1}+5\cdot\frac{1}{5}y^5\right]_{1}^{2}=\left[-\frac{1}{y}+y^5\right]_{1}^{2}=\left(-\frac{1}{2}+32\right)-(-1+1)=\frac{63}{2}$

27. $\int_{0}^{1}x\left(\sqrt[3]{x}+\sqrt[4]{x}\right)dx=\int_{0}^{1}(x^{4/3}+x^{5/4})\,dx=\left[\frac{3}{7}x^{7/3}+\frac{4}{9}x^{9/4}\right]_{0}^{1}=\left(\frac{3}{7}+\frac{4}{9}\right)-0=\frac{55}{63}$

28. $\int_{0}^{5}\left(2e^x+4\cos x\right)dx=\left[2e^x+4\sin x\right]_{0}^{5}=\left(2e^5+4\sin 5\right)-\left(2e^0+4\sin 0\right)=2e^5+4\sin 5-2\approx 290.99$

29. $\int_{1}^{4}\sqrt{5/x}\,dx=\sqrt{5}\int_{1}^{4}x^{-1/2}\,dx=\sqrt{5}\left[2\sqrt{x}\,\right]_{1}^{4}=\sqrt{5}\,(2\cdot 2-2\cdot 1)=2\sqrt{5}$

30.

$$\int_{1}^{9}\frac{3x-2}{\sqrt{x}}\,dx=\int_{1}^{9}(3x^{1/2}-2x^{-1/2})\,dx=\left[3\cdot\frac{2}{3}x^{3/2}-2\cdot 2x^{1/2}\right]_{1}^{9}=\left[2x^{3/2}-4x^{1/2}\right]_{1}^{9}$$

$=(54-12)-(2-4)=44$

31. $\int_0^{\pi}(4\sin\theta-3\cos\theta)\,d\theta=[-4\cos\theta-3\sin\theta]_0^{\pi}=(4-0)-(-4-0)=8$

32. $\int_{\pi/4}^{\pi/3}\sec\theta\ \tan\theta\ d\theta=[\sec\theta]_{\pi/4}^{\pi/3}=\sec\ \frac{\pi}{3}-\sec\ \frac{\pi}{4}=2-\sqrt{2}$

33.

$$\int_0^{\pi/4}\frac{1+\cos^2\theta}{\cos^2\theta}\ d\theta=\int_0^{\pi/4}\left(\frac{1}{\cos^2\theta}+\frac{\cos^2\theta}{\cos^2\theta}\right)d\theta=\int_0^{\pi/4}\left(\sec^2\theta+1\right)d\theta$$

$$=[\tan\theta+\theta]_0^{\pi/4}=\left(\tan\ \frac{\pi}{4}+\frac{\pi}{4}\right)-(0+0)=1+\frac{\pi}{4}$$

34.

$$\int_0^{\pi/3}\frac{\sin\theta+\sin\theta\ \tan^2\theta}{\sec^2\theta}\ d\theta=\int_0^{\pi/3}\frac{\sin\theta\ (1+\tan^2\theta)}{\sec^2\theta}\ d\theta=\int_0^{\pi/3}\frac{\sin\theta\ \sec^2\theta}{\sec^2\theta}\ d\theta=\int_0^{\pi/3}\sin\theta\ d\theta$$

$$=[-\cos\theta]_0^{\pi/3}=-\frac{1}{2}-(-1)=\frac{1}{2}$$

35.

$$\int_1^{64}\frac{1+\sqrt[3]{x}}{\sqrt{x}}\ dx=\int_1^{64}\left(\frac{1}{x^{1/2}}+\frac{x^{1/3}}{x^{1/2}}\right)dx=\int_1^{64}(x^{-1/2}+x^{(1/3)-(1/2)})dx=\int_1^{64}(x^{-1/2}+x^{-1/6})dx$$

$$=\left[2x^{1/2}+\frac{6}{5}x^{5/6}\right]_1^{64}=\left(16+\frac{192}{5}\right)-\left(2+\frac{6}{5}\right)=14+\frac{186}{5}=\frac{256}{5}$$

36. $\int_0^1(1+x^2)^3\,dx=\int_0^1(1+3x^2+3x^4+x^6)dx=\left[x+x^3+\frac{3}{5}x^5+\frac{1}{7}x^7\right]_0^1=\left(1+1+\frac{3}{5}+\frac{1}{7}\right)-0=\frac{96}{35}$

37.

$$\int_1^e\frac{x^2+x+1}{x}\ dx=\int_1^e\left(x+1+\frac{1}{x}\right)dx=\left[\frac{1}{2}x^2+x+\ln|x|\right]_1^e=\left(\frac{1}{2}e^2+e+\ln e\right)-\left(\frac{1}{2}+1+\ln 1\right)=\frac{1}{2}e^2+e-\frac{1}{2}$$

38.

$$\int_4^9\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)^2dx=\int_4^9\left(x+2+\frac{1}{x}\right)dx=\left[\frac{1}{2}x^2+2x+\ln|x|\right]_4^9=\frac{81}{2}+18+\ln 9-(8+8+\ln 4)=\frac{85}{2}+\ln\ \frac{9}{4}$$

39.

$$\int_{-1}^{2}(x-2|x|)\,dx = \int_{-1}^{0}[x-2(-x)]\,dx + \int_{0}^{2}[x-2(x)]\,dx = \int_{-1}^{0}3x\,dx + \int_{0}^{2}(-x)\,dx = 3\left[\frac{1}{2}x^2\right]_{-1}^{0} - \left[\frac{1}{2}x^2\right]_{0}^{2}$$

$$= 3\left(0-\frac{1}{2}\right)-(2-0) = -\frac{7}{2} = -3.5$$

40.

$$\int_{0}^{3\pi/2}|\sin x|\,dx = \int_{0}^{\pi}\sin x\,dx + \int_{\pi}^{3\pi/2}(-\sin x)\,dx = [-\cos x]_{0}^{\pi} + [\cos x]_{\pi}^{3\pi/2}$$

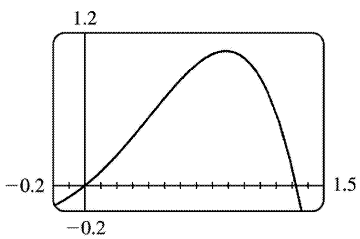
$$= [1-(-1)] + [0-(-1)] = 2+1 = 3$$

41. The graph shows that $y=x+x^2-x^4$ has $x-$ intercepts at $x=0$ and at $x=a\approx 1.32$. So the area of the region that lies under the curve and above the $x-$ axis is

$$\int_{0}^{a}\left(x+x^2-x^4\right)dx = \left[\frac{1}{2}x^2+\frac{1}{3}x^3-\frac{1}{5}x^5\right]_{0}^{a}$$

$$= \left(\frac{1}{2}a^2+\frac{1}{3}a^3-\frac{1}{5}a^5\right)-0$$

$$\approx 0.84$$

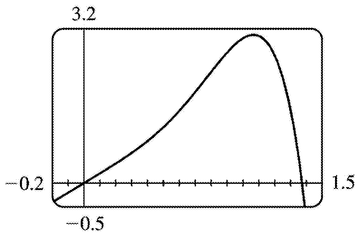


42. The graph shows that $y=2x+3x^4-2x^6$ has $x-$ intercepts at $x=0$ and at $x=a\approx 1.37$. So the area of the region that lies under the curve and above the $x-$ axis is

$$\int_{0}^{a}\left(2x+3x^4-2x^6\right)dx = \left[x^2+\frac{3}{5}x^5-\frac{2}{7}x^7\right]_{0}^{a}$$

$$= \left(a^2+\frac{3}{5}a^5-\frac{2}{7}a^7\right)-0$$

$$\approx 2.18$$



43. $A=\int_0^2 \left(2y-y^2\right) dy=\left[y^2-\frac{1}{3}y^3 \right]_0^2=\left(4-\frac{8}{3}\right)-0=\frac{4}{3}$

44. $y=\sqrt[4]{x} \Rightarrow x=y^4$, so $A=\int_0^1 y^4 dy=\left[\frac{1}{5}y^5 \right]_0^1=\frac{1}{5}$.

45. If $w'(t)$ is the rate of change of weight in pounds per year, then $w(t)$ represents the weight in pounds of the child at age t . We know from the Net Change Theorem that $\int_5^{10} w'(t)dt=w(10)-w(5)$, so the integral represents the increase in the child's weight (in pounds) between the ages of 5 and 10 .

46. $\int_a^b I(t)dt=\int_a^b Q'(t)dt=Q(b)-Q(a)$ by the Net Change Theorem, so it represents the change in the charge Q from time $t=a$ to $t=b$.

47. Since $r(t)$ is the rate at which oil leaks, we can write $r(t)=-V'(t)$, where $V(t)$ is the volume of oil at time t . Thus, by the Net Change Theorem,
$\int_0^{120} r(t)dt=-\int_0^{120} V'(t)dt=-[V(120)-V(0)]=V(0)-V(120)$, which is the number of gallons of oil that leaked from the tank in the first two hours (120 minutes).

48. By the Net Change Theorem, $\int_0^{15} n'(t)dt=n(15)-n(0)=n(15)-100$ represents the increase in the bee population in 15 weeks. So $100+\int_0^{15} n'(t)dt=n(15)$ represents the total bee population after 15 weeks.

49. By the Net Change Theorem, $\int_{1000}^{5000} R'(x)dx=R(5000)-R(1000)$, so it represents the increase in revenue when production is increased from 1000 units to 5000 units.

50. The slope of the trail is the rate of change of the elevation E , so $f(x)=E'(x)$. By the Net Change Theorem, $\int_3^5 f(x)dx=\int_3^5 E'(x)dx=E(5)-E(3)$ is the change in the elevation E between $x=3$ miles and $x=5$ miles from the start of the trail.

51. In general, the unit of measurement for $\int_a^b f(x)\,dx$ is the product of the unit for $f(x)$ and the unit for x . Since
$f(x)$ is measured in newtons and x is measured in meters, the units for $\int_0^{100} f(x)\,dx$ are newton–meters. (A newton–meter is abbreviated N–m and is called a joule.)

52. The units for $a(x)$ are pounds per foot and the units for x are feet, so the units for da/dx are pounds per foot per foot, denoted (lb / ft) / ft. The unit of measurement for $\int_2^8 a(x)\,dx$ is the product of pounds per foot and feet; that is, pounds.

53. **(a)** displacement $=\int_0^3 (3t-5)\,dt=\left[\frac{3}{2}t^2-5t\right]_0^3=\frac{27}{2}-15=-\frac{3}{2}$ m

(b)

$$\text{distance traveled } =\int_0^3 |3t-5|\,dt=\int_0^{5/3}(5-3t)\,dt+\int_{5/3}^3 (3t-5)\,dt$$

$$=\left[5t-\frac{3}{2}t^2\right]_0^{5/3}+\left[\frac{3}{2}t^2-5t\right]_{5/3}^3=\frac{25}{3}-\frac{3}{2}\cdot\frac{25}{9}+\frac{27}{2}-15-\left(\frac{3}{2}\cdot\frac{25}{9}-\frac{25}{3}\right)=\frac{41}{6}\text{ m}$$

54. **(a)** displacement $=\int_1^6 \left(t^2-2t-8\right)dt=\left[\frac{1}{3}t^3-t^2-8t\right]_1^6=(72-36-48)-\left(\frac{1}{3}-1-8\right)=-\frac{10}{3}$ m

(b)

$$\text{distance traveled } =\int_1^6 |t^2-2t-8|\,dt=\int_1^6 |(t-4)(t+2)|\,dt$$

$$=\int_1^4\left(-t^2+2t+8\right)dt+\int_4^6\left(t^2-2t-8\right)dt=\left[-\frac{1}{3}t^3+t^2+8t\right]_1^4+\left[\frac{1}{3}t^3-t^2-8t\right]_4^6$$

$$=\left(-\frac{64}{3}+16+32\right)-\left(-\frac{1}{3}+1+8\right)+(72-36-48)-\left(\frac{64}{3}-16-32\right)=\frac{98}{3}\text{ m}$$

55. **(a)** $v'(t)=a(t)=t+4\Rightarrow v(t)=\frac{1}{2}t^2+4t+C\Rightarrow v(0)=C=5\Rightarrow v(t)=\frac{1}{2}t^2+4t+5$ m / s

(b)

$$\text{distance traveled } =\int_0^{10}|v(t)|\,dt=\int_0^{10}\left|\frac{1}{2}t^2+4t+5\right|dt=\int_0^{10}\left(\frac{1}{2}t^2+4t+5\right)dt$$

$$=\left[\frac{1}{6}t^3+2t^2+5t\right]_0^{10}=\frac{500}{3}+200+50=416\frac{2}{3}\text{ m}$$

56. **(a)** $v'(t)=a(t)=2t+3\Rightarrow v(t)=t^2+3t+C\Rightarrow v(0)=C=-4\Rightarrow v(t)=t^2+3t-4$

(b)

$$\text{distance traveled} = \int_0^3 \left| t^2+3t-4 \right| dt = \int_0^3 |(t+4)(t-1)|\, dt$$
$$= \int_0^1 \left(-t^2-3t+4\right) dt + \int_1^3 \left(t^2+3t-4\right) dt$$
$$= \left[-\frac{1}{3} t^3 - \frac{3}{2} t^2 + 4t \right]_0^1 + \left[\frac{1}{3} t^3 + \frac{3}{2} t^2 - 4t \right]_1^3$$
$$= \left(-\frac{1}{3} - \frac{3}{2} + 4 \right) + \left(9 + \frac{27}{2} - 12 \right) - \left(\frac{1}{3} + \frac{3}{2} - 4 \right) = \frac{89}{6} \text{ m}$$

57. Since $m'(x) = \rho(x)$, $m = \int_0^4 \rho(x)\,dx = \int_0^4 \left(9+2\sqrt{x}\right) dx = \left[9x + \frac{4}{3} x^{3/2} \right]_0^4 = 36 + \frac{32}{3} - 0 = \frac{140}{3} = 46\frac{2}{3}$ kg.

58. By the Net Change Theorem, the amount of water that flows from the tank is $\int_0^{10} r(t)dt = \int_0^{10} (200-4t)\,dt = \left[200t - 2t^2 \right]_0^{10} = (2000-200) - 0 = 1800$ liters.

59. Let s be the position of the car. We know from Equation 2 that $s(100) - s(0) = \int_0^{100} v(t)\,dt$. We use the Midpoint Rule for $0 \le t \le 100$ with $n=5$. Note that the length of each of the five time intervals is 20 seconds $= \frac{20}{3600}$ hour$= \frac{1}{180}$ hour. So the distance traveled is

$$\int_0^{100} v(t)\,dt \approx \frac{1}{180} [v(10)+v(30)+v(50)+v(70)+v(90)]$$
$$= \frac{1}{180} (38+58+51+53+47)$$
$$= \frac{247}{180} \approx 1.4 \text{miles}$$

60. **(a)** By the Net Change Theorem, the total amount spewed into the atmosphere is $Q(6) - Q(0) = \int_0^6 r(t)\,dt = Q(6)$ since $Q(0)=0$. The rate $r(t)$ is positive, so Q is an increasing function. Thus, an upper estimate for $Q(6)$ is R_6 and a lower estimate for $Q(6)$ is L_6. $\Delta t = \frac{b-a}{n} = \frac{6-0}{6} = 1$.

$R_6 = \sum_{i=1}^{6} r\left(t_i\right) \Delta t = 10+24+36+46+54+60 = 230$ tonnes.

$L_6 = \sum_{i=1}^{6} r\left(t_{i-1}\right) \Delta t = R_6 + r(0) - r(6) = 230+2-60 = 172$ tonnes.

(b) $\Delta t = \frac{b-a}{n} = \frac{6-0}{3} = 2$. $Q(6) \approx M_3 = 2[r(1)+r(3)+r(5)] = 2(10+36+54) = 2(100) = 200$ tonnes.

61. From the Net Change Theorem, the increase in cost if the production level is raised from 2000 yards to 4000 yards is $C(4000)-C(2000)=\int_{2000}^{4000} C'(x)\,dx$.

$$\int_{2000}^{4000} C'(x)\,dx = \int_{2000}^{4000} \left(3-0.01x+0.000006x^2\right) dx$$
$$= \left[3x-0.005x^2+0.000002x^3\right]_{2000}^{4000} = 60{,}000-2{,}000=\$58{,}000$$

62. By the Net Change Theorem, the amount of water after four days is

$$25{,}000+\int_0^4 r(t)\,dt \approx 25{,}000+M_4$$
$$=25{,}000+\frac{4-0}{4}\left[r(0.5)+r(1.5)+r(2.5)+r(3.5)\right]$$
$$\approx 25{,}000+\left[1500+1770+740+(-690)\right]=28{,}320 \text{ liters}$$

63. **(a)** We can find the area between the Lorenz curve and the line $y=x$ by subtracting the area under $y=L(x)$ from the area under $y=x$. Thus,

$$\text{coefficient of inequality} = \frac{\text{area between Lorenz curve and line } y=x}{\text{area under line } y=x} = \frac{\int_0^1 [x-L(x)]\,dx}{\int_0^1 x\,dx}$$
$$= \frac{\int_0^1 [x-L(x)]\,dx}{\left[x^2/2\right]_0^1} = \frac{\int_0^1 [x-L(x)]\,dx}{1/2} = 2\int_0^1 [x-L(x)]\,dx$$

(b) $L(x)=\frac{5}{12}x^2+\frac{7}{12}x \Rightarrow L(50\%)=L\left(\frac{1}{2}\right)=\frac{5}{48}+\frac{7}{24}=\frac{19}{48}=0.3958\overline{3}$, so the bottom 50% of the households receive at most about 40% of the income. Using the result in part (a),

$$\text{coefficient of inequality} = 2\int_0^1 [x-L(x)]\,dx = 2\int_0^1 \left(x-\frac{5}{12}x^2-\frac{7}{12}x\right) dx$$
$$= 2\int_0^1 \left(\frac{5}{12}x-\frac{5}{12}x^2\right) dx = 2\int_0^1 \frac{5}{12}\left(x-x^2\right) dx$$
$$= \frac{5}{6}\left[\frac{1}{2}x^2-\frac{1}{3}x^3\right]_0^1 = \frac{5}{6}\left(\frac{1}{2}-\frac{1}{3}\right) = \frac{5}{6}\left(\frac{1}{6}\right) = \frac{5}{36}$$

64. **(a)** From Exercise 4.1. (a), $v(t)=0.00146t^3-0.11553t^2+24.98169t-21.26872$.

(b) $h(125)-h(0)=\int_0^{125} v(t)\,dt=\left[0.000365t^4-0.03851t^3+12.490845t^2-21.26872t\right]_0^{125} \approx 206{,}407$ ft

1. Let $u=3x$. Then $du=3dx$, so $dx=\frac{1}{3}\,du$. Thus,
$\int \cos 3x\,dx=\int \cos u\left(\frac{1}{3}\,du\right)=\frac{1}{3}\int \cos u\,du=\frac{1}{3}\sin u+C=\frac{1}{3}\sin 3x+C$. Don't forget that it is often very easy to check an indefinite integration by differentiating your answer. In this case,
$\frac{d}{dx}\left(\frac{1}{3}\sin 3x+C\right)=\frac{1}{3}(\cos 3x)\cdot 3=\cos 3x$, the desired result.

2. Let $u=4+x^2$. Then $du=2x\,dx$ and $x\,dx=\frac{1}{2}\,du$, so
$\int x\left(4+x^2\right)^{10}dx=\int u^{10}\left(\frac{1}{2}\,du\right)=\frac{1}{2}\cdot\frac{1}{11}u^{11}+C=\frac{1}{22}\left(4+x^2\right)^{11}+C$.

3. Let $u=x^3+1$. Then $du=3x^2\,dx$ and $x^2\,dx=\frac{1}{3}\,du$, so
$\int x^2\sqrt{x^3+1}\,dx=\int \sqrt{u}\left(\frac{1}{3}\,du\right)=\frac{1}{3}\,\frac{u^{3/2}}{3/2}+C=\frac{1}{3}\cdot\frac{2}{3}u^{3/2}+C=\frac{2}{9}(x^3+1)^{3/2}+C$.

4. Let $u=\sqrt{x}$. Then $du=\frac{1}{2\sqrt{x}}\,dx$ and $\frac{1}{\sqrt{x}}\,dx=2\,du$, so
$\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx=\int \sin u(2\,du)=2(-\cos u)+C=-2\cos\sqrt{x}+C$.

5. Let $u=1+2x$. Then $du=2\,dx$ and $dx=\frac{1}{2}\,du$, so
$\int \frac{4}{(1+2x)^3}\,dx=4\int u^{-3}\left(\frac{1}{2}\,du\right)=2\,\frac{u^{-2}}{-2}+C=-\frac{1}{u^2}+C=-\frac{1}{(1+2x)^2}+C$.

6. Let $u=\sin\theta$. Then $du=\cos\theta\,d\theta$, so $\int e^{\sin\theta}\cos\theta\,d\theta=\int e^u\,du=e^u+C=e^{\sin\theta}+C$.

7. Let $u=x^2+3$. Then $du=2x\,dx$, so $\int 2x(x^2+3)^4\,dx=\int u^4\,du=\frac{1}{5}u^5+C=\frac{1}{5}(x^2+3)^5+C$.

8. Let $u=x^3+5$. Then $du=3x^2\,dx$ and $x^2\,dx=\frac{1}{3}\,du$, so
$\int x^2(x^3+5)^9\,dx=\int u^9\left(\frac{1}{3}\,du\right)=\frac{1}{3}\cdot\frac{1}{10}u^{10}+C=\frac{1}{30}(x^3+5)^{10}+C$.

9. Let $u=3x-2$. Then $du=3\,dx$ and

$dx=\frac{1}{3}\,du$, so $\int(3x-2)^{20}dx=\int u^{20}\left(\frac{1}{3}\,du\right)=\frac{1}{3}\cdot\frac{1}{21}\,u^{21}+C=\frac{1}{63}\,(3x-2)^{21}+C$.

10. Let $u=2-x$. Then $du=-dx$ and $dx=-du$, so $\int(2-x)^{6}dx=\int u^{6}(-du)=-\frac{1}{7}\,u^{7}+C=-\frac{1}{7}\,(2-x)^{7}+C$.

11. Let $u=1+x+2x^{2}$. Then $du=(1+4x)dx$, so

$$\int\frac{1+4x}{\sqrt{1+x+2x^{2}}}\,dx=\int\frac{du}{\sqrt{u}}=\int u^{-1/2}du=\frac{u^{1/2}}{1/2}+C=2\sqrt{1+x+2x^{2}}+C\ .$$

12. Let $u=x^{2}+1$. Then $du=2x\,dx$ and $x\,dx=\frac{1}{2}\,du$, so

$$\int\frac{x}{(x^{2}+1)^{2}}\,dx=\int u^{-2}\left(\frac{1}{2}\,du\right)=\frac{1}{2}\cdot\frac{-1}{u}+C=\frac{-1}{2u}+C=\frac{-1}{2(x^{2}+1)}+C\ .$$

13. Let $u=5-3x$. Then $du=-3\,dx$ and $dx=-\frac{1}{3}\,du$, so

$$\int\frac{dx}{5-3x}=\int\frac{1}{u}\left(-\frac{1}{3}\,du\right)=-\frac{1}{3}\ln|u|+C=-\frac{1}{3}\ln|5-3x|+C\ .$$

14. Let $u=x^{2}+1$. Then $du=2x\,dx$ and $x\,dx=\frac{1}{2}\,du$, so

$$\int\frac{x}{x^{2}+1}\,dx=\int\frac{\frac{1}{2}\,du}{u}=\frac{1}{2}\ln|u|+C=\frac{1}{2}\ln\left|x^{2}+1\right|+C=\frac{1}{2}\ln\left(x^{2}+1\right)+C\ \text{[since } x^{2}+1>0\text{]}$$

or $\ln(x^{2}+1)^{1/2}+C=\ln\sqrt{x^{2}+1}+C$.

15. Let $u=2y+1$. Then $du=2\,dy$ and $dy=\frac{1}{2}\,du$, so

$$\int\frac{3}{(2y+1)^{5}}\,dy=\int 3u^{-5}\left(\frac{1}{2}\,du\right)=\frac{3}{2}\cdot\frac{1}{-4}\,u^{-4}+C=\frac{-3}{8(2y+1)^{4}}+C\ .$$

16. Let $u=5t+4$. Then $du=5\,dt$ and $dt=\frac{1}{5}\,du$, so

$$\int\frac{1}{(5t+4)^{2.7}}\,dt=\int u^{-2.7}\left(\frac{1}{5}\,du\right)=\frac{1}{5}\cdot\frac{1}{-1.7}\,u^{-1.7}+C=\frac{-1}{8.5}\,u^{-1.7}+C=\frac{-2}{17(5t+4)^{1.7}}+C\ .$$

17. Let $u=4-t$. Then $du=-dt$ and $dt=-du$, so $\int\sqrt{4-t}\,dt=\int u^{1/2}(-du)=-\frac{2}{3}u^{3/2}+C=-\frac{2}{3}(4-t)^{3/2}+C$.

18. Let $u=2y^4-1$. Then $du=8y^3\,dy$ and $y^3\,dy=\frac{1}{8}\,du$, so
$\int y^3\sqrt{2y^4-1}\,dy=\int u^{1/2}\left(\frac{1}{8}\,du\right)=\frac{1}{8}\cdot\frac{2}{3}u^{3/2}+C=\frac{1}{12}(2y^4-1)^{3/2}+C$.

19. Let $u=\pi t$. Then $du=\pi\,dt$ and $dt=\frac{1}{\pi}\,du$, so
$\int\sin\pi t\,dt=\int\sin u\left(\frac{1}{\pi}\,du\right)=\frac{1}{\pi}(-\cos u)+C=-\frac{1}{\pi}\cos\pi t+C$.

20. Let $u=2\theta$. Then $du=2\,d\theta$ and $d\theta=\frac{1}{2}\,du$, so $\int\sec 2\theta\tan 2\theta\,d\theta=\int\sec u$
$\tan u\left(\frac{1}{2}\,du\right)=\frac{1}{2}\sec u+C=\frac{1}{2}\sec 2\theta+C$.

21. Let $u=\ln x$. Then $du=\frac{dx}{x}$, so $\int\frac{(\ln x)^2}{x}\,dx=\int u^2\,du=\frac{1}{3}u^3+C=\frac{1}{3}(\ln x)^3+C$.

22. Let $u=\tan^{-1}x$. Then $du=\frac{dx}{1+x^2}$, so $\int\frac{\tan^{-1}x}{1+x^2}\,dx=\int u\,du=\frac{u^2}{2}+C=\frac{\left(\tan^{-1}x\right)^2}{2}+C$.

23. Let $u=\sqrt{t}$. Then $du=\frac{dt}{2\sqrt{t}}$ and $\frac{1}{\sqrt{t}}\,dt=2\,du$, so $\int\frac{\cos\sqrt{t}}{\sqrt{t}}\,dt=\int\cos u\,(2\,du)=2\sin u+C=2\sin\sqrt{t}+C$
.

24. Let $u=1+x^{3/2}$. Then $du=\frac{3}{2}x^{1/2}dx$ and $\sqrt{x}\,dx=\frac{2}{3}\,du$, so
$\int\sqrt{x}\sin(1+x^{3/2})dx=\int\sin u\left(\frac{2}{3}\,du\right)=\frac{2}{3}\cdot(-\cos u)+C=-\frac{2}{3}\cos(1+x^{3/2})+C$.

25. Let $u=\sin\theta$. Then $du=\cos\theta\,d\theta$, so $\int\cos\theta\sin^6\theta\,d\theta=\int u^6\,du=\frac{1}{7}u^7+C=\frac{1}{7}\sin^7\theta+C$.

26. Let $u=1+\tan\theta$. Then $du=\sec^2\theta\,d\theta$, so $\int(1+\tan\theta)^5\sec^2\theta\,d\theta=\int u^5\,du=\frac{1}{6}u^6+C=\frac{1}{6}(1+\tan\theta)^6+C$.

27. Let $u=1+e^{x}$. Then $du=e^{x}dx$, so $\int e^{x}\sqrt{1+e^{x}}\,dx=\int\sqrt{u}\,du=\frac{2}{3}u^{3/2}+C=\frac{2}{3}(1+e^{x})^{3/2}+C$.

Or: Let $u=\sqrt{1+e^{x}}$. Then $u^{2}=1+e^{x}$ and $2u\,du=e^{x}dx$, so
$\int e^{x}\sqrt{1+e^{x}}\,dx=\int u\cdot 2u\,du=\frac{2}{3}u^{3}+C=\frac{2}{3}(1+e^{x})^{3/2}+C$.

28. Let $u=\cos t$. Then $du=-\sin t\,dt$ and $\sin t\,dt=-du$, so $\int e^{\cos t}\sin t\,dt=\int e^{u}(-du)=-e^{u}+C=-e^{\cos t}+C$.

29. Let $u=1+z^{3}$. Then $du=3z^{2}dz$ and $z^{2}dz=\frac{1}{3}du$, so
$\int\frac{z^{2}}{\sqrt[3]{1+z^{3}}}dz=\int u^{-1/3}\left(\frac{1}{3}du\right)=\frac{1}{3}\cdot\frac{3}{2}u^{2/3}+C=\frac{1}{2}(1+z^{3})^{2/3}+C$.

30. Let $u=ax^{2}+2bx+c$. Then $du=2(ax+b)dx$ and $(ax+b)dx=\frac{1}{2}du$, so
$\int\frac{(ax+b)dx}{\sqrt{ax^{2}+2bx+c}}=\int\frac{\frac{1}{2}du}{\sqrt{u}}=\frac{1}{2}\int u^{-1/2}du=u^{1/2}+C=\sqrt{ax^{2}+2bx+c}+C$.

31. Let $u=\ln x$. Then $du=\frac{dx}{x}$, so $\int\frac{dx}{x\ln x}=\int\frac{du}{u}=\ln|u|+C=\ln|\ln x|+C$.

32. Let $u=e^{x}+1$. Then $du=e^{x}dx$, so $\int\frac{e^{x}}{e^{x}+1}dx=\int\frac{du}{u}=\ln|u|+C=\ln\left(e^{x}+1\right)+C$.

33. Let $u=\cot x$. Then $du=-\csc^{2}x\,dx$ and $\csc^{2}x\,dx=-du$, so
$\int\sqrt{\cot x}\csc^{2}x\,dx=\int\sqrt{u}(-du)=-\frac{u^{3/2}}{3/2}+C=-\frac{2}{3}(\cot x)^{3/2}+C$.

34. Let $u=\frac{\pi}{x}$. Then $du=-\frac{\pi}{x^{2}}dx$ and $\frac{1}{x^{2}}dx=-\frac{1}{\pi}du$, so
$\int\frac{\cos(\pi/x)}{x^{2}}dx=\int\cos u\left(-\frac{1}{\pi}du\right)=-\frac{1}{\pi}\sin u+C=-\frac{1}{\pi}\sin\frac{\pi}{x}+C$.

35.

$\int \cot x\,dx=\int \frac{\cos x}{\sin x}\,dx$. Let $u=\sin x$. Then $du=\cos x dx$, so $\int \cot x\,dx=\int \frac{1}{u}\,du=\ln |u|+C=\ln |\sin x|+C$.

36. Let $u=\cos x$. Then $du=-\sin x\,dx$ and $\sin x\,dx=-du$, so
$\int \frac{\sin x}{1+\cos^2 x}\,dx=\int \frac{-du}{1+u^2}=-\tan^{-1}u+C=-\tan^{-1}(\cos x)+C$.

37. Let $u=\sec x$. Then $du=\sec x\tan x\,dx$, so
$\int \sec^3 x\tan x\,dx=\int \sec^2 x(\sec x\tan x)\,dx=\int u^2\,du=\frac{1}{3}u^3+C=\frac{1}{3}\sec^3 x+C$.

38. Let $u=x^3+1$. Then $x^3=u-1$ and $du=3x^2dx$, so

$$\int \sqrt[3]{x^3+1}\,x^5dx = \int \sqrt[3]{x^3+1}\cdot x^3\cdot x^2dx=\int u^{1/3}(u-1)\left(\frac{1}{3}\,du\right)=\frac{1}{3}\int (u^{4/3}-u^{1/3})\,du$$
$$= \frac{1}{3}\left(\frac{3}{7}u^{7/3}-\frac{3}{4}u^{4/3}\right)+C=\frac{1}{7}(x^3+1)^{7/3}-\frac{1}{4}(x^3+1)^{4/3}+C$$

39. Let $u=b+cx^{a+1}$. Then $du=(a+1)cx^a dx$, so
$\int x^a\sqrt{b+cx^{a+1}}\,dx=\int u^{1/2}\frac{1}{(a+1)c}\,du=\frac{1}{(a+1)c}\left(\frac{2}{3}u^{3/2}\right)+C=\frac{2}{3c(a+1)}\left(b+cx^{a+1}\right)^{3/2}+C$.

40. Let $u=\cos$ t. Then $du=-\sin t\,dt$ and $\sin t\,dt=-du$, so
$\int \sin t\sec^2(\cos t)\,dt=\int \sec^2 u\cdot(-du)=-\tan u+C=-\tan(\cos t)+C$.

41. Let $u=1+x^2$. Then $du=2x\,dx$, so

$$\int \frac{1+x}{1+x^2}\,dx=\int \frac{1}{1+x^2}\,dx+\int \frac{x}{1+x^2}\,dx=\tan^{-1}x+\int \frac{\frac{1}{2}du}{u}=\tan^{-1}x+\frac{1}{2}\ln |u|+C$$
$=\tan^{-1}x+\frac{1}{2}\ln \left|1+x^2\right|+C=\tan^{-1}x+\frac{1}{2}\ln\left(1+x^2\right)+C$ [since $1+x^2>0$].

42. Let $u=x^2$. Then $du=2x\,dx$, so $\int \frac{x}{1+x^4}\,dx=\int \frac{\frac{1}{2}du}{1+u^2}=\frac{1}{2}\tan^{-1}u+C=\frac{1}{2}\tan^{-1}\left(x^2\right)+C$.

43. Let $u=x+2$. Then $du=dx$, so

$$\int \frac{x}{\sqrt[4]{x+2}}\,dx = \int \frac{u-2}{\sqrt[4]{u}}\,du = \int \left(u^{3/4}-2u^{-1/4}\right)du = \frac{4}{7}u^{7/4}-2\cdot\frac{4}{3}u^{3/4}+C$$
$$= \frac{4}{7}(x+2)^{7/4}-\frac{8}{3}(x+2)^{3/4}+C$$

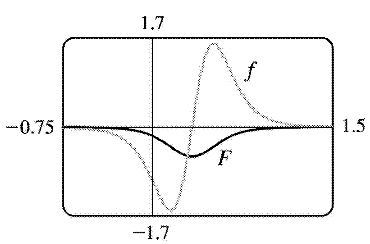
44. Let $u=1-x$. Then $x=1-u$ and $dx=-du$, so

$$\int \frac{x^2}{\sqrt{1-x}}\,dx = \int \frac{(1-u)^2}{\sqrt{u}}(-du) = -\int \frac{1-2u+u^2}{\sqrt{u}}\,du = -\int \left(u^{-1/2}-2u^{1/2}+u^{3/2}\right)du$$
$$= -\left(2u^{1/2}-2\cdot\frac{2}{3}u^{3/2}+\frac{2}{5}u^{5/2}\right)+C = -2\sqrt{1-x}+\frac{4}{3}(1-x)^{3/2}-\frac{2}{5}(1-x)^{5/2}+C$$

45. $f(x)=\dfrac{3x-1}{\left(3x^2-2x+1\right)^4}$.

$u=3x^2-2x+1 \Rightarrow du=(6x-2)dx=2(3x-1)dx$, so

$$\int \frac{3x-1}{\left(3x^2-2x+1\right)^4}\,dx = \int \frac{1}{u^4}\left(\frac{1}{2}\,du\right) = \frac{1}{2}\int u^{-4}\,du$$
$$= -\frac{1}{6}u^{-3}+C = -\frac{1}{6\left(3x^2-2x+1\right)^3}+C$$



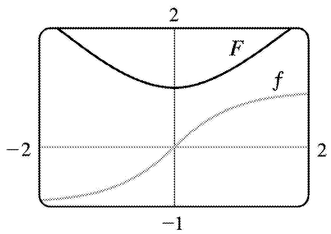
Notice that at $x=\frac{1}{3}$, f changes from negative to positive, and F has a local minimum.

46. $f(x)=\dfrac{x}{\sqrt{x^2+1}}$. $u=x^2+1 \Rightarrow du=2x\,dx$, so

$$\int \frac{x}{\sqrt{x^2+1}}\,dx = \int \frac{1}{\sqrt{u}}\left(\frac{1}{2}\,du\right) = \frac{1}{2}\int u^{-1/2}\,du$$

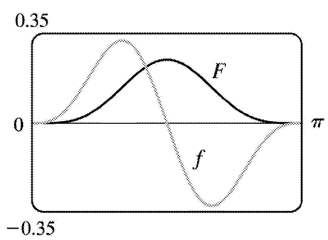
$$= u^{1/2} + C = \sqrt{x^2+1} + C.$$

Note that at $x=0$, f changes from negative to positive and F has a local minimum.



47. $f(x)=\sin^3 x \cos x$. $u=\sin x \Rightarrow du=\cos x\,dx$, so $\left[\int \sin^3 x \cos x\,dx = \int u^3\,du = \frac{1}{4}u^4 + C = \frac{1}{4}\sin^4 x + C\right]$

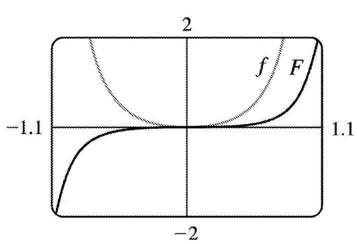
Note that at $x=\frac{\pi}{2}$, f changes from positive to negative and F has a local maximum. Also, both f and F are periodic with period π , so at $x=0$ and at $x=\pi$, f changes from negative to positive and F has local minima.



48. $f(\theta)=\tan^2\theta \sec^2\theta$. $u=\tan\theta \Rightarrow du=\sec^2\theta\,d\theta$, so

$$\int \tan^2\theta \sec^2\theta\,d\theta = \int u^2\,du = \frac{1}{3}u^3 + C = \frac{1}{3}\tan^3\theta + C$$

Note that f is positive and F is increasing. At $x=0$, $f=0$ and F has a horizontal tangent.



49. Let $u=x-1$, so $du=dx$. When $x=0$, $u=-1$; when $x=2$, $u=1$. Thus, $\int_0^2 (x-1)^{25}\,dx = \int_{-1}^1 u^{25}\,du = 0$ by Theorem 7(b), since $f(u)=u^{25}$ is an odd function.

50. Let $u=4+3x$, so $du=3dx$. When $x=0$, $u=4$; when $x=7$, $u=25$. Thus,

$$\int_0^7\sqrt{4+3x}\,dx=\int_4^{25}\sqrt{u}\left(\frac{1}{3}\,du\right)=\frac{1}{3}\left[\frac{u^{3/2}}{3/2}\right]_4^{25}=\frac{2}{9}\left(25^{3/2}-4^{3/2}\right)=\frac{2}{9}(125-8)=\frac{234}{9}=26$$

51. Let $u=1+2x^3$, so $du=6x^2dx$. When $x=0$, $u=1$; when $x=1$, $u=3$. Thus,

$$\int_0^1 x^2\left(1+2x^3\right)^5dx=\int_1^3 u^5\left(\frac{1}{6}\,du\right)=\frac{1}{6}\left[\frac{1}{6}u^6\right]_1^3=\frac{1}{36}\left(3^6-1^6\right)=\frac{1}{36}(729-1)=\frac{728}{36}=\frac{182}{9}$$

52. Let $u=x^2$, so $du=2x\,dx$. When $x=0$, $u=0$; when $x=\sqrt{\pi}$, $u=\pi$. Thus,
$$\int_0^{\sqrt{\pi}} x\cos\left(x^2\right)dx=\int_0^{\pi}\cos u\left(\frac{1}{2}\,du\right)=\frac{1}{2}\left[\sin u\right]_0^{\pi}=\frac{1}{2}(\sin\pi-\sin 0)=\frac{1}{2}(0-0)=0 .$$

53. Let $u=t/4$, so $du=\frac{1}{4}\,dt$. When $t=0$, $u=0$; when $t=\pi$, $u=\pi/4$. Thus,
$$\int_0^{\pi}\sec^2(t/4)\,dt=\int_0^{\pi/4}\sec^2u\,(4\,du)=4\left[\tan u\right]_0^{\pi/4}=4\left(\tan\frac{\pi}{4}-\tan 0\right)=4(1-0)=4 .$$

54. Let $u=\pi t$, so $du=\pi\,dt$. When $t=\frac{1}{6}$, $u=\frac{\pi}{6}$; when $t=\frac{1}{2}$, $u=\frac{\pi}{2}$. Thus,
$$\int_{1/6}^{1/2}\csc\pi t\cot\pi t\,dt=\int_{\pi/6}^{\pi/2}\csc u\cot u\left(\frac{1}{\pi}\,du\right)=\frac{1}{\pi}\left[-\csc u\right]_{\pi/6}^{\pi/2}=-\frac{1}{\pi}(1-2)=\frac{1}{\pi} .$$

55. $\int_{-\pi/6}^{\pi/6}\tan^3\theta\,d\theta=0$ by Theorem (b), since $f(\theta)=\tan^3\theta$ is an odd function.

56. $\int_0^2\frac{dx}{(2x-3)^2}$ does not exist since $f(x)=\frac{1}{(2x-3)^2}$ has an infinite discontinuity at $x=\frac{3}{2}$.

57. Let $u=1/x$, so $du=-1/x^2dx$. When $x=1$, $u=1$; when $x=2$, $u=\frac{1}{2}$. Thus,
$$\int_1^2\frac{e^{1/x}}{x^2}\,dx=\int_1^{1/2}e^u(-du)=-\left[e^u\right]_1^{1/2}=-(e^{1/2}-e)=e-\sqrt{e} .$$

58. Let $u=-x^2$, so $du=-2x\,dx$. When $x=0$, $u=0$; when $x=1$, $u=-1$. Thus,
$$\int_0^1 xe^{-x^2}dx=\int_0^{-1}e^u\left(-\frac{1}{2}\,du\right)=-\frac{1}{2}\left[e^u\right]_0^{-1}=-\frac{1}{2}\left(e^{-1}-e^0\right)=\frac{1}{2}(1-1/e) .$$

59. Let $u=\cos\theta$, so $du=-\sin\theta\, d\theta$. When $\theta=0$, $u=1$; when $\theta=\frac{\pi}{3}$, $u=\frac{1}{2}$. Thus,

$$\int_0^{\pi/3}\frac{\sin\theta}{\cos^2\theta}\,d\theta=\int_1^{1/2}\frac{-du}{u^2}=\int_{1/2}^{1}u^{-2}du=\left[-\frac{1}{u}\right]_{1/2}^{1}=-1-(-2)=1 .$$

60. $\int_{-\pi/2}^{\pi/2}\frac{x^2\sin x}{1+x^6}\,dx=0$ by Theorem (b), since $f(x)=\frac{x^2\sin x}{1+x^6}$ is an odd function.

61. Let $u=1+2x$, so $du=2dx$. When $x=0$, $u=1$; when $x=13$, $u=27$. Thus,

$$\int_0^{13}\frac{dx}{\sqrt[3]{(1+2x)^2}}=\int_1^{27}u^{-2/3}\left(\frac{1}{2}\,du\right)=\left[\frac{1}{2}\cdot 3u^{1/3}\right]_1^{27}=\frac{3}{2}(3-1)=3 .$$

62. Let $u=\sin x$, so $du=\cos x\,dx$. When $x=0$, $u=0$; when $x=\frac{\pi}{2}$, $u=1$. Thus,

$$\int_0^{\pi/2}\cos x\sin(\sin x)dx=\int_0^1\sin u\,du=[-\cos u]_0^1=-(\cos 1-1)=1-\cos 1 .$$

63. Let $u=x-1$, so $u+1=x$ and $du=dx$. When $x=1$, $u=0$; when $x=2$, $u=1$. Thus,

$$\int_1^2 x\sqrt{x-1}\,dx=\int_0^1(u+1)\sqrt{u}\,du=\int_0^1(u^{3/2}+u^{1/2})du=\left[\frac{2}{5}u^{5/2}+\frac{2}{3}u^{3/2}\right]_0^1=\frac{2}{5}+\frac{2}{3}=\frac{16}{15} .$$

64. Let $u=1+2x$, so $x=\frac{1}{2}(u-1)$ and $du=2dx$. When $x=0$, $u=1$; when $x=4$, $u=9$. Thus,

$$\int_0^4\frac{x\,dx}{\sqrt{1+2x}}=\int_1^9\frac{\frac{1}{2}(u-1)}{\sqrt{u}}\,\frac{du}{2}=\frac{1}{4}\int_1^9(u^{1/2}-u^{-1/2})du=\frac{1}{4}\left[\frac{2}{3}u^{3/2}-2u^{1/2}\right]_1^9$$

$$=\frac{1}{4}\cdot\frac{2}{3}\left[u^{3/2}-3u^{1/2}\right]_1^9=\frac{1}{6}\,[(27-9)-(1-3)]=\frac{20}{6}=\frac{10}{3}$$

65. Let $u=\ln x$, so $du=\frac{dx}{x}$. When $x=e$, $u=1$; when $x=e^4$; $u=4$. Thus,

$$\int_e^{e^4}\frac{dx}{x\sqrt{\ln x}}=\int_1^4u^{-1/2}du=2\left[u^{1/2}\right]_1^4=2(2-1)=2 .$$

66. Let $u=\sin^{-1}x$, so $du=\frac{dx}{\sqrt{1-x^2}}$. When $x=0$, $u=0$; when $x=\frac{1}{2}$, $u=\frac{\pi}{6}$. Thus,

$$\int_0^{1/2} \frac{\sin^{-1} x}{\sqrt{1-x^2}}\, dx = \int_0^{\pi/6} u\, du = \left[\frac{u^2}{2} \right]_0^{\pi/6} = \frac{\pi^2}{72} .$$

67. $\int_0^4 \frac{dx}{(x-2)^3}$ does not exist since $f(x) = \frac{1}{(x-2)^3}$ has an infinite discontinuity at $x=2$.

68. Assume $a>0$. Let $u=a^2-x^2$, so $du=-2x\,dx$. When $x=0$, $u=a^2$; when $x=a$, $u=0$. Thus,

$$\int_0^a x\sqrt{a^2-x^2}\, dx = \int_{a^2}^0 u^{1/2}\left(-\frac{1}{2}\, du\right) = \frac{1}{2}\int_0^{a^2} u^{1/2} du = \frac{1}{2}\cdot\left[\frac{2}{3}u^{3/2}\right]_0^{a^2} = \frac{1}{3}a^3 .$$

69. Let $u=x^2+a^2$, so $du=2x\,dx$ and $x\,dx = \frac{1}{2}\, du$. When $x=0$, $u=a^2$; when $x=a$, $u=2a^2$. Thus,

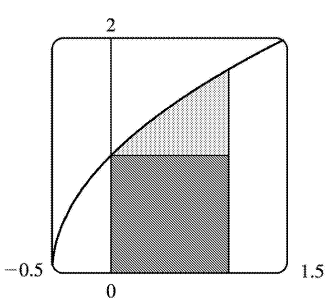
$$\int_0^a x\sqrt{x^2+a^2}\, dx = \int_{a^2}^{2a^2} u^{1/2}\left(\frac{1}{2}\, du\right) = \frac{1}{2}\left[\frac{2}{3}u^{3/2}\right]_{a^2}^{2a^2} = \left[\frac{1}{3}u^{3/2}\right]_{a^2}^{2a^2}$$

$$= \frac{1}{3}\left[(2a^2)^{3/2} - (a^2)^{3/2}\right] = \frac{1}{3}\left(2\sqrt{2}-1\right)a^3$$

70. $\int_{-a}^a x\sqrt{x^2+a^2}\, dx = 0$ by Theorem 7(b), since $f(x)=x\sqrt{x^2+a^2}$ is an odd function.

71. From the graph, it appears that the area under the curve is about $1+\left(\text{a little more than } \frac{1}{2}\cdot 1\cdot 0.7\right)$, or about 1.4 . The exact area is given by $A=\int_0^1\sqrt{2x+1}\, dx$. Let $u=2x+1$, so $du=2\,dx$. The limits change to $2\cdot 0+1=1$ and $2\cdot 1+1=3$, and

$$A=\int_1^3\sqrt{u}\left(\frac{1}{2}\, du\right) = \frac{1}{2}\left[\frac{2}{3}u^{3/2}\right]_1^3 = \frac{1}{3}\left(3\sqrt{3}-1\right) = \sqrt{3}-\frac{1}{3} \approx 1.399 .$$

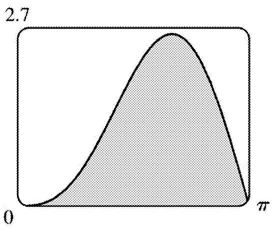


72. From the graph, it appears that the area under the curve is almost $\frac{1}{2}\cdot\pi\cdot 2.6$, or about 4 . The

exact area is given by

$$A = \int_0^{\pi} (2\sin x - \sin 2x)dx = -2[\cos x]_0^{\pi} - \int_0^{\pi} \sin 2x dx$$
$$= -2(-1-1) - 0 = 4$$

Note: $\int_0^{\pi} \sin 2x dx = 0$ since it is clear from the graph of $y = \sin 2x$ that $\int_{\pi/2}^{\pi} \sin 2x dx = -\int_0^{\pi/2} \sin 2x dx$.



73. First write the integral as a sum of two integrals: $I = \int_{-2}^{2} (x+3)\sqrt{4-x^2}\, dx = I_1 + I_2 = \int_{-2}^{2} x\sqrt{4-x^2}\, dx + \int_{-2}^{2} 3\sqrt{4-x^2}\, dx$. $I_1 = 0$ by Theorem 7(b), since $f(x) = x\sqrt{4-x^2}$ is an odd function and we are integrating from $x = -2$ to $x = 2$. We interpret I_2 as three times the area of a semicircle with radius 2 , so $I = 0 + 3 \cdot \frac{1}{2}\left(\pi \cdot 2^2\right) = 6\pi$.

74. Let $u = x^2$. Then $du = 2x dx$ and the limits are unchanged ($0^2 = 0$ and $1^2 = 1$), so $I = \int_0^1 x\sqrt{1-x^4}\, dx = \frac{1}{2}\int_0^1 \sqrt{1-u^2}\, du$. But this integral can be interpreted as the area of a quarter–circle with radius 1 . So $I = \frac{1}{2} \cdot \frac{1}{4}\left(\pi \cdot 1^2\right) = \frac{1}{8}\pi$.

75. **First Figure**

Let $u = \sqrt{x}$, so $x = u^2$ and $dx = 2u du$. When $x = 0$, $u = 0$; when $x = 1$, $u = 1$. Thus, $A_1 = \int_0^1 e^{\sqrt{x}}\, dx = \int_0^1 e^u (2u du) = 2\int_0^1 u e^u du$.

Second Figure

$A_2 = \int_0^1 2x e^x dx = 2\int_0^1 u e^u du$.

Third Figure

Let $u = \sin x$, so $du = \cos x dx$. When $x = 0$, $u = 0$; when

$x=\frac{\pi}{2}$, $u=1$. Thus, $A_3=\int_0^{\pi/2} e^{\sin x}\sin 2x\,dx=\int_0^{\pi/2} e^{\sin x}(2\sin x\cos x)\,dx=\int_0^1 e^u(2u\,du)=2\int_0^1 ue^u\,du$.

Since $A_1=A_2=A_3$, all three areas are equal.

76. Let $r(t)=ae^{bt}$ with $a=450.268$ and $b=1.12567$, and $n(t)=$ population after t hours. Since $r(t)=n'(t)$, $\int_0^3 r(t)\,dt=n(3)-n(0)$ is the total change in the population after three hours. Since we start with 400 bacteria, the population will be

$n(3)=400+\int_0^3 r(t)\,dt=400+\int_0^3 ae^{bt}\,dt=400+\frac{a}{b}\left[e^{bt}\right]_0^3=400+\frac{a}{b}\left(e^{3b}-1\right)\approx 400+11{,}313=11{,}713$ bacteria

77. The volume of inhaled air in the lungs at time t is

$$V(t) = \int_0^t f(u)\,du=\int_0^t \frac{1}{2}\sin\left(\frac{2\pi}{5}u\right)du=\int_0^{2\pi t/5}\frac{1}{2}\sin v\left(\frac{5}{2\pi}\,dv\right) v=\frac{2\pi}{5}u, dv=\frac{2\pi}{5}du]$$
$$= \frac{5}{4\pi}\left[-\cos v\right]_0^{2\pi t/5}=\frac{5}{4\pi}\left[-\cos\left(\frac{2\pi}{5}t\right)+1\right]=\frac{5}{4\pi}\left[1-\cos\left(\frac{2\pi}{5}t\right)\right]\text{ liters}$$

78.

$$\text{Number of calculators } =x(4)-x(2)=\int_2^4 5000\left[1-100(t+10)^{-2}\right]dt$$
$$=5000\left[t+100(t+10)^{-1}\right]_2^4=5000\left[\left(4+\frac{100}{14}\right)-\left(2+\frac{100}{12}\right)\right]\approx 4048$$

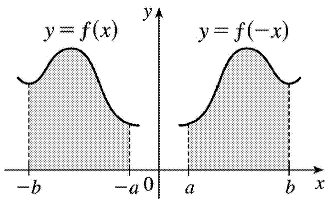
79. Let $u=2x$. Then $du=2dx$, so $\int_0^2 f(2x)\,dx=\int_0^4 f(u)\left(\frac{1}{2}du\right)=\frac{1}{2}\int_0^4 f(u)\,du=\frac{1}{2}(10)=5$.

80. Let $u=x^2$. Then $du=2x\,dx$, so $\int_0^3 xf\left(x^2\right)dx=\int_0^9 f(u)\left(\frac{1}{2}du\right)=\frac{1}{2}\int_0^9 f(u)\,du=\frac{1}{2}(4)=2$.

81. Let $u=-x$. Then $du=-dx$, so

$$\int_a^b f(-x)\,dx=\int_{-a}^{-b} f(u)(-du)=\int_{-b}^{-a} f(u)\,du=\int_{-b}^{-a} f(x)\,dx$$

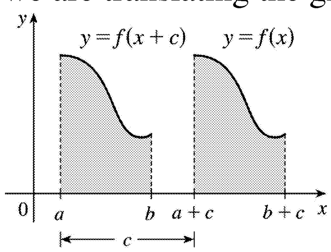
From the diagram, we see that the equality follows from the fact that we are reflecting the graph of f , and the limits of integration, about the $y-$ axis.



82. Let $u=x+c$. Then $du=dx$, so

$$\int_a^b f(x+c)\,dx=\int_{a+c}^{b+c} f(u)\,du=\int_{a+c}^{b+c} f(x)\,dx$$

From the diagram, we see that the equality follows from the fact that
we are translating the graph of f , and the limits of integration, by a distance c .



83. Let $u=1-x$. Then $x=1-u$ and $dx=-du$, so
$\int_0^1 x^a(1-x)^b\,dx=\int_1^0 (1-u)^a u^b(-du)=\int_0^1 u^b(1-u)^a\,du=\int_0^1 x^b(1-x)^a\,dx$.

84. Let $u=\pi-x$. Then $du=-dx$. When $x=\pi$, $u=0$ and when $x=0$, $u=\pi$. So

$$\int_0^\pi xf(\sin x)\,dx = -\int_\pi^0 (\pi-u)f(\sin(\pi-u))\,du=\int_0^\pi (\pi-u)f(\sin u)\,du$$

$$= \pi\int_0^\pi f(\sin u)\,du-\int_0^\pi uf(\sin u)\,du=\pi\int_0^\pi f(\sin x)\,dx-\int_0^\pi xf(\sin x)\,dx$$

$$\Rightarrow 2\int_0^\pi xf(\sin x)\,dx=\pi\int_0^\pi f(\sin x)\,dx\Rightarrow \int_0^\pi xf(\sin x)\,dx=\frac{\pi}{2}\int_0^\pi f(\sin x)\,dx .$$

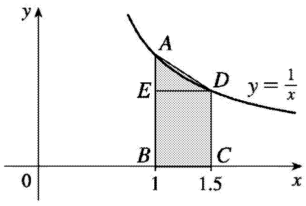
85. $\dfrac{x\sin x}{1+\cos^2 x}=x\cdot\dfrac{\sin x}{2-\sin^2 x}=xf(\sin x)$, where $f(t)=\dfrac{t}{2-t^2}$. By Exercise 84,

$$\int_0^\pi \frac{x\sin x}{1+\cos^2 x}\,dx=\int_0^\pi xf(\sin x)\,dx=\frac{\pi}{2}\int_0^\pi f(\sin x)\,dx=\frac{\pi}{2}\int_0^\pi \frac{\sin x}{1+\cos^2 x}\,dx$$

Let $u=\cos x$. Then $du=-\sin x\,dx$. When $x=\pi$, $u=-1$ and when $x=0$, $u=1$. So

$$\frac{\pi}{2}\int_0^\pi \frac{\sin x}{1+\cos^2 x}\,dx=-\frac{\pi}{2}\int_1^{-1}\frac{du}{1+u^2}=\frac{\pi}{2}\int_{-1}^1\frac{du}{1+u^2}=\frac{\pi}{2}\left[\tan^{-1}u\right]_{-1}^1=$$

$$\frac{\pi}{2}\left[\tan^{-1}1-\tan^{-1}(-1)\right]=\frac{\pi}{2}\left[\frac{\pi}{4}-\left(-\frac{\pi}{4}\right)\right]=\frac{\pi^2}{4}$$



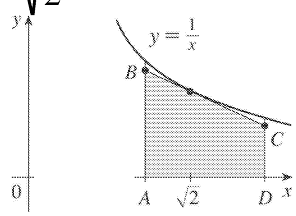
1. **(a)**
We interpret ln 1.5 as the area under the curve $y=1/x$ from $x=1$ to $x=1.5$. The area of the rectangle $BCDE$ is $\frac{1}{2}\cdot\frac{2}{3}=\frac{1}{3}$. The area of the trapezoid $ABCD$ is $\frac{1}{2}\cdot\frac{1}{2}\left(1+\frac{2}{3}\right)=\frac{5}{12}$. Thus, by comparing areas, we observe that $\frac{1}{3}<\ln 1.5<\frac{5}{12}$.

(b) With $f(t)=1/t$, $n=10$, and $\Delta x=0.05$, we have

$$\ln 1.5=\int_1^{1.5}(1/t)\,dt\approx(0.05)[f(1.025)+f(1.075)+\cdots+f(1.475)]$$

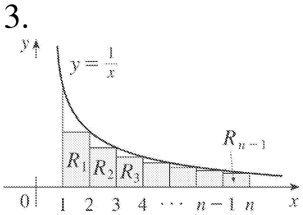
$$=(0.05)\left[\frac{1}{1.025}+\frac{1}{1.075}+\cdots+\frac{1}{1.475}\right]\approx 0.4054$$

2. **(a)** $y=\frac{1}{t}$, $y'=-\frac{1}{t^2}$. The slope of AD is $\frac{1/2-1}{2-1}=-\frac{1}{2}$. Let c be the $t-$ coordinate of the point on $y=\frac{1}{t}$ with slope $-\frac{1}{2}$. Then $-\frac{1}{c^2}=-\frac{1}{2}\Rightarrow c^2=2\Rightarrow c=\sqrt{2}$ since $c>0$. Therefore, the tangent line is given by $y-\frac{1}{\sqrt{2}}=-\frac{1}{2}\left(t-\sqrt{2}\right)\Rightarrow y=-\frac{1}{2}t+\sqrt{2}$.

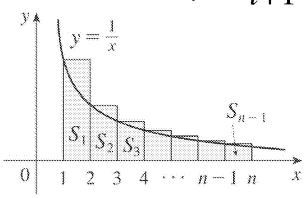


(b)
Since the graph of $y=1/t$ is concave upward, the graph lies above the tangent line, that is, above the line segment BC . Now $|AB|=-\frac{1}{2}+\sqrt{2}$ and $|CD|=-1+\sqrt{2}$. So the area of the trapezoid $ABCD$ is $\frac{1}{2}\left[\left(-\frac{1}{2}+\sqrt{2}\right)+\left(-1+\sqrt{2}\right)1\right]=-\frac{3}{4}+\sqrt{2}\approx 0.6642$. So ln 2> area of trapezoid $ABCD>0.66$.

3.



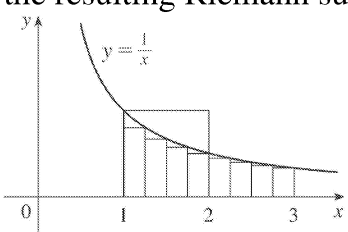
The area of R_i is $\frac{1}{i+1}$ and so $\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}<\int_1^n \frac{1}{t}\,dt=\ln n$.



The area of S_i is $\frac{1}{i}$ and so $1+\frac{1}{2}+\cdots+\frac{1}{n-1}>\int_1^n \frac{1}{t}\,dt=\ln n$.

Thus, $\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}<\ln n<1+\frac{1}{2}+\cdots+\frac{1}{n-1}$.

4. **(a)** From the diagram, we see that the area under the graph of $y=1/x$ between $x=1$ and $x=2$ is less than the area of the square, which is 1 . So $\ln 2=\int_1^2 (1/x)\,dx<1$. To show the other side of the inequality, we must find an area larger than 1 which lies under the graph of $y=1/x$ between $x=1$ and $x=3$. One way to do this is to partition the interval $[1,3]$ into 8 intervals of equal length and calculate the resulting Riemann sum, using the right endpoints:



$$\frac{1}{4}\left(\frac{1}{5/4}+\frac{1}{3/2}+\frac{1}{7/4}+\frac{1}{2}+\frac{1}{9/4}+\frac{1}{5/2}+\frac{1}{11/4}+\frac{1}{3}\right)=\frac{28{,}271}{27{,}720}>1$$

and therefore $1<\int_1^3 (1/x)\,dx=\ln 3$.

A slightly easier method uses the fact that since $y=1/x$ is concave upward, it lies above all its tangent lines. Drawing two such tangent lines at the points $\left(\frac{3}{2},\frac{2}{3}\right)$ and $\left(\frac{5}{2},\frac{2}{5}\right)$, we see that the area under the curve from $x=1$ to $x=3$ is more than the sum of the areas of the two trapezoids, that is, $\frac{2}{3}+\frac{2}{5}=\frac{16}{15}$. Thus, $1<\frac{16}{15}<\int_1^3 (1/x)\,dx=\ln 3$.

(b) By part (a), $\ln 2<1<\ln 3$. But e is defined such that $\ln e=1$, and because the natural logarithm function is increasing, we have $\ln 2<\ln e<\ln 3\Leftrightarrow 2<e<3$.

5. If $f(x)=\ln\left(x^r\right)$, then $f'(x)=\left(1/x^r\right)\left(rx^{r-1}\right)=r/x$. But if $g(x)=r\ln x$, then $g'(x)=r/x$. So f and g must differ by a constant: $\ln\left(x^r\right)=r\ln x+C$. Put $x=1$: $\ln\left(1^r\right)=r\ln 1+C\Rightarrow C=0$, so $\ln\left(x^r\right)=r\ln x$.

6. Using the second law of logarithms and Equation 10, we have $\ln\left(e^x/e^y\right)=\ln e^x-\ln e^y=x-y=\ln\left(e^{x-y}\right)$. Since ln is a one–to–one function, it follows that $e^x/e^y=e^{x-y}$.

7. Using the third law of logarithms and Equation 10, we have $\ln e^{rx}=rx=r\ln e^x=\ln\left(e^x\right)^r$. Since ln is a one–to–one function, it follows that $e^{rx}=\left(e^x\right)^r$.

8. Using Definition 13 and the second law of exponents for e^x , we have
$a^{x-y}=e^{(x-y)\ln a}=e^{x\ln a-y\ln a}=\dfrac{e^{x\ln a}}{e^{y\ln a}}=\dfrac{a^x}{a^y}$.

9. Using Definition 13, the first law of logarithms, and the first law of exponents for e^x , we have $(ab)^x=e^{x\ln(ab)}=e^{x(\ln a+\ln b)}=e^{x\ln a+x\ln b}=e^{x\ln a}e^{x\ln b}=a^xb^x$.

10. Let $\log_a x=r$ and $\log_a y=s$. Then $a^r=x$ and $a^s=y$.

(a) $xy=a^ra^s=a^{r+s}\Rightarrow\log_a(xy)=r+s=\log_a x+\log_a y$

(b) $\dfrac{x}{y}=\dfrac{a^r}{a^s}=a^{r-s}\Rightarrow\log_a\dfrac{x}{y}=r-s=\log_a x-\log_a y$

(c) $x^y=\left(a^r\right)^y=a^{ry}\Rightarrow\log_a\left(x^y\right)=ry=y\log_a x$

1.

$$A=\int_{x=0}^{x=4}\left(y_T-y_B\right)dx=\int_0^4\left[\left(5x-x^2\right)-x\right]dx=\int_0^4\left(4x-x^2\right)dx$$

$$=\left[2x^2-\frac{1}{3}x^3\right]_0^4=\left(32-\frac{64}{3}\right)-(0)=\frac{32}{3}$$

2.

$$A=\int_0^2\left(\sqrt{x+2}-\frac{1}{x+1}\right)dx=\left[\frac{2}{3}(x+2)^{3/2}-\ln(x+1)\right]_0^2$$

$$=\left[\frac{2}{3}(4)^{3/2}-\ln 3\right]-\left[\frac{2}{3}(2)^{3/2}-\ln 1\right]=\frac{16}{3}-\ln 3-\frac{4}{3}\sqrt{2}$$

3.

$$A=\int_{y=-1}^{y=1}\left(x_R-x_L\right)dy=\int_{-1}^1\left[e^y-\left(y^2-2\right)\right]dy$$

$$=\int_{-1}^1\left(e^y-y^2+2\right)dy=\left[e^y-\frac{1}{3}y^3+2y\right]_{-1}^1=\left(e^1-\frac{1}{3}+2\right)-\left(e^{-1}+\frac{1}{3}-2\right)=e-\frac{1}{e}+\frac{10}{3}$$

4.

$$A=\int_0^3\left[\left(2y-y^2\right)-\left(y^2-4y\right)\right]dy=\int_0^3\left(-2y^2+6y\right)dy$$

$$=\left[-\frac{2}{3}y^3+3y^2\right]_0^3=(-18+27)-0=9$$

5.

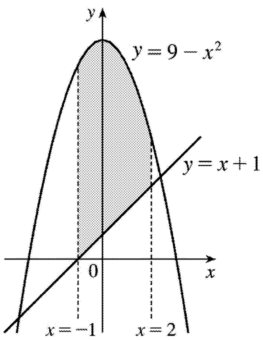
$$A=\int_{-1}^2\left[\left(9-x^2\right)-(x+1)\right]dx$$

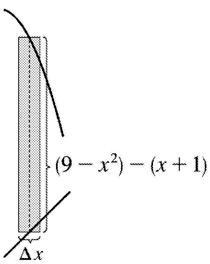
$$=\int_{-1}^2\left(8-x-x^2\right)dx$$

$$=\left[8x-\frac{x^2}{2}-\frac{x^3}{3}\right]_{-1}^2$$

$$=\left(16-2-\frac{8}{3}\right)-\left(-8-\frac{1}{2}+\frac{1}{3}\right)$$

$$=22-3+\frac{1}{2}=\frac{39}{2}$$





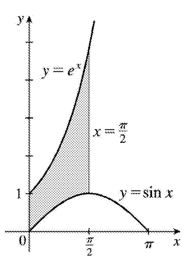
6.

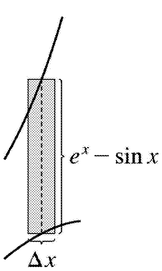
$$A=\int_0^{\pi/2}\left(e^x-\sin x\right)dx$$

$$=\left[e^x+\cos x\right]_0^{\pi/2}$$

$$=\left(e^{\pi/2}+0\right)-(1+1)$$

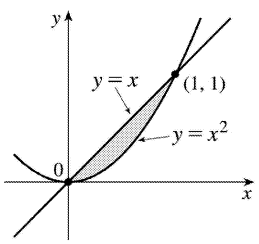
$$=e^{\pi/2}-2$$

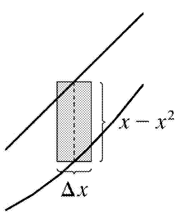




7. The curves intersect when $x=x^2 \Rightarrow x^2-x=0 \Leftrightarrow x(x-1)=0 \Leftrightarrow x=0\ ,\ 1$.

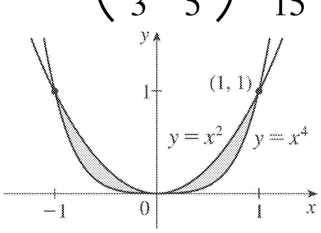
$$A = \int_0^1 \left(x-x^2\right)dx$$
$$= \left[\ \frac{1}{2}x^2 - \frac{1}{3}x^3\ \right]_0^1$$
$$= \frac{1}{2} - \frac{1}{3}$$
$$= \frac{1}{6}$$

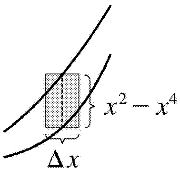




8.

$$A = \int_{-1}^1 \left(x^2-x^4\right)dx$$
$$=2\int_0^1 \left(x^2-x^4\right)dx$$
$$=2\left[\ \frac{1}{3}x^3 - \frac{1}{5}x^5\ \right]_0^1$$
$$=2\left(\ \frac{1}{3} - \frac{1}{5}\ \right) = \frac{4}{15}$$



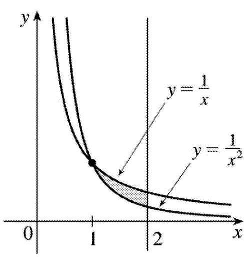


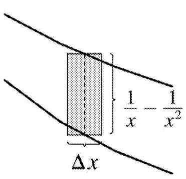
9.

$$A = \int_1^2 \left(\frac{1}{x} - \frac{1}{x^2} \right) dx = \left[\ln x + \frac{1}{x} \right]_1^2$$

$$= \left(\ln 2 + \frac{1}{2} \right) - (\ln 1 + 1)$$

$$= \ln 2 - \frac{1}{2} \approx 0.19$$

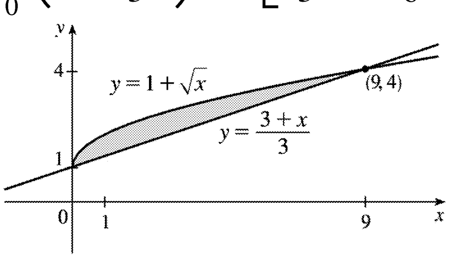


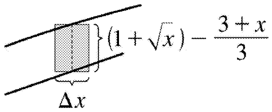


10. $1+\sqrt{x} = \frac{3+x}{3} = 1 + \frac{x}{3} \Rightarrow \sqrt{x} = \frac{x}{3} \Rightarrow x = \frac{x^2}{9} \Rightarrow 9x - x^2 = 0 \Rightarrow x(9-x) = 0 \Rightarrow x = 0$ or 9 , so

$$A = \int_0^9 \left[(1+\sqrt{x}) - \left(\frac{3+x}{3} \right) \right] dx = \int_0^9 \left[(1+\sqrt{x}) - \left(1 + \frac{x}{3} \right) \right] dx =$$

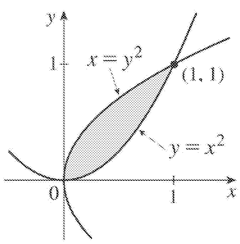
$$\int_0^9 \left(\sqrt{x} - \frac{1}{3} x \right) dx = \left[\frac{2}{3} x^{3/2} - \frac{1}{6} x^2 \right]_0^9 = 18 - \frac{27}{2} = \frac{9}{2}$$

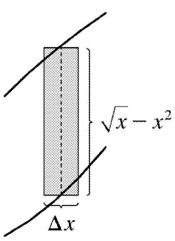




11.

$$A=\int_0^1 \left(\sqrt{x}-x^2\right)dx$$
$$=\left[\frac{2}{3}x^{3/2}-\frac{1}{3}x^3\right]_0^1$$
$$=\frac{2}{3}-\frac{1}{3}$$
$$=\frac{1}{3}$$



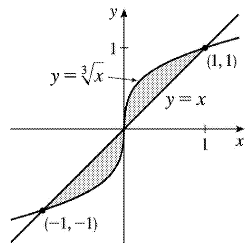


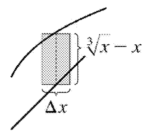
12. $x=\sqrt[3]{x}\Rightarrow x^3=x\Rightarrow x^3-x=0\Rightarrow x(x^2-1)=0\Rightarrow x(x+1)(x-1)=0\Rightarrow$ $x=-1$, 0 , or 1 , so

$$A=\int_{-1}^{1}\left|\sqrt[3]{x}-x\right|dx=\int_{-1}^{0}\left(x-\sqrt[3]{x}\right)dx+\int_0^1\left(\sqrt[3]{x}-x\right)dx=2\int_0^1\left(x^{1/3}-x\right)dx$$

[by symmetry]

$$=2\left[\frac{3}{4}x^{4/3}-\frac{1}{2}x^2\right]_0^1=2\left(\frac{3}{4}-\frac{1}{2}\right)=\frac{1}{2}$$

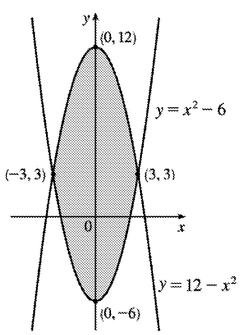


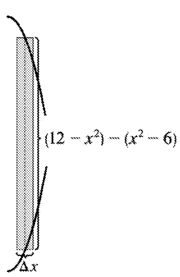


13. $12 - x^2 = x^2 - 6 \Leftrightarrow 2x^2 = 18 \Leftrightarrow x^2 = 9 \Leftrightarrow x^2 = \pm 3$, so

$$A = \int_{-3}^{3} \left[\left(12 - x^2\right) - \left(x^2 - 6\right) \right] dx = 2\int_{0}^{3} \left(18 - 2x^2\right) dx \text{ [by symmetry]}$$

$$= 2\left[18x - \frac{2}{3} x^3 \right]_0^3 = 2[\,(54 - 18) - 0] = 2(36) = 72$$





14. $x^3 - x = 3x \Rightarrow x^3 - 4x = 0 \Rightarrow x(x^2 - 4) = 0 \Rightarrow x(x+2)(x-2) = 0 \Rightarrow x = 0$, -2 , or 2 .
By symmetry,

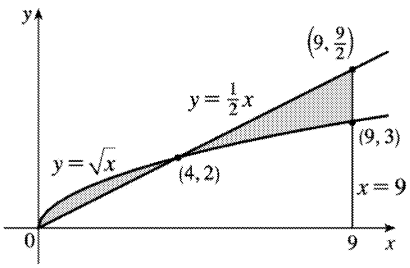
$$A = \int_{-2}^{2} \left| 3x - \left(x^3 - x\right) \right| dx = 2\int_{0}^{2} \left[3x - \left(x^3 - x\right) \right] dx = 2\int_{0}^{2} \left(4x - x^3\right) dx = 2\left[2x^2 - \frac{1}{4} x^4 \right]_0^2$$

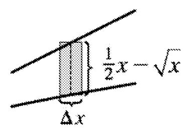
$= 2(8 - 4) = 8$]

15. $\frac{1}{2} x = \sqrt{x} \Rightarrow \frac{1}{4} x^2 = x \Rightarrow x^2 - 4x = 0 \Rightarrow x(x-4) = 0 \Rightarrow x = 0$ or 4 , so

$$A=\int_0^4\left(\sqrt{x}-\frac{1}{2}x\right)dx+\int_4^9\left(\frac{1}{2}x-\sqrt{x}\right)dx=\left[\frac{2}{3}x^{3/2}-\frac{1}{4}x^2\right]_0^4+\left[\frac{1}{4}x^2-\frac{2}{3}x^{3/2}\right]_4^9$$

$$=\left[\left(\frac{16}{3}-4\right)-0\right]+\left[\left(\frac{81}{4}-18\right)-\left(4-\frac{16}{3}\right)\right]=\frac{81}{4}+\frac{32}{3}-26=\frac{59}{12}$$





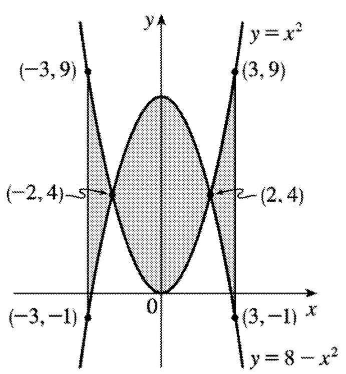
For $4<x<9$

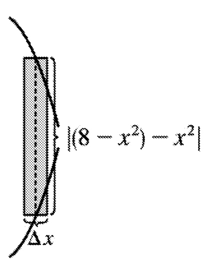
16.

$$A=\int_{-3}^{3}\left|\left(8-x^2\right)-x^2\right|dx=2\int_0^3\left|8-2x^2\right|dx=2\int_0^2\left(8-2x^2\right)dx+2\int_2^3\left(2x^2-8\right)dx$$

$$=2\left[8x-\frac{2}{3}x^3\right]_0^2+2\left[\frac{2}{3}x^3-8x\right]_2^3=2\left[\left(16-\frac{16}{3}\right)-0\right]+2\left[(18-24)-\left(\frac{16}{3}-16\right)\right]$$

$$=32-\frac{32}{3}+20-\frac{32}{3}=52-\frac{64}{3}=\frac{92}{3}$$

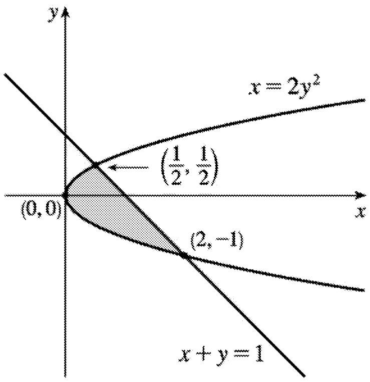


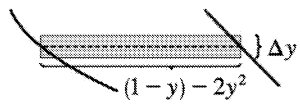


17. $2y^2=1-y\Leftrightarrow 2y^2+y-1=0\Leftrightarrow(2y-1)(y+1)=0\Leftrightarrow y=\frac{1}{2}$ or -1 , so $x=\frac{1}{2}$ or 2 and

$$A=\int_{-1}^{1/2}\left[(1-y)-2y^2\right]dy=\int_{-1}^{1/2}\left(1-y-2y^2\right)dy=\left[y-\frac{1}{2}y^2-\frac{2}{3}y^3\right]_{-1}^{1/2}$$

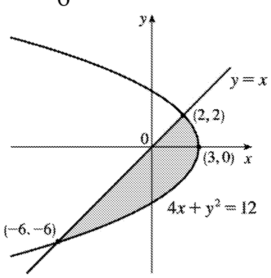
$$=\left(\frac{1}{2}-\frac{1}{8}-\frac{1}{12}\right)-\left(-1-\frac{1}{2}+\frac{2}{3}\right)=\frac{7}{24}-\left(-\frac{5}{6}\right)=\frac{7}{24}+\frac{20}{24}=\frac{27}{24}=\frac{9}{8}$$

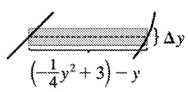




18. $4x+x^2=12 \Leftrightarrow (x+6)(x-2)=0 \Leftrightarrow x=-6$ or $x=2$, so $y=-6$ or $y=2$ and

$$A=\int_{-6}^{2}\left[\left(-\frac{1}{4}y^2+3\right)-y\right]dy=\left[-\frac{1}{12}y^3-\frac{1}{2}y^2+3y\right]_{-6}^{2}=\left(-\frac{2}{3}-2+6\right)-(18-18-18)=22-\frac{2}{3}=\frac{64}{3}\ .$$





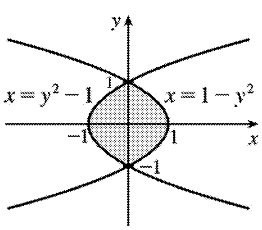
19. The curves intersect when $1-y^2=y^2-1 \Leftrightarrow 2=2y^2 \Leftrightarrow y^2=1 \Leftrightarrow y=\pm 1$.

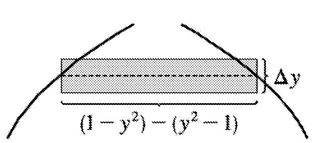
$$A=\int_{-1}^{1}\left[\left(1-y^2\right)-\left(y^2-1\right)\right]dy$$

$$=\int_{-1}^{1}2\left(1-y^2\right)dy$$

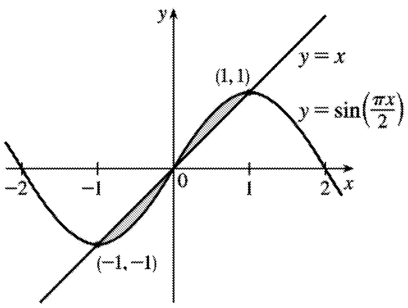
$$=2\cdot 2\int_{0}^{1}\left(1-y^2\right)dy$$

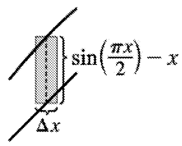
$$=4\left[y-\frac{1}{3}y^3\right]_0^1=4\left(1-\frac{1}{3}\right)=\frac{8}{3}$$





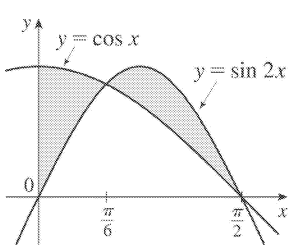
20. $A=2\int_0^1 \left[\sin\left(\frac{\pi x}{2}\right)-x\right]dx=2\left[-\frac{2}{\pi}\cos\left(\frac{\pi x}{2}\right)-\frac{x^2}{2}\right]_0^1=2\left[\left(0-\frac{1}{2}\right)-\left(-\frac{2}{\pi}-0\right)\right]=\frac{4}{\pi}-1$





21. Notice that $\cos x=\sin 2x=2\sin x\cos x \Leftrightarrow$
$2\sin x\cos x-\cos x=0 \Leftrightarrow \cos x(2\sin x-1)=0 \Leftrightarrow$
$2\sin x=1$ or $\cos x=0 \Leftrightarrow x=\frac{\pi}{6}$ or $\frac{\pi}{2}$.

$$A = \int_0^{\pi/6} (\cos x-\sin 2x)\,dx+\int_{\pi/6}^{\pi/2} (\sin 2x-\cos x)\,dx$$
$$= \left[\sin x+\frac{1}{2}\cos 2x\right]_0^{\pi/6}+\left[-\frac{1}{2}\cos 2x-\sin x\right]_{\pi/6}^{\pi/2}$$
$$= \frac{1}{2}+\frac{1}{2}\cdot\frac{1}{2}-\left(0+\frac{1}{2}\cdot 1\right)+\left(\frac{1}{2}-1\right)-\left(-\frac{1}{2}\cdot\frac{1}{2}-\frac{1}{2}\right)=\frac{1}{2}$$

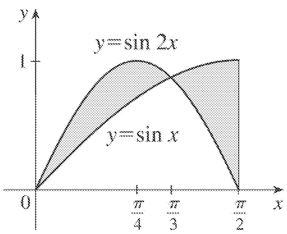


22. $\sin x=\sin 2x=2\sin x\cos x$ when $\sin x=0$ and when $\cos x=\frac{1}{2}$;

that is, when $x=0$ or $\frac{\pi}{3}$.

$$A = \int_0^{\pi/3} (\sin 2x-\sin x)\,dx+\int_{\pi/3}^{\pi/2} (\sin x-\sin 2x)\,dx$$
$$= \left[-\frac{1}{2}\cos 2x+\cos x\right]_0^{\pi/3}+\left[\frac{1}{2}\cos 2x-\cos x\right]_{\pi/3}^{\pi/2}$$

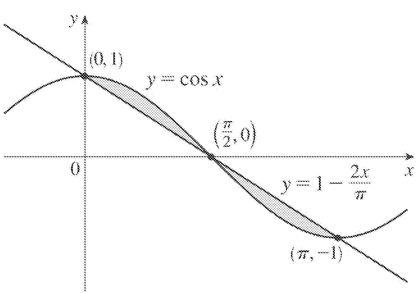
$$=\left[-\frac{1}{2}\left(-\frac{1}{2}\right)+\frac{1}{2}\right]-\left(-\frac{1}{2}+1\right)$$
$$+\left(-\frac{1}{2}-0\right)-\left[\frac{1}{2}\left(-\frac{1}{2}\right)-\frac{1}{2}\right]=\frac{1}{2}$$

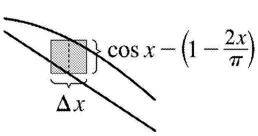


23. From the graph, we see that the curves intersect at $x=0$, $x=\frac{\pi}{2}$, and $x=\pi$. By symmetry,

$$A=\int_0^{\pi}\left|\cos x-\left(1-\frac{2x}{\pi}\right)\right|dx=2\int_0^{\pi/2}\left[\cos x-\left(1-\frac{2x}{\pi}\right)\right]dx=2\int_0^{\pi/2}\left(\cos x-1+\frac{2x}{\pi}\right)dx$$

$$=2\left[\sin x-x+\frac{1}{\pi}x^2\right]_0^{\pi/2}=2\left[\left(1-\frac{\pi}{2}+\frac{1}{\pi}\cdot\frac{\pi^2}{4}\right)-0\right]=2\left(1-\frac{\pi}{2}+\frac{\pi}{4}\right)=2-\frac{\pi}{2}$$

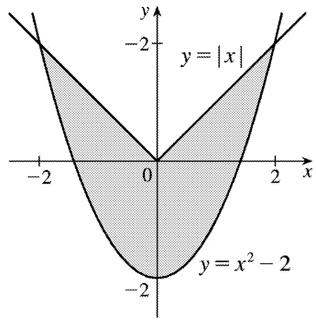


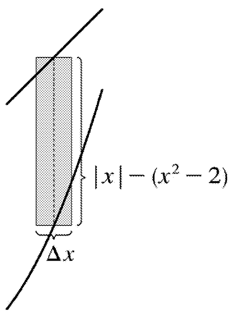


24. For $x>0$, $x=x^2-2\Rightarrow 0=x^2-x-2\Rightarrow 0=(x-2)(x+1)\Rightarrow x=2$. By symmetry,

$$\int_{-2}^{2}\left[|x|-\left(x^2-2\right)\right]dx=2\int_0^2\left[x-\left(x^2-2\right)\right]dx=2\int_0^2\left(x-x^2+2\right)dx=2\left[\frac{1}{2}x^2-\frac{1}{3}x^3+2x\right]_0^2=$$

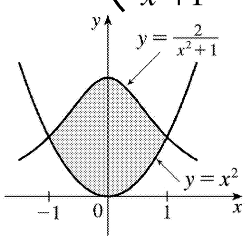
$$2\left(2-\frac{8}{3}+4\right)=\frac{20}{3}$$

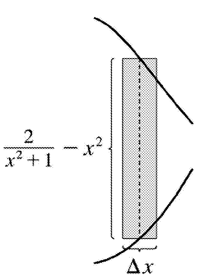




25. The curves intersect when $x^2 = \dfrac{2}{x^2+1} \Leftrightarrow x^4+x^2=2 \Leftrightarrow x^4+x^2-2=0 \Leftrightarrow \left(x^2+2\right)\left(x^2-1\right)=0 \Leftrightarrow x^2=1 \Leftrightarrow x=\pm 1$.

$$A=\int_{-1}^{1}\left(\frac{2}{x^2+1}-x^2\right)dx=2\int_{0}^{1}\left(\frac{2}{x^2+1}-x^2\right)dx=2\left[2\tan^{-1}x-\frac{1}{3}x^3\right]_0^1=2\left(2\cdot\frac{\pi}{4}-\frac{1}{3}\right)=\pi-\frac{2}{3}\approx 2.47$$





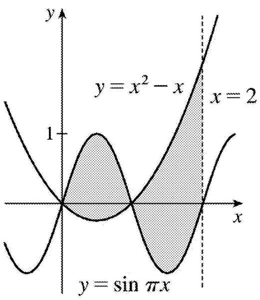
26.

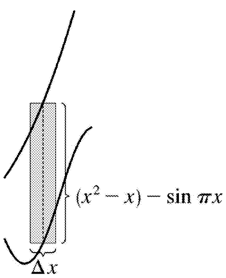
$$A=\int_0^1\left[\sin\pi x-\left(x^2-x\right)\right]dx+\int_1^2\left[\left(x^2-x\right)-\sin\pi x\right]dx$$

$$=\left[-\frac{1}{\pi}\cos\pi x-\frac{1}{3}x^3+\frac{1}{2}x^2\right]_0^1+\left[\frac{1}{3}x^3-\frac{1}{2}x^2+\frac{1}{\pi}\cos\pi x\right]_1^2$$

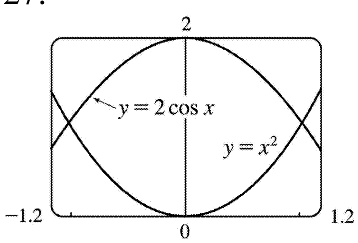
$$=\left(\frac{1}{\pi}-\frac{1}{3}+\frac{1}{2}\right)-\left(-\frac{1}{\pi}\right)+\left(\frac{8}{3}-2+\frac{1}{\pi}\right)-\left(\frac{1}{3}-\frac{1}{2}-\frac{1}{\pi}\right)$$

$$=\frac{4}{\pi}+1$$





27.



From the graph, we see that the curves intersect at $x=\pm a\approx\pm 1.02$, with $2\cos x>x^2$ on $(-a,a)$. So the area of the region bounded by the curves is

$$A=\int_{-a}^{a}\left(2\cos x-x^2\right)dx=2\int_0^a\left(2\cos x-x^2\right)dx$$

$$=2\left[2\sin x-\frac{1}{3}x^3\right]_0^a\approx 2.70$$

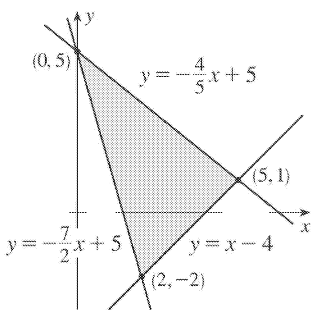
28.

$$A=\int_{0}^{2}\left[\left(-\frac{4}{5}x+5\right)-\left(-\frac{7}{2}x+5\right)\right]dx+\int_{2}^{5}\left[\left(-\frac{4}{5}x+5\right)-(x-4)\right]dx$$

$$=\int_{0}^{2}\frac{27}{10}x\,dx+\int_{2}^{5}\left(-\frac{9}{5}x+9\right)dx$$

$$=\left[\frac{27}{20}x^{2}\right]_{0}^{2}+\left[-\frac{9}{10}x^{2}+9x\right]_{2}^{5}$$

$$=\left(\frac{27}{5}-0\right)+\left(-\frac{45}{2}+45\right)-\left(-\frac{18}{5}+18\right)=\frac{27}{2}$$



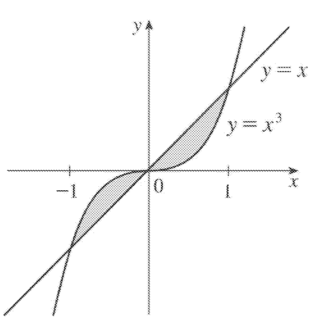
29.

$$A=\int_{-1}^{1}\left|x^{3}-x\right|dx$$

$$=2\int_{0}^{1}\left(x-x^{3}\right)dx \text{ [by symmetry]}$$

$$=2\left[\frac{1}{2}x^{2}-\frac{1}{4}x^{4}\right]_{0}^{1}$$

$$=2\left(\frac{1}{2}-\frac{1}{4}\right)=\frac{1}{2}$$



30. The curves intersect when $\sqrt{x+2}=x \Rightarrow x+2=x^2 \Rightarrow x^2-x-2=0 \Rightarrow$ $(x-2)(x+1)=0 \Rightarrow x=-1$ or 2 .

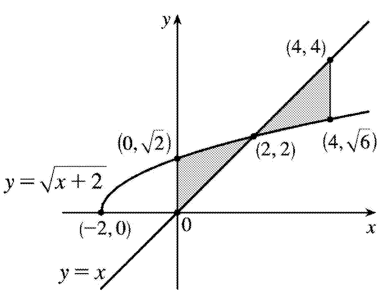
$$A=\int_0^4 \left|\sqrt{x+2}-x\right| dx$$

$$=\int_0^2 \left(\sqrt{x+2}-x\right)dx+\int_2^4 \left(x-\sqrt{x+2}\right)dx$$

$$=\left[\frac{2}{3}(x+2)^{3/2}-\frac{1}{2}x^2\right]_0^2+\left[\frac{1}{2}x^2-\frac{2}{3}(x+2)^{3/2}\right]_2^4$$

$$=\left(\frac{16}{3}-2\right)-\left(\frac{2}{3}\left(2\sqrt{2}\right)-0\right)+\left(8-\frac{2}{3}\left(6\sqrt{6}\right)\right)-\left(2-\frac{16}{3}\right)$$

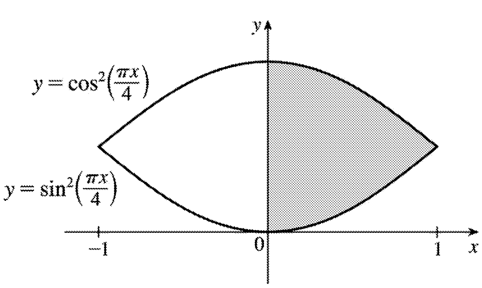
$$=4+\frac{32}{3}-\frac{4}{3}\sqrt{2}-4\sqrt{6}=\frac{44}{3}-4\sqrt{6}-\frac{4}{3}\sqrt{2}$$



31. Let $f(x)=\cos^2\left(\frac{\pi x}{4}\right)-\sin^2\left(\frac{\pi x}{4}\right)$ and $\Delta x=\frac{1-0}{4}$.

The shaded area is given by

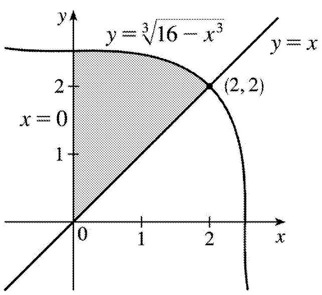
$$A=\int_0^1 f(x)dx \approx M_4=\frac{1}{4}\left[f\left(\frac{1}{8}\right)+f\left(\frac{3}{8}\right)+f\left(\frac{5}{8}\right)+f\left(\frac{7}{8}\right)\right]\approx 0.6407$$



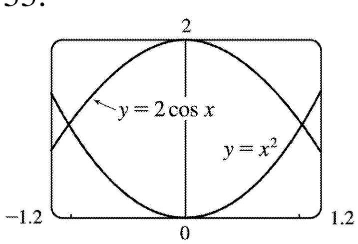
32. The curves intersect when $\sqrt[3]{16-x^3}=x \Rightarrow$

$16-x^3=x^3 \Rightarrow 2x^3=16 \Rightarrow x^3=8 \Rightarrow x=2.$
Let $f(x)=\sqrt[3]{16-x^3}-x$ and $\Delta x=\dfrac{2-0}{4}$.
The shaded area is given by

$$A=\int_0^2 f(x)\,dx \approx M_4$$
$$=\frac{2}{4}\left[f\left(\frac{1}{4}\right)+f\left(\frac{3}{4}\right)+f\left(\frac{5}{4}\right)+f\left(\frac{7}{4}\right)\right]$$
$$\approx 2.8144$$



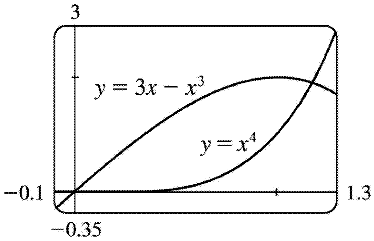
33.



From the graph, we see that the curves intersect at $x=\pm a\approx\pm 1.02$, with $2\cos x>x^2$ on $(-a,a)$. So the area of the region bounded by the curves is

$$A = \int_{-a}^{a}\left(2\cos x-x^2\right)dx=2\int_0^a\left(2\cos x-x^2\right)dx$$
$$= 2\left[2\sin x-\frac{1}{3}x^3\right]_0^a\approx 2.70$$

34.

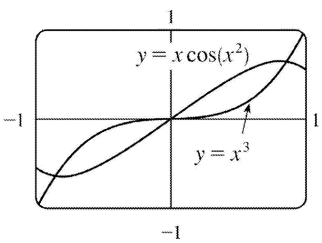


From the graph, we see that the curves intersect at $x=0$ and at $x=a\approx 1.17$, with $3x-x^3>x^4$ on $(0,a)$. So the area of the region bounded by the curves is

$$A = \int_0^a \left[\left(3x-x^3\right)-x^4\right]dx=\left[\frac{3}{2}x^2-\frac{1}{4}x^4-\frac{1}{5}x^5\right]_0^a$$

$$\approx 1.15$$

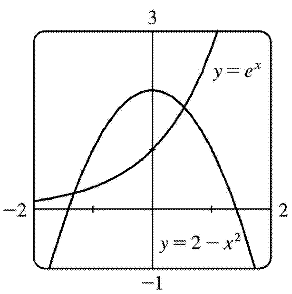
35.



From the graph, we see that the curves intersect at $x=\pm a\approx\pm 0.86$.
So the area of the region bounded by the curves is

$$A=2\int_0^a\left[x\cos\left(x^2\right)-x^3\right]dx=2\left[\frac{1}{2}\sin\left(x^2\right)-\frac{1}{4}x^4\right]_0^a\approx 0.40$$

36.



From the graph, we see that the curves intersect at $x=a\approx -1.32$ and $x=b\approx 0.54$, with $2-x^2>e^x$ on (a,b). So the area of the region bounded by the curves is

$$A=\int_a^b\left[\left(2-x^2\right)-e^x\right]\mathrm{dx}=\left[2x-\frac{1}{3}x^3-e^x\right]_a^b\approx 1.45$$

37. As the figure illustrates, the curves $y=x$ and

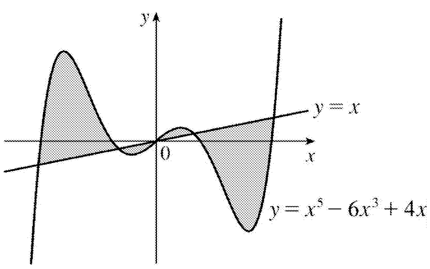
$y=x^5-6x^3+4x$ enclose a four–part region symmetric about the origin (since x^5-6x^3+4x and x are odd functions of x). The curves intersect at values of x where $x^5-6x^3+4x=x$; that is, where $x(x^4-6x^2+3)=0$. That happens at $x=0$ and where $x^2=\frac{6\pm\sqrt{36-12}}{2}=3\pm\sqrt{6}$; that is, at $x=-\sqrt{3+\sqrt{6}}$, $-\sqrt{3-\sqrt{6}}$, 0 , $\sqrt{3-\sqrt{6}}$, and $\sqrt{3+\sqrt{6}}$.

The exact area is

$$2\int_0^{\sqrt{3+\sqrt{6}}}\left|\left(x^5-6x^3+4x\right)-x\right|dx=2\int_0^{\sqrt{3+\sqrt{6}}}\left|x^5-6x^3+3x\right|dx$$

$$=2\int_0^{\sqrt{3-\sqrt{6}}}\left(x^5-6x^3+3x\right)dx+2\int_{\sqrt{3-\sqrt{6}}}^{\sqrt{3+\sqrt{6}}}\left(-x^5+6x^3-3x\right)dx$$

$$=12\sqrt{6}-9$$



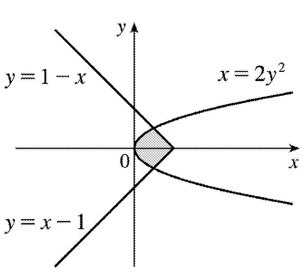
38. The inequality $x\geq 2y^2$ describes the region that lies on, or to the right of, the parabola $x=2y^2$. The inequality $x\leq 1-|y|$ describes the region

that lies on, or to the left of, the curve $x=1-|y|=\begin{cases}1-y \text{ if } y\geq 0\\ 1+y \text{ if } y<0\end{cases}$.

So the given region is the shaded region that lies between the curves.

The graphs of $x=1-y$ and $x=2y^2$ intersect when $1-y=2y^2\Leftrightarrow$

$2y^2+y-1=0\Leftrightarrow(2y-1)(y+1)=0\Rightarrow y=\frac{1}{2}$ (for $y\geq 0$).



By symmetry,

$$A=2\int_0^{1/2}\left[(1-y)-2y^2\right]dy=2\left[-\frac{2}{3}y^3-\frac{1}{2}y^2+y\right]_0^{1/2}=2\left[\left(-\frac{1}{12}-\frac{1}{8}+\frac{1}{2}\right)-0\right]=2\left(\frac{7}{24}\right)=\frac{7}{12}\ .$$

39. 1 second $=\frac{1}{3600}$ hour, so 10 s $=\frac{1}{360}$ h. With the given data, we can take $n=5$ to use the Midpoint Rule. $\Delta t=\frac{1/360-0}{5}=\frac{1}{1800}$, so

$$\begin{aligned}\text{distance}_{\text{Kelly}}-\text{distance}_{\text{Chris}} &= \int_0^{1/360} v_K dt-\int_0^{1/360} v_C dt=\int_0^{1/360}\left(v_K-v_C\right)dt\\ &\approx M_5=\frac{1}{1800}\left[\left(v_K-v_C\right)(1)+\left(v_K-v_C\right)(3)+\left(v_K-v_C\right)(5)\right.\\ &\quad\left.+\left(v_K-v_C\right)(7)+\left(v_K-v_C\right)(9)\right]\\ &= \frac{1}{1800}\left[(22-20)+(52-46)+(71-62)+(86-75)+(98-86)\right]\\ &= \frac{1}{1800}(2+6+9+11+12)=\frac{1}{1800}(40)=\frac{1}{45}\text{ mile,or}117\frac{1}{3}\text{ feet}\end{aligned}$$

40. If x= distance from left end of pool and $w=w(x)$= width at x , then the Midpoint Rule with $n=4$ and $\Delta x=\frac{b-a}{n}=\frac{8\cdot 2-0}{4}=4$ gives Area $=\int_0^{16} w\,dx\approx 4(6.2+6.8+5.0+4.8)=4(22.8)=91.2\text{ m}^2$.

41. We know that the area under curve A between $t=0$ and $t=x$ is $\int_0^x v_A(t)\,dt=s_A(x)$, where $v_A(t)$ is the velocity of car A and s_A is its displacement. Similarly, the area under curve B between $t=0$ and $t=x$ is $\int_0^x v_B(t)\,dt=s_B(x)$.

(a) After one minute, the area under curve A is greater than the area under curve B . So car A is ahead after one minute.

(b) The area of the shaded region has numerical value $s_A(1)-s_B(1)$, which is the distance by which A is ahead of B after 1 minute.

(c) After two minutes, car B is traveling faster than car A and has gained some ground, but the area under curve A from $t=0$ to $t=2$ is still greater than the corresponding area for curve B , so car A is still ahead.

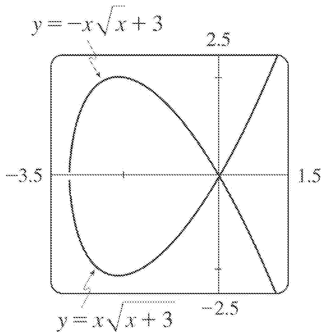
(d) From the graph, it appears that the area between curves A and B for $0\leq t\leq 1$ (when car A is going faster), which corresponds to the distance by which car A is ahead, seems to be about 3 squares. Therefore, the cars will be side by side at the time x where the area between the curves for $1 \le t \le x$ (when car B is going faster) is the same as the area for $0 \le t \le 1$. From the graph, it appears that this time is $x \approx 2.2$. So the cars are side by side when $t \approx 2.2$ minutes.

42. The area under $R'(x)$ from $x=50$ to $x=100$ represents the change in revenue, and the area under $C'(x)$ from $x=50$ to $x=100$ represents the change in cost. The shaded region represents the difference between these two values; that is, the increase in profit as the production level increases from 50 units to 100 units. We use the
Midpoint Rule with $n=5$ and $\Delta x=10$:

$$\begin{aligned} M_5 &= \Delta x\{+[R'(65)-C'(65)]+[R'(75)-C'(75)] \\ &\quad +[R'(85)-C'(85)]+[R'(95)-C'(95)]\} \\ &\approx 10(2.40-0.85+2.20-0.90+2.00-1.00+1.80-1.10+1.70-1.20) \\ &= 10(5.05)=50.5 \text{ thousand dollars} \end{aligned}$$

Using M_1 would give us $50(2-1)=50$ thousand dollars.

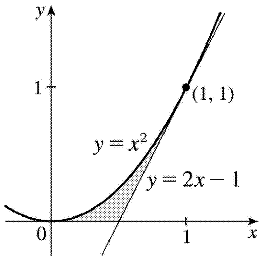
43.



To graph this function, we must first express it as a combination of explicit functions of y ; namely, $y=\pm x\sqrt{x+3}$. We can see from the graph that the loop extends from $x=-3$ to $x=0$, and that by symmetry, the area we seek is just twice the area under the top half of the curve on this interval, the equation of the top half being $y=-x\sqrt{x+3}$. So the area is $A=2\int_{-3}^{0}\left(-x\sqrt{x+3}\right)dx$. We substitute $u=x+3$, so $du=dx$ and the limits change to 0 and 3 , and we get

$$\begin{aligned} A &= -2\int_0^3 \left[(u-3)\sqrt{u}\,\right]du=-2\int_0^3\left(u^{3/2}-3u^{1/2}\right)du \\ &= -2\left[\frac{2}{5}u^{5/2}-2u^{3/2}\right]_0^3=-2\left[\frac{2}{5}\left(3^2\sqrt{3}\right)-2\left(3\sqrt{3}\right)\right]=\frac{24}{5}\sqrt{3} \end{aligned}$$

44.

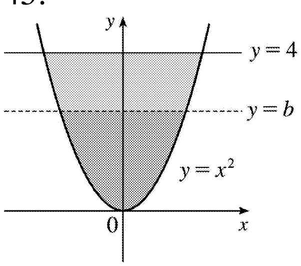


We start by finding the equation of the tangent line to $y=x^2$ at the point $(1,1)$: $y'=2x$, so the slope of the tangent is $2(1)=2$, and its equation is $y-1=2(x-1)$, or $y=2x-1$. We would need two integrals to integrate with respect to x , but only one to integrate with respect to y .

$$A=\int_0^1\left[\frac{1}{2}(y+1)-\sqrt{y}\right]dy=\left[\frac{1}{4}y^2+\frac{1}{2}y-\frac{2}{3}y^{3/2}\right]_0^1$$

$$=\frac{1}{4}+\frac{1}{2}-\frac{2}{3}=\frac{1}{12}$$

45.



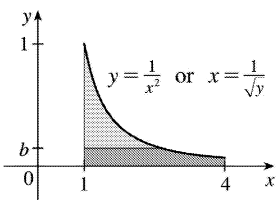
By the symmetry of the problem, we consider only the first quadrant, where $y=x^2\Rightarrow x=\sqrt{y}$. We are looking for a number b such that $\int_0^b\sqrt{y}\,dy=\int_b^4\sqrt{y}\,dy\Rightarrow\frac{2}{3}\left[y^{3/2}\right]_0^b=\frac{2}{3}\left[y^{3/2}\right]_b^4\Rightarrow b^{3/2}=4^{3/2}-b^{3/2}\Rightarrow 2b^{3/2}=8$ $\Rightarrow b^{3/2}=4\Rightarrow b=4^{2/3}\approx 2.52$.

46. **(a)** We want to choose a so that $\int_1^a\frac{1}{x^2}\,dx=\int_a^4\frac{1}{x^2}\,dx\Rightarrow\left[\frac{-1}{x}\right]_1^a=\left[\frac{-1}{x}\right]_a^4\Rightarrow-\frac{1}{a}+1=-\frac{1}{4}+\frac{1}{a}\Rightarrow$ $\frac{5}{4}=\frac{2}{a}\Rightarrow a=\frac{8}{5}$.

(b) The area under the curve $y=1/x^2$ from $x=1$ to $x=4$ is $\frac{3}{4}$. Now the line $y=b$ must intersect the curve $x=1/\sqrt{y}$ and not the line $x=4$, since the area under the line $y=1/4^2$ from $x=1$ to $x=4$ is only $\frac{3}{16}$, which is less than half of

$\frac{3}{4}$. We want to choose b so that the upper area in the diagram is half of the total area under the curve $y=\frac{1}{x^2}$ from $x=1$ to $x=4$. This implies that

$\int_b^1 (1/\sqrt{y}-1)\,dy=\frac{1}{2}\cdot\frac{3}{4} \Rightarrow \left[2\sqrt{y}-y\right]_b^1=\frac{3}{8} \Rightarrow 1-2\sqrt{b}+b=\frac{3}{8} \Rightarrow b-2\sqrt{b}+\frac{5}{8}=0$. Letting $c=\sqrt{b}$, we get $c^2-2c+\frac{5}{8}=0 \Rightarrow 8c^2-16c+5=0$. Thus, $c=\frac{16\pm\sqrt{256-160}}{16}=1\pm\frac{\sqrt{6}}{4}$. But $c=\sqrt{b}<1 \Rightarrow c=1-\frac{\sqrt{6}}{4} \Rightarrow$ $b=c^2=1+\frac{3}{8}-\frac{\sqrt{6}}{2}=\frac{1}{8}\left(11-4\sqrt{6}\right)\approx 0.1503$.

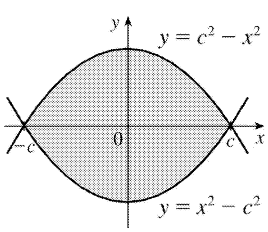


47. We first assume that $c>0$, since c can be replaced by $-c$ in both equations without changing the graphs, and if $c=0$ the curves do not enclose a region. We see from the graph that the enclosed area A lies between $x=-c$ and $x=c$, and by symmetry, it is equal to four times the area in the first quadrant. The enclosed area is

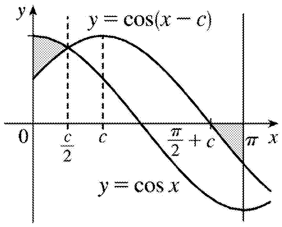
$$A = 4\int_0^c \left(c^2-x^2\right)dx=4\left[c^2x-\frac{1}{3}x^3\right]_0^c$$

$$= 4\left(c^3-\frac{1}{3}c^3\right)=4\left(\frac{2}{3}c^3\right)=\frac{8}{3}c^3$$

So $A=576 \Leftrightarrow \frac{8}{3}c^3=576 \Leftrightarrow c^3=216 \Leftrightarrow c=\sqrt[3]{216}=6$. Note that $c=-6$ is another solution, since the graphs are the same.



48.



It appears from the diagram that the curves $y=\cos x$ and $y=\cos(x-c)$ intersect halfway between 0 and c, namely, when $x=c/2$. We can verify that this is indeed true by noting that $\cos(c/2-c)=\cos(-c/2)=\cos(c/2)$.

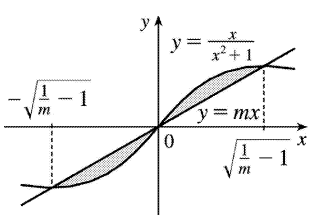
The point where $\cos(x-c)$ crosses the $x-$ axis is $x=\frac{\pi}{2}+c$. So we require that

$\int_0^{c/2}[\cos x-\cos(x-c)]dx=-\int_{\pi/2+c}^{\pi}\cos(x-c)dx$ (the negative sign on the RHS is needed since the second area is beneath the $x-$ axis) $\Leftrightarrow$

$[\sin x-\sin(x-c)]_0^{c/2}=-[\sin(x-c)]_{\pi/2+c}^{\pi}\Rightarrow$

$[\sin(c/2)-\sin(-c/2)]-[-\sin(-c)]=-\sin(\pi-c)+\sin\left[\left(\frac{\pi}{2}+c\right)-c\right]\Leftrightarrow 2\sin(c/2)-\sin c=-\sin c+1$ . . So $2\sin(c/2)=1\Leftrightarrow\sin(c/2)=\frac{1}{2}\Leftrightarrow c/2=\frac{\pi}{6}\Leftrightarrow c=\frac{\pi}{3}$.

49. The curve and the line will determine a region when they intersect at two or more points. So we solve the equation $x/\left(x^2+1\right)=mx\Rightarrow x=x\left(mx^2+m\right)\Rightarrow$
$x\left(mx^2+m\right)-x=0\Rightarrow x\left(mx^2+m-1\right)=0\Rightarrow$
$x=0$ or $mx^2+m-1=0\Rightarrow x=0$ or $x^2=\frac{1-m}{m}\Rightarrow$

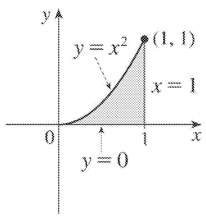


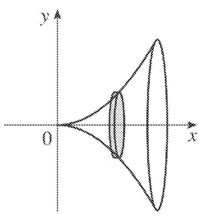
$x=0$ or $x=\pm\sqrt{\frac{1}{m}-1}$. Note that if $m=1$, this has only the solution $x=0$, and no region is determined. But if $1/m-1>0\Leftrightarrow 1/m>1\Leftrightarrow 0<m<1$, then there are two solutions. [Another way of seeing this is to observe that the slope of the tangent to $y=x/(x^2+1)$ at the origin is $y'=1$ and therefore we must have $0<m<1$.] Note that we cannot just integrate between the positive and negative roots, since the curve and the line cross at the origin. Since mx and $x/(x^2+1)$ are both odd functions, the total area is twice the area between the curves on the interval $\left[0,\sqrt{1/m-1}\,\right]$. So the total area enclosed is

$$2\int_0^{\sqrt{1/m-1}}\left[\frac{x}{x^2+1}-mx\right]dx=2\left[\frac{1}{2}\ln\left(x^2+1\right)-\frac{1}{2}mx^2\right]_0^{\sqrt{1/m-1}}=\left[\ln(1/m-1+1)-m(1/m-1)\right]-(\ln 1-0)=$$
$$\ln(1/m)-1+m=m-\ln m-1$$

1. A cross–section is circular with radius x^2 , so its area is $A(x)=\pi\left(x^2\right)^2$.

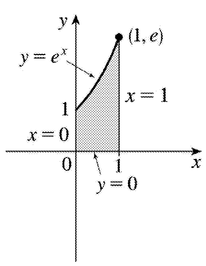
$$V=\int_0^1 A(x)dx=\int_0^1 \pi\left(x^2\right)^2dx=\pi\int_0^1 x^4dx=\pi\left[\frac{1}{5}x^5\right]_0^1=\frac{\pi}{5}$$

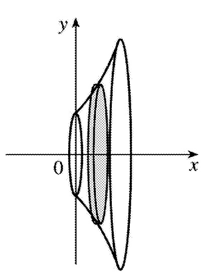




2. A cross–section is a disk with radius e^x , so its area is $A(x)=\pi\left(e^x\right)^2$.

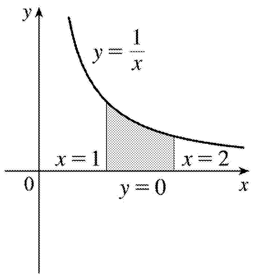
$$V=\int_0^1 A(x)dx=\int_0^1 \pi\left(e^x\right)^2dx=\pi\int_0^1 e^{2x}dx=\frac{1}{2}\pi\left[e^{2x}\right]_0^1=\frac{\pi}{2}\left(e^2-1\right)$$

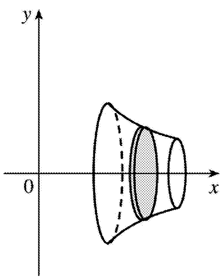




3. A cross–section is a disk with radius $1/x$, so its area is $A(x)=\pi(1/x)^2$.

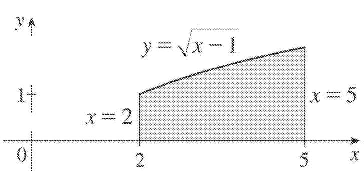
$$V=\int_1^2 A(x)\,dx=\int_1^2 \pi\left(\frac{1}{x}\right)^2 dx=\pi\int_1^2 \frac{1}{x^2}\,dx=\pi\left[-\frac{1}{x}\right]_1^2=\pi\left[-\frac{1}{2}-(-1)\right]=\frac{\pi}{2}$$

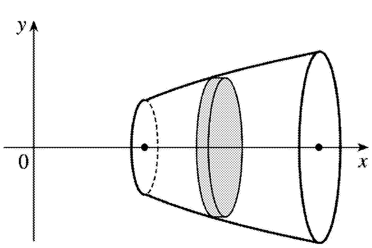




4. A cross–section is circular with radius $\sqrt{x-1}$, so its area is $A(x)=\pi\left(\sqrt{x-1}\right)^2=\pi(x-1)$.

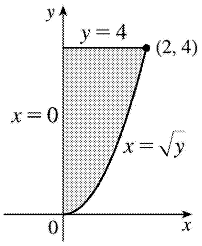
$$V=\int_2^5 A(x)\,dx=\int_2^5 \pi(x-1)\,dx=\pi\left[\frac{1}{2}x^2-x\right]_2^5=\pi\left(\frac{25}{2}-5-\frac{4}{2}+2\right)=\frac{15}{2}\pi$$

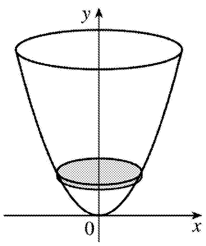




5. A cross–section is a disk with radius $\sqrt{y}$, so its area is $A(y)=\pi\left(\sqrt{y}\right)^2$.

$$V=\int_0^4 A(y)dy=\int_0^4 \pi\left(\sqrt{y}\right)^2 dy=\pi\int_0^4 y dy=\pi\left[\frac{1}{2}y^2\right]_0^4=8\pi$$

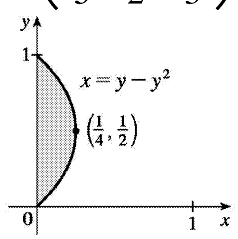


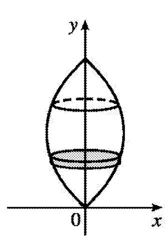


6. A cross–section is a disk with radius $y-y^2$, so its area is $A(y)=\pi\left(y-y^2\right)^2$.

$$V=\int_0^1 A(y)dy=\int_0^1 \pi\left(y-y^2\right)^2 dy=\pi\int_0^1\left(y^4-2y^3+y^2\right)dy=\pi\left[\frac{1}{5}y^5-\frac{1}{2}y^4+\frac{1}{3}y^3\right]_0^1$$

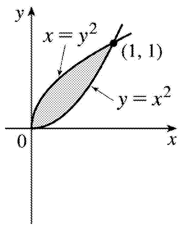
$$\pi\left(\frac{1}{5}-\frac{1}{2}+\frac{1}{3}\right)=\frac{\pi}{30}$$

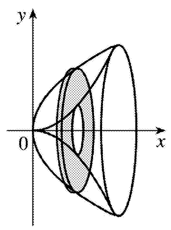




7. A cross–section is a washer (annulus) with inner radius x^2 and outer radius $\sqrt{x}$, so its area is $A(x)=\pi\left(\sqrt{x}\right)^2-\pi\left(x^2\right)^2=\pi\left(x-x^4\right)$.

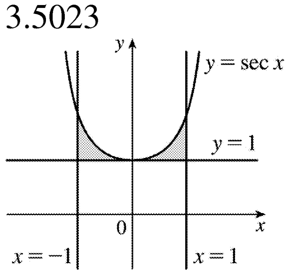
$$V=\int_0^1 A(x)\,dx=\pi\int_0^1 \left(x-x^4\right)dx=\pi\left[\frac{1}{2}x^2-\frac{1}{5}x^5\right]_0^1=\pi\left(\frac{1}{2}-\frac{1}{5}\right)=\frac{3\pi}{10}$$

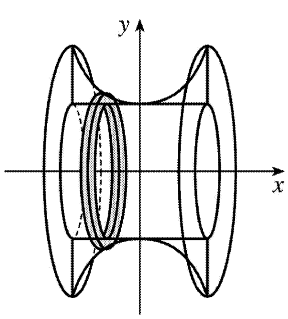




8. A cross–section is a washer with inner radius 1 and outer radius sec x , so its area is
$A(x)=\pi(\sec x)^2-\pi(1)^2=\pi\left(\sec^2 x-1\right)$.

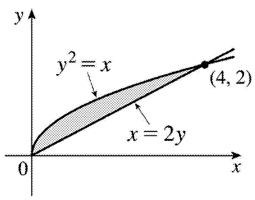
$$V=\int_{-1}^{1} A(x)\,dx=\int_{-1}^{1}\pi\left(\sec^2 x-1\right)dx=2\pi\int_0^1\left(\sec^2 x-1\right)dx=2\pi\left[\tan x-x\right]_0^1=2\pi(\tan 1-1)\approx 3.5023$$

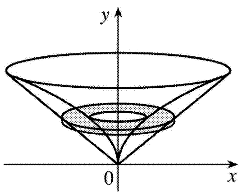




9. A cross–section is a washer with inner radius y^2 and outer radius $2y$, so its area is
$A(y)=\pi(2y)^2-\pi\left(y^2\right)^2=\pi\left(4y^2-y^4\right)$.

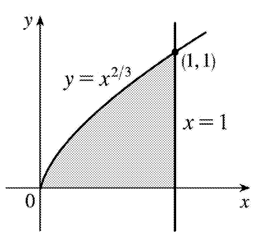
$$V=\int_0^2 A(y)\,dy=\pi\int_0^2\left(4y^2-y^4\right)dy=\pi\left[\frac{4}{3}y^3-\frac{1}{5}y^5\right]_0^2=\pi\left(\frac{32}{3}-\frac{32}{5}\right)=\frac{64\pi}{15}$$

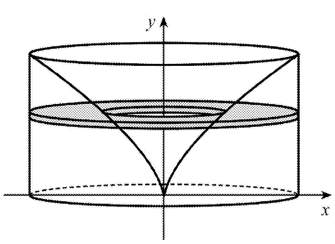




10. $y=x^{2/3}\Leftrightarrow x=y^{3/2}$, so a cross–section is a washer with inner radius $y^{3/2}$ and outer radius 1 , and its area is $A(y)=\pi(1)^2-\pi\left(y^{3/2}\right)^2=\pi\left(1-y^3\right)$.

$$V=\int_0^1 A(y)\,dy=\pi\int_0^1\left(1-y^3\right)dy=\pi\left[y-\frac{1}{4}y^4\right]_0^1=\frac{3}{4}\pi$$

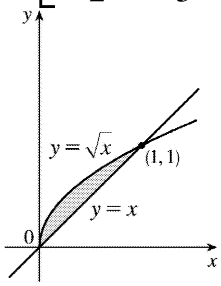


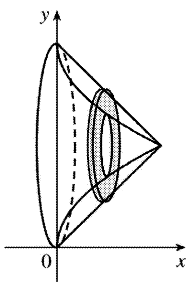


11. A cross–section is a washer with inner radius $1-\sqrt{x}$ and outer radius $1-x$, so its area is $A(x)=\pi(1-x)^2-\pi\left(1-\sqrt{x}\right)^2=\pi\left[\left(1-2x+x^2\right)-\left(1-2\sqrt{x}+x\right)\right]=\pi\left(-3x+x^2+2\sqrt{x}\right)$.

$$V=\int_0^1 A(x)\,dx=\pi\int_0^1\left(-3x+x^2+2\sqrt{x}\right)dx=$$

$$\pi\left[-\frac{3}{2}x^{2}+\frac{1}{3}x^{3}+\frac{4}{3}x^{3/2}\right]_{0}^{1}=\pi\left(-\frac{3}{2}+\frac{5}{3}\right)=\frac{\pi}{6}$$

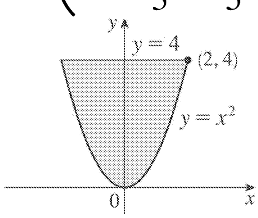


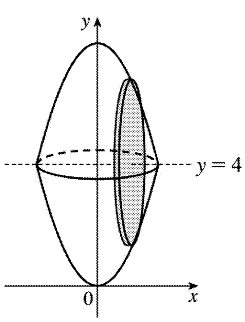


12. A cross–section is circular with radius $4–x^{2}$, so its area is $A(x)=\pi\left(4–x^{2}\right)^{2}=\pi\left(16–8x^{2}+x^{4}\right)$.

$$V=\int_{-2}^{2}A(x)dx=2\int_{0}^{2}A(x)dx=2\pi\int_{0}^{2}\left(16–8x^{2}+x^{4}\right)dx=2\pi\left[16x–\frac{8}{3}x^{3}+\frac{1}{5}x^{5}\right]_{0}^{2}=$$

$$2\pi\left(32–\frac{64}{3}+\frac{32}{5}\right)=64\pi\left(1–\frac{2}{3}+\frac{1}{5}\right)=64\pi\cdot\frac{8}{15}=\frac{512\pi}{15}$$



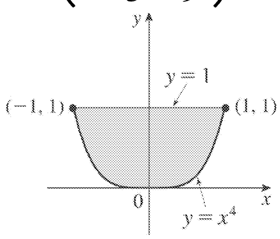


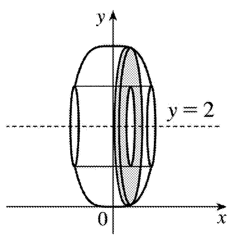
13. A cross–section is an annulus with inner radius 2–1 and outer radius $2–x^{4}$, so its area is

$A(x)=\pi\left(2-x^4\right)^2-\pi\,(2-1)^2=\pi\left(3-4x^4+x^8\right)$.

$$V=\int_{-1}^{1}A(x)dx=2\int_{0}^{1}A(x)dx=2\pi\int_{0}^{1}\left(3-4x^4+x^8\right)dx=2\pi\left[3x-\frac{4}{5}x^5+\frac{1}{9}x^9\right]_0^1=$$

$$2\pi\left(3-\frac{4}{5}+\frac{1}{9}\right)=\frac{208}{45}\pi$$



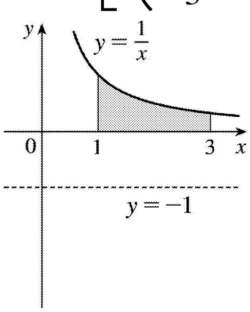


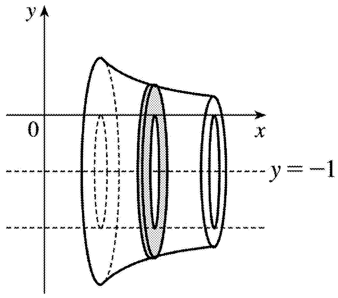
14.

$$V=\int_1^3\pi\left\{\left[\frac{1}{x}-(-1)\right]^2-[0-(-1)]^2\right\}dx=\pi\int_1^3\left[\left(\frac{1}{x}+1\right)^2-1^2\right]dx$$

$$=\pi\int_1^3\left(\frac{1}{x^2}+\frac{2}{x}\right)dx=\pi\left[-\frac{1}{x}+2\ln x\right]_1^3$$

$$=\pi\left[\left(-\frac{1}{3}+2\ln 3\right)-(-1+0)\right]=\pi\left(2\ln 3+\frac{2}{3}\right)=2\pi\left(\ln 3+\frac{1}{3}\right)$$

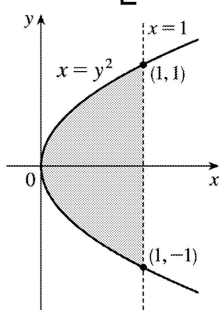


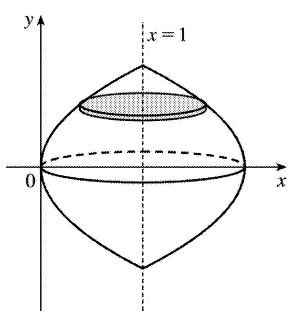


15.

$$V = \int_{-1}^{1} \pi \left(1-y^2\right)^2 dy = 2\int_{0}^{1} \pi \left(1-y^2\right)^2 dy = 2\pi \int_{0}^{1} \left(1-2y^2+y^4\right) dy$$

$$= 2\pi \left[y - \frac{2}{3} y^3 + \frac{1}{5} y^5 \right]_0^1 = 2\pi \cdot \frac{8}{15} = \frac{16}{15} \pi$$

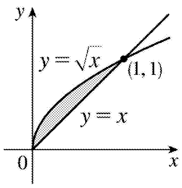


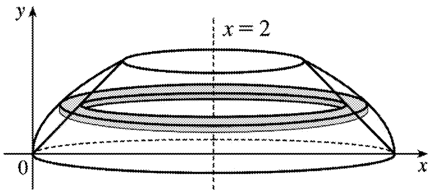


16. $y=\sqrt{x} \Rightarrow x=y^2$, so the outer radius is $2-y^2$.

$$V = \int_{0}^{1} \pi \left[\left(2-y^2\right)^2 - (2-y)^2 \right] dy = \pi \int_{0}^{1} \left[\left(4-4y^2+y^4\right) - \left(4-4y+y^2\right) \right] dy$$

$$= \pi \int_{0}^{1} \left(y^4-5y^2+4y\right) dy = \pi \left[\frac{1}{5} y^5 - \frac{5}{3} y^3 + 2y^2 \right]_0^1 = \pi \left(\frac{1}{5} - \frac{5}{3} + 2 \right) = \frac{8}{15} \pi$$



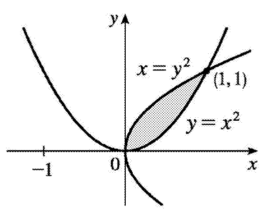


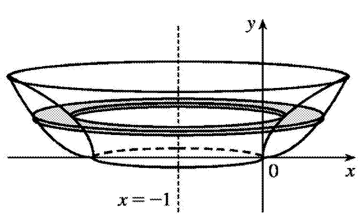
17. $y=x^2 \Rightarrow x=\sqrt{y}$ for $x\geq 0$. The outer radius is the distance from $x=-1$ to $x=\sqrt{y}$ and the inner radius is the distance from $x=-1$ to $x=y^2$.

$$V=\int_0^1 \pi \left\{ \left[\sqrt{y}-(-1)\right]^2-\left[y^2-(-1)\right]^2\right\} dy=\pi\int_0^1 \left[\left(\sqrt{y}+1\right)^2-\left(y^2+1\right)^2\right]dy$$

$$=\pi\int_0^1 \left(y+2\sqrt{y}+1-y^4-2y^2-1\right)dy=\pi\int_0^1 \left(y+2\sqrt{y}-y^4-2y^2\right)dy$$

$$=\pi\left[\frac{1}{2}y^2+\frac{4}{3}y^{3/2}-\frac{1}{5}y^5-\frac{2}{3}y^3\right]_0^1=\pi\left(\frac{1}{2}+\frac{4}{3}-\frac{1}{5}-\frac{2}{3}\right)=\frac{29}{30}\pi$$



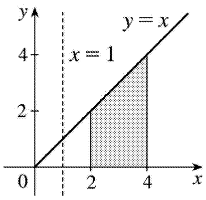


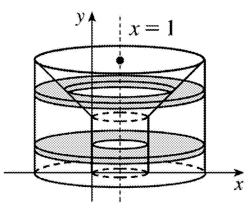
18. For $0\leq y<2$, a cross–section is an annulus with inner radius $2-1$ and outer radius $4-1$, the area of which is $A_1(y)=\pi\,(4-1)^2-\pi\,(2-1)^2$. For $2\leq y\leq 4$, a cross–section is an annulus with inner radius $y-1$ and outer radius $4-1$, the area of which is $A_2(y)=\pi\,(4-1)^2-\pi\,(y-1)^2$.

$$V = \int_0^4 A(y)\,dy = \pi\int_0^2 \left[(4-1)^2-(2-1)^2\right]dy + \pi\int_2^4 \left[(4-1)^2-(y-1)^2\right]dy$$

$$= \pi\,[8y]_0^2 + \pi\int_2^4 \left(8+2y-y^2\right)dy = 16\pi + \pi\left[8y+y^2-\frac{1}{3}y^3\right]_2^4$$

$$= 16\pi + \pi\left[\left(32+16-\frac{64}{3}\right)-\left(16+4-\frac{8}{3}\right)\right] = \frac{76}{3}\pi$$





19. R_1 about OA (the line $y=0$): $V = \int_0^1 A(x)\,dx = \int_0^1 \pi (x^3)^2\,dx = \pi\int_0^1 x^6\,dx = \pi\left[\frac{1}{7}x^7\right]_0^1 = \frac{\pi}{7}$

20. R_1 about OC (the line $x=0$):

$$V = \int_0^1 A(y)\,dy = \int_0^1 \left[\pi(1)^2 - \pi\left(\sqrt[3]{y}\right)^2\right]dy = \pi\int_0^1 (1-y^{2/3})\,dy = \pi\left[y-\frac{3}{5}y^{5/3}\right]_0^1 = \pi\left(1-\frac{3}{5}\right) = \frac{2\pi}{5}$$

21. R_1 about AB (the line $x=1$):

$$V = \int_0^1 A(y)\,dy = \int_0^1 \pi\left(1-\sqrt[3]{y}\right)^2 dy = \pi\int_0^1 (1-2y^{1/3}+y^{2/3})\,dy$$

$$= \pi\left[y-\frac{3}{2}y^{4/3}+\frac{3}{5}y^{5/3}\right]_0^1 = \pi\left(1-\frac{3}{2}+\frac{3}{5}\right) = \frac{\pi}{10}$$

22. R_1 about BC (the line $y=1$):

$$V = \int_0^1 A(x)\,dx = \int_0^1 \left[\pi(1)^2 - \pi(1-x^3)^2\right]dx = \pi\int_0^1 \left[1-(1-2x^3+x^6)\right]dx$$

$$=\pi\int_0^1(2x^3-x^6)dx=\pi\left[\frac{1}{2}x^4-\frac{1}{7}x^7\right]_0^1=\pi\left(\frac{1}{2}-\frac{1}{7}\right)=\frac{5\pi}{14}$$

23. R_2 about OA (the line $y=0$):

$$V=\int_0^1 A(x)dx=\int_0^1\left[\pi(1)^2-\pi\left(\sqrt{x}\right)^2\right]dx=\pi\int_0^1(1-x)dx=\pi\left[x-\frac{1}{2}x^2\right]_0^1=\pi\left(1-\frac{1}{2}\right)=\frac{\pi}{2}$$

24. R_2 about OC (the line $x=0$): $V=\int_0^1 A(y)dy=\int_0^1\pi(y^2)^2dy=\pi\int_0^1 y^4dy=\pi\left[\frac{1}{5}y^5\right]_0^1=\frac{\pi}{5}$

25. R_2 about AB (the line $x=1$):

$$V=\int_0^1 A(y)dy=\int_0^1\left[\pi(1)^2-\pi(1-y^2)^2\right]dy=\pi\int_0^1\left[1-(1-2y^2+y^4)\right]dy$$

$$=\pi\int_0^1(2y^2-y^4)dy=\pi\left[\frac{2}{3}y^3-\frac{1}{5}y^5\right]_0^1=\pi\left(\frac{2}{3}-\frac{1}{5}\right)=\frac{7\pi}{15}$$

26. R_2 about BC (the line $y=1$):

$$V=\int_0^1 A(x)dx=\int_0^1\pi\left(1-\sqrt{x}\right)^2dx=\pi\int_0^1(1-2x^{1/2}+x)dx$$

$$=\pi\left[x-\frac{4}{3}x^{3/2}+\frac{1}{2}x^2\right]_0^1=\pi\left(1-\frac{4}{3}+\frac{1}{2}\right)=\frac{\pi}{6}$$

27. R_3 about OA (the line $y=0$):

$$V=\int_0^1 A(x)dx=\int_0^1\left[\pi\left(\sqrt{x}\right)^2-\pi(x^3)^2\right]dx=\pi\int_0^1(x-x^6)dx=\pi\left[\frac{1}{2}x^2-\frac{1}{7}x^7\right]_0^1=\pi\left(\frac{1}{2}-\frac{1}{7}\right)=\frac{5\pi}{14}\ .$$

Note: Let $=_1+$ mathpi5$_2$ $+R_3$. If we rotate about any of the segments OA , OC , AB , or BC , we obtain a right circular cylinder of height 1 and radius 1 . Its volume is $\pi r^2h=\pi(1)^2\cdot 1=\pi$. As a check for Exercises 19, 23, and 27, we can add the answers, and that sum must equal π . Thus,

$$\frac{\pi}{7}+\frac{\pi}{2}+\frac{5\pi}{14}=\left(\frac{2+7+5}{14}\right)\pi=\pi\ .$$

28.

R_3 about *OC* (the line $x=0$):

$$V=\int_0^1 A(y)\,dy=\int_0^1\left[\pi\left(\sqrt[3]{y}\right)^2-\pi\left(y^2\right)^2\right]dy=\pi\int_0^1\left(y^{2/3}-y^4\right)dy$$

$$=\pi\left[\frac{3}{5}y^{5/3}-\frac{1}{5}y^5\right]_0^1=\pi\left(\frac{3}{5}-\frac{1}{5}\right)=\frac{2\pi}{5}$$

Note: See the note in Exercise 27. For Exercises 20, 24, and 28, we have $\frac{2\pi}{5}+\frac{\pi}{5}+\frac{2\pi}{5}=\pi$.

29. R_3 about *AB* (the line $x=1$):

$$V=\int_0^1 A(y)\,dy=\int_0^1\left[\pi\left(1-y^2\right)^2-\pi\left(1-\sqrt[3]{y}\right)^2\right]dy=\pi\int_0^1\left[\left(1-2y^2+y^4\right)-\left(1-2y^{1/3}+y^{2/3}\right)\right]dy$$

$$=\pi\int_0^1\left(-2y^2+y^4+2y^{1/3}-y^{2/3}\right)dy=\pi\left[-\frac{2}{3}y^3+\frac{1}{5}y^5+\frac{3}{2}y^{4/3}-\frac{3}{5}y^{5/3}\right]_0^1$$

$$=\pi\left(-\frac{2}{3}+\frac{1}{5}+\frac{3}{2}-\frac{3}{5}\right)=\frac{13\pi}{30}$$

Note: See the note in Exercise 27. For Exercises 21, 25, and 29, we have
$\frac{\pi}{10}+\frac{7\pi}{15}+\frac{13\pi}{30}=\left(\frac{3+14+13}{30}\right)\pi=\pi$.

30. R_3 about *BC* (the line $y=1$):

$$V=\int_0^1 A(x)\,dx=\int_0^1\left[\pi\left(1-x^3\right)^2-\pi\left(1-\sqrt{x}\right)^2\right]dx$$

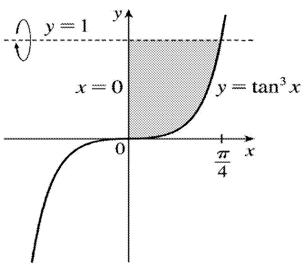
$$=\pi\int_0^1\left[\left(1-2x^3+x^6\right)-\left(1-2x^{1/2}+x\right)\right]dx=\pi\int_0^1\left(-2x^3+x^6+2x^{1/2}-x\right)dx$$

$$=\pi\left[-\frac{1}{2}x^4+\frac{1}{7}x^7+\frac{4}{3}x^{3/2}-\frac{1}{2}x^2\right]_0^1=\pi\left(-\frac{1}{2}+\frac{1}{7}+\frac{4}{3}-\frac{1}{2}\right)=\frac{10\pi}{21}$$

Note: See the note in Exercise 27. For Exercises 22, 26, and 30, we have
$\frac{5\pi}{14}+\frac{\pi}{6}+\frac{10\pi}{21}=\left(\frac{15+7+20}{42}\right)\pi=\pi$.

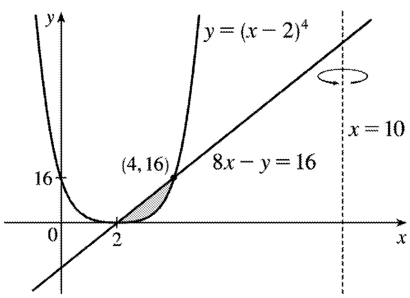
31.

$$V=\pi\int_0^{\pi/4}(1-\tan^3 x)^2\,dx$$



32. $y=(x-2)^4$ and $8x-y=16$ intersect when $(x-2)^4=8x-16=8(x-2)\Leftrightarrow(x-2)^4-8(x-2)=0\Leftrightarrow$ $(x-2)\left[(x-2)^3-8\right]=0\Leftrightarrow x-2=0$ or $x-2=2\Leftrightarrow x=2$ or 4 . $y=(x-2)^4\Rightarrow x-2=\pm\sqrt[4]{y}\Rightarrow x=2+\sqrt[4]{y}$ [since $x\geq 2$]. $8x-y=16\Rightarrow 8x=y+16\Rightarrow x=\frac{1}{8}y+2$.

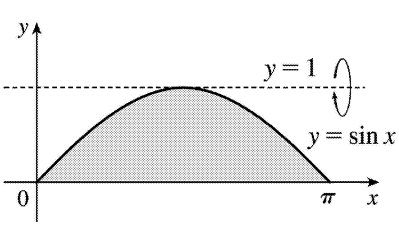
$$V=\pi\int_0^{16}\left\{\left[10-\left(\frac{1}{8}y+2\right)\right]^2-\left[10-\left(2+\sqrt[4]{y}\right)\right]^2\right\}dy$$



33.

$$V=\pi\int_0^{\pi}\left[(1-0)^2-(1-\sin x)^2\right]dx$$

$$=\pi\int_0^{\pi}\left[1^2-(1-\sin x)^2\right]dx$$



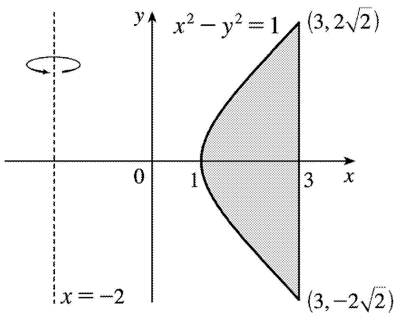
34.

$$V=\pi\int_0^{\pi}\left[(\sin x+2)^2-2^2\right]dx$$

35.

$$V=\pi\int_{-\sqrt{8}}^{\sqrt{8}}\left\{[3-(-2)]^2-\left[\sqrt{y^2+1}-(-2)\right]^2\right\}dy$$

$$=\pi\int_{-2\sqrt{2}}^{2\sqrt{2}}\left[5^2-\left(\sqrt{1+y^2}+2\right)^2\right]dy$$

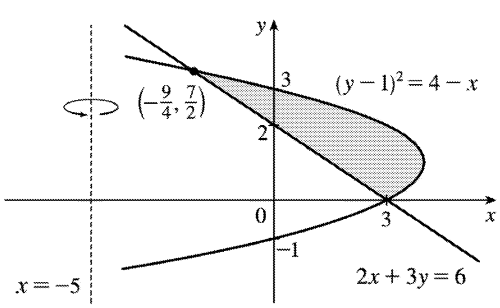


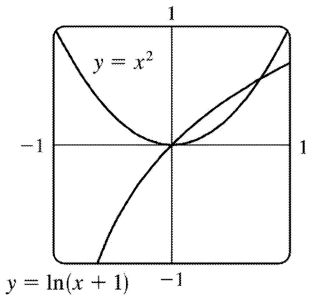
36. Solve the equations for x : $(y-1)^2=4-x \Leftrightarrow x=4-(y-1)^2$ and $2x+3y=6 \Leftrightarrow x=3-\frac{3}{2}y$.

The points of intersection of the two curves are $(3,0)$ and $\left(-\frac{9}{4},\frac{7}{2}\right)$. Therefore,

$$V=\pi\int_0^{7/2}\left\{\left[4-(y-1)^2-(-5)\right]^2-\left[3-\frac{3}{2}y-(-5)\right]^2\right\}dy$$

$$=\pi\int_0^{7/2}\left\{\left[9-(y-1)^2\right]^2-\left(8-\frac{3}{2}y\right)^2\right\}dy$$



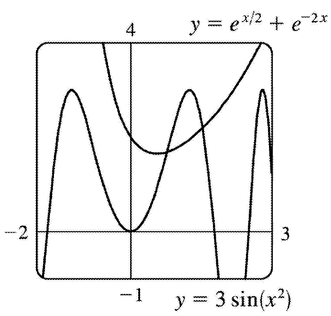


37.

$y=x^2$ and $y=\ln(x+1)$ intersect at $x=0$ and at $x=a\approx 0.747$.

$$V=\pi\int_0^a \left\{ [\ln(x+1)]^2-\left(x^2\right)^2 \right\} dx\approx 0.132$$

38.



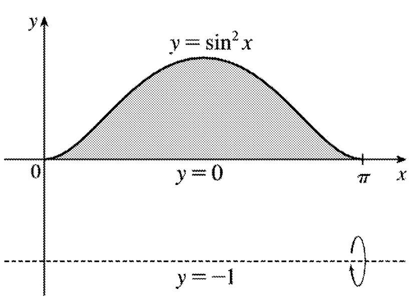
$y=3\sin(x^2)$ and $y=e^{x/2}+e^{-2x}$ intersect at $x=a\approx 0.772$ and at $x=b\approx 1.524$.

$$V=\pi\int_a^b \left\{ \left[3\sin(x^2)\right]^2-\left(e^{x/2}+e^{-2x}\right)^2 \right\} dx\approx 7.519$$

39.

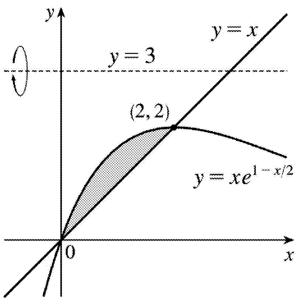
$$V=\pi\int_0^{\pi} \left\{ \left[\sin^2 x-(-1)\right]^2-[0-(-1)]^2 \right\} dx$$

$$=\frac{11}{8}\pi^2$$



40. V

$$=\pi\int_0^2\left[(3-x)^2-(3-xe^{1-x/2})^2\right]dx \overset{\text{CAS}}{=} \pi\left(-2e^2+24e-\frac{142}{3}\right)$$



41. $\pi\int_0^{\pi/2}\cos^2 x\,dx$ describes the volume of the solid obtained by rotating the region $R=\left\{(x,y)\mid 0\le x\le\frac{\pi}{2}, 0\le y\le\cos x\right\}$ of the xy–plane about the x–axis.

42. $\pi\int_2^5 y\,dy=\pi\int_2^5\left(\sqrt{y}\right)^2dy$ describes the volume of the solid obtained by rotating the region $R=\left\{(x,y)\mid 2\le y\le 5, 0\le x\le\sqrt{y}\right\}$ of the xy–plane about the y–axis.

43. $\pi\int_0^1\left(y^4-y^8\right)dy=\pi\int_0^1\left[\left(y^2\right)^2-\left(y^4\right)^2\right]dy$ describes the volume of the solid obtained by rotating the region $R=\left\{(x,y)\mid 0\le y\le 1, y^4\le x\le y^2\right\}$ of the xy–plane about the y–axis.

44. $\pi\int_0^{\pi/2}\left[(1+\cos x)^2-1^2\right]dx$ describes the volume of the solid obtained by rotating the region $=\left\{(x,y)\mid 0\le x\le\frac{\pi}{2}, 1\le y\le 1+\cos x\right\}$ of the xy–plane about the x–axis.
Or: The solid could be obtained by rotating the region $'=\left\{(x,y)\mid 0\le x\le\frac{\pi}{2}, 0\le y\,\square\, x\right\}$ about the line $y=-1$.

45. There are 10 subintervals over the 15–cm length, so we'll use $n=10/2=5$ for the Midpoint Rule.

$$V=\int_0^{15}A(x)\,dx\approx M_5=\frac{15-0}{5}\left[A(1.5)+A(4.5)+A(7.5)+A(10.5)+A(13.5)\right]$$

$$=3(18+79+106+128+39)=3\cdot 370=1110\text{ cm}^3$$

46.

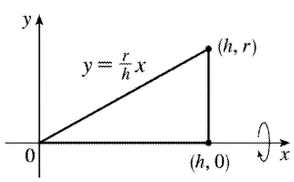
$$V=\int_0^{10} A(x)\,dx\approx M_5=\frac{10-0}{5}\left[A(1)+A(3)+A(5)+A(7)+A(9)\right]$$

$$=2(0.65+0.61+0.59+0.55+0.50)=2(2.90)=5.80\text{m}^3$$

47. We'll form a right circular cone with height h and base radius r by revolving the line $y=\frac{r}{h}x$ about the x-axis.

$$V=\pi\int_0^h\left(\frac{r}{h}x\right)^2dx=\pi\int_0^h\frac{r^2}{h^2}x^2dx=\pi\frac{r^2}{h^2}\left[\frac{1}{3}x^3\right]_0^h$$

$$=\pi\frac{r^2}{h^2}\left(\frac{1}{3}h^3\right)=\frac{1}{3}\pi r^2h$$



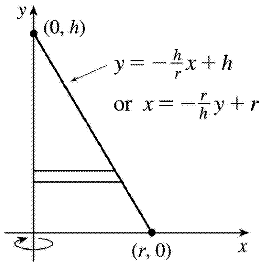
Another solution: Revolve $x=-\frac{r}{h}y+r$ about the y-axis.

$$V=\pi\int_0^h\left(-\frac{r}{h}y+r\right)^2dy=\pi\int_0^h\left[\frac{r^2}{h^2}y^2-\frac{2r^2}{h}y+r^2\right]dy$$

$$=\pi\left[\frac{r^2}{3h^2}y^3-\frac{r^2}{h}y^2+r^2y\right]_0^h=\pi\left(\frac{1}{3}r^2h-r^2h+r^2h\right)=\frac{1}{3}\pi r^2h$$

* Or use substitution with $u=r-\frac{r}{h}y$ and $du=-\frac{r}{h}dy$ to get

$$\pi\int_r^0u^2\left(-\frac{h}{r}\right)du=-\pi\frac{h}{r}\left[\frac{1}{3}u^3\right]_r^0=-\pi\frac{h}{r}\left(-\frac{1}{3}r^3\right)=\frac{1}{3}\pi r^2h\ .$$



48.

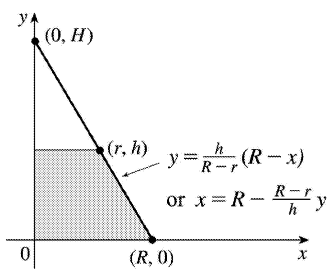
$$V = \pi \int_0^h \left(R - \frac{R-r}{h} y \right)^2 dy$$

$$= \pi \int_0^h \left[R^2 - \frac{2R(R-r)}{h} y + \left(\frac{R-r}{h} \right)^2 y^2 \right] dy$$

$$= \pi \left[R^2 y - \frac{R(R-r)}{h} y^2 + \frac{1}{3} \left(\frac{R-r}{h} \right)^2 y^3 \right]_0^h$$

$$= \pi \left[R^2 h - R(R-r)h + \frac{1}{3} (R-r)^2 h \right] \square$$

$$= \frac{1}{3} \pi h \left[3Rr + (R^2 - 2Rr + r^2) \right] = \frac{1}{3} \pi h (R^2 + Rr + r^2)$$



Another solution: $\frac{H}{R} = \frac{H-h}{r}$ by similar triangles. Therefore,

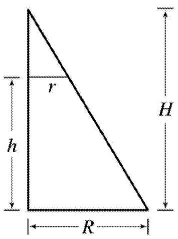
$Hr = HR - hR \Rightarrow hR = H(R-r) \Rightarrow H = \frac{hR}{R-r}$. Now

$$V = \frac{1}{3} \pi R^2 H - \frac{1}{3} \pi r^2 (H-h) \text{ [by Exercise 47]}$$

$$= \frac{1}{3} \pi R^2 \frac{hR}{R-r} - \frac{1}{3} \pi r^2 \frac{rh}{R-r} \left[H - h = \frac{rH}{R} = \frac{rhR}{R(R-r)} \right]$$

$$= \frac{1}{3} \pi h \frac{R^3 - r^3}{R-r} = \frac{1}{3} \pi h \left(R^2 + Rr + r^2 \right)$$

$$=\frac{1}{3}\left[\pi R^2+\pi r^2+\sqrt{\left(\pi R^2\right)\left(\pi r^2\right)}\right]h=\frac{1}{3}\left(A_1+A_2+\sqrt{A_1A_2}\right)h$$



where A_1 and A_2 are the areas of the bases of the frustum. (See Exercise 50 for a related result.)

49. $x^2+y^2=r^2 \Leftrightarrow x^2=r^2-y^2$

$$V=\pi\int_{r-h}^{r}\left(r^2-y^2\right)dy=\pi\left[r^2y-\frac{y^3}{3}\right]_{r-h}^{r}$$

$$=\pi\left\{\left[r^3-\frac{r^3}{3}\right]-\left[r^2(r-h)-\frac{(r-h)^3}{3}\right]\right\}$$

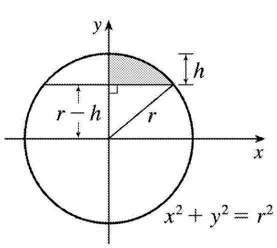
$$=\pi\left\{\frac{2}{3}r^3-\frac{1}{3}(r-h)\left[3r^2-(r-h)^2\right]\right\}$$

$$=\frac{1}{3}\pi\left\{2r^3-(r-h)\left[3r^2-\left(r^2-2rh+h^2\right)\right]\right\}$$

$$=\frac{1}{3}\pi\left\{2r^3-(r-h)\left[2r^2+2rh-h^2\right]\right\}$$

$$=\frac{1}{3}\pi\left(2r^3-2r^3-2r^2h+rh^2+2r^2h+2rh^2-h^3\right)$$

$$=\frac{1}{3}\pi\left(3rh^2-h^3\right)=\frac{1}{3}\pi h^2(3r-h)\text{ , or, equivalently, }\pi h^2\left(r-\frac{h}{3}\right)$$



50. An equation of the line is

$$x=\frac{\Delta x}{\Delta y}\,y+(x\text{ –intercept })=\frac{a/2-b/2}{h-0}\,y+\frac{b}{2}=\frac{a-b}{2h}\,y+\frac{b}{2}\ .$$

$$V=\int_0^h A(y)\,dy=\int_0^h (2x)^2\,dy$$

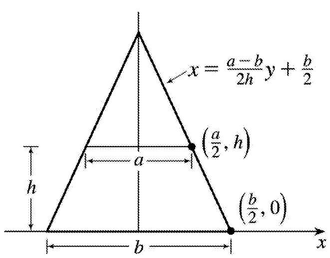
$$=\int_0^h\left[2\left(\frac{a-b}{2h}\,y+\frac{b}{2}\right)\right]^2dy=\int_0^h\left[\frac{a-b}{h}\,y+b\right]^2dy$$

$$=\int_0^h\left[\frac{(a-b)^2}{h^2}\,y^2+\frac{2b(a-b)}{h}\,y+b^2\right]dy$$

$$=\left[\frac{(a-b)^2}{3h^2}\,y^3+\frac{b(a-b)}{h}\,y^2+b^2y\right]_0^h$$

$$=\frac{1}{3}(a-b)^2h+b(a-b)h+b^2h=\frac{1}{3}\left(a^2-2ab+b^2+3ab\right)h$$

$$=\frac{1}{3}\left(a^2+ab+b^2\right)h$$



Note that this can be written as $\frac{1}{3}\left(A_1+A_2+\sqrt{A_1A_2}\right)h$, as in Exercise 48.

If $a=b$, we get a rectangular solid with volume b^2h . If $a=0$, we get a square pyramid with volume $\frac{1}{3}\,b^2h$.

51. For a cross–section at height y , we see from similar triangles that $\frac{\alpha/2}{b/2}=\frac{h-y}{h}$, so $\alpha=b\left(1-\frac{y}{h}\right)$.

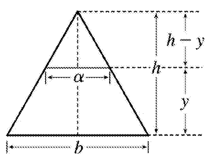
Similarly, for cross–sections having $2b$ as their base and β replacing α , $\beta=2b\left(1-\frac{y}{h}\right)$. So

$$V = \int_0^h A(y)\,dy = \int_0^h \left[b\left(1-\frac{y}{h}\right)\right]\left[2b\left(1-\frac{y}{h}\right)\right] dy$$

$$= \int_0^h 2b^2\left(1-\frac{y}{h}\right)^2 dy = 2b^2\int_0^h \left(1-\frac{2y}{h}+\frac{y^2}{h^2}\right) dy$$

$$= 2b^2\left[y-\frac{y^2}{h}+\frac{y^3}{3h^2}\right]_0^h = 2b^2\left[h-h+\frac{1}{3}h\right]$$

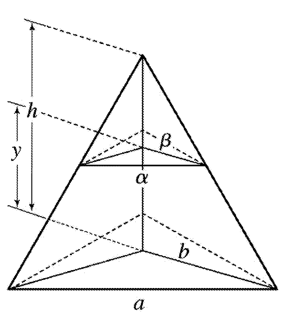
$= \frac{2}{3}b^2h$ [$=\frac{1}{3}Bh$ where B is the area of the base, as with any pyramid.]



52. Consider the triangle consisting of two vertices of the base and the center of the base. This triangle is similar to the corresponding triangle at a height y, so $a/b=\alpha/\beta \Rightarrow \alpha = a\beta/b$. Also by similar triangles, $b/h=\beta/(h-y)$
$\Rightarrow \beta = b(h-y)/h$. These two equations imply that $\alpha = a(1-y/h)$, and since the cross–section is an equilateral triangle, it has area $A(y)=\frac{1}{2}\cdot\alpha\cdot\frac{\sqrt{3}}{2}\alpha = \frac{a^2(1-y/h)^2}{4}\sqrt{3}$, so

$$V = \int_0^h A(y)\,dy = \frac{a^2\sqrt{3}}{4}\int_0^h \left(1-\frac{y}{h}\right)^2 dy$$

$$= \frac{a^2\sqrt{3}}{4}\left[-\frac{h}{3}\left(1-\frac{y}{h}\right)^3\right]_0^h = -\frac{\sqrt{3}}{12}a^2h(-1) = \frac{\sqrt{3}}{12}a^2h$$



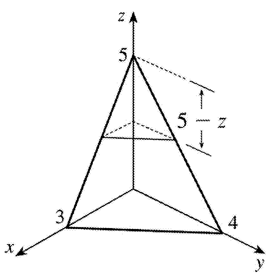
53. A cross–section at height z is a triangle similar to the base, so we'll multiply the legs of the base triangle, 3 and 4, by a proportionality factor of $(5-z)/5$. Thus, the triangle at height z has area

$A(z)=\frac{1}{2}\cdot 3\left(\frac{5-z}{5}\right)\cdot 4\left(\frac{5-z}{5}\right)=6\left(1-\frac{z}{5}\right)^2$, so

$$V=\int_0^5 A(z)\,dz=6\int_0^5 (1-z/5)^2\,dz$$

$$=6\int_1^0 u^2(-5\,du)\left[u=1-z/5,du=-\frac{1}{5}\,dz\right]$$

$$=-30\left[\frac{1}{3}u^3\right]_1^0=-30\left(-\frac{1}{3}\right)=10\text{cm}^3$$



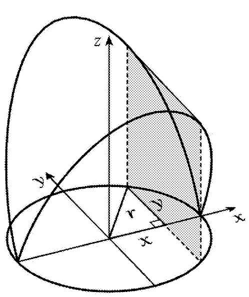
54. A cross–section is shaded in the diagram.

$A(x)=(2y)^2=\left(2\sqrt{r^2-x^2}\right)^2$, so

$$V=\int_{-r}^{r} A(x)\,dx=2\int_0^r 4\left(r^2-x^2\right)dx$$

$$=8\left[r^2x-\frac{1}{3}x^3\right]_0^r$$

$$=8\left(\frac{2}{3}r^3\right)=\frac{16}{3}r^3$$

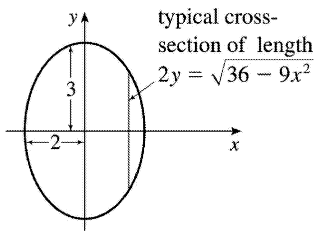


55. If l is a leg of the isosceles right triangle and $2y$ is the hypotenuse, then $l^2+l^2=(2y)^2 \Rightarrow 2l^2=4y^2 \Rightarrow l^2=2y^2$.

$$V=\int_{-2}^{2} A(x)\,dx=2\int_{0}^{2} A(x)\,dx=2\int_{0}^{2} \frac{1}{2}(l)(l)\,dx=2\int_{0}^{2} y^2\,dx$$

$$=2\int_{0}^{2} \frac{1}{4}\left(36-9x^2\right)dx=\frac{9}{2}\int_{0}^{2}\left(4-x^2\right)dx$$

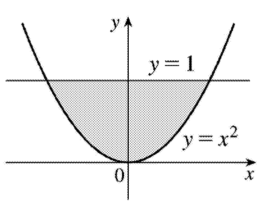
$$=\frac{9}{2}\left[4x-\frac{1}{3}x^3\right]_0^2=\frac{9}{2}\left(8-\frac{8}{3}\right)=24$$



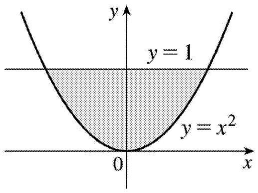
56. The cross–section of the base corresponding to the coordinate y has length $2x=2\sqrt{y}$. The corresponding equilateral triangle with side s has area

$A(y)=s^2\left(\frac{\sqrt{3}}{4}\right)=(2x)^2\left(\frac{\sqrt{3}}{4}\right)=\left(2\sqrt{y}\right)^2\left(\frac{\sqrt{3}}{4}\right)=y\sqrt{3}$. Therefore,

$$V=\int_{0}^{1} A(y)\,dy=\int_{0}^{1} y\sqrt{3}\,dy=\sqrt{3}\left[\frac{1}{2}y^2\right]_0^1=\frac{\sqrt{3}}{2}\ .$$



57. The cross–section of the base corresponding to the coordinate y has length $2x=2\sqrt{y}$. The square has area $A(y)=\left(2\sqrt{y}\right)^2=4y$, so $V=\int_{0}^{1} A(y)\,dy=\int_{0}^{1} 4y\,dy=\left[2y^2\right]_0^1=2$.

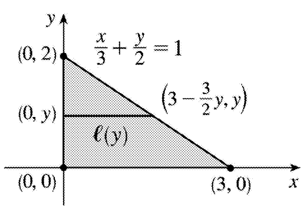


58. A typical cross–section perpendicular to the y –axis in the base has length $\ell(y)=3-\frac{3}{2}$ y. This length is the diameter of a cross–sectional semicircle in S , so

$$V=\int_0^2 A(y)\,dy=\int_0^2 \frac{\pi}{2}\left[\frac{\ell(y)}{2}\right]^2 dy=\frac{\pi}{8}\int_0^2\left(3-\frac{3}{2}y\right)^2 dy$$

$$=\frac{\pi}{8}\int_3^0 u^2\left(-\frac{2}{3}\right)du\left[u=3-\frac{3}{2}y,du=-\frac{3}{2}dy\right]$$

$$=-\frac{\pi}{12}\left[\frac{1}{3}u^3\right]_3^0=-\frac{\pi}{12}(-9)=\frac{3\pi}{4}$$



59. A typical cross–section perpendicular to the y –axis in the base has length $\ell(y)=3-\frac{3}{2}$ y. This length is the leg of an isosceles right triangle, so

$$A(y)=\frac{1}{2}[\ell(y)]^2\left[\frac{1}{2}bh\text{ with base = height}\right]$$

$$=\frac{1}{2}\left[3\left(1-\frac{1}{2}y\right)\right]^2=\frac{9}{2}\left(1-\frac{1}{2}y\right)^2$$

Thus,

$$V=\int_0^2 A(y)\,dy=\frac{9}{2}\int_1^0 u^2(-2du)\quad\left[u=1-\frac{1}{2}y,du=-\frac{1}{2}dy\right]$$

$$=-9\left[\frac{1}{3}u^3\right]_1^0=-9\left(-\frac{1}{3}\right)=3$$

60. **(a)** $V=\int_{-r}^{r} A(x)\,dx=2\int_{0}^{r} A(x)\,dx=2\int_{0}^{r} \frac{1}{2}\,h\left(2\sqrt{r^2-x^2}\right)dx=2h\int_{0}^{r}\sqrt{r^2-x^2}\,dx$

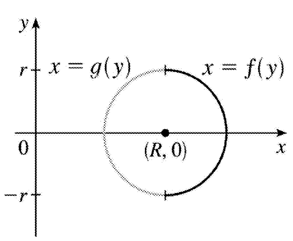
(b) Observe that the integral represents one quarter of the area of a circle of radius r , so $V=2h\cdot \frac{1}{4}\pi r^2=\frac{1}{2}\pi h r^2$.

61. **(a)** The torus is obtained by rotating the circle $(x-R)^2+y^2=r^2$ about the $y-$ axis. Solving for x , we see that the right half of the circle is given by $x=R+\sqrt{r^2-y^2}=f(y)$ and the left half by $x=R-\sqrt{r^2-y^2}=g(y)$. So

$$V = \pi\int_{-r}^{r}\left\{[f(y)]^2-[g(y)]^2\right\}dy$$

$$= 2\pi\int_{0}^{r}\left[\left(R^2+2R\sqrt{r^2-y^2}+r^2-y^2\right)-\left(R^2-2R\sqrt{r^2-y^2}+r^2-y^2\right)\right]dy$$

$$= 2\pi\int_{0}^{r}4R\sqrt{r^2-y^2}\,dy=8\pi R\int_{0}^{r}\sqrt{r^2-y^2}\,dy$$



(b) Observe that the integral represents a quarter of the area of a circle with radius r , so $8\pi R\int_{0}^{r}\sqrt{r^2-y^2}\,dy=8\pi R\cdot \frac{1}{4}\pi r^2=2\pi^2 r^2 R$.

62. The cross–sections perpendicular to the $y-$ axis in Figure 17 are rectangles. The rectangle corresponding to the coordinate y has a base of length $2\sqrt{16-y^2}$ in the $xy-$ plane and a height of $\frac{1}{\sqrt{3}}\,y$, since $\angle BAC=30^\circ$ and $|BC|=\frac{1}{\sqrt{3}}\,|AB|$. Thus, $A(y)=\frac{2}{\sqrt{3}}\,y\sqrt{16-y^2}$ and

$$V = \int_{0}^{4} A(y)\,dy=\frac{2}{\sqrt{3}}\int_{0}^{4}\sqrt{16-y^2}\,y\,dy$$

$$= \frac{2}{\sqrt{3}} \int_{16}^{0} u^{1/2} \left(-\frac{1}{2} \right) du \text{[Put } u=16-y^2 \text{, so } du=-2y\,dy\text{]}$$

$$= \frac{1}{\sqrt{3}} \int_{0}^{16} u^{1/2} du = \frac{1}{\sqrt{3}} \frac{2}{3} \left[u^{3/2} \right]_0^{16} = \frac{2}{3\sqrt{3}} (64) = \frac{128}{3\sqrt{3}}$$

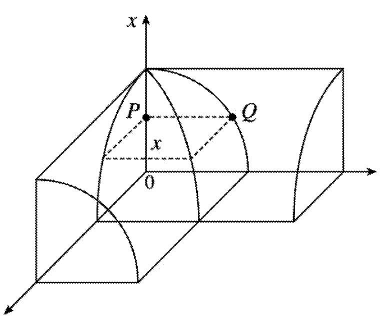
63. **(a)** Volume $\left(S_1\right) = \int_0^h A(z)\,dz =$ Volume $\left(S_2\right)$ since the cross–sectional area $A(z)$ at height z is the same for both solids.

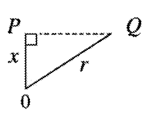
(b) By Cavalieri's Principle, the volume of the cylinder in the figure is the same as that of a right circular cylinder with radius r and height h , that is, $\pi r^2 h$.

64. Each cross–section of the solid S in a plane perpendicular to the x– axis is a square (since the edges of the cut lie on the cylinders, which are perpendicular). One–quarter of this square and one –eighth of S are shown. The area of this quarter –square is $|PQ|^2 = r^2 - x^2$. Therefore, $A(x) = 4\left(r^2 - x^2\right)$ and the volume of S is

$$V = \int_{-r}^{r} A(x)\,dx = 4\int_{-r}^{r} \left(r^2 - x^2\right) dx$$

$$= 8\int_0^r \left(r^2 - x^2\right) dx = 8\left[r^2 x - \frac{1}{3} x^3 \right]_0^r = \frac{16}{3} r^3$$



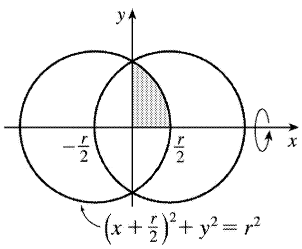


65. The volume is obtained by rotating the area common to two circles of radius r , as shown. The volume of the right half is

$$V_{\text{right}} = \pi \int_0^{r/2} y^2\,dx = \pi \int_0^{r/2} \left[r^2 - \left(\frac{1}{2} r + x \right)^2 \right] dx$$

$$= \pi \left[r^2 x - \frac{1}{3} \left(\frac{1}{2} r + x \right)^3 \right]_0^{r/2} = \pi \left[\left(\frac{1}{2} r^3 - \frac{1}{3} r^3 \right) - \left(0 - \frac{1}{24} r^3 \right) \right] = \frac{5}{24} \pi r^3$$

So by symmetry, the total volume is twice this, or $\frac{5}{12} \pi r^3$.



Another solution: We observe that the volume is the twice the volume of a cap of a sphere, so we can use the formula from Exercise 49 with $h = \frac{1}{2} r$:

$$V = 2 \cdot \frac{1}{3} \pi h^2 (3r - h) = \frac{2}{3} \pi \left(\frac{1}{2} r \right)^2 \left(3r - \frac{1}{2} r \right) = \frac{5}{12} \pi r^3 .$$

66. We consider two cases: one in which the ball is not completely submerged and the other in which it is.

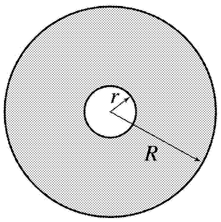
Case 1: $0 \le h \le 10$ The ball will not be completely submerged, and so a cross–section of the water parallel to the surface will be the shaded area shown in the first diagram. We can find the area of the cross–section at height x above the bottom of the bowl by using the Pythagorean Theorem: $R^2 = 15^2 - (15 - x)^2$ and $r^2 = 5^2 - (x - 5)^2$, so $A(x) = \pi \left(R^2 - r^2 \right) = 20\pi x$. The volume of water when it has depth h is then $V(h) = \int_0^h A(x)\,dx = \int_0^h 20\pi x\,dx = \left[10\pi x^2 \right]_0^h = 10\pi h^2 \text{ cm}^3$, $0 \le h \le 10$.

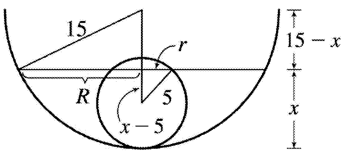
Case 2: $10 < h \le 15$ In this case we can find the volume by simply subtracting the volume displaced by the ball from the total volume inside the bowl underneath the surface of the water. The total volume underneath the
surface is just the volume of a cap of the bowl, so we use the formula from Exercise 49:

$V_{\text{cap}}(h) = \frac{1}{3} \pi h^2 (45 - h)$. The volume of

the small sphere is $V_{\text{ball}} = \frac{4}{3} \pi (5)^3 = \frac{500}{3} \pi$,

so the total volume is $V_{\text{cap}} - V_{\text{ball}} = \frac{1}{3} \pi \left(45h^2 - h^3 - 500 \right) \text{ cm}^3$.



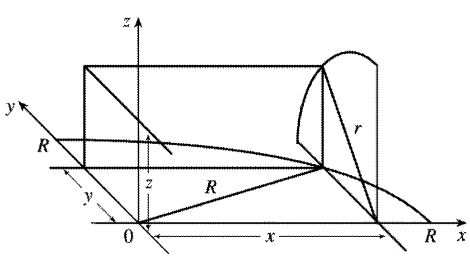


67. Take the x– axis to be the axis of the cylindrical hole of radius r . A quarter of the cross–section through y, perpendicular to the y– axis, is the rectangle shown. Using the Pythagorean Theorem twice, we see that the dimensions of this rectangle are $x=\sqrt{R^2-y^2}$ and $z=\sqrt{r^2-y^2}$, so

$\frac{1}{4}A(y)=xz=\sqrt{r^2-y^2}\sqrt{R^2-y^2}$, and

$$V=\int_{-r}^{r}A(y)\,dy=\int_{-r}^{r}4\sqrt{r^2-y^2}\sqrt{R^2-y^2}\,dy$$

$$=8\int_{0}^{r}\sqrt{r^2-y^2}\sqrt{R^2-y^2}\,dy$$



68. The line $y=r$ intersects the semicircle $y=\sqrt{R^2-x^2}$ when $r=\sqrt{R^2-x^2}\Rightarrow r^2=R^2-x^2\Rightarrow$ $x^2=R^2-r^2\Rightarrow x=\pm\sqrt{R^2-r^2}$. Rotating the shaded region about the x –axis gives us

$$V=\int_{-\sqrt{R^2-r^2}}^{\sqrt{R^2-r^2}}\pi\left[\left(\sqrt{R^2-x^2}\right)^2-r^2\right]dx$$

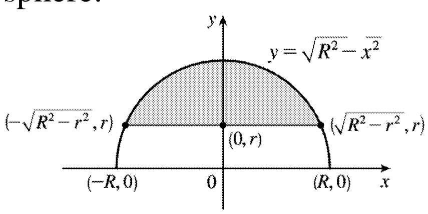
$$= 2\pi \int_0^{\sqrt{R^2-r^2}} \left(R^2-x^2-r^2\right) dx \text{ [by symmetry]}$$

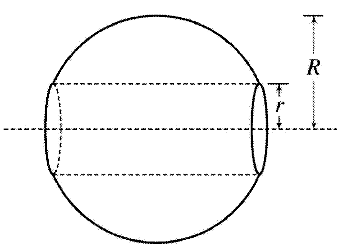
$$= 2\pi \int_0^{\sqrt{R^2-r^2}} \left[\left(R^2-r^2\right)-x^2\right] dx = 2\pi \left[\left(R^2-r^2\right)x - \frac{1}{3}x^3\right]_0^{\sqrt{R^2-r^2}}$$

$$= 2\pi \left[\left(R^2-r^2\right)^{3/2} - \frac{1}{3}\left(R^2-r^2\right)^{3/2}\right]$$

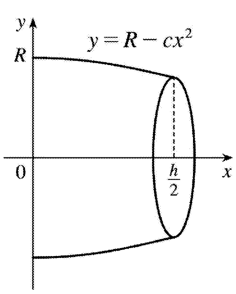
$$= 2\pi \cdot \frac{2}{3}\left(R^2-r^2\right)^{3/2} = \frac{4\pi}{3}\left(R^2-r^2\right)^{3/2}$$

Our answer makes sense in limiting cases. As $r \to 0$, $V \to \frac{4}{3}\pi R^3$, which is the volume of the full sphere. As $r \to R$, $V \to 0$, which makes sense because the hole's radius is approaching that of the sphere.





69. **(a)** The radius of the barrel is the same at each end by symmetry, since the function $y=R-cx^2$ is even. Since the barrel is obtained by rotating the graph of the function y about the $x-$ axis, this radius is equal to the value of y at $x=\frac{1}{2}h$, which is $R-c\left(\frac{1}{2}h\right)^2=R-d=r$.



(b) The barrel is symmetric about the $y-$ axis, so its volume is twice the volume of that part of the barrel for $x>0$. Also, the barrel is a volume of rotation, so

$$V = 2\int_0^{h/2} \pi y^2 dx = 2\pi \int_0^{h/2} \left(R-cx^2\right)^2 dx = 2\pi \left[R^2 x - \frac{2}{3} Rcx^3 + \frac{1}{5} c^2 x^5 \right]_0^{h/2}$$

$$= 2\pi \left(\frac{1}{2} R^2 h - \frac{1}{12} Rch^3 + \frac{1}{160} c^2 h^5 \right)$$

Trying to make this look more like the expression we want, we rewrite it as

$V = \frac{1}{3} \pi h \left[2R^2 + \left(R^2 - \frac{1}{2} Rch^2 + \frac{3}{80} c^2 h^4 \right) \right]$. But

$R^2 - \frac{1}{2} Rch^2 + \frac{3}{80} c^2 h^4 = \left(R - \frac{1}{4} ch^2 \right)^2 - \frac{1}{40} c^2 h^4 = (R-d)^2 - \frac{2}{5} \left(\frac{1}{4} ch^2 \right)^2 = r^2 - \frac{2}{5} d^2$.

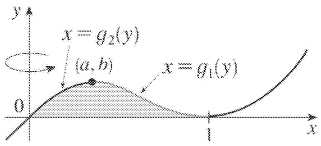
Substituting this back into V , we see that $V = \frac{1}{3} \pi h \left(2R^2 + r^2 - \frac{2}{5} d^2 \right)$, as required.

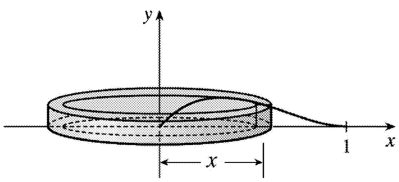
70. It suffices to consider the case where is bounded by the curves $y=f(x)$ and $y=g(x)$ for $a \leq x \leq b$, where $g(x) \leq f(x)$ for all x in $[a,b]$, since other regions can be decomposed into subregions of this type. We are concerned with the volume obtained when is rotated about the line $y=-k$, which is equal to

$$V_2 = \pi \int_a^b \left([f(x)+k]^2 - [g(x)+k]^2 \right) dx = \pi \int_a^b \left([f(x)]^2 - [g(x)]^2 \right) dx + 2\pi k \int_a^b [f(x)-g(x)] dx$$

$$= V_1 + 2\pi kA$$

1.



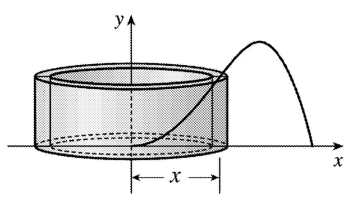


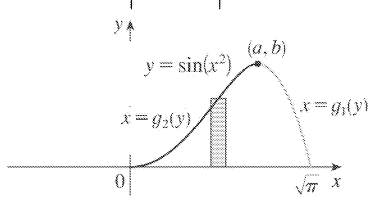
If we were to use the "washer'' method, we would first have to locate the local maximum point (a,b) *of* $y=x(x-1)^2$ using the methods of Chapter 4. Then we would have to solve the equation $y=x(x-1)^2$ for x in terms of y to obtain the functions $x=g_1(y)$ and $x=g_2(y)$ shown in the first figure. This step would be difficult because it involves the cubic formula. Finally we would find the volume using $V=\pi\int_0^b \left\{ \left[g_1(y)\right]^2-\left[g_2(y)\right]^2 \right\} dy$.

Using shells, we find that a typical approximating shell has radius x , so its circumference is $2\pi x$. Its height is y , that is, $x(x-1)^2$. So the total volume is

$$V=\int_0^1 2\pi x\left[x(x-1)^2\right]dx=2\pi\int_0^1 \left(x^4-2x^3+x^2\right)dx=2\pi\left[\frac{x^5}{5}-2\frac{x^4}{4}+\frac{x^3}{3}\right]_0^1=\frac{\pi}{15}$$

2.





A typical cylindrical shell has circumference $2\pi x$ and height $\sin\left(x^2\right)$. $V=\int_0^{\sqrt{\pi}} 2\pi x\sin\left(x^2\right)dx$. Let

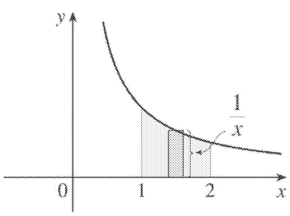
$u=x^2$. Then $du=2x\,dx$, so $V=\pi\int_0^{\pi}\sin u\,du=\pi\left[-\cos u\right]_0^{\pi}=\pi[1-(-1)]=2\pi$.

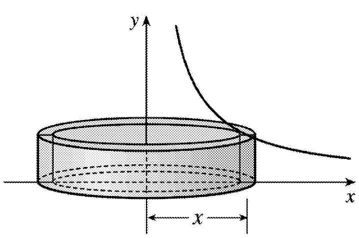
For slicing, we would first have to locate the local maximum point (a,b) of $y=\sin\left(x^2\right)$ using the methods of Chapter 4. Then we would have to solve the equation $y=\sin\left(x^2\right)$ for x in terms of y to obtain the functions $x=g_1(y)$ and $x=g_2(y)$ shown in the second figure. Finally we would find the volume using ▯ . Using shells is definitely preferable to slicing.

3.

$$V=\int_1^2 2\pi x\cdot\frac{1}{x}\,dx=2\pi\int_1^2 1\,dx$$

$$=2\pi\left[x\right]_1^2=2\pi(2-1)=2\pi$$

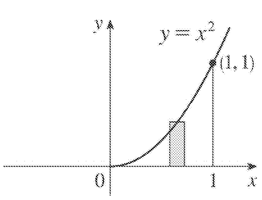


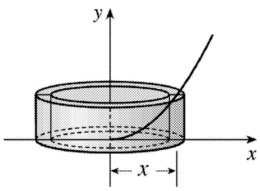


4.

$$V=\int_0^1 2\pi x\cdot x^2\,dx=2\pi\int_0^1 x^3\,dx$$

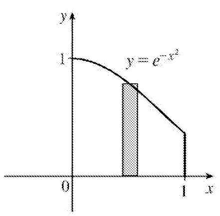
$$=2\pi\left[\frac{1}{4}x^4\right]_0^1=2\pi\cdot\frac{1}{4}=\frac{\pi}{2}$$

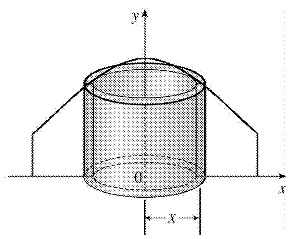




5. $V=\int_0^1 2\pi x e^{-x^2}\,dx$. Let $u=x^2$. Thus, $du=2x\,dx$, so

$$V=\pi\int_0^1 e^{-u}\,du=\pi\left[-e^{-u}\right]_0^1=\pi(1-1/e)$$

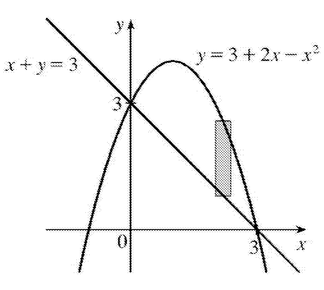


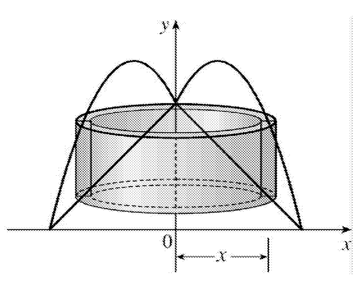


6.

$$V=2\pi\int_0^3\left\{x[(3+2x-x^2)-(3-x)]\right\}dx=2\pi\int_0^3\left[x(3x-x^2)\right]dx$$

$$=2\pi\int_0^3(3x^2-x^3)dx=2\pi\left[x^3-\frac{1}{4}x^4\right]_0^3=2\pi\left(27-\frac{81}{4}\right)=2\pi\left(\frac{27}{4}\right)=\frac{27\pi}{2}$$



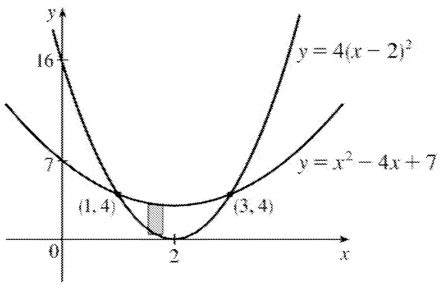


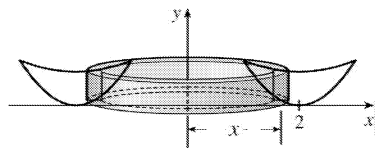
7. The curves intersect when $4(x-2)^2=x^2-4x+7\Leftrightarrow 4x^2-16x+16=x^2-4x+7\Leftrightarrow 3x^2-12x+9=0\Leftrightarrow$ $3(x^2-4x+3)=0\Leftrightarrow 3(x-1)(x-3)=0$, so $x=1$ or 3 .

$$V=2\pi\int_1^3\left\{x\left[\left(x^2-4x+7\right)-4(x-2)^2\right]\right\}dx=2\pi\int_1^3\left[x(x^2-4x+7-4x^2+16x-16)\right]dx$$

$$=2\pi\int_1^3\left[x(-3x^2+12x-9)\right]dx=2\pi(-3)\int_1^3(x^3-4x^2+3x)\,dx=-6\pi\left[\frac{1}{4}x^4-\frac{4}{3}x^3+\frac{3}{2}x^2\right]_1^3$$

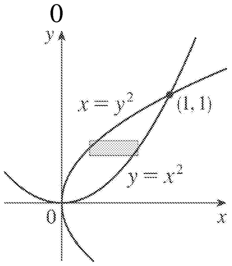
$$=-6\pi\left[\left(\frac{81}{4}-36+\frac{27}{2}\right)-\left(\frac{1}{4}-\frac{4}{3}+\frac{3}{2}\right)\right]=-6\pi\left(20-36+12+\frac{4}{3}\right)=-6\pi\left(-\frac{8}{3}\right)=16\pi$$

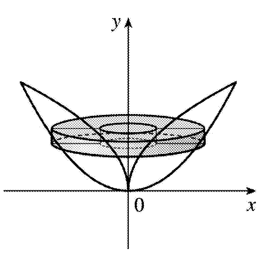




8. By slicing:

$$V=\int_0^1\pi\left[(\sqrt{y})^2-(y^2)^2\right]dy=\pi\int_0^1\left(y-y^4\right)dy=\pi\left[\frac{1}{2}y^2-\frac{1}{5}y^5\right]_0^1=\pi\left(\frac{1}{2}-\frac{1}{5}\right)=\frac{3\pi}{10}$$



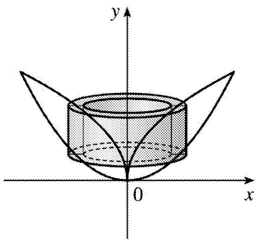


By cylindrical shells:

$$V=\int_0^1 2\pi x\left(\sqrt{x}-x^2\right)dx=2\pi\int_0^1(x^{3/2}-x^3)\,dx$$

$$=2\pi\left[\frac{2}{5}x^{5/2}-\frac{1}{4}x^4\right]_0^1=2\pi\left(\frac{2}{5}-\frac{1}{4}\right)$$

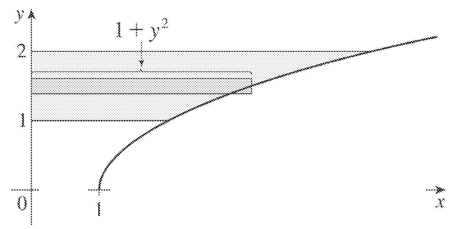
$$=2\pi\left(\frac{3}{20}\right)=\frac{3\pi}{10}$$

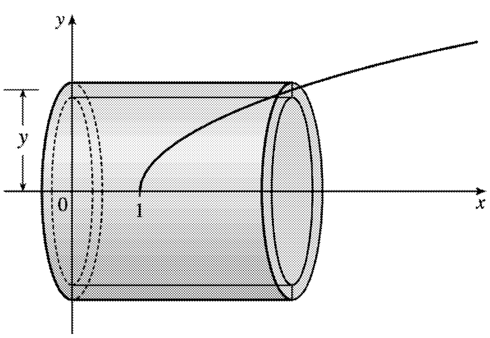


9.

$$V=\int_1^2 2\pi y\left(1+y^2\right)dy=2\pi\int_1^2\left(y+y^3\right)dy=2\pi\left[\frac{1}{2}y^2+\frac{1}{4}y^4\right]_1^2$$

$$=2\pi\left[(2+4)-\left(\frac{1}{2}+\frac{1}{4}\right)\right]=2\pi\left(\frac{21}{4}\right)=\frac{21\pi}{2}$$

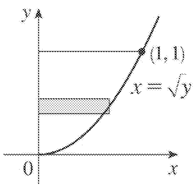


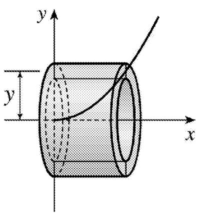


10.

$$V=\int_0^1 2\pi y\sqrt{y}\,dy=2\pi\int_0^1 y^{3/2}dy$$

$$=2\pi\left[\frac{2}{5}y^{5/2}\right]_0^1=\frac{4\pi}{5}$$



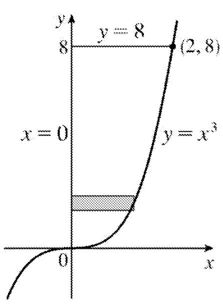


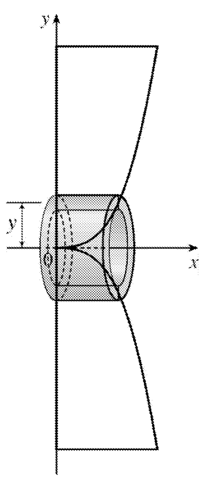
11.

$$V = 2\pi\int_0^8 \left[y(\sqrt[3]{y}-0)\right] dy$$

$$= 2\pi\int_0^8 y^{4/3}\,dy = 2\pi\left[\frac{3}{7}y^{7/3}\right]_0^8$$

$$= \frac{6\pi}{7}(8^{7/3}) = \frac{6\pi}{7}(2^7) = \frac{768\pi}{7}$$



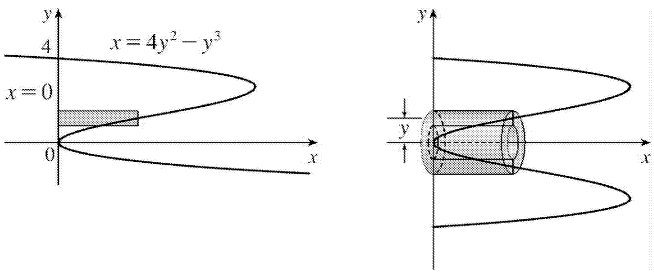


12.

$$V = 2\pi\int_0^4 \left[y(4y^2-y^3)\right] dy$$

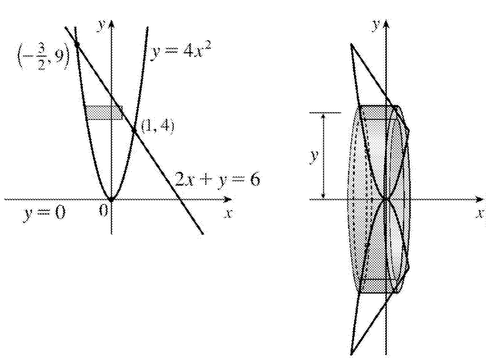
$$= 2\pi\int_0^4 (4y^3-y^4)\,dy$$

$$=2\pi\left[y^4-\frac{1}{5}y^5\right]_0^4=2\pi\left(256-\frac{1024}{5}\right)$$
$$=2\pi\left(\frac{256}{5}\right)=\frac{512\pi}{5}$$



13. The curves intersect when $4x^2=6-2x\Leftrightarrow 2x^2+x-3=0\Leftrightarrow(2x+3)(x-1)=0\Leftrightarrow x=-\frac{3}{2}$ or 1 . Solving the equations for x gives us $y=4x^2\Rightarrow x=\pm\frac{1}{2}\sqrt{y}$ and $2x+y=6\Rightarrow x=-\frac{1}{2}y+3$.

$$V=2\pi\int_0^4\left\{y\left[\left(\frac{1}{2}\sqrt{y}\right)-\left(-\frac{1}{2}\sqrt{y}\right)\right]\right\}dy+2\pi\int_4^9\left\{y\left[\left(-\frac{1}{2}y+3\right)-\left(-\frac{1}{2}\sqrt{y}\right)\right]\right\}dy$$
$$=2\pi\int_0^4\left(y\sqrt{y}\right)dy+2\pi\int_4^9\left(-\frac{1}{2}y^2+3y+\frac{1}{2}y^{3/2}\right)dy=2\pi\left[\frac{2}{5}y^{5/2}\right]_0^4+2\pi\left[-\frac{1}{6}y^3+\frac{3}{2}y^2+\frac{1}{5}y^{5/2}\right]_4^9$$
$$=2\pi\left(\frac{2}{5}\cdot 32\right)+2\pi\left[\left(-\frac{243}{2}+\frac{243}{2}+\frac{243}{5}\right)-\left(-\frac{32}{3}+24+\frac{32}{5}\right)\right]$$
$$=\frac{128}{5}\pi+2\pi\left(\frac{433}{15}\right)=\frac{1250}{15}\pi=\frac{250}{3}\pi$$



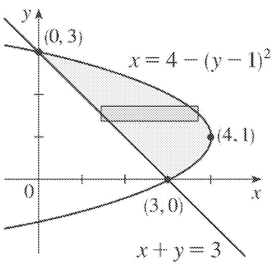
14.

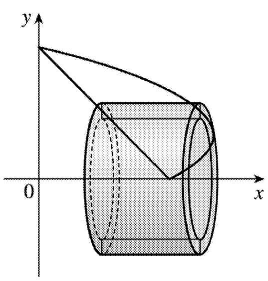
$$V=\int_0^3 2\pi y\left[4-(y-1)^2-(3-y)\right]dy$$

$$=2\pi\int_0^3 y\left(-y^2+3y\right)dy$$

$$=2\pi\int_0^3\left(-y^3+3y^2\right)dy=2\pi\left[-\frac{1}{4}y^4+y^3\right]_0^3$$

$$=2\pi\left(-\frac{81}{4}+27\right)=2\pi\left(\frac{27}{4}\right)=\frac{27\pi}{2}$$

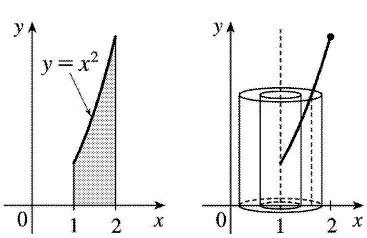




15.

$$V=\int_1^2 2\pi(x-1)x^2\,dx=2\pi\left[\frac{1}{4}x^4-\frac{1}{3}x^3\right]_1^2$$

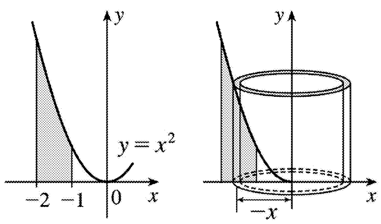
$$=2\pi\left[\left(4-\frac{8}{3}\right)-\left(\frac{1}{4}-\frac{1}{3}\right)\right]=\frac{17}{6}\pi$$



16.

$$V = \int_{-2}^{-1} 2\pi(-x)\cdot x^2\,dx = 2\pi\left[-\frac{1}{4}x^4\right]_{-2}^{-1}$$

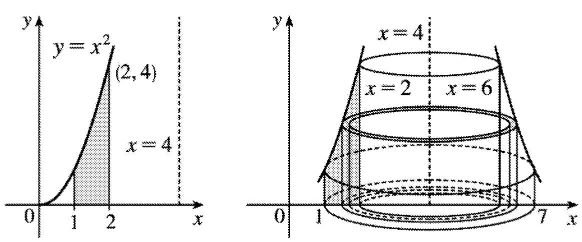
$$= 2\pi\left[\left(-\frac{1}{4}\right)-(-4)\right] = \frac{15}{2}\pi$$



17.

$$V = \int_{1}^{2} 2\pi(4-x)x^2\,dx = 2\pi\left[\frac{4}{3}x^3-\frac{1}{4}x^4\right]_{1}^{2}$$

$$= 2\pi\left[\left(\frac{32}{3}-4\right)-\left(\frac{4}{3}-\frac{1}{4}\right)\right] = \frac{67}{6}\pi$$



18.

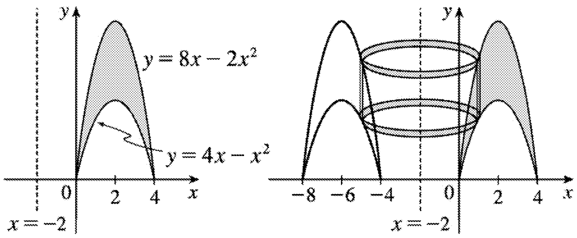
$$V = \int_{0}^{4} 2\pi[x-(-2)][(8x-2x^2)-(4x-x^2)]\,dx$$

$$= \int_{0}^{4} 2\pi(2+x)(4x-x^2)\,dx$$

$$= 2\pi\int_{0}^{4} (8x+2x^2-x^3)\,dx$$

$$= 2\pi\left[4x^2+\frac{2}{3}x^3-\frac{1}{4}x^4\right]_{0}^{4}$$

$$= 2\pi\left(64+\frac{128}{3}-64\right) = \frac{256}{3}\pi$$



19.

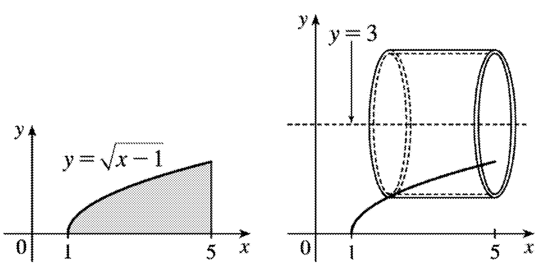
$$V=\int_0^2 2\pi(3-y)(5-x)dy$$

$$=\int_0^2 2\pi\,(3-y)\left(5-y^2-1\right)dy$$

$$=\int_0^2 2\pi\left(12-4y-3y^2+y^3\right)dy$$

$$=2\pi\left[12y-2y^2-y^3+\frac{1}{4}y^4\right]_0^2$$

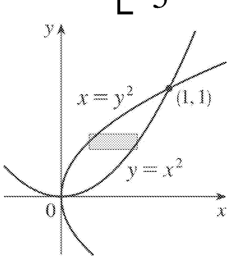
$$=2\pi(24-8-8+4)=24\pi$$

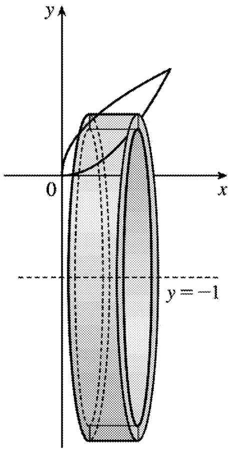


20.

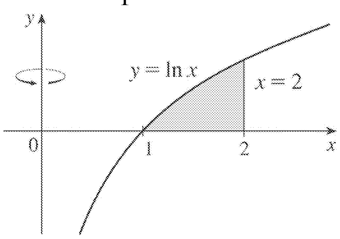
$$V=\int_0^1 2\pi\,(y+1)\left(\sqrt{y}-y^2\right)dy=2\pi\int_0^1(y^{3/2}+y^{1/2}-y^3-y^2)dy$$

$$=2\pi\left[\frac{2}{5}y^{5/2}+\frac{2}{3}y^{3/2}-\frac{1}{4}y^4-\frac{1}{3}y^3\right]_0^1=2\pi\left(\frac{2}{5}+\frac{2}{3}-\frac{1}{4}-\frac{1}{3}\right)=2\pi\left(\frac{29}{60}\right)=\frac{29\pi}{30}$$

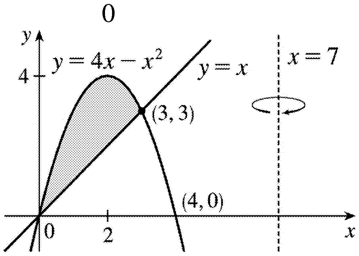




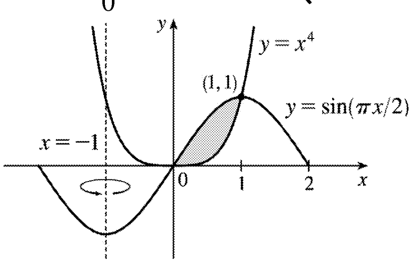
21. $V = \int_1^2 2\pi x \ln x \, dx$



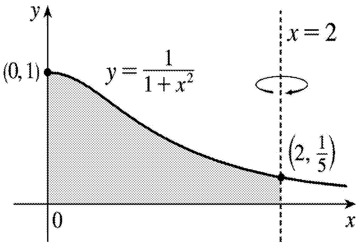
22. $V = \int_0^3 2\pi\,(7 - x)\left[\left(4x - x^2\right) - x\right] dx$



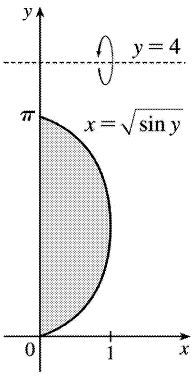
23. $V = \int_0^1 2\pi\,[x - (-1)]\left(\sin \frac{\pi}{2} x - x^4\right) dx$



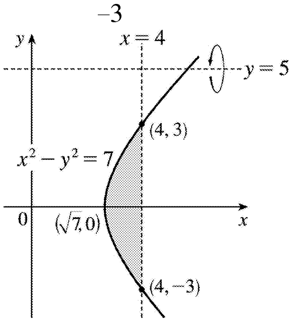
24. $V = \int_0^2 2\pi\,(2 - x)\left(\frac{1}{1 + x^2}\right) dx$



25. $V=\int_{0}^{\pi} 2\pi(4-y)\sqrt{\sin y}\,dy$



26. $V=\int_{-3}^{3} 2\pi(5-y)\left(4-\sqrt{y^2+7}\right)dy$



27. $\Delta x=\dfrac{\pi/4-0}{4}=\dfrac{\pi}{16}$.

$$V=\int_{0}^{\pi/4} 2\pi x\tan x\,dx\approx 2\pi\cdot\frac{\pi}{16}\left(\frac{\pi}{32}\tan\frac{\pi}{32}+\frac{3\pi}{32}\tan\frac{3\pi}{32}+\frac{5\pi}{32}\tan\frac{5\pi}{32}+\frac{7\pi}{32}\tan\frac{7\pi}{32}\right)\approx 1.142$$

28. $\Delta x=\dfrac{12-2}{5}=2$, $n=5$ and $x_i^*=2+(2i+1)$, where $i=0$, 1 , 2 , 3 , 4 . The values of $f(x)$ are taken directly from the diagram.

$$\begin{aligned} V &= \int_{2}^{12} 2\pi x f(x)\,dx\approx 2\pi\left[3f(3)+5f(5)+7f(7)+9f(9)+11f(11)\right]\cdot 2 \\ &\approx 2\pi\left[3(2)+5(4)+7(4)+9(2)+11(1)\right]2=332\pi \end{aligned}$$

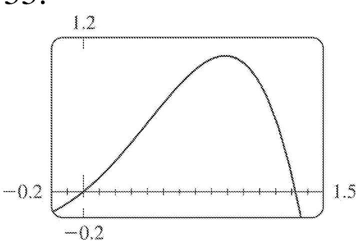
29. $\int_0^3 2\pi x^5 dx = 2\pi \int_0^3 x(x^4) dx$. The solid is obtained by rotating the region $0 \le y \le x^4$, $0 \le x \le 3$ about the y –axis using cylindrical shells.

30. $2\pi \int_0^2 \frac{y}{1+y^2} dy = 2\pi \int_0^2 y \left(\frac{1}{1+y^2} \right) dy$. The solid is obtained by rotating the region $0 \le x \le \frac{1}{1+y^2}$, $0 \le y \le 2$ about the x –axis using cylindrical shells.

31. $\int_0^1 2\pi (3-y)(1-y^2) dy$. The solid is obtained by rotating the region bounded by (i) $x=1-y^2$, $x=0$, and $y=0$ or (ii) $x=y^2$, $x=1$, and $y=0$ about the line $y=3$ using cylindrical shells.

32. $\int_0^{\pi/4} 2\pi(\pi-x)(\cos x - \sin x) dx$. The solid is obtained by rotating the region bounded by (i) $0 \le y \le \cos x - \sin x$, $0 \le x \le \frac{\pi}{4}$ or (ii) $\sin x \le y \le \cos x$, $0 \le x \le \frac{\pi}{4}$ about the line $x=\pi$ using cylindrical shells.

33.

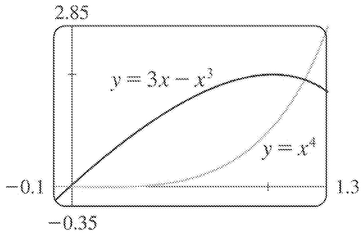


From the graph, the curves intersect at $x=0$ and at $x=a \approx 1.32$, with $x+x^2-x^4>0$ on the interval $(0,a)$. So the volume of the solid obtained by rotating the region about the y– axis is

$$V = 2\pi \int_0^a \left[x(x+x^2-x^4) \right] dx = 2\pi \int_0^a (x^2+x^3-x^5) dx$$

$$= 2\pi \left[\frac{1}{3} x^3 + \frac{1}{4} x^4 - \frac{1}{6} x^6 \right]_0^a \approx 4.05$$

34.



From the graph, the curves intersect at $x=0$ and at $x=a\approx 1.17$, with $3x-x^3>x^4$ on the interval $(0,a)$. So the volume of the solid obtained by rotating the region about the $y-$ axis is

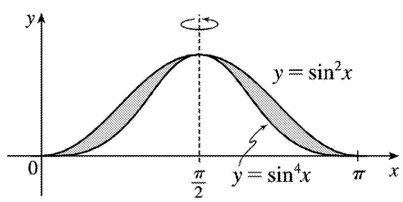
$$V = 2\pi\int_0^a \left\{ x[(3x-x^3)-x^4]\right\} dx=2\pi\int_0^a (3x^2-x^4-x^5)dx$$

$$= 2\pi \left[x^3-\frac{1}{5}x^5-\frac{1}{6}x^6\right]_0^a \approx 4.62$$

35.

$$V =2\pi\int_0^{\pi/2}\left[\left(\frac{\pi}{2}-x\right) \left(\sin^2 x-\sin^4 x\right) \right] dx$$

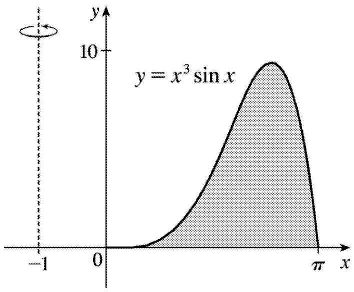
$$=\frac{1}{32}\pi^3$$



36.

$$V =2\pi\int_0^{\pi} \left\{ [x-(-1)](x^3\sin x)\right\} dx=2\pi(\pi^4+\pi^3-12\pi^2-6\pi+48)$$

$$=2\pi^5+2\pi^4-24\pi^3-12\pi^2+96\pi$$



37. Use disks:

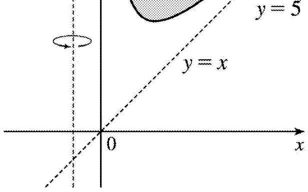
$$V = \int_{-2}^{1} \pi \left(x^2+x-2\right)^2 dx = \pi \int_{-2}^{1} \left(x^4+2x^3-3x^2-4x+4\right) dx$$
$$= \pi \left[\frac{1}{5}x^5+\frac{1}{2}x^4-x^3-2x^2+4x \right]_{-2}^{1} = \pi \left[\left(\frac{1}{5}+\frac{1}{2}-1-2+4 \right) - \left(-\frac{32}{5}+8+8-8-8 \right) \right]$$
$$= \pi \left(\frac{33}{5}+\frac{3}{2} \right) = \frac{81}{10}\pi$$

38. Use shells:

$$V = \int_{1}^{2} 2\pi x \left(-x^2+3x-2\right) dx = 2\pi \int_{1}^{2} \left(-x^3+3x^2-2x\right) dx$$
$$= 2\pi \left[-\frac{1}{4}x^4+x^3-x^2 \right]_{1}^{2} = 2\pi \left[(-4+8-4) - \left(-\frac{1}{4}+1-1 \right) \right] = \frac{\pi}{2}$$

39. Use shells:

$$V = \int_1^4 2\pi\,[x-(-1)]\,[5-(x+4/x)]\,dx = 2\pi \int_1^4 (x+1)(5-x-4/x)\,dx = 2\pi \int_1^4 \left(5x-x^2-4+5-x-4/x\right) dx =$$
$$2\pi \int_1^4 \left(-x^2+4x+1-4/x\right) dx = 2\pi \left[-\frac{1}{3}x^3+2x^2+x-4\ln x \right]_1^4 =$$
$$2\pi \left[\left(-\frac{64}{3}+32+4-4\ln 4 \right) - \left(-\frac{1}{3}+2+1-0 \right) \right] = 2\pi(12-4\ln 4) = 8\pi(3-\ln 4)$$



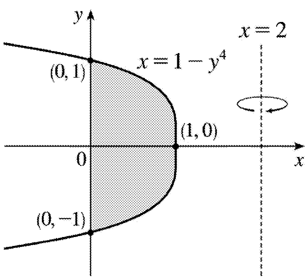
40. Use washers:

$$V=\int_{-1}^{1}\pi\left\{[2-0]^2-\left[2-\left(1-y^4\right)\right]^2\right\}dy$$

$$=2\pi\int_0^1\left[4-\left(1+y^4\right)^2\right]dy \text{ [by symmetry]}$$

$$=2\pi\int_0^1\left[4-\left(1+2y^4+y^8\right)\right]dy=2\pi\int_0^1\left(3-2y^4-y^8\right)dy$$

$$=2\pi\left[3y-\frac{2}{5}y^5-\frac{1}{9}y^9\right]_0^1=2\pi\left(3-\frac{2}{5}-\frac{1}{9}\right)=2\pi\left(\frac{112}{45}\right)=\frac{224\pi}{45}$$



41. Use disks: $V=\pi\int_0^2\left[\sqrt{1-(y-1)^2}\right]^2dy=\pi\int_0^2\left(2y-y^2\right)dy=\pi\left[y^2-\frac{1}{3}y^3\right]_0^2=\pi\left(4-\frac{8}{3}\right)=\frac{4}{3}\pi$

42. Using shells, we have

$$V=\int_0^2 2\pi y\left[\sqrt{1-(y-1)^2}-\left(-\sqrt{1-(y-1)^2}\right)\right]dy$$

$$=2\pi\int_0^2 y\cdot 2\sqrt{1-(y-1)^2}\,dy=4\pi\int_{-1}^{1}(u+1)\sqrt{1-u^2}\,du \quad [\text{let } u=y-1]$$

$$=4\pi\int_{-1}^{1}u\sqrt{1-u^2}\,du+4\pi\int_{-1}^{1}\sqrt{1-u^2}\,du$$

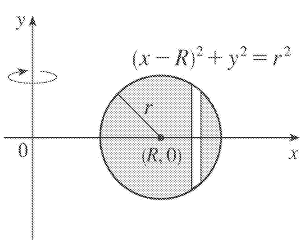
The first definite integral equals zero because its integrand is an odd function. The second is the area of a semicircle of radius 1 , that is, $\frac{\pi}{2}$. Thus, $V=4\pi\cdot 0+4\pi\cdot\frac{\pi}{2}=2\pi^2$.

43.

$$V = 2\int_0^r 2\pi x\sqrt{r^2-x^2}\,dx = -2\pi\int_0^r \left(r^2-x^2\right)^{1/2}(-2x)\,dx = \left[-2\pi\cdot\frac{2}{3}\left(r^2-x^2\right)^{3/2}\right]_0^r$$
$$= -\frac{4}{3}\pi\left(0-r^3\right) = \frac{4}{3}\pi r^3$$

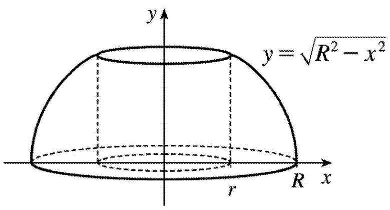
44.
$$V = \int_{R-r}^{R+r} 2\pi x\cdot 2\sqrt{r^2-(x-R)^2}\,dx$$
$$= \int_{-r}^{r} 4\pi(u+R)\sqrt{r^2-u^2}\,du \quad [u=x-R]$$
$$= 4\pi R\int_{-r}^{r}\sqrt{r^2-u^2}\,du + 4\pi\int_{-r}^{r} u\sqrt{r^2-u^2}\,du$$

The first integral is the area of a semicircle of radius r, that is, $\frac{1}{2}\pi r^2$,
and the second is zero since the integrand is an odd function. Thus, $V=4\pi R\left(\frac{1}{2}\pi r^2\right)+4\pi\cdot 0=2\pi Rr^2$
.



45. $V=2\pi\int_0^r x\left(-\frac{h}{r}x+h\right)dx=2\pi h\int_0^r\left(-\frac{x^2}{r}+x\right)dx=2\pi h\left[-\frac{x^3}{3r}+\frac{x^2}{2}\right]_0^r=2\pi h\frac{r^2}{6}=\frac{\pi r^2 h}{3}$

46. By symmetry, the volume of a napkin ring obtained by drilling a hole of radius r through a sphere with radius R is twice the volume obtained by rotating the area above the x– axis and below the curve $y=\sqrt{R^2-x^2}$ (the equation of the top half of the cross–section of the sphere), between $x=r$ and $x=R$, about the y– axis.
This volume is equal to



$$2\int_{\text{innerradius}}^{\text{outerradius}} 2\pi rh\,dx=2\cdot 2\pi\int_r^R x\sqrt{R^2-x^2}\,dx=4\pi\left[-\frac{1}{3}\left(R^2-x^2\right)^{3/2}\right]_r^R=\frac{4}{3}\pi\left(R^2-r^2\right)^{3/2}$$

But by the Pythagorean Theorem, $R^2-r^2=\left(\frac{1}{2}h\right)^2$, so the volume of the napkin ring is $\frac{4}{3}\pi\left(\frac{1}{2}h\right)^3=\frac{1}{6}\pi h^3$, which is independent of both R and r; that is, the amount of wood in a napkin ring of height h is the same regardless of the size of the sphere used. Note that most of this calculation has been done already, but with more difficulty, in Exercise 6.2.68.

Another solution: The height of the missing cap is the radius of the sphere minus half the height of the cut–out cylinder, that is, $R-\frac{1}{2}h$. Using Exercise 6.2.49,

$$V_{\text{napkinring}}=V_{\text{sphere}}-V_{\text{cylinder}}-2V_{\text{cap}}=\frac{4}{3}\pi R^3-\pi r^2h-2\cdot\frac{\pi}{3}\left(R-\frac{1}{2}h\right)^2\left[3R-\left(R-\frac{1}{2}h\right)\right]=\frac{1}{6}\pi h^3$$

1. By Equation 2, $W=Fd=(900)(8)=7200$ J.
2. $F=mg=(60)(9.8)=588$ N; $W=Fd=588\cdot 2=1176$ J

3.

$$W=\int_a^b f(x)dx=\int_0^9 \frac{10}{(1+x)^2}dx=10\int_1^{10}\frac{1}{u^2}du\ [u=1+x, du=dx]$$

$$=10\left[-\frac{1}{u}\right]_1^{10}=10\left(-\frac{1}{10}+1\right)=9\text{ft–lb}$$

4. $W=\int_1^2 \cos\left(\frac{1}{3}\pi x\right)dx=\frac{3}{\pi}\left[\sin\left(\frac{1}{3}\pi x\right)\right]_1^2=\frac{3}{\pi}\left(\frac{\sqrt{3}}{2}-\frac{\sqrt{3}}{2}\right)=0$ N–m $=0$ J.

Interpretation: From $x=1$ to $x=\frac{3}{2}$, the force does work equal to $\int_1^{3/2}\cos\left(\frac{1}{3}\pi x\right)dx=\frac{3}{\pi}\left(1-\frac{\sqrt{3}}{2}\right)$ J in accelerating the particle and increasing its kinetic energy. From $x=\frac{3}{2}$ to $x=2$, the force opposes the motion of the particle, decreasing its kinetic energy. This is negative work, equal in magnitude but opposite in sign to the work done from $x=1$ to $x=\frac{3}{2}$.

5. The force function is given by $F(x)$ (in newtons) and the work (in joules) is the area under the curve, given by $\int_0^8 F(x)dx=\int_0^4 F(x)dx+\int_4^8 F(x)dx=\frac{1}{2}(4)(30)+(4)(30)=180$ J.

6. $W=\int_4^{20} f(x)dx\approx M_4=\Delta x[f(6)+f(10)+f(14)+f(18)]=\frac{20-4}{4}[5.8+8.8+8.2+5.2]=4(28)=112$ J

7. $10=f(x)=kx=\frac{1}{3}k$ [4 inches $=\frac{1}{3}$ foot], so $k=30$ lb / ft and $f(x)=30x$. Now 6 inches $=\frac{1}{2}$ foot, so $W=\int_0^{1/2}30x\,dx=\left[15x^2\right]_0^{1/2}=\frac{15}{4}$ ft–lb.

8. $25=f(x)=kx=k(0.1)$ [10 cm $=0.1$ m], so $k=250$ N / m and $f(x)=250x$. Now 5 cm $=0.05$ m, so $W=\int_0^{0.05}250x\,dx=\left[125x^2\right]_0^{0.05}=125(0.0025)=0.3125\approx 0.31$ J.

9. If $\int_0^{0.12} kx\,dx = 2$ J, then $2 = \left[\frac{1}{2}kx^2\right]_0^{0.12} = \frac{1}{2}k(0.0144) = 0.0072k$ and $k = \frac{2}{0.0072} = \frac{2500}{9} \approx 277.78$ N / m. Thus, the work needed to stretch the spring from 35 cm to 40 cm is

$$\int_{0.05}^{0.10} \frac{2500}{9}x\,dx = \left[\frac{1250}{9}x^2\right]_{1/20}^{1/10} = \frac{1250}{9}\left(\frac{1}{100} - \frac{1}{400}\right) = \frac{25}{24} \approx 1.04 \text{ J.}$$

10. If $12 = \int_0^1 kx\,dx = \left[\frac{1}{2}kx^2\right]_0^1 = \frac{1}{2}k$, then $k = 24$ lb / ft and the work required is

$$\int_0^{3/4} 24x\,dx = \left[12x^2\right]_0^{3/4} = 12 \cdot \frac{9}{16} = \frac{27}{4} = 6.75 \text{ ft–lb.}$$

11. $f(x) = kx$, so $30 = \frac{2500}{9}x$ and $x = \frac{270}{2500}$ m $= 10.8$ cm

12. Let L be the natural length of the spring in meters. Then

$$6 = \int_{0.10-L}^{0.12-L} kx\,dx = \left[\frac{1}{2}kx^2\right]_{0.10-L}^{0.12-L} = \frac{1}{2}k\left[(0.12-L)^2 - (0.10-L)^2\right]$$ and

$$10 = \int_{0.12-L}^{0.14-L} kx\,dx = \left[\frac{1}{2}kx^2\right]_{0.12-L}^{0.14-L} = \frac{1}{2}k\left[(0.14-L)^2 - (0.12-L)^2\right]$$. Simplifying gives us $12 = k(0.0044 - 0.04L)$ and $20 = k(0.0052 - 0.04L)$. Subtracting the first equation from the second gives $8 = 0.0008k$, so $k = 10,000$. Now the second equation becomes $20 = 52 - 400L$, so $L = \frac{32}{400}$ m $= 8$ cm.

13. **(a)** The portion of the rope from x ft to $(x + \Delta x)$ ft below the top of the building weighs $\frac{1}{2}\Delta x$ lb and must be lifted x_i^* ft, so its contribution to the total work is $\frac{1}{2}x_i^*\Delta x$ ft–lb. The total work is

$$W = \lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{2}x_i^*\Delta x = \int_0^{50}\frac{1}{2}x\,dx = \left[\frac{1}{4}x^2\right]_0^{50} = \frac{2500}{4} = 625 \text{ ft–lb}$$ Notice that the exact height of the building does not matter (as long as it is more than 50 ft).

(b) When half the rope is pulled to the top of the building, the work to lift the top half of the rope is $W_1 = \int_0^{25}\frac{1}{2}x\,dx = \left[\frac{1}{4}x^2\right]_0^{25} = \frac{625}{4}$ ft–lb. The bottom half of the rope is lifted 25 ft and the work needed to accomplish that is $W_2 = \int_{25}^{50}\frac{1}{2}\cdot 25\,dx = \frac{25}{2}\left[x\right]_{25}^{50} = \frac{625}{2}$ ft–lb. The total work done in pulling half the rope to the top of the building is $W=W_1+W_2=\frac{625}{2}+\frac{625}{4}=\frac{3}{4}\cdot 625=\frac{1875}{4}$ ft–lb.

14. *Assumptions* : 1. After lifting, the chain is L–shaped, with 4 m of the chain lying along the ground.
2. The chain slides effortlessly and without friction along the ground while its end is lifted.
3. The weight density of the chain is constant throughout its length and therefore equals $(8\text{kg/m})(9.8\text{m/s}^2)=78.4\text{N/m}$.
The part of the chain xm from the lifted end is raised $6-x$m if $0\le x\le 6$ m, and it is lifted 0 m if $x>6$m .
Thus, the work needed is

$$W=\lim_{n\to\infty}\sum_{i=1}^{n}(6-x_i^*)\cdot 78.4\,\Delta x=\int_0^6(6-x)78.4\,dx=78.4\left[6x-\frac{1}{2}x^2\right]_0^6=(78.4)(18)=1411.2\text{ J.}$$

15. The work needed to lift the cable is $\lim_{n\to\infty}\sum_{i=1}^{n}2x_i^*\,\Delta x=\int_0^{500}2x\,dx=\left[x^2\right]_0^{500}=250,000$ ft–lb. The work needed to lift the coal is 800 lb · 500 ft =400 , 000 ft–lb. Thus, the total work required is 250 , 000+400 , 000=650 , 000 ft–lb.

16. The work needed to lift the bucket itself is 4 lb · 80 ft =320 ft–lb. At time t (in seconds) the bucket is $x_i^*=2t$ ft above its original 80 ft depth, but it now holds only $(40-0.2t)$ lb of water. In terms of distance, the bucket holds $\left[40-0.2\left(\frac{1}{2}x_i^*\right)\right]$ lb of water when it is x_i^* ft above its original 80 ft depth. Moving this amount of water a distance Δx requires $\left(40-\frac{1}{10}x_i^*\right)\Delta x$ ft–lb of work. Thus, the work needed to lift the water is

$$W=\lim_{n\to\infty}\sum_{i=1}^{n}\left(40-\frac{1}{10}x_i^*\right)\Delta x=\int_0^{80}\left(40-\frac{1}{10}x\right)dx=\left[40x-\frac{1}{20}x^2\right]_0^{80}=(3200-320)\text{ ft–lb}$$

Adding the work of lifting the bucket gives a total of 3200 ft–lb of work.

17. At a height of x meters ($0\le x\le 12$), the mass of the rope is (0.8 kg / m)(12–x m)=(9.6–0.8x) kg and the mass of the water is $\left(\frac{36}{12}\text{ kg/m}\right)(12-x\text{m})=(36-3x)$ kg. The mass of the bucket is 10 kg, so the total mass is $(9.6-0.8x)+(36-3x)+10=(55.6-3.8x)$ kg, and hence, the total force is $9.8(55.6-3.8x)$ N.

The work needed to lift the bucket Δx m through the i th subinterval of $[0,12]$ is $9.8(55.6-3.8x_i^*)\Delta x$, so the total work is

$$W=\lim_{n\to\infty}\sum_{i=1}^{n}9.8(55.6-3.8x_i^*)\Delta x=\int_0^{12}(9.8)(55.6-3.8x)dx=9.8\left[55.6x-1.9x^2\right]_0^{12}$$
$$=9.8(393.6)\approx 3857\text{ J}$$

18. The chain's weight density is $\frac{25\text{lb}}{10\text{ft}}=2.5$ lb / ft. The part of the chain x ft below the ceiling (for $5\le x\le 10$) has to be lifted $2(x-5)$ ft, so the work needed to lift the i th subinterval of the chain is 2 ($x_i^*-5)(2.5\Delta x)$. The total work needed is

$$W=\lim_{n\to\infty}\sum_{i=1}^{n}2(x_i^*-5)(2.5)\Delta x=\int_5^{10}[2(x-5)(2.5)]dx=5\int_5^{10}(x-5)dx$$
$$=5\left[\frac{1}{2}x^2-5x\right]_5^{10}=5\left[(50-50)-\left(\frac{25}{2}-25\right)\right]=5\left(\frac{25}{2}\right)=62.5\text{ ft–lb}$$

19. A ''slice'' of water Δx m thick and lying at a depth of x_i^* m (where $0\le x_i^*\le\frac{1}{2}$) has volume $(2\times 1\times\Delta x)$ m ^{3} , a mass of $2000\Delta x$ kg, weighs about $(9.8)(2000\Delta x)=19,600\Delta x$ N, and thus requires about $19,600x_i^*\Delta x$ J of work for its removal. So $W=\lim_{n\to\infty}\sum_{i=1}^{n}19,600x_i^*\Delta x=\int_0^{1/2}19,600x\,dx=\left[9800x^2\right]_0^{1/2}=2450$ J.

20. A horizontal cylindrical slice of water Δx ft thick has a volume of $\pi r^2h=\pi\cdot 12^2\cdot\Delta x$ ft ^{3} and weighs about $\left(62.5\text{lb/ft}^3\right)\left(144\pi\Delta x\text{ft}^3\right)=9000\pi\Delta x$ lb. If the slice lies x_i^* ft below the edge of the pool (where $1\le x_i^*\le 5$), then the work needed to pump it out is about $9000\pi x_i^*\Delta x$. Thus,

$$W=\lim_{n\to\infty}\sum_{i=1}^{n}9000\pi x_i^*\Delta x=\int_1^{5}9000\pi x\,dx=\left[4500\pi x^2\right]_1^5=4500\pi(25-1)=108,000\pi\text{ ft–lb}$$

21. A rectangular ''slice'' of water Δx m thick and lying x ft above the bottom has width x ft and volume $8x\Delta x$ m ^{3} . It weighs about $(9.8\times 1000)(8x\Delta x)$ N, and must be lifted $(5-x)$ m by the pump, so the work needed is about $\left(9.8\times 10^3\right)(5-x)(8x\Delta x)$ J. The total work required is

$$W \approx \int_0^3 \left(9.8\times 10^3\right)(5-x)8x\,dx=\left(9.8\times 10^3\right)\int_0^3 \left(40x-8x^2\right)dx=\left(9.8\times 10^3\right)\left[20x^2-\frac{8}{3}x^3\right]_0^3$$

$$= \left(9.8\times 10^3\right)(180-72)=\left(9.8\times 10^3\right)(108)=1058.4\times 10^3\approx 1.06\times 10^6\,\mathrm{J}$$

22. For convenience, measure depth x from the middle of the tank, so that $-1.5\leq x\leq 1.5$ m. Lifting a slice of water of thickness Δx at depth x requires a work contribution of

$\Delta W\approx\left(9.8\times 10^3\right)\left(2\sqrt{(1.5)^2-x^2}\right)(6\,\Delta x)(2.5+x)$, so

$$W \approx \int_{-1.5}^{1.5}\left(9.8\times 10^3\right)12\sqrt{2.25-x^2}\,(2.5+x)\,dx$$

$$= \left(9.8\times 10^3\right)\left[60\int_0^{3/2}\sqrt{\frac{9}{4}-x^2}\,dx+12\int_{-3/2}^{3/2}x\sqrt{\frac{9}{4}-x^2}\,dx\right]$$

The second integral is 0 because its integrand is an odd function, and the first integral represents the area of a quarter–circle of radius

$\frac{3}{2}$ .Therefore,[$W\approx\left(9.8\times 10^3\right)60\int_0^{3/2}\sqrt{\frac{9}{4}-x^2}\,dx=\left(9.8\times 10^3\right)(60)\left(\frac{1}{4}\pi\right)\left(\frac{3}{2}\right)^2=330{,}750\pi\approx 1.04\times 10^6$J]

23. Measure depth x downward from the flat top of the tank, so that $0\leq x\leq 2$ ft. Then

$\Delta W=(62.5)\left(2\sqrt{4-x^2}\right)(8\,\Delta x)(x+1)$ ft–lb, so

$$W \approx (62.5)(16)\int_0^2(x+1)\sqrt{4-x^2}\,dx=1000\left(\int_0^2 x\sqrt{4-x^2}\,dx+\int_0^2\sqrt{4-x^2}\,dx\right)$$

$$= 1000\left[\int_0^4 u^{1/2}\left(\frac{1}{2}\right)du+\frac{1}{4}\pi\left(2^2\right)\right]\quad [\text{Put } u=4-x^2, \text{so } du=-2x\,dx]$$

$$= 1000\left(\left[\frac{1}{2}\cdot\frac{2}{3}u^{3/2}\right]_0^4+\pi\right)=1000\left(\frac{8}{3}+\pi\right)\approx 5.8\times 10^3\,\text{ft–lb}$$

Note: The second integral represents the area of a quarter–circle of radius 2 .

24. Let x be depth in feet, so that $0\leq x\leq 5$. Then $\Delta W=(62.5)\pi\left(\sqrt{5^2-x^2}\right)^2\Delta x\cdot x$ ft–lb and

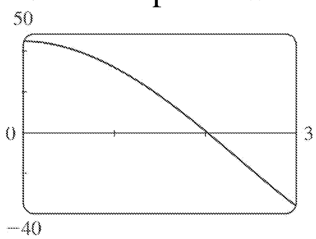
$$W \approx 62.5\pi\int_0^5 x\left(25-x^2\right)dx=62.5\pi\left[\frac{25}{2}x^2-\frac{1}{4}x^4\right]_0^5=62.5\pi\left(\frac{625}{2}-\frac{625}{4}\right)=62.5\pi\left(\frac{625}{4}\right)$$

$$\approx 3.07\times 10^4\,\text{ft–lb}$$

25. If only 4.7×10^5 J of work is done, then only the water above a certain level (call it h) will be pumped out. So we use the same formula as in Exercise 21 , except that the work is fixed, and we are trying to find the lower limit of integration:

$$4.7\times 10^5 \approx \int_h^3 \left(9.8\times 10^3\right)(5-x)8x\,dx = \left(9.8\times 10^3\right)\left[20x^2-\frac{8}{3}x^3\right]_h^3 \Leftrightarrow$$

$\frac{4.7}{9.8}\times 10^2 \approx 48 = \left(20\cdot 3^2-\frac{8}{3}\cdot 3^3\right)-\left(20h^2-\frac{8}{3}h^3\right) \Leftrightarrow 2h^3-15h^2+45=0$. To find the solution of this equation, we plot $2h^3-15h^2+45$ between $h=0$ and $h=3$. We see that the equation is satisfied for $h\approx 2.0$. So the depth of water remaining in the tank is about 2.0 m.



26. $W\approx(9.8\times 920)\int_0^{3/2} 12\sqrt{\frac{9}{4}-x^2}\left(\frac{5}{2}+x\right)dx = 9016\left[30\int_0^{3/2}\sqrt{\frac{9}{4}-x^2}\,dx+12\int_0^{3/2}x\sqrt{\frac{9}{4}-x^2}\,dx\right]$.

Here $\int_0^{3/2}\sqrt{\frac{9}{4}-x^2}\,dx=\frac{1}{4}\pi\left(\frac{3}{2}\right)^2=\frac{9\pi}{16}$ and $\int_0^{3/2}x\sqrt{\frac{9}{4}-x^2}\,dx=\int_0^{9/4}\frac{1}{2}u^{1/2}du$ [where $u=\frac{9}{4}-x^2$, so $du=-2x\,dx$] $=\left[\frac{1}{3}u^{3/2}\right]_0^{9/4}=\frac{1}{3}\left(\frac{27}{8}\right)=\frac{9}{8}$, so

$W\approx 9016\left[30\cdot\frac{9}{16}\pi+12\cdot\frac{9}{8}\right]=9016\left(\frac{135}{8}\pi+\frac{27}{2}\right)\approx 6.00\times 10^5$ J.

27. $V=\pi r^2 x$, so V is a function of x and P can also be regarded as a function of x . If $V_1=\pi r^2 x_1$ and $V_2=\pi r^2 x_2$, then

$$W = \int_{x_1}^{x_2} F(x)\,dx=\int_{x_1}^{x_2}\pi r^2 P(V(x))\,dx$$

$$= \int_{x_1}^{x_2} P(V(x))\,dV(x) \text{ [Let } V(x)=\pi r^2 x\text{, so } dV(x)=\pi r^2 dx.]$$

$$= \int_{V_1}^{V_2} P(V)\,dV \text{ by the Substitution Rule.}$$

28. $160 \text{ lb / in}^2 = 160 \cdot 144 \text{ lb / ft}^2$, $100 \text{ in}^3 = \frac{100}{1728} \text{ ft}^3$, and $800 \text{ in}^3 = \frac{800}{1728} \text{ ft}^3$.

$k = PV^{1.4} = (160 \cdot 144)\left(\frac{100}{1728}\right)^{1.4} = 23,040\left(\frac{25}{432}\right)^{1.4} \approx 426.5$. Therefore, $P \approx 426.5V^{-1.4}$ and

$$\begin{aligned} W &= \int_{100/1728}^{800/1728} 426.5V^{-1.4}\,dV = 426.5\left[\frac{1}{-0.4}V^{-0.4}\right]_{25/432}^{25/54} \\ &= (426.5)(2.5)\left[\left(\frac{432}{25}\right)^{0.4} - \left(\frac{54}{25}\right)^{0.4}\right] \\ &\approx 1.88 \times 10^3 \text{ ft–lb} \end{aligned}$$

29. $W = \int_a^b F(r)\,dr = \int_a^b G\frac{m_1 m_2}{r^2}\,dr = Gm_1 m_2\left[\frac{-1}{r}\right]_a^b = Gm_1 m_2\left(\frac{1}{a} - \frac{1}{b}\right)$

30. By Exercise 29, $W = GMm\left(\frac{1}{R} - \frac{1}{R + 1{,}000{,}000}\right)$ where M= mass of Earth in kg, R= radius of Earth in m, and m= mass of satellite in kg. (Note that 1000 km =1,000,000 m.) Thus,

$$W = \left(6.67 \times 10^{-11}\right)\left(5.98 \times 10^{24}\right)(1000) \times \left(\frac{1}{6.37 \times 10^6} - \frac{1}{7.37 \times 10^6}\right) \approx 8.50 \times 10^9 \text{ J}$$

1. $f_{\text{ave}}=\dfrac{1}{b-a}\int_a^b f(x)dx=\dfrac{1}{1-(-1)}\int_{-1}^{1}x^2dx=\dfrac{1}{2}\cdot 2\int_0^1 x^2dx=\left[\dfrac{1}{3}x^3\right]_0^1=\dfrac{1}{3}$

2. $f_{\text{ave}}=\dfrac{1}{4-1}\int_1^4(1/x)dx=\dfrac{1}{3}\left[\ln x\right]_1^4=\dfrac{1}{3}\ln 4\approx 0.46$

3. $g_{\text{ave}}=\dfrac{1}{\frac{\pi}{2}-0}\int_0^{\pi/2}\cos x\,dx=\dfrac{2}{\pi}\left[\sin x\right]_0^{\pi/2}=\dfrac{2}{\pi}(1-0)=\dfrac{2}{\pi}$

4.

$$g_{\text{ave}}=\frac{1}{2-0}\int_0^2 x^2\sqrt{1+x^3}\,dx=\frac{1}{2}\int_1^9\sqrt{u}\cdot\frac{1}{3}du\ [\,u=1+x^3\ ,du=3x^2dx\,]$$

$$=\frac{1}{6}\left[\frac{2}{3}u^{3/2}\right]_1^9=\frac{1}{9}(27-1)=\frac{26}{9}$$

5. $f_{\text{ave}}=\dfrac{1}{5-0}\int_0^5 te^{-t^2}dt=\dfrac{1}{5}\int_0^{-25}e^u\left(-\dfrac{1}{2}du\right)\ [\,u=-t^2\ ,du=-2t\,dt\ ,t\,dt=-\dfrac{1}{2}du\,]$

$=-\dfrac{1}{10}\left[e^u\right]_0^{-25}=-\dfrac{1}{10}\left(e^{-25}-1\right)=\dfrac{1}{10}\left(1-e^{-25}\right)$

6. $f_{\text{ave}}=\dfrac{1}{\frac{\pi}{4}-0}\int_0^{\pi/4}\sec\theta\tan\theta\,d\theta=\dfrac{4}{\pi}\left[\sec\theta\right]_0^{\pi/4}=\dfrac{4}{\pi}\left(\sqrt{2}-1\right)$

7.

$$h_{\text{ave}}=\frac{1}{\pi-0}\int_0^{\pi}\cos^4x\sin x\,dx=\frac{1}{\pi}\int_1^{-1}u^4(-du)\ [\,u=\cos x\ ,du=-\sin x\,dx\,]$$

$$=\frac{1}{\pi}\int_{-1}^{1}u^4du=\frac{1}{\pi}\cdot 2\int_0^1u^4du=\frac{2}{\pi}\left[\frac{1}{5}u^5\right]_0^1=\frac{2}{5\pi}$$

8.

$$h_{\text{ave}}=\frac{1}{6-1}\int_1^6\frac{3}{(1+r)^2}dr=\frac{1}{5}\int_2^7 3u^{-2}du\ [\,u=1+r\ ,du=dr\,]$$

$$=-\frac{3}{5}\left[u^{-1}\right]_2^7=-\frac{3}{5}\left(\frac{1}{7}-\frac{1}{2}\right)=\frac{3}{5}\left(\frac{1}{2}-\frac{1}{7}\right)=\frac{3}{5}\cdot\frac{5}{14}=\frac{3}{14}$$

9.

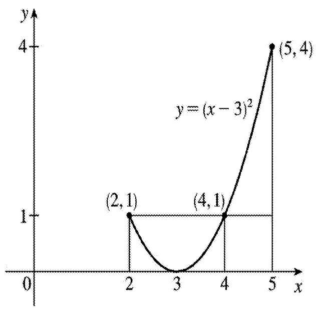
(a) $$f_{\text{ave}}=\frac{1}{5-2}\int_2^5(x-3)^2\,dx=\frac{1}{3}\left[\frac{1}{3}(x-3)^3\right]_2^5$$

$$=\frac{1}{9}\left[2^3-(-1)^3\right]=\frac{1}{9}(8+1)=1$$

(b) $f(c)=f_{\text{ave}}\Leftrightarrow(c-3)^2=1\Leftrightarrow c-3=\pm 1$

$\Leftrightarrow c=2$ or 4

(c)



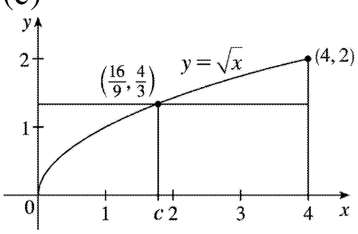
10.

(a) $$f_{\text{ave}}=\frac{1}{4-0}\int_0^4\sqrt{x}\,dx=\frac{1}{4}\left[\frac{2}{3}x^{3/2}\right]_0^4$$

$$=\frac{1}{6}\left[x^{3/2}\right]_0^4=\frac{1}{6}[8-0]=\frac{4}{3}$$

(b) $f(c)=f_{\text{ave}}\Leftrightarrow\sqrt{c}=\frac{4}{3}\Leftrightarrow c=\frac{16}{9}$

(c)



11.

(a)

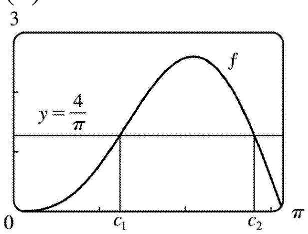
$$f_{\text{ave}} = \frac{1}{\pi - 0}\int_0^{\pi}(2\sin x - \sin 2x)\,dx$$

$$= \frac{1}{\pi}\left[-2\cos x + \frac{1}{2}\cos 2x\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\left(2+\frac{1}{2}\right) - \left(-2+\frac{1}{2}\right)\right] = \frac{4}{\pi}$$

(b) $f(c)=f_{\text{ave}} \Leftrightarrow 2\sin c - \sin 2c = \frac{4}{\pi} \Leftrightarrow$
$c_1 \approx 1.238$ or $c_2 \approx 2.808$

(c)



12.

(a)

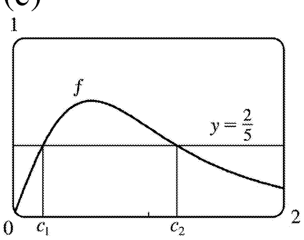
$$f_{\text{ave}} = \frac{1}{2-0}\int_0^2 \frac{2x}{(1+x^2)^2}\,dx$$

$$= \frac{1}{2}\int_1^5 \frac{1}{u^2}\,du \quad [\,u=1+x^2,\ du=2x\,dx\,]$$

$$= \frac{1}{2}\left[-\frac{1}{u}\right]_1^5 = -\frac{1}{2}\left(\frac{1}{5}-1\right) = \frac{2}{5}$$

(b) $f(c)=f_{\text{ave}} \Leftrightarrow \frac{2c}{(1+c^2)^2} = \frac{2}{5} \Leftrightarrow 5c=(1+c^2)^2$
$\Leftrightarrow c_1 \approx 0.220$ or $c_2 \approx 1.207$

(c)



13. f is continuous on $[1,3]$, so by the Mean Value Theorem for Integrals there exists a number c in $[1,3]$ such that $\int_1^3 f(x)\,dx = f(c)(3-1) \Rightarrow 8 = 2f(c)$; that is, there is a number c such that $f(c) = \frac{8}{2} = 4$.

14. The requirement is that

$\frac{1}{b-0}\int_0^b f(x)dx=3$. The LHS of this equation is equal to

$\frac{1}{b}\int_0^b \left(2+6x-3x^2\right)dx=\frac{1}{b}\left[2x+3x^2-x^3\right]_0^b=2+3b-b^2$, so we solve the equation $2+3b-b^2=3 \Leftrightarrow b^2-3b+1=0 \Leftrightarrow$

$b=\frac{3\pm\sqrt{(-3)^2-4\cdot 1\cdot 1}}{2\cdot 1}=\frac{3\pm\sqrt{5}}{2}$. Both roots are valid since they are positive.

15.

$$f_{\text{ave}}=\frac{1}{50-20}\int_{20}^{50} f(x)dx\approx\frac{1}{30}M_3=\frac{1}{30}\cdot\frac{50-20}{3}[f(25)+f(35)+f(45)]$$
$$=\frac{1}{3}(38+29+48)=\frac{115}{3}=38\frac{1}{3}$$

16. **(a)** $v_{\text{ave}}=\frac{1}{12-0}\int_0^{12} v(t)dt=\frac{1}{12}I$. Use the Midpoint Rule with $n=3$ and $\Delta t=\frac{12-0}{3}=4$ to estimate I .

$I\approx M_3=4[v(2)+v(6)+v(10)]=4[21+50+66]=4(137)=548$. Thus, $v_{\text{ave}}\approx\frac{1}{12}(548)=45\frac{2}{3}$ km / h.

(b) Estimating from the graph, $v(t)=45\frac{2}{3}$ when $t\approx 5.2$ s.

17. Let $t=0$ and $t=12$ correspond to 9 A.M. and 9 P.M., respectively.

$$T_{\text{ave}}=\frac{1}{12-0}\int_0^{12}\left[50+14\sin\frac{1}{12}\pi t\right]dt=\frac{1}{12}\left[50t-14\cdot\frac{12}{\pi}\cos\frac{1}{12}\pi t\right]_0^{12}$$
$$=\frac{1}{12}\left[50\cdot 12+14\cdot\frac{12}{\pi}+14\cdot\frac{12}{\pi}\right]=\left(50+\frac{28}{\pi}\right)\text{ F}\approx 59^\circ\text{ F}$$

18.

$$T_{\text{ave}}=\frac{1}{30-0}\int_0^{30}\left(20+75e^{-t/50}\right)dt=\frac{1}{30}\left[20t-50\cdot 75e^{-t/50}\right]_0^{30}=\frac{1}{30}\left[\left(600-3750e^{-3/5}\right)-(-3750)\right]$$
$$=\frac{1}{30}\left(4350-3750e^{-3/5}\right)=145-125e^{-3/5}\approx 76.4^\circ\text{ C}$$

19. $\rho_{\text{ave}}=\frac{1}{8}\int_0^8\frac{12}{\sqrt{x+1}}dx=\frac{3}{2}\int_0^8(x+1)^{-1/2}dx=\left[3\sqrt{x+1}\right]_0^8=9-3=6$ kg / m

20. $s=\frac{1}{2}gt^2 \Rightarrow t=\sqrt{2s/g}$ [since $t\geq 0$]. Now $v=\frac{ds}{dt}=gt=g\sqrt{2s/g}=\sqrt{2gs} \Rightarrow v^2=2gs \Rightarrow s=\frac{v^2}{2g}$. We see that v can be regarded as a function of t or of s : $v=F(t)=gt$ and $v=G(s)=\sqrt{2gs}$. Note that $v_T=F(T)=gT$.

Displacement can be viewed as a function of t : $s=s(t)=\frac{1}{2}gt^2$; also $s(t)=\frac{v^2}{2g}=\frac{[F(t)]^2}{2g}$.
When $t=T$, these two formulas for $s(t)$ imply that
$\sqrt{2gs(T)}=F(T)=v_T=gT=2\left(\frac{1}{2}gT^2\right)/T=2s(T)/T\,(*)$
The average of the velocities with respect to time t during the interval $[0,T]$ is

$$v_{t\text{–ave}}=F_{\text{ave}}=\frac{1}{T-0}\int_0^T F(t)\,dt=\frac{1}{T}[s(T)-s(0)] \text{ [by FTC]}$$
$$=\frac{s(T)}{T} \text{ [since } s(0)=0 \text{]} = \frac{1}{2}v_T \text{ [by (*)]}$$

But the average of the velocities with respect to displacement s during the corresponding displacement interval $[s(0),s(T)]=[0,s(T)]$ is

$$v_{s\text{–ave}}=G_{\text{ave}}=\frac{1}{s(T)-0}\int_0^{s(T)} G(s)\,ds=\frac{1}{s(T)}\int_0^{s(T)}\sqrt{2gs}\,ds=\frac{\sqrt{2g}}{s(T)}\int_0^{s(T)} s^{1/2}\,ds$$
$$=\frac{\sqrt{2g}}{s(T)}\cdot\frac{2}{3}\left[s^{3/2}\right]_0^{s(T)}=\frac{2}{3}\cdot\frac{\sqrt{2g}}{s(T)}\cdot[s(T)]^{3/2}=\frac{2}{3}\sqrt{2gs(T)}=\frac{2}{3}v_T[\text{by}(*)]$$

21.

$$V_{\text{ave}}=\frac{1}{5}\int_0^5 V(t)\,dt=\frac{1}{5}\int_0^5\frac{5}{4\pi}\left[1-\cos\left(\frac{2}{5}\pi t\right)\right]dt=\frac{1}{4\pi}\int_0^5\left[1-\cos\left(\frac{2}{5}\pi t\right)\right]dt$$
$$=\frac{1}{4\pi}\left[t-\frac{5}{2\pi}\sin\left(\frac{2}{5}\pi t\right)\right]_0^5=\frac{1}{4\pi}[(5-0)-0]=\frac{5}{4\pi}\approx 0.4 \text{ L}$$

22. $v_{\text{ave}}=\frac{1}{R-0}\int_0^R v(r)\,dr=\frac{1}{R}\int_0^R\frac{P}{4\eta l}\left(R^2-r^2\right)dr=\frac{P}{4\eta lR}\left[R^2r-\frac{1}{3}r^3\right]_0^R=\frac{P}{4\eta lR}\left(\frac{2}{3}\right)R^3=\frac{PR^2}{6\eta l}$.

Since $v(r)$ is decreasing on $(0,R]$, $v_{\max}=v(0)=\frac{PR^2}{4\eta l}$. Thus, $v_{\text{ave}}=\frac{2}{3}v_{\max}$.

23. Let

$F(x)=\int_a^x f(t)\,dt$ for x in $[a,b]$. Then F is continuous on $[a,b]$ and differentiable on (a,b) , so by the Mean Value Theorem there is a number c in (a,b) such that $F(b)-F(a)=F'(c)(b-a)$. But $F'(x)=f(x)$ by the Fundamental Theorem of Calculus. Therefore, $\int_a^b f(t)\,dt-0=f(c)(b-a)$.

24.

$$f_{\text{ave}}[a,b] = \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{b-a}\int_a^c f(x)\,dx + \frac{1}{b-a}\int_c^b f(x)\,dx$$

$$= \frac{c-a}{b-a}\left[\frac{1}{c-a}\int_a^c f(x)\,dx\right] + \frac{b-c}{b-a}\left[\frac{1}{b-c}\int_c^b f(x)\,dx\right] = \frac{c-a}{b-a}\, f_{\text{ave}}[a,c] + \frac{b-c}{b-a}\, f_{\text{ave}}[c,b]$$

1. Let $u=\ln x$, $dv=x\,dx\Rightarrow du=dx/x$, $v=\frac{1}{2}x^2$. Then by Equation 2, $\int u\,dv=uv-\int v\,du$,

$$\int x\ln x\,dx = \frac{1}{2}x^2\ln x-\int\frac{1}{2}x^2(dx/x)=\frac{1}{2}x^2\ln x-\frac{1}{2}\int x\,dx=\frac{1}{2}x^2\ln x-\frac{1}{2}\cdot\frac{1}{2}x^2+C$$
$$= \frac{1}{2}x^2\ln x-\frac{1}{4}x^2+C$$

2. Let $u=\theta$, $dv=\sec^2\theta\,d\theta\Rightarrow du=d\theta$, $v=\tan\theta$. Then
$\int\theta\sec^2\theta\,d\theta=\theta\tan\theta-\int\tan\theta\,d\theta=\theta\tan\theta-\ln|\sec\theta|+C$.

3. Let $u=x$, $dv=\cos 5x\,dx\Rightarrow du=dx$, $v=\frac{1}{5}\sin 5x$. Then by Equation 2,
$\int x\cos 5x\,dx=\frac{1}{5}x\sin 5x-\int\frac{1}{5}\sin 5x\,dx=\frac{1}{5}x\sin 5x+\frac{1}{25}\cos 5x+C$.

4. Let $u=x$, $dv=e^{-x}dx\Rightarrow du=dx$, $v=-e^{-x}$. Then $\int xe^{-x}dx=-xe^{-x}+\int e^{-x}dx=-xe^{-x}-e^{-x}+C$.

5. Let $u=r$, $dv=e^{r/2}dr\Rightarrow du=dr$, $v=2e^{r/2}$. Then $\int re^{r/2}dr=2re^{r/2}-\int 2e^{r/2}dr=2re^{r/2}-4e^{r/2}+C$.

6. Let $u=t$, $dv=\sin 2t\,dt\Rightarrow du=dt$, $v=-\frac{1}{2}\cos 2t$. Then $\int t\sin 2t\,dt=-\frac{1}{2}t\cos 2t+\frac{1}{2}\int\cos 2t$
$dt=-\frac{1}{2}t\cos 2t+\frac{1}{4}\sin 2t+C$.

7. Let $u=x^2$, $dv=\sin\pi x\,dx\Rightarrow du=2x\,dx$ and $v=-\frac{1}{\pi}\cos\pi x$. Then
$I=\int x^2\sin\pi x\,dx=-\frac{1}{\pi}x^2\cos\pi x+\frac{2}{\pi}\int x\cos\pi x\,dx$ (*).
Next let $U=x$, $dV=\cos\pi x\,dx\Rightarrow dU=dx$, $V=\frac{1}{\pi}\sin\pi x$, so
$\int x\cos\pi x\,dx=\frac{1}{\pi}x\sin\pi x-\frac{1}{\pi}\int\sin\pi x\,dx=\frac{1}{\pi}x\sin\pi x+\frac{1}{\pi^2}\cos\pi x+C_1$. Substituting for $\int x\cos\pi x\,dx$ in (*), we get
$I=-\frac{1}{\pi}x^2\cos\pi x+\frac{2}{\pi}\left(\frac{1}{\pi}x\sin\pi x+\frac{1}{\pi^2}\cos\pi x+C_1\right)=-\frac{1}{\pi}x^2\cos\pi x+\frac{2}{\pi^2}x\sin\pi x+\frac{2}{\pi^3}\cos\pi x+C$,
where $C=\frac{2}{\pi}C_1$.

8. Let $u=x^2$, $dv=\cos mx\,dx \Rightarrow du=2x\,dx$, $v=\frac{1}{m}\sin mx$. Then $I=\int x^2\cos mx\,dx=\frac{1}{m}x^2\sin mx-\frac{2}{m}\int x\sin mx\,dx$ (*). Next let $U=x$, $dV=\sin mx\,dx \Rightarrow dU=dx$, $V=-\frac{1}{m}\cos mx$, so $\int x\sin mx\,dx=-\frac{1}{m}x\cos mx+\frac{1}{m}\int \cos mx\,dx=-\frac{1}{m}x\cos mx+\frac{1}{m^2}\sin mx+C_1$. Substituting for $\int x\sin mx\,dx$ in (*), we get $I=\frac{1}{m}x^2\sin mx-\frac{2}{m}\left(-\frac{1}{m}x\cos mx+\frac{1}{m^2}\sin mx+C_1\right)=\frac{1}{m}x^2\sin mx+\frac{2}{m^2}x\cos mx-\frac{2}{m^3}\sin mx+C$, where $C=-\frac{2}{m}C_1$.

9. Let $u=\ln(2x+1)$, $dv=dx \Rightarrow du=\frac{2}{2x+1}dx$, $v=x$. Then

$$\begin{aligned}\int \ln(2x+1)dx &= x\ln(2x+1)-\int\frac{2x}{2x+1}dx=x\ln(2x+1)-\int\frac{(2x+1)-1}{2x+1}dx\\ &=x\ln(2x+1)-\int\left(1-\frac{1}{2x+1}\right)dx=x\ln(2x+1)-x+\frac{1}{2}\ln(2x+1)+C\\ &=\frac{1}{2}(2x+1)\ln(2x+1)-x+C\end{aligned}$$

10. Let $u=\sin^{-1}x$, $dv=dx \Rightarrow du=\frac{dx}{\sqrt{1-x^2}}$, $v=x$. Then $\int \sin^{-1}x\,dx=x\sin^{-1}x-\int\frac{x}{\sqrt{1-x^2}}dx$. Setting $t=1-x^2$, we get $dt=-2x\,dx$, so $-\int\frac{x\,dx}{\sqrt{1-x^2}}=-\int t^{-1/2}\left(-\frac{1}{2}dt\right)=\frac{1}{2}\left(2t^{1/2}\right)+C=t^{1/2}+C=\sqrt{1-x^2}+C$. Hence, $\int \sin^{-1}x\,dx=x\sin^{-1}x+\sqrt{1-x^2}+C$.

11. Let $u=\arctan 4t$, $dv=dt \Rightarrow du=\frac{4}{1+(4t)^2}dt=\frac{4}{1+16t^2}dt$, $v=t$. Then

$$\begin{aligned}\int \arctan 4t\,dt &= t\arctan 4t-\int\frac{4t}{1+16t^2}dt=t\arctan 4t-\frac{1}{8}\int\frac{32t}{1+16t^2}dt\\ &=t\arctan 4t-\frac{1}{8}\ln(1+16t^2)+C\end{aligned}$$

12. Let $u=\ln p$, $dv=p^5dp \Rightarrow$

$du=\frac{1}{p}\,dp$, $v=\frac{1}{6}\,p^6$. Then $\int p^5 \ln p\,dp=\frac{1}{6}\,p^6 \ln p-\frac{1}{6}\int p^5 dp=\frac{1}{6}\,p^6 \ln p-\frac{1}{36}\,p^6+C$.

13. First let $u=(\ln x)^2$, $dv=dx \Rightarrow du=2\ln x\cdot \frac{1}{x}\,dx$, $v=x$. Then by Equation 2,
$I=\int (\ln x)^2 dx=x(\ln x)^2-2\int x\ln x\cdot \frac{1}{x}\,dx=x(\ln x)^2-2\int \ln x\,dx$. Next let $U=\ln x$, $dV=dx\Rightarrow dU=1/x\,dx$, $V=x$ to get $\int \ln x\,dx=x\ln x-\int x\cdot (1/x)\,dx=x\ln x-\int dx=x\ln x-x+C_1$. Thus,
$I=x(\ln x)^2-2\left(x\ln x-x+C_1\right)=x(\ln x)^2-2x\ln x+2x+C$, where $C=-2C_1$.

14. Let $u=t^3$, $dv=e^t dt \Rightarrow du=3t^2 dt$, $v=e^t$. Then $I=\int t^3 e^t dt=t^3 e^t-\int 3t^2 e^t dt$. Integrate by parts twice more with $dv=e^t dt$.

$$\begin{aligned} I &= t^3 e^t-\left(3t^2 e^t-\int 6te^t dt\right)=t^3 e^t-3t^2 e^t+6te^t-\int 6e^t dt \\ &= t^3 e^t-3t^2 e^t+6te^t-6e^t+C=\left(t^3-3t^2+6t-6\right) e^t+C \end{aligned}$$

More generally, if $p(t)$ is a polynomial of degree n in t , then repeated integration by parts shows that $\int p(t)\,e^t dt=\left[\,p(t)-p'(t)+p''(t)-p'''(t)+\cdots+(-1)^n p^{(n)}(t)\right] e^t+C$.

15. First let $u=\sin 3\theta$, $dv=e^{2\theta}\,d\theta \Rightarrow du=3\cos 3\theta\,d\theta$, $v=\frac{1}{2}\,e^{2\theta}$. Then
$I=\int e^{2\theta}\sin 3\theta\,d\theta=\frac{1}{2}\,e^{2\theta}\sin 3\theta-\frac{3}{2}\int e^{2\theta}\cos 3\theta\,d\theta$. Next let $U=\cos 3\theta$, $dV=e^{2\theta}\,d\theta$
$\Rightarrow dU=-3\sin 3\theta\,d\theta$, $V=\frac{1}{2}\,e^{2\theta}$ to get
$\int e^{2\theta}\cos 3\theta\,d\theta=\frac{1}{2}\,e^{2\theta}\cos 3\theta+\frac{3}{2}\int e^{2\theta}\sin 3\theta\,d\theta$. Substituting in the previous formula gives
$I=\frac{1}{2}\,e^{2\theta}\sin 3\theta-\frac{3}{4}\,e^{2\theta}\cos 3\theta-\frac{9}{4}\int e^{2\theta}\sin 3\theta\,d\theta=\frac{1}{2}\,e^{2\theta}\sin 3\theta-\frac{3}{4}\,e^{2\theta}\cos 3\theta-\frac{9}{4}\,I\Rightarrow$
$\frac{13}{4}\,I=\frac{1}{2}\,e^{2\theta}\sin 3\theta-\frac{3}{4}\,e^{2\theta}\cos 3\theta+C_1$. Hence, $I=\frac{1}{13}\,e^{2\theta}\,(2\sin 3\theta-3\cos 3\theta)+C$, where $C=\frac{4}{13}\,C_1$.

16. First let $u=e^{-\theta}$, $dv=\cos 2\theta\,d\theta \Rightarrow du=-e^{-\theta}\,d\theta$, $v=\frac{1}{2}\sin 2\theta$. Then
$I=\int e^{-\theta}\cos 2\theta\,d\theta=\frac{1}{2}\,e^{-\theta}\sin 2\theta-\int \frac{1}{2}\sin 2\theta\left(-e^{-\theta}\,d\theta\right)=\frac{1}{2}\,e^{-\theta}\sin 2\theta+\frac{1}{2}\int e^{-\theta}\sin 2\theta\,d\theta$.
Next let $U=e^{-\theta}$, $dV=\sin 2\theta\,d\theta \Rightarrow dU=-e^{-\theta}\,d\theta$, $V=-\frac{1}{2}\cos 2\theta$, so

$\int e^{-\theta}\sin 2\theta\, d\theta = -\frac{1}{2}e^{-\theta}\cos 2\theta - \int\left(-\frac{1}{2}\right)\cos 2\theta\left(-e^{-\theta}\,d\theta\right) = -\frac{1}{2}e^{-\theta}\cos 2\theta - \frac{1}{2}\int e^{-\theta}\cos 2\theta\, d\theta$.

So $I = \frac{1}{2}e^{-\theta}\sin 2\theta + \frac{1}{2}\left[\left(-\frac{1}{2}e^{-\theta}\cos 2\theta\right) - \frac{1}{2}I\right] = \frac{1}{2}e^{-\theta}\sin 2\theta - \frac{1}{4}e^{-\theta}\cos 2\theta - \frac{1}{4}I \Rightarrow$

$\frac{5}{4}I = \frac{1}{2}e^{-\theta}\sin 2\theta - \frac{1}{4}e^{-\theta}\cos 2\theta + C_1 \Rightarrow$

$I = \frac{4}{5}\left(\frac{1}{2}e^{-\theta}\sin 2\theta - \frac{1}{4}e^{-\theta}\cos 2\theta + C_1\right) = \frac{2}{5}e^{-\theta}\sin 2\theta - \frac{1}{5}e^{-\theta}\cos 2\theta + C$.

17. Let $u=y$, $dv=\sinh y\,dy \Rightarrow du=dy$, $v=\cosh y$. Then
$\int y\sinh y\,dy = y\cosh y - \int\cosh y\,dy = y\cosh y - \sinh y + C$.

18. Let $u=y$, $dv=\cosh ay\,dy \Rightarrow du=dy$, $v=\frac{\sinh ay}{a}$. Then
$\int y\cosh ay\,dy = \frac{y\sinh ay}{a} - \frac{1}{a}\int\sinh ay\,dy = \frac{y\sinh ay}{a} - \frac{\cosh ay}{a^2} + C$.

19. Let $u=t$, $dv=\sin 3t\,dt \Rightarrow du=dt$, $v=-\frac{1}{3}\cos 3t$. Then
$\int_0^\pi t\sin 3t\,dt = \left[-\frac{1}{3}t\cos 3t\right]_0^\pi + \frac{1}{3}\int_0^\pi \cos 3t\,dt = \left(\frac{1}{3}\pi - 0\right) + \frac{1}{9}\left[\sin 3t\right]_0^\pi = \frac{\pi}{3}$.

20. First let $u=x^2+1$, $dv=e^{-x}dx \Rightarrow du=2x\,dx$, $v=-e^{-x}$. By (6),
$\int_0^1\left(x^2+1\right)e^{-x}dx = \left[-\left(x^2+1\right)e^{-x}\right]_0^1 + \int_0^1 2xe^{-x}dx = -2e^{-1}+1+2\int_0^1 xe^{-x}dx$. Next let $U=x$, $dV=e^{-x}dx \Rightarrow dU=dx$, $V=-e^{-x}$. By (6) again, $\int_0^1 xe^{-x}dx = \left[-xe^{-x}\right]_0^1 + \int_0^1 e^{-x}dx = -e^{-1} + \left[-e^{-x}\right]_0^1 = -e^{-1} - e^{-1} + 1 = -2e^{-1}+1$. So
$\int_0^1\left(x^2+1\right)e^{-x}dx = -2e^{-1}+1+2\left(-2e^{-1}+1\right) = -2e^{-1}+1-4e^{-1}+2 = -6e^{-1}+3$.

21. Let $u=\ln x$, $dv=x^{-2}dx \Rightarrow du=\frac{1}{x}dx$, $v=-x^{-1}$. By (6),
$$\int_1^2 \frac{\ln x}{x^2}dx = \left[-\frac{\ln x}{x}\right]_1^2 + \int_1^2 x^{-2}dx = -\frac{1}{2}\ln 2 + \ln 1 + \left[-\frac{1}{x}\right]_1^2 = -\frac{1}{2}\ln 2 + 0 - \frac{1}{2} + 1 = \frac{1}{2} - \frac{1}{2}\ln 2 .$$

22. Let $u=\ln t$, $dv=\sqrt{t}\,dt \Rightarrow du=dt/t$, $v=\frac{2}{3}t^{3/2}$. By Formula 6,

$\int_1^4 \sqrt{t}\ln t\,dt = \left[\frac{2}{3}t^{3/2}\ln t\right]_1^4 - \frac{2}{3}\int_1^4 \sqrt{t}\,dt = \frac{2}{3}\cdot 8\cdot \ln 4 - 0 - \left[\frac{2}{3}\cdot\frac{2}{3}t^{3/2}\right]_1^4 = \frac{16}{3}\ln 4 - \frac{4}{9}(8-1) = \frac{16}{3}\ln 4 - \frac{28}{9}$

.

23. Let $u=y$, $dv=\frac{dy}{e^{2y}}=e^{-2y}dy \Rightarrow du=dy$, $v=-\frac{1}{2}e^{-2y}$. Then

$\int_0^1 \frac{y}{e^{2y}}dy = \left[-\frac{1}{2}ye^{-2y}\right]_0^1 + \frac{1}{2}\int_0^1 e^{-2y}dy = \left(-\frac{1}{2}e^{-2}+0\right) - \frac{1}{4}\left[e^{-2y}\right]_0^1 = -\frac{1}{2}e^{-2} - \frac{1}{4}e^{-2} + \frac{1}{4} = \frac{1}{4} - \frac{3}{4}e^{-2}$.

24. Let $u=x$, $dv=\csc^2 x\,dx \Rightarrow du=dx$, $v=-\cot x$. Then

$\int_{\pi/4}^{\pi/2} x\csc^2 x\,dx = [-x\cot x]_{\pi/4}^{\pi/2} + \int_{\pi/4}^{\pi/2}\cot x\,dx = -\frac{\pi}{2}\cdot 0 + \frac{\pi}{4}\cdot 1 + [\ln|\sin x|]_{\pi/4}^{\pi/2}$ [see Exercise 5.5.]

$= \frac{\pi}{4} + \ln 1 - \ln\frac{1}{\sqrt{2}} = \frac{\pi}{4} + 0 - \ln 2^{-1/2} = \frac{\pi}{4} + \frac{1}{2}\ln 2$

25. Let $u=\cos^{-1}x$, $dv=dx \Rightarrow du=-\frac{dx}{\sqrt{1-x^2}}$, $v=x$. Then

$I=\int_0^{1/2}\cos^{-1}x\,dx = \left[x\cos^{-1}x\right]_0^{1/2} + \int_0^{1/2}\frac{x\,dx}{\sqrt{1-x^2}} = \frac{1}{2}\cdot\frac{\pi}{3} + \int_1^{3/4} t^{-1/2}\left[-\frac{1}{2}\right]dt$, where $t=1-x^2 \Rightarrow dt=-2x\,dx$.

Thus, $I=\frac{\pi}{6}+\frac{1}{2}\int_{3/4}^{1} t^{-1/2}dt = \frac{\pi}{6} + \left[\sqrt{t}\right]_{3/4}^{1} = \frac{\pi}{6} + 1 - \frac{\sqrt{3}}{2} = \frac{1}{6}\left(\pi + 6 - 3\sqrt{3}\right)$.

26. Let $u=x$, $dv=5^x dx \Rightarrow du=dx$, $v=\left(5^x/\ln 5\right)$. Then

$\int_0^1 x5^x dx = \left[\frac{x5^x}{\ln 5}\right]_0^1 - \int_0^1 \frac{5^x}{\ln 5}dx = \frac{5}{\ln 5} - 0 - \frac{1}{\ln 5}\left[\frac{5^x}{\ln 5}\right]_0^1 = \frac{5}{\ln 5} - \frac{5}{(\ln 5)^2} + \frac{1}{(\ln 5)^2}$

$= \frac{5}{\ln 5} - \frac{4}{(\ln 5)^2}$

27. Let $u=\ln(\sin x)$, $dv=\cos x\,dx \Rightarrow du=\frac{\cos x}{\sin x}dx$, $v=\sin x$. Then

$I=\int \cos x\ln(\sin x)dx = \sin x\ln(\sin x) - \int\cos x\,dx = \sin x\ln(\sin x) - \sin x + C$.

Another method: Substitute $t=\sin x$, so $dt=\cos x\,dx$. Then $I=\int \ln t\,dt=t\ln t-t+C$ (see Example 2) and so $I=\sin x(\ln\sin x-1)+C$.

28. Let $u=\arctan(1/x)$, $dv=dx\Rightarrow du=\frac{1}{1+(1/x)^2}\cdot\frac{-1}{x^2}\,dx=\frac{-dx}{x^2+1}$, $v=x$. Then

$$\int_1^{\sqrt{3}}\arctan(1/x)\,dx=\left[x\arctan\left(\frac{1}{x}\right)\right]_1^{\sqrt{3}}+\int_1^{\sqrt{3}}\frac{x}{dx}\,x^2+1=\sqrt{3}\,\frac{\pi}{6}-1\cdot\frac{\pi}{4}+\frac{1}{2}\left[\ln(x^2+1)\right]_1^{\sqrt{3}}$$

$$=\frac{\pi\sqrt{3}}{6}-\frac{\pi}{4}+\frac{1}{2}(\ln 4-\ln 2)=\frac{\pi\sqrt{3}}{6}-\frac{\pi}{2}+\frac{1}{2}\ln\frac{4}{2}=\frac{\pi\sqrt{3}}{6}-\frac{\pi}{2}+\frac{1}{2}\ln 2$$

29. Let $w=\ln x\Rightarrow dw=dx/x$. Then $x=e^w$ and $dx=e^w dw$, so

$$\int\cos(\ln x)\,dx=\int e^w\cos w\,dw=\frac{1}{2}e^w(\sin w+\cos w)+C \qquad \text{[by the method of Example 4]}$$

$$=\frac{1}{2}x[\sin(\ln x)+\cos(\ln x)]+C$$

30. Let $u=r^2$, $dv=\frac{r}{\sqrt{4+r^2}}\,dr\Rightarrow du=2r\,dr$, $v=\sqrt{4+r^2}$. By (6),

$$\int_0^1\frac{r^3}{\sqrt{4+r^2}}\,dr=\left[r^2\sqrt{4+r^2}\right]_0^1-2\int_0^1 r\sqrt{4+r^2}\,dr=\sqrt{5}-\frac{2}{3}\left[\left(4+r^2\right)^{3/2}\right]_0^1$$

$$=\sqrt{5}-\frac{2}{3}(5)^{3/2}+\frac{2}{3}(8)=\sqrt{5}\left(1-\frac{10}{3}\right)+\frac{16}{3}=\frac{16}{3}-\frac{7}{3}\sqrt{5}$$

31. Let $u=(\ln x)^2$, $dv=x^4dx\Rightarrow du=2\frac{\ln x}{x}\,dx$, $v=\frac{x^5}{5}$. By (6),

$$\int_1^2 x^4(\ln x)^2dx=\left[\frac{x^5}{5}(\ln x)^2\right]_1^2-2\int_1^2\frac{x^4}{5}\ln x\,dx=\frac{32}{5}(\ln 2)^2-0-2\int_1^2\frac{x^4}{5}\ln x\,dx .$$

Let $U=\ln x$, $dV=\frac{x^4}{5}\,dx\Rightarrow dU=\frac{1}{x}\,dx$, $V=\frac{x^5}{25}$.

Then $\int_1^2\frac{x^4}{5}\ln x\,dx=\left[\frac{x^5}{25}\ln x\right]_1^2-\int_1^2\frac{x^4}{25}\,dx=\frac{32}{25}\ln 2-0-\left[\frac{x^5}{125}\right]_1^2=\frac{32}{25}\ln 2-\left(\frac{32}{125}-\frac{1}{125}\right)$.

So $\int_1^2 x^4(\ln x)^2 dx = \frac{32}{5}(\ln 2)^2 - 2\left(\frac{32}{25}\ln 2 - \frac{31}{125}\right) = \frac{32}{5}(\ln 2)^2 - \frac{64}{25}\ln 2 + \frac{62}{125}$.

32. Let $u=\sin(t-s)$, $dv=e^s ds \Rightarrow du=-\cos(t-s)ds$, $v=e^s$. Then $I=\int_0^t e^s \sin(t-s)ds = \left[e^s \sin(t-s)\right]_0^t + \int_0^t e^s \cos(t-s)ds = e^t \sin 0 - e^0 \sin t + I_1$. For I_1 , let $U=\cos(t-s)$, $dV=e^s ds$ $\Rightarrow dU=\sin(t-s)ds$, $V=e^s$. So $I_1=\left[e^s\cos(t-s)\right]_0^t - \int_0^t e^s \sin(t-s)ds = e^t\cos 0 - e^0 \cos t - I$. Thus, $I=-\sin t + e^t - \cos t - I \Rightarrow 2I = e^t - \cos t - \sin t \Rightarrow I = \frac{1}{2}\left(e^t - \cos t - \sin t\right)$.

33. Let $w=\sqrt{x}$, so that $x=w^2$ and $dx=2w\,dw$. Thus, $\int \sin\sqrt{x}\,dx = \int 2w\sin w\,dw$. Now use parts with $u=2w$, $dv=\sin w\,dw$, $du=2dw$, $v=-\cos w$ to get

$$\begin{aligned}\int 2w\sin w\,dw &= -2w\cos w + \int 2\cos w\,dw = -2w\cos w + 2\sin w + C \\ &= -2\sqrt{x}\cos\sqrt{x} + 2\sin\sqrt{x} + C = 2\left(\sin\sqrt{x} - \sqrt{x}\cos\sqrt{x}\right) + C\end{aligned}$$

34. Let $w=\sqrt{x}$, so that $x=w^2$ and $dx=2w\,dw$. Thus, $\int_1^4 e^{\sqrt{x}}\,dx = \int_1^2 e^w 2w\,dw$. Now use parts with $u=2w$, $dv=e^w dw$, $du=2dw$, $v=e^w$ to get $\int_1^2 e^w 2w\,dw = \left[2we^w\right]_1^2 - 2\int_1^2 e^w dw = 4e^2 - 2e - 2\left(e^2 - e\right) = 2e^2$.

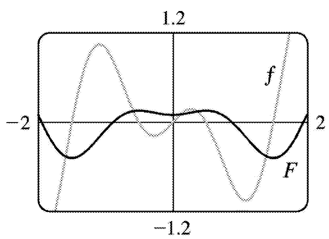
35. Let $x=\theta^2$, so that $dx=2\theta\,d\theta$. Thus, $\int_{\sqrt{\pi/2}}^{\sqrt{\pi}} \theta^3 \cos\left(\theta^2\right) d\theta = \int_{\sqrt{\pi/2}}^{\sqrt{\pi}} \theta^2 \cos\left(\theta^2\right) \cdot \frac{1}{2}(2\theta\,d\theta) = \frac{1}{2}\int_{\pi/2}^{\pi} x\cos x\}\{dx\}$. Now use parts with $u=x$, $dv=\cos x\,dx$, $du=dx$, $v=\sin x$ to get

$$\begin{aligned}\frac{1}{2}\int_{\pi/2}^{\pi} x\cos x\,dx &= \frac{1}{2}\left(\left[x\sin x\right]_{\pi/2}^{\pi} - \int_{\pi/2}^{\pi}\sin x\,dx\right) = \frac{1}{2}\left[x\sin x + \cos x\right]_{\pi/2}^{\pi} \\ &= \frac{1}{2}(\pi\sin\pi + \cos\pi) - \frac{1}{2}\left(\frac{\pi}{2}\sin\frac{\pi}{2} + \cos\frac{\pi}{2}\right) = \frac{1}{2}(\pi\cdot 0 - 1) - \frac{1}{2}\left(\frac{\pi}{2}\cdot 1 + 0\right) = -\frac{1}{2} - \frac{\pi}{4}\end{aligned}$$

36.

$$\begin{aligned}\int x^5 e^{x^2} dx &= \int \left(x^2\right)^2 e^{x^2} x\,dx = \int t^2 e^t \frac{1}{2}\,dt \text{ [where } t=x^2 \Rightarrow \frac{1}{2}\,dt = x\,dx \text{]} \\ &= \frac{1}{2}\left(t^2 - 2t + 2\right)e^t + C \text{ [by Example 3] } = \frac{1}{2}\left(x^4 - 2x^2 + 2\right)e^{x^2} + C\end{aligned}$$

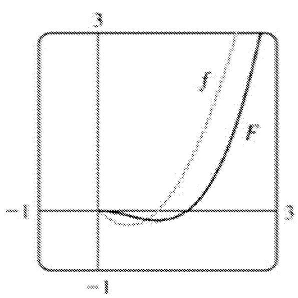
37. Let $u=x$, $dv=\cos \pi x dx \Rightarrow du=dx$, $v=(\sin \pi x)/\pi$. Then
$\int x\cos \pi x dx = x\cdot \frac{\sin \pi x}{\pi} - \int \frac{\sin \pi x}{\pi} dx = \frac{x\sin \pi x}{\pi} + \frac{\cos \pi x}{\pi^2} + C$. We see from the graph that this is reasonable, since F has extreme values where f is 0 .



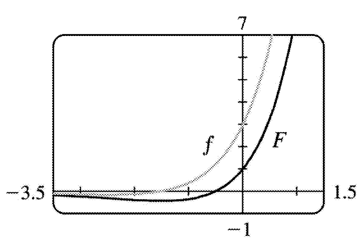
38. Let $u=\ln x$, $dv=x^{3/2}dx \Rightarrow du=\frac{1}{x} dx$, $v=\frac{2}{5} x^{5/2}$. Then

$$\int x^{3/2}\ln x dx = \frac{2}{5} x^{5/2}\ln x - \frac{2}{5}\int x^{3/2} dx = \frac{2}{5} x^{5/2}\ln x - \left(\frac{2}{5} \right)^2 x^{5/2} + C$$
$$= \frac{2}{5} x^{5/2}\ln x - \frac{4}{25} x^{5/2} + C.$$

We see from the graph that this is reasonable, since F has a minimum where f changes from negative to positive.



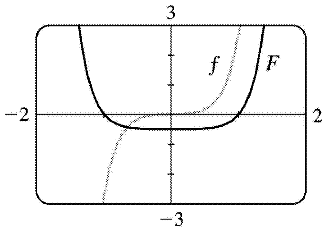
39. Let $u=2x+3$, $dv=e^x dx \Rightarrow du=2dx$, $v=e^x$. Then
$\int (2x+3)e^x dx=(2x+3)e^x - 2\int e^x dx=(2x+3)e^x - 2e^x + C=(2x+1)e^x + C$. We see from the graph that this is reasonable, since F has a minimum where f changes from negative to positive.



40. $\int x^3 e^{x^2} dx = \int x^2 \cdot x e^{x^2} dx = I$.

Let $u = x^2$, $dv = x e^{x^2} dx \Rightarrow du = 2x\,dx$, $v = \frac{1}{2} e^{x^2}$. Then

$I = \frac{1}{2} x^2 e^{x^2} - \int x e^{x^2} dx = \frac{1}{2} x^2 e^{x^2} - \frac{1}{2} e^{x^2} + C = \frac{1}{2} e^{x^2} \left(x^2 - 1 \right) + C$. We see from the graph that this is reasonable, since F has a minimum where f changes from negative to positive.



41. **(a)** Take $n=2$ in Example 6 to get $\int \sin^2 x\,dx = -\frac{1}{2} \cos x \sin x + \frac{1}{2} \int 1\,dx = \frac{x}{2} - \frac{\sin 2x}{4} + C$.

(b) $\int \sin^4 x\,dx = -\frac{1}{4} \cos x \sin^3 x + \frac{3}{4} \int \sin^2 x\,dx = -\frac{1}{4} \cos x \sin^3 x + \frac{3}{8} x - \frac{3}{16} \sin 2x + C$.

42. **(a)** Let $u = \cos^{n-1} x$, $dv = \cos x\,dx \Rightarrow du = -(n-1)\cos^{n-2} x \sin x\,dx$, $v = \sin x$ in (2):

$$\begin{aligned} \int \cos^n x\,dx &= \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x \sin^2 x\,dx \\ &= \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x \left(1 - \cos^2 x \right) dx \\ &= \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x\,dx - (n-1) \int \cos^n x\,dx \end{aligned}$$

Rearranging terms gives $n \int \cos^n x\,dx = \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x\,dx$ or

$$\int \cos^n x\,dx = \frac{1}{n} \cos^{n-1} x \sin x + \frac{n-1}{n} \int \cos^{n-2} x\,dx$$

(b) Take $n=2$ in part (a) to get $\int \cos^2 x\,dx = \frac{1}{2} \cos x \sin x + \frac{1}{2} \int 1\,dx = \frac{x}{2} + \frac{\sin 2x}{4} + C$.

(c) $\int \cos^4 x\,dx = \frac{1}{4} \cos^3 x \sin x + \frac{3}{4} \int \cos^2 x\,dx = \frac{1}{4} \cos^3 x \sin x + \frac{3}{8} x + \frac{3}{16} \sin 2x + C$

43. **(a)** From Example 6, $\int \sin^n x\,dx = -\frac{1}{n} \cos x \sin^{n-1} x + \frac{n-1}{n} \int \sin^{n-2} x\,dx$. Using (6),

$$\int_0^{\pi/2} \sin^n x\,dx = \left[-\frac{\cos x \sin^{n-1} x}{n} \right]_0^{\pi/2} + \frac{n-1}{n} \int_0^{\pi/2} \sin^{n-2} x\,dx$$
$$= (0-0) + \frac{n-1}{n} \int_0^{\pi/2} \sin^{n-2} x\,dx = \frac{n-1}{n} \int_0^{\pi/2} \sin^{n-2} x\,dx$$

(b) Using $n=3$ in part (a), we have $\int_0^{\pi/2} \sin^3 x\,dx = \frac{2}{3} \int_0^{\pi/2} \sin x\,dx = \left[-\frac{2}{3} \cos x \right]_0^{\pi/2} = \frac{2}{3}$.

Using $n=5$ in part (a), we have $\int_0^{\pi/2} \sin^5 x\,dx = \frac{4}{5} \int_0^{\pi/2} \sin^3 x\,dx = \frac{4}{5} \cdot \frac{2}{3} = \frac{8}{15}$.

(c) The formula holds for $n=1$ (that is, $2n+1=3$) by (b). Assume it holds for some $k \geq 1$. Then $\int_0^{\pi/2} \sin^{2k+1} x\,dx = \frac{2\cdot 4\cdot 6\cdot \cdots \cdot (2k)}{3\cdot 5\cdot 7\cdot \cdots \cdot (2k+1)}$. By Example 6,

$$\int_0^{\pi/2} \sin^{2k+3} x\,dx = \frac{2k+2}{2k+3} \int_0^{\pi/2} \sin^{2k+1} x\,dx = \frac{2k+2}{2k+3} \cdot \frac{2\cdot 4\cdot 6\cdot \cdots \cdot (2k)}{3\cdot 5\cdot 7\cdot \cdots \cdot (2k+1)}$$
$$= \frac{2\cdot 4\cdot 6\cdot \cdots \cdot (2k)[2(k+1)]}{3\cdot 5\cdot 7\cdot \cdots \cdot (2k+1)[2(k+1)+1]},$$

so the formula holds for $n=k+1$. By induction, the formula holds for all $n \geq 1$.

44. Using Exercise 43 (a), we see that the formula holds for $n=1$, because $\int_0^{\pi/2} \sin^2 x\,dx = \frac{1}{2} \int_0^{\pi/2} 1\,dx = \frac{1}{2} [x]_0^{\pi/2} = \frac{1}{2} \cdot \frac{\pi}{2}$.

Now assume it holds for some $k \geq 1$. Then $\int_0^{\pi/2} \sin^{2k} x\,dx = \frac{1\cdot 3\cdot 5\cdot \cdots \cdot (2k-1)}{2\cdot 4\cdot 6\cdot \cdots \cdot (2k)} \frac{\pi}{2}$. By Exercise 43 (a),

$$\int_0^{\pi/2} \sin^{2(k+1)} x\,dx = \frac{2k+1}{2k+2} \int_0^{\pi/2} \sin^{2k} x\,dx = \frac{2k+1}{2k+2} \cdot \frac{1\cdot 3\cdot 5\cdot \cdots \cdot (2k-1)}{2\cdot 4\cdot 6\cdot \cdots \cdot (2k)} \frac{\pi}{2}$$
$$= \frac{1\cdot 3\cdot 5\cdot \cdots \cdot (2k-1)(2k+1)}{2\cdot 4\cdot 6\cdot \cdots \cdot (2k)(2k+2)} \cdot \frac{\pi}{2},$$

so the formula holds for $n=k+1$. By induction, the formula holds for all $n \geq 1$.

45. Let $u=(\ln x)^n$, $dv=dx \Rightarrow du=n(\ln x)^{n-1}(dx/x)$, $v=x$. By Equation 2, $\int (\ln x)^n dx = x(\ln x)^n - \int nx(\ln x)^{n-1}(dx/x) = x(\ln x)^n - n\int (\ln x)^{n-1} dx$.

46. Let $u=x^n$, $dv=e^x dx \Rightarrow du=nx^{n-1} dx$, $v=e^x$. By Equation 2, $\int x^n e^x dx = x^n e^x - n\int x^{n-1} e^x dx$.

47. Let $u=\left(x^2+a^2\right)^n$, $dv=dx \Rightarrow du=n\left(x^2+a^2\right)^{n-1} 2x\,dx$, $v=x$. Then

$$\int \left(x^2+a^2\right)^n dx = x\left(x^2+a^2\right)^n - 2n\int x^2\left(x^2+a^2\right)^{n-1} dx$$

$$= x\left(x^2+a^2\right)^n - 2n\left[\int \left(x^2+a^2\right)^n dx - a^2\int \left(x^2+a^2\right)^{n-1} dx\right] \quad \left[\text{since } x^2=\left(x^2+a^2\right)-a^2\right]$$

$\Rightarrow (2n+1)\int \left(x^2+a^2\right)^n dx = x\left(x^2+a^2\right)^n + 2na^2\int \left(x^2+a^2\right)^{n-1} dx$, and

$\int \left(x^2+a^2\right)^n dx = \dfrac{x\left(x^2+a^2\right)^n}{2n+1} + \dfrac{2na^2}{2n+1}\int \left(x^2+a^2\right)^{n-1} dx$ [provided $2n+1\neq 0$].

48. Let $u=\sec^{n-2} x$, $dv=\sec^2 x\,dx \Rightarrow du=(n-2)\sec^{n-3} x \sec x \tan x\,dx$, $v=\tan x$. Then by Equation 2,

$$\int \sec^n x\,dx = \tan x \sec^{n-2} x - (n-2)\int \sec^{n-2} x \tan^2 x\,dx$$

$$= \tan x \sec^{n-2} x - (n-2)\int \sec^{n-2} x\left(\sec^2 x - 1\right) dx$$

$$= \tan x \sec^{n-2} x - (n-2)\int \sec^n x\,dx + (n-2)\int \sec^{n-2} x\,dx$$

so $(n-1)\int \sec^n x\,dx = \tan x \sec^{n-2} x + (n-2)\int \sec^{n-2} x\,dx$. If $n-1\neq 0$, then

$\int \sec^n x\,dx = \dfrac{\tan x \sec^{n-2} x}{n-1} + \dfrac{n-2}{n-1}\int \sec^{n-2} x\,dx$.

49. Take $n=3$ in Exercise 45 to get $\int (\ln x)^3 dx = x(\ln x)^3 - 3\int (\ln x)^2 dx = x(\ln x)^3 - 3x(\ln x)^2 + 6x\ln x - 6x + C$ [by Exercise 13].

Or : Instead of using Exercise 13 , apply Exercise 45 again with $n=2$.

50. Take $n=4$ in Exercise 46 to get

$$\int x^4 e^x dx = x^4 e^x - 4\int x^3 e^x dx = x^4 e^x - 4\left(x^3 - 3x^2 + 6x - 6\right)e^x + C \qquad \text{[by Exercise 14]}$$

$$= e^x\left(x^4 - 4x^3 + 12x^2 - 24x + 24\right) + C$$

Or: Instead of using Exercise 14 , apply Exercise 46 with $n=3$, then $n=2$, then $n=1$.

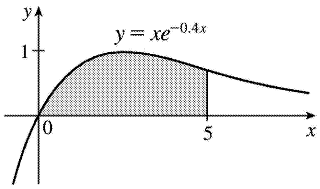
51. Area $=\int_0^5 xe^{-0.4x} dx$. Let $u=x$, $dv=e^{-0.4x} dx \Rightarrow$

$du=dx$, $v=-2.5e^{-0.4x}$. Then

$$\text{area} = \left[-2.5xe^{-0.4x}\right]_0^5 + 2.5\int_0^5 e^{-0.4x} dx$$

$$= -12.5e^{-2} + 0 + 2.5\left[-2.5e^{-0.4x}\right]_0^5$$

$$= -12.5e^{-2} - 6.25(e^{-2} - 1) = 6.25 - 18.75e^{-2} \text{ or } \frac{25}{4} - \frac{75}{4}e^{-2}$$



52. The curves $y = x\ln x$ and $y = 5\ln x$ intersect when $x\ln x = 5\ln x \Leftrightarrow x\ln x - 5\ln x = 0 \Leftrightarrow (x-5)\ln x = 0$; that is, when $x = 1$ or $x = 5$. For $1 < x < 5$, we have $5\ln x > x\ln x$ since $\ln x > 0$. Thus, area $= \int_1^5 (5\ln x - x\ln x)\,dx = \int_1^5 [(5-x)\ln x]\,dx$. Let $u = \ln x$, $dv = (5-x)\,dx \Rightarrow du = dx/x$, $v = 5x - \frac{1}{2}x^2$. Then

$$\text{area} = \left[(\ln x)\left(5x - \frac{1}{2}x^2\right) \right]_1^5 - \int_1^5 \left[\left(5x - \frac{1}{2}x^2\right)\frac{1}{x} \right] dx = (\ln 5)\left(\frac{25}{2}\right) - 0 - \int_1^5 \left(5 - \frac{1}{2}x\right) dx$$

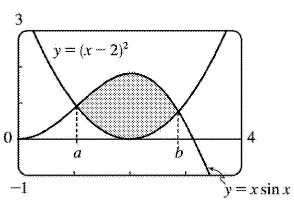
$$= \frac{25}{2}\ln 5 - \left[5x - \frac{1}{4}x^2 \right]_1^5 = \frac{25}{2}\ln 5 - \left[\left(25 - \frac{25}{4}\right) - \left(5 - \frac{1}{4}\right) \right] = \frac{25}{2}\ln 5 - 14$$

53. The curves $y = x\sin x$ and $y = (x-2)^2$ intersect at $a \approx 1.04748$ and $b \approx 2.87307$, so

$$\text{area} = \int_a^b \left[x\sin x - (x-2)^2 \right] dx$$

$$= \left[-x\cos x + \sin x - \frac{1}{3}(x-2)^3 \right]_a^b \text{ [by Example 1]}$$

$$\approx 2.81358 - 0.63075 = 2.18283$$

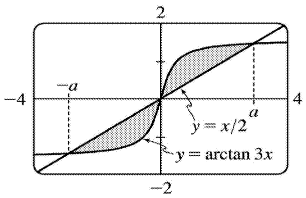


54. The curves $y = \arctan 3x$ and $y = x/2$ intersect at $x = \pm a \approx \pm 2.91379$, so

$$\text{area} = \int_{-a}^{a} \left| \arctan 3x - \frac{1}{2}x \right| dx = 2\int_0^a \left(\arctan 3x - \frac{1}{2}x \right) dx$$

$$= 2\left[x\arctan 3x - \frac{1}{6}\ln(1+9x^2) - \frac{1}{4}x^2 \right]_0^a \text{ [see Example 5]}$$

$$\approx 2(1.39768) = 2.79536 .$$



55. $V=\int_0^1 2\pi x\cos(\pi x/2)dx$. Let $u=x$, $dv=\cos(\pi x/2)dx \Rightarrow du=dx$, $v=\frac{2}{\pi}\sin(\pi x/2)$.

$$V=2\pi\left[\frac{2}{\pi}x\sin\left(\frac{\pi x}{2}\right)\right]_0^1-2\pi\cdot\frac{2}{\pi}\int_0^1\sin\left(\frac{\pi x}{2}\right)dx=2\pi\left(\frac{2}{\pi}-0\right)-4\left[-\frac{2}{\pi}\cos\left(\frac{\pi x}{2}\right)\right]_0^1$$

$$=4+\frac{8}{\pi}(0-1)=4-\frac{8}{\pi}\ .$$

56.

$$\text{Volume}=\int_0^1 2\pi x\left(e^x-e^{-x}\right)dx=2\pi\int_0^1\left(xe^x-xe^{-x}\right)dx$$

$$=2\pi\left[\int_0^1 xe^x dx-\int_0^1 xe^{-x}dx\right]\text{ [both integrals by parts]}$$

$$=2\pi\left[\left(xe^x-e^x\right)-\left(-xe^{-x}-e^{-x}\right)\right]_0^1=2\pi[2/e-0]=4\pi/e$$

57. Volume $=\int_{-1}^0 2\pi(1-x)e^{-x}dx$. Let $u=1-x$, $dv=e^{-x}dx \Rightarrow du=-dx$, $v=-e^{-x}$.

$$V=2\pi\left[(1-x)(-e^{-x})\right]_{-1}^0-2\pi\int_{-1}^0 e^{-x}dx=2\pi\left[(x-1)(e^{-x})+e^{-x}\right]_{-1}^0$$

$$=2\pi\left[xe^{-x}\right]_{-1}^0=2\pi(0+e)=2\pi e$$

58.

$$\text{Volume}=\int_1^\pi 2\pi y\cdot\ln y\,dy=2\pi\left[\frac{1}{2}y^2\ln y-\frac{1}{4}y^2\right]_1^\pi$$

$$=2\pi\left[\frac{1}{4}y^2(2\ln y-1)\right]_1^\pi=2\pi\left[\frac{\pi^2(2\ln\pi-1)}{4}-\frac{(0-1)}{4}\right]=\pi^3\ln\pi-\frac{\pi^3}{2}+\frac{\pi}{2}$$

59. The average value of $f(x)=x^2\ln x$ on the interval $[1,3]$ is $f_{\text{ave}}=\frac{1}{3-1}\int_1^3 x^2\ln x\,dx=\frac{1}{2}I$.

Let $u=\ln x$, $dv=x^2dx \Rightarrow du=(1/x)dx$, $v=\frac{1}{3}x^3$. So

$I=\left[\frac{1}{3}x^3\ln x\right]_1^3-\int_1^3\frac{1}{3}x^2dx=(9\ln 3-0)-\left[\frac{1}{9}x^3\right]_1^3=9\ln 3-\left(3-\frac{1}{9}\right)=9\ln 3-\frac{26}{9}$.

Thus, $f_{\text{ave}}=\frac{1}{2}I=\frac{1}{2}\left(9\ln 3-\frac{26}{9}\right)=\frac{9}{2}\ln 3-\frac{13}{9}$.

60. The rocket will have height $H=\int_0^{60}v(t)dt$ after 60 seconds.

$$H=\int_0^{60}\left[-gt-v_e\ln\left(\frac{m-rt}{m}\right)\right]dt=-g\left[\frac{1}{2}t^2\right]_0^{60}-v_e\left[\int_0^{60}\ln(m-rt)dt-\int_0^{60}\ln m\,dt\right]$$

$$=-g(1800)+v_e(\ln m)(60)-v_e\int_0^{60}\ln(m-rt)dt$$

Let $u=\ln(m-rt)$, $dv=dt\Rightarrow du=\frac{1}{m-rt}(-r)dt$, $v=t$. Then

$$\int_0^{60}\ln(m-rt)dt=[t\ln(m-rt)]_0^{60}+\int_0^{60}\frac{rt}{m-rt}dt=60\ln(m-60r)+\int_0^{60}\left(-1+\frac{m}{m-rt}\right)dt$$

$$=60\ln(m-60r)+\left[-t-\frac{m}{r}\ln(m-rt)\right]_0^{60}$$

$$=60\ln(m-60r)-60-\frac{m}{r}\ln(m-60r)+\frac{m}{r}\ln m$$

So $H=-1800g+60v_e\ln m-60v_e\ln(m-60r)+60v_e+\frac{m}{r}v_e\ln(m-60r)-\frac{m}{r}v_e\ln m$. Substituting $g=9.8$, $m=30,000$, $r=160$, and $v_e=3000$ gives us $H\approx 14,844$ m.

61. Since $v(t)>0$ for all t , the desired distance is $s(t)=\int_0^t v(w)dw=\int_0^t w^2e^{-w}dw$.

First let $u=w^2$, $dv=e^{-w}dw\Rightarrow du=2w\,dw$, $v=-e^{-w}$. Then $s(t)=\left[-w^2e^{-w}\right]_0^t+2\int_0^t we^{-w}dw$.

Next let $U=w$, $dV=e^{-w}dw\Rightarrow dU=dw$, $V=-e^{-w}$. Then

$$s(t)=-t^2e^{-t}+2\left(\left[-we^{-w}\right]_0^t+\int_0^t e^{-w}dw\right)=-t^2e^{-t}+2\left(-te^{-t}+0+\left[-e^{-w}\right]_0^t\right)$$

$$=-t^2e^{-t}+2\left(-te^{-t}-e^{-t}+1\right)=-t^2e^{-t}-2te^{-t}-2e^{-t}+2$$

$$=2-e^{-t}\left(t^2+2t+2\right)\text{ meters}$$

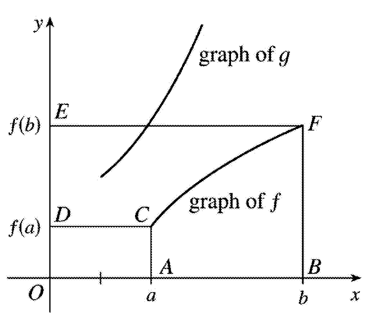
62. Suppose $f(0)=g(0)=0$ and let $u=f(x)$, $dv=g''(x)dx\Rightarrow du=f'(x)dx$, $v=g'(x)$. Then $\int_0^a f(x)g''(x)dx=\left[f(x)g'(x)\right]_0^a-\int_0^a f'(x)g'(x)dx=f(a)g'(a)-\int_0^a f'(x)g'(x)dx$. Now let $U=f'(x)$,

$dV=g'(x)dx \Rightarrow dU=f''(x)dx$ and $V=g(x)$, so
$\int_0^a f'(x)g'(x)dx=\left[f'(x)g(x)\right]_0^a-\int_0^a f''(x)g(x)dx=f'(a)g(a)-\int_0^a f''(x)g(x)dx$.
Combining the two results, we get $\int_0^a f(x)g''(x)dx=f(a)g'(a)-f'(a)g(a)+\int_0^a f''(x)g(x)dx$.

63. For $I=\int_1^4 xf''(x)dx$, let $u=x$, $dv=f''(x)dx \Rightarrow du=dx$, $v=f'(x)$. Then
$I=\left[xf'(x)\right]_1^4-\int_1^4 f'(x)dx=4f'(4)-1\cdot f'(1)-[f(4)-f(1)]=4\cdot 3-1\cdot 5-(7-2)=12-5-5=2$.
We used the fact that f'' is continuous to guarantee that I exists.

64. **(a)** Take $g(x)=x$ and $g'(x)=1$ in Equation 1.
(b) By part (a), $\int_a^b f(x)dx=bf(b)-af(a)-\int_a^b xf'(x)dx$. Now let $y=f(x)$, so that $x=g(y)$ and $dy=f'(x)dx$. Then $\int_a^b xf'(x)dx=\int_{f(a)}^{f(b)} g(y)dy$. The result follows.
(c) Part (b) says that the area of region $ABFC$ is

$= bf(b)$	$-\ af(a)$	$-\ \int_{f(a)}^{f(b)} g(y)dy$
= (area of rectangle $OBFE$)	− (area of rectangle $OACD$)	− (area of region $DCFE$)



(d) We have $f(x)=\ln x$, so $f^{-1}(x)=e^x$, and since $g=f^{-1}$, we have $g(y)=e^y$. By part (b),
$\int_1^e \ln x\,dx=e\ln e-1\ln 1-\int_{\ln 1}^{\ln e} e^y dy=e-\int_0^1 e^y dy=e-\left[e^y\right]_0^1=e-(e-1)=1$.

65. Using the formula for volumes of rotation and the figure, we see that Volume
$=\int_0^d \pi b^2 dy-\int_0^c \pi a^2 dy-\int_c^d \pi\,[g(y)]^2 dy=\pi b^2 d-\pi a^2 c-\int_c^d \pi\,[g(y)]^2 dy$. Let $y=f(x)$, which gives $dy=f'(x)dx$ and $g(y)=x$, so that $V=\pi b^2 d-\pi a^2 c-\pi\int_a^b x^2 f'(x)dx$. Now integrate by parts with $u=x^2$, and
$dv=f'(x)dx \Rightarrow du=2x\,dx$, $v=f(x)$, and

$\int_a^b x^2 f'(x)dx = \left[x^2 f(x)\right]_a^b - \int_a^b 2x f(x)dx = b^2 f(b) - a^2 f(a) - \int_a^b 2x f(x)dx$, but $f(a)=c$ and $f(b)=d \Rightarrow$
$V = \pi b^2 d - \pi a^2 c - \pi \left[b^2 d - a^2 c - \int_a^b 2xf(x)dx \right] = \int_a^b 2\pi x f(x)dx$.

66. **(a)** We note that for $0 \le x \le \frac{\pi}{2}$, $0 \le \sin x \le 1$, so $\sin^{2n+2} x \le \sin^{2n+1} x \le \sin^{2n} x$. So by the second Comparison Property of the Integral, $I_{2n+2} \le I_{2n+1} \le I_{2n}$.
(b) Substituting directly into the result from Exercise 44 , we get

$$\frac{I_{2n+2}}{I_{2n}} = \frac{\dfrac{1\cdot 3\cdot 5\cdot \cdots \cdot [2(n+1)-1]}{2\cdot 4\cdot 6\cdot \cdots \cdot [2(n+1)]}\dfrac{\pi}{2}}{\dfrac{1\cdot 3\cdot 5\cdot \cdots \cdot (2n-1)}{2\cdot 4\cdot 6\cdot \cdots \cdot (2n)}\dfrac{\pi}{2}} = \frac{2(n+1)-1}{2(n+1)} = \frac{2n+1}{2n+2}$$

(c) We divide the result from part (a) by I_{2n} . The inequalities are preserved since I_{2n} is positive:
$\frac{I_{2n+2}}{I_{2n}} \le \frac{I_{2n+1}}{I_{2n}} \le \frac{I_{2n}}{I_{2n}}$. Now from part (b), the left term is equal to $\frac{2n+1}{2n+2}$, so the expression becomes
$\frac{2n+1}{2n+2} \le \frac{I_{2n+1}}{I_{2n}} \le 1$. Now $\lim_{n\to\infty} \frac{2n+1}{2n+2} = \lim_{n\to\infty} 1 = 1$, so by the Squeeze Theorem, $\lim_{n\to\infty} \frac{I_{2n+1}}{I_{2n}} = 1$.
(d) We substitute the results from Exercises 43 and 44 into the result from part (c):

$$\begin{aligned} 1 &= \lim_{n\to\infty} \frac{I_{2n+1}}{I_{2n}} = \lim_{n\to\infty} \frac{\dfrac{2\cdot 4\cdot 6\cdot \cdots \cdot (2n)}{3\cdot 5\cdot 7\cdot \cdots \cdot (2n+1)}}{\dfrac{1\cdot 3\cdot 5\cdot \cdots \cdot (2n-1)}{2\cdot 4\cdot 6\cdot \cdots \cdot (2n)}\dfrac{\pi}{2}} \\ &= \lim_{n\to\infty} \left[\frac{2\cdot 4\cdot 6\cdot \cdots \cdot (2n)}{3\cdot 5\cdot 7\cdot \cdots \cdot (2n+1)} \right] \left[\frac{2\cdot 4\cdot 6\cdot \cdots \cdot (2n)}{1\cdot 3\cdot 5\cdot \cdots \cdot (2n-1)} \left(\frac{2}{\pi} \right) \right] \\ &= \lim_{n\to\infty} \frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdot \cdots \cdot \frac{2n}{2n-1}\cdot\frac{2n}{2n+1}\cdot\frac{2}{\pi} \qquad \text{[rearrange terms]} \end{aligned}$$

Multiplying both sides by $\frac{\pi}{2}$ gives us the *Wallis product* :

$$\frac{\pi}{2} = \frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdot \cdots$$

(e) The area of the k th rectangle is k . At the $2n$ th step, the area is increased from $2n-1$ to $2n$ by multiplying the width by $\frac{2n}{2n-1}$, and at the $(2n+1)$ th step, the area is increased from $2n$ to $2n+1$ by multiplying the height by $\frac{2n+1}{2n}$. These two steps multiply the ratio of width to height by $\frac{2n}{2n-1}$ and $\frac{1}{(2n+1)/(2n)}=\frac{2n}{2n+1}$ respectively. So, by part (d), the limiting ratio is $\frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdot\cdots=\frac{\pi}{2}$.

1.

$$\int \sin^3 x\cos^2 x\,dx = \int \sin^2 x\cos^2 x\sin x\,dx = \int \left(1-\cos^2 x\right)\cos^2 x\sin x\,dx = \int \left(1-u^2\right)u^2(-du)$$

$$= \int \left(u^2-1\right)u^2\,du = \int \left(u^4-u^2\right)du = \frac{1}{5}u^5 - \frac{1}{3}u^3 + C = \frac{1}{5}\cos^5 x - \frac{1}{3}\cos^3 x + C$$

2.

$$\int \sin^6 x\cos^3 x\,dx = \int \sin^6 x\cos^2 x\cos x\,dx = \int \sin^6 x\left(1-\sin^2 x\right)\cos x\,dx = \int u^6\left(1-u^2\right)du$$

$$= \int \left(u^6-u^8\right)du = \frac{1}{7}u^7 - \frac{1}{9}u^9 + C = \frac{1}{7}\sin^7 x - \frac{1}{9}\sin^9 x + C$$

3.

$$\int_{\pi/2}^{3\pi/4} \sin^5 x\cos^3 x\,dx = \int_{\pi/2}^{3\pi/4} \sin^5 x\cos^2 x\cos x\,dx = \int_{\pi/2}^{3\pi/4} \sin^5 x\left(1-\sin^2 x\right)\cos x\,dx$$

$$\overset{s}{=} \int_{1}^{\sqrt{2}/2} u^5\left(1-u^2\right)du = \int_{1}^{\sqrt{2}/2} \left(u^5-u^7\right)du = \left[\frac{1}{6}u^6 - \frac{1}{8}u^8\right]_1^{\sqrt{2}/2}$$

$$= \left(\frac{1/8}{6} - \frac{1/16}{8}\right) - \left(\frac{1}{6} - \frac{1}{8}\right) = -\frac{11}{384}$$

4.

$$\int_0^{\pi/2} \cos^5 x\,dx = \int_0^{\pi/2} \left(\cos^2 x\right)^2\cos x\,dx = \int_0^{\pi/2} \left(1-\sin^2 x\right)^2\cos x\,dx = \int_0^1 \left(1-u^2\right)^2 du$$

$$= \int_0^1 \left(1-2u^2+u^4\right)du = \left[u - \frac{2}{3}u^3 + \frac{1}{5}u^5\right]_0^1 = \left(1 - \frac{2}{3} + \frac{1}{5}\right) - 0 = \frac{8}{15}$$

5.

$$\int \cos^5 x\sin^4 x\,dx = \int \cos^4 x\sin^4 x\cos x\,dx = \int \left(1-\sin^2 x\right)^2\sin^4 x\cos x\,dx = \int \left(1-u^2\right)^2 u^4\,du$$

$$= \int \left(1-2u^2+u^4\right)u^4\,du = \int \left(u^4-2u^6+u^8\right)du = \frac{1}{5}u^5 - \frac{2}{7}u^7 + \frac{1}{9}u^9 + C$$

$$= \frac{1}{5}\sin^5 x - \frac{2}{7}\sin^7 x + \frac{1}{9}\sin^9 x + C$$

6.

$$\int \sin^3(mx)dx = \int \left(1-\cos^2 mx\right) \sin mx\,dx = -\frac{1}{m}\int \left(1-u^2\right) du \ [\ u=\cos mx\ ,\ du=-m\sin mx\,dx\ ]$$
$$= -\frac{1}{m}\left(u-\frac{1}{3}u^3\right)+C = -\frac{1}{m}\left(\cos mx-\frac{1}{3}\cos^3 mx\right)+C$$
$$= \frac{1}{3m}\cos^3 mx-\frac{1}{m}\cos mx+C$$

7.

$$\int_0^{\pi/2}\cos^2\theta\, d\theta = \int_0^{\pi/2}\frac{1}{2}(1+\cos 2\theta)d\theta \ \text{[half–angle identity]}$$
$$= \frac{1}{2}\left[\theta+\frac{1}{2}\sin 2\theta\right]_0^{\pi/2} = \frac{1}{2}\left[\left(\frac{\pi}{2}+0\right)-(0+0)\right] = \frac{\pi}{4}$$

8. $\int_0^{\pi/2}\sin^2(2\theta)d\theta = \int_0^{\pi/2}\frac{1}{2}(1-\cos 4\theta)d\theta = \frac{1}{2}\left[\theta-\frac{1}{4}\sin 4\theta\right]_0^{\pi/2} = \frac{1}{2}\left[\left(\frac{\pi}{2}-0\right)-(0-0)\right] = \frac{\pi}{4}$

9.

$$\int_0^{\pi}\sin^4(3t)dt = \int_0^{\pi}\left[\sin^2(3t)\right]^2 dt = \int_0^{\pi}\left[\frac{1}{2}(1-\cos 6t)\right]^2 dt = \frac{1}{4}\int_0^{\pi}(1-2\cos 6t+\cos^2 6t)dt$$
$$= \frac{1}{4}\int_0^{\pi}\left[1-2\cos 6t+\frac{1}{2}(1+\cos 12t)\right]dt = \frac{1}{4}\int_0^{\pi}\left(\frac{3}{2}-2\cos 6t+\frac{1}{2}\cos 12t\right)dt$$
$$= \frac{1}{4}\left[\frac{3}{2}t-\frac{1}{3}\sin 6t+\frac{1}{24}\sin 12t\right]_0^{\pi} = \frac{1}{4}\left[\left(\frac{3\pi}{2}-0+0\right)-(0-0+0)\right] = \frac{3\pi}{8}$$

10.

$$\int_0^{\pi}\cos^6\theta\, d\theta = \int_0^{\pi}(\cos^2\theta)^3 d\theta = \int_0^{\pi}\left[\frac{1}{2}(1+\cos 2\theta)\right]^3 d\theta = \frac{1}{8}\int_0^{\pi}(1+3\cos 2\theta+3\cos^2 2\theta+\cos^3 2\theta)d\theta$$
$$= \frac{1}{8}\left[\theta+\frac{3}{2}\sin 2\theta\right]_0^{\pi}+\frac{1}{8}\int_0^{\pi}\left[\frac{3}{2}(1+\cos 4\theta)\right]d\theta+\frac{1}{8}\int_0^{\pi}\left[(1-\sin^2 2\theta)\cos 2\theta\right]d\theta$$
$$= \frac{1}{8}\pi+\frac{3}{16}\left[\theta+\frac{1}{4}\sin 4\theta\right]_0^{\pi}+\frac{1}{8}\int_0^{0}(1-u^2)\left(\frac{1}{2}du\right) \ [\ u=\sin 2\theta\ ,\ du=2\cos 2\theta\, d\theta\ ]$$
$$= \frac{\pi}{8}+\frac{3\pi}{16}+0 = \frac{5\pi}{16}$$

11.

$$\int (1+\cos\theta)^2 d\theta = \int (1+2\cos\theta+\cos^2\theta)\,d\theta = \theta+2\sin\theta+\frac{1}{2}\int(1+\cos 2\theta)\,d\theta$$

$$=\theta+2\sin\theta+\frac{1}{2}\theta+\frac{1}{4}\sin 2\theta+C=\frac{3}{2}\theta+2\sin\theta+\frac{1}{4}\sin 2\theta+C$$

12. Let $u=x$, $dv=\cos^2 x\,dx \Rightarrow du=dx$, $v=\int\cos^2 x\,dx=\int\frac{1}{2}(1+\cos 2x)dx=\frac{1}{2}x+\frac{1}{4}\sin 2x$, so

$$\int x\cos^2 x\,dx = x\left(\frac{1}{2}x+\frac{1}{4}\sin 2x\right)-\int\left(\frac{1}{2}x+\frac{1}{4}\sin 2x\right)dx=\frac{1}{2}x^2+\frac{1}{4}x\sin 2x-\frac{1}{4}x^2+\frac{1}{8}\cos 2x+C$$

$$=\frac{1}{4}x^2+\frac{1}{4}x\sin 2x+\frac{1}{8}\cos 2x+C$$

13.

$$\int_0^{\pi/4}\sin^4 x\cos^2 x\,dx=\int_0^{\pi/4}\sin^2 x(\sin x\cos x)^2dx=\int_0^{\pi/4}\frac{1}{2}(1-\cos 2x)\left(\frac{1}{2}\sin 2x\right)^2dx$$

$$=\frac{1}{8}\int_0^{\pi/4}(1-\cos 2x)\sin^2 2x\,dx=\frac{1}{8}\int_0^{\pi/4}\sin^2 2x\,dx-\frac{1}{8}\int_0^{\pi/4}\sin^2 2x\cos 2x\,dx$$

$$=\frac{1}{16}\int_0^{\pi/4}(1-\cos 4x)dx-\frac{1}{16}\left[\frac{1}{3}\sin^3 2x\right]_0^{\pi/4}=\frac{1}{16}\left[x-\frac{1}{4}\sin 4x-\frac{1}{3}\sin^3 2x\right]_0^{\pi/4}$$

$$=\frac{1}{16}\left(\frac{\pi}{4}-0-\frac{1}{3}\right)=\frac{1}{192}(3\pi-4)$$

14.

$$\int_0^{\pi/2}\sin^2 x\cos^2 x\,dx=\int_0^{\pi/2}\frac{1}{4}\left(4\sin^2 x\cos^2 x\right)dx=\int_0^{\pi/2}\frac{1}{4}(2\sin x\cos x)^2dx=\frac{1}{4}\int_0^{\pi/2}\sin^2 2x\,dx$$

$$=\frac{1}{4}\int_0^{\pi/2}\frac{1}{2}(1-\cos 4x)dx=\frac{1}{8}\int_0^{\pi/2}(1-\cos 4x)dx=\frac{1}{8}\left[x-\frac{1}{4}\sin 4x\right]_0^{\pi/2}$$

$$=\frac{1}{8}\left(\frac{\pi}{2}\right)=\frac{\pi}{16}$$

15.

$$\int\sin^3 x\sqrt{\cos x}\,dx=\int\left(1-\cos^2 x\right)\sqrt{\cos x}\sin x\,dx=\int\left(1-u^2\right)u^{1/2}(-du)=\int\left(u^{5/2}-u^{1/2}\right)du$$

$$=\frac{2}{7}u^{7/2}-\frac{2}{3}u^{3/2}+C=\frac{2}{7}(\cos x)^{7/2}-\frac{2}{3}(\cos x)^{3/2}+C$$

$$=\left(\frac{2}{7}\cos^3 x-\frac{2}{3}\cos x\right)\sqrt{\cos x}+C$$

16. Let $u=\sin\theta$. Then $du=\cos\theta\, d\theta$ and

$$\int \cos\theta \cos^5(\sin\theta)d\theta =\int \cos^5 u\, du=\int (\cos^2 u)^2 \cos u\, du=\int (1-\sin^2 u)^2 \cos u\, du$$
$$=\int (1-2\sin^2 u+\sin^4 u)\cos u\, du=I$$

Now let $x=\sin u$. Then $dx=\cos u\,du$ and

$$I =\int (1-2x^2+x^4)dx=x-\frac{2}{3}x^3+\frac{1}{5}x^5+C=\sin u-\frac{2}{3}\sin^3 u+\frac{1}{5}\sin^5 u+C$$
$$=\sin(\sin\theta)-\frac{2}{3}\sin^3(\sin\theta)+\frac{1}{5}\sin^5(\sin\theta)+C$$

17.

$$\int \cos^2 x\tan^3 x\,dx =\int \frac{\sin^3 x}{\cos x}\,dx=\int \frac{\left(1-u^2\right)(-du)}{u} =\int \left[\frac{-1}{u}+u \right] du$$
$$=-\ln|u|+\frac{1}{2}u^2+C=\frac{1}{2}\cos^2 x-\ln|\cos x|+C$$

18.

$$\int \cot^5\theta \sin^4\theta\, d\theta =\int \frac{\cos^5\theta}{\sin^5\theta}\sin^4\theta\, d\theta=\int \frac{\cos^5\theta}{\sin\theta}\, d\theta=\int \frac{\cos^4\theta}{\sin\theta}\cos\theta\, d\theta=\int \frac{\left(1-\sin^2\theta\right)^2}{\sin\theta}\cos\theta\, d\theta$$
$$=\int \frac{\left(1-u^2\right)^2}{u}\,du=\int \frac{1-2u^2+u^4}{u}\,du=\int \left(\frac{1}{u}-2u+u^3\right) du$$
$$=\ln|u|-u^2+\frac{1}{4}u^4+C=\ln|\sin\theta|-\sin^2\theta+\frac{1}{4}\sin^4\theta+C$$

19.

$$\int \frac{1-\sin x}{\cos x}\,dx =\int (\sec x-\tan x)dx=\ln|\sec x+\tan x|-\ln|\sec x|+C \quad \left[\text{by (1) and the boxed formula above it}\right]$$
$$=\ln|(\sec x+\tan x)\cos x|+C=\ln|1+\sin x|+C$$
$$=\ln(1+\sin x)+C \text{ since } 1+\sin x\geq 0$$

Or:

$$\int \frac{1-\sin x}{\cos x}\,dx =\int \frac{1-\sin x}{\cos x}\cdot\frac{1+\sin x}{1+\sin x}\,dx=\int \frac{\left(1-\sin^2 x\right)dx}{\cos x(1+\sin x)}=\int \frac{\cos x\,dx}{1+\sin x}$$

$=\int \frac{dw}{w}$ [where $w=1+\sin x$, $dw=\cos x\,dx$]

$=\ln|w|+C=\ln|1+\sin x|+C=\ln(1+\sin x)+C$

20. $\int \cos^2 x \sin 2x\,dx=2\int \cos^3 x \sin x\,dx=-2\int u^3 du=-\frac{1}{2}u^4+C=-\frac{1}{2}\cos^4 x+C$

21. Let $u=\tan x$, $du=\sec^2 x\,dx$. Then $\int \sec^2 x \tan x\,dx=\int u\,du=\frac{1}{2}u^2+C=\frac{1}{2}\tan^2 x+C$.

Or: Let $v=\sec x$, $dv=\sec x\tan x\,dx$. Then $\int \sec^2 x\tan x\,dx=\int v\,dv=\frac{1}{2}v^2+C=\frac{1}{2}\sec^2 x+C$.

22.

$\int_0^{\pi/2}\sec^4(t/2)\,dt=\int_0^{\pi/4}\sec^4 x(2\,dx)$ [$x=t/2$, $dx=\frac{1}{2}dt$] $=2\int_0^{\pi/4}\sec^2 x(1+\tan^2 x)\,dx$

$=2\int_0^1(1+u^2)\,du$ [$u=\tan x$, $du=\sec^2 x\,dx$] $=2\left[u+\frac{1}{3}u^3\right]_0^1=2\left(1+\frac{1}{3}\right)=\frac{8}{3}$

23. $\int \tan^2 x\,dx=\int\left(\sec^2 x-1\right)dx=\tan x-x+C$

24. $\int \tan^4 x\,dx=\int \tan^2 x\left(\sec^2 x-1\right)dx=\int \tan^2 x\sec^2 x\,dx-\int \tan^2 x\,dx=\frac{1}{3}\tan^3 x-\tan x+x+C$

(Set $u=\tan x$ in the first integral and use Exercise 23 for the second.)

25.

$\int \sec^6 t\,dt=\int \sec^4 t\cdot\sec^2 t\,dt=\int(\tan^2 t+1)^2\sec^2 t\,dt=\int(u^2+1)^2\,du$

$=\int(u^4+2u^2+1)\,du=\frac{1}{5}u^5+\frac{2}{3}u^3+u+C=\frac{1}{5}\tan^5 t+\frac{2}{3}\tan^3 t+\tan t+C$

26.

$\int_0^{\pi/4}\sec^4\theta\tan^4\theta\,d\theta=\int_0^{\pi/4}(\tan^2\theta+1)\tan^4\theta\sec^2\theta\,d\theta=\int_0^1(u^2+1)u^4\,du$

$=\int_0^1(u^6+u^4)\,du=\left[\frac{1}{7}u^7+\frac{1}{5}u^5\right]_0^1=\frac{1}{7}+\frac{1}{5}=\frac{12}{35}$

27.

$\int_0^{\pi/3}\tan^5 x\sec^4 x\,dx=\int_0^{\pi/3}\tan^5 x(\tan^2 x+1)\sec^2 x\,dx$

$$=\int_0^{\sqrt{3}} u^5(u^2+1)\,du$$

$$=\int_0^{\sqrt{3}} (u^7+u^5)\,du=\left[\frac{1}{8}u^8+\frac{1}{6}u^6\right]_0^{\sqrt{3}}=\frac{81}{8}+\frac{27}{6}=\frac{81}{8}+\frac{9}{2}=\frac{81}{8}+\frac{36}{8}=\frac{117}{8}$$

Alternate solution:

$$\int_0^{\pi/3}\tan^5 x\sec^4 x\,dx=\int_0^{\pi/3}\tan^4 x\sec^3 x\sec x\tan x\,dx=\int_0^{\pi/3}(\sec^2 x-1)^2\sec^3 x\sec x\tan x\,dx$$

$$=\int_1^2 (u^2-1)^2u^3\,du$$

$$=\int_1^2 (u^4-2u^2+1)u^3\,du=\int_1^2 (u^7-2u^5+u^3)\,du$$

$$=\left[\frac{1}{8}u^8-\frac{1}{3}u^6+\frac{1}{4}u^4\right]_1^2=\left(32-\frac{64}{3}+4\right)-\left(\frac{1}{8}-\frac{1}{3}+\frac{1}{4}\right)=\frac{117}{8}$$

28.

$$\int\tan^3(2x)\sec^5(2x)\,dx=\int\tan^2(2x)\sec^4(2x)\cdot\sec(2x)\tan(2x)\,dx$$

$$=\int(u^2-1)u^4\left(\frac{1}{2}\,du\right)\ [\,u=\sec(2x), du=2\sec(2x)\tan(2x)\,dx\,]$$

$$=\frac{1}{2}\int(u^6-u^4)\,du=\frac{1}{14}u^7-\frac{1}{10}u^5+C=\frac{1}{14}\sec^7(2x)-\frac{1}{10}\sec^5(2x)+C$$

29.

$$\int\tan^3 x\sec x\,dx=\int\tan^2 x\sec x\tan x\,dx=\int\left(\sec^2 x-1\right)\sec x\tan x\,dx$$

$$=\int(u^2-1)\,du$$

$$=\frac{1}{3}u^3-u+C=\frac{1}{3}\sec^3 x-\sec x+C$$

30.

$$\int_0^{\pi/3}\tan^5 x\sec^6 x\,dx=\int_0^{\pi/3}\tan^5 x\sec^4 x\sec^2 x\,dx=\int_0^{\pi/3}\tan^5 x\left(1+\tan^2 x\right)^2\sec^2 x\,dx$$

$$=\int_0^{\sqrt{3}} u^5\left(1+u^2\right)^2du\ [\,u=\tan x,\ du=\sec^2 x\,dx\,]=\int_0^{\sqrt{3}} u^5\left(1+2u^2+u^4\right)du$$

$$=\int_0^{\sqrt{3}}\left(u^5+2u^7+u^9\right)du=\left[\frac{1}{6}u^6+\frac{1}{4}u^8+\frac{1}{10}u^{10}\right]_0^{\sqrt{3}}=\frac{27}{6}+\frac{81}{4}+\frac{243}{10}=\frac{981}{20}$$

Alternate solution:

$$\int_0^{\pi/3}\tan^5 x\sec^6 x\,dx=\int_0^{\pi/3}\tan^4 x\sec^5 x\sec x\tan x\,dx=\int_0^{\pi/3}\left(\sec^2 x-1\right)^2\sec^5 x\sec x\tan x\,dx$$

$$=\int_1^2\left(u^2-1\right)^2u^5\,du\ [\ u=\sec x\ ,\ du=\sec x\tan x\,dx\ ]$$

$$=\int_1^2\left(u^4-2u^2+1\right)u^5\,du=\int_1^2\left(u^9-2u^7+u^5\right)du$$

$$=\left[\frac{1}{10}u^{10}-\frac{1}{4}u^8+\frac{1}{6}u^6\right]_1^2=\left(\frac{512}{5}-64+\frac{32}{3}\right)-\left(\frac{1}{10}-\frac{1}{4}+\frac{1}{6}\right)=\frac{981}{20}$$

31.

$$\int\tan^5 x\,dx=\int\left(\sec^2 x-1\right)^2\tan x\,dx=\int\sec^4 x\tan x\,dx-2\int\sec^2 x\tan x\,dx+\int\tan x\,dx$$

$$=\int\sec^3 x\sec x\tan x\,dx-2\int\tan x\sec^2 x\,dx+\int\tan x\,dx$$

$$=\frac{1}{4}\sec^4 x-\tan^2 x+\ln|\sec x|+C\ [\text{ or }\ \frac{1}{4}\sec^4 x-\sec^2 x+\ln|\sec x|+C\ ]$$

32.

$$\int\tan^6 ay\,dy=\int\tan^4 ay\left(\sec^2 ay-1\right)dy=\int\tan^4 ay\sec^2 ay\,dy-\int\tan^4 ay\,dy$$

$$=\frac{1}{5a}\tan^5 ay-\int\tan^2 ay\left(\sec^2 ay-1\right)dy$$

$$=\frac{1}{5a}\tan^5 ay-\int\tan^2 ay\sec^2 ay\,dy+\int\left(\sec^2 ay-1\right)dy$$

$$=\frac{1}{5a}\tan^5 ay-\frac{1}{3a}\tan^3 ay+\frac{1}{a}\tan ay-y+C$$

33.

$$\int\frac{\tan^3\theta}{\cos^4\theta}\,d\theta=\int\tan^3\theta\sec^4\theta\,d\theta=\int\tan^3\theta\cdot(\tan^2\theta+1)\cdot\sec^2\theta\,d\theta$$

$$=\int u^3(u^2+1)\,du u=\tan\theta\ ,\ du=\sec^2\theta\,d\theta\ ]$$

$$=\int(u^5+u^3)\,du=\frac{1}{6}u^6+\frac{1}{4}u^4+C=\frac{1}{6}\tan^6\theta+\frac{1}{4}\tan^4\theta+C$$

34.

$$\int\tan^2 x\sec x\,dx=\int\left(\sec^2 x-1\right)\sec x\,dx=\int\sec^3 x\,dx-\int\sec x\,dx$$

$$=\frac{1}{2}\left(\sec x\tan x+\ln\left|\sec x+\tan x\right|\right)-\ln\left|\sec x+\tan x\right|+C \text{ [by Example 8 and (1)]}$$

$$=\frac{1}{2}\left(\sec x\tan x-\ln\left|\sec x+\tan x\right|\right)+C$$

35. $\int_{\pi/6}^{\pi/2}\cot^2 x\,dx=\int_{\pi/6}^{\pi/2}\left(\csc^2 x-1\right)dx=[-\cot x-x]_{\pi/6}^{\pi/2}=\left(0-\frac{\pi}{2}\right)-\left(-\sqrt{3}-\frac{\pi}{6}\right)=\sqrt{3}-\frac{\pi}{3}$

36.

$$\int_{\pi/4}^{\pi/2}\cot^3 x\,dx=\int_{\pi/4}^{\pi/2}\cot x\left(\csc^2 x-1\right)dx=\int_{\pi/4}^{\pi/2}\cot x\csc^2 x\,dx-\int_{\pi/4}^{\pi/2}\frac{\cos x}{\sin x}\,dx$$

$$=\left[-\frac{1}{2}\cot^2 x-\ln\left|\sin x\right|\right]_{\pi/4}^{\pi/2}=(0-\ln 1)-\left[-\frac{1}{2}-\ln\frac{1}{\sqrt{2}}\right]=\frac{1}{2}+\ln\frac{1}{\sqrt{2}}=\frac{1}{2}(1-\ln 2)$$

37.

$$\int\cot^3\alpha\csc^3\alpha\,d\alpha=\int\cot^2\alpha\csc^2\alpha\cdot\csc\alpha\cot\alpha\,d\alpha=\int(\csc^2\alpha-1)\csc^2\alpha\cdot\csc\alpha\cot\alpha\,d\alpha$$

$$=\int(u^2-1)u^2\cdot(-du)\ [\,u=\csc\alpha,\ du=-\csc\alpha\cot\alpha\,d\alpha\,]$$

$$=\int(u^2-u^4)du=\frac{1}{3}u^3-\frac{1}{5}u^5+C=\frac{1}{3}\csc^3\alpha-\frac{1}{5}\csc^5\alpha+C$$

38.

$$\int\csc^4 x\cot^6 x\,dx=\int\cot^6 x(\cot^2 x+1)\csc^2 x\,dx$$

$$=\int u^6(u^2+1)\cdot(-du)\ [\,u=\cot x,\ du=-\csc^2 x\,dx\,]$$

$$=\int(-u^8-u^6)du=-\frac{1}{9}u^9-\frac{1}{7}u^7+C=-\frac{1}{9}\cot^9 x-\frac{1}{7}\cot^7 x+C$$

39. $I=\int\csc x\,dx=\int\frac{\csc x(\csc x-\cot x)}{\csc x-\cot x}\,dx=\int\frac{-\csc x\cot x+\csc^2 x}{\csc x-\cot x}\,dx$. Let $u=\csc x-\cot x\Rightarrow$ $du=\left(-\csc x\cot x+\csc^2 x\right)dx$. Then $I=\int du/u=\ln|u|=\ln\left|\csc x-\cot x\right|+C$.

40.

$$\int\csc^4 x\cot^6 x\,dx=\int\cot^6 x(\cot^2 x+1)\csc^2 x\,dx$$

$$=\int u^6(u^2+1)\cdot(-du)\ [\,u=\cot x,\ du=-\csc^2 x\,dx\,]$$

$$=\int(-u^8-u^6)du=-\frac{1}{9}u^9-\frac{1}{7}u^7+C=-\frac{1}{9}\cot^9 x-\frac{1}{7}\cot^7 x+C$$

41. Use Equation 2(b):

$$\int \sin 5x\sin 2x\,dx=\int\frac{1}{2}\left[\cos(5x-2x)-\cos(5x+2x)\right]dx=\frac{1}{2}\int(\cos 3x-\cos 7x)\,dx$$
$$=\frac{1}{6}\sin 3x-\frac{1}{14}\sin 7x+C$$

42. Use Equation 2(a):

$$\int \sin 3x\cos x\,dx=\int\frac{1}{2}\left[\sin(3x+x)+\sin(3x-x)\right]dx=\frac{1}{2}\int(\sin 4x+\sin 2x)\,dx$$
$$=-\frac{1}{8}\cos 4x-\frac{1}{4}\cos 2x+C$$

43. Use Equation 2(c):

$$\int\cos 7\theta\cos 5\theta\,d\theta=\int\frac{1}{2}\left[\cos(7\theta-5\theta)+\cos(7\theta+5\theta)\right]d\theta=\frac{1}{2}\int(\cos 2\theta+\cos 12\theta)\,d\theta$$
$$=\frac{1}{2}\left(\frac{1}{2}\sin 2\theta+\frac{1}{12}\sin 12\theta\right)+C=\frac{1}{4}\sin 2\theta+\frac{1}{24}\sin 12\theta+C$$

44.

$$\int\frac{\cos x+\sin x}{\sin 2x}\,dx=\frac{1}{2}\int\frac{\cos x+\sin x}{\sin x\cos x}\,dx=\frac{1}{2}\int(\csc x+\sec x)\,dx$$
$$=\frac{1}{2}\left(\ln|\csc x-\cot x|+\ln|\sec x+\tan x|\right)+C\quad\text{[by Exercise 39 and (1)]}$$

45. $\displaystyle\int\frac{1-\tan^2 x}{\sec^2 x}\,dx=\int\left(\cos^2 x-\sin^2 x\right)dx=\int\cos 2x\,dx=\frac{1}{2}\sin 2x+C$

46.

$$\int\frac{dx}{\cos x-1}=\int\frac{1}{\cos x-1}\cdot\frac{\cos x+1}{\cos x+1}\,dx=\int\frac{\cos x+1}{\cos^2 x-1}\,dx=\int\frac{\cos x+1}{-\sin^2 x}\,dx$$
$$=\int\left(-\cot x\csc x-\csc^2 x\right)dx=\csc x+\cot x+C$$

47. Let $u=\tan(t^2)\Rightarrow du=2t\sec^2(t^2)\,dt$. Then

$$\int t\sec^2\left(t^2\right)\tan^4\left(t^2\right)dt=\int u^4\left(\frac{1}{2}\,du\right)=\frac{1}{10}\,u^5+C=\frac{1}{10}\,\tan^5(t^2)+C\ .$$

48. Let $u=\tan^7 x$, $dv=\sec x\tan x dx\Rightarrow du=7\tan^6 x\sec^2 x dx$, $v=\sec x$. Then

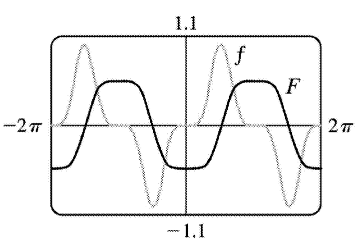
$$\begin{aligned}\int\tan^8 x\sec x dx &=\int\tan^7 x\cdot\sec x\tan x dx=\tan^7 x\sec x-\int 7\tan^6 x\sec^2 x\sec x dx\\ &=\tan^7 x\sec x-7\int\tan^6 x\left(\tan^2 x+1\right)\sec x dx\\ &=\tan^7 x\sec x-7\int\tan^8 x\sec x dx-7\int\tan^6 x\sec x dx\ .\end{aligned}$$

Thus, $8\int\tan^8 x\sec x dx=\tan^7 x\sec x-7\int\tan^6 x\sec x dx$ and $\int_0^{\pi/4}\tan^8 x\sec x$ $dx=\frac{1}{8}\left[\tan^7 x\sec x\right]_0^{\pi/4}-\frac{7}{8}\int_0^{\pi/4}\tan^6 x\sec x dx=\frac{\sqrt{2}}{8}-\frac{7}{8}I$.

49. Let $u=\cos x\Rightarrow du=-\sin x dx$. Then

$$\begin{aligned}\int\sin^5 x dx &= \int\left(1-\cos^2 x\right)^2\sin x dx=\int\left(1-u^2\right)^2(-du)\\ &= \int\left(-1+2u^2-u^4\right)du=-\frac{1}{5}u^5+\frac{2}{3}u^3-u+C\\ &= -\frac{1}{5}\cos^5 x+\frac{2}{3}\cos^3 x-\cos x+C\end{aligned}$$

Notice that F is increasing when $f(x)>0$, so the graphs serve as a check on our work.

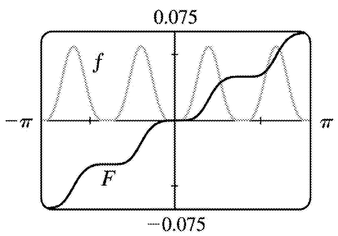


50.

$$\begin{aligned}\int\sin^4 x\cos^4 x dx &=\int\left(\frac{1}{2}\sin 2x\right)^4 dx=\frac{1}{16}\int\sin^4 2x dx=\frac{1}{16}\int\left[\frac{1}{2}\,(1-\cos 4x)\right]^2 dx\\ &=\frac{1}{64}\int\left(1-2\cos 4x+\cos^2 4x\right)dx\\ &=\frac{1}{64}\left(x-\frac{1}{2}\sin 4x\right)+\frac{1}{128}\int(1+\cos 8x)dx\end{aligned}$$

$$=\frac{1}{64}\left(x-\frac{1}{2}\sin 4x\right)+\frac{1}{128}\left(x+\frac{1}{8}\sin 8x\right)+C$$
$$=\frac{3}{128}x-\frac{1}{128}\sin 4x+\frac{1}{1024}\sin 8x+C$$

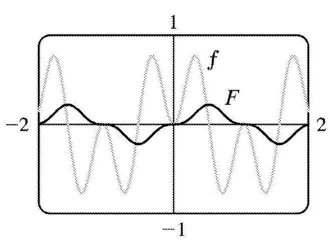
Notice that $f(x)=0$ whenever F has a horizontal tangent.



51.

$$\int \sin 3x\sin 6x\,dx=\int \frac{1}{2}\left[\cos(3x-6x)-\cos(3x+6x)\right]dx$$
$$=\frac{1}{2}\int(\cos 3x-\cos 9x)dx$$
$$=\frac{1}{6}\sin 3x-\frac{1}{18}\sin 9x+C$$

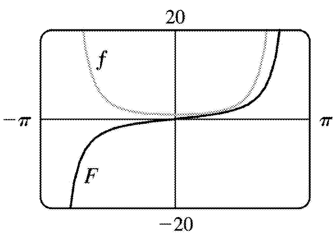
Notice that $f(x)=0$ whenever F has a horizontal tangent.



52.

$$\int \sec^4\frac{x}{2}\,dx=\int\left(\tan^2\frac{x}{2}+1\right)\sec^2\frac{x}{2}\,dx$$
$$=\int\left(u^2+1\right)2du\ \left[u=\tan\frac{x}{2},\ du=\frac{1}{2}\sec^2\frac{x}{2}\,dx\right]$$
$$=\frac{2}{3}u^3+2u+C=\frac{2}{3}\tan^3\frac{x}{2}+2\tan\frac{x}{2}+C$$

Notice that F is increasing and f is positive on the intervals on which they are defined. Also, F has no horizontal tangent and f is never zero.



53.

$$f_{\text{ave}} = \frac{1}{2\pi}\int_{-\pi}^{\pi}\sin^2 x\cos^3 x\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi}\sin^2 x\left(1-\sin^2 x\right)\cos x\,dx$$

$$= \frac{1}{2\pi}\int_0^0 u^2\left(1-u^2\right)du \text{ [where } u=\sin x \text{]}$$

$$=0$$

54. **(a)** Let $u=\cos x$. Then $du=-\sin x\,dx \Rightarrow \int \sin x\cos x\,dx = \int u(-du) = -\frac{1}{2}u^2+C = -\frac{1}{2}\cos^2 x+C_1$.

(b) Let $u=\sin x$. Then $du=\cos x\,dx \Rightarrow \int \sin x\cos x\,dx = \int u\,du = \frac{1}{2}u^2+C = \frac{1}{2}\sin^2 x+C_2$.

(c) $\int \sin x\cos x\,dx = \int \frac{1}{2}\sin 2x\,dx = -\frac{1}{4}\cos 2x+C_3$

(d) Let $u=\sin x$, $dv=\cos x\,dx$. Then $du=\cos x\,dx$, $v=\sin x$,

so $\int \sin x\cos x\,dx = \sin^2 x - \int \sin x\cos x\,dx$, by Equation .1.2, so $\int \sin x\cos x\,dx = \frac{1}{2}\sin^2 x+C_4$.

The answers differ from one another by constants. Since
$\cos 2x = 1-2\sin^2 x = 2\cos^2 x-1$, we find that $-\frac{1}{4}\cos 2x = \frac{1}{2}\sin^2 x-\frac{1}{4} = -\frac{1}{2}\cos^2 x+\frac{1}{4}$.

55. For $0<x<\frac{\pi}{2}$, we have $0<\sin x<1$, so $\sin^3 x<\sin x$. Hence the area is
$\int_0^{\pi/2}\left(\sin x-\sin^3 x\right)dx = \int_0^{\pi/2}\sin x\left(1-\sin^2 x\right)dx = \int_0^{\pi/2}\cos^2 x\sin x\,dx$. Now let $u=\cos x \Rightarrow du=-\sin x\,dx$.
Then area $=\int_1^0 u^2(-du) = \int_0^1 u^2\,du = \left[\frac{1}{3}u^3\right]_0^1 = \frac{1}{3}$.

56. $\sin x>0$ for $0<x<\frac{\pi}{2}$, so the sign of $2\sin^2 x-\sin x$ [which equals $2\sin x\left(\sin x-\frac{1}{2}\right)$] is the same as that of $\sin x-\frac{1}{2}$. Thus $2\sin^2 x-\sin x$ is positive on $\left(\frac{\pi}{6},\frac{\pi}{2}\right)$ and negative on $\left(0,\frac{\pi}{6}\right)$. The desired area is

$$\int_0^{\pi/6}\left(\sin x - 2\sin^2 x\right)dx + \int_{\pi/6}^{\pi/2}\left(2\sin^2 x - \sin x\right)dx$$

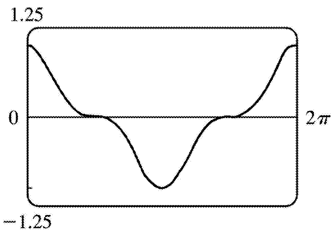
$$= \int_0^{\pi/6}(\sin x - 1 + \cos 2x)\,dx + \int_{\pi/6}^{\pi/2}(1 - \cos 2x - \sin x)\,dx$$

$$= \left[-\cos x - x + \frac{1}{2}\sin 2x\right]_0^{\pi/6} + \left[x - \frac{1}{2}\sin 2x + \cos x\right]_{\pi/6}^{\pi/2}$$

$$= -\frac{\sqrt{3}}{2} - \frac{\pi}{6} + \frac{\sqrt{3}}{4} - (-1) + \frac{\pi}{2} - \left(\frac{\pi}{6} - \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{2}\right)$$

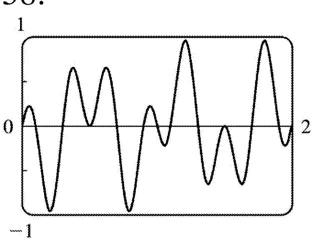
$$= 1 + \frac{\pi}{6} - \frac{\sqrt{3}}{2}$$

57.



It seems from the graph that $\int_0^{2\pi}\cos^3 x\,dx = 0$, since the area below the x –axis and above the graph looks about equal to the area above the axis and below the graph. By Example 1, the integral is $\left[\sin x - \frac{1}{3}\sin^3 x\right]_0^{2\pi} = 0$. Note that due to symmetry, the integral of any odd power of $\sin x$ or $\cos x$ between limits which differ by $2n\pi$ (n any integer) is 0 .

58.



It seems from the graph that $\int_0^2 \sin 2\pi x \cos 5\pi x\,dx = 0$, since each bulge above the x– axis seems to have a corresponding depression below the x– axis. To evaluate the integral, we use a trigonometric identity:

$$\int_0^1 \sin 2\pi x \cos 5\pi x\,dx = \frac{1}{2}\int_0^2\left[\sin(2\pi x - 5\pi x) + \sin(2\pi x + 5\pi x)\right]dx$$

$$= \frac{1}{2}\int_0^2\left[\sin(-3\pi x) + \sin 7\pi x\right]dx$$

$$= \frac{1}{2}\left[\frac{1}{3\pi}\cos(-3\pi x) - \frac{1}{7\pi}\cos 7\pi x\right]_0^2$$
$$= \frac{1}{2}\left[\frac{1}{3\pi}(1-1) - \frac{1}{7\pi}(1-1)\right] = 0$$

59. $V = \int_{\pi/2}^{\pi} \pi \sin^2 x\,dx = \pi \int_{\pi/2}^{\pi} \frac{1}{2}(1-\cos 2x)\,dx = \pi\left[\frac{1}{2}x - \frac{1}{4}\sin 2x\right]_{\pi/2}^{\pi} = \pi\left(\frac{\pi}{2} - 0 - \frac{\pi}{4} + 0\right) = \frac{\pi^2}{4}$

60.

Volume $= \int_0^{\pi/4} \pi\left(\tan^2 x\right)^2 dx = \pi\int_0^{\pi/4} \tan^2 x\left(\sec^2 x - 1\right)dx = \pi\int_0^{\pi/4}\tan^2 x\sec^2 x\,dx - \pi\int_0^{\pi/4}\tan^2 x\,dx$

$$= \pi\int_0^{\pi/4} u^2\,du - \pi\int_0^{\pi/4}\left(\sec^2 x - 1\right)dx$$
$$= \pi\left[\frac{1}{3}u^3\right]_{x=0}^{\pi/4} - \pi\left[\tan x - x\right]_0^{\pi/4} = \pi\left[\frac{1}{3}\tan^3 x - \tan x + x\right]_0^{\pi/4} = \pi\left[\frac{1}{3} - 1 + \frac{\pi}{4}\right] = \pi\left(\frac{\pi}{4} - \frac{2}{3}\right)$$

61.

Volume $= \pi\int_0^{\pi/2}\left[(1+\cos x)^2 - 1^2\right]dx = \pi\int_0^{\pi/2}\left(2\cos x + \cos^2 x\right)dx$

$$= \pi\left[2\sin x + \frac{1}{2}x + \frac{1}{4}\sin 2x\right]_0^{\pi/2} = \pi\left(2 + \frac{\pi}{4}\right) = 2\pi + \frac{\pi^2}{4}$$

62.

Volume $= \pi\int_0^{\pi/2}\left[1^2 - (1-\cos x)^2\right]dx = \pi\int_0^{\pi/2}\left(2\cos x - \cos^2 x\right)dx$

$$= \pi\left[2\sin x - \frac{1}{2}x - \frac{1}{4}\sin 2x\right]_0^{\pi/2} = \pi\left[\left(2 - \frac{\pi}{4} - 0\right) - 0\right] = 2\pi - \frac{\pi^2}{4}$$

63. $s = f(t) = \int_0^t \sin\omega u\cos^2\omega u\,du$. Let $y = \cos\omega u \Rightarrow dy = -\omega\sin\omega u\,du$. Then
$s = -\frac{1}{\omega}\int_1^{\cos\omega t} y^2\,dy = -\frac{1}{\omega}\left[\frac{1}{3}y^3\right]_1^{\cos\omega t} = \frac{1}{3\omega}\left(1 - \cos^3\omega t\right)$.

64. **(a)** We want to calculate the square root of the average value of $[E(t)]^2 = [155\sin(120\pi t)]^2 = 155^2\sin^2(120\pi t)$. First, we calculate the average value itself, by integrating $[E(t)]^2$ over one cycle (between $t=0$ and $t=\frac{1}{60}$, since there are 60 cycles per second) and

dividing by $\left(\frac{1}{60}-0\right)$:

$$[E(t)]^2_{\text{ave}} = \frac{1}{1/60}\int_0^{1/60}\left[155^2\sin^2(120\pi t)\right]dt = 60\cdot 155^2\int_0^{1/60}\frac{1}{2}\,[1-\cos(240\pi t)]\,dt$$

$$= 60\cdot 155^2\left(\frac{1}{2}\right)\left[t-\frac{1}{240\pi}\sin(240\pi t)\right]_0^{1/60} = 60\cdot 155^2\left(\frac{1}{2}\right)\left[\left(\frac{1}{60}-0\right)-(0-0)\right] = \frac{155^2}{2}$$

The RMS value is just the square root of this quantity, which is $\frac{155}{\sqrt{2}}\approx 110$ V.

(b) $220=\sqrt{[E(t)]^2_{\text{ave}}}\Rightarrow$

$$220^2 = [E(t)]^2_{\text{ave}} = \frac{1}{1/60}\int_0^{1/60}A^2\sin^2(120\pi t)\,dt = 60A^2\int_0^{1/60}\frac{1}{2}\,[1-\cos(240\pi t)]\,dt$$

$$= 30A^2\left[t-\frac{1}{240\pi}\sin(240\pi t)\right]_0^{1/60} = 30A^2\left[\left(\frac{1}{60}-0\right)-(0-0)\right] = \frac{1}{2}A^2$$

Thus, $220^2=\frac{1}{2}A^2\Rightarrow A=220\sqrt{2}\approx 311$ V.

65. Just note that the integrand is odd .
Or: If $m\neq n$, calculate

$$\int_{-\pi}^{\pi}\sin mx\cos nx\,dx = \int_{-\pi}^{\pi}\frac{1}{2}\,[\sin(m-n)x+\sin(m+n)x]\,dx$$

$$= \frac{1}{2}\left[-\frac{\cos(m-n)x}{m-n}-\frac{\cos(m+n)x}{m+n}\right]_{-\pi}^{\pi}=0$$

If $m=n$, then the first term in each set of brackets is zero.

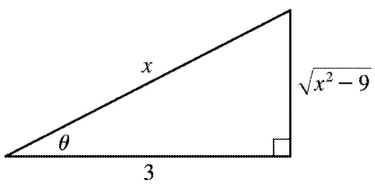
66. $\int_{-\pi}^{\pi}\sin mx\sin nx\,dx=\int_{-\pi}^{\pi}\frac{1}{2}\,[\cos(m-n)x-\cos(m+n)x]\,dx$. If $m\neq n$, this is equal to
$\frac{1}{2}\left[\frac{\sin(m-n)x}{m-n}-\frac{\sin(m+n)x}{m+n}\right]_{-\pi}^{\pi}=0$. If $m=n$, we get
$\int_{-\pi}^{\pi}\frac{1}{2}\,[1-\cos(m+n)x]\,dx=\left[\frac{1}{2}x\right]_{-\pi}^{\pi}-\left[\frac{\sin(m+n)x}{2(m+n)}\right]_{-\pi}^{\pi}=\pi-0=\pi$.

67. $\int_{-\pi}^{\pi}\cos mx\cos nx\,dx=\int_{-\pi}^{\pi}\frac{1}{2}\,[\cos(m-n)x+\cos(m+n)x]\,dx$. If $m\neq n$, this is equal to
$\frac{1}{2}\left[\frac{\sin(m-n)x}{m-n}+\frac{\sin(m+n)x}{m+n}\right]_{-\pi}^{\pi}=0$. If $m=n$, we get
$\int_{-\pi}^{\pi}\frac{1}{2}\,[1+\cos(m+n)x]\,dx=\left[\frac{1}{2}x\right]_{-\pi}^{\pi}+\left[\frac{\sin(m+n)x}{2(m+n)}\right]_{-\pi}^{\pi}=\pi+0=\pi$.

68. $\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin mx\,dx=\frac{1}{\pi}\int_{-\pi}^{\pi}\left[\left(\sum_{n=1}^{m} a_n \sin nx\right)\sin mx\right]dx=\sum_{n=1}^{m}\frac{a_n}{\pi}\int_{-\pi}^{\pi}\sin mx\sin nx\,dx$. By Exercise 66 , every term is zero except the m th one, and that term is $\frac{a_m}{\pi}\cdot\pi=a_m$.

1. Let $x = 3\sec\theta$, where $0 \le \theta < \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$. Then $dx = 3\sec\theta \tan\theta \, d\theta$ and

$$\sqrt{x^2 - 9} = \sqrt{9\sec^2\theta - 9} = \sqrt{9\left(\sec^2\theta - 1\right)} = \sqrt{9\tan^2\theta}$$
$$= 3|\tan\theta| = 3\tan\theta \text{ for the relevant values of } \theta$$

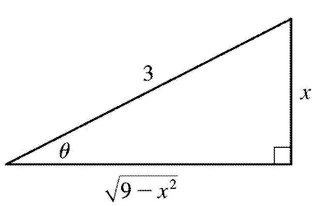


$$\int \frac{1}{x^2\sqrt{x^2-9}} \, dx = \int \frac{1}{9\sec^2\theta \cdot 3\tan\theta} 3\sec\theta \tan\theta \, d\theta = \frac{1}{9}\int \cos\theta \, d\theta = \frac{1}{9}\sin\theta + C = \frac{1}{9} \frac{\sqrt{x^2-9}}{x} + C$$

Note that $-\sec(\theta + \pi) = \sec\theta$, so the figure is sufficient for the case $\pi \le \theta < \frac{3\pi}{2}$.

2. Let $x = 3\sin\theta$, where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. Then $dx = 3\cos\theta \, d\theta$ and

$$\sqrt{9 - x^2} = \sqrt{9 - 9\sin^2\theta} = \sqrt{9\left(1 - \sin^2\theta\right)} = \sqrt{9\cos^2\theta}$$
$= 3|\cos\theta| = 3\cos\theta$ for the relevant values of \italic{\theta}\,\,.



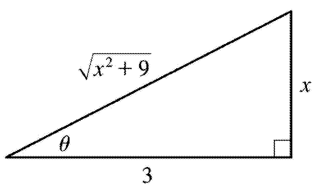
$$\int x^3\sqrt{9-x^2} \, dx = \int 3^3\sin^3\theta \cdot 3\cos\theta \cdot 3\cos\theta \, d\theta = 3^5\int \sin^3\theta \cos^2\theta \, d\theta$$
$$= 3^5\int \sin^2\theta \cos^2\theta \sin\theta \, d\theta = 3^5\int \left(1 - \cos^2\theta\right)\cos^2\theta \sin\theta \, d\theta$$
$$= 3^5\int \left(1 - u^2\right) u^2 (-du) [u = \cos\theta , du = -\sin\theta \, d\theta]$$
$$= 3^5\int \left(u^4 - u^2\right) du = 3^5\left(\frac{1}{5}u^5 - \frac{1}{3}u^3\right) + C = 3^5\left(\frac{1}{5}\cos^5\theta - \frac{1}{3}\cos^3\theta\right) + C$$
$$= 3^5\left[\frac{1}{5}\frac{\left(9-x^2\right)^{5/2}}{3^5} - \frac{1}{3}\frac{\left(9-x^2\right)^{3/2}}{3^3}\right] + C$$

$$= \frac{1}{5}\left(9-x^2\right)^{5/2} - 3\left(9-x^2\right)^{3/2} + C \text{ or} - \frac{1}{5}\left(x^2+6\right)(9-x)^{3/2} + C$$

3. Let $x=3\tan\theta$, where $-\frac{\pi}{2}<\theta<\frac{\pi}{2}$. Then $dx=3\sec^2\theta\ d\theta$ and

$$\sqrt{x^2+9} = \sqrt{9\tan^2\theta+9} = \sqrt{9\left(\tan^2\theta+1\right)} = \sqrt{9\sec^2\theta}$$

$= 3|\sec\theta| = 3\sec\theta$ for the relevant values of θ .



$$\int \frac{x^3}{\sqrt{x^2+9}}\,dx = \int \frac{3^3\tan^3\theta}{3\sec\theta}\,3\sec^2\theta\ d\theta = 3^3\int \tan^3\theta\ \sec\theta\ d\theta = 3^3\int \tan^2\theta\ \tan\theta\ \sec\theta\ d\theta$$

$$= 3^3\int\left(\sec^2\theta-1\right)\tan\theta\ \sec\theta\ d\theta = 3^3\int\left(u^2-1\right)du\,[u=\sec\theta\ ,du=\sec\theta\ \tan\theta\ d\theta\ ]$$

$$= 3^3\left(\frac{1}{3}u^3-u\right)+C = 3^3\left(\frac{1}{3}\sec^3\theta-\sec\theta\right)+C = 3^3\left[\frac{1}{3}\ \frac{\left(x^2+9\right)^{3/2}}{3^3} - \frac{\sqrt{x^2+9}}{3}\right]+C$$

$$= \frac{1}{3}\left(x^2+9\right)^{3/2} - 9\sqrt{x^2+9}+C \text{ or } \frac{1}{3}\left(x^2-18\right)\sqrt{x^2+9}+C$$

4. Let $x=4\sin\theta$, where $-\pi/2\leq\theta\leq\pi/2$. Then $dx=4\cos\theta\ d\theta$ and $\sqrt{16-x^2} = \sqrt{16-16\sin^2\theta} = \sqrt{16\cos^2\theta} = 4|\cos\theta| = 4\cos\theta$. When $x=0$, $4\sin\theta=0\Rightarrow\theta=0$, and when $x=2\sqrt{3}$, $4\sin\theta=2\sqrt{3}\Rightarrow\sin\theta=\frac{\sqrt{3}}{2}\ \Rightarrow\theta=\frac{\pi}{3}$. Thus, substitution gives

$$\int_0^{2\sqrt{3}} \frac{x^3}{\sqrt{16-x^2}}\,dx = \int_0^{\pi/3} \frac{4^3\sin^3\theta}{4\cos\theta}\,4\cos\theta\,d\theta = 4^3\int_0^{\pi/3}\sin^3\theta\,d\theta$$

$$=4^3\int_0^{\pi/3}\left(1-\cos^2\theta\right)\sin\theta\,d\theta$$

$$=-4^3\int_1^{1/2}\left(1-u^2\right)du=-64\left[u-\frac{1}{3}u^3\right]_1^{1/2}$$

$$=-64\left[\left(\frac{1}{2}-\frac{1}{24}\right)-\left(1-\frac{1}{3}\right)\right]=-64\left(-\frac{5}{24}\right)=\frac{40}{3}$$

Or: Let $u=16-x^2$, $x^2=16-u$, $du=-2x dx$.

5. Let $t=\sec\theta$, so $dt=\sec\theta\tan\theta\,d\theta$, $t=\sqrt{2}\Rightarrow\theta=\frac{\pi}{4}$, and $t=2\Rightarrow\theta=\frac{\pi}{3}$. Then

$$\int_{\sqrt{2}}^{2}\frac{1}{t^3\sqrt{t^2-1}}\,dt = \int_{\pi/4}^{\pi/3}\frac{1}{\sec^3\theta\tan\theta}\sec\theta\tan\theta\,d\theta=\int_{\pi/4}^{\pi/3}\frac{1}{\sec^2\theta}\,d\theta=\int_{\pi/4}^{\pi/3}\cos^2\theta\,d\theta$$

$$=\int_{\pi/4}^{\pi/3}\frac{1}{2}(1+\cos 2\theta)\,d\theta=\frac{1}{2}\left[\theta+\frac{1}{2}\sin 2\theta\right]_{\pi/4}^{\pi/3}$$

$$=\frac{1}{2}\left[\left(\frac{\pi}{3}+\frac{1}{2}\,\frac{\sqrt{3}}{2}\right)-\left(\frac{\pi}{4}+\frac{1}{2}\cdot 1\right)\right]=\frac{1}{2}\left(\frac{\pi}{12}+\frac{\sqrt{3}}{4}-\frac{1}{2}\right)=\frac{\pi}{24}+\frac{\sqrt{3}}{8}-\frac{1}{4}$$

6. Let $x=2\tan\theta$, so $dx=2\sec^2\theta\ d\theta$, $x=0\Rightarrow\theta=0$, and $x=2\Rightarrow\theta=\frac{\pi}{4}$. Then

$$\int_0^2 x^3\sqrt{x^2+4}\,dx=\int_0^{\pi/4}2^3\tan^3\theta\cdot 2\sec\theta\cdot 2\sec^2\theta\ d\theta=2^5\int_0^{\pi/4}\tan^2\theta\ \sec^2\theta\ \sec\theta\ \tan\theta\ d\theta$$

$$=2^5\int_0^{\pi/4}(\sec^2\theta-1)\sec^2\theta\ \sec\theta\ \tan\theta\ d\theta$$

$$=2^5\int_1^{\sqrt{2}}(u^2-1)u^2\,du\ [u=\sec\theta\ ,du=\sec\theta\ \tan\theta\ d\theta\ ]$$

$$=2^5\int_1^{\sqrt{2}}(u^4-u^2)\,du=2^5\left[\frac{1}{5}u^5-\frac{1}{3}u^3\right]_1^{\sqrt{2}}=2^5\left[\left(\frac{1}{5}\cdot 4\sqrt{2}-\frac{1}{3}\cdot 2\sqrt{2}\right)-\left(\frac{1}{5}-\frac{1}{3}\right)\right]$$

$$=32\left(\frac{2}{15}\sqrt{2}+\frac{2}{15}\right)=\frac{64}{15}\left(\sqrt{2}+1\right)$$

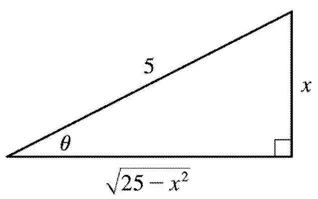
Or: Let $u=x^2+4$, $x^2=u-4$, $du=2x\,dx$.

7. Let $x=5\sin\theta$, so $dx=5\cos\theta\ d\theta$. Then

$$\int \frac{1}{x^2\sqrt{25-x^2}}\,dx = \int \frac{1}{5^2\sin^2\theta\cdot 5\cos\theta}\,5\cos\theta\ d\theta$$

$$= \frac{1}{25}\int \csc^2\theta\ d\theta = -\frac{1}{25}\cot\theta + C$$

$$= -\frac{1}{25}\,\frac{\sqrt{25-x^2}}{x} + C$$

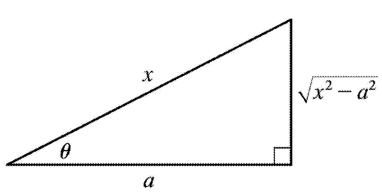


8. Let $x=a\sec\theta$, where $0\le\theta<\frac{\pi}{2}$ or $\pi\le\theta<\frac{3\pi}{2}$. Then $dx=a\sec\theta\ \tan\theta\ d\theta$ and $\sqrt{x^2-a^2}=a\tan\theta$, so

$$\int \frac{\sqrt{x^2-a^2}}{x^4}\,dx = \int \frac{a\tan\theta}{a^4\sec^4\theta}\,a\sec\theta\ \tan\theta\ d\theta$$

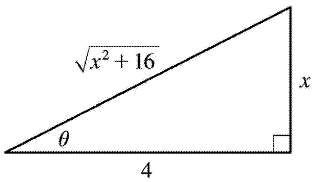
$$= \frac{1}{a^2}\int \sin^2\theta\ \cos\theta\ d\theta$$

$$= \frac{1}{3a^2}\sin^3\theta + C = \frac{\left(x^2-a^2\right)^{3/2}}{3a^2x^3} + C$$



9. Let $x=4\tan\theta$, where $-\frac{\pi}{2}<\theta<\frac{\pi}{2}$. Then $dx=4\sec^2\theta\ d\theta$ and

$$\sqrt{x^2+16} = \sqrt{16\tan^2\theta + 16} = \sqrt{16(\tan^2\theta + 1)}$$
$$= \sqrt{16\sec^2\theta} = 4|\sec\theta|$$
$$= 4\sec\theta \text{ for the relevant values of } \theta .$$

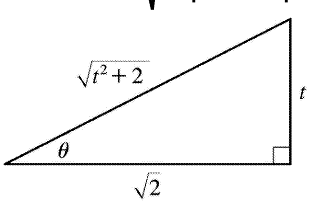


$$\int \frac{dx}{\sqrt{x^2+16}} = \int \frac{4\sec^2\theta\, d\theta}{4\sec\theta} = \int \sec\theta\, d\theta = \ln|\sec\theta + \tan\theta| + C_1$$
$$= \ln\left|\frac{\sqrt{x^2+16}}{4} + \frac{x}{4}\right| + C_1 = \ln\left|\sqrt{x^2+16} + x\right| - \ln|4| + C_1$$
$$= \ln\left(\sqrt{x^2+16} + x\right) + C \text{ , where } C = C_1 - \ln 4 .$$

(Since $\sqrt{x^2+16} + x > 0$, we don't need the absolute value.)

10. Let $t = \sqrt{2}\tan\theta$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Then $dt = \sqrt{2}\sec^2\theta\, d\theta$ and

$$\sqrt{t^2+2} = \sqrt{2\tan^2\theta + 2} = \sqrt{2(\tan^2\theta + 1)} = \sqrt{2\sec^2\theta}$$
$$= \sqrt{2}|\sec\theta| = \sqrt{2}\sec\theta \text{ for the relevant values of } \theta .$$



$$\int \frac{t^5}{\sqrt{t^2+2}}\, dt = \int \frac{4\sqrt{2}\tan^5\theta}{\sqrt{2}\sec\theta} \sqrt{2}\sec^2\theta\, d\theta = 4\sqrt{2}\int \tan^5\theta \sec\theta\, d\theta = 4\sqrt{2}\int \left(\sec^2\theta - 1\right)^2 \sec\theta \tan\theta\, d\theta$$
$$= 4\sqrt{2}\int \left(u^2-1\right)^2 du \; [\, u = \sec\theta \text{ , } du = \sec\theta \tan\theta\, d\theta \,] = 4\sqrt{2}\int \left(u^4 - 2u^2 + 1\right) du$$

$$=4\sqrt{2}\left(\frac{1}{5}u^5-\frac{2}{3}u^3+u\right)+C=\frac{4\sqrt{2}}{15}u(3u^4-10u^2+15)+C$$

$$=\frac{4\sqrt{2}}{15}\cdot\frac{\sqrt{t^2+2}}{\sqrt{2}}\left[3\cdot\frac{(t^2+2)^2}{2^2}-10\frac{t^2+2}{2}+15\right]+C$$

$$=\frac{4}{15}\sqrt{t^2+2}\cdot\frac{1}{4}\left[3(t^4+4t^2+4)-20(t^2+2)+60\right]+C$$

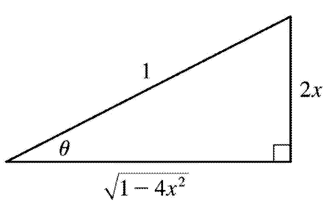
$$=\frac{1}{15}\sqrt{t^2+2}\,(3t^4-8t^2+32)+C$$

11. Let $2x=\sin\theta$, where $-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$. Then $x=\frac{1}{2}\sin\theta$, $dx=\frac{1}{2}\cos\theta\ d\theta$, and $\sqrt{1-4x^2}=\sqrt{1-(2x)^2}=\cos\theta$.

$$\int\sqrt{1-4x^2}\,dx=\int\cos\theta\left(\frac{1}{2}\cos\theta\right)d\theta=\frac{1}{4}\int(1+\cos 2\theta)\,d\theta$$

$$=\frac{1}{4}\left(\theta+\frac{1}{2}\sin 2\theta\right)+C=\frac{1}{4}(\theta+\sin\theta\cos\theta)+C$$

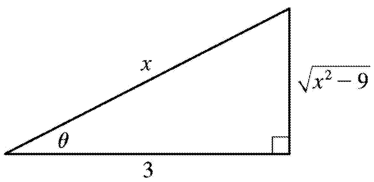
$$=\frac{1}{4}\left[\sin^{-1}(2x)+2x\sqrt{1-4x^2}\right]+C$$



12. $\int_0^1 x\sqrt{x^2+4}\,dx=\int_4^5\sqrt{u}\left(\frac{1}{2}\,du\right)$ [$u=x^2+4$, $du=2x\,dx$] $=\frac{1}{2}\cdot\frac{2}{3}\left[u^{3/2}\right]_4^5=\frac{1}{3}\left(5\sqrt{5}-8\right)$

13. Let $x=3\sec\theta$, where $0\le\theta<\frac{\pi}{2}$ or $\pi\le\theta<\frac{3\pi}{2}$. Then $dx=3\sec\theta\tan\theta\ d\theta$ and $\sqrt{x^2-9}=3\tan\theta$, so

$$\int\frac{\sqrt{x^2-9}}{x^3}\,dx=\int\frac{3\tan\theta}{27\sec^3\theta}\,3\sec\theta\tan\theta\ d\theta=\frac{1}{3}\int\frac{\tan^2\theta}{\sec^2\theta}\,d\theta$$



$$=\frac{1}{3}\int \sin^2\theta \; d\theta = \frac{1}{3}\int \frac{1}{2}(1-\cos 2\theta)\,d\theta = \frac{1}{6}\theta - \frac{1}{12}\sin 2\theta + C = \frac{1}{6}\theta - \frac{1}{6}\sin\theta\cos\theta + C$$

$$=\frac{1}{6}\sec^{-1}\left(\frac{x}{3}\right) - \frac{1}{6}\,\frac{\sqrt{x^2-9}}{x}\,\frac{3}{x} + C = \frac{1}{6}\sec^{-1}\left(\frac{x}{3}\right) - \frac{\sqrt{x^2-9}}{2x^2} + C$$

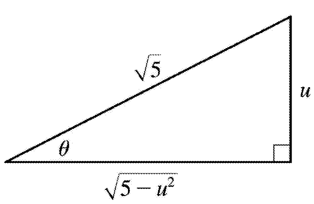
14. Let $u=\sqrt{5}\sin\theta$, so $du=\sqrt{5}\cos\theta \; d\theta$. Then

$$\int \frac{du}{u\sqrt{5-u^2}} = \int \frac{1}{\sqrt{5}\sin\theta \cdot \sqrt{5}\cos\theta}\sqrt{5}\cos\theta \; d\theta = \frac{1}{\sqrt{5}}\int \csc\theta \; d\theta$$

$$= \frac{1}{\sqrt{5}}\ln\left|\csc\theta - \cot\theta\right| + C \text{ [byExercise2.39}$$

$$= \frac{1}{\sqrt{5}}\ln\left|\frac{\sqrt{5}}{u} - \frac{\sqrt{5-u^2}}{u}\right| + C$$

$$= \frac{1}{\sqrt{5}}\ln\left|\frac{\sqrt{5}-\sqrt{5-u^2}}{u}\right| + C$$

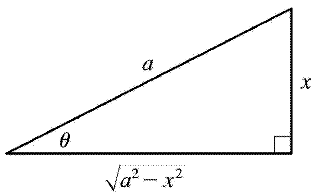


15. Let $x=a\sin\theta$, where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. Then $dx=a\cos\theta \, d\theta$ and

$$\frac{x^2\,dx}{\left(a^2-x^2\right)^{3/2}} = \int \frac{a^2\sin^2\theta \; a\cos\theta \, d\theta}{a^3\cos^3\theta} = \int \tan^2\theta \, d\theta$$

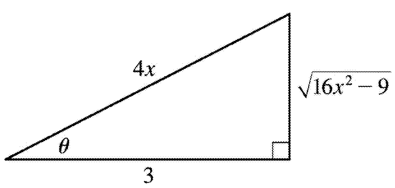
$$= \int\left(\sec^2\theta - 1\right)d\theta = \tan\theta - \theta + C$$

$$= \frac{x}{\sqrt{a^2 - x^2}} - \sin^{-1}\frac{x}{a} + C$$



16. Let $4x = 3\sec\theta$, where $0 \le \theta < \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$. Then $dx = \frac{3}{4}\sec\theta\tan\theta\,d\theta$ and $\sqrt{16x^2 - 9} = 3\tan\theta$, so

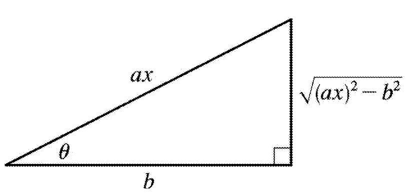
$$\int \frac{dx}{x^2\sqrt{16x^2-9}} = \int \frac{\frac{3}{4}\sec\theta\tan\theta\,d\theta}{\left(\frac{3}{4}\right)^2\sec^2\theta\;3\tan\theta}$$



$$= \frac{4}{9}\int \cos\theta\,d\theta = \frac{4}{9}\sin\theta + C = \frac{4}{9}\,\frac{\sqrt{16x^2-9}}{4x} + C = \frac{\sqrt{16x^2-9}}{9x} + C$$

17. Let $u = x^2 - 7$, so $du = 2x\,dx$. Then $\int \frac{x}{\sqrt{x^2-7}}\,dx = \frac{1}{2}\int \frac{1}{\sqrt{u}}\,du = \frac{1}{2}\cdot 2\sqrt{u} + C = \sqrt{x^2-7} + C$.

18. Let $ax = b\sec\theta$, so $(ax)^2 = b^2\sec^2\theta \Rightarrow (ax)^2 - b^2 = b^2\sec^2\theta - b^2 = b^2\left(\sec^2\theta - 1\right) = b^2\tan^2\theta$. So $\sqrt{\left(ax^2\right) - b^2} = b\tan\theta$, $dx = \frac{b}{a}\sec\theta\tan\theta\,d\theta$, and



$$\int \frac{dx}{\left[(ax)^2-b^2\right]^{3/2}} = \int \frac{\frac{b}{a}\sec\theta\ \tan\theta}{b^3\tan^3\theta}\,d\theta = \frac{1}{ab^2}\int \frac{\sec\theta}{\tan^2\theta}\,d\theta = \frac{1}{ab^2}\int \frac{\cos\theta}{\sin^2\theta}\,d\theta = \frac{1}{ab^2}\int \csc\theta\ \cot\theta\ d\theta$$

$$= -\frac{1}{ab^2}\csc\theta + C = -\frac{1}{ab^2}\frac{ax}{\sqrt{(ax)^2-b^2}} + C = -\frac{x}{b^2\sqrt{(ax)^2-b^2}} + C$$

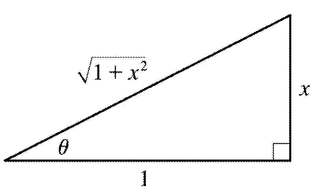
19. Let $x=\tan\theta$, where $-\frac{\pi}{2}<\theta<\frac{\pi}{2}$. Then $dx=\sec^2\theta\ d\theta$ and $\sqrt{1+x^2}=\sec\theta$, so

$$\int \frac{\sqrt{1+x^2}}{x}\,dx = \int \frac{\sec\theta}{\tan\theta}\sec^2\theta\ d\theta = \int \frac{\sec\theta}{\tan\theta}(1+\tan^2\theta)\,d\theta$$

$$= \int(\csc\theta+\sec\theta\ \tan\theta)\,d\theta$$

$$= \ln\left|\csc\theta-\cot\theta\right| + \sec\theta + C \quad [\text{ by Exercise 8.2.39}]$$

$$= \ln\left|\frac{\sqrt{1+x^2}}{x}-\frac{1}{x}\right| + \frac{\sqrt{1+x^2}}{1} + C = \ln\left|\frac{\sqrt{1+x^2}-1}{x}\right| + \sqrt{1+x^2} + C$$

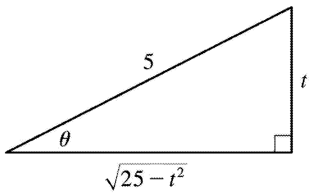


20. Let $t=5\sin\theta$, where $-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$. Then $dt=5\cos\theta\ d\theta$ and $\sqrt{25-t^2}=5\cos\theta$, so

$$\int \frac{t}{\sqrt{25-t^2}}\,dt = \int \frac{5\sin\theta}{5\cos\theta}5\cos\theta\ d\theta = 5\int \sin\theta\ d\theta$$

$$= -5\cos\theta + C = -5\cdot\frac{\sqrt{25-t^2}}{5} + C = -\sqrt{25-t^2} + C$$

Or: Let $u=25-t^2$, so $du=-2t\,dt$.



21. Let $u=4-9x^2 \Rightarrow du=-18x\,dx$. Then $x^2=\frac{1}{9}(4-u)$ and

$$\int_0^{2/3} x^3\sqrt{4-9x^2}\,dx = \int_4^0 \frac{1}{9}(4-u)u^{1/2}\left(-\frac{1}{18}\right)du=\frac{1}{162}\int_0^4\left(4u^{1/2}-u^{3/2}\right)du$$

$$= \frac{1}{162}\left[\frac{8}{3}u^{3/2}-\frac{2}{5}u^{5/2}\right]_0^4=\frac{1}{162}\left[\frac{64}{3}-\frac{64}{5}\right]=\frac{64}{1215}$$

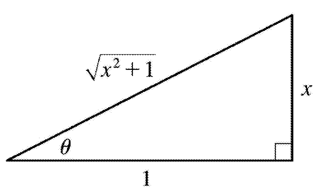
Or: Let $3x=2\sin\theta$, where $-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$.

22. Let $x=\tan\theta$, where $-\frac{\pi}{2}<\theta<\frac{\pi}{2}$. Then $dx=\sec^2\theta\,d\theta$, $\sqrt{x^2+1}=\sec\theta$ and $x=0\Rightarrow\theta=0$, $x=1\Rightarrow$ $\theta=\frac{\pi}{4}$, so

$$\int_0^1\sqrt{x^2+1}\,dx = \int_0^{\pi/4}\sec\theta\sec^2\theta\,d\theta=\int_0^{\pi/4}\sec^3\theta\,d\theta$$

$$= \frac{1}{2}\left[\sec\theta\tan\theta+\ln|\sec\theta+\tan\theta|\right]_0^{\pi/4} \quad \text{[by Example 8.2.8]}$$

$$= \frac{1}{2}\left[\sqrt{2}\cdot 1+\ln\left(1+\sqrt{2}\right)-0-\ln(1+0)\right]=\frac{1}{2}\left[\sqrt{2}+\ln\left(1+\sqrt{2}\right)\right]$$



23. $5+4x-x^2=-(x^2-4x+4)+9=-(x-2)^2+9$. Let $x-2=3\sin\theta$, $-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$, so $dx=3\cos\theta\,d\theta$. Then

$$\int\sqrt{5+4x-x^2}\,dx=\int\sqrt{9-(x-2)^2}\,dx=\int\sqrt{9-9\sin^2\theta}\;3\cos\theta\,d\theta$$

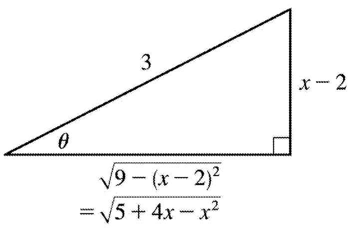
$$=\int\sqrt{9\cos^2\theta}\;3\cos\theta\,d\theta=\int 9\cos^2\theta\,d\theta$$

$$=\frac{9}{2}\int(1+\cos 2\theta)\,d\theta=\frac{9}{2}\left(\theta+\frac{1}{2}\sin 2\theta\right)+C$$

$$=\frac{9}{2}\theta+\frac{9}{4}\sin 2\theta+C=\frac{9}{2}\theta+\frac{9}{4}(2\sin\theta\cos\theta)+C$$

$$=\frac{9}{2}\sin^{-1}\left(\frac{x-2}{3}\right)+\frac{9}{2}\cdot\frac{x-2}{3}\cdot\frac{\sqrt{5+4x-x^2}}{3}+C$$

$$=\frac{9}{2}\sin^{-1}\left(\frac{x-2}{3}\right)+\frac{1}{2}(x-2)\sqrt{5+4x-x^2}+C$$



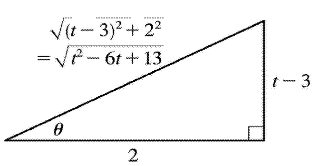
24. $t^2-6t+13=\left(t^2-6t+9\right)+4=(t-3)^2+2^2$. Let $t-3=2\tan\theta$, so $dt=2\sec^2\theta\,d\theta$. Then

$$\int\frac{dt}{\sqrt{t^2-6t+13}}=\int\frac{1}{\sqrt{(2\tan\theta)^2+2^2}}\,2\sec^2\theta\,d\theta=\int\frac{2\sec^2\theta}{2\sec\theta}\,d\theta$$

$$=\int\sec\theta\,d\theta=\ln|\sec\theta+\tan\theta|+C_1\qquad\text{[by Formula 8.2.1]}$$

$$=\ln\left|\frac{\sqrt{t^2-6t+13}}{2}+\frac{t-3}{2}\right|+C_1$$

$$=\ln\left|\sqrt{t^2-6t+13}+t-3\right|+C\text{ where }C=C_1-\ln 2$$



25. $9x^2+6x-8=(3x+1)^2-9$, so let $u=3x+1$, $du=3dx$. Then $\displaystyle\int\frac{dx}{\sqrt{9x^2+6x-8}}=\int\frac{\frac{1}{3}\,du}{\sqrt{u^2-9}}$. Now let

$u=3\sec\theta$, where

$0 \le \theta < \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$. Then $du = 3\sec\theta\,\tan\theta\,d\theta$ and $\sqrt{u^2-9} = 3\tan\theta$, so

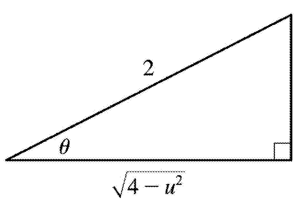
$$\int \frac{\frac{1}{3}\,du}{\sqrt{u^2-9}} = \int \frac{\sec\theta\,\tan\theta\,d\theta}{3\tan\theta} = \frac{1}{3}\int \sec\theta\,d\theta = \frac{1}{3}\ln\left|\sec\theta + \tan\theta\right| + C_1 = \frac{1}{3}\ln\left|\frac{u+\sqrt{u^2-9}}{3}\right| + C_1$$

$$= \frac{1}{3}\ln\left|u+\sqrt{u^2-9}\right| + C = \frac{1}{3}\ln\left|3x+1+\sqrt{9x^2+6x-8}\right| + C$$

26. $4x - x^2 = -\left(x^2-4x+4\right)+4 = 4-(x-2)^2$, so let $u = x-2$. Then $x = u+2$ and $dx = du$, so

$$\int \frac{x^2\,dx}{\sqrt{4x-x^2}} = \int \frac{(u+2)^2\,du}{\sqrt{4-u^2}} = \int \frac{(2\sin\theta+2)^2}{2\cos\theta}\,2\cos\theta\,d\theta$$

$$= 4\int\left(\sin^2\theta + 2\sin\theta + 1\right)d\theta$$
$$= 2\int(1-\cos 2\theta)\,d\theta + 8\int \sin\theta\,d\theta + 4\int d\theta$$
$$= 2\theta - \sin 2\theta - 8\cos\theta + 4\theta + C$$
$$= 6\theta - 8\cos\theta - 2\sin\theta\,\cos\theta + C$$
$$= 6\sin^{-1}\left(\frac{1}{2}u\right) - 4\sqrt{4-u^2} - \frac{1}{2}u\sqrt{4-u^2} + C$$
$$= 6\sin^{-1}\left(\frac{x-2}{2}\right) - 4\sqrt{4x-x^2} - \left(\frac{x-2}{2}\right)\sqrt{4x-x^2} + C$$



27. $x^2+2x+2 = (x+1)^2+1$. Let $u = x+1$, $du = dx$. Then

$$\int \frac{dx}{\left(x^2+2x+2\right)^2} = \int \frac{du}{\left(u^2+1\right)^2} = \int \frac{\sec^2\theta\,d\theta}{\sec^4\theta} \quad \left[\begin{array}{c}\text{where } u=\tan\theta,\,du=\sec^2\theta\,d\theta,\\ \text{and } u^2+1=\sec^2\theta\text{ x}\end{array}\right]$$

$$= \int \cos^2\theta\,d\theta = \frac{1}{2}\int(1+\cos 2\theta)\,d\theta = \frac{1}{2}(\theta + \sin\theta\,\cos\theta) + C$$

$$= \frac{1}{2}\left[\tan^{-1}u + \frac{u}{1+u^2}\right] + C = \frac{1}{2}\left[\tan^{-1}(x+1) + \frac{x+1}{x^2+2x+2}\right] + C$$

28. $5-4x-x^2 = -\left(x^2+4x+4\right)+9 = 9-(x+2)^2$. Let $u=x+2 \Rightarrow du=dx$. Then

$$\int \frac{dx}{\left(5-4x-x^2\right)^{5/2}} = \int \frac{du}{\left(9-u^2\right)^{5/2}} = \int \frac{3\cos\theta\, d\theta}{(3\cos\theta)^5} \quad \left[\begin{array}{c}\text{where } u=3\sin\theta,\ du=3\cos\theta\, d\theta, \\ \text{and } \sqrt{9-u^2}=3\cos\theta\end{array}\right]$$

$$= \frac{1}{81}\int \sec^4\theta\, d\theta = \frac{1}{81}\int\left(\tan^2\theta+1\right)\sec^2\theta\, d\theta = \frac{1}{81}\left[\frac{1}{3}\tan^3\theta + \tan\theta\right] + C$$

$$= \frac{1}{243}\left[\frac{u^3}{\left(9-u^2\right)^{3/2}} + \frac{3u}{\sqrt{9-u^2}}\right] + C = \frac{1}{243}\left[\frac{(x+2)^3}{\left(5-4x-x^2\right)^{3/2}} + \frac{3(x+2)}{\sqrt{5-4x-x^2}}\right] + C$$

29. Let $u=x^2$, $du=2x\,dx$. Then

$$\int x\sqrt{1-x^4}\, dx = \int\sqrt{1-u^2}\left(\frac{1}{2}\, du\right) = \frac{1}{2}\int\cos\theta\cdot\cos\theta\, d\theta \quad \left[\begin{array}{c}\text{where} u=\sin\theta,\ du=\cos\theta\, d\theta, \\ \text{and} \sqrt{1-u^2}=\cos\theta\end{array}\right]$$

$$= \frac{1}{2}\int\frac{1}{2}(1+\cos 2\theta)d\theta = \frac{1}{4}\theta + \frac{1}{8}\sin 2\theta + C = \frac{1}{4}\theta + \frac{1}{4}\sin\theta\cos\theta + C$$

$$= \frac{1}{4}\sin^{-1}u + \frac{1}{4}u\sqrt{1-u^2} + C = \frac{1}{4}\sin^{-1}(x^2) + \frac{1}{4}x^2\sqrt{1-x^4} + C$$

30. Let $u=\sin t$, $du=\cos t\,dt$. Then

$$\int_0^{\pi/2}\frac{\cos t}{\sqrt{1+\sin^2 t}}\, dt = \int_0^1\frac{1}{\sqrt{1+u^2}}\, dt = \int_0^{\pi/4}\frac{1}{\sec\theta}\sec^2\theta\, d\theta \quad \left[\begin{array}{c}\text{where } u=\tan\theta,\ du=\sec^2\theta\, d\theta, \\ \text{and} \sqrt{1+u^2}=\sec\theta\end{array}\right]$$

$$= \int_0^{\pi/4}\sec\theta\, d\theta = \left[\ln\left|\sec\theta+\tan\theta\right|\right]_0^{\pi/4} \quad \text{[by (1) in Section 8.2]}$$

$$= \ln\left(\sqrt{2}+1\right) - \ln(1+0) = \ln\left(\sqrt{2}+1\right)$$

31. **(a)** Let $x=a\tan\theta$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Then $\sqrt{x^2+a^2} = a\sec\theta$ and

$$\int\frac{dx}{\sqrt{x^2+a^2}} = \int\frac{a\sec^2\theta\, d\theta}{a\sec\theta} = \int\sec\theta\, d\theta = \ln\left|\sec\theta+\tan\theta\right| + C_1 = \ln\left|\frac{\sqrt{x^2+a^2}}{a} + \frac{x}{a}\right| + C_1$$

$$= \ln\left(x+\sqrt{x^2+a^2}\right)+C \text{ where } C=C_1-\ln|a|$$

(b) Let $x=a\sinh t$, so that $dx=a\cosh t\,dt$ and $\sqrt{x^2+a^2}=a\cosh t$. Then

$$\int \frac{dx}{\sqrt{x^2+a^2}}=\int \frac{a\cosh t\,dt}{a\cosh t}=t+C=\sinh^{-1}\frac{x}{a}+C.$$

32. **(a)** Let $x=a\tan\theta$, $-\frac{\pi}{2}<\theta<\frac{\pi}{2}$. Then

$$I=\int \frac{x^2}{\left(x^2+a^2\right)^{3/2}}\,dx=\int \frac{a^2\tan^2\theta}{a^3\sec^3\theta}\,a\sec^2\theta\,d\theta=\int \frac{\tan^2\theta}{\sec\theta}\,d\theta=\int \frac{\sec^2\theta-1}{\sec\theta}\,d\theta$$

$$=\int(\sec\theta-\cos\theta)\,d\theta=\ln|\sec\theta+\tan\theta|-\sin\theta+C$$

$$=\ln\left|\frac{\sqrt{x^2+a^2}}{a}+\frac{x}{a}\right|-\frac{x}{\sqrt{x^2+a^2}}+C=\ln\left(x+\sqrt{x^2+a^2}\right)-\frac{x}{\sqrt{x^2+a^2}}+C_1$$

(b) Let $x=a\sinh t$. Then

$$I=\int \frac{a^2\sinh^2 t}{a^3\cosh^3 t}\,a\cosh t\,dt=\int \tanh^2 t\,dt=\int\left(1-\operatorname{sech}^2 t\right)dt=t-\tanh t+C$$

$$=\sinh^{-1}\frac{x}{a}-\frac{x}{\sqrt{a^2+x^2}}+C$$

33. The average value of $f(x)=\sqrt{x^2-1}/x$ on the interval $[1,7]$ is

$$\frac{1}{7-1}\int_1^7 \frac{\sqrt{x^2-1}}{x}\,dx=\frac{1}{6}\int_0^\alpha \frac{\tan\theta}{\sec\theta}\cdot\sec\theta\tan\theta\,d\theta\quad\left[\begin{array}{c}\text{where } x=\sec\theta, dx=\sec\theta\tan\theta\,d\theta,\\ \sqrt{x^2-1}=\tan\theta, \text{and } \alpha=\sec^{-1}7\end{array}\right]$$

$$=\frac{1}{6}\int_0^\alpha \tan^2\theta\,d\theta=\frac{1}{6}\int_0^\alpha(\sec^2\theta-1)\,d\theta$$

$$=\frac{1}{6}\left[\tan\theta-\theta\right]_0^\alpha=\frac{1}{6}(\tan\alpha-\alpha)$$

$$=\frac{1}{6}\left(\sqrt{48}-\sec^{-1}7\right)$$

34. $9x^2-4y^2=36 \Rightarrow y=\pm\frac{3}{2}\sqrt{x^2-4} \Rightarrow$

$$\text{area} = 2\int_2^3 \frac{3}{2}\sqrt{x^2-4}\,dx = 3\int_2^3\sqrt{x^2-4}\,dx$$

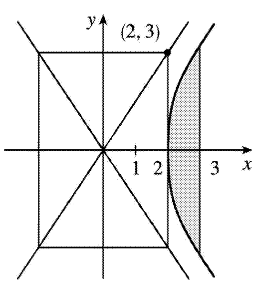
$$= 3\int_0^\alpha 2\tan\theta\, 2\sec\theta\tan\theta\, d\theta \quad \left[\begin{array}{c}\text{where } x=2\sec\theta, \\ dx=2\sec\theta\tan\theta\, d\theta, \\ \alpha=\sec^{-1}\frac{3}{2}\end{array}\right]$$

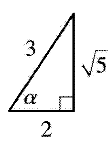
$$= 12\int_0^\alpha\left(\sec^2\theta-1\right)\sec\theta\, d\theta = 12\int_0^\alpha\left(\sec^3\theta-\sec\theta\right)d\theta$$

$$= 12\left[\frac{1}{2}(\sec\theta\tan\theta+\ln|\sec\theta+\tan\theta|)-\ln|\sec\theta+\tan\theta|\right]_0^\alpha$$

$$= 6\left[\sec\theta\tan\theta-\ln|\sec\theta+\tan\theta|\right]_0^\alpha$$

$$= 6\left[\frac{3\sqrt{5}}{4}-\ln\left(\frac{3}{2}+\frac{\sqrt{5}}{2}\right)\right] = \frac{9\sqrt{5}}{2}-6\ln\left(\frac{3+\sqrt{5}}{2}\right)$$





35. Area of $\triangle POQ = \frac{1}{2}(r\cos\theta)(r\sin\theta) = \frac{1}{2}r^2\sin\theta\cos\theta$. Area of region $PQR = \int_{r\cos\theta}^{r}\sqrt{r^2-x^2}\,dx$. Let $x=r\cos u \Rightarrow dx=-r\sin u\,du$ for $\theta \le u \le \frac{\pi}{2}$. Then we obtain

$$\int\sqrt{r^2-x^2}\,dx = \int r\sin u(-r\sin u)du = -r^2\int\sin^2 u\,du = -\frac{1}{2}r^2(u-\sin u\cos u)+C$$

$$= -\frac{1}{2}r^2\cos^{-1}(x/r)+\frac{1}{2}x\sqrt{r^2-x^2}+C$$

so

$$\text{area of region} = \frac{1}{2}\left[-r^2\cos^{-1}(x/r)+x\sqrt{r^2-x^2}\right]_{r\cos\theta}^{r}$$

$$= \frac{1}{2}\left[0-\left(-r^2\theta+r\cos\theta\, r\sin\theta\right)\right]$$

$$= \frac{1}{2} r^2 \theta - \frac{1}{2} r^2 \sin\theta \cos\theta$$

and thus, (area of sector POR)=(area of $\triangle POQ$)+(area of region PQR)=$\frac{1}{2} r^2 \theta$.

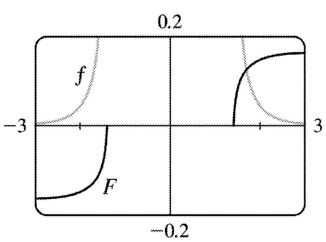
36. Let $x=\sqrt{2}\sec\theta$, where $0 \le \theta < \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$, so $dx=\sqrt{2}\sec\theta \tan\theta \, d\theta$. Then

$$\int \frac{dx}{x^4\sqrt{x^2-2}} = \int \frac{\sqrt{2}\sec\theta \tan\theta \, d\theta}{4\sec^4\theta \sqrt{2}\tan\theta}$$

$$= \frac{1}{4}\int \cos^3\theta \, d\theta = \frac{1}{4}\int \left(1-\sin^2\theta\right)\cos\theta \, d\theta$$

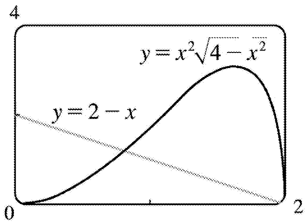
$$= \frac{1}{4}\left[\sin\theta - \frac{1}{3}\sin^3\theta\right]+C \quad [\text{substitute } u=\sin\theta]$$

$$= \frac{1}{4}\left[\frac{\sqrt{x^2-2}}{x} - \frac{\left(x^2-2\right)^{3/2}}{3x^3}\right]+C$$



From the graph, it appears that our answer is reasonable.

37. From the graph, it appears that the curve $y=x^2\sqrt{4-x^2}$ and the line $y=2-x$ intersect at about $x=0.81$ and $x=2$, with $x^2\sqrt{4-x^2}>2-x$ on $(0.81,2)$. So the area bounded by the curve and the line is $A \approx \int_{0.81}^{2}\left[x^2\sqrt{4-x^2}-(2-x)\right]dx=\int_{0.81}^{2} x^2\sqrt{4-x^2}\,dx-\left[2x-\frac{1}{2}x^2\right]_{0.81}^{2}$. To evaluate the integral, we put $x=2\sin\theta$, where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. Then



$dx=2\cos\theta\ d\theta$, $x=2\Rightarrow\theta=\sin^{-1}1=\dfrac{\pi}{2}$, and $x=0.81\Rightarrow\theta=\sin^{-1}0.405\approx0.417$. So

$$\int_{0.81}^{2}x^2\sqrt{4-x^2}\,dx\approx\int_{0.417}^{\pi/2}4\sin^2\theta\ (2\cos\theta)(2\cos\theta\ d\theta)=4\int_{0.417}^{\pi/2}\sin^2 2\theta\ d\theta=4\int_{0.417}^{\pi/2}\frac{1}{2}(1-\cos4\theta)d\theta$$

$$=2\left[\theta-\frac{1}{4}\sin4\theta\right]_{0.417}^{\pi/2}=2\left[\left(\frac{\pi}{2}-0\right)-\left(0.417-\frac{1}{4}(0.995)\right)\right]\approx2.81$$

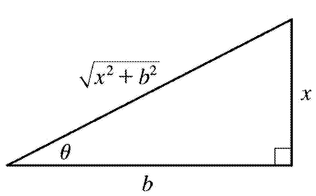
Thus, $A\approx2.81-\left[\left(2\cdot2-\frac{1}{2}\cdot2^2\right)-\left(2\cdot0.81-\frac{1}{2}\cdot0.81^2\right)\right]\approx2.10$.

38. Let $x=b\tan\theta$, so that $dx=b\sec^2\theta\ d\theta$ and $\sqrt{x^2+b^2}=b\sec\theta$.

$$E(P)=\int_{-a}^{L-a}\frac{\lambda b}{4\pi\varepsilon_0\left(x^2+b^2\right)^{3/2}}\,dx=\frac{\lambda b}{4\pi\varepsilon_0}\int_{\theta_1}^{\theta_2}\frac{1}{(b\sec\theta)^3}\,b\sec^2\theta\,d\theta$$

$$=\frac{\lambda}{4\pi\varepsilon_0 b}\int_{\theta_1}^{\theta_2}\frac{1}{\sec\theta}\,d\theta=\frac{\lambda}{4\pi\varepsilon_0 b}\int_{\theta_1}^{\theta_2}\cos\theta\,d\theta=\frac{\lambda}{4\pi\varepsilon_0 b}\left[\sin\theta\right]_{\theta_1}^{\theta_2}$$

$$=\frac{\lambda}{4\pi\varepsilon_0 b}\left[\frac{x}{\sqrt{x^2+b^2}}\right]_{-a}^{L-a}=\frac{\lambda}{4\pi\varepsilon_0 b}\left(\frac{L-a}{\sqrt{(L-a)^2+b^2}}+\frac{a}{\sqrt{a^2+b^2}}\right)$$



39. Let the equation of the large circle be $x^2+y^2=R^2$. Then the equation of the small circle is $x^2+(y-b)^2=r^2$, where $b=\sqrt{R^2-r^2}$ is the distance between the centers of the circles. The desired area is

$$A=\int_{-r}^{r}\left[\left(b+\sqrt{r^2-x^2}\right)-\sqrt{R^2-x^2}\right]dx=2\int_0^r\left(b+\sqrt{r^2-x^2}-\sqrt{R^2-x^2}\right)dx$$
$$=2\int_0^r b\,dx+2\int_0^r\sqrt{r^2-x^2}\,dx-2\int_0^r\sqrt{R^2-x^2}\,dx$$

The first integral is just $2br=2r\sqrt{R^2-r^2}$. To evaluate the other two integrals, note that

$$\int\sqrt{a^2-x^2}\,dx=\int a^2\cos^2\theta\ d\theta\ \ [x=a\sin\theta\ ,dx=a\cos\theta\ d\theta\ ]=\frac{1}{2}a^2\int(1+\cos 2\theta\ )d\theta$$
$$=\frac{1}{2}a^2\left(\theta+\frac{1}{2}\sin 2\theta\right)+C=\frac{1}{2}a^2(\theta+\sin\theta\ \cos\theta\ )+C$$
$$=\frac{a^2}{2}\arcsin\left(\frac{x}{a}\right)+\frac{a^2}{2}\left(\frac{x}{a}\right)\frac{\sqrt{a^2-x^2}}{a}+C=\frac{a^2}{2}\arcsin\left(\frac{x}{a}\right)+\frac{x}{2}\sqrt{a^2-x^2}+C$$

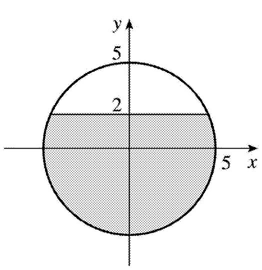
so the desired area is

$$A=2r\sqrt{R^2-r^2}+\left[r^2\arcsin(x/r)+x\sqrt{r^2-x^2}\right]_0^r-\left[R^2\arcsin(x/R)+x\sqrt{R^2-x^2}\right]_0^r$$
$$=2r\sqrt{R^2-r^2}+r^2\left(\frac{\pi}{2}\right)-\left[R^2\arcsin(r/R)+r\sqrt{R^2-r^2}\right]=r\sqrt{R^2-r^2}+\frac{\pi}{2}r^2-R^2\arcsin(r/R)$$

40. Note that the circular cross–sections of the tank are the same everywhere, so the percentage of the total capacity that is being used is equal to the percentage of any cross–section that is under water. The underwater area is

$$A=2\int_{-5}^{2}\sqrt{25-y^2}\,dy$$
$$=\left[25\arcsin(y/5)+y\sqrt{25-y^2}\right]_{-5}^{2}\ y=5\sin\theta\ ]$$
$$=25\arcsin\frac{2}{5}+2\sqrt{21}+\frac{25}{2}\pi\approx 58.72\text{ft}^2$$

so the fraction of the total capacity in use is $\dfrac{A}{\pi(5)^2}\approx\dfrac{58.72}{25\pi}\approx 0.748$ or 74.8% .



41. We use cylindrical shells and assume that $R>r$. $x^2=r^2-(y-R)^2 \Rightarrow x=\pm\sqrt{r^2-(y-R)^2}$, so $g(y)=2\sqrt{r^2-(y-R)^2}$ and

$$V = \int_{R-r}^{R+r} 2\pi y \cdot 2\sqrt{r^2-(y-R)^2}\, dy = \int_{-r}^{r} 4\pi(u+R)\sqrt{r^2-u^2}\, du \quad [\text{where } u=y-R]$$

$$= 4\pi\int_{-r}^{r} u\sqrt{r^2-u^2}\, du + 4\pi R\int_{-r}^{r}\sqrt{r^2-u^2}\, du \quad \left[\begin{array}{l}\text{where } u=r\sin\theta,\, du=r\cos\theta\, d\theta \\ \text{in the second tegral}\end{array}\right]$$

$$= 4\pi\left[-\frac{1}{3}\left(r^2-u^2\right)^{3/2}\right]_{-r}^{r} + 4\pi R\int_{-\pi/2}^{\pi/2} r^2\cos^2\theta\, d\theta = -\frac{4\pi}{3}(0-0) + 4\pi Rr^2\int_{-\pi/2}^{\pi/2}\cos^2\theta\, d\theta$$

$$= 2\pi Rr^2\int_{-\pi/2}^{\pi/2}(1+\cos 2\theta)\, d\theta = 2\pi Rr^2\left[\theta + \frac{1}{2}\sin 2\theta\right]_{-\pi/2}^{\pi/2} = 2\pi^2 Rr^2$$

Another method: Use washers instead of shells, so $V=8\pi R\int_0^r\sqrt{r^2-y^2}\, dy$ as in Exercise 6.2.61(a) , but evaluate the integral using $y=r\sin\theta$.

1. **(a)** $\dfrac{2x}{(x+3)(3x+1)} = \dfrac{A}{x+3} + \dfrac{B}{3x+1}$

(b) $\dfrac{1}{x^3+2x^2+x} = \dfrac{1}{x(x^2+2x+1)} = \dfrac{1}{x(x+1)^2} = \dfrac{A}{x} + \dfrac{B}{x+1} + \dfrac{C}{(x+1)^2}$

2. **(a)** $\dfrac{x-1}{x^3+x^2} = \dfrac{x-1}{x^2(x+1)} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x+1}$

(b) $\dfrac{x-1}{x^3+x} = \dfrac{x-1}{x\left(x^2+1\right)} = \dfrac{A}{x} + \dfrac{Bx+C}{x^2+1}$

3. **(a)** $\dfrac{2}{x^2+3x-4} = \dfrac{2}{(x+4)(x-1)} = \dfrac{A}{x+4} + \dfrac{B}{x-1}$

(b) x^2+x+1 is irreducible, so $\dfrac{x^2}{(x-1)\left(x^2+x+1\right)} = \dfrac{A}{x-1} + \dfrac{Bx+C}{x^2+x+1}$.

4. **(a)** $\dfrac{x^3}{x^2+4x+3} = x-4+\dfrac{13x+12}{x^2+4x+3} = x-4+\dfrac{13x+12}{(x+1)(x+3)} = x-4+\dfrac{A}{x+1} + \dfrac{B}{x+3}$

(b) $\dfrac{2x+1}{(x+1)^3(x^2+4)^2} = \dfrac{A}{x+1} + \dfrac{B}{(x+1)^2} + \dfrac{C}{(x+1)^3} + \dfrac{Dx+E}{x^2+4} + \dfrac{Fx+G}{(x^2+4)^2}$

5. **(a)**

$\dfrac{x^4}{x^4-1} = \dfrac{(x^4-1)+1}{x^4-1} = 1+\dfrac{1}{x^4-1}$ [or use long division] $= 1+\dfrac{1}{(x^2-1)(x^2+1)}$

$= 1+\dfrac{1}{(x-1)(x+1)(x^2+1)} = 1+\dfrac{A}{x-1} + \dfrac{B}{x+1} + \dfrac{Cx+D}{x^2+1}$

(b) $\dfrac{t^4+t^2+1}{\left(t^2+1\right)\left(t^2+4\right)^2} = \dfrac{At+B}{t^2+1} + \dfrac{Ct+D}{t^2+4} + \dfrac{Et+F}{\left(t^2+4\right)^2}$

6. **(a)** $\dfrac{x^4}{(x^3+x)(x^2-x+3)} = \dfrac{x^4}{x(x^2+1)(x^2-x+3)} = \dfrac{x^3}{(x^2+1)(x^2-x+3)} = \dfrac{Ax+B}{x^2+1} + \dfrac{Cx+D}{x^2-x+3}$

(b)

$$\frac{1}{x^6-x^3}=\frac{1}{x^3\left(x^3-1\right)}=\frac{1}{x^3(x-1)\left(x^2+x+1\right)}=\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x^3}+\frac{D}{x-1}+\frac{Ex+F}{x^2+x+1}$$

7. $\int \frac{x}{x-6}\,dx=\int \frac{(x-6)+6}{x-6}\,dx=\int \left(1+\frac{6}{x-6}\right)dx=x+6\ln|x-6|+C$

8.
$$\int \frac{r^2}{r+4}\,dr=\int \left(\frac{r^2-16}{r+4}+\frac{16}{r+4}\right)dr=\int \left(r-4+\frac{16}{r+4}\right)dr \text{ [or use long division]}$$
$$=\frac{1}{2}r^2-4r+16\ln|r+4|+C$$

9. $\frac{x-9}{(x+5)(x-2)}=\frac{A}{x+5}+\frac{B}{x-2}$. Multiply both sides by $(x+5)(x-2)$ to get $x-9=A(x-2)+B(x+5)$. Substituting 2 for x gives $-7=7B\Leftrightarrow B=-1$. Substituting -5 for x gives $-14=-7A\Leftrightarrow A=2$. Thus,
$$\int \frac{x-9}{(x+5)(x-2)}\,dx=\int \left(\frac{2}{x+5}+\frac{-1}{x-2}\right)dx=2\ln|x+5|-\ln|x-2|+C$$

10. $\frac{1}{(t+4)(t-1)}=\frac{A}{t+4}+\frac{B}{t-1}\Rightarrow 1=A(t-1)+B(t+4)$.
$t=1\Rightarrow 1=5B\Rightarrow B=\frac{1}{5}$. $t=-4\Rightarrow 1=-5A\Rightarrow A=-\frac{1}{5}$. Thus,
$$\int \frac{1}{(t+4)(t-1)}\,dt=\int \left(\frac{-1/5}{t+4}+\frac{1/5}{t-1}\right)dt=-\frac{1}{5}\ln|t+4|+\frac{1}{5}\ln|t-1|+C \text{ or } \frac{1}{5}\ln\left|\frac{t-1}{t+4}\right|+C$$

11. $\frac{1}{x^2-1}=\frac{1}{(x+1)(x-1)}=\frac{A}{x+1}+\frac{B}{x-1}$. Multiply both sides by $(x+1)(x-1)$ to get $1=A(x-1)+B(x+1)$.
Substituting 1 for x gives $1=2B\Leftrightarrow B=\frac{1}{2}$. Substituting -1 for x gives $1=-2A\Leftrightarrow A=-\frac{1}{2}$. Thus,
$$\int_2^3 \frac{1}{x^2-1}\,dx=\int_2^3 \left(\frac{-1/2}{x+1}+\frac{1/2}{x-1}\right)dx=\left[-\frac{1}{2}\ln|x+1|+\frac{1}{2}\ln|x-1|\right]_2^3$$
$$=\left(-\frac{1}{2}\ln 4+\frac{1}{2}\ln 2\right)-\left(-\frac{1}{2}\ln 3+\frac{1}{2}\ln 1\right)=\frac{1}{2}(\ln 2+\ln 3-\ln 4)\left[\text{ or } \frac{1}{2}\ln\frac{3}{2}\right]$$

12. $\frac{x-1}{x^2+3x+2}=\frac{A}{x+1}+\frac{B}{x+2}$. Multiply both sides by $(x+1)(x+2)$ to get $x-1=A(x+2)+B(x+1)$.
Substituting -2 for x gives $-3=-B\Leftrightarrow B=3$. Substituting -1 for x gives $-2=A$. Thus,

$$\int_0^1 \frac{x-1}{x^2+3x+2}\,dx = \int_0^1 \left(\frac{-2}{x+1} + \frac{3}{x+2} \right) dx = \left[-2\ln|x+1| + 3\ln|x+2| \right]_0^1$$

$$= (-2\ln 2 + 3\ln 3) - (-2\ln 1 + 3\ln 2) = 3\ln 3 - 5\ln 2 \left[\text{or} \ln \frac{27}{32} \right]$$

13. $\int \frac{ax}{x^2-bx}\,dx = \int \frac{ax}{x(x-b)}\,dx = \int \frac{a}{x-b}\,dx = a\ln|x-b| + C$

14. If $a \neq b$, $\frac{1}{(x+a)(x+b)} = \frac{1}{b-a} \left(\frac{1}{x+a} - \frac{1}{x+b} \right)$, so if $a \neq b$, then

$$\int \frac{dx}{(x+a)(x+b)} = \frac{1}{b-a} (\ln|x+a| - \ln|x+b|) + C = \frac{1}{b-a} \ln \left| \frac{x+a}{x+b} \right| + C$$

If $a=b$, then $\int \frac{dx}{(x+a)^2} = -\frac{1}{x+a} + C$.

15. $\frac{2x+3}{(x+1)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2} \Rightarrow 2x+3 = A(x+1) + B$. Take $x=-1$ to get $B=1$, and equate coefficients of x to get $A=2$. Now

$$\int_0^1 \frac{2x+3}{(x+1)^2}\,dx = \int_0^1 \left[\frac{2}{x+1} + \frac{1}{(x+1)^2} \right] dx = \left[2\ln(x+1) - \frac{1}{x+1} \right]_0^1$$

$$= 2\ln 2 - \frac{1}{2} - (2\ln 1 - 1) = 2\ln 2 + \frac{1}{2}$$

16. $\frac{x^3-4x-10}{x^2-x-6} = x+1 + \frac{3x-4}{(x-3)(x+2)}$. Write $\frac{3x-4}{(x-3)(x+2)} = \frac{A}{x-3} + \frac{B}{x+2}$. Then $3x-4 = A(x+2) + B(x-3)$. Taking $x=3$ and $x=-2$, we get $5=5A \Leftrightarrow A=1$ and $-10=-5B \Leftrightarrow B=2$, so

$$\int_0^1 \frac{x^3-4x-10}{x^2-x-6}\,dx = \int_0^1 \left(x+1 + \frac{1}{x-3} + \frac{2}{x+2} \right) dx = \left[\frac{1}{2}x^2 + x + \ln|x-3| + 2\ln(x+2) \right]_0^1$$

$$= \left(\frac{1}{2} + 1 + \ln 2 + 2\ln 3 \right) - (0 + 0 + \ln 3 + 2\ln 2) = \frac{3}{2} + \ln 3 - \ln 2 = \frac{3}{2} + \ln \frac{3}{2}$$

17. $\frac{4y^2-7y-12}{y(y+2)(y-3)}=\frac{A}{y}+\frac{B}{y+2}+\frac{C}{y-3}\Rightarrow 4y^2-7y-12=A(y+2)(y-3)+By(y-3)+Cy(y+2)$. Setting $y=0$ gives $-12=-6A$, so $A=2$. Setting $y=-2$ gives $18=10B$, so $B=\frac{9}{5}$. Setting $y=3$ gives $3=15C$, so $C=\frac{1}{5}$. Now

$$\int_1^2 \frac{4y^2-7y-12}{y(y+2)(y-3)}\,dy=\int_1^2\left(\frac{2}{y}+\frac{9/5}{y+2}+\frac{1/5}{y-3}\right)dy=\left[2\ln|y|+\frac{9}{5}\ln|y+2|+\frac{1}{5}\ln|y-3|\right]_1^2$$

$$=2\ln 2+\frac{9}{5}\ln 4+\frac{1}{5}\ln 1-2\ln 1-\frac{9}{5}\ln 3-\frac{1}{5}\ln 2$$

$$=2\ln 2+\frac{18}{5}\ln 2-\frac{1}{5}\ln 2-\frac{9}{5}\ln 3=\frac{27}{5}\ln 2-\frac{9}{5}\ln 3=\frac{9}{5}(3\ln 2-\ln 3)=\frac{9}{5}\ln\frac{8}{3}$$

18. $\frac{x^2+2x-1}{x^3-x}=\frac{x^2+2x-1}{x(x+1)(x-1)}=\frac{A}{x}+\frac{B}{x+1}+\frac{C}{x-1}$. Multiply both sides by $x(x+1)(x-1)$ to get $x^2+2x-1=A(x+1)(x-1)+Bx(x-1)+Cx(x+1)$. Substituting 0 for x gives $-1=-A\Leftrightarrow A=1$. Substituting -1 for x gives $-2=2B\Leftrightarrow B=-1$. Substituting 1 for x gives $2=2C\Leftrightarrow C=1$. Thus,

$$\int\frac{x^2+2x-1}{x^3-x}\,dx=\int\left(\frac{1}{x}-\frac{1}{x+1}+\frac{1}{x-1}\right)dx=\ln|x|-\ln|x+1|+\ln|x-1|+C=\ln\left|\frac{x(x-1)}{x+1}\right|+C\ .$$

19. $\frac{1}{(x+5)^2(x-1)}=\frac{A}{x+5}+\frac{B}{(x+5)^2}+\frac{C}{x-1}\Rightarrow 1=A(x+5)(x-1)+B(x-1)+C(x+5)^2$. Setting $x=-5$ gives $1=-6B$, so $B=-\frac{1}{6}$. Setting $x=1$ gives $1=36C$, so $C=\frac{1}{36}$. Setting $x=-2$ gives $1=A(3)(-3)+B(-3)+C\left(3^2\right)=-9A-3B+9C=-9A+\frac{1}{2}+\frac{1}{4}=-9A+\frac{3}{4}$, so $9A=-\frac{1}{4}$ and $A=-\frac{1}{36}$. Now

$$\int\frac{1}{(x+5)^2(x-1)}\,dx=\int\left[\frac{-1/36}{x+5}-\frac{1/6}{(x+5)^2}+\frac{1/36}{x-1}\right]dx$$

$$=-\frac{1}{36}\ln|x+5|+\frac{1}{6(x+5)}+\frac{1}{36}\ln|x-1|+C$$

20. $\frac{x^2}{(x-3)(x+2)^2}=\frac{A}{x-3}+\frac{B}{x+2}+\frac{C}{(x+2)^2}\Rightarrow x^2=A(x+2)^2+B(x-3)(x+2)+C(x-3)$.

Setting $x=3$ gives $A=\frac{9}{25}$. Take $x=-2$ to get $C=-\frac{4}{5}$, and equate the coefficients of x^2 to get $1=A+B$

$\Rightarrow B=\frac{16}{25}$. Then

$$\int \frac{x^2}{(x-3)(x+2)^2}\,dx = \int\left[\frac{9/25}{x-3}+\frac{16/25}{x+2}-\frac{4/5}{(x+2)^2}\right]dx$$
$$= \frac{9}{25}\ln|x-3|+\frac{16}{25}\ln|x+2|+\frac{4}{5(x+2)}+C$$

21. $\frac{5x^2+3x-2}{x^3+2x^2}=\frac{5x^2+3x-2}{x^2(x+2)}=\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x+2}$. Multiply by $x^2(x+2)$ to get $5x^2+3x-2=Ax(x+2)+B(x+2)+Cx^2$. Set $x=-2$ to get $C=3$, and take $x=0$ to get $B=-1$. Equating the coefficients of x^2 gives $5=A+C\Rightarrow A=2$. So

$$\int\frac{5x^2+3x-2}{x^3+2x^2}\,dx=\int\left(\frac{2}{x}-\frac{1}{x^2}+\frac{3}{x+2}\right)dx=2\ln|x|+\frac{1}{x}+3\ln|x+2|+C .$$

22. $\frac{1}{s^2(s-1)^2}=\frac{A}{s}+\frac{B}{s^2}+\frac{C}{s-1}+\frac{D}{(s-1)^2}\Rightarrow 1=As(s-1)^2+B(s-1)^2+Cs^2(s-1)+Ds^2$. Set $s=0$, giving $B=1$. Then set $s=1$ to get $D=1$. Equate the coefficients of s^3 to get $0=A+C$ or $A=-C$, and finally set $s=2$ to get $1=2A+1-4A+4$ or $A=2$. Now

$$\int\frac{ds}{s^2(s-1)^2}=\int\left[\frac{2}{s}+\frac{1}{s^2}-\frac{2}{s-1}+\frac{1}{(s-1)^2}\right]ds=2\ln|s|-\frac{1}{s}-2\ln|s-1|-\frac{1}{s-1}+C .$$

23. $\frac{x^2}{(x+1)^3}=\frac{A}{x+1}+\frac{B}{(x+1)^2}+\frac{C}{(x+1)^3}$. Multiply by $(x+1)^3$ to get $x^2=A(x+1)^2+B(x+1)+C$. Setting $x=-1$ gives $C=1$. Equating the coefficients of x^2 gives $A=1$, and setting $x=0$ gives $B=-2$. Now

$$\int\frac{x^2\,dx}{(x+1)^3}=\int\left[\frac{1}{x+1}-\frac{2}{(x+1)^2}+\frac{1}{(x+1)^3}\right]dx=\ln|x+1|+\frac{2}{x+1}-\frac{1}{2(x+1)^2}+C .$$

24. $\frac{x}{x+1}=\frac{(x+1)-1}{x+1}=1-\frac{1}{x+1}$, so $\frac{x^3}{(x+1)^3}=\left[1-\frac{1}{x+1}\right]^3=1-\frac{3}{x+1}+\frac{3}{(x+1)^2}-\frac{1}{(x+1)^3}$. Thus,

$$\int\frac{x^3}{(x+1)^3}\,dx=\int\left[1-\frac{3}{x+1}+\frac{3}{(x+1)^2}-\frac{1}{(x+1)^3}\right]dx=x-3\ln|x+1|-\frac{3}{x+1}+\frac{1}{2(x+1)^2}+C .$$

25. $\frac{10}{(x-1)\left(x^2+9\right)}=\frac{A}{x-1}+\frac{Bx+C}{x^2+9}$. Multiply both sides by $(x-1)\left(x^2+9\right)$ to get $10=A\left(x^2+9\right)+(Bx+C)(x-1)$ (*). Substituting 1 for x gives $10=10A\Leftrightarrow A=1$. Substituting 0 for x gives $10=9A-C\Rightarrow C=9(1)-10=-1$. The coefficients of the x^2 –terms in (*) must be equal, so $0=A+B$ $\Rightarrow B=-1$. Thus,

$$\int \frac{10}{(x-1)(x^2+9)}\,dx=\int\left(\frac{1}{x-1}+\frac{-x-1}{x^2+9}\right)dx=\int\left(\frac{1}{x-1}-\frac{x}{x^2+9}-\frac{1}{x^2+9}\right)dx$$

$$=\ln|x-1|-\frac{1}{2}\ln\left(x^2+9\right)\ [\text{let } u=x^2+9]-\frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right)\ [\textit{Formula}\,10]+C$$

26. $\frac{x^2-x+6}{x^3+3x}=\frac{x^2-x+6}{x\left(x^2+3\right)}=\frac{A}{x}+\frac{Bx+C}{x^2+3}$. Multiply by $x\left(x^2+3\right)$ to get $x^2-x+6=A\left(x^2+3\right)+(Bx+C)x$. Substituting 0 for x gives $6=3A\Leftrightarrow A=2$. The coefficients of the x^2 –terms must be equal, so $1=A+B\Rightarrow$ $B=1-2=-1$. The coefficients of the x –terms must be equal, so $-1=C$. Thus,

$$\int \frac{x^2-x+6}{x^3+3x}\,dx=\int\left(\frac{2}{x}+\frac{-x-1}{x^2+3}\right)dx=\int\left(\frac{2}{x}-\frac{x}{x^2+3}-\frac{1}{x^2+3}\right)dx$$

$$=2\ln|x|-\frac{1}{2}\ln\left(x^2+3\right)-\frac{1}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}}+C$$

27. $\frac{x^3+x^2+2x+1}{\left(x^2+1\right)\left(x^2+2\right)}=\frac{Ax+B}{x^2+1}+\frac{Cx+D}{x^2+2}$. Multiply both sides by $\left(x^2+1\right)\left(x^2+2\right)$ to get

$x^3+x^2+2x+1=(Ax+B)\left(x^2+2\right)+(Cx+D)\left(x^2+1\right)\Leftrightarrow$

$x^3+x^2+2x+1=\left(Ax^3+Bx^2+2Ax+2B\right)+\left(Cx^3+Dx^2+Cx+D\right)\Leftrightarrow$

$x^3+x^2+2x+1=(A+C)x^3+(B+D)x^2+(2A+C)x+(2B+D)$. Comparing coefficients gives us the following system of equations:

$A+C=1$	**(1)**	$B+D=1$	**(2)**
$2A+C=2$	**(3)**	$2B+D=1$	**(4)**

Subtracting equation **(1)** from equation **(3)** gives us $A=1$, so $C=0$. Subtracting equation **(2)** from equation **(4)** gives us $B=0$, so $D=1$. Thus, $I=\int \frac{x^3+x^2+2x+1}{\left(x^2+1\right)\left(x^2+2\right)}\,dx=\int\left(\frac{x}{x^2+1}+\frac{1}{x^2+2}\right)dx$. For

$\int \frac{x}{x^2+1}\,dx$, let $u=x^2+1$ so $du=2x\,dx$ and then $\int \frac{x}{x^2+1}\,dx=\frac{1}{2}\int \frac{1}{u}\,du=\frac{1}{2}\ln|u|+C=\frac{1}{2}\ln\left(x^2+1\right)+C$. For $\int \frac{1}{x^2+2}\,dx$, use Formula 10 with $a=\sqrt{2}$. So $\int \frac{1}{x^2+2}\,dx=\int \frac{1}{x^2+\left(\sqrt{2}\right)^2}\,dx=\frac{1}{\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}}+C$. Thus, $I=\frac{1}{2}\ln\left(x^2+1\right)+\frac{1}{\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}}+C$.

28. $\frac{x^2-2x-1}{(x-1)^2\left(x^2+1\right)}=\frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{Cx+D}{x^2+1}\Rightarrow x^2-2x-1=A(x-1)\left(x^2+1\right)+B\left(x^2+1\right)+(Cx+D)(x-1)^2$.

Setting $x=1$ gives $B=-1$. Equating the coefficients of x^3 gives $A=-C$. Equating the constant terms gives $-1=-A-1+D$, so $D=A$,
and setting $x=2$ gives $-1=5A-5-2A+A$ or $A=1$. We have

$$\int \frac{x^2-2x-1}{(x-1)^2\left(x^2+1\right)}\,dx=\int\left[\frac{1}{x-1}-\frac{1}{(x-1)^2}-\frac{x-1}{x^2+1}\right]dx$$
$$=\ln|x-1|+\frac{1}{x-1}-\frac{1}{2}\ln\left(x^2+1\right)+\tan^{-1}x+C$$

29.

$$\int \frac{x+4}{x^2+2x+5}\,dx=\int \frac{x+1}{x^2+2x+5}\,dx+\int \frac{3}{x^2+2x+5}\,dx=\frac{1}{2}\int \frac{(2x+2)\,dx}{x^2+2x+5}+\int \frac{3\,dx}{(x+1)^2+4}$$
$$=\frac{1}{2}\ln\left|x^2+2x+5\right|+3\int \frac{2\,du}{4(u^2+1)}\quad\left[\begin{array}{c}\text{where } x+1=2u,\\ \text{and } dx=2\,du\end{array}\right]$$
$$=\frac{1}{2}\ln(x^2+2x+5)+\frac{3}{2}\tan^{-1}u+C=\frac{1}{2}\ln(x^2+2x+5)+\frac{3}{2}\tan^{-1}\left(\frac{x+1}{2}\right)+C$$

30. $\frac{x^3-2x^2+x+1}{x^4+5x^2+4}=\frac{x^3-2x^2+x+1}{\left(x^2+1\right)\left(x^2+4\right)}=\frac{Ax+B}{x^2+1}+\frac{Cx+D}{x^2+4}\Rightarrow x^3-2x^2+x+1=(Ax+B)\left(x^2+4\right)+(Cx+D)\left(x^2+1\right)$. Equating coefficients gives $A+C=1$, $B+D=-2$, $4A+C=1$, $4B+D=1\Rightarrow A=0$, $C=1$, $B=1$, $D=-3$. Now $\int \frac{x^3-2x^2+x+1}{x^4+5x^2+4}\,dx=\int \frac{dx}{x^2+1}+\int \frac{x-3}{x^2+4}\,dx=\tan^{-1}x+\frac{1}{2}\ln\left(x^2+4\right)-\frac{3}{2}\tan^{-1}(x/2)+C$.

31. $\frac{1}{x^3-1}=\frac{1}{(x-1)\left(x^2+x+1\right)}=\frac{A}{x-1}+\frac{Bx+C}{x^2+x+1}\Rightarrow 1=A\left(x^2+x+1\right)+(Bx+C)(x-1)$.

Take $x=1$ to get $A=\frac{1}{3}$. Equating coefficients of x^2 and then comparing the constant terms, we get $0=\frac{1}{3}+B$, $1=\frac{1}{3}-C$, so $B=-\frac{1}{3}$, $C=-\frac{2}{3}\Rightarrow$

$$\int\frac{1}{x^3-1}\,dx = \int\frac{\frac{1}{3}}{x-1}\,dx+\int\frac{-\frac{1}{3}x-\frac{2}{3}}{x^2+x+1}\,dx=\frac{1}{3}\ln|x-1|-\frac{1}{3}\int\frac{x+2}{x^2+x+1}\,dx$$

$$=\frac{1}{3}\ln|x-1|-\frac{1}{3}\int\frac{x+1/2}{x^2+x+1}\,dx-\frac{1}{3}\int\frac{(3/2)\,dx}{(x+1/2)^2+3/4}$$

$$=\frac{1}{3}\ln|x-1|-\frac{1}{6}\ln\left(x^2+x+1\right)-\frac{1}{2}\left(\frac{2}{\sqrt{3}}\right)\tan^{-1}\left(\frac{x+\frac{1}{2}}{\sqrt{3}/2}\right)+K$$

$$=\frac{1}{3}\ln|x-1|-\frac{1}{6}\ln\left(x^2+x+1\right)-\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{1}{\sqrt{3}}(2x+1)\right)+K$$

32.

$$\int_0^1\frac{x}{x^2+4x+13}\,dx=\int_0^1\frac{\frac{1}{2}(2x+4)}{x^2+4x+13}\,dx-2\int_0^1\frac{dx}{(x+2)^2+9}$$

$$=\frac{1}{2}\int_{13}^{18}\frac{dy}{y}-2\int_{2/3}^{1}\frac{3}{du}\,9u^2+9\left[\begin{array}{c}\text{where } y=x^2+4x+13, dy=(2x+4)\,dx,\\ x+2=3u,\text{ and } dx=3\,du\end{array}\right]$$

$$=\frac{1}{2}\left[\ln y\right]_{13}^{18}-\frac{2}{3}\left[\tan^{-1}u\right]_{2/3}^{1}=\frac{1}{2}\ln\frac{18}{13}-\frac{2}{3}\left(\frac{\pi}{4}-\tan^{-1}\left(\frac{2}{3}\right)\right)$$

$$=\frac{1}{2}\ln\frac{18}{13}-\frac{\pi}{6}+\frac{2}{3}\tan^{-1}\left(\frac{2}{3}\right)$$

33. Let $u=x^3+3x^2+4$. Then $du=3\left(x^2+2x\right)dx\Rightarrow$

$$\int_2^5\frac{x^2+2x}{x^3+3x^2+4}\,dx=\frac{1}{3}\int_{24}^{204}\frac{}{du}\,u=\frac{1}{3}\left[\ln u\right]_{24}^{204}=\frac{1}{3}(\ln 204-\ln 24)=\frac{1}{3}\ln\frac{204}{24}=\frac{1}{3}\ln\frac{17}{2}\ .$$

34. $\frac{x^3}{x^3+1}=\frac{\left(x^3+1\right)-1}{x^3+1}=1-\frac{1}{x^3+1}=1-\left(\frac{A}{x+1}+\frac{Bx+C}{x^2-x+1}\right)\Rightarrow 1=A\left(x^2-x+1\right)+(Bx+C)(x+1)$. Equate the terms of degree 2 , 1 and 0 to get $0=A+B$, $0=-A+B+C$, $1=A+C$. Solve the three equations to get

$A=\frac{1}{3}$, $B=-\frac{1}{3}$, and $C=\frac{2}{3}$. So

$$\int \frac{x^3}{x^3+1}\,dx = \int \left[1-\frac{\frac{1}{3}}{x+1}+\frac{\frac{1}{3}x-\frac{2}{3}}{x^2-x+1} \right] dx$$

$$= x-\frac{1}{3}\ln|x+1|+\frac{1}{6}\int \frac{2x-1}{x^2-x+1}\,dx-\frac{1}{2}\int \frac{dx}{\left(x-\frac{1}{2}\right)^2+\frac{3}{4}}$$

$$= x-\frac{1}{3}\ln|x+1|+\frac{1}{6}\ln\left(x^2-x+1\right)-\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{1}{\sqrt{3}}(2x-1)\right)+K$$

35. $\frac{1}{x^4-x^2}=\frac{1}{x^2(x-1)(x+1)}=\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x-1}+\frac{D}{x+1}$. Multiply by $x^2(x-1)(x+1)$ to get $1=Ax(x-1)(x+1)+B(x-1)(x+1)+Cx^2(x+1)+Dx^2(x-1)$. Setting $x=1$ gives $C=\frac{1}{2}$, taking $x=-1$ gives $D=-\frac{1}{2}$. Equating the coefficients of x^3 gives $0=A+C+D=A$. Finally, setting $x=0$ yields $B=-1$. Now $\int \frac{dx}{x^4-x^2}=\int \left[\frac{-1}{x^2}+\frac{1/2}{x-1}-\frac{1/2}{x+1} \right] dx=\frac{1}{x}+\frac{1}{2}\ln\left|\frac{x-1}{x+1}\right|+C$.

36. Let $u=x^4+5x^2+4 \Rightarrow du=\left(4x^3+10x\right)dx=2\left(2x^3+5x\right)dx$, so

$$\int_0^1 \frac{2x^3+5x}{x^4+5x^2+4}\,dx=\frac{1}{2}\int_4^{10}\frac{du}{u}=\frac{1}{2}\left[\ln|u|\right]_4^{10}=\frac{1}{2}(\ln 10-\ln 4)=\frac{1}{2}\ln\frac{5}{2} .$$

37. $\int \frac{x-3}{\left(x^2+2x+4\right)^2}\,dx=\int \frac{x-3}{\left[(x+1)^2+3\right]^2}\,dx=\int \frac{u-4}{\left(u^2+3\right)^2}\,du$ [with $u=x+1$]

$$=\int \frac{u\,du}{\left(u^2+3\right)^2}-4\int \frac{du}{\left(u^2+3\right)^2}=\frac{1}{2}\int \frac{dv}{v^2}-4\int \frac{\sqrt{3}\sec^2\theta\,d\theta}{9\sec^4\theta} \quad \left[\begin{array}{l} v=u^2+3 \text{ in the first integral;} \\ u=\sqrt{3}\tan\theta \text{ in the second} \end{array} \right]$$

$$=\frac{-1}{(2v)}-\frac{4\sqrt{3}}{9}\int \cos^2\theta\,d\theta=\frac{-1}{2\left(u^2+3\right)}-\frac{2\sqrt{3}}{9}(\theta+\sin\theta\cos\theta)+C$$

$$=\frac{-1}{2\left(x^2+2x+4\right)}-\frac{2\sqrt{3}}{9}\left[\tan^{-1}\left(\frac{x+1}{\sqrt{3}}\right)+\frac{\sqrt{3}(x+1)}{x^2+2x+4}\right]+C$$

$$=\frac{-1}{2\left(x^2+2x+4\right)}-\frac{2\sqrt{3}}{9}\tan^{-1}\left(\frac{x+1}{\sqrt{3}}\right)-\frac{2(x+1)}{3\left(x^2+2x+4\right)}+C$$

38. $\dfrac{x^4+1}{x\left(x^2+1\right)^2}=\dfrac{A}{x}+\dfrac{Bx+C}{x^2+1}+\dfrac{Dx+E}{\left(x^2+1\right)^2}\Rightarrow x^4+1=A\left(x^2+1\right)^2+(Bx+C)x\left(x^2+1\right)+(Dx+E)x$. Setting $x=0$ gives $A=1$, and equating the coefficients of x^4 gives $1=A+B$, so $B=0$. Now

$\dfrac{C}{x^2+1}+\dfrac{Dx+E}{\left(x^2+1\right)^2}=\dfrac{x^4+1}{x\left(x^2+1\right)^2}-\dfrac{1}{x}=\dfrac{1}{x}\left[\dfrac{x^4+1-\left(x^4+2x^2+1\right)}{\left(x^2+1\right)^2}\right]=\dfrac{-2x}{\left(x^2+1\right)^2}$, so we can take $C=0$, $D=-2$, and $E=0$. Hence, $\displaystyle\int\frac{x^4+1}{x\left(x^2+1\right)^2}dx=\int\left[\frac{1}{x}-\frac{2x}{\left(x^2+1\right)^2}\right]dx=\ln|x|+\frac{1}{x^2+1}+C$.

39. Let $u=\sqrt{x+1}$. Then $x=u^2-1$, $dx=2u\,du\Rightarrow$

$\displaystyle\int\frac{dx}{x\sqrt{x+1}}=\int\frac{2u\,du}{\left(u^2-1\right)u}=2\int\frac{du}{u^2-1}=\ln\left|\frac{u-1}{u+1}\right|+C=\ln\left|\frac{\sqrt{x+1}-1}{\sqrt{x+1}+1}\right|+C$.

40. Let $u=\sqrt{x+2}$. Then $x=u^2-2$, $dx=2u\,du\Rightarrow I=\displaystyle\int\frac{dx}{x-\sqrt{x+2}}=\int\frac{2u\,du}{u^2-2-u}=2\int\frac{u\,du}{u^2-u-2}$ and

$\dfrac{u}{u^2-u-2}=\dfrac{A}{u-2}+\dfrac{B}{u+1}\Rightarrow u=A(u+1)+B(u-2)$. Substituting -1 for u gives $-1=-3B\Leftrightarrow B=\dfrac{1}{3}$ and substituting 2 for u gives $2=3A\Leftrightarrow A=\dfrac{2}{3}$. Thus,

$$I=\frac{2}{3}\int\left[\frac{2}{u-2}+\frac{1}{u+1}\right]du=\frac{2}{3}\left(2\ln|u-2|+\ln|u+1|\right)+C$$

$$=\frac{2}{3}\left[2\ln\left|\sqrt{x+2}-2\right|+\ln\left(\sqrt{x+2}+1\right)\right]+C$$

41. Let $u=\sqrt{x}$, so $u^2=x$ and $dx=2u\,du$. Thus,

$$\int_9^{16}\frac{\sqrt{x}}{x-4}dx=\int_3^4\frac{u}{u^2-4}2u\,du=2\int_3^4\frac{u^2}{u^2-4}du=2\int_3^4\left(1+\frac{4}{u^2-4}\right)du\quad\text{[by long division]}$$

$$=2+8\int_3^4\frac{}{du}(u+2)(u-2).\ (\ *\ )$$

Multiply $\frac{1}{(u+2)(u-2)} = \frac{A}{u+2} + \frac{B}{u-2}$ by $(u+2)(u-2)$ to get $1=A(u-2)+B(u+2)$. Equating coefficients we get $A+B=0$ and $-2A+2B=1$. Solving gives us $B=\frac{1}{4}$ and $A=-\frac{1}{4}$, so $\frac{1}{(u+2)(u-2)} = \frac{-1/4}{u+2} + \frac{1/4}{u-2}$ and (*) is

$$2+8\int_3^4 \left(\frac{-1/4}{u+2} + \frac{1/4}{u-2} \right) du = 2+8\left[-\frac{1}{4}\ln|u+2| + \frac{1}{4}\ln|u-2| \right]_3^4$$

$$= 2+[2\ln|u-2| - 2\ln|u+2|]_3^4 = 2+2\left[\ln\left| \frac{u-2}{u+2} \right| \right]_3^4$$

$$= 2+2\left(\ln\frac{2}{6} - \ln\frac{1}{5} \right) = 2+2\ln\frac{2/6}{1/5}$$

$$= 2+2\ln\frac{5}{3} \text{ or } 2+\ln\left(\frac{5}{3}\right)^2 = 2+\ln\frac{25}{9}$$

42. Let $u=\sqrt[3]{x}$. Then $x=u^3$, $dx=3u^2\,du \Rightarrow$

$$\int_0^1 \frac{1}{1+\sqrt[3]{x}}\,dx = \int_0^1 \frac{3u^2}{du}\, 1+u = \int_0^1 \left(3u-3+\frac{3}{1+u} \right) du = \left[\frac{3}{2}u^2 - 3u + 3\ln(1+u) \right]_0^1$$

$$= 3\left(\ln 2 - \frac{1}{2} \right)$$

43. Let $u=\sqrt[3]{x^2+1}$. Then $x^2=u^3-1$, $2x\,dx=3u^2\,du \Rightarrow$

$$\int \frac{x^3\,dx}{\sqrt[3]{x^2+1}} = \int \frac{\left(u^3-1\right)\frac{3}{2}u^2\,du}{u} = \frac{3}{2}\int\left(u^4-u\right)du = \frac{3}{10}u^5 - \frac{3}{4}u^2 + C$$

$$= \frac{3}{10}\left(x^2+1\right)^{5/3} - \frac{3}{4}\left(x^2+1\right)^{2/3} + C$$

44. Let $u=\sqrt{x}$. Then $x=u^2$, $dx=2u\,du \Rightarrow \int_{1/3}^{3} \frac{\sqrt{x}}{x^2+x}\,dx = \int_{1/\sqrt{3}}^{\sqrt{3}} \frac{u\cdot 2u\,du}{u^4+u^2} = 2\int_{1/\sqrt{3}}^{\sqrt{3}} \frac{du}{u^2+1}$

$$=2\left[\tan^{-1}u\right]_{1/\sqrt{3}}^{\sqrt{3}} = 2\left(\frac{\pi}{3} - \frac{\pi}{6} \right) = \frac{\pi}{3}$$

45. If we were to substitute $u=\sqrt{x}$, then the square root would disappear but a cube root would remain. On the other hand, the substitution

$u=\sqrt[3]{x}$ would eliminate the cube root but leave a square root. We can eliminate both roots by means of the substitution $u=\sqrt[6]{x}$. (Note that 6 is the least common multiple of 2 and 3 .)
Let $u=\sqrt[6]{x}$. Then $x=u^6$, so $dx=6u^5\,du$ and $\sqrt{x}=u^3$, $\sqrt[3]{x}=u^2$. Thus,

$$\int \frac{dx}{\sqrt{x}-\sqrt[3]{x}} = \int \frac{6u^5\,du}{u^3-u^2} = 6\int \frac{u^5}{u^2(u-1)}\,du = 6\int \frac{u^3}{u-1}\,du$$

$$= 6\int \left(u^2+u+1+\frac{1}{u-1} \right) du \quad \text{[by long division]}$$

$$= 6\left(\frac{1}{3}u^3+\frac{1}{2}u^2+u+\ln|u-1| \right)+C = 2\sqrt{x}+3\sqrt[3]{x}+6\sqrt[6]{x}+6\ln\left|\sqrt[6]{x}-1\right|+C$$

46. Let $u=\sqrt[12]{x}$. Then $x=u^{12}$, $dx=12u^{11}\,du \Rightarrow$

$$\int \frac{dx}{\sqrt[3]{x}+\sqrt[4]{x}} = \int \frac{12u^{11}\,du}{u^4+u^3} = 12\int \frac{u^8\,du}{u+1} = 12\int \left(u^7-u^6+u^5-u^4+u^3-u^2+u-1+\frac{1}{u+1} \right) du$$

$$= \frac{3}{2}u^8-\frac{12}{7}u^7+2u^6-\frac{12}{5}u^5+3u^4-4u^3+6u^2-12u+12\ln|u+1|+C$$

$$= \frac{3}{2}x^{2/3}-\frac{12}{7}x^{7/12}+2\sqrt{x}-\frac{12}{5}x^{5/12}+3\sqrt[3]{x}-4\sqrt[4]{x}+6\sqrt[6]{x}-12\sqrt[12]{x}+12\ln\left(\sqrt[12]{x}+1\right)+C$$

47. Let $u=e^x$. Then $x=\ln u$, $dx=\frac{du}{u} \Rightarrow$

$$\int \frac{e^{2x}\,dx}{e^{2x}+3e^x+2} = \int \frac{u^2\,(du/u)}{u^2+3u+2} = \int \frac{u\,du}{(u+1)(u+2)} = \int \left[\frac{-1}{u+1}+\frac{2}{u+2} \right] du$$

$$= 2\ln|u+2|-\ln|u+1|+C = \ln\left[(e^x+2)^2/(e^x+1)\right]+C$$

48. Let $u=\sin x$. Then $du=\cos x\,dx \Rightarrow$

$$\int \frac{\cos x\,dx}{\sin^2 x+\sin x} = \int \frac{du}{u^2+u} = \int \frac{du}{u(u+1)} = \int \left[\frac{1}{u}-\frac{1}{u+1} \right] du$$

$$= \ln\left| \frac{u}{u+1} \right| +C = \ln\left| \frac{\sin x}{1+\sin x} \right| +C$$

49. Let $u=\ln\left(x^2-x+2\right)$, $dv=dx$. Then $du=\frac{2x-1}{x^2-x+2}\,dx$, $v=x$, and (by integration by parts)

$$\int \ln\left(x^2-x+2\right)dx = x\ln\left(x^2-x+2\right)-\int \frac{2x^2-x}{x^2-x+2}\,dx = x\ln\ x^2-x+2\ -\int\left(2+\frac{x-4}{x^2-x+2}\right)dx$$

$$= x\ln\left(x^2-x+2\right)-2x-\int \frac{\frac{1}{2}(2x-1)}{x^2-x+2}\,dx+\frac{7}{2}\int \frac{dx}{\left(x-\frac{1}{2}\right)^2+\frac{7}{4}}$$

$$= x\ln\left(x^2-x+2\right)-2x-\frac{1}{2}\ln\left(x^2-x+2\right)+\frac{7}{2}\int \frac{\frac{\sqrt{7}}{2}\,du}{\frac{7}{4}(u^2+1)}$$

$$\left[\begin{array}{c}\text{where } x-\frac{1}{2}=\frac{\sqrt{7}}{2}u,\\ dx=\frac{\sqrt{7}}{2}du,\\ \left(x-\frac{1}{2}\right)^2+\frac{7}{4}=\frac{7}{4}(u^2+1)\end{array}\right]$$

$$=\left(x-\frac{1}{2}\right)\ln\left(x^2-x+2\right)-2x+\sqrt{7}\tan^{-1}u+C$$

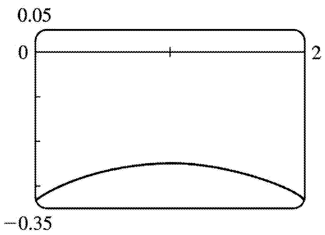
$$=\left(x-\frac{1}{2}\right)\ln\left(x^2-x+2\right)-2x+\sqrt{7}\tan^{-1}\frac{2x-1}{\sqrt{7}}+C$$

50. Let $u=\tan^{-1}x$, $dv=x\,dx \Rightarrow du=dx/\left(1+x^2\right)$, $v=\frac{1}{2}x^2$.

Then $\int x\tan^{-1}x\,dx=\frac{1}{2}x^2\tan^{-1}x-\frac{1}{2}\int \frac{x^2}{1+x^2}\,dx$. To evaluate the last integral, use long division or observe that $\int \frac{x^2}{1+x^2}\,dx=\int \frac{\left(1+x^2\right)-1}{1+x^2}\,dx=\int 1\,dx-\int \frac{1}{1+x^2}\,dx=x-\tan^{-1}x+C_1$. So

$$\int x\tan^{-1}x\,dx=\frac{1}{2}x^2\tan^{-1}x-\frac{1}{2}\left(x-\tan^{-1}x+C_1\right)=\frac{1}{2}\left(x^2\tan^{-1}x+\tan^{-1}x-x\right)+C .$$

51.



From the graph, we see that the integral will be negative, and we guess that the area is about the same as that of a rectangle with width 2 and height 0.3 , so we estimate the integral to be $-(2\cdot 0.3)=-0.6$.

Now $\dfrac{1}{x^2-2x-3}=\dfrac{1}{(x-3)(x+1)}=\dfrac{A}{x-3}+\dfrac{B}{x+1}\Leftrightarrow 1=(A+B)x+A-3B$, so $A=-B$ and $A-3B=1\Leftrightarrow A=\dfrac{1}{4}$ and $B=-\dfrac{1}{4}$, so the integral becomes

$$\int_0^2 \frac{dx}{x^2-2x-3}=\frac{1}{4}\int_0^2\frac{dx}{x-3}-\frac{1}{4}\int_0^2\frac{dx}{x+1}=\frac{1}{4}\Big[\ln|x-3|-\ln|x+1|\Big]_0^2$$

$$=\frac{1}{4}\left[\ln\left|\frac{x-3}{x+1}\right|\right]_0^2=\frac{1}{4}\left(\ln\frac{1}{3}-\ln 3\right)=-\frac{1}{2}\ln 3\approx -0.55$$

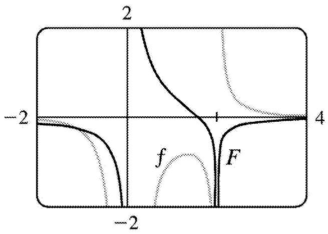
52. $\dfrac{1}{x^3-2x^2}=\dfrac{1}{x^2(x-2)}=\dfrac{A}{x}+\dfrac{B}{x^2}+\dfrac{C}{x-2}\Rightarrow 1=(A+C)x^2+(B-2A)x-2B$, so $A+C=B-2A=0$ and $-2B=1\Rightarrow B=-\dfrac{1}{2}$, $A=-\dfrac{1}{4}$, and $C=\dfrac{1}{4}$. So the general antiderivative of $\dfrac{1}{x^3-2x^2}$ is

$$\int\frac{dx}{x^3-2x^2}=-\frac{1}{4}\int\frac{dx}{x}-\frac{1}{2}\int\frac{dx}{x^2}+\frac{1}{4}\int\frac{dx}{x-2}$$

$$=-\frac{1}{4}\ln|x|-\frac{1}{2}(-1/x)+\frac{1}{4}\ln|x-2|+C$$

$$=\frac{1}{4}\ln\left|\frac{x-2}{x}\right|+\frac{1}{2x}+C$$

We plot this function with $C=0$ on the same screen as $y=\dfrac{1}{x^3-2x^2}$.



53.

$$\int \frac{dx}{x^2-2x} = \int \frac{dx}{(x-1)^2-1} = \int \frac{du}{u^2-1}$$

$$= \frac{1}{2}\ln\left|\frac{u-1}{u+1}\right| + C \text{ [by Equation 6] } = \frac{1}{2}\ln\left|\frac{x-2}{x}\right| + C$$

54.

$$\int \frac{(2x+1)dx}{4x^2+12x-7} = \frac{1}{4}\int \frac{(8x+12)dx}{4x^2+12x-7} - \int \frac{2dx}{(2x+3)^2-16}$$

$$= \frac{1}{4}\ln\left|4x^2+12x-7\right| - \int \frac{du}{u^2-16}$$

$$= \frac{1}{4}\ln\left|4x^2+12x-7\right| - \frac{1}{8}\ln\left|(u-4)/(u+4)\right| + C$$

$$= \frac{1}{4}\ln\left|4x^2+12x-7\right| - \frac{1}{8}\ln\left|(2x-1)/(2x+7)\right| + C$$

55. **(a)** If $t=\tan\left(\frac{x}{2}\right)$, then $\frac{x}{2}=\tan^{-1}t$. The figure gives $\cos\left(\frac{x}{2}\right)=\frac{1}{\sqrt{1+t^2}}$ and $\sin\left(\frac{x}{2}\right)=\frac{t}{\sqrt{1+t^2}}$.

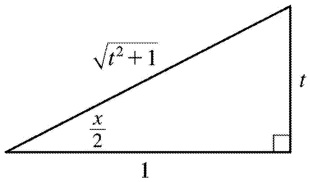
(b) $\cos x=\cos\left(2\cdot\frac{x}{2}\right)=2\cos^2\left(\frac{x}{2}\right)-1$

$$=2\left(\frac{1}{\sqrt{1+t^2}}\right)^2-1=\frac{2}{1+t^2}-1=\frac{1-t^2}{1+t^2}$$

$$\sin x=\sin\left(2\cdot\frac{x}{2}\right)=2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)=2\frac{t}{\sqrt{1+t^2}}\frac{1}{\sqrt{1+t^2}}=\frac{2t}{1+t^2}$$

(c)

$$\frac{x}{2}=\arctan t \Rightarrow x=2\arctan t \Rightarrow dx=\frac{2}{1+t^2}\,dt$$



56. Let $t=\tan(x/2)$. Then, using Exercise 55 , $dx=\frac{2}{1+t^2}\,dt$, $\sin x=\frac{2t}{1+t^2}\Rightarrow$

$$\int\frac{dx}{3-5\sin x}=\int\frac{2dt/\left(1+t^2\right)}{3-10t/\left(1+t^2\right)}=\int\frac{2dt}{3\left(1+t^2\right)-10t}=2\int\frac{dt}{3t^2-10t+3}$$

$$=\frac{1}{4}\int\left[\frac{1}{t-3}-\frac{3}{3t-1}\right]dt=\frac{1}{4}\left(\ln|t-3|-\ln|3t-1|\right)+C=\frac{1}{4}\ln\left|\frac{\tan(x/2)-3}{3\tan(x/2)-1}\right|+C$$

57. Let $t=\tan(x/2)$. Then, the expressions in Exercise 55 , we have

$$\int\frac{1}{3\sin x-4\cos x}\,dx=\int\frac{1}{3\left(\frac{2t}{1+t^2}\right)-4\left(\frac{1-t^2}{1+t^2}\right)}\frac{2dt}{1+t^2}=2\int\frac{dt}{3(2t)-4\left(1-t^2\right)}=\int\frac{dt}{2t^2+3t-2}$$

$$=\int\frac{dt}{(2t-1)(t+2)}=\int\left[\frac{2}{5}\frac{1}{2t-1}-\frac{1}{5}\frac{1}{t+2}\right]dt\quad\text{[using partial fractions]}$$

$$=\frac{1}{5}\left[\ln|2t-1|-\ln|t+2|\right]+C=\frac{1}{5}\ln\left|\frac{2t-1}{t+2}\right|+C=\frac{1}{5}\ln\left|\frac{2\tan(x/2)-1}{\tan(x/2)+2}\right|+C$$

58. Let $t=\tan(x/2)$. Then, by Exercise 55 ,

$$\int_{\pi/3}^{\pi/2}\frac{dx}{1+\sin x-\cos x}=\int_{1/\sqrt{3}}^{1}\frac{2dt/\left(1+t^2\right)}{1+2t/\left(1+t^2\right)-\left(1-t^2\right)/\left(1+t^2\right)}=\int_{1/\sqrt{3}}^{1}\frac{2dt}{1+t^2+2t-1+t^2}$$

$$=\int_{1/\sqrt{3}}^{1}\left[\frac{1}{t}-\frac{1}{t+1}\right]dt=\left[\ln t-\ln(t+1)\right]_{1/\sqrt{3}}^{1}=\ln\frac{1}{2}-\ln\frac{1}{\sqrt{3}+1}=\ln\frac{\sqrt{3}+1}{2}$$

59. Let $t=\tan(x/2)$. Then, by Exercise 55 ,

$$\int \frac{dx}{2\sin x+\sin 2x} = \frac{1}{2}\int \frac{dx}{\sin x+\sin x\cos x} = \frac{1}{2}\int \frac{2\,dt/\left(1+t^2\right)}{2t/\left(1+t^2\right)+2t\left(1-t^2\right)/\left(1+t^2\right)^2}$$

$$= \frac{1}{2}\int \frac{\left(1+t^2\right)dt}{t\left(1+t^2\right)+t\left(1-t^2\right)} = \frac{1}{4}\int \frac{\left(1+t^2\right)dt}{t} = \frac{1}{4}\int\left(\frac{1}{t}+t\right)dt$$

$$= \frac{1}{4}\ln|t|+\frac{1}{8}t^2+C = \frac{1}{4}\ln\left|\tan\left(\frac{1}{2}x\right)\right|+\frac{1}{8}\tan^2\left(\frac{1}{2}x\right)+C$$

60. $x^2-6x+8=(x-3)^2-1$ is positive for $5\le x\le 10$, so

$$\text{area} = \int_5^{10}\frac{dx}{(x-3)^2-1}=\int_2^7\frac{du}{u^2-1}\ [\text{put}\ u=x-3]=\left[\frac{1}{2}\ln\left|\frac{u-1}{u+1}\right|\right]_2^7$$

$$= \frac{1}{2}\ln\frac{3}{4}-\frac{1}{2}\ln\frac{1}{3}=\frac{1}{2}(\ln 3-2\ln 2+\ln 3)=\ln 3-\ln 2=\ln\frac{3}{2}$$

61. $\frac{x+1}{x-1}=1+\frac{2}{x-1}>0$ for $2\le x\le 3$, so area

$$=\int_2^3\left[1+\frac{2}{x-1}\right]dx=\left[x+2\ln|x-1|\right]_2^3=(3+2\ln 2)-(2+2\ln 1)=1+2\ln 2.$$

62. **(a)** We use disks, so the volume is $V=\pi\int_0^1\left[\frac{1}{x^2+3x+2}\right]^2dx=\pi\int_0^1\frac{dx}{(x+1)^2(x+2)^2}$. To evaluate the integral, we use partial fractions: $\frac{1}{(x+1)^2(x+2)^2}=\frac{A}{x+1}+\frac{B}{(x+1)^2}+\frac{C}{x+2}+\frac{D}{(x+2)^2}\Rightarrow$ $1=A(x+1)(x+2)^2+B(x+2)^2+C(x+1)^2(x+2)+D(x+1)^2$. We set $x=-1$, giving $B=1$, then set $x=-2$, giving $D=1$. Now equating coefficients of x^3 gives $A=-C$, and then equating constants gives $1=4A+4+2(-A)+1\Rightarrow A=-2\Rightarrow C=2$. So the expression becomes

$$V = \pi\int_0^1\left[\frac{-2}{x+1}+\frac{1}{(x+1)^2}+\frac{2}{(x+2)}+\frac{1}{(x+2)^2}\right]dx=\pi\left[2\ln\left|\frac{x+2}{x+1}\right|-\frac{1}{x+1}-\frac{1}{x+2}\right]_0^1$$

$$= \pi\left[\left(2\ln\frac{3}{2}-\frac{1}{2}-\frac{1}{3}\right)-\left(2\ln 2-1-\frac{1}{2}\right)\right]=\pi\left(2\ln\frac{3/2}{2}+\frac{2}{3}\right)=\pi\left(\frac{2}{3}+\ln\frac{9}{16}\right)$$

(b) In this case, we use cylindrical shells, so the volume is $V=2\pi\int_0^1\frac{x\,dx}{x^2+3x+2}=2\pi\int_0^1\frac{x\,dx}{(x+1)(x+2)}$. We use partial fractions to simplify the integrand: $\dfrac{x}{(x+1)(x+2)}=\dfrac{A}{x+1}+\dfrac{B}{x+2}\Rightarrow x=(A+B)x+2A+B$. So $A+B=1$ and $2A+B=0\Rightarrow A=-1$ and $B=2$. So the volume is

$$2\pi\int_0^1\left[\frac{-1}{x+1}+\frac{2}{x+2}\right]dx = 2\pi\left[-\ln|x+1|+2\ln|x+2|\right]_0^1$$

$$= 2\pi(-\ln 2+2\ln 3+\ln 1-2\ln 2)=2\pi(2\ln 3-3\ln 2)=2\pi\ln\frac{9}{8}$$

63. $\dfrac{P+S}{P[(r-1)P-S]}=\dfrac{A}{P}+\dfrac{B}{(r-1)P-S}\Rightarrow P+S=A[(r-1)P-S]+BP=[(r-1)A+B]P-AS\Rightarrow(r-1)A+B=1$, $-A=1$ $\Rightarrow A=-1$, $B=r$. Now

$$t=\int\frac{P+S}{P[(r-1)P-S]}\,dP=\int\left[\frac{-1}{P}+\frac{r}{(r-1)P-S}\right]dP=-\int\frac{dP}{P}+\frac{r}{r-1}\int\frac{r-1}{(r-1)P-S}\,dP$$

so $t=-\ln P+\dfrac{r}{r-1}\ln|(r-1)P-S|+C$. Here $r=0.10$ and $S=900$, so

$$t=-\ln P+\frac{0.1}{-0.9}\ln|-0.9P-900|+C=-\ln P-\frac{1}{9}\ln(|-1|\,|0.9P+900|)$$

$$=-\ln P-\frac{1}{9}\ln(0.9P+900)+C$$

When $t=0$, $P=10,000$, so $0=-\ln 10,000-\frac{1}{9}\ln(9900)+C$. Thus, $C=\ln 10,000+\frac{1}{9}\ln 9900$, so our equation becomes

$$t=\ln 10{,}000-\ln P+\frac{1}{9}\ln 9900-\frac{1}{9}\ln(0.9P+900)=\ln\frac{10{,}000}{P}+\frac{1}{9}\ln\frac{9900}{0.9P+900}$$

$$=\ln\frac{10{,}000}{P}+\frac{1}{9}\ln\frac{1100}{0.1P+100}=\ln\frac{10{,}000}{P}+\frac{1}{9}\ln\frac{11{,}000}{P+1000}$$

64. If we subtract and add $2x^2$, we get

$$x^4+1 = x^4+2x^2+1-2x^2=\left(x^2+1\right)^2-2x^2=\left(x^2+1\right)^2-\left(\sqrt{2}\,x\right)^2$$

$$=\left[\left(x^2+1\right)-\sqrt{2}\,x\right]\left[\left(x^2+1\right)+\sqrt{2}\,x\right]=\left(x^2-\sqrt{2}\,x+1\right)\left(x^2+\sqrt{2}\,x+1\right)$$

So we can decompose $\dfrac{1}{x^4+1}=\dfrac{Ax+B}{x^2+\sqrt{2}\,x+1}+\dfrac{Cx+D}{x^2-\sqrt{2}\,x+1}\Rightarrow$

$1=(Ax+B)\left(x^2-\sqrt{2}\,x+1\right)+(Cx+D)\left(x^2+\sqrt{2}\,x+1\right)$. Setting the constant terms equal gives $B+D=1$, then from the coefficients of x^3 we get $A+C=0$. Now from the coefficients of x we get $A+C+(B-D)\sqrt{2}=0$ $\Leftrightarrow [(1-D)-D]\sqrt{2}=0 \Rightarrow D=\frac{1}{2} \Rightarrow B=\frac{1}{2}$, and finally, from the coefficients of x^2 we get $\sqrt{2}(C-A)+B+D=0 \Rightarrow C-A=-\frac{1}{\sqrt{2}} \Rightarrow C=-\frac{\sqrt{2}}{4}$ and $A=\frac{\sqrt{2}}{4}$.

So we rewrite the integrand, splitting the terms into forms which we know how to integrate:

$$\frac{1}{x^4+1} = \frac{\frac{\sqrt{2}}{4}x+\frac{1}{2}}{x^2+\sqrt{2}x+1} + \frac{-\frac{\sqrt{2}}{4}x+\frac{1}{2}}{x^2-\sqrt{2}x+1} = \frac{1}{4\sqrt{2}}\left[\frac{2x+2\sqrt{2}}{x^2+\sqrt{2}x+1} - \frac{2x-2\sqrt{2}}{x^2-\sqrt{2}x+1}\right]$$

$$= \frac{\sqrt{2}}{8}\left[\frac{2x+\sqrt{2}}{x^2+\sqrt{2}x+1} - \frac{2x-\sqrt{2}}{x^2-\sqrt{2}x+1}\right] + \frac{1}{4}\left[\frac{1}{\left(x+\frac{1}{\sqrt{2}}\right)^2+\frac{1}{2}} + \frac{1}{\left(x-\frac{1}{\sqrt{2}}\right)^2+\frac{1}{2}}\right]$$

Now we integrate: $\int \frac{dx}{x^4+1} = \frac{\sqrt{2}}{8}\ln\left(\frac{x^2+\sqrt{2}x+1}{x^2-\sqrt{2}x+1}\right) + \frac{\sqrt{2}}{4}\left[\tan^{-1}(\sqrt{2}x+1)+\tan^{-1}(\sqrt{2}x-1)\right]+C$.

65. **(a)** In Maple, we define $f(x)$, and then use `convert(f,parfrac,x)` ; to obtain

$$f(x)=\frac{24{,}110/4879}{5x+2} - \frac{668/323}{2x+1} - \frac{9438/80{,}155}{3x-7} + \frac{(22{,}098x+48{,}935)/260{,}015}{x^2+x+5} .$$

In Mathematica, we use the command Apart, and in Derive, we use Expand.

(b)

$$\int f(x)dx = \frac{24{,}110}{4879}\cdot\frac{1}{5}\ln|5x+2| - \frac{668}{323}\cdot\frac{1}{2}\ln|2x+1| - \frac{9438}{80{,}155}\cdot\frac{1}{3}\ln|3x-7|$$

$$+\frac{1}{260{,}015}\int\frac{22{,}098\left(x+\frac{1}{2}\right)+37{,}886}{\left(x+\frac{1}{2}\right)^2+\frac{19}{4}}dx+C$$

$$=\frac{24{,}110}{4879}\cdot\frac{1}{5}\ln|5x+2| - \frac{668}{323}\cdot\frac{1}{2}\ln|2x+1| - \frac{9438}{80{,}155}\cdot\frac{1}{3}\ln|3x-7|$$

$$+\frac{1}{260{,}015}\left[22{,}098\cdot\frac{1}{2}\ln\left(x^2+x+5\right)+37{,}886\cdot\sqrt{\frac{4}{19}}\tan^{-1}\left(\frac{1}{\sqrt{19/4}}\left(x+\frac{1}{2}\right)\right)\right]+C$$

$$=\frac{4822}{4879}\ln|5x+2| - \frac{334}{323}\ln|2x+1| - \frac{3146}{80{,}155}\ln|3x-7| + \frac{11{,}049}{260{,}015}\ln\left(x^2+x+5\right)$$

$$+\frac{75{,}772}{260{,}015\sqrt{19}}\tan^{-1}\left[\frac{1}{\sqrt{19}}(2x+1)\right]+C$$

Using a CAS, we get

$$\frac{4822\ln(5x+2)}{4879}-\frac{334\ln(2x+1)}{323}-\frac{3146\ln(3x-7)}{80{,}155}$$
$$+\frac{11{,}049\ln\left(x^2+x+5\right)}{260{,}015}+\frac{3988\sqrt{19}}{260{,}015}\tan^{-1}\left[\frac{\sqrt{19}}{19}(2x+1)\right]$$

The main difference in this answer is that the absolute value signs and the constant of integration have been omitted. Also, the fractions have been reduced and the denominators rationalized.

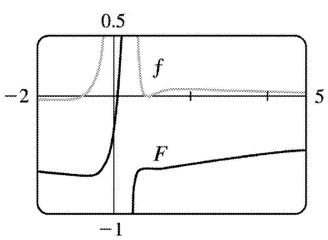
66. **(a)** In Maple, we define $f(x)$, and then use `convert(f,parfrac,x);` to get

$$f(x)=\frac{5828/1815}{(5x-2)^2}-\frac{59{,}096/19{,}965}{5x-2}+\frac{2(2843x+816)/3993}{2x^2+1}+\frac{(313x-251)/363}{\left(2x^2+1\right)^2}\ .$$

In Mathematica, we use the command Apart, and in Derive, we use Expand.

(b) As we saw in Exercise 65, computer algebra systems omit the absolute value signs in $\int(1/y)\,dy=\ln|y|$. So we use the CAS to integrate the expression in part (a) and add the necessary absolute value signs and constant of integration to get

$$\int f(x)dx=-\frac{5828}{9075(5x-2)}-\frac{59{,}096\ln|5x-2|}{99{,}825}+\frac{2843\ln\left(2x^2+1\right)}{7986}$$
$$+\frac{503}{15{,}972}\sqrt{2}\tan^{-1}\left(\sqrt{2}\,x\right)-\frac{1}{2904}\,\frac{1004x+626}{2x^2+1}+C$$



(c) From the graph, we see that f goes from negative to positive at $x\approx-0.78$, then back to negative at $x\approx0.8$, and finally back to positive at $x=1$. Also, ${}_{x\to0.4}f(x)=\infty$. So we see (by the First Derivative Test) that $\int f(x)dx$ has minima at $x\approx-0.78$ and $x=1$, and a maximum at $x\approx0.80$, and that $\int f(x)dx$ is unbounded as $x\to0.4$. Note also that just to the right of $x=0.4$, f has large values, so $\int f(x)dx$ increases rapidly, but slows down as f drops toward 0 . $\int f(x)dx$ decreases from about 0.8 to 1 , then increases slowly since f stays small and positive.

67. There are only finitely many values of x where $Q(x)=0$ (assuming that Q is not the zero

polynomial). At all other values of x , $F(x)/Q(x)=G(x)/Q(x)$, so $F(x)=G(x)$. In other words, the values of F and G agree at all except perhaps finitely many values of x . By continuity of F and G , the polynomials F and G must agree at those values of x too.
More explicitly: if a is a value of x such that $Q(a)=0$, then $Q(x)\neq 0$ for all x sufficiently close to a . Thus,

$$F(a)=\lim_{x\to a} F(x)\,[\text{by continuity of } F] = \lim_{x\to a} G(x)\,[\text{whenever } Q(x)\neq 0]$$
$$= G(a)\,[\text{by continuity of } G]$$

68. Let $f(x)=ax^2+bx+c$. We calculate the partial fraction decomposition of $\frac{f(x)}{x^2(x+1)^3}$. Since $f(0)=1$, we must have $c=1$, so $\frac{f(x)}{x^2(x+1)^3}=\frac{ax^2+bx+1}{x^2(x+1)^3}=\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x+1}+\frac{D}{(x+1)^2}+\frac{E}{(x+1)^3}$. Now in order for the integral not to contain any logarithms (that is, in order for it to be a rational function), we must have $A=C=0$, so $ax^2+bx+1=B(x+1)^3+Dx^2(x+1)+Ex^2$. Equating constant terms gives $B=1$, then equating coefficients of x gives $3B=b\Rightarrow b=3$. This is the quantity we are looking for, since $f'(0)=b$.

1. $\int \frac{\sin x+\sec x}{\tan x}\,dx=\int\left(\frac{\sin x}{\tan x}+\frac{\sec x}{\tan x}\right)dx=\int(\cos x+\csc x)dx=\sin x+\ln|\csc x-\cot x|+C$

2.

$$\int \tan^3\theta\,d\theta=\int(\sec^2\theta-1)\tan\theta\,d\theta=\int\tan\theta\sec^2\theta\,d\theta-\int\frac{\sin\theta}{\cos\theta}\,d\theta$$

$$=\int u\,du+\int\frac{dv}{v}\left[\begin{array}{ll} u=\tan\theta, & v=\cos\theta, \\ du=\sec^2\theta\,d\theta & dv=-\sin\theta\,d\theta\end{array}\right]$$

$$=\frac{1}{2}u^2+\ln|v|+C=\frac{1}{2}\tan^2\theta+\ln|\cos\theta|+C$$

3.

$$\int_0^2\frac{2t}{(t-3)^2}\,dt=\int_{-3}^{-1}\frac{2(u+3)}{u^2}\,du=\int_{-3}^{-1}\left(\frac{2}{u}+\frac{6}{u^2}\right)du=\left[2\ln|u|-\frac{6}{u}\right]_{-3}^{-1}$$

$$=(2\ln 1+6)-(2\ln 3+2)=4-2\ln 3\text{ or }4-\ln 9$$

4. Let $u=x^2$. Then $du=2x\,dx\Rightarrow\int\frac{x\,dx}{\sqrt{3-x^4}}=\frac{1}{2}\int\frac{du}{\sqrt{3-u^2}}=\frac{1}{2}\sin^{-1}\frac{u}{\sqrt{3}}+C=\frac{1}{2}\sin^{-1}\frac{x^2}{\sqrt{3}}+C$.

5. Let $u=\arctan y$. Then $du=\frac{dy}{1+y^2}\Rightarrow\int_{-1}^{1}\frac{e^{\arctan y}}{1+y^2}\,dy=\int_{-\pi/4}^{\pi/4}e^u\,du=\left[e^u\right]_{-\pi/4}^{\pi/4}=e^{\pi/4}-e^{-\pi/4}$.

6.

$$\int x\csc x\cot x\,dx\left[\begin{array}{ll} u=x, & dv=\csc x\cot x\,dx, \\ du=dx & v=-\csc x\end{array}\right]=-x\csc x-\int(-\csc x)dx$$

$$=-x\csc x+\ln|\csc x-\cot x|+C$$

7.

$$\int_1^3 r^4\ln r\,dr\left[\begin{array}{ll} u=\ln r, & dv=r^4\,dr \\ du=\frac{dr}{r} & v=\frac{1}{5}r^5\end{array}\right]=\left[\frac{1}{5}r^5\ln r\right]_1^3-\int_1^3\frac{1}{5}r^4\,dr=\frac{243}{5}\ln 3-0-\left[\frac{1}{25}r^5\right]_1^3$$

$$=\frac{243}{5}\ln 3-\left(\frac{243}{25}-\frac{1}{25}\right)=\frac{243}{5}\ln 3-\frac{242}{25}$$

8. $\dfrac{x-1}{x^2-4x-5}=\dfrac{x-1}{(x-5)(x+1)}=\dfrac{A}{x-5}+\dfrac{B}{x+1}\Rightarrow x-1=A(x+1)+B(x-5)$. Setting $x=-1$ gives $-2=-6B$, so $B=\dfrac{1}{3}$. Setting $x=5$ gives $4=6A$, so $A=\dfrac{2}{3}$. Now

$$\int_0^4 \frac{x-1}{x^2-4x-5}\,dx=\int_0^4\left(\frac{2/3}{x-5}+\frac{1/3}{x+1}\right)dx=\left[\frac{2}{3}\ln|x-5|+\frac{1}{3}\ln|x+1|\right]_0^4$$

$$=\frac{2}{3}\ln 1+\frac{1}{3}\ln 5-\frac{2}{3}\ln 5-\frac{1}{3}\ln 1=-\frac{1}{3}\ln 5$$

9.

$$\int\frac{x-1}{x^2-4x+5}\,dx=\int\frac{(x-2)+1}{(x-2)^2+1}\,dx=\int\left(\frac{u}{u^2+1}+\frac{1}{u^2+1}\right)du\ [\,u=x-2\ ,\ du=dx\,]$$

$$=\frac{1}{2}\ln\left(u^2+1\right)+\tan^{-1}u+C=\frac{1}{2}\ln\left(x^2-4x+5\right)+\tan^{-1}(x-2)+C$$

10.

$$\int\frac{x}{x^4+x^2+1}\,dx=\int\frac{\frac{1}{2}\,du}{u^2+u+1}\ [\,u=x^2\ ,\ du=2x\,dx\,]=\frac{1}{2}\int\frac{du}{\left(u+\frac{1}{2}\right)^2+\frac{3}{4}}$$

$$=\frac{1}{2}\int\frac{\frac{\sqrt{3}}{2}\,dv}{\frac{3}{4}\left(v^2+1\right)}\ \left[\,u+\frac{1}{2}=\frac{\sqrt{3}}{2}\,v\ ,\ du=\frac{\sqrt{3}}{2}\,dv\,\right]=\frac{\sqrt{3}}{4}\cdot\frac{4}{3}\int\frac{dv}{v^2+1}$$

$$=\frac{1}{\sqrt{3}}\tan^{-1}v+C=\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2}{\sqrt{3}}\left(x^2+\frac{1}{2}\right)\right)+C$$

11.

$$\int\sin^3\theta\,\cos^5\theta\,d\theta=\int\cos^5\theta\,\sin^2\theta\,\sin\theta\,d\theta=-\int\cos^5\theta\,(1-\cos^2\theta)(-\sin\theta)\,d\theta$$

$$=-\int u^5(1-u^2)\,du\left[\begin{array}{l}\mathrm{u}=\cos\theta\ ,\\ \mathrm{du}=-\sin\theta\ \mathrm{d}\theta\end{array}\right]$$

$$=\int(u^7-u^5)\,du=\frac{1}{8}u^8-\frac{1}{6}u^6+C=\frac{1}{8}\cos^8\theta-\frac{1}{6}\cos^6\theta+C$$

Another solution:

$$\int \sin^3\theta \cos^5\theta \, d\theta = \int \sin^3\theta (\cos^2\theta)^2 \cos\theta \, d\theta = \int \sin^3\theta (1-\sin^2\theta)^2 \cos\theta \, d\theta$$

$$= \int u^3(1-u^2)^2 du \left[\begin{array}{l} u=\sin\theta, \\ du=\cos\theta \, d\theta \end{array} \right] = \int u^3(1-2u^2+u^4) du$$

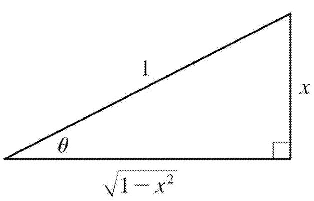
$$= \int (u^3-2u^5+u^7) du = \frac{1}{4}u^4 - \frac{1}{3}u^6 + \frac{1}{8}u^8 + C = \frac{1}{4}\sin^4\theta - \frac{1}{3}\sin^6\theta + \frac{1}{8}\sin^8\theta + C$$

12. Let $u=\cos x$. Then $du=-\sin x\,dx \Rightarrow \int \sin x \cos(\cos x)\,dx = -\int \cos u\,du = -\sin u + C = -\sin(\cos x) + C$.

13. Let $x=\sin\theta$, where $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$. Then $dx=\cos\theta \, d\theta$ and $(1-x^2)^{1/2}=\cos\theta$, so

$$\int \frac{dx}{(1-x^2)^{3/2}} = \int \frac{\cos\theta \, d\theta}{(\cos\theta)^3} = \int \sec^2\theta \, d\theta$$

$$= \tan\theta + C = \frac{x}{\sqrt{1-x^2}} + C$$



14. Let $u=\ln x$. Then $du=dx/x \Rightarrow$

$$\int \frac{\sqrt{1+\ln x}}{x\ln x} dx = \int \frac{\sqrt{1+u}}{u} du = \int \frac{v}{v^2-1} 2v\,dv \quad [\text{put } v=\sqrt{1+u}, u=v^2-1, du=2v\,dv]$$

$$= 2\int \left(1+\frac{1}{v^2-1}\right) dv = 2v + \ln\left|\frac{v-1}{v+1}\right| + C = 2\sqrt{1+\ln x} + \ln\left(\frac{\sqrt{1+\ln x}-1}{\sqrt{1+\ln x}+1}\right) + C$$

15. Let $u=1-x^2 \Rightarrow du=-2x\,dx$. Then

$$\int_0^{1/2} \frac{x}{\sqrt{1-x^2}} dx = -\frac{1}{2}\int_1^{3/4} \frac{1}{\sqrt{u}} du = \frac{1}{2}\int_{3/4}^{1} u^{-1/2} du = \frac{1}{2}\left[2u^{1/2}\right]_{3/4}^{1} = \left[\sqrt{u}\right]_{3/4}^{1} = 1 - \frac{\sqrt{3}}{2}$$

16.

$$\int_0^{\sqrt{2}/2} \frac{x^2}{\sqrt{1-x^2}}\,dx = \int_0^{\pi/4} \frac{\sin^2\theta}{\cos\theta}\cos\theta\,d\theta \quad [\,x=\sin\theta\,,\,\}\{dx\}=\cos\theta\,d\theta\,]$$

$$=\int_0^{\pi/4}\frac{1}{2}(1-\cos 2\theta)\,d\theta=\frac{1}{2}\left[\theta-\frac{1}{2}\sin 2\theta\right]_0^{\pi/4}=\frac{1}{2}\left[\left(\frac{\pi}{4}-\frac{1}{2}\right)-(0-0)\right]=\frac{\pi}{8}-\frac{1}{4}$$

17.

$$\int x\sin^2 x\,dx \left[\begin{array}{ll} \mathrm{u=x,} & \mathrm{dv=\sin^2 xdx} \\ \mathrm{du=dx} & \mathrm{v=\int \sin^2 xdx=\int \frac{1}{2}(1-\cos 2x)dx=\frac{1}{2}x-\frac{1}{2}\sin x\cos x} \end{array}\right]$$

$$=\frac{1}{2}x^2-\frac{1}{2}x\sin x\cos x-\int\left(\frac{1}{2}x-\frac{1}{2}\sin x\cos x\right)dx$$

$$=\frac{1}{2}x^2-\frac{1}{2}x\sin x\cos x-\frac{1}{4}x^2+\frac{1}{4}\sin^2 x+C=\frac{1}{4}x^2-\frac{1}{2}x\sin x\cos x+\frac{1}{4}\sin^2 x+C$$

Note: $\int \sin x\cos x\,dx=\int s\,ds=\frac{1}{2}s^2+C$ [where $s=\sin x, ds=\cos x\,dx$] .

A slightly different method is to write $\int x\sin^2 x\,dx=\int x\cdot\frac{1}{2}(1-\cos 2x)\,dx=\frac{1}{2}\int x\,dx-\frac{1}{2}\int x\cos 2x\,dx$. If we evaluate the second integral by parts, we arrive at the equivalent answer $\frac{1}{4}x^2-\frac{1}{4}x\sin 2x-\frac{1}{8}\cos 2x+C$.

18. Let $u=e^{2t}$, $du=2e^{2t}\,dt$. Then

$$\int\frac{e^{2t}}{1+e^{4t}}\,dt=\int\frac{\frac{1}{2}(2e^{2t})\,dt}{1+(e^{2t})^2}=\int\frac{\frac{1}{2}\,du}{1+u^2}=\frac{1}{2}\tan^{-1}u+C=\frac{1}{2}\tan^{-1}(e^{2t})+C\ .$$

19. Let $u=e^x$. Then $\int e^{x+e^x}dx=\int e^{e^x}e^x\,dx=\int e^u\,du=e^u+C=e^{e^x}+C$.

20. Let $u=\sqrt[3]{x}$. Then $x=u^3\Rightarrow\int e^{\sqrt[3]{x}}dx=\int e^u\cdot 3u^2\,du$. Now use parts: let $w=u^2$, $dv=e^u\,du\Rightarrow dw=2u\,du$, $v=e^u\Rightarrow 3\int e^u u^2\,du=3\left(u^2e^u-2\int ue^u\,du\right)$. Now use parts again with $W=u$, $dV=e^u\,du$ to get

$\int e^u 3u^2 du = e^u \left(3u^2 - 6u + 6\right) + C = 3e^{\sqrt[3]{x}} \left(x^{2/3} - 2\sqrt[3]{x} + 2\right) + C$.

21. Integrate by parts three times, first with $u = t^3$, $dv = e^{-2t} dt$:

$$\int t^3 e^{-2t} dt = -\frac{1}{2} t^3 e^{-2t} + \frac{1}{2} \int 3t^2 e^{-2t} dt = -\frac{1}{2} t^3 e^{-2t} - \frac{3}{4} t^2 e^{-2t} + \frac{1}{2} \int 3te^{-2t} dt$$

$$= -e^{-2t} \left[\frac{1}{2} t^3 + \frac{3}{4} t^2 \right] - \frac{3}{4} te^{-2t} + \frac{3}{4} \int e^{-2t} dt = -e^{-2t} \left[\frac{1}{2} t^3 + \frac{3}{4} t^2 + \frac{3}{4} t + \frac{3}{8} \right] + C$$

$$= -\frac{1}{8} e^{-2t} \left(4t^3 + 6t^2 + 6t + 3\right) + C$$

22. Integrate by parts: $u = \sin^{-1} x$, $dv = x dx \Rightarrow du = \left(1 \big/ \sqrt{1-x^2}\right) dx$, $v = \frac{1}{2} x^2$, so

$$\int x \sin^{-1} x dx = \frac{1}{2} x^2 \sin^{-1} x - \frac{1}{2} \int \frac{x^2 dx}{\sqrt{1-x^2}} = \frac{1}{2} x^2 \sin^{-1} x - \frac{1}{2} \int \frac{\sin^2\theta \cos\theta \, d\theta}{\cos\theta}$$ where $x = \sin\theta$ for $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$$= \frac{1}{2} x^2 \sin^{-1} x - \frac{1}{4} \int (1 - \cos 2\theta) d\theta = \frac{1}{2} x^2 \sin^{-1} x - \frac{1}{4} (\theta - \sin\theta \cos\theta) + C$$

$$= \frac{1}{2} x^2 \sin^{-1} x - \frac{1}{4} \left[\sin^{-1} x - x\sqrt{1-x^2}\right] + C = \frac{1}{4} \left[\left(2x^2 - 1\right) \sin^{-1} x + x\sqrt{1-x^2}\right] + C$$

23. Let $u = 1 + \sqrt{x}$. Then $x = (u-1)^2$, $dx = 2(u-1) du \Rightarrow$

$$\int_0^1 \left(1 + \sqrt{x}\right)^8 dx = \int_1^2 u^8 \cdot 2(u-1) du = 2\int_1^2 \left(u^9 - u^8\right) du = \left[\frac{1}{5} u^{10} - 2 \cdot \frac{1}{9} u^9\right]_1^2$$

$$= \frac{1024}{5} - \frac{1024}{9} - \frac{1}{5} + \frac{2}{9} = \frac{4097}{45}$$

24. Let $u = \ln\left(x^2 - 1\right)$, $dv = dx \Leftrightarrow du = \frac{2x}{x^2 - 1}$, $v = x$. Then

$$\int \ln\left(x^2 - 1\right) dx = x \ln\left(x^2 - 1\right) - \int \frac{2x^2}{x^2 - 1} dx = x \ln\left(x^2 - 1\right) - \int \left[2 + \frac{2}{(x-1)(x+1)}\right] dx$$

$$= x \ln\left(x^2 - 1\right) - \int \left[2 + \frac{1}{x-1} - \frac{1}{x+1}\right] dx$$

$$=x\ln\left(x^2-1\right)-2x-\ln|x-1|+\ln|x+1|+C$$

25. $\dfrac{3x^2-2}{x^2-2x-8}=3+\dfrac{6x+22}{(x-4)(x+2)}=3+\dfrac{A}{x-4}+\dfrac{B}{x+2}\Rightarrow 6x+22=A(x+2)+B(x-4)$. Setting $x=4$ gives $46=6A$, so $A=\dfrac{23}{3}$. Setting $x=-2$ gives $10=-6B$, so $B=-\dfrac{5}{3}$. Now

$$\int\frac{3x^2-2}{x^2-2x-8}\,dx=\int\left(3+\frac{23/3}{x-4}-\frac{5/3}{x+2}\right)dx=3x+\frac{23}{3}\ln|x-4|-\frac{5}{3}\ln|x+2|+C\ .$$

26. $\displaystyle\int\frac{3x^2-2}{x^3-2x-8}\,dx=\int\frac{du}{u}\left[\begin{array}{l}\mathrm{u=x^3-2x-8,}\\ \mathrm{du=\left(3x^2-2\right)dx}\end{array}\right]=\ln|u|+C=\ln\left|x^3-2x-8\right|+C$

27. Let $u=\ln(\sin x)$. Then $du=\cot x\,dx\Rightarrow\int\cot x\ln(\sin x)\,dx=\int u\,du=\dfrac{1}{2}u^2+C=\dfrac{1}{2}[\ln(\sin x)]^2+C$.

28.

$$\int\sin\sqrt{at}\,dt=\int\sin u\cdot\frac{2}{a}u\,du\ \left[u=\sqrt{at}\ ,\ u^2=at\ ,\ 2u\,du=a\,dt\right]=\frac{2}{a}\int u\sin u\,du$$

$$=\frac{2}{a}[-u\cos u+\sin u]+C\ [\text{ integration by parts}]=-\frac{2}{a}\sqrt{at}\cos\sqrt{at}+\frac{2}{a}\sin\sqrt{at}+C$$

$$=-2\sqrt{\frac{t}{a}}\cos\sqrt{at}+\frac{2}{a}\sin\sqrt{at}+C$$

29.

$$\int_0^5\frac{3w-1}{w+2}\,dw=\int_0^5\left(3-\frac{7}{w+2}\right)dw=\left[3w-7\ln|w+2|\right]_0^5$$

$$=15-7\ln 7+7\ln 2=15+7(\ln 2-\ln 7)=15+7\ln\frac{2}{7}$$

30. $x^2-4x<0$ on $[0,4]$, so

$$\int_{-2}^{2}\left|x^2-4x\right|dx=\int_{-2}^{0}(x^2-4x)\,dx+\int_0^2\left(4x-x^2\right)dx=\left[\frac{1}{3}x^3-2x^2\right]_{-2}^{0}+\left[2x^2-\frac{1}{3}x^3\right]_0^2$$

$$=0-\left(-\frac{8}{3}-8\right)+\left(8-\frac{8}{3}\right)-0=16$$

31. As in Example 5,

$$\int \sqrt{\frac{1+x}{1-x}}\,dx = \int \frac{\sqrt{1+x}}{\sqrt{1-x}} \cdot \frac{\sqrt{1+x}}{\sqrt{1+x}}\,dx = \int \frac{1+x}{\sqrt{1-x^2}}\,dx = \int \frac{dx}{\sqrt{1-x^2}} + \int \frac{x\,dx}{\sqrt{1-x^2}}$$

$$= \sin^{-1}x - \sqrt{1-x^2} + C$$

Another method: Substitute $u=\sqrt{(1+x)/(1-x)}$.

32.

$$\int \frac{\sqrt{2x-1}}{2x+3}\,dx = \int \frac{u \cdot u\,du}{u^2+4} \quad u=\sqrt{2x-1}, 2x+3=u^2+4, u^2=2x-1, u\,du=dx = \int \left(1-\frac{4}{u^2+4}\right)du$$

$$= u - 4 \cdot \frac{1}{2}\tan^{-1}\left(\frac{1}{2}u\right) + C = \sqrt{2x-1} - 2\tan^{-1}\left(\frac{1}{2}\sqrt{2x-1}\right) + C$$

33. $3-2x-x^2 = -(x^2+2x+1)+4 = 4-(x+1)^2$. Let $x+1=2\sin\theta$, where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. Then $dx=2\cos\theta\,d\theta$ and

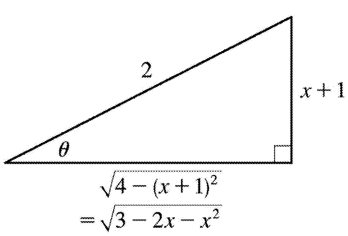
$$\int \sqrt{3-2x-x^2}\,dx = \int \sqrt{4-(x+1)^2}\,dx = \int \sqrt{4-4\sin^2\theta}\;2\cos\theta\,d\theta$$

$$= 4\int \cos^2\theta\,d\theta = 2\int (1+\cos 2\theta)\,d\theta$$

$$= 2\theta + \sin 2\theta + C = 2\theta + 2\sin\theta\cos\theta + C$$

$$= 2\sin^{-1}\left(\frac{x+1}{2}\right) + 2 \cdot \frac{x+1}{2} \cdot \frac{\sqrt{3-2x-x^2}}{2} + C$$

$$= 2\sin^{-1}\left(\frac{x+1}{2}\right) + \frac{x+1}{2}\sqrt{3-2x-x^2} + C$$



34.

$$\int_{\pi/4}^{\pi/2} \frac{1+4\cot x}{4-\cot x}\,dx = \int_{\pi/4}^{\pi/2} \left[\frac{(1+4\cos x/\sin x)}{(4-\cos x/\sin x)} \cdot \frac{\sin x}{\sin x}\right] dx = \int_{\pi/4}^{\pi/2} \frac{\sin x + 4\cos x}{4\sin x - \cos x}\,dx$$

$$=\int_{3/\sqrt{2}}^{4}\frac{1}{u}\,du\left[\begin{array}{l}u=4\sin x-\cos x,\\ du=(4\cos x+\sin x)\,dx\end{array}\right]=\left[\ln|u|\right]_{3/\sqrt{2}}^{4}$$

$$=\ln 4-\ln\frac{3}{\sqrt{2}}=\ln\frac{4}{3/\sqrt{2}}=\ln\left(\frac{4}{3}\sqrt{2}\right)$$

35. Because $f(x)=x^8\sin x$ is the product of an even function and an odd function, it is odd. Therefore, $\int_{-1}^{1}x^8\sin x\,dx=0$ [by (5.5.)(b)].

36. $\sin 4x\cos 3x=\frac{1}{2}(\sin x+\sin 7x)$ by Formula .2.2(a), so

$$\int\sin 4x\cos 3x\,dx=\frac{1}{2}\int(\sin x+\sin 7x)\,dx=\frac{1}{2}\left[-\cos x-\frac{1}{7}\cos 7x\right]+C=-\frac{1}{2}\cos x-\frac{1}{14}\cos 7x+C\ .$$

37.

$$\int_0^{\pi/4}\cos^2\theta\,\tan^2\theta\,d\theta=\int_0^{\pi/4}\sin^2\theta\,d\theta=\int_0^{\pi/4}\frac{1}{2}(1-\cos 2\theta)\,d\theta=\left[\frac{1}{2}\theta-\frac{1}{4}\sin 2\theta\right]_0^{\pi/4}$$

$$=\left(\frac{\pi}{8}-\frac{1}{4}\right)-(0-0)=\frac{\pi}{8}-\frac{1}{4}$$

38.

$$\int_0^{\pi/4}\tan^5\theta\,\sec^3\theta\,d\theta=\int_0^{\pi/4}\left(\tan^2\theta\right)^2\sec^2\theta\cdot\sec\theta\,\tan\theta\,d\theta=\int_1^{\sqrt{2}}(u^2-1)^2u^2\,du\left[\begin{array}{l}u=\sec\theta\\ du=\sec\theta\,\tan\theta\,d\theta\end{array}\right]$$

$$=\int_1^{\sqrt{2}}(u^6-2u^4+u^2)\,du=\left[\frac{1}{7}u^7-\frac{2}{5}u^5+\frac{1}{3}u^3\right]_1^{\sqrt{2}}$$

$$=\left(\frac{8}{7}\sqrt{2}-\frac{8}{5}\sqrt{2}+\frac{2}{3}\sqrt{2}\right)-\left(\frac{1}{7}-\frac{2}{5}+\frac{1}{3}\right)=\frac{22}{105}\sqrt{2}-\frac{8}{105}=\frac{2}{105}(11\sqrt{2}-4)$$

39. Let $u=1-x^2$. Then $du=-2x\,dx\Rightarrow$

$$\int\frac{x\,dx}{1-x^2+\sqrt{1-x^2}}=-\frac{1}{2}\int\frac{du}{u+\sqrt{u}}=-\int\frac{v\,dv}{v^2+v}\ [v=\sqrt{u},u=v^2,du=2v\,dv]$$

$$=-\int\frac{dv}{v+1}=-\ln|v+1|+C=-\ln\left(\sqrt{1-x^2}+1\right)+C$$

40. $4y^2-4y-3=(2y-1)^2-2^2$, so let $u=2y-1 \Rightarrow du=2dy$. Thus,

$$\int \frac{dy}{\sqrt{4y^2-4y-3}} = \int \frac{dy}{\sqrt{(2y-1)^2-2^2}} = \frac{1}{2}\int \frac{du}{\sqrt{u^2-2^2}}$$
$$= \frac{1}{2}\ln\left|u+\sqrt{u^2-2^2}\right|$$
$$= \frac{1}{2}\ln\left|2y-1+\sqrt{4y^2-4y-3}\right|+C$$

41. Let $u=\theta$, $dv=\tan^2\theta\ d\theta=\left(\sec^2\theta-1\right)d\theta \Rightarrow du=d\theta$ and $v=\tan\theta-\theta$. So

$$\int\theta\tan^2\theta\ d\theta = \theta(\tan\theta-\theta)-\int(\tan\theta-\theta)d\theta=\theta\tan\theta-\theta^2-\ln|\sec\theta|+\frac{1}{2}\theta^2+C$$
$$= \theta\tan\theta-\frac{1}{2}\theta^2-\ln|\sec\theta|+C$$

42. Integrate by parts with $u=\tan^{-1}x$, $dv=x^2dx \Rightarrow du=dx/\left(1+x^2\right)$, $v=\frac{1}{3}x^3$:

$$\int x^2\tan^{-1}x\,dx = \frac{1}{3}x^3\tan^{-1}x-\int\frac{x^3}{3}\frac{dx}{1+x^2}=\frac{1}{3}x^3\tan^{-1}x-\frac{1}{3}\int\left[x-\frac{x}{x^2+1}\right]dx$$
$$= \frac{1}{3}x^3\tan^{-1}x-\frac{1}{6}x^2+\frac{1}{6}\ln\left(x^2+1\right)+C$$

43. Let $u=1+e^x$, so that $du=e^x dx$. Then

$\int e^x\sqrt{1+e^x}\,dx=\int u^{1/2}du=\frac{2}{3}u^{3/2}+C=\frac{2}{3}(1+e^x)^{3/2}+C$.

Or: Let $u=\sqrt{1+e^x}$, so that $u^2=1+e^x$ and $2udu=e^x dx$. Then

$\int e^x\sqrt{1+e^x}\,dx=\int u\cdot 2udu=\int 2u^2du=\frac{2}{3}u^3+C=\frac{2}{3}(1+e^x)^{3/2}+C$.

44. Let $u=\sqrt{1+e^x}$. Then $u^2=1+e^x$, $2udu=e^x dx=(u^2-1)dx$, and $dx=\frac{2u}{u^2-1}du$, so

$$\int \sqrt{1+e^x}\,dx = \int u\cdot \frac{2u}{u^2-1}\,du = \int \frac{2u^2}{u^2-1}\,du = \int \left(2+\frac{2}{u^2-1}\right)du = \int \left(2+\frac{1}{u-1}-\frac{1}{u+1}\right)du$$

$$=2u+\ln|u-1|-\ln|u+1|+C = 2\sqrt{1+e^x}+\ln\left(\sqrt{1+e^x}-1\right)-\ln\left(\sqrt{1+e^x}+1\right)+C$$

45. Let $t=x^3$. Then $dt=3x^2dx \Rightarrow I=\int x^5e^{-x^3}dx=\frac{1}{3}\int te^{-t}dt$. Now integrate by parts with $u=t$, $dv=e^{-t}dt$:

$I=-\frac{1}{3}te^{-t}+\frac{1}{3}\int e^{-t}dt=-\frac{1}{3}te^{-t}-\frac{1}{3}e^{-t}+C=-\frac{1}{3}e^{-x^3}\left(x^3+1\right)+C$.

46. Let $u=e^x$. Then $x=\ln u$, $dx=du/u \Rightarrow$

$$\int \frac{1+e^x}{1-e^x}\,dx = \int \frac{(1+u)du}{(1-u)u} = -\int \frac{(u+1)du}{(u-1)u} = -\int\left(\frac{2}{u-1}-\frac{1}{u}\right)du$$

$$= \ln|u|-2\ln|u-1|+C = \ln e^x-2\ln\left|e^x-1\right|+C = x-2\ln\left|e^x-1\right|+C$$

47.

$$\int \frac{x+a}{x^2+a^2}\,dx = \frac{1}{2}\int \frac{2x\,dx}{x^2+a^2}+a\int\frac{dx}{x^2+a^2} = \frac{1}{2}\ln\left(x^2+a^2\right)+a\cdot\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+C$$

$$=\ln\sqrt{x^2+a^2}+\tan^{-1}(x/a)+C$$

48. Let $u=x^2$. Then $du=2x\,dx \Rightarrow$

$$\int\frac{x\,dx}{x^4-a^4} = \int\frac{\frac{1}{2}du}{u^2-\left(a^2\right)^2} = \frac{1}{4a^2}\ln\left|\frac{u-a^2}{u+a^2}\right|+C = \frac{1}{4a^2}\ln\left|\frac{x^2-a^2}{x^2+a^2}\right|+C.$$

49. Let $u=\sqrt{4x+1} \Rightarrow u^2=4x+1 \Rightarrow 2u\,du=4\,dx \Rightarrow dx=\frac{1}{2}u\,du$. So

$$\int\frac{1}{x\sqrt{4x+1}}\,dx = \int\frac{\frac{1}{2}u\,du}{\frac{1}{4}\left(u^2-1\right)u} = 2\int\frac{du}{u^2-1} = 2\left(\frac{1}{2}\right)\ln\left|\frac{u-1}{u+1}\right|+C$$

$$= \ln\left|\frac{\sqrt{4x+1}-1}{\sqrt{4x+1}+1}\right|+C$$

50. As in Exercise 49, let $u=\sqrt{4x+1}$. Then $\int \frac{dx}{x^2\sqrt{4x+1}} = \int \frac{\frac{1}{2}u\,du}{\left[\frac{1}{4}\left(u^2-1\right)\right]^2 u} = 8\int \frac{du}{\left(u^2-1\right)^2}$. Now

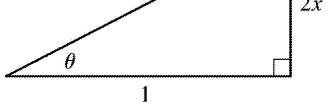
$\frac{1}{\left(u^2-1\right)^2} = \frac{1}{(u+1)^2(u-1)^2} = \frac{A}{u+1} + \frac{B}{(u+1)^2} + \frac{C}{u-1} + \frac{D}{(u-1)^2} \Rightarrow$

$1=A(u+1)(u-1)^2+B(u-1)^2+C(u-1)(u+1)^2+D(u+1)^2$. $u=1 \Rightarrow D=\frac{1}{4}$, $u=-1 \Rightarrow B=\frac{1}{4}$. Equating coefficients of u^3 gives $A+C=0$, and equating coefficients of 1 gives $1=A+B-C+D \Rightarrow 1=A+\frac{1}{4}-C+\frac{1}{4}$ $\Rightarrow \frac{1}{2}=A-C$. So $A=\frac{1}{4}$ and $C=-\frac{1}{4}$. Therefore,

$$
\begin{aligned}
\int \frac{dx}{x^2\sqrt{4x+1}} &= 8\int \left[\frac{1/4}{u+1} + \frac{1/4}{(u+1)^2} + \frac{-1/4}{u-1} + \frac{1/4}{(u-1)^2} \right] du \\
&= \int \left[\frac{2}{u+1} + 2(u+1)^{-2} - \frac{2}{u-1} + 2(u-1)^{-2} \right] du \\
&= 2\ln|u+1| - \frac{2}{u+1} - 2\ln|u-1| - \frac{2}{u-1} + C \\
&= 2\ln\left(\sqrt{4x+1}+1\right) - \frac{2}{\sqrt{4x+1}+1} - 2\ln\left|\sqrt{4x+1}-1\right| - \frac{2}{\sqrt{4x+1}-1} + C
\end{aligned}
$$

51. Let $2x=\tan\theta \Rightarrow x=\frac{1}{2}\tan\theta$, $dx=\frac{1}{2}\sec^2\theta\; d\theta$, $\sqrt{4x^2+1}=\sec\theta$, so

$$
\begin{aligned}
&\int \frac{dx}{x\sqrt{4x^2+1}} = \int \frac{\frac{1}{2}\sec^2\theta\; d\theta}{\frac{1}{2}\tan\theta\;\sec\theta} = \int \frac{\sec\theta}{\tan\theta}\,d\theta = \int \csc\theta\; d\theta \\
&= -\ln|\csc\theta+\cot\theta|+C \\
&= -\ln\left| \frac{\sqrt{4x^2+1}}{2x} + \frac{1}{2x} \right| + C \left[\text{or} \ln\left| \frac{\sqrt{4x^2+1}}{2x} - \frac{1}{2x} \right| + C \right]
\end{aligned}
$$



52. Let $u=x^2$. Then $du=2x\,dx\Rightarrow$

$$\int \frac{dx}{x\left(x^4+1\right)} = \int \frac{x\,dx}{x^2\left(x^4+1\right)} = \frac{1}{2}\int \frac{du}{u\left(u^2+1\right)} = \frac{1}{2}\int \left[\frac{1}{u} - \frac{u}{u^2+1} \right] du = \frac{1}{2}\ln |u| - \frac{1}{4}\ln \left(u^2+1\right)+C$$

$$= \frac{1}{2}\ln \left(x^2\right) - \frac{1}{4}\ln \left(x^4+1\right)+C = \frac{1}{4}\left[\ln \left(x^4\right) - \ln \left(x^4+1\right)\right]+C = \frac{1}{4}\ln \left(\frac{x^4}{x^4+1}\right)+C$$

Or: Write $I=\int \frac{x^3\,dx}{x^4\left(x^4+1\right)}$ and let $u=x^4$.

53. $\int x^2\sinh (mx)dx = \frac{1}{m}x^2\cosh (mx) - \frac{2}{m}\int x\cosh (mx)\,dx$ $\left[\begin{array}{ll} u=x^2, & dv=\sinh (mx)\,dx \\ du=2x\,dx & v=\frac{1}{m}\cosh (mx) \end{array}\right]$

$= \frac{1}{m}x^2\cosh (mx) - \frac{2}{m}\left(\frac{1}{m}x\sinh (mx) - \frac{1}{m}\int \sinh (mx)\,dx\right)$ $\left[\begin{array}{ll} U=x, & dV=\cosh (mx)\,dx \\ dU=dx & V=\frac{1}{m}\sinh (mx) \end{array}\right]$

$= \frac{1}{m}x^2\cosh (mx) - \frac{2}{m^2}x\sinh (mx) + \frac{2}{m^3}\cosh (mx)+C$

54.

$$\int (x+\sin x)^2 dx = \int \left(x^2+2x\sin x+\sin^2 x\right)dx = \frac{1}{3}x^3+2(\sin x - x\cos x) + \frac{1}{2}(x - \sin x\cos x)+C$$

$$= \frac{1}{3}x^3 + \frac{1}{2}x + 2\sin x - \frac{1}{2}\sin x\cos x - 2x\cos x + C$$

55. Let $u=\sqrt{x+1}$. Then $x=u^2-1\Rightarrow$

$$\int \frac{dx}{x+4+4\sqrt{x+1}} = \int \frac{2u\,du}{u^2+3+4u} = \int \left[\frac{-1}{u+1} + \frac{3}{u+3} \right] du$$

$$= 3\ln |u+3| - \ln |u+1| + C = 3\ln \left(\sqrt{x+1}+3\right) - \ln \left(\sqrt{x+1}+1\right)+C$$

56. Let $t=\sqrt{x^2-1}$. Then $dt=\left(x\Big/\sqrt{x^2-1}\right)dx$, $x^2-1=t^2$, $x=\sqrt{t^2+1}$, so
$I=\int \frac{x\ln x}{\sqrt{x^2-1}}\,dx = \int \ln \sqrt{t^2+1}\,dt = \frac{1}{2}\int \ln \left(t^2+1\right)dt$. Now use parts with $u=\ln \left(t^2+1\right)$, $dv=dt$:

$$I=\frac{1}{2}t\ln\left(t^2+1\right)-\int\frac{t^2}{t^2+1}\,dt=\frac{1}{2}t\ln\left(t^2+1\right)-\int\left[1-\frac{1}{t^2+1}\right]dt$$

$$=\frac{1}{2}t\ln\left(t^2+1\right)-t+\tan^{-1}t+C=\sqrt{x^2-1}\ln x-\sqrt{x^2-1}+\tan^{-1}\sqrt{x^2-1}+C$$

Another method: First integrate by parts with $u=\ln x$, $dv=\left(x\Big/\sqrt{x^2-1}\right)dx$ and then use substitution $\left(x=\sec\theta \text{ or } u=\sqrt{x^2-1}\right)$.

57. Let $u=\sqrt[3]{x+c}$. Then $x=u^3-c\Rightarrow$

$$\int x\sqrt[3]{x+c}\,dx=\int\left(u^3-c\right)u\cdot 3u^2du=3\int\left(u^6-cu^3\right)du=\frac{3}{7}u^7-\frac{3}{4}cu^4+C$$

$$=\frac{3}{7}(x+c)^{7/3}-\frac{3}{4}c(x+c)^{4/3}+C$$

58. Integrate by parts with $u=\ln(1+x)$, $dv=x^2dx\Rightarrow du=dx/(1+x)$, $v=\frac{1}{3}x^3$:

$$\int x^2\ln(1+x)dx=\frac{1}{3}x^3\ln(1+x)-\int\frac{x^3dx}{3(1+x)}=\frac{1}{3}x^3\ln(1+x)-\frac{1}{3}\int\left(x^2-x+1-\frac{1}{x+1}\right)dx$$

$$=\frac{1}{3}x^3\ln(1+x)-\frac{1}{9}x^3+\frac{1}{6}x^2-\frac{1}{3}x+\frac{1}{3}\ln(1+x)+C$$

59. Let $u=e^x$. Then $x=\ln u$, $dx=du/u\Rightarrow$

$$\int\frac{dx}{e^{3x}-e^x}=\int\frac{du/u}{u^3-u}=\int\frac{du}{(u-1)u^2(u+1)}=\int\left[\frac{1/2}{u-1}-\frac{1}{u^2}-\frac{1/2}{u+1}\right]du$$

$$=\frac{1}{u}+\frac{1}{2}\ln\left|\frac{u-1}{u+1}\right|+C=e^{-x}+\frac{1}{2}\ln\left|\frac{e^x-1}{e^x+1}\right|+C$$

60. Let $u=\sqrt[3]{x}$. Then $x=u^3$, $dx=3u^2du\Rightarrow$

$$\int\frac{dx}{x+\sqrt[3]{x}}=\int\frac{3u^2du}{u^3+u}=\frac{3}{2}\int\frac{2u\,du}{u^2+1}=\frac{3}{2}\ln\left(u^2+1\right)+C=\frac{3}{2}\ln\left(x^{2/3}+1\right)+C\ .$$

61. Let $u=x^5$. Then $du=5x^4dx\Rightarrow$

$$\int \frac{x^4\,dx}{x^{10}+16} = \int \frac{\frac{1}{5}\,du}{u^2+16} = \frac{1}{5}\cdot\frac{1}{4}\tan^{-1}\left(\frac{1}{4}u\right)+C = \frac{1}{20}\tan^{-1}\left(\frac{1}{4}x^5\right)+C\ .$$

62. Let $u=x+1$. Then $du=dx \Rightarrow$

$$\int \frac{x^3}{(x+1)^{10}}\,dx = \int \frac{(u-1)^3}{u^{10}}\,du = \int\left(u^{-7}-3u^{-8}+3u^{-9}-u^{-10}\right)du$$

$$= -\frac{1}{6}u^{-6}+\frac{3}{7}u^{-7}-\frac{3}{8}u^{-8}+\frac{1}{9}u^{-9}+C$$

$$= (x+1)^{-9}\left[-\frac{1}{6}(x+1)^3+\frac{3}{7}(x+1)^2-\frac{3}{8}(x+1)+\frac{1}{9}\right]+C$$

63. Let $y=\sqrt{x}$ so that $dy=\frac{1}{2\sqrt{x}}\,dx \Rightarrow dx=2\sqrt{x}\,dy=2y\,dy$. Then

$$\int \sqrt{x}\,e^{\sqrt{x}}\,dx = \int ye^{y}(2ydy) = \int 2y^2e^{y}dy \left[\begin{array}{ll} u=2y^2, & dv=e^{y}dy, \\ du=4ydy & v=e^{y} \end{array}\right]$$

$$=2y^2e^{y}-\int 4ye^{y}dy \left[\begin{array}{ll} U=4y, & dV=e^{y}dy, \\ dU=4dy & V=e^{y} \end{array}\right]$$

$$=2y^2e^{y}-\left(4ye^{y}-\int 4e^{y}dy\right)=2y^2e^{y}-4ye^{y}+4e^{y}+C$$

$$=2(y^2-2y+2)e^{y}+C=2\left(x-2\sqrt{x}+2\right)e^{\sqrt{x}}+C$$

64. Let $u=\tan x$. Then

$$\int_{\pi/4}^{\pi/3} \frac{\ln(\tan x)}{dx}\sin x\cos x = \int_{\pi/4}^{\pi/3}\frac{\ln(\tan x)}{\tan x}\sec^2 x\,dx = \int_1^{\sqrt{3}}\frac{\ln u}{u}\,du$$

$$= \left[\frac{1}{2}(\ln u)^2\right]_1^{\sqrt{3}} = \frac{1}{2}\left(\ln\sqrt{3}\right)^2 = \frac{1}{8}(\ln 3)^2$$

65.

$$\int \frac{dx}{\sqrt{x+1}+\sqrt{x}} = \int\left(\frac{1}{\sqrt{x+1}+\sqrt{x}}\cdot\frac{\sqrt{x+1}-\sqrt{x}}{\sqrt{x+1}-\sqrt{x}}\right)dx = \int\left(\sqrt{x+1}-\sqrt{x}\right)dx$$

$$= \frac{2}{3}\left[(x+1)^{3/2}-x^{3/2}\right]+C$$

66. $\int \frac{u^3+1}{u^3-u^2}\,du=\int \left[1+\frac{u^2+1}{(u-1)u^2}\right]du=u+\int \left[\frac{2}{u-1}-\frac{1}{u}-\frac{1}{u^2}\right]du=u+2\ln|u-1|-\ln|u|+\frac{1}{u}+C$. Thus,

$$\int_2^3 \frac{u^3+1}{u^3-u^2}\,du = \left[u+2\ln(u-1)-\ln u+\frac{1}{u}\right]_2^3=\left(3+2\ln 2-\ln 3+\frac{1}{3}\right)-\left(2+2\ln 1-\ln 2+\frac{1}{2}\right)$$

$$= 1+3\ln 2-\ln 3-\frac{1}{6}=\frac{5}{6}+\ln\frac{8}{3}$$

67. Let $u=\sqrt{t}$. Then $du=dt/\left(2\sqrt{t}\right)\Rightarrow$

$$\int_1^3 \frac{\arctan\sqrt{t}}{\sqrt{t}}\,dt=\int_1^{\sqrt{3}} \tan^{-1}u(2\,du)=2\left[u\tan^{-1}u-\frac{1}{2}\ln(1+u^2)\right]_1^{\sqrt{3}}$$ [Example 5 in Section 8.1]

$$=2\left[\left(\sqrt{3}\tan^{-1}\sqrt{3}-\frac{1}{2}\ln 4\right)-\left(\tan^{-1}1-\frac{1}{2}\ln 2\right)\right]$$

$$=2\left[\left(\sqrt{3}\cdot\frac{\pi}{3}-\ln 2\right)-\left(\frac{\pi}{4}-\frac{1}{2}\ln 2\right)\right]=\frac{2}{3}\sqrt{3}\,\pi-\frac{1}{2}\pi-\ln 2$$

68. Let $u=e^x$. Then $x=\ln u$, $dx=du/u\Rightarrow$

$$\int \frac{dx}{1+2e^x-e^{-x}} = \int \frac{du/u}{1+2u-1/u}=\int \frac{du}{2u^2+u-1}=\int \left[\frac{2/3}{2u-1}-\frac{1/3}{u+1}\right]du$$

$$= \frac{1}{3}\ln|2u-1|-\frac{1}{3}\ln|u+1|+C=\frac{1}{3}\ln\left|\left(2e^x-1\right)/\left(e^x+1\right)\right|+C$$

69. Let $u=e^x$. Then $x=\ln u$, $dx=du/u\Rightarrow$

$$\int \frac{e^{2x}}{1+e^x}\,dx = \int \frac{u^2}{1+u}\frac{du}{u}=\int \frac{u}{1+u}\,du=\int \left(1-\frac{1}{1+u}\right)du$$

$$= u-\ln|1+u|+C=e^x-\ln\left(1+e^x\right)+C$$

70. Use parts with $u=\ln(x+1)$, $dv=dx/x^2$:

$$\int \frac{\ln(x+1)}{x^2}\,dx = -\frac{1}{x}\ln(x+1)+\int \frac{dx}{x(x+1)}=-\frac{1}{x}\ln(x+1)+\int \left[\frac{1}{x}-\frac{1}{x+1}\right]dx$$

$$= -\frac{1}{x}\ln(x+1)+\ln|x|-\ln(x+1)+C=-\left(1+\frac{1}{x}\right)\ln(x+1)+\ln|x|+C$$

71. $\dfrac{x}{x^4+4x^2+3}=\dfrac{x}{\left(x^2+3\right)\left(x^2+1\right)}=\dfrac{Ax+B}{x^2+3}+\dfrac{Cx+D}{x^2+1}\Rightarrow$

$x=(Ax+B)\left(x^2+1\right)+(Cx+D)\left(x^2+3\right)=\left(Ax^3+Bx^2+Ax+B\right)+\left(Cx^3+Dx^2+3Cx+3D\right)$

$=(A+C)x^3+(B+D)x^2+(A+3C)x+(B+3D)\Rightarrow$

$A+C=0$, $B+D=0$, $A+3C=1$, $B+3D=0\Rightarrow A=-\dfrac{1}{2}$, $C=\dfrac{1}{2}$, $B=0$, $D=0$.Thus,

$$\int\frac{x}{x^4+4x^2+3}\,dx=\int\left(\frac{-\frac{1}{2}x}{x^2+3}+\frac{\frac{1}{2}x}{x^2+1}\right)dx$$

$$=-\frac{1}{4}\ln\left(x^2+3\right)+\frac{1}{4}\ln\left(x^2+1\right)+C\ \text{ or }\ \frac{1}{4}\ln\left(\frac{x^2+1}{x^2+3}\right)+C$$

72. Let $u=\sqrt[6]{t}$. Then $t=u^6$, $dt=6u^5\,du\Rightarrow$

$$\int\frac{\sqrt{t}\,dt}{1+\sqrt[3]{t}}=\int\frac{u^3\cdot 6u^5\,du}{1+u^2}=6\int\frac{u^8}{u^2+1}\,du=6\int\left(u^6-u^4+u^2-1+\frac{1}{u^2+1}\right)du$$

$$=6\left(\frac{1}{7}u^7-\frac{1}{5}u^5+\frac{1}{3}u^3-u+\tan^{-1}u\right)+C$$

$$=6\left(\frac{1}{7}t^{7/6}-\frac{1}{5}t^{5/6}+\frac{1}{3}t^{1/2}-t^{1/6}+\tan^{-1}t^{1/6}\right)+C$$

73. $\dfrac{1}{(x-2)\left(x^2+4\right)}=\dfrac{A}{x-2}+\dfrac{Bx+C}{x^2+4}\Rightarrow 1=A\left(x^2+4\right)+(Bx+C)(x-2)=(A+B)x^2+(C-2B)x+(4A-2C)$. So

$0=A+B=C-2B$, $1=4A-2C$. Setting $x=2$ gives $A=\dfrac{1}{8}\Rightarrow B=-\dfrac{1}{8}$ and $C=-\dfrac{1}{4}$. So

$$\int\frac{1}{(x-2)\left(x^2+4\right)}\,dx=\int\left(\frac{\frac{1}{8}}{x-2}+\frac{-\frac{1}{8}x-\frac{1}{4}}{x^2+4}\right)dx=\frac{1}{8}\int\frac{dx}{x-2}-\frac{1}{16}\int\frac{2x\,dx}{x^2+4}-\frac{1}{4}\int\frac{dx}{x^2+4}$$

$$=\frac{1}{8}\ln|x-2|-\frac{1}{16}\ln\left(x^2+4\right)-\frac{1}{8}\tan^{-1}(x/2)+C$$

74. Let $u=e^x$. Then $x=\ln u$, $dx=du/u\Rightarrow$

$$\int\frac{dx}{e^x-e^{-x}}=\int\frac{e^x\,dx}{e^{2x}-1}=\int\frac{u}{u^2-1}\,\frac{du}{u}=\int\frac{du}{u^2-1}=\frac{1}{2}\ln\left|\frac{u-1}{u+1}\right|+C=\frac{1}{2}\ln\left(\frac{\left|e^x-1\right|}{e^x+1}\right)+C\ .$$

75.

$$\int \sin x \sin 2x \sin 3x\,dx = \int \sin x \cdot \frac{1}{2}\left[\cos(2x-3x)-\cos(2x+3x)\right]dx = \frac{1}{2}\int(\sin x\cos x - \sin x\cos 5x)\,dx$$

$$= \frac{1}{4}\int \sin 2x\,dx - \frac{1}{2}\int \frac{1}{2}\left[\sin(x+5x)+\sin(x-5x)\right]dx$$

$$= -\frac{1}{8}\cos 2x - \frac{1}{4}\int(\sin 6x - \sin 4x)\,dx = -\frac{1}{8}\cos 2x + \frac{1}{24}\cos 6x - \frac{1}{16}\cos 4x + C$$

76.

$$\int\left(x^2-bx\right)\sin 2x\,dx = -\frac{1}{2}\left(x^2-bx\right)\cos 2x + \frac{1}{2}\int(2x-b)\cos 2x\,dx$$

$$= -\frac{1}{2}\left(x^2-bx\right)\cos 2x + \frac{1}{2}\left[\frac{1}{2}(2x-b)\sin 2x - \int \sin 2x\,dx\right]$$

$$= -\frac{1}{2}\left(x^2-bx\right)\cos 2x + \frac{1}{4}(2x-b)\sin 2x + \frac{1}{4}\cos 2x + C$$

77. Let $u=x^{3/2}$ so that $u^2=x^3$ and $du=\frac{3}{2}x^{1/2}dx \Rightarrow \sqrt{x}\,dx=\frac{2}{3}\,du$. Then

$$\int \frac{\sqrt{x}}{1+x^3}\,dx = \int \frac{\frac{2}{3}}{1+u^2}\,du = \frac{2}{3}\tan^{-1}u + C = \frac{2}{3}\tan^{-1}(x^{3/2}) + C\ .$$

78.

$$\int \frac{\sec x\cos 2x}{\sin x+\sec x}\,dx = \int \frac{\sec x\cos 2x}{\sin x+\sec x}\cdot\frac{2\cos x}{2\cos x}\,dx = \int \frac{2\cos 2x}{2\sin x\cos x+2}\,dx$$

$$= \int \frac{2\cos 2x}{\sin 2x+2}\,dx = \int \frac{1}{u}\,du \left[\begin{array}{c} u=\sin 2x+2, \\ du=2\cos 2x dx \end{array}\right]$$

$$= \ln|u| + C = \ln|\sin 2x+2| + C = \ln(\sin 2x+2) + C$$

79. Let $u=x$, $dv=\sin^2 x\cos x\,dx \Rightarrow du=dx$, $v=\frac{1}{3}\sin^3 x$. Then

$$\int x\sin^2 x\cos x\,dx = \frac{1}{3}x\sin^3 x - \int \frac{1}{3}\sin^3 x\,dx = \frac{1}{3}x\sin^3 x - \frac{1}{3}\int(1-\cos^2 x)\sin x\,dx$$

$$= \frac{1}{3}x\sin^3 x + \frac{1}{3}\int(1-y^2)\,dy \left[\begin{array}{c} y=\cos x, \\ dy=-\sin x dx \end{array}\right]$$

$$=\frac{1}{3}x\sin^3 x+\frac{1}{3}y-\frac{1}{9}y^3+C=\frac{1}{3}x\sin^3 x+\frac{1}{3}\cos x-\frac{1}{9}\cos^3 x+C$$

80.

$$\int\frac{\sin x\cos x}{\sin^4 x+\cos^4 x}dx=\int\frac{\sin x\cos x}{(\sin^2 x)^2+(\cos^2 x)^2}dx=\int\frac{\sin x\cos x}{(\sin^2 x)^2+(1-\sin^2 x)^2}dx$$

$$=\int\frac{1}{u^2+(1-u)^2}\left(\frac{1}{2}du\right)\left[\begin{array}{l}u=\sin^2 x,\\ du=2\sin x\cos x\,dx\end{array}\right]$$

$$=\int\frac{1}{4u^2-4u+2}du=\int\frac{1}{(4u^2-4u+1)+1}du$$

$$=\int\frac{1}{(2u-1)^2+1}du=\frac{1}{2}\int\frac{1}{y^2+1}dy\left[\begin{array}{l}y=2u-1,\\ dy=2\,du\end{array}\right]$$

$$=\frac{1}{2}\tan^{-1}y+C=\frac{1}{2}\tan^{-1}(2u-1)+C=\frac{1}{2}\tan^{-1}(2\sin^2 x-1)+C$$

Another solution:

$$\int\frac{\sin x\cos x}{\sin^4 x+\cos^4 x}\,\mathrm{dx}=\int\frac{(\sin x\cos x)/\cos^4 x}{(\sin^4 x+\cos^4 x)/\cos^4 x}\,\mathrm{dx}=\int\frac{\tan x\sec^2 x}{\tan^4 x+1}\,\mathrm{dx}$$

$$=\int\frac{1}{u^2+1}\left(\frac{1}{2}du\right)\left[\begin{array}{l}\mathrm{u=\tan^2 x,}\\ \mathrm{du=2\tan x\sec^2 x\,dx}\end{array}\right]$$

$$=\frac{1}{2}\tan^{-1}\mathrm{u+C}=\frac{1}{2}\tan^{-1}(\tan^2\mathrm{x})+\mathrm{C}$$

81. The function $y=2xe^{x^2}$ *does* have an elementary antiderivative, so we'll use this fact to help evaluate the integral.

$$\int(2x^2+1)e^{x^2}dx=\int 2x^2e^{x^2}dx+\int e^{x^2}dx=\int x(2xe^{x^2})dx+\int e^{x^2}dx$$

$$=xe^{x^2}-\int e^{x^2}dx+\int e^{x^2}dx\left[\begin{array}{ll}\mathrm{u=x,} & \mathrm{dv=2xe^{x^2}dx,}\\ \mathrm{du=dx} & \mathrm{v=e^{x^2}}\end{array}\right]=xe^{x^2}+C$$

1. We could make the substitution $u=\sqrt{2}\,x$ to obtain the radical $\sqrt{7-u^2}$ and then use Formula 33 with $a=\sqrt{7}$. Alternatively, we will factor $\sqrt{2}$ out of the radical and use $a=\sqrt{\frac{7}{2}}$.

$$\int \frac{\sqrt{7-2x^2}}{x^2}\,dx = \sqrt{2}\int \frac{\sqrt{\frac{7}{2}-x^2}}{x^2}\,dx=\sqrt{2}\left[-\frac{1}{x}\sqrt{\frac{7}{2}-x^2}-\sin^{-1}\frac{x}{\sqrt{\frac{7}{2}}}\right]+C$$

$$= -\frac{1}{x}\sqrt{7-2x^2}-\sqrt{2}\sin^{-1}\left(\sqrt{\frac{2}{7}}\,x\right)+C$$

2.

$$\int \frac{3x}{\sqrt{3-2x}}\,dx=3\int \frac{x}{\sqrt{3+(-2)x}}\,dx=3\left[\frac{2}{3(-2)^2}(-2x-2\cdot 3)\sqrt{3+(-2)x}\right]+C$$

$$=\frac{1}{2}(-2x-6)\sqrt{3-2x}+C=-(x+3)\sqrt{3-2x}+C$$

3. Let $u=\pi x \Rightarrow du=\pi\,dx$, so

$$\int \sec^3(\pi x)dx = \frac{1}{\pi}\int \sec^3 u\,du=\frac{1}{\pi}\left(\frac{1}{2}\sec u\tan u+\frac{1}{2}\ln|\sec u+\tan u|\right)+C$$

$$=\frac{1}{2\pi}\sec \pi x\tan \pi x+\frac{1}{2\pi}\ln|\sec \pi x+\tan \pi x|+C$$

4. $\int e^{2\theta}\sin 3\theta\,d\theta=\frac{e^{2\theta}}{2^2+3^2}(2\sin 3\theta-3\cos 3\theta)+C=\frac{2}{13}e^{2\theta}\sin 3\theta-\frac{3}{13}e^{2\theta}\cos 3\theta+C$

5. $\int_0^1 2x\cos^{-1}x\,dx=2\left[\frac{2x^2-1}{4}\cos^{-1}x-\frac{x\sqrt{1-x^2}}{4}\right]_0^1=2\left[\left(\frac{1}{4}\cdot 0-0\right)-\left(-\frac{1}{4}\cdot\frac{\pi}{2}-0\right)\right]=2\left(\frac{\pi}{8}\right)=\frac{\pi}{4}$

6.

$$\int_2^3 \frac{1}{x^2\sqrt{4x^2-7}}\,dx = \int_4^6 \frac{1}{\left(\frac{1}{2}u^2\sqrt{u^2-7}\right)}\left(\frac{1}{2}\,du\right)[u=2x,\ du=2\,dx]$$

$$=2\int_4^6 \frac{du}{u^2\sqrt{u^2-7}}=2\left[\frac{\sqrt{u^2-7}}{7u}\right]_4^6$$

$$=2\left(\frac{\sqrt{29}}{42}-\frac{3}{28}\right)=\frac{\sqrt{29}}{21}-\frac{3}{14}$$

7. By Formula 99 with $a=-3$ and $b=4$,

$$\int e^{-3x}\cos 4x\,dx=\frac{e^{-3x}}{(-3)^2+4^2}(-3\cos 4x+4\sin 4x)+C=\frac{e^{-3x}}{25}(-3\cos 4x+4\sin 4x)+C\ .$$

8. Let $u=x/2$, so $dx=2\,du$, and we use Formula 72:

$$\begin{aligned}\int \csc^3(x/2)\,dx &= 2\int \csc^3 u\,du=-\csc u\cot u+\ln|\csc u-\cot u|+C\\ &= -\csc(x/2)\cot(x/2)+\ln|\csc(x/2)-\cot(x/2)|+C\end{aligned}$$

9. Let $u=2x$ and $a=3$. Then $du=2\,dx$ and

$$\int \frac{dx}{x^2\sqrt{4x^2+9}}=\int \frac{\frac{1}{2}\,du}{\frac{u^2}{4}\sqrt{u^2+a^2}}=2\int \frac{du}{u^2\sqrt{a^2+u^2}}=-2\,\frac{\sqrt{a^2+u^2}}{a^2u}+C$$

$$=-2\,\frac{\sqrt{4x^2+9}}{9\cdot 2x}+C=-\frac{\sqrt{4x^2+9}}{9x}+C$$

10. Let $u=\sqrt{2}\,y$ and $a=\sqrt{3}$. Then $du=\sqrt{2}\,dy$ and

$$\int \frac{\sqrt{2y^2-3}}{y^2}\,dy = \int \frac{\sqrt{u^2-a^2}}{\frac{1}{2}u^2}\,\frac{du}{\sqrt{2}} = \sqrt{2}\int \frac{\sqrt{u^2-a^2}}{u^2}\,du$$
$$= \sqrt{2}\left(-\frac{\sqrt{u^2-a^2}}{u} + \ln\left|u+\sqrt{u^2-a^2}\right|\right) + C$$
$$= \sqrt{2}\left(-\frac{\sqrt{2y^2-3}}{\sqrt{2}\,y} + \ln\left|\sqrt{2}\,y+\sqrt{2y^2-3}\right|\right) + C$$
$$= -\frac{\sqrt{2y^2-3}}{y} + \sqrt{2}\ln\left|\sqrt{2}\,y+\sqrt{2y^2-3}\right| + C$$

11.

$$\int_{-1}^{0} t^2 e^{-t}\,dt = \left[\frac{1}{-1}t^2 e^{-t}\right]_{-1}^{0} - \frac{2}{-1}\int_{-1}^{0} te^{-t}\,dt = e+2\int_{-1}^{0} te^{-t}\,dt = e+2\left[\frac{1}{(-1)^2}(-t-1)e^{-t}\right]_{-1}^{0}$$
$$= e+2\left[-e^{0}+0\right] = e-2$$

12. Let $u=3x$. Then $du=3\,dx$, so

$$\int x^2\cos 3x\,dx = \frac{1}{27}\int u^2\cos u\;du = \frac{1}{27}\left(u^2\sin u - 2\int u\sin u\;du\right)$$
$$= \frac{1}{3}x^2\sin 3x - \frac{2}{27}(\sin 3x - 3x\cos 3x) + C$$
$$= \frac{1}{27}\left[\left(9x^2-2\right)\sin 3x + 6x\cos 3x\right] + C$$

Thus, $\int_0^{\pi} x^2\cos 3x\,dx = \frac{1}{27}\left[\left(9x^2-2\right)\sin 3x+6x\cos 3x\right]_0^{\pi} = \frac{1}{27}\left[(0+6\pi(-1))-(0+0)\right] = -\frac{6\pi}{27} = -\frac{2\pi}{9}$.

13.

$$\int \frac{\tan^3(1/z)}{z^2}\,dz \begin{bmatrix} u=1/z, \\ du=-dz/z^2 \end{bmatrix} = -\int \tan^3 u\;du = -\frac{1}{2}\tan^2 u - \ln|\cos u| + C$$
$$= -\frac{1}{2}\tan^2\left(\frac{1}{z}\right) - \ln\left|\cos\left(\frac{1}{z}\right)\right| + C$$

14. Let $u=\sqrt{x}$. Then $u^2=x$ and $2u\;du=dx$, so

$$\int \sin^{-1}\sqrt{x}\,dx = 2\int u\sin^{-1}u\,du = \frac{2u^2-1}{2}\sin^{-1}u + \frac{u\sqrt{1-u^2}}{2} + C$$
$$= \frac{2x-1}{2}\sin^{-1}\sqrt{x} + \frac{\sqrt{x(1-x)}}{2} + C$$

15. Let $u = e^x$. Then $du = e^x dx$, so
$$\int e^x \operatorname{sech}\left(e^x\right)dx = \int \operatorname{sech} u\,du \overset{107}{=} \tan^{-1}|\sinh u| + C = \tan^{-1}\left[\sinh\left(e^x\right)\right] + C$$

16. Let $u = x^2$, so that $du = 2x\,dx$. Then
$$\int x\sin\left(x^2\right)\cos\left(3x^2\right)dx = \frac{1}{2}\int \sin u\cos 3u\,du = -\frac{1}{2}\frac{\cos(1-3)u}{2(1-3)} - \frac{1}{2}\frac{\cos(1+3)u}{2(1+3)} + C$$
$$= \frac{1}{8}\cos 2u - \frac{1}{16}\cos 4u + C = \frac{1}{8}\cos\left(2x^2\right) - \frac{1}{16}\cos\left(4x^2\right) + C$$

17. Let $z = 6+4y-4y^2 = 6-(4y^2-4y+1)+1 = 7-(2y-1)^2$, $u = 2y-1$, and $a = \sqrt{7}$. Then $z = a^2-u^2$, $du = 2\,dy$, and

$$\int y\sqrt{6+4y-4y^2}\,dy = \int y\sqrt{z}\,dy = \int \frac{1}{2}(u+1)\sqrt{a^2-u^2}\,\frac{1}{2}\,du$$
$$= \frac{1}{4}\int u\sqrt{a^2-u^2}\,du + \frac{1}{4}\int\sqrt{a^2-u^2}\,du$$
$$= \frac{1}{4}\int\sqrt{a^2-u^2}\,du - \frac{1}{8}\int(-2u)\sqrt{a^2-u^2}\,du$$
$$= \frac{u}{8}\sqrt{a^2-u^2} + \frac{a^2}{8}\sin^{-1}\left(\frac{u}{a}\right) - \frac{1}{8}\int\sqrt{w}\,dw \quad \left[\begin{array}{l} w=a^2-u^2, \\ dw=-2u\,du \end{array}\right]$$
$$= \frac{2y-1}{8}\sqrt{6+4y-4y^2} + \frac{7}{8}\sin^{-1}\frac{2y-1}{\sqrt{7}} - \frac{1}{8}\cdot\frac{2}{3}w^{3/2} + C$$
$$= \frac{2y-1}{8}\sqrt{6+4y-4y^2} + \frac{7}{8}\sin^{-1}\frac{2y-1}{\sqrt{7}} - \frac{1}{12}(6+4y-4y^2)^{3/2} + C.$$

This can be rewritten as

$$\sqrt{6+4y-4y^2}\left[\frac{1}{8}(2y-1)-\frac{1}{12}(6+4y-4y^2)\right]+\frac{7}{8}\sin^{-1}\frac{2y-1}{\sqrt{7}}+C$$
$$=\left(\frac{1}{3}y^2-\frac{1}{12}y-\frac{5}{8}\right)\sqrt{6+4y-4y^2}+\frac{7}{8}\sin^{-1}\left(\frac{2y-1}{\sqrt{7}}\right)+C$$
$$=\frac{1}{24}(8y^2-2y-15)\sqrt{6+4y-4y^2}+\frac{7}{8}\sin^{-1}\left(\frac{2y-1}{\sqrt{7}}\right)+C$$

18. Let $u=x^2$. Then $du=2x\,dx$, so by Formula 48,

$$\int\frac{x^5\,dx}{x^2+\sqrt{2}}=\frac{1}{2}\int\frac{u^2}{u+\sqrt{2}}\,du=\frac{1}{2}\cdot\frac{1}{2}\left[\left(u+\sqrt{2}\right)^2-4\sqrt{2}\left(u+\sqrt{2}\right)+4\ln\left|u+\sqrt{2}\right|\right]+C$$
$$=\frac{1}{4}\left[\left(x^2+\sqrt{2}\right)^2-4\sqrt{2}\left(x^2+\sqrt{2}\right)+4\ln\left(x^2+\sqrt{2}\right)\right]+C$$
$$=\frac{1}{4}x^4-\frac{1}{\sqrt{2}}x^2+\ln\left(x^2+\sqrt{2}\right)+K$$

Or: Let $u=x^2+\sqrt{2}$.

19. Let $u=\sin x$. Then $du=\cos x\,dx$, so

$$\int\sin^2 x\cos x\ln(\sin x)\,dx=\int u^2\ln u\,du=\frac{u^{2+1}}{(2+1)^2}\left[(2+1)\ln u-1\right]+C=\frac{1}{9}u^3(3\ln u-1)+C$$
$$=\frac{1}{9}\sin^3 x\left[3\ln(\sin x)-1\right]+C$$

20. Let $u=e^x$. Then $x=\ln u$, $dx=du/u$, so

$$\int\frac{dx}{e^x\left(1+2e^x\right)}=\int\frac{du/u}{u(1+2u)}=\int\frac{du}{u^2(1+2u)}=-\frac{1}{u}+2\ln\left|\frac{1+2u}{u}\right|+C$$
$$=-e^{-x}+2\ln\left(e^{-x}+2\right)+C$$

21. Let $u=e^x$ and $a=\sqrt{3}$. Then $du=e^x\,dx$ and

$$\int\frac{e^x}{3-e^{2x}}\,dx=\int\frac{du}{a^2-u^2}=\frac{1}{2a}\ln\left|\frac{u+a}{u-a}\right|+C=\frac{1}{2\sqrt{3}}\ln\left|\frac{e^x+\sqrt{3}}{e^x-\sqrt{3}}\right|+C.$$

22. Let

$u = x^2$ and $a=2$. Then $du=2x\,dx$ and

$$\int_0^2 x^3\sqrt{4x^2-x^4}\,dx = \frac{1}{2}\int_0^2 x^2\sqrt{2\cdot 2\cdot x^2-(x^2)^2}\cdot 2x\,dx = \frac{1}{2}\int_0^4 u\sqrt{2au-u^2}\,du$$
$$= \left[\frac{2u^2-au-3a^2}{12}\sqrt{2au-u^2}+\frac{a^3}{4}\cos^{-1}\left(\frac{a-u}{a}\right)\right]_0^4$$
$$= \left[\frac{2u^2-2u-12}{12}\sqrt{4u-u^2}+\frac{8}{4}\cos^{-1}\left(\frac{2-u}{2}\right)\right]_0^4$$
$$= \left[\frac{u^2-u-6}{6}\sqrt{4u-u^2}+2\cos^{-1}\left(\frac{2-u}{2}\right)\right]_0^4$$
$$= -(0+2\cos^{-1}1)=2\cdot\pi-2\cdot 0=2\pi$$

23.

$$\int \sec^5 x\,dx = \frac{1}{4}\tan x\sec^3 x+\frac{3}{4}\int \sec^3 x\,dx=\frac{1}{4}\tan x\sec^3 x+\frac{3}{4}\left(\frac{1}{2}\tan x\sec x+\frac{1}{2}\int \sec x\,dx\right)$$
$$= \frac{1}{4}\tan x\sec^3 x+\frac{3}{8}\tan x\sec x+\frac{3}{8}\ln\left|\sec x+\tan x\right|+C$$

24. Let $u=2x$. Then $du=2dx$, so

$$\int \sin^6 2x\,dx = \frac{1}{2}\int \sin^6 u\,du=\frac{1}{2}\left(-\frac{1}{6}\sin^5 u\cos u+\frac{5}{6}\int \sin^4 u\,du\right)$$
$$=-\frac{1}{12}\sin^5 u\cos u+\frac{5}{12}\left(-\frac{1}{4}\sin^3 u\cos u+\frac{3}{4}\int \sin^2 u\,du\right)$$
$$=-\frac{1}{12}\sin^5 u\cos u-\frac{5}{48}\sin^3 u\cos u+\frac{5}{16}\left(\frac{1}{2}u-\frac{1}{4}\sin 2u\right)+C$$
$$= -\frac{1}{12}\sin^5 2x\cos 2x-\frac{5}{48}\sin^3 2x\cos 2x-\frac{5}{64}\sin 4x+\frac{5}{16}x+C$$

25. Let $u=\ln x$ and $a=2$. Then $du=\dfrac{dx}{x}$ and

$$\int \frac{\sqrt{4+(\ln x)^2}}{x}\,dx = \int\sqrt{a^2+u^2}\,du=\frac{u}{2}\sqrt{a^2+u^2}+\frac{a^2}{2}\ln\left(u+\sqrt{a^2+u^2}\right)+C$$
$$=\frac{1}{2}(\ln x)\sqrt{4+(\ln x)^2}+2\ln\left[\ln x+\sqrt{4+(\ln x)^2}\right]+C$$

26.

$$\begin{aligned}\int x^4 e^{-x}dx &= -x^4 e^{-x} + 4\int x^3 e^{-x}dx = -x^4 e^{-x} + 4\left(-x^3 e^{-x} + 3\int x^2 e^{-x}dx\right)\\ &= -\left(x^4 + 4x^3\right)e^{-x} + 12\left(-x^2 e^{-x} + 2\int xe^{-x}dx\right)\\ &= -\left(x^4 + 4x^3 + 12x^2\right)e^{-x} + 24\left[(-x-1)e^{-x}\right] + C\\ &= -\left(x^4 + 4x^3 + 12x^2 + 24x + 24\right)e^{-x} + C\end{aligned}$$

$$\begin{aligned}\text{So } \int_0^1 x^4 e^{-x}dx &= \left[-\left(x^4 + 4x^3 + 12x^2 + 24x + 24\right)e^{-x}\right]_0^1\\ &= -(1+4+12+24+24)e^{-1} + 24e^0 = 24 - 65e^{-1}.\end{aligned}$$

27. Let $u = e^x$. Then $x = \ln u$, $dx = du/u$, so

$$\int \sqrt{e^{2x}-1}\,dx = \int \frac{\sqrt{u^2-1}}{u}\,du = \sqrt{u^2-1} - \cos^{-1}(1/u) + C = \sqrt{e^{2x}-1} - \cos^{-1}\left(e^{-x}\right) + C.$$

28. Let $u = \alpha t - 3$ and assume that $\alpha \neq 0$. Then $du = \alpha\, dt$ and

$$\begin{aligned}\int e^t \sin(\alpha t - 3)\,dt &= \frac{1}{\alpha}\int e^{(u+3)/\alpha}\sin u\,du = \frac{1}{\alpha}e^{3/\alpha}\int e^{(1/\alpha)u}\sin u\,du\\ &= \frac{1}{\alpha}e^{3/\alpha}\frac{e^{(1/\alpha)u}}{(1/\alpha)^2+1^2}\left(\frac{1}{\alpha}\sin u - \cos u\right) + C\\ &= \frac{1}{\alpha}e^{3/\alpha}e^{(1/\alpha)u}\frac{\alpha^2}{1+\alpha^2}\left(\frac{1}{\alpha}\sin u - \cos u\right) + C\\ &= \frac{1}{1+\alpha^2}e^{(u+3)/\alpha}(\sin u - \alpha\cos u) + C\\ &= \frac{1}{1+\alpha^2}e^t\left[\sin(\alpha t - 3) - \alpha\cos(\alpha t - 3)\right] + C\end{aligned}$$

29.

$$\begin{aligned}\int \frac{x^4\,dx}{\sqrt{x^{10}-2}} &= \int \frac{x^4\,dx}{\sqrt{(x^5)^2-2}} = \frac{1}{5}\int \frac{du}{\sqrt{u^2-2}} \quad [\,u = x^5,\ du = 5x^4\,dx\,]\\ &= \frac{1}{5}\ln\left|u + \sqrt{u^2-2}\right| + C = \frac{1}{5}\ln\left|x^5 + \sqrt{x^{10}-2}\right| + C\end{aligned}$$

30. Let $u=\tan\theta$ and $a=3$. Then $du=\sec^2\theta\ d\theta$ and

$$\int\frac{\sec^2\theta\ \tan^2\theta}{\sqrt{9-\tan^2\theta}}\,d\theta=\int\frac{u^2}{\sqrt{a^2-u^2}}\,du=-\frac{u}{2}\sqrt{a^2-u^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{u}{a}\right)+C$$
$$=-\frac{1}{2}\tan\theta\sqrt{9-\tan^2\theta}+\frac{9}{2}\sin^{-1}\left(\frac{\tan\theta}{3}\right)+C$$

31. Using cylindrical shells, we get

$$V=2\pi\int_0^2 x\cdot x\sqrt{4-x^2}\,dx=2\pi\int_0^2 x^2\sqrt{4-x^2}\,dx=2\pi\left[\frac{x}{8}(2x^2-4)\sqrt{4-x^2}+\frac{16}{8}\sin^{-1}\frac{x}{2}\right]_0^2$$
$$=2\pi\left[(0+2\sin^{-1}1)-(0+2\sin^{-1}0\right]=2\pi\left(2\cdot\frac{\pi}{2}\right)=2\pi^2$$

32. Using disks, we get

$$\text{Volume}=\int_0^{\pi/4}\pi\tan^4 x\,dx=\pi\left(\left[\frac{1}{3}\tan^3 x\right]_0^{\pi/4}-\int_0^{\pi/4}\tan^2 x\,dx\right)=\pi\left[\frac{1}{3}\tan^3 x-\tan x+x\right]_0^{\pi/4}$$
$$=\pi\left(\frac{1}{3}-1+\frac{\pi}{4}\right)=\pi\left(\frac{\pi}{4}-\frac{2}{3}\right)$$

33. **(a)** $$\frac{d}{du}\left[\frac{1}{b^3}\left(a+bu-\frac{a^2}{a+bu}-2a\ln|a+bu|\right)+C\right]=\frac{1}{b^3}\left[b+\frac{ba^2}{(a+bu)^2}-\frac{2ab}{(a+bu)}\right]$$
$$=\frac{1}{b^3}\left[\frac{b(a+bu)^2+ba^2-(a+bu)2ab}{(a+bu)^2}\right]=\frac{1}{b^3}\left[\frac{b^3u^2}{(a+bu)^2}\right]=\frac{u^2}{(a+bu)^2}$$

(b) Let $t=a+bu\Rightarrow dt=b\,du$. Note that $u=\frac{t-a}{b}$ and $du=\frac{1}{b}\,dt$.

$$\int\frac{u^2\,du}{(a+bu)^2}=\frac{1}{b^3}\int\frac{(t-a)^2}{t^2}\,dt=\frac{1}{b^3}\int\frac{t^2-2at+a^2}{t^2}\,dt$$
$$=\frac{1}{b^3}\int\left(1-\frac{2a}{t}+\frac{a^2}{t^2}\right)dt=\frac{1}{b^3}\left(t-2a\ln|t|-\frac{a^2}{t}\right)+C$$

$$= \frac{1}{b^3}\left(a+bu-\frac{a^2}{a+bu}-2a\ln|a+bu|\right)+C$$

34. **(a)** $\frac{d}{du}\left[\frac{u}{8}\left(2u^2-a^2\right)\sqrt{a^2-u^2}+\frac{a^4}{8}\sin^{-1}\frac{u}{a}+C\right]$

$$=\frac{u}{8}\left(2u^2-a^2\right)\frac{-u}{\sqrt{a^2-u^2}}+\sqrt{a^2-u^2}\left[\frac{u}{8}(4u)+\left(2u^2-a^2\right)\frac{1}{8}\right]+\frac{a^4}{8}\frac{1/a}{\sqrt{1-u^2/a^2}}$$

$$=-\frac{u^2\left(2u^2-a^2\right)}{8\sqrt{a^2-u^2}}+\sqrt{a^2-u^2}\left[\frac{u^2}{2}+\frac{2u^2-a^2}{8}\right]+\frac{a^4}{8\sqrt{a^2-u^2}}$$

$$=\frac{1}{2}\left(a^2-u^2\right)^{-1/2}\left[-\frac{u^2}{4}\left(2u^2-a^2\right)+u^2\left(a^2-u^2\right)+\frac{1}{4}\left(a^2-u^2\right)\left(2u^2-a^2\right)+\frac{a^4}{4}\right]$$

$$=\frac{1}{2}\left(a^2-u^2\right)^{-1/2}\left[2u^2a^2-2u^4\right]=\frac{u^2\left(a^2-u^2\right)}{\sqrt{a^2-u^2}}=u^2\sqrt{a^2-u^2}$$

(b) Let $u=a\sin\theta \Rightarrow du=a\cos\theta\, d\theta$. Then

$$\int u^2\sqrt{a^2-u^2}\,du=\int a^2\sin^2\theta\; a\sqrt{1-\sin^2\theta}\; a\cos\theta\, d\theta=a^4\int\sin^2\theta\cos^2\theta\, d\theta$$

$$=a^4\int\frac{1}{2}(1+\cos 2\theta)\frac{1}{2}(1-\cos 2\theta)d\theta=\frac{1}{4}a^4\int\left(1-\cos^2 2\theta\right)d\theta$$

$$=\frac{1}{4}a^4\int\left[1-\frac{1}{2}(1+\cos 4\theta)\right]d\theta=\frac{1}{4}a^4\left(\frac{1}{2}\theta-\frac{1}{8}\sin 4\theta\right)+C$$

$$=\frac{1}{4}a^4\left(\frac{1}{2}\theta-\frac{1}{8}2\sin 2\theta\cos 2\theta\right)+C=\frac{1}{4}a^4\left[\frac{1}{2}\theta-\frac{1}{2}\sin\theta\cos\theta\left(1-2\sin^2\theta\right)\right]+C$$

$$=\frac{a^4}{8}\left[\sin^{-1}\frac{u}{a}-\frac{u}{a}\frac{\sqrt{a^2-u^2}}{a}\left(1-\frac{2u^2}{a^2}\right)\right]+C$$

$$=\frac{a^4}{8}\left[\sin^{-1}\frac{u}{a}-\frac{u}{a}\frac{\sqrt{a^2-u^2}}{a}\frac{a^2-2u^2}{a^2}\right]+C$$

$$=\frac{u}{8}\left(2u^2-a^2\right)\sqrt{a^2-u^2}+\frac{a^4}{8}\sin^{-1}\frac{u}{a}+C$$

35. Maple, Mathematica and Derive all give

$\int x^2\sqrt{5-x^2}\,dx=-\frac{1}{4}x\left(5-x^2\right)^{3/2}+\frac{5}{8}x\sqrt{5-x^2}+\frac{25}{8}\sin^{-1}\left(\frac{1}{\sqrt{5}}x\right)$. Using Formula 31, we get $\int x^2\sqrt{5-x^2}\,dx=\frac{1}{8}x\left(2x^2-5\right)\sqrt{5-x^2}+\frac{1}{8}\left(5^2\right)\sin^{-1}\left(\frac{1}{\sqrt{5}}x\right)+C$. But $-\frac{1}{4}x\left(5-x^2\right)^{3/2}+\frac{5}{8}x\sqrt{5-x^2}=\frac{1}{8}x\sqrt{5-x^2}\left[5-2\left(5-x^2\right)\right]=\frac{1}{8}x\left(2x^2-5\right)\sqrt{5-x^2}$, and the $\sin^{-1}$ terms are the same in each expression, so the answers are equivalent.

36. Maple and Mathematica both give $\int x^2\left(1+x^3\right)^4dx=\frac{1}{15}x^{15}+\frac{1}{3}x^{12}+\frac{2}{3}x^9+\frac{2}{3}x^6+\frac{1}{3}x^3$, while Derive gives $\int x^2\left(1+x^3\right)^4dx=\frac{1}{15}\left(x^3+1\right)^5$. Using the substitution $u=1+x^3\Rightarrow du=3x^2dx$, we get $\int x^2\left(1+x^3\right)^4dx=\int u^4\left(\frac{1}{3}du\right)=\frac{1}{15}u^5+C=\frac{1}{15}\left(1+x^3\right)^5+C$. We can use the Binomial Theorem or a CAS to expand this expression, and we get $\frac{1}{15}\left(1+x^3\right)^5+C=\frac{1}{15}+\frac{1}{3}x^3+\frac{2}{3}x^6+\frac{2}{3}x^9+\frac{1}{3}x^{12}+\frac{1}{15}x^{15}+C$.

37. Maple and Derive both give $\int\sin^3x\cos^2x\,dx=-\frac{1}{5}\sin^2x\cos^3x-\frac{2}{15}\cos^3x$ (although Derive factors the expression), and Mathematica gives $\int\sin^3x\cos^2x\,dx=-\frac{1}{8}\cos x-\frac{1}{48}\cos 3x+\frac{1}{80}\cos 5x$. We can use a CAS to show that both of these expressions are equal to $-\frac{1}{3}\cos^3x+\frac{1}{5}\cos^5x$. Using Formula 86, we write

$$\begin{aligned}\int\sin^3x\cos^2x\,dx &= -\frac{1}{5}\sin^2x\cos^3x+\frac{2}{5}\int\sin x\cos^2x\,dx=-\frac{1}{5}\sin^2x\cos^3x+\frac{2}{5}\left(-\frac{1}{3}\cos^3x\right)+C\\ &= -\frac{1}{5}\sin^2x\cos^3x-\frac{2}{15}\cos^3x+C\end{aligned}$$

38. Maple gives $\int\tan^2x\sec^4dx=\frac{1}{5}\frac{\sin^3x}{\cos^5x}+\frac{2}{15}\frac{\sin^3x}{\cos^3x}$, Mathematica gives $\int\tan^2x\sec^4dx=-\frac{1}{120}\sec^5x(-20\sin x+5\sin 3x+\sin 5x)$, and Derive gives $\int\tan^2x\sec^4dx=-\frac{2}{15}\tan x-\frac{\sin x}{15\cos^3x}+\frac{\sin x}{5\cos^5x}$. All of these expressions can be ‘‘simplified’’ to $-\frac{1}{15}\frac{\sin x\left(\cos^2x-2\cos^4x-3\right)}{\cos^5x}$ using Maple. Using the identity $1+\tan^2x=\sec^2x$, we write

$\int \tan^2 x \sec^4 x\,dx = \int \tan^2 x\left(1+\tan^2 x\right)\sec^2 x\,dx = \int \left(\tan^2 x+\tan^4 x\right)\sec^2 x\,dx$. Now we substitute $u = \tan x$ $\Rightarrow du = \sec^2 x\,dx$, and the integral becomes $\int \left(u^2+u^4\right)du = \frac{1}{3}u^3+\frac{1}{5}u^5+C = \frac{1}{3}\tan^3 x+\frac{1}{5}\tan^5 x+C$. If we write $\sin^5 x = \sin^3 x\left(1-\cos^2 x\right)$ and substitute into the numerator of the $\tan^5 x$ term, this becomes $\frac{1}{3}\frac{\sin^3 x}{\cos^3 x}+\frac{1}{5}\frac{\sin^3 x\left(1-\cos^2 x\right)}{\cos^5 x}+C = \frac{1}{5}\frac{\sin^3 x}{\cos^5 x}+\left(\frac{1}{3}-\frac{1}{5}\right)\frac{\sin^3 x}{\cos^3 x}+C = \frac{1}{5}\frac{\sin^3 x}{\cos^5 x}+\frac{2}{15}\frac{\sin^3 x}{\cos^3 x}+C$, which is the same as Maple's expression.

39. Maple gives $\int x\sqrt{1+2x}\,dx = \frac{1}{10}(1+2x)^{5/2}-\frac{1}{6}(1+2x)^{3/2}$, Mathematica gives $\sqrt{1+2x}\left(\frac{2}{5}x^2+\frac{1}{15}x-\frac{1}{15}\right)$, and Derive gives $\frac{1}{15}(1+2x)^{3/2}(3x-1)$. The first two expressions can be simplified to Derive's result. If we use Formula 54, we get

$$\begin{aligned}\int x\sqrt{1+2x}\,dx &= \frac{2}{15(2)^2}(3\cdot 2x-2\cdot 1)(1+2x)^{3/2}+C = \frac{1}{30}(6x-2)(1+2x)^{3/2}+C\\ &= \frac{1}{15}(3x-1)(1+2x)^{3/2}\end{aligned}$$

40. Maple and Derive both give $\int \sin^4 x\,dx = -\frac{1}{4}\sin^3 x\cos x-\frac{3}{8}\cos x\sin x+\frac{3}{8}x$, while Mathematica gives $\frac{1}{32}(12x-8\sin 2x+\sin 4x)$, which can be expanded and simplified to give the other expression. Now

$$\begin{aligned}\int \sin^4 x\,dx &= -\frac{1}{4}\sin^3 x\cos x+\frac{3}{4}\int \sin^2 x\,dx = -\frac{1}{4}\sin^3 x\cos x+\frac{3}{4}\left(\frac{1}{2}x-\frac{1}{4}\sin 2x\right)+C\\ &= -\frac{1}{4}\sin^3 x\cos x-\frac{3}{8}\sin x\cos x+\frac{3}{8}x+C \quad \text{since } \sin 2x = 2\sin x\cos x\end{aligned}$$

41. Maple gives $\int \tan^5 x\,dx = \frac{1}{4}\tan^4 x-\frac{1}{2}\tan^2 x+\frac{1}{2}\ln\left(1+\tan^2 x\right)$, Mathematica gives $\int \tan^5 x\,dx = \frac{1}{4}[-1-2\cos(2x)]\sec^4 x-\ln(\cos x)$, and Derive gives $\int \tan^5 x\,dx = \frac{1}{4}\tan^4 x-\frac{1}{2}\tan^2 x-\ln(\cos x)$. These expressions are equivalent, and none includes absolute value bars or a constant of integration. Note that Mathematica's and Derive's expressions suggest that the integral is undefined where $\cos x<0$, which is not the case.
Using Formula 75,

$\int \tan^5 x\,dx = \frac{1}{5-1}\tan^{5-1}x - \int \tan^{5-2}x\,dx = \frac{1}{4}\tan^4 x - \int \tan^3 x\,dx$. Using Formula 69,
$\int \tan^3 x\,dx = \frac{1}{2}\tan^2 x + \ln|\cos x| + C$, so $\int \tan^5 x\,dx = \frac{1}{4}\tan^4 x - \frac{1}{2}\tan^2 x - \ln|\cos x| + C$.

42. Maple gives $\int x^5\sqrt{x^2+1}\,dx = \frac{1}{35}x^4\sqrt{1+x^2} - \frac{4}{105}x^2\sqrt{1+x^2} + \frac{8}{105}\sqrt{1+x^2} + \frac{1}{7}x^6\sqrt{1+x^2}$. When we use the factor command on this expression, it becomes $\frac{1}{105}\left(1+x^2\right)^{3/2}\left(15x^4-12x^2+8\right)$. Mathematica gives $\sqrt{1+x^2}\left(\frac{8}{105} - \frac{4}{105}x^2 + \frac{1}{35}x^4 + \frac{1}{7}x^6\right)$, which again factors to give the above expression, and Derive gives the factored form immediately. If we substitute $u = \sqrt{x^2+1} \Rightarrow x^4 = \left(u^2-1\right)^2$, $x\,dx = u\,du$, then the integral
becomes

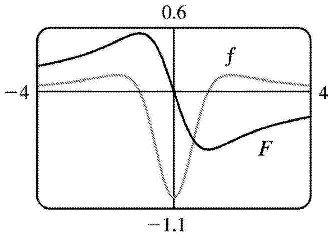
$$\begin{aligned}
\int \left(u^2-1\right)^2 u\,(u\,du) &= \int \left(u^4-2u^2+1\right)u^2\,du = \frac{1}{7}u^7 - \frac{2}{5}u^5 + \frac{1}{3}u^3 + C \\
&= \left(x^2+1\right)^{3/2}\left[\frac{1}{7}\left(x^2+1\right)^2 - \frac{2}{5}\left(x^2+1\right) + \frac{1}{3}\right] + C \\
&= \frac{1}{105}\left(x^2+1\right)^{3/2}\left[15\left(x^2+1\right)^2 - 42\left(x^2+1\right) + 35\right] + C \\
&= \frac{1}{105}\left(x^2+1\right)^{3/2}\left(15x^4-12x^2+8\right) + C
\end{aligned}$$

43. Derive gives $I = \int 2^x\sqrt{4^x-1}\,dx = \frac{2^{x-1}\sqrt{2^{2x}-1}}{\ln 2} - \frac{\ln\left(\sqrt{2^{2x}-1}+2^x\right)}{2\ln 2}$ immediately. Neither Maple nor Mathematica is able to evaluate I in its given form. However, if we instead write I as $\int 2^x\sqrt{(2^x)^2-1}\,dx$, both systems give the same answer as Derive (after minor simplification). Our trick works because the CAS now recognizes 2^x as a promising substitution.

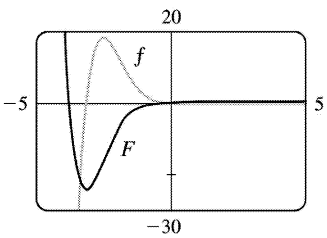
44. None of Maple, Mathematica and Derive is able to evaluate $\int (1+\ln x)\sqrt{1+(x\ln x)^2}\,dx$. However, if we let $u = x\ln x$, then $du = (1+\ln x)dx$ and the integral is simply $\int \sqrt{1+u^2}\,du$, which any CAS can evaluate. The antiderivative is $\frac{1}{2}\ln\left(x\ln x + \sqrt{1+(x\ln x)^2}\right) + \frac{1}{2}x\ln x\sqrt{1+(x\ln x)^2} + C$.

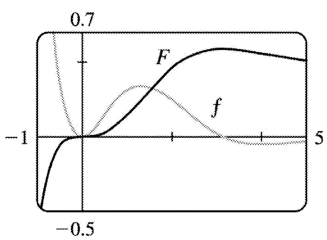
45. Maple gives the antiderivative $F(x) = \int \frac{x^2-1}{x^4+x^2+1}\,dx = -\frac{1}{2}\ln\left(x^2+x+1\right) + \frac{1}{2}\ln\left(x^2-x+1\right)$. We can see that at 0 , this antiderivative is 0 . From the graphs, it appears that F has a maximum at $x=-1$ and

a minimum at $x=1$, and that F has inflection points at $x\approx-1.7$, $x=0$, and $x\approx 1.7$.



46. Maple gives the antiderivative which, after we use the simplify command, becomes $\int xe^{-x}\sin x\,dx=-\frac{1}{2}e^{-x}(\cos x+x\cos x+x\sin x)$. At $x=0$, this antiderivative has the value $-\frac{1}{2}$, so we use $F(x)=-\frac{1}{2}e^{-x}(\cos x+x\cos x+x\sin x)+\frac{1}{2}$ to make $F(0)=0$.



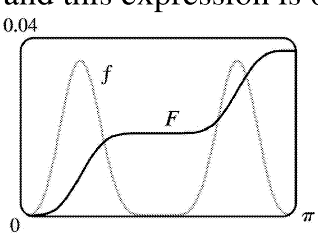


From the graphs, it appears that F has a minimum at $x\approx-3.1$ and a maximum at $x\approx 3.1$, and that F has inflection points where f' changes sign, at $x\approx-2.5$, $x=0$, $x\approx 1.3$ and $x\approx 4.1$.

47. Since $f(x)=\sin^4 x\cos^6 x$ is everywhere positive, we know that its antiderivative F is increasing. Maple gives

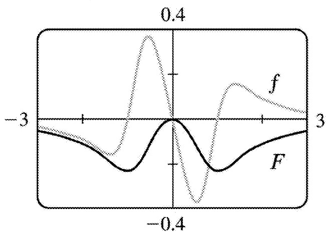
$$\int f(x)\,dx=-\frac{1}{10}\sin^3 x\cos^7 x-\frac{3}{80}\sin x\cos^7 x+\frac{1}{160}\cos^5 x\sin x+\frac{1}{128}\cos^3 x\sin x+\frac{3}{256}\cos x\sin x+\frac{3}{256}x$$

and this expression is 0 at $x=0$.



F has a minimum at $x=0$ and a maximum at $x=\pi$. F has inflection points where f' changes sign, that is, at $x \approx 0.7$, $x = \pi/2$, and $x \approx 2.5$.

48. From the graph of $f(x) = \dfrac{x^3 - x}{x^6 + 1}$, we can see that F has a maximum at $x = 0$, and minima at $x \approx \pm 1$. The antiderivative given by Maple is $F(x) = -\dfrac{1}{3} \ln\left(x^2 + 1\right) + \dfrac{1}{6} \ln\left(x^4 - x^2 + 1\right)$, and $F(0) = 0$. Note that f is odd, and its antiderivative F is even.



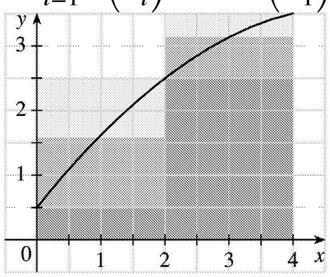
F has inflection points where f' changes sign, that is, at $x \approx \pm 0.5$ and $x \approx \pm 1.4$.

1. **(a)** $\Delta x=(b-a)/n=(4-0)/2=2$

$L_2=\sum_{i=1}^{2} f\left(x_{i-1}\right)\Delta x=f\left(x_0\right)\cdot 2+f\left(x_1\right)\cdot 2=2[f(0)+f(2)]=2(0.5+2.5)=6$

$R_2=\sum_{i=1}^{2} f\left(x_i\right)\Delta x=f\left(x_1\right)\cdot 2+f\left(x_2\right)\cdot 2=2[f(2)+f(4)]=2(2.5+3.5)=12$

$M_2=\sum_{i=1}^{2} f\left(\bar{x}_i\right)\Delta x=f\left(\bar{x}_1\right)\cdot 2+f\left(\bar{x}_2\right)\cdot 2=2[f(1)+f(3)]\approx 2(1.6+3.2)=9.6$



(b)

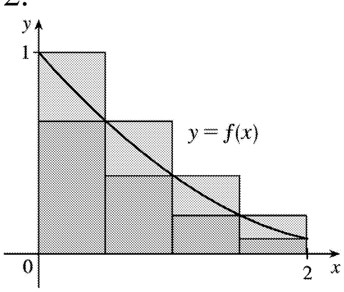
L_2 is an underestimate, since the area under the small rectangles is less than the area under the curve, and R_2 is an overestimate, since the area under the large rectangles is greater than the area under the curve. It appears that M_2 is an overestimate, though it is fairly close to I . See the solution to Exercise 45 for a proof of the fact that if f is concave down on $[a,b]$, then the Midpoint Rule is an overestimate of $\int_a^b f(x)dx$.

(c) $T_2=\left(\frac{1}{2}\Delta x\right)\left[f\left(x_0\right)+2f\left(x_1\right)+f\left(x_2\right)\right]=\frac{2}{2}[f(0)+2f(2)+f(4)]=0.5+2(2.5)+3.5=9$.

This approximation is an underestimate, since the graph is concave down. Thus, $T_2=9<I$. See the solution to Exercise 45 for a general proof of this conclusion.

(d) For any n , we will have $L_n<T_n<I<M_n<R_n$.

2.



The diagram shows that $L_4>T_4>\int_0^2 f(x)dx>R_4$, and it appears that M_4 is a bit less than $\int_0^2 f(x)dx$. In fact, for any function that is concave upward, it can be shown that $L_n>T_n>\int_0^2 f(x)dx>M_n>R_n$.

(a) Since 0.9540>0.8675>0.8632>0.7811 , it follows that $L_n=0.9540 T_n=0.8675$, $M_n=0.8632$, and

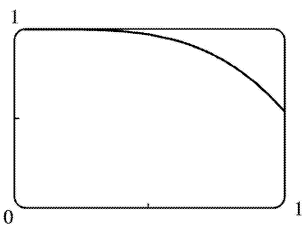
$R_n = 0.7811$.

(b) Since $M_n < \int_0^2 f(x)\,dx < T_n$, we have $0.8632 < \int_0^2 f(x)\,dx < 0.8675$.

3. $f(x) = \cos\left(x^2\right)$, $\Delta x = \frac{1-0}{4} = \frac{1}{4}$

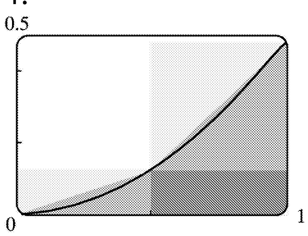
(a) $T_4 = \frac{1}{4\cdot 2}\left[f(0) + 2f\left(\frac{1}{4}\right) + 2f\left(\frac{2}{4}\right) + 2f\left(\frac{3}{4}\right) + f(1)\right] \approx 0.895759$

(b) $M_4 = \frac{1}{4}\left[f\left(\frac{1}{8}\right) + f\left(\frac{3}{8}\right) + f\left(\frac{5}{8}\right) + f\left(\frac{7}{8}\right)\right] \approx 0.908907$



The graph shows that f is concave down on $[0,1]$. So T_4 is an underestimate and M_4 is an overestimate. We can conclude that $0.895759 < \int_0^1 \cos\left(x^2\right)dx < 0.908907$.

4.



(a) Since f is increasing on $[0,1]$, L_2 will underestimate I (since the area of the darkest rectangle is less than the area under the curve), and R_2 will overestimate I . Since f is concave upward on $[0,1]$, M_2 will underestimate I and T_2 will overestimate I (the area under the straight line segments is greater than the area under the curve).

(b) For any n , we will have $L_n < M_n < I < T_n < R_n$.

(c) $L_5 = \sum_{i=1}^{5} f\left(x_{i-1}\right)\Delta x = \frac{1}{5}[f(0.0) + f(0.2) + f(0.4) + f(0.6) + f(0.8)] \approx 0.1187$

$R_5 = \sum_{i=1}^{5} f\left(x_i\right)\Delta x = \frac{1}{5}[f(0.2) + f(0.4) + f(0.6) + f(0.8) + f(1)] \approx 0.2146$

$M_5 = \sum_{i=1}^{5} f\left(\bar{x}_i\right)\Delta x = \frac{1}{5}[f(0.1) + f(0.3) + f(0.5) + f(0.7) + f(0.9)] \approx 0.1622$

$T_5=\left(\frac{1}{2}\Delta x\right)\approx 0.1666$

From the graph, it appears that the Midpoint Rule gives the best approximation. (This is in fact the case, since $I\approx 0.16371405$.)

5. $f(x)=x^2\sin x$, $\Delta x=\frac{b-a}{n}=\frac{\pi-0}{8}=\frac{\pi}{8}$

(a) $M_8=\frac{\pi}{8}\left[f\left(\frac{\pi}{16}\right)+f\left(\frac{3\pi}{16}\right)+f\left(\frac{5\pi}{16}\right)+\cdots+f\left(\frac{15\pi}{16}\right)\right]\approx 5.932957$

(b)

$$S_8=\frac{\pi}{8\cdot 3}\left[f(0)+4f\left(\frac{\pi}{8}\right)+2f\left(\frac{2\pi}{8}\right)+4f\left(\frac{3\pi}{8}\right)+2f\left(\frac{4\pi}{8}\right)+4f\left(\frac{5\pi}{8}\right)+2f\left(\frac{6\pi}{8}\right)+4f\left(\frac{7\pi}{8}\right)\right.$$

$$\approx 5.869247$$

Actual:

$$\int_0^\pi x^2\sin x\,dx=\left[-x^2\cos x\right]_0^\pi+2\int_0^\pi x\cos x\,dx=\left[-\pi^2(-1)-0\right]+2\left[\cos x+x\sin x\right]_0^\pi$$

$$=\pi^2+2[(-1+0)-(1+0)]=\pi^2-4\approx 5.869604$$

Errors: $E_M=$ actual $-M_8=\int_0^\pi x^2\sin x\,dx-M_8\approx -0.063353$

$E_S=$ actual $-S_8=\int_0^\pi x^2\sin x\,dx-S_8\approx 0.000357$

6. $f(x)=e^{-\sqrt{x}}$, $\Delta x=\frac{b-a}{n}=\frac{1-0}{6}=\frac{1}{6}$

(a) $M_6=\frac{1}{6}\left[f\left(\frac{1}{12}\right)+f\left(\frac{3}{12}\right)+f\left(\frac{5}{12}\right)+f\left(\frac{7}{12}\right)+f\left(\frac{9}{12}\right)+f\left(\frac{11}{12}\right)\right]\approx 0.525100$

(b) $S_6=\frac{1}{6\cdot 3}\left[f(0)+4f\left(\frac{1}{6}\right)+2f\left(\frac{2}{6}\right)+4f\left(\frac{3}{6}\right)+2f\left(\frac{4}{6}\right)+4f\left(\frac{5}{6}\right)+f(1)\right]\approx 0.533979$

Actual:

$$\int_0^1 e^{-\sqrt{x}}\,dx=\int_0^{-1} e^u 2u\,du u=-\sqrt{x}\ ,\ u^2=x\ ,\ 2udu=dx\ ]$$

$$=2\left[(u-1)e^u\right]_0^{-1}=2\left[-2e^{-1}-\left(-1e^0\right)\right]=2-4e^{-1}\approx 0.528482$$

Errors: $E_M=$ actual $-M_6=\int_0^1 e^{-\sqrt{x}}\,dx-M_6\approx 0.003382$

$E_S=$ actual $-S_6=\int_0^1 e^{-\sqrt{x}}\,dx-S_6\approx -0.005497$

7. $f(x)=\sqrt[4]{1+x^2}$, $\Delta x=\frac{2-0}{8}=\frac{1}{4}$

(a) $T_8=\frac{1}{4\cdot 2}\left[f(0)+2f\left(\frac{1}{4}\right)+2f\left(\frac{1}{2}\right)+\cdot\cdot\cdot+2f\left(\frac{3}{2}\right)+2f\left(\frac{7}{4}\right)+f(2)\right]\approx 2.413790$

(b) $M_8=\frac{1}{4}\left[f\left(\frac{1}{8}\right)+f\left(\frac{3}{8}\right)+\cdot\cdot\cdot+f\left(\frac{13}{8}\right)+f\left(\frac{15}{8}\right)\right]\approx 2.411453$

(c)

$S_8=\frac{1}{4\cdot 3}\left[f(0)+4f\left(\frac{1}{4}\right)+2f\left(\frac{1}{2}\right)+4f\left(\frac{3}{4}\right)+2f(1)+4f\left(\frac{5}{4}\right)+2f\left(\frac{3}{2}\right)+4f\left(\frac{7}{4}\right)+f(2)\right]\approx 2.4122$

8. $f(x)=\sin\left(x^2\right)$, $\Delta x=\frac{\frac{1}{2}-0}{4}=\frac{1}{8}$

(a) $T_4=\frac{1}{8\cdot 2}\left[f(0)+2f\left(\frac{1}{8}\right)+2f\left(\frac{2}{8}\right)+2f\left(\frac{3}{8}\right)+f\left(\frac{1}{2}\right)\right]\approx 0.042743$

(b) $M_4=\frac{1}{8}\left[f\left(\frac{1}{16}\right)+f\left(\frac{3}{16}\right)+f\left(\frac{5}{16}\right)+f\left(\frac{7}{16}\right)\right]\approx 0.040850$

(c) $S_4=\frac{1}{8\cdot 3}\left[f(0)+4f\left(\frac{1}{8}\right)+2f\left(\frac{2}{8}\right)+4f\left(\frac{3}{8}\right)+f\left(\frac{1}{2}\right)\right]\approx 0.041478$

9. $f(x)=\frac{\ln x}{1+x}$, $\Delta x=\frac{2-1}{10}=\frac{1}{10}$

(a) $T_{10}=\frac{1}{10\cdot 2}\left[f(1)+2f(1.1)+2f(1.2)+\cdot\cdot\cdot+2f(1.8)+2f(1.9)+f(2)\right]\approx 0.146879$

(b) $M_{10}=\frac{1}{10}\left[f(1.05)+f(1.15)+\cdot\cdot\cdot+f(1.85)+f(1.95)\right]\approx 0.147391$

(c)

$$S_{10}=\frac{1}{10\cdot 3}[f(1)+4f(1.1)+2f(1.2)+4f(1.3)+2f(1.4)+4f(1.5)+2f(1.6)+4f(1.7)]$$
$$+\ 2f(1.8)+4f(1.9)+f(2)]$$
$$\approx 0.147219$$

10. $f(t)=\frac{1}{1+t^2+t^4}$, $\Delta t=\frac{3-0}{6}=\frac{1}{2}$

(a) $T_6=\frac{1}{2\cdot 2}\left[f(0)+2f\left(\frac{1}{2}\right)+2f(1)+2f\left(\frac{3}{2}\right)+2f(2)+2f\left(\frac{5}{2}\right)+f(3)\right]\approx 0.895122$

(b) $M_6=\frac{1}{2}\left[f\left(\frac{1}{4}\right)+f\left(\frac{3}{4}\right)+f\left(\frac{5}{4}\right)+f\left(\frac{7}{4}\right)+f\left(\frac{9}{4}\right)+f\left(\frac{11}{4}\right)\right]\approx 0.895478$

(c) $S_6=\frac{1}{2\cdot 3}\left[f(0)+4f\left(\frac{1}{2}\right)+2f(1)+4f\left(\frac{3}{2}\right)+2f(2)+4f\left(\frac{5}{2}\right)+f(3)\right]\approx 0.898014$

11.

$f(t)=\sin(e^{t/2})$, $\Delta t=\dfrac{\frac{1}{2}-0}{8}=\dfrac{1}{16}$

(a) $T_8=\dfrac{1}{16\cdot 2}\left[f(0)+2f\left(\dfrac{1}{16}\right)+2f\left(\dfrac{2}{16}\right)+\cdots+2f\left(\dfrac{7}{16}\right)+f\left(\dfrac{1}{2}\right)\right]\approx 0.451948$

(b) $M_8=\dfrac{1}{16}\left[f\left(\dfrac{1}{32}\right)+f\left(\dfrac{3}{32}\right)+f\left(\dfrac{5}{32}\right)+\cdots+f\left(\dfrac{13}{32}\right)+f\left(\dfrac{15}{32}\right)\right]\approx 0.451991$

(c) $S_8=\dfrac{1}{16\cdot 3}\left[f(0)+4f\left(\dfrac{1}{16}\right)+2f\left(\dfrac{2}{16}\right)+\cdots+4f\left(\dfrac{7}{16}\right)+f\left(\dfrac{1}{2}\right)\right]\approx 0.451976$

12. $f(x)=\sqrt{1+\sqrt{x}}$, $\Delta x=\dfrac{4-0}{8}=\dfrac{1}{2}$

(a) $T_8=\dfrac{1}{2\cdot 2}\left[f(0)+2f\left(\dfrac{1}{2}\right)+2f(1)+\cdots+2f(3)+2f\left(\dfrac{7}{2}\right)+f(4)\right]\approx 6.042985$

(b) $M_8=\dfrac{1}{2}\left[f\left(\dfrac{1}{4}\right)+f\left(\dfrac{3}{4}\right)+\cdots+f\left(\dfrac{13}{4}\right)+f\left(\dfrac{15}{4}\right)\right]\approx 6.084778$

(c)

$S_8=\dfrac{1}{2\cdot 3}\left[f(0)+4f\left(\dfrac{1}{2}\right)+2f(1)+4f\left(\dfrac{3}{2}\right)+2f(2)+4f\left(\dfrac{5}{2}\right)+2f(3)+4f\left(\dfrac{7}{2}\right)+f(4)\right]\approx 6.061678$

13. $f(x)=e^{1/x}$, $\Delta x=\dfrac{2-1}{4}=\dfrac{1}{4}$

(a) $T_4=\dfrac{1}{4\cdot 2}[f(1)+2f(1.25)+2f(1.5)+2f(1.75)+f(2)]\approx 2.031893$

(b) $M_4=\dfrac{1}{4}[f(1.125)+f(1.375)+f(1.625)+f(1.875)]\approx 2.014207$

(c) $S_4=\dfrac{1}{4\cdot 3}[f(1)+4f(1.25)+2f(1.5)+4f(1.75)+f(2)]\approx 2.020651$

14. $f(x)=\sqrt{x}\sin x$, $\Delta x=\dfrac{4-0}{8}=\dfrac{1}{2}$

(a)

$T_8=\dfrac{1}{2\cdot 2}\left\{f(0)+2\left[f\left(\dfrac{1}{2}\right)+f(1)+f\left(\dfrac{3}{2}\right)+f(2)+f\left(\dfrac{5}{2}\right)+f(3)+f\left(\dfrac{7}{2}\right)\right]+f(4)\right\}\approx 1.732865$

(b) $M_8=\dfrac{1}{2}\left[f\left(\dfrac{1}{4}\right)+f\left(\dfrac{3}{4}\right)+f\left(\dfrac{5}{4}\right)+f\left(\dfrac{7}{4}\right)+\cdots+f\left(\dfrac{13}{4}\right)+f\left(\dfrac{15}{4}\right)\right]\approx 1.787427$

(c)

$S_8=\dfrac{1}{2\cdot 3}\left[f(0)+4f\left(\dfrac{1}{2}\right)+2f(1)+4f\left(\dfrac{3}{2}\right)+2f(2)+4f\left(\dfrac{5}{2}\right)+2f(3)+4f\left(\dfrac{7}{2}\right)+f(4)\right]\approx 1.772142$

15. $f(x)=\dfrac{\cos x}{x}$, $\Delta x=\dfrac{5-1}{8}=\dfrac{1}{2}$

(a) $T_8 = \frac{1}{2\cdot 2}\left[f(1)+2f\left(\frac{3}{2}\right)+2f(2)+\cdots+2f(4)+2f\left(\frac{9}{2}\right)+f(5)\right]\approx -0.495333$

(b)

$M_8 = \frac{1}{2}\left[f\left(\frac{5}{4}\right)+f\left(\frac{7}{4}\right)+f\left(\frac{9}{4}\right)+f\left(\frac{11}{4}\right)+f\left(\frac{13}{4}\right)+f\left(\frac{15}{4}\right)+f\left(\frac{17}{4}\right)+f\left(\frac{19}{4}\right)\right]\approx -0.5433$

(c)

$$S_8 = \frac{1}{2\cdot 3}\left[f(1)+4f\left(\frac{3}{2}\right)+2f(2)+4f\left(\frac{5}{2}\right)+2f(3)+4f\left(\frac{7}{2}\right)+2f(4)+4f\left(\frac{9}{2}\right)+f(5)\right]$$
$$\approx -0.526123$$

16. $f(x)=\ln\left(x^3+2\right)$, $\Delta x=\frac{6-4}{10}=\frac{1}{5}$

(a) $T_{10}=\frac{1}{5\cdot 2}\left[f(4)+2f(4.2)+2f(4.4)+\cdots+2f(5.6)+2f(5.8)+f(6)\right]\approx 9.649753$

(b) $M_{10}=\frac{1}{5}\left[f(4.1)+f(4.3)+\cdots+f(5.7)+f(5.9)\right]\approx 9.650912$

(c)

$$S_{10}=\frac{1}{5\cdot 3}[f(4)+4f(4.2)+2f(4.4)+4f(4.6)+2f(4.8)+4f(5)]$$
$$+\ 2f(5.2)+4f(5.4)+2f(5.6)+4f(5.8)+f(6)]$$
$$\approx 9.650526$$

17. $f(y)=\frac{1}{1+y^5}$, $\Delta y=\frac{3-0}{6}=\frac{1}{2}$

(a) $T_6=\frac{1}{2\cdot 2}\left[f(0)+2f\left(\frac{1}{2}\right)+2f\left(\frac{2}{2}\right)+2f\left(\frac{3}{2}\right)+2f\left(\frac{4}{2}\right)+2f\left(\frac{5}{2}\right)+f(3)\right]\approx 1.064275$

(b) $M_6=\frac{1}{2}\left[f\left(\frac{1}{4}\right)+f\left(\frac{3}{4}\right)+f\left(\frac{5}{4}\right)+f\left(\frac{7}{4}\right)+f\left(\frac{9}{4}\right)+f\left(\frac{11}{4}\right)\right]\approx 1.067416$

(c) $S_6=\frac{1}{2\cdot 3}\left[f(0)+4f\left(\frac{1}{2}\right)+2f\left(\frac{2}{2}\right)+4f\left(\frac{3}{2}\right)+2f\left(\frac{4}{2}\right)+4f\left(\frac{5}{2}\right)+f(3)\right]\approx 1.074915$

18. $f(x)=\frac{e^x}{x}$, $\Delta x=\frac{4-2}{10}=\frac{1}{5}$

(a) $T_{10}=\frac{1}{5\cdot 2}\{f(2)+2[f(2.2)+f(2.4)+f(2.6)+\cdots+f(3.8)]+f(4)\}\approx 14.704592$

(b) $M_{10}=\frac{1}{5}[f(2.1)+f(2.3)+f(2.5)+f(2.7)+\cdots+f(3.7)+f(3.9)]\approx 14.662669$

(c) $S_{10}=\frac{1}{5\cdot 3}[f(2)+4f(2.2)+2f(2.4)+4f(2.6)+\cdots+2f(3.6)+4f(3.8)+f(4)]\approx 14.676696$

19. $f(x)=e^{-x^2}$, $\Delta x=\frac{2-0}{10}=\frac{1}{5}$

(a) $T_{10}=\frac{1}{5\cdot 2}\{f(0)+2[f(0.2)+f(0.4)+\cdots+f(1.8)]+f(2)\}\approx 0.881839$

$M_{10}=\frac{1}{5}[f(0.1)+f(0.3)+f(0.5)+\cdots+f(1.7)+f(1.9)]\approx 0.882202$

(b) $f(x)=e^{-x^2}$, $f'(x)=-2xe^{-x^2}$, $f''(x)=\left(4x^2-2\right)e^{-x^2}$, $f'''(x)=4x\left(3-2x^2\right)e^{-x^2}$.

$f'''(x)=0\Leftrightarrow x=0$ or $x=\pm\sqrt{\frac{3}{2}}$. So to find the maximum value of $\left|f''(x)\right|$ on $[0,2]$, we need only consider its values at $x=0$, $x=2$, and $x=\sqrt{\frac{3}{2}}$. $\left|f''(0)\right|=2$, $\left|f''(2)\right|\approx 0.2564$ and $\left|f''\left(\sqrt{\frac{3}{2}}\right)\right|\approx 0.8925$. Thus, taking $K=2$, $a=0$, $b=2$, and $n=10$ in Theorem 3, we get $\left|E_T\right|\leq 2\cdot 2^3/\left(12\cdot 10^2\right)=\frac{1}{75}=0.01\overline{3}$, and $\left|E_M\right|\leq\left(\left|E_T\right|\right)/2\leq 0.00\overline{6}$.

(c) Take $K=2$ in Theorem 3. $\left|E_T\right|\leq\frac{K(b-a)^3}{12n^2}\leq 10^{-5}\Leftrightarrow\frac{2(2-0)^3}{12n^2}\leq 10^{-5}\Leftrightarrow\frac{3}{4}n^2\geq 10^5\Leftrightarrow n\geq 365.1\ldots\Leftrightarrow$ $n\geq 366$. Take $n=366$ for T_n . For E_M , again take $K=2$ in Theorem 3 to get $\left|E_M\right|\leq 10^{-5}\Leftrightarrow\frac{3}{2}n^2\geq 10^5$ $\Leftrightarrow n\geq 258.2\Rightarrow n\geq 259$. Take $n=259$ for M_n .

20. **(a)** $T_8=\frac{1}{8\cdot 2}\left\{f(0)+2\left[f\left(\frac{1}{8}\right)+f\left(\frac{2}{8}\right)+\cdots+f\left(\frac{7}{8}\right)\right]+f(1)\right\}\approx 0.902333$

$M_8=\frac{1}{8}\left[f\left(\frac{1}{16}\right)+f\left(\frac{3}{16}\right)+f\left(\frac{5}{16}\right)+\cdots+f\left(\frac{15}{16}\right)\right]=0.905620$

(b) $f(x)=\cos\left(x^2\right)$, $f'(x)=-2x\sin\left(x^2\right)$, $f''(x)=-2\sin\left(x^2\right)-4x^2\cos\left(x^2\right)$. For $0\leq x\leq 1$, sin and cos are positive, so $\left|f''(x)\right|=2\sin\left(x^2\right)+4x^2\cos\left(x^2\right)\leq 2\cdot 1+4\cdot 1\cdot 1=6$ since $\sin\left(x^2\right)\leq 1$ and $\cos\left(x^2\right)\leq 1$ for all x , and $x^2\leq 1$ for $0\leq x\leq 1$. So for $n=8$, we take $K=6$, $a=0$, and $b=1$ in Theorem 3, to get $\left|E_T\right|\leq 6\cdot 1^3/\left(12\cdot 8^2\right)=\frac{1}{128}=0.0078125$ and $\left|E_M\right|\leq\frac{1}{256}=0.00390625$.

(c) Using $K=6$ as in part (b), we have $\left|E_T\right|\leq 6\cdot 1^3/\left(12n^2\right)=1/\left(2n^2\right)\leq 10^{-5}\Rightarrow 2n^2\geq 10^5\Rightarrow$ $n\geq\sqrt{\frac{1}{2}\cdot 10^5}$ or $n\geq 224$. To guarantee that $\left|E_M\right|\leq 0.00001$, we need $6\cdot 1^3/\left(24n^2\right)\leq 10^{-5}\Rightarrow$ $4n^2\geq 10^5\Rightarrow n\geq\sqrt{\frac{1}{4}\cdot 10^5}$ or $n\geq 159$.

21. **(a)** $T_{10}=\frac{1}{10\cdot 2}\{f(0)+2[f(0.1)+f(0.2)+\cdots+f(0.9)]+f(1)\}\approx 1.71971349$

$S_{10}=\frac{1}{10\cdot 3}[f(0)+4f(0.1)+2f(0.2)+4f(0.3)+\cdots+4f(0.9)+f(1)]\approx 1.71828278$

Since $I=\int_0^1 e^x dx=\left[e^x\right]_0^1=e-1\approx 1.71828183$, $E_T=I-T_{10}\approx -0.00143166$ and $E_S=I-S_{10}\approx -0.00000095$.

(b) $f(x)=e^x\Rightarrow f''(x)=e^x\leq e$ for $0\leq x\leq 1$. Taking $K=e$, $a=0$, $b=1$, and $n=10$ in Theorem 3, we get $\left|E_T\right|\leq e(1)^3/\left(12\cdot 10^2\right)\approx 0.002265>0.00143166$. $f^{(4)}(x)=e^x<e$ for $0\leq x\leq 1$. Using Theorem 4, we have $\left|E_S\right|\leq e(1)^5/\left(180\cdot 10^4\right)\approx 0.0000015>0.00000095$. We see that the actual errors are about two–thirds the size of the error estimates.

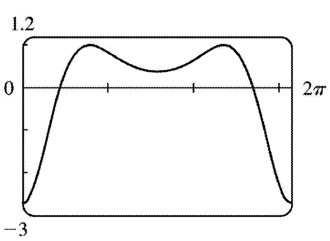
(c) From part (b), we take $K=e$ to get $\left|E_T\right|\leq \frac{K(b-a)^3}{12n^2}\leq 0.00001\Rightarrow n^2\geq \frac{e\left(1^3\right)}{12(0.00001)}\Rightarrow n\geq 150.5$.

Take $n=151$ for T_n . Now $\left|E_M\right|\leq \frac{K(b-a)^3}{24n^2}\leq 0.00001\Rightarrow n\geq 106.4$. Take $n=107$ for M_n . Finally,

$\left|E_S\right|\leq \frac{K(b-a)^5}{180n^4}\leq 0.00001\Rightarrow n^4\geq \frac{e\left(1^5\right)}{180(0.00001)}\Rightarrow n\geq 6.23$. Take $n=8$ for S_n (since n has to be even for Simpson's Rule).

22. From Example 7(b), we take $K=76e$ to get $\left|E_S\right|\leq 76e(1)^5/\left(180n^4\right)\leq 0.00001\Rightarrow$ $n^4\geq 76e/[180(0.00001)]\Rightarrow n\geq 18.4$. Take $n=20$ (since n must be even).

23. **(a)** Using a CAS, we differentiate $f(x)=e^{\cos x}$ twice, and find that $f''(x)=e^{\cos x}\left(\sin^2 x-\cos x\right)$. From the graph, we see that the maximum value of $\left|f''(x)\right|$ occurs at the endpoints of the interval $[0,2\pi]$. Since $f''(0)=-e$, we can use $K=e$ or $K=2.8$.



(b) A CAS gives $M_{10}\approx 7.954926518$. (In Maple, use `student[middlesum]`.)

(c) Using Theorem 3 for the Midpoint Rule, with $K=e$, we get

$\left|E_M\right| \le \dfrac{e(2\pi-0)^3}{24\cdot 10^2} \approx 0.280945995$. With $K=2.8$, we get $\left|E_M\right| \le \dfrac{2.8(2\pi-0)^3}{24\cdot 10^2} = 0.289391916$.

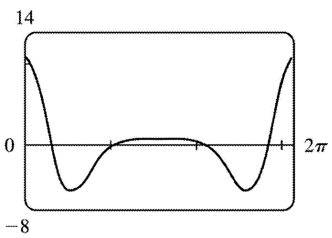
(d) A CAS gives $I \approx 7.954926521$.

(e) The actual error is only about 3×10^{-9} , much less than the estimate in part (c).

(f) We use the CAS to differentiate twice more, and then graph

$f^{(4)}(x)=e^{\cos x}\left(\sin^4 x-6\sin^2 x\cos x+3-7\sin^2 x+\cos x\right)$.

From the graph, we see that the maximum value of $\left|f^{(4)}(x)\right|$ occurs at the endpoints of the interval $[0,2\pi]$. Since $f^{(4)}(0)=4e$, we can use $K=4e$ or $K=10.9$.



(g) A CAS gives $S_{10} \approx 7.953789422$. (In Maple, use student[simpson]. .)

(h) Using Theorem 4 with $K=4e$, we get $\left|E_S\right| \le \dfrac{4e(2\pi-0)^5}{180\cdot 10^4} \approx 0.059153618$. With $K=10.9$, we get

$\left|E_S\right| \le \dfrac{10.9(2\pi-0)^5}{180\cdot 10^4} \approx 0.059299814$.

(i) The actual error is about $7.954926521-7.953789422 \approx 0.00114$. This is quite a bit smaller than the estimate in part (h), though the difference is not nearly as great as it was in the case of the Midpoint Rule.

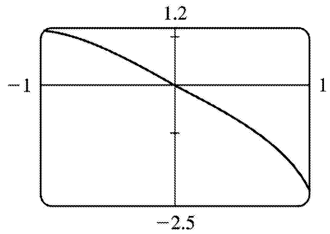
(j) To ensure that $\left|E_S\right| \le 0.0001$, we use Theorem 4: $\left|E_S\right| \le \dfrac{4e(2\pi)^5}{180\cdot n^4} \le 0.0001 \Rightarrow \dfrac{4e(2\pi)^5}{180\cdot 0.0001} \le n^4$

$\Rightarrow n^4 \ge 5,915,362 \Leftrightarrow n \ge 49.3$. So we must take $n \ge 50$ to ensure that $\left|I-S_n\right| \le 0.0001$. ($K=10.9$ leads to the same value of n .)

24. **(a)** Using the CAS, we differentiate $f(x)=\sqrt{4-x^3}$ twice,

and find that $f''(x)=-\dfrac{9x^4}{4\left(4-x^3\right)^{3/2}}-\dfrac{3x}{\left(4-x^3\right)^{1/2}}$.

From the graph, we see that $\left|f''(x)\right| < 2.2$ on $[-1,1]$.



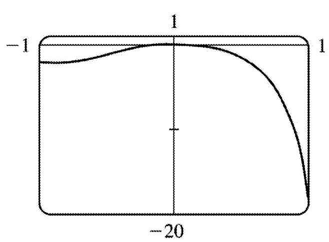
(b) A CAS gives $M_{10} \approx 3.995804152$. (In Maple, use `student[middlesum]` .)

(c) Using Theorem 3 for the Midpoint Rule, with $K=2.2$, we get $\left|E_M\right| \le \dfrac{2.2[1-(-1)]^3}{24 \cdot 10^2} \approx 0.00733$.

(d) A CAS gives $I \approx 3.995487677$.
(e) The actual error is about -0.0003165 , much less than the estimate in part (c).

(f) We use the CAS to differentiate twice more, and then graph $f^{(4)}(x) = \dfrac{9}{16}\dfrac{x^2\left(x^6-224x^3-1280\right)}{\left(4-x^3\right)^{7/2}}$.

From the graph, we see that $\left|f^{(4)}(x)\right| < 18.1$ on $[-1,1]$.



(g) A CAS gives $S_{10} \approx 3.995449790$. (In Maple, use `student[simpson]` .)

(h) Using Theorem 4 with $K=18.1$, we get $\left|E_S\right| \le \dfrac{18.1[1-(-1)]^5}{180 \cdot 10^4} \approx 0.000322$.

(i) The actual error is about $3.995487677 - 3.995449790 \approx 0.0000379$. This is quite a bit smaller than the estimate in part (h).

(j) To ensure that $\left|E_S\right| \le 0.0001$,we use Theorem 4: $\left|E_S\right| \le \dfrac{18.1(2)^5}{180 \cdot n^4} \le 0.0001 \Rightarrow \dfrac{18.1(2)^5}{180 \cdot 0.0001} \le n^4$

$\Rightarrow n^4 \ge 32$, $178 \Rightarrow n \ge 13.4$. So we must take $n \ge 14$ to ensure that $\left|I - S_n\right| \le 0.0001$.

25. $I = \int_0^1 x^3\,dx = \left[\ \frac{1}{4}x^4\right]_0^1 = 0.25$. $f(x) = x^3$.

$n=4$:

$$L_4=\frac{1}{4}\left[0^3+\left(\frac{1}{4}\right)^3+\left(\frac{2}{4}\right)^3+\left(\frac{3}{4}\right)^3\right]=0.140625$$

$$R_4=\frac{1}{4}\left[\left(\frac{1}{4}\right)^3+\left(\frac{2}{4}\right)^3+\left(\frac{3}{4}\right)^3+1^3\right]=0.390625$$

$$T_4=\frac{1}{4\cdot 2}\left[0^3+2\left(\frac{1}{4}\right)^3+2\left(\frac{2}{4}\right)^3+2\left(\frac{3}{4}\right)^3+1^3\right]=0.265625\ ,$$

$$M_4=\frac{1}{4}\left[\left(\frac{1}{8}\right)^3+\left(\frac{3}{8}\right)^3+\left(\frac{5}{8}\right)^3+\left(\frac{7}{8}\right)^3\right]=0.2421875\ ,$$

$$E_L=I-L_4=\frac{1}{4}-0.140625=0.109375\ ,\ E_R=\frac{1}{4}-0.390625=-0.140625\ ,$$

$$E_T=\frac{1}{4}-0.265625=-0.015625\ ,\ E_M=\frac{1}{4}-0.2421875=0.0078125$$

$n=8$:

$$L_8=\frac{1}{8}\left[f(0)+f\left(\frac{1}{8}\right)+f\left(\frac{2}{8}\right)+\cdots+f\left(\frac{7}{8}\right)\right]\approx 0.191406$$

$$R_8=\frac{1}{8}\left[f\left(\frac{1}{8}\right)+f\left(\frac{2}{8}\right)+\cdots+f\left(\frac{7}{8}\right)+f(1)\right]\approx 0.316406$$

$$T_8=\frac{1}{8\cdot 2}\left\{f(0)+2\left[f\left(\frac{1}{8}\right)+f\left(\frac{2}{8}\right)+\cdots+f\left(\frac{7}{8}\right)\right]+f(1)\right\}\approx 0.253906$$

$$M_8=\frac{1}{8}\left[f\left(\frac{1}{16}\right)+f\left(\frac{3}{16}\right)+\cdots+f\left(\frac{13}{16}\right)+f\left(\frac{15}{16}\right)\right]=0.248047$$

$$E_L\approx\frac{1}{4}-0.191406\approx 0.058594\ ,\ E_R\approx\frac{1}{4}-0.316406\approx -0.066406\ ,$$

$$E_T\approx\frac{1}{4}-0.253906\approx -0.003906\ ,\ E_M\approx\frac{1}{4}-0.248047\approx 0.001953\ .$$

$n=16$:

$$L_{16}=\frac{1}{16}\left[f(0)+f\left(\frac{1}{16}\right)+f\left(\frac{2}{16}\right)+\cdots+f\left(\frac{15}{16}\right)\right]\approx 0.219727$$

$$R_{16}=\frac{1}{16}\left[f\left(\frac{1}{16}\right)+f\left(\frac{2}{16}\right)+\cdots+f\left(\frac{15}{16}\right)+f(1)\right]\approx 0.282227$$

$$T_{16}=\frac{1}{16\cdot 2}\left\{f(0)+2\left[f\left(\frac{1}{16}\right)+f\left(\frac{2}{16}\right)+\cdots+f\left(\frac{15}{16}\right)\right]+f(1)\right\}\approx 0.250977$$

$$M_{16}=\frac{1}{16}\left[f\left(\frac{1}{32}\right)+f\left(\frac{3}{32}\right)+\cdots+f\left(\frac{31}{32}\right)\right]\approx 0.249512$$

$$E_L\approx\frac{1}{4}-0.219727\approx 0.030273\ ,\ E_R\approx\frac{1}{4}-0.282227\approx -0.032227\ ,$$

$E_T \approx \frac{1}{4} - 0.250977 \approx -0.000977$, $E_M \approx \frac{1}{4} - 0.249512 \approx 0.000488$.

n	L_n	R_n	T_n	M_n
4	0.140625	0.390625	0.265625	0.242188
8	0.191406	0.316406	0.253906	0.248047
16	0.219727	0.282227	0.250977	0.249512

n	E_L	E_R	E_T	E_M
4	0.109375	−0.140625	−0.015625	0.007813
8	0.058594	−0.066406	−0.003906	0.001953
16	0.030273	−0.032227	−0.000977	0.000488

Observations:
(a) E_L and E_R are always opposite in sign, as are E_T and E_M .
(b) As n is doubled, E_L and E_R are decreased by about a factor of 2 , and E_T and E_M are decreased by a factor of about 4 .
(c) The Midpoint approximation is about twice as accurate as the Trapezoidal approximation.
(d) All the approximations become more accurate as the value of n increases.
(e) The Midpoint and Trapezoidal approximations are much more accurate than the endpoint approximations.

26. $\int_0^2 e^x dx = \left[e^x\right]_0^2 = e^2 - 1 \approx 6.389056$. $f(x) = e^x$
$n=4$:
$\Delta x = (2-0)/4 = \frac{1}{2}$
$L_4 = \frac{1}{2}\left[e^0 + e^{1/2} + e^1 + e^{3/2}\right] \approx 4.924346$
$R_4 = \frac{1}{2}\left[e^{1/2} + e^1 + e^{3/2} + e^2\right] \approx 8.118874$
$T_4 = \frac{1}{2 \cdot 2}\left[e^0 + 2e^{1/2} + 2e^1 + 2e^{3/2} + e^2\right] \approx 6.521610$
$M_4 = \frac{1}{2}\left[e^{1/4} + e^{3/4} + e^{5/4} + e^{7/4}\right] \approx 6.322986$.
$E_L \approx 6.389056 - 4.924346 \approx 1.464710$, $E_R \approx 6.389056 - 8.118874 = -1.729818$,

$E_T \approx 6.389056 - 6.521610 \approx -0.132554$, $E_M \approx 6.389056 - 6.322986 = 0.0660706$.

$n=8$:

$\Delta x = (2-0)/8 = \dfrac{1}{4}$

$L_8 = \dfrac{1}{4}\left[e^0 + e^{1/4} + e^{1/2} + e^{3/4} + e^1 + e^{5/4} + e^{3/2} + e^{7/4}\right] \approx 5.623666$

$R_8 = \dfrac{1}{4}\left[e^{1/4} + e^{1/2} + e^{3/4} + e^1 + e^{5/4} + e^{3/2} + e^{7/4} + e^2\right] \approx 7.220930$

$T_8 = \dfrac{1}{4 \cdot 2}\left[e^0 + 2e^{1/4} + 2e^{1/2} + 2e^{3/4} + 2e^1 + 2e^{5/4} + 2e^{3/2} + 2e^{7/4} + e^2\right] \approx 6.422298$

$M_8 = \dfrac{1}{4}\left[e^{1/8} + e^{3/8} + e^{5/8} + e^{7/8} + e^{9/8} + e^{11/8} + e^{13/8} + e^{15/8}\right] \approx 6.372448$

$E_L \approx 6.389056 - 5.623666 \approx 0.765390$, $E_R \approx 6.389056 - 7.220930 \approx -0.831874$,

$E_T \approx 6.389056 - 6.422298 \approx -0.033242$, $E_M \approx 6.389056 - 6.372448 \approx 0.016608$.

$n=16$:

$\Delta x = (2-0)/16 = \dfrac{1}{8}$

$L_{16} = \dfrac{1}{8}\left[f(0) + f\left(\dfrac{1}{8}\right) + f\left(\dfrac{2}{8}\right) + \cdots + f\left(\dfrac{14}{8}\right) + f\left(\dfrac{15}{8}\right)\right] \approx 5.998057$

$R_{16} = \dfrac{1}{8}\left[f\left(\dfrac{1}{8}\right) + f\left(\dfrac{2}{8}\right) + f\left(\dfrac{3}{8}\right) + \cdots + f\left(\dfrac{15}{8}\right) + f(2)\right] \approx 6.796689$

$T_{16} = \dfrac{1}{8 \cdot 2}\left(f(0) + 2\left[f\left(\dfrac{1}{8}\right) + f\left(\dfrac{2}{8}\right) + f\left(\dfrac{3}{8}\right) + \cdots + f\left(\dfrac{15}{8}\right)\right] + f(2)\right) \approx 6.397373$

$M_{16} = \dfrac{1}{8}\left[f\left(\dfrac{1}{16}\right) + f\left(\dfrac{3}{16}\right) + f\left(\dfrac{5}{16}\right) + \cdots + f\left(\dfrac{29}{16}\right) + f\left(\dfrac{31}{16}\right)\right] \approx 6.384899$

$E_L \approx 6.389056 - 5.998057 \approx 0.390999$, $E_R \approx 6.389056 - 6.796689 \approx -0.407633$,

$E_T \approx 6.389056 - 6.397373 \approx -0.008317$, $E_M \approx 6.389056 - 6.384899 \approx 0.004158$.

n	E_L	E_R	E_T	E_M
4	1.464710	−1.729818	−0.132554	0.066071
8	0.765390	−0.831874	−0.033242	0.016608
16	0.390999	−0.407633	−0.008317	0.004158

Observations:

(a) E_L and E_R are always opposite in sign, as are E_T and E_M .

(b) As n is doubled,

E_L and E_R are decreased by a factor of about 2 , and E_T and E_M are decreased by a factor of about 4 .
(c) The Midpoint approximation is about twice as accurate as the Trapezoidal approximation.
(d) All the approximations become more accurate as the value of n increases.
(e) The Midpoint and Trapezoidal approximations are much more accurate than the endpoint approximations.

27. $\int_1^4 \sqrt{x}\, dx = \left[\frac{2}{3} x^{3/2} \right]_1^4 = \frac{2}{3} (8-1) = \frac{14}{3} \approx 4.666667$

$n=6$:

$\Delta x = (4-1)/6 = \frac{1}{2}$

$T_6 = \frac{1}{2\cdot 2} \left[\sqrt{1} + 2\sqrt{1.5} + 2\sqrt{2} + 2\sqrt{2.5} + 2\sqrt{3} + 2\sqrt{3.5} + \sqrt{4} \right] \approx 4.661488$

$M_6 = \frac{1}{2} \left[\sqrt{1.25} + \sqrt{1.75} + \sqrt{2.25} + \sqrt{2.75} + \sqrt{3.25} + \sqrt{3.75} \right] \approx 4.669245$

$S_6 = \frac{1}{2\cdot 3} \left[\sqrt{1} + 4\sqrt{1.5} + 2\sqrt{2} + 4\sqrt{2.5} + 2\sqrt{3} + 4\sqrt{3.5} + \sqrt{4} \right] \approx 4.666563$

$E_T \approx \frac{14}{3} - 4.661488 \approx 0.005178$, $E_M \approx \frac{14}{3} - 4.669245 \approx -0.002578$,

$E_S \approx \frac{14}{3} - 4.666563 \approx 0.000104$.

$n=12$:

$\Delta x = (4-1)/12 = \frac{1}{4}$

$T_{12} = \frac{1}{4\cdot 2} \left(f(1) + 2[f(1.25) + f(1.5) + \cdots + f(3.5) + f(3.75)] + f(4) \right) \approx 4.665367$

$M_{12} = \frac{1}{4} \left[f(1.125) + f(1.375) + f(1.625) + \cdots + f(3.875) \right] \approx 4.667316$

$S_{12} = \frac{1}{4\cdot 3} \approx 4.666659$

$E_T \approx \frac{14}{3} - 4.665367 \approx 0.001300$, $E_M \approx \frac{14}{3} - 4.667316 \approx -0.000649$,

$E_S \approx \frac{14}{3} - 4.666659 \approx 0.000007$.

Note: These errors were computed more precisely and then rounded to six places. That is, they were not computed by comparing the rounded values of T_n , M_n , and S_n with the rounded value of the actual integral.

n	T_n	M_n	S_n
6	4.661488	4.669245	4.666563
12	4.665367	4.667316	4.666659

n	E_T	E_M	E_S
6	0.005178	−0.002578	0.000104
12	0.001300	−0.000649	0.000007

Observations:

(a) E_T and E_M are opposite in sign and decrease by a factor of about 4 as n is doubled.

(b) The Simpson's approximation is much more accurate than the Midpoint and Trapezoidal approximations, and seems to decrease by a factor of about 16 as n is doubled.

28. $I=\int_{-1}^{2} xe^{x}dx=\left[xe^{x}-e^{x}\right]_{-1}^{2}=e^{2}+2/e\approx 8.124815$. $f(x)=xe^{x}$.

$n=6$:

$\Delta x=[2-(-1)]/6=\frac{1}{2}$

$T_6=\frac{1}{2\cdot 2}\ \{f(-1)+2[f(-0.5)+f(0)+\cdot\cdot\cdot+f(1.5)]+f(2)\}\approx 8.583514$

$M_6=\frac{1}{2}\ [f(-0.75)+f(-0.25)+\cdot\cdot\cdot+f(1.75)]\approx 7.896632$

$S_6=\frac{1}{2\cdot 3}\ [f(-1)+4f(-0.5)+2f(0)+4f(0.5)+2f(1)+4f(1.5)+f(2)]\approx 8.136885$

$E_T\approx I-8.583514\approx -0.458699$, $E_M\approx I-7.896632\approx 0.228183$,

$E_S\approx I-8.136885\approx -0.012070$.

$n=12$:

$\Delta x=[2-(-1)]/12=\frac{1}{4}$

$T_{12}=\frac{1}{4\cdot 2}\ \{f(-1)+2[f(-0.75)+f(-0.5)+\cdot\cdot\cdot+f(1.75)]+f(2)\}\approx 8.240073$

$M_{12}=\frac{1}{4}\left[f\left(-\frac{7}{8}\right)+f\left(-\frac{5}{8}\right)+\cdot\cdot\cdot+f\left(\frac{13}{8}\right)+f\left(\frac{15}{8}\right)\right]\approx 8.067259$

$S_{12}=\frac{1}{4\cdot 3}\ [f(-1)+4f(-0.75)+2f(-0.5)+\cdot\cdot\cdot+2f(1.5)+4f(1.75)+f(2)]\approx 8.125593$

$E_T\approx I-8.240073\approx -0.115258$, $E_M\approx I-8.067259\approx 0.057556$,

$E_S \approx I - 8.125593 \approx -0.000778$

n	T_n	M_n	S_n
6	8.583514	7.896632	8.136885
12	8.240073	8.067259	8.125593

n	E_T	E_M	E_S
6	−0.458699	0.228183	−0.012070
12	−0.115258	0.057556	−0.000778

Observations:

(a) E_T and E_M are opposite in sign and decrease by a factor of about 4 as n is doubled.

(b) The Simpson's approximation is much more accurate than the Midpoint and Trapezoidal approximations, and seems to decrease by a factor of about 16 as n is doubled.

29. $\Delta x = (4-0)/4 = 1$

(a) $T_4 = \frac{1}{2}\left[f(0)+2f(1)+2f(2)+2f(3)+f(4)\right] \approx \frac{1}{2}\left[0+2(3)+2(5)+2(3)+1\right] = 11.5$

(b) $M_4 = 1\cdot\left[f(0.5)+f(1.5)+f(2.5)+f(3.5)\right] \approx 1+4.5+4.5+2 = 12$

(c) $S_4 = \frac{1}{3}\left[f(0)+4f(1)+2f(2)+4f(3)+f(4)\right] \approx \frac{1}{3}\left[0+4(3)+2(5)+4(3)+1\right] = 11.\overline{6}$

30. If x= distance from left end of pool and $w = w(x)$= width at x , then Simpson's Rule with $n=8$ and $\Delta x = 2$ gives Area $= \int_0^{16} w\,dx \approx \frac{2}{3}\left[0+4(6.2)+2(7.2)+4(6.8)+2(5.6)+4(5.0)+2(4.8)+4(4.8)+0\right] \approx 84\ \text{m}^2$.

31. **(a)** We are given the function values at the endpoints of 8 intervals of length 0.4, so we'll use the Midpoint Rule with $n = 8/2 = 4$ and $\Delta x = (3.2-0)/4 = 0.8$.

$$\begin{aligned}\int_0^{3.2} f(x)\,dx \approx M_4 &= 0.8[f(0.4)+f(1.2)+f(2.0)+f(2.8)] \\ &= 0.8[6.5+6.4+7.6+8.8] \\ &= 0.8(29.3) = 23.44\end{aligned}$$

(b) $-4 \le f''(x) \le 1 \Rightarrow \left|f''(x)\right| \le 4$, so use $K=4$, $a=0$, $b=3.2$, and $n=4$ in Theorem 3. So

$\left|E_M\right| \leq \frac{4(3.2-0)^3}{24(4)^2} = \frac{128}{375} = 0.341\overline{3}$.

32. We use Simpson's Rule with $n=10$ and $\Delta x = \frac{1}{2}$:

$$\begin{aligned} \text{distance} = \int_0^5 \text{v(t)}\,\text{dt} \approx S_{10} &= \frac{1}{2\cdot 3}\left[f(0)+4f(0.5)+2f(1)+\cdots+4f(4.5)+f(5)\right] \\ &= \frac{1}{6} \\ &= \frac{1}{6}(268.41) = 44.735\text{m} \end{aligned}$$

33. By the Net Change Theorem, the increase in velocity is equal to $\int_0^6 a(t)\,dt$. We use Simpson's Rule with $n=6$ and $\Delta t=(6-0)/6=1$ to estimate this integral:

$$\begin{aligned} \int_0^6 a(t)\,dt \approx S_6 &= \frac{1}{3}[a(0)+4a(1)+2a(2)+4a(3)+2a(4)+4a(5)+a(6)] \\ &\approx \frac{1}{3}[0+4(0.5)+2(4.1)+4(9.8)+2(12.9)+4(9.5)+0] = \frac{1}{3}(113.2) = 37.7\overline{3}\,ft/s \end{aligned}$$

34. By the Net Change Theorem, the total amount of water that leaked out during the first six hours is equal to $\int_0^6 r(t)\,dt$. We use Simpson's Rule with $n=6$ and $\Delta t = \frac{6-0}{6} = 1$ to estimate this integral:

$$\begin{aligned} \int_0^6 \text{r(t)}\,\text{dt} \approx S_6 &= \frac{1}{3}\left[r(0)+4r(1)+2r(2)+4r(3)+2r(4)+4r(5)+r(6)\right] \\ &\approx \frac{1}{3}\left[4+4(3)+2(2.4)+4(1.9)+2(1.4)+4(1.1)+1\right] \\ &= \frac{1}{3}(36.6) = 12.2\text{liters} \end{aligned}$$

The function values were obtained from a high–resolution graph.

35. By the Net Change Theorem, the energy used is equal to $\int_0^6 P(t)\,dt$. We use Simpson's Rule with $n=12$ and $\Delta t=(6-0)/12=\frac{1}{2}$ to estimate this integral:

$$\int_0^6 P(t)\,dt \approx S_{12} = \frac{1/2}{3}\,[P(0)+4P(0.5)+2P(1)+4P(1.5)+2P(2)+4P(2.5)]$$
$$+2P(3)+4P(3.5)+2P(4)+4P(4.5)+2P(5)+4P(5.5)+P(6)]$$
$$=\frac{1}{6}\,[1814+4(1735)+2(1686)+4(1646)+2(1637)+4(1609)+2(1604)]$$
$$+4(1611)+2(1621)+4(1666)+2(1745)+4(1886)+2052]$$
$$=\frac{1}{6}\,(61{,}064)=10{,}177.\overline{3} \text{ megawatt–hours.}$$

36. By the Net Change Theorem, the total amount of data transmitted is equal to $\int_0^8 D(t)\,dt \times 3600$. We use Simpson's Rule with $n=8$ and $\Delta t=(8-0)/8=1$ to estimate this integral:

$$\int_0^8 D(t)\,dt \approx S_8 = \frac{1}{3}\,[D(0)+4D(1)+2D(2)+4D(3)+2D(4)+4D(5)+2D(6)+4D(7)+D(8)]$$
$$\approx \frac{1}{3}\,[0.35+4(0.32)+2(0.41)+4(0.50)+2(0.51)+4(0.56)+2(0.56)+4(0.83)+0.88]$$
$$=\frac{1}{3}\,(13.03)=4.34\overline{3}$$

Now multiply by 3600 to obtain 15 , 636 megabits.

37. Let $y=f(x)$ denote the curve. Using cylindrical shells, $V=\int_2^{10} 2\pi x f(x)\,dx = 2\pi\int_2^{10} x f(x)\,dx = 2\pi I$. Now use Simpson's Rule to approximate I :

$$I \approx S_8 = \frac{10-2}{3(8)}\,[2f(2)+4\cdot 3f(3)+2\cdot 4f(4)+4\cdot 5f(5)+2\cdot 6f(6)$$
$$+4\cdot 7f(7)+2\cdot 8f(8)+4\cdot 9f(9)+10f(10)]$$
$$\approx \frac{1}{3}\,[2(0)+12(1.5)+8(1.9)+20(2.2)+12(3.0)+28(3.8)+16(4.0)+36(3.1)+10(0)]$$
$$=\frac{1}{3}\,(395.2)$$

Thus, $V \approx 2\pi\cdot\frac{1}{3}\,(395.2) \approx 827.7$ or 828 cubic units.

38.

$$\text{Work} = \int_0^{18} f(x)\,dx \approx S_6 = \frac{18-0}{6\cdot 3}\,[f(0)+4f(3)+2f(6)+4f(9)+2f(12)+4f(15)+f(18)]$$
$$=1\cdot[9.8+4(9.1)+2(8.5)+4(8.0)+2(7.7)+4(7.5)+7.4]=148 \text{ joules}$$

39. Volume $=\pi\int_0^2\left(\sqrt[3]{1+x^3}\right)^2dx=\pi\int_0^2\left(1+x^3\right)^{2/3}dx$. $V\approx\pi\cdot S_{10}$ where $f(x)=\left(1+x^3\right)^{2/3}$ and $\Delta x=(2-0)/10=\frac{1}{5}$. Therefore,

$$V\approx\pi\cdot S_{10}=\pi\frac{1}{5\cdot 3}[f(0)+4f(0.2)+2f(0.4)+4f(0.6)+2f(0.8)+4f(1)$$
$$+2f(1.2)+4f(1.4)+2f(1.6)+4f(1.8)+f(2)]\approx 12.325078$$

40. Using Simpson's Rule with $n=10$, $\Delta x=\frac{\pi/2}{10}$, $L=1$, $\theta_0=\frac{42\pi}{180}$ radians, $g=9.8\text{ m / s}^2$, $k^2=\sin^2\left(\frac{1}{2}\theta_0\right)$, and $f(x)=1/\sqrt{1-k^2\sin^2x}$, we get

$$T=4\sqrt{\frac{L}{g}}\int_0^{\pi/2}\frac{dx}{\sqrt{1-k^2\sin^2x}}\approx 4\sqrt{\frac{L}{g}}S_{10}$$
$$=4\sqrt{\frac{1}{9.8}}\left(\frac{\pi/2}{10\cdot 3}\right)\left[f(0)+4f\left(\frac{\pi}{20}\right)+2f\left(\frac{2\pi}{20}\right)+\cdots+4f\left(\frac{9\pi}{20}\right)+f\left(\frac{\pi}{2}\right)\right]\approx 2.07665$$

41. $I(\theta)=\frac{N^2\sin^2k}{k^2}$, where $k=\frac{\pi Nd\sin\theta}{\lambda}$, $N=10,000$, $d=10^{-4}$, and $\lambda=632.8\times 10^{-9}$. So $I(\theta)=\frac{\left(10^4\right)^2\sin^2k}{k^2}$, where $k=\frac{\pi\left(10^4\right)\left(10^{-4}\right)\sin\theta}{632.8\times 10^{-9}}$. Now $n=10$ and $\Delta\theta=\frac{10^{-6}-\left(-10^{-6}\right)}{10}=2\times 10^{-7}$, so $M_{10}=2\times 10^{-7}[I(-0.0000009)+I(-0.0000007)+\cdots+I(0.0000009)]\approx 59.4$.

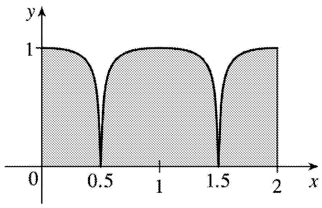
42. $f(x)=\cos(\pi x)$, $\Delta x=\frac{20-0}{10}=2\Rightarrow$

$$T_{10}=\frac{2}{2}\{f(0)+2[f(2)+f(4)+\cdots+f(18)]+f(20)\}$$
$$=1[\cos 0+2(\cos 2\pi+\cos 4\pi+\cdots+\cos 18\pi)+\cos 20\pi]$$
$$=1+2(1+1+1+1+1+1+1+1+1)+1=20$$

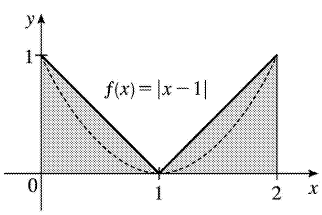
The actual value is $\int_0^{20}\cos(\pi x)dx=\frac{1}{\pi}[\sin\pi x]_0^{20}=\frac{1}{\pi}(\sin 20\pi-\sin 0)=0$. The discrepancy is due to the fact that the function is sampled only at points of the form $2n$, where its value is $f(2n)=\cos(2n\pi)=1$.

43. Consider the function f whose graph is shown. The area

$\int_0^2 f(x)dx$ is close to 2 . The Trapezoidal Rule gives $T_2=\frac{2-0}{2\cdot 2}[f(0)+2f(1)+f(2)]=\frac{1}{2}[1+2\cdot 1+1]=2$.
The Midpoint Rule gives $M_2=\frac{2-0}{2}[f(0.5)+f(1.5)]=1[0+0]=0$,
so the Trapezoidal Rule is more accurate.



44. Consider the function $f(x)=|x-1|$, $0\leq x\leq 2$. The area $\int_0^2 f(x)dx$ is exactly 1 . So is the right endpoint approximation: $R_2=f(1)\Delta x+f(2)\Delta x=0\cdot 1+1\cdot 1=1$. But Simpson's Rule approximates f with the parabola $y=(x-1)^2$, shown dashed, and $S_2=\frac{\Delta x}{3}[f(0)+4f(1)+f(2)]=\frac{1}{3}[1+4\cdot 0+1]=\frac{2}{3}$.



45. Since the Trapezoidal and Midpoint approximations on the interval $[a,b]$ are the sums of the Trapezoidal and Midpoint approximations on the subintervals $\left[x_{i-1},x_i\right]$, $i=1,2,\ldots,n$, we can focus our attention on one such interval. The condition $f''(x)<0$ for $a\leq x\leq b$ means that the graph of f is concave down as in Figure 5. In that figure, T_n is the area of the trapezoid $AQRD$, $\int_a^b f(x)dx$ is the area of the region $AQPRD$, and M_n is the area of the trapezoid $ABCD$, so $T_n<\int_a^b f(x)dx<M_n$. In general, the condition $f''<0$ implies that the graph of f on $[a,b]$ lies above the chord joining the points $(a,f(a))$ and $(b,f(b))$. Thus, $\int_a^b f(x)dx>T_n$. Since M_n is the area under a tangent to the graph, and since $f''<0$ implies that the tangent lies above the graph, we also have $M_n>\int_a^b f(x)dx$. Thus, $T_n<\int_a^b f(x)dx<M_n$.

46. Let f be a polynomial of degree ≤ 3 ; say $f(x)=Ax^3+Bx^2+Cx+D$. It will suffice to show that Simpson's estimate is exact when there are two subintervals ($n=2$), because for a larger even number

of subintervals the sum of exact estimates is exact. As in the derivation of Simpson's Rule, we can assume that $x_0=-h$, $x_1=0$, and $x_2=h$. Then Simpson's approximation is

$$\begin{aligned}\int_{-h}^{h} f(x)dx \approx{} & \frac{1}{3}h[f(-h)+4f(0)+f(h)] \\ ={} & \frac{1}{3}h\Big[\Big(-Ah^3+Bh^2-Ch+D\Big)+4D+\Big(Ah^3+Bh^2+Ch+D\Big)\Big] \\ ={} & \frac{1}{3}h\Big[2Bh^2+6D\Big]=\frac{2}{3}Bh^3+2Dh\end{aligned}$$

The exact value of the integral is

$$\begin{aligned}\int_{-h}^{h}\Big(Ax^3+Bx^2+Cx+D\Big)dx ={} & 2\int_0^h\Big(Bx^2+D\Big)dx \quad \text{[by Theorem 5.5. (a) and (b)]} \\ ={} & 2\left[\frac{1}{3}Bx^3+Dx\right]_0^h=\frac{2}{3}Bh^3+2Dh\end{aligned}$$

Thus, Simpson's Rule is exact.

47. $T_n=\frac{1}{2}\Delta x\Big[f\big(x_0\big)+2f\big(x_1\big)+\cdots+2f\big(x_{n-1}\big)+f\big(x_n\big)\Big]$ and $M_n=\Delta x\Big[f\big(\bar{x}_1\big)+f\big(\bar{x}_2\big)+\cdots+f\big(\bar{x}_{n-1}\big)+f\big(\bar{x}_n\big)\Big]$, where $\bar{x}_i=\frac{1}{2}\big(x_{i-1}+x_i\big)$. Now

$$\begin{aligned}T_{2n} ={} & \frac{1}{2}\left(\frac{1}{2}\Delta x\right)\Big[f\big(x_0\big)+2f\big(\bar{x}_1\big)+2f\big(x_1\big)+2f\big(\bar{x}_2\big)+2f\big(x_2\big)+\cdots \\ & +2f\big(\bar{x}_{n-1}\big)+2f\big(x_{n-1}\big)+2f\big(\bar{x}_n\big)+f\big(x_n\big)\Big]\end{aligned}$$

$$\begin{aligned}\text{so } \frac{1}{2}\big(T_n+M_n\big) ={} & \frac{1}{2}T_n+\frac{1}{2}M_n \\ ={} & \frac{1}{4}\Delta x\Big[f\big(x_0\big)+2f\big(x_1\big)+\cdots+2f\big(x_{n-1}\big)+f\big(x_n\big)\Big] \\ & +\frac{1}{4}\Delta x\Big[2f\big(\bar{x}_1\big)+2f\big(\bar{x}_2\big)+\cdots+2f\big(\bar{x}_{n-1}\big)+2f\big(\bar{x}_n\big)\Big] \\ ={} & T_{2n}\end{aligned}$$

48. $T_n=\frac{\Delta x}{2}\Big[f\big(x_0\big)+2\sum_{i=1}^{n-1}f\big(x_i\big)+f\big(x_n\big)\Big]$ and $M_n=\Delta x\sum_{i=1}^{n}f\left(x_i-\frac{\Delta x}{2}\right)$, so

$$\frac{1}{3}T_n+\frac{2}{3}M_n=\frac{1}{3}\big(T_n+2M_n\big)=\frac{\Delta x}{3\cdot 2}\left[f\big(x_0\big)+2\sum_{i=1}^{n-1}f\big(x_i\big)+f\big(x_n\big)+4\sum_{i=1}^{n}f\left(x_i-\frac{\Delta x}{2}\right)\right]$$

where

$\Delta x=\frac{b-a}{n}$. Let $\delta x=\frac{b-a}{2n}$. Then $\Delta x=2\delta x$, so

$$\frac{1}{3}T_n+\frac{2}{3}M_n = \frac{\delta x}{3}\left[f\left(x_0\right)+2\sum_{i=1}^{n-1}f\left(x_i\right)+f\left(x_n\right)+4\sum_{i=1}^{n}f\left(x_i-\delta x\right)\right]$$

$$= \frac{1}{3}\delta x\left[f\left(x_0\right)+4f\left(x_1-\delta x\right)+2f\left(x_1\right)+4f\left(x_2-\delta x\right)\right.$$

$$\left.+2f\left(x_2\right)+\cdots+2f\left(x_{n-1}\right)+4f\left(x_n-\delta x\right)+f\left(x_n\right)\right]$$

Since $x_0, x_1-\delta x, x_1, x_2-\delta x, x_2, \ldots, x_{n-1}, x_n-\delta x, x_n$ are the subinterval endpoints for S_{2n} , and since $\delta x=\frac{b-a}{2n}$ is the width of the subintervals for S_{2n} , the last expression for $\frac{1}{3}T_n+\frac{2}{3}M_n$ is the usual expression for S_{2n} . Therefore, $\frac{1}{3}T_n+\frac{2}{3}M_n=S_{2n}$.

1. **(a)** Since $\int_1^{\infty} x^4 e^{-x^4}\,dx$ has an infinite interval of integration, it is an improper integral of Type I.

(b) Since $y=\sec x$ has an infinite discontinuity at $x=\frac{\pi}{2}$, $\int_0^{\pi/2} \sec x\,dx$ is a Type II improper integral.

(c) Since $y=\frac{x}{(x-2)(x-3)}$ has an infinite discontinuity at $x=2$, $\int_0^2 \frac{x}{x^2-5x+6}\,dx$ is a Type II improper integral.

(d) Since $\int_{-\infty}^0 \frac{1}{x^2+5}\,dx$ has an infinite interval of integration, it is an improper integral of Type I.

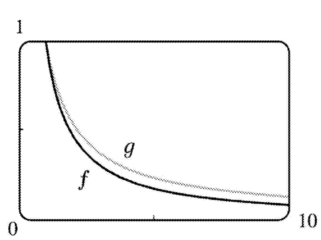
2. **(a)** Since $y=1/(2x-1)$ is defined and continuous on $[1,2]$, the integral is proper.

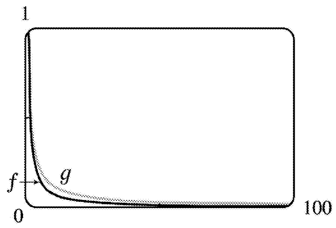
(b) Since $y=\frac{1}{2x-1}$ has an infinite discontinuity at $x=\frac{1}{2}$, $\int_0^1 \frac{1}{2x-1}\,dx$ is a Type II improper integral.

(c) Since $\int_{-\infty}^{\infty} \frac{\sin x}{1+x^2}\,dx$ has an infinite interval of integration, it is an improper integral of Type I.

(d) Since $y=\ln(x-1)$ has an infinite discontinuity at $x=1$, $\int_1^2 \ln(x-1)\,dx$ is a Type II improper integral.

3. The area under the graph of $y=1/x^3=x^{-3}$ between $x=1$ and $x=t$ is
$A(t)=\int_1^t x^{-3}\,dx=\left[-\frac{1}{2}x^{-2}\right]_1^t=-\frac{1}{2}t^{-2}-\left(-\frac{1}{2}\right)=\frac{1}{2}-1\Big/\left(2t^2\right)$. So the area for $1\le x\le 10$ is $A(10)=0.5-0.005=0.495$, the area for $1\le x\le 100$ is $A(100)=0.5-0.00005=0.49995$, and the area for $1\le x\le 1000$ is $A(1000)=0.5-0.0000005=0.4999995$. The total area under the curve for $x\ge 1$ is
$\lim_{t\to\infty} A(t)=\lim_{t\to\infty}\left[\frac{1}{2}-1\Big/\left(2t^2\right)\right]=\frac{1}{2}$.

4. **(a)** 



(b) The area under the graph of f from $x=1$ to $x=t$ is

$$F(t) = \int_1^t f(x)\,dx = \int_1^t x^{-1.1}\,dx = \left[-\frac{1}{0.1}\,x^{-0.1} \right]_1^t$$
$$= -10\left(t^{-0.1}-1\right) = 10\left(1-t^{-0.1}\right)$$

and the area under the graph of g is

$$G(t) = \int_1^t g(x)\,dx = \int_1^t x^{-0.9}\,dx = \left[\frac{1}{0.1}\,x^{0.1} \right]_1^t$$
$$= 10\left(t^{0.1}-1\right)$$

t	$F(t)$	$G(t)$
10	2.06	2.59
100	3.69	5.85
10^4	6.02	15.12
10^6	7.49	29.81
10^{10}	9	90
10^{20}	9.9	990

(c) The total area under the graph of f is $\lim_{t\to\infty} F(t) = \lim_{t\to\infty} 10\left(1-t^{-0.1}\right) = 10$.

The total area under the graph of g does not exist, since $\lim_{t\to\infty} G(t) = \lim_{t\to\infty} 10\left(t^{0.1}-1\right) = \infty$.

5. $I = \int_1^\infty \frac{1}{(3x+1)^2}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{(3x+1)^2}\,dx$. Now

$$\int \frac{1}{(3x+1)^2}\,dx = \frac{1}{3}\int \frac{1}{u^2}\,du \quad [\,u=3x+1\ ,\ du=3\,dx\,]$$

$$=-\frac{1}{3u}+C=-\frac{1}{3(3x+1)}+C\ ,$$

so $I=\lim_{t\to\infty}\left[-\frac{1}{3(3x+1)}\right]_1^t=\lim_{t\to\infty}\left[-\frac{1}{3(3t+1)}+\frac{1}{12}\right]=0+\frac{1}{12}=\frac{1}{12}$. Convergent

6. $\int_{-\infty}^{0}\frac{1}{2x-5}\,dx=\lim_{t\to-\infty}\int_t^0\frac{1}{2x-5}\,dx=\lim_{t\to-\infty}\left[\frac{1}{2}\ln|2x-5|\right]_t^0=\lim_{t\to\infty}\left[\frac{1}{2}\ln 5-\frac{1}{2}\ln|2t-5|\right]=-\infty$. Divergent

7.

$$\int_{-\infty}^{-1}\frac{1}{\sqrt{2-w}}\,dw=\lim_{t\to-\infty}\int_t^{-1}\frac{1}{\sqrt{2-w}}\,dw=\lim_{t\to-\infty}\left[-2\sqrt{2-w}\right]_t^{-1}$$
$$=\lim_{t\to-\infty}\left[-2\sqrt{3}+2\sqrt{2-t}\right]=\infty\ \text{. Divergent}$$

8.

$$\int_0^{\infty}\frac{x}{(x^2+2)^2}\,dx=\lim_{t\to\infty}\int_0^t\frac{x}{(x^2+2)^2}\,dx=\lim_{t\to\infty}\frac{1}{2}\left[\frac{-1}{x^2+2}\right]_0^t=\frac{1}{2}\lim_{t\to\infty}\left(\frac{-1}{t^2+2}+\frac{1}{2}\right)$$
$$=\frac{1}{2}\left(0+\frac{1}{2}\right)=\frac{1}{4}\ \text{. Convergent}$$

9. $\int_4^{\infty}e^{-y/2}dy=\lim_{t\to\infty}\int_4^t e^{-y/2}dy=\lim_{t\to\infty}\left[-2e^{-y/2}\right]_4^t=\lim_{t\to\infty}(-2e^{-t/2}+2e^{-2})=0+2e^{-2}=2e^{-2}$. Convergent

10. $\int_{-\infty}^{-1}e^{-2t}dt=\lim_{x\to-\infty}\int_x^{-1}e^{-2t}dt=\lim_{x\to-\infty}\left[-\frac{1}{2}e^{-2t}\right]_x^{-1}=\lim_{x\to-\infty}\left[-\frac{1}{2}e^{2}+\frac{1}{2}e^{-2x}\right]=\infty$. Divergent

11. $\int_{-\infty}^{\infty}\frac{x}{dx}1+x^2=\int_{-\infty}^{0}\frac{x}{dx}1+x^2+\int_0^{\infty}\frac{x}{dx}1+x^2$ and

$$\int_{-\infty}^{0}\frac{x}{dx}1+x^2=\lim_{t\to-\infty}\left[\frac{1}{2}\ln\left(1+x^2\right)\right]_{\text{t}}^0=\lim_{t\to-\infty}\left[0-\frac{1}{2}\ln\left(1+t^2\right)\right]=-\infty\ \text{. Divergent}$$

12. $I=\int_{-\infty}^{\infty}\left(2-v^4\right)dv=I_1+I_2=\int_{-\infty}^{0}\left(2-v^4\right)dv+\int_{0}^{\infty}\left(2-v^4\right)dv$, but

$I_1=\lim_{t\to-\infty}\left[2v-\frac{1}{5}v^5\right]_t^0=\lim_{t\to-\infty}\left(-2t+\frac{1}{5}t^5\right)=-\infty$. Since I_1 is divergent, I is divergent, and there is no need to evaluate I_2 . Divergent

13. $\int_{-\infty}^{\infty} xe^{-x^2}dx=\int_{-\infty}^{0} xe^{-x^2}dx+\int_{0}^{\infty} xe^{-x^2}dx$.

$\int_{-\infty}^{0} xe^{-x^2}dx=\lim_{t\to-\infty}\left(-\frac{1}{2}\right)\left[e^{-x^2}\right]_t^0=\lim_{t\to\infty}\left(-\frac{1}{2}\right)\left(e^{-t^2}-1\right)=-\frac{1}{2}\cdot 1=-\frac{1}{2}$, and

$\int_{0}^{\infty} xe^{-x^2}dx=\lim_{t\to\infty}\left(-\frac{1}{2}\right)\left[e^{-x^2}\right]_0^t=\lim_{t\to\infty}\left(-\frac{1}{2}\right)\left(e^{-t^2}-1\right)=-\frac{1}{2}\cdot(-1)=\frac{1}{2}$.

Therefore, $\int_{-\infty}^{\infty} xe^{-x^2}dx=-\frac{1}{2}+\frac{1}{2}=0$. Convergent

14. $\int_{-\infty}^{\infty} x^2e^{-x^3}dx=\int_{-\infty}^{0} x^2e^{-x^3}dx+\int_{0}^{\infty} x^2e^{-x^3}dx$, and

$\int_{-\infty}^{0} x^2e^{-x^3}dx=\lim_{t\to-\infty}\left[-\frac{1}{3}e^{-x^3}\right]_t^0=-\frac{1}{3}+\frac{1}{3}\left(\lim_{t\to-\infty} e^{-t^3}\right)=\infty$. Divergent

15. $\int_{2\pi}^{\infty}\sin\theta\, d\theta=\lim_{t\to\infty}\int_{2\pi}^{t}\sin\theta\, d\theta=\lim_{t\to\infty}\left[-\cos\theta\right]_{2\pi}^{t}=\lim_{t\to\infty}(-\cos t+1)$. This limit does not exist, so the integral is divergent. Divergent

16. $\int_0^{\infty}\cos^2\alpha\, d\alpha=\lim_{t\to\infty}\int_0^t\frac{1}{2}(1+\cos 2\alpha)d\alpha=\lim_{t\to\infty}\left[\frac{1}{2}\alpha+\frac{1}{4}\sin 2\alpha\right]_0^t=\lim_{t\to\infty}\left[\frac{1}{2}t+\frac{1}{4}\sin 2t\right]=\infty$ since $\left|\frac{1}{4}\sin 2t\right|\le\frac{1}{4}$ for all t , but $\frac{1}{2}t\to\infty$ as $t\to\infty$. Divergent

17.

$$\int_1^{\infty}\frac{x+1}{x^2+2x}dx=\lim_{t\to\infty}\int_1^t\frac{\frac{1}{2}(2x+2)}{x^2+2x}dx=\frac{1}{2}\lim_{t\to\infty}\left[\ln(x^2+2x)\right]_1^t=\frac{1}{2}\lim_{t\to\infty}\left[\ln(t^2+2t)-\ln 3\right]$$

$=\infty$. Divergent

18.

$$\int_0^\infty \frac{dz}{z^2+3z+2} = \lim_{t\to\infty}\int_0^t \left[\frac{1}{z+1}-\frac{1}{z+2}\right]dz = \lim_{t\to\infty}\left[\ln\left(\frac{z+1}{z+2}\right)\right]_0^t$$

$$= \lim_{t\to\infty}\left[\ln\left(\frac{t+1}{t+2}\right) - \ln\left(\frac{1}{2}\right)\right] = \ln 1 + \ln 2 = \ln 2 \text{ . Convergent}$$

19.

$$\int_0^\infty se^{-5s}\,ds = \lim_{t\to\infty}\int_0^t se^{-5s}\,ds = \lim_{t\to\infty}\left[-\frac{1}{5}se^{-5s}-\frac{1}{25}e^{-5s}\right]_0^t \left[\begin{array}{c}\text{by integration by}\\ \text{parts with } u=s\end{array}\right]$$

$$= \lim_{t\to\infty}\left(-\frac{1}{5}te^{-5t}-\frac{1}{25}e^{-5t}+\frac{1}{25}\right) = 0-0+\frac{1}{25} \text{ [by l'Hospital's Rule]}$$

$$= \frac{1}{25} \text{ . Convergent}$$

20.

$$\int_{-\infty}^6 re^{r/3}\,dr = \lim_{t\to-\infty}\int_t^6 re^{r/3}\,dr = \lim_{t\to-\infty}\left[3re^{r/3}-9e^{r/3}\right]_t^6 \left[\begin{array}{c}\text{by integration } by\\ \text{parts with } u=r\end{array}\right]$$

$$= \lim_{t\to-\infty}\left(18e^2-9e^2-3te^{t/3}+9e^{t/3}\right) = 9e^2-0+0 \text{ [by l'Hospital's Rule]}$$

$$= 9e^2 \text{ . Convergent}$$

21. $\int_1^\infty \frac{\ln x}{x}\,dx = \lim_{t\to\infty}\left[\frac{(\ln x)^2}{2}\right]_1^t$ (by substitution with $u=\ln x$, $du=dx/x$) $=\lim_{t\to\infty}\frac{(\ln t)^2}{2}=\infty$. Divergent

22. $\int_{-\infty}^\infty e^{-|x|}\,dx = \int_{-\infty}^0 e^x\,dx + \int_0^\infty e^{-x}\,dx$, $\int_{-\infty}^0 e^x\,dx = \lim_{t\to-\infty}\left[e^x\right]_t^0 = \lim_{t\to-\infty}\left(1-e^t\right)=1$, and $\int_0^\infty e^{-x}\,dx = \lim_{t\to\infty}\left[-e^{-x}\right]_0^t = \lim_{t\to\infty}\left(1-e^{-t}\right)=1$. Therefore, $\int_{-\infty}^\infty e^{-|x|}\,dx = 1+1=2$. Convergent

23.

$$\int_{-\infty}^{\infty} \frac{x^2}{9+x^6}\,dx = \int_{-\infty}^{0} \frac{x^2}{9+x^6}\,dx + \int_{0}^{\infty} \frac{x^2}{9+x^6}\,dx = 2\int_{0}^{\infty} \frac{x^2}{9+x^6}\,dx \text{ [since the integrand is even].}$$

$$\text{Now } \int \frac{x^2\,dx}{9+x^6} \left[\begin{array}{l} u=x^3 \\ du=3x^2\,dx \end{array} \right] = \int \frac{\frac{1}{3}\,du}{9+u^2} \left[\begin{array}{l} u=3v \\ du=3\,dv \end{array} \right] = \int \frac{\frac{1}{3}(3\,dv)}{9+9v^2} = \frac{1}{9}\int \frac{dv}{1+v^2}$$

$$= \frac{1}{9}\tan^{-1} v + C = \frac{1}{9}\tan^{-1}\left(\frac{u}{3}\right) + C = \frac{1}{9}\tan^{-1}\left(\frac{x^3}{3}\right) + C\ ,$$

$$\text{so } 2\int_{0}^{\infty} \frac{x^2}{9+x^6}\,dx = 2\lim_{t\to\infty} \int_{0}^{t} \frac{x^2}{9+x^6}\,dx = 2\lim_{t\to\infty} \left[\frac{1}{9}\tan^{-1}\left(\frac{x^3}{3}\right) \right]_0^t$$

$$= 2\lim_{t\to\infty} \frac{1}{9}\tan^{-1}\left(\frac{t^3}{3}\right) = \frac{2}{9}\cdot\frac{\pi}{2} = \frac{\pi}{9} \text{ . Convergent}$$

24. Integrate by parts with $u=\ln x$, $dv=dx/x^3 \Rightarrow du=dx/x$, $v=-1/\left(2x^2\right)$.

$$\int_{1}^{\infty} \frac{\ln x}{x^3}\,dx = \lim_{t\to\infty} \int_{1}^{t} \frac{\ln x}{x^3}\,dx = \lim_{t\to\infty} \left(\left[-\frac{1}{2x^2}\ln x \right]_1^t + \frac{1}{2}\int_{1}^{t} \frac{1}{x^3}\,dx \right)$$

$$= \lim_{t\to\infty} \left(-\frac{1}{2}\,\frac{\ln t}{t^2} + 0 - \frac{1}{4t^2} + \frac{1}{4} \right) = \frac{1}{4}$$

since $\lim_{t\to\infty} \frac{\ln t}{t^2} = \lim_{t\to\infty} \frac{1/t}{2t} = \lim_{t\to\infty} \frac{1}{2t^2} = 0$. Convergent

25. Integrate by parts with $u=\ln x$, $dv=dx/x^2 \Rightarrow du=dx/x$, $v=-1/x$.

$$\int_{1}^{\infty} \frac{\ln x}{x^2}\,dx = \lim_{t\to\infty} \int_{1}^{t} \frac{\ln x}{x^2}\,dx = \lim_{t\to\infty} \left[-\frac{\ln x}{x} - \frac{1}{x} \right]_1^t = \lim_{t\to\infty} \left(-\frac{\ln t}{t} - \frac{1}{t} + 0 + 1 \right)$$

$$= -0-0+0+1=1$$

since $\lim_{t\to\infty} \frac{\ln t}{t} = \lim_{t\to\infty} \frac{1/t}{1} = 0$. Convergent

26. $\int_0^\infty \frac{x\arctan x}{(1+x^2)^2}\,dx=\lim_{t\to\infty}\int_0^t \frac{x\arctan x}{(1+x^2)^2}\,dx$. Let $u=\arctan x$, $dv=\frac{x\,dx}{(1+x^2)^2}$. Then $du=\frac{dx}{1+x^2}$,

$v=\frac{1}{2}\int \frac{2x\,dx}{(1+x^2)^2}=\frac{-1/2}{1+x^2}$, and

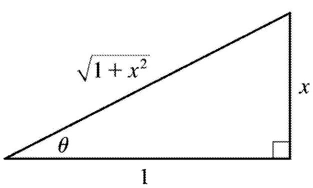
$$\int \frac{x\arctan x}{(1+x^2)^2}\,dx=-\frac{1}{2}\frac{\arctan x}{1+x^2}+\frac{1}{2}\int \frac{dx}{(1+x^2)^2}\left[\begin{array}{l}x=\tan\theta ,\\ dx=\sec^2\theta\,d\theta\end{array}\right]$$

$$=-\frac{1}{2}\frac{\arctan x}{1+x^2}+\frac{1}{2}\int \frac{\sec^2\theta\,d\theta}{(\sec^2\theta)^2}$$

$$=-\frac{1}{2}\frac{\arctan x}{1+x^2}+\frac{1}{2}\int \cos^2\theta\,d\theta$$

$$=-\frac{1}{2}\frac{\arctan x}{1+x^2}+\frac{\theta}{4}+\frac{\sin\theta\cos\theta}{4}+C$$

$$=-\frac{1}{2}\frac{\arctan x}{1+x^2}+\frac{1}{4}\arctan x+\frac{1}{4}\frac{x}{1+x^2}+C$$



It follows that

$$\int_0^\infty \frac{x\arctan x}{(1+x^2)^2}\,dx=\lim_{t\to\infty}\left[-\frac{1}{2}\frac{\arctan x}{1+x^2}+\frac{1}{4}\arctan x+\frac{1}{4}\frac{x}{1+x^2}\right]_0^t$$

$$=\lim_{t\to\infty}\left(-\frac{1}{2}\frac{\arctan t}{1+t^2}+\frac{1}{4}\arctan t+\frac{1}{4}\frac{t}{1+t^2}\right)=0+\frac{1}{4}\cdot\frac{\pi}{2}+0=\frac{\pi}{8}\ .$$

Convergent.

27. There is an infinite discontinuity at the left endpoint of $[0,3]$.

$$\int_0^3 \frac{dx}{\sqrt{x}} = \lim_{t\to 0^+} \int_t^3 \frac{dx}{\sqrt{x}} = \lim_{t\to 0^+} \left[2\sqrt{x}\,\right]_t^3 = \lim_{t\to 0^+} \left(2\sqrt{3}-2\sqrt{t}\,\right) = 2\sqrt{3}$$. Convergent

28. There is an infinite discontinuity at the left endpoint of $[0,3]$.

$$\int_0^3 \frac{dx}{x\sqrt{x}} = \lim_{t\to 0^+} \int_t^3 \frac{dx}{\sqrt{x^{3/2}}} = \lim_{t\to 0^+} \left[\frac{-2}{\sqrt{x}}\right]_t^3 = \frac{-2}{\sqrt{3}} + \lim_{t\to 0^+} \frac{2}{\sqrt{t}} = \infty$$. Divergent

29. There is an infinite discontinuity at the right endpoint of $[-1,0]$.

$$\int_{-1}^0 \frac{dx}{x^2} = \lim_{t\to 0^-} \int_{-1}^t \frac{dx}{x^2} = \lim_{t\to 0^-} \left[\frac{-1}{x}\right]_{-1}^t = \lim_{t\to 0^-} \left[-\frac{1}{t} + \frac{1}{-1}\right] = \infty$$. Divergent

30. $$\int_1^9 \frac{dx}{\sqrt[3]{x-9}} = \lim_{t\to 9^-} \int_1^t \frac{}{dx}\sqrt[3]{x-9} = \lim_{t\to 9^-} \left[\frac{3}{2}(x-9)^{2/3}\right]_1^t = \lim_{t\to 9^-} \left[\frac{3}{2}(t-9)^{2/3} - \frac{3}{2}(4)\right] = 0-6=-6$$. Convergent

31. $$\int_{-2}^3 \frac{}{dx} x^4 = \int_{-2}^0 \frac{dx}{x^4} + \int_0^3 \frac{dx}{x^4}$$, but $$\int_{-2}^0 \frac{dx}{x^4} = \lim_{t\to 0^-} \left[-\frac{x^{-3}}{3}\right]_{-2}^t = \lim_{t\to 0^-} \left[-\frac{1}{3t^3} - \frac{1}{24}\right] = \infty$$. Divergent

32. $$\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \lim_{t\to 1^-} \int_0^t \frac{dx}{\sqrt{1-x^2}} = \lim_{t\to 1^-} \left[\sin^{-1}x\right]_0^t = \lim_{t\to 1^-} \sin^{-1}t = \frac{\pi}{2}$$. Convergent

33. There is an infinite discontinuity at $x=1$. $\int_0^{33} (x-1)^{-1/5} dx = \int_0^1 (x-1)^{-1/5} dx + \int_1^{33} (x-1)^{-1/5} dx$. Here

$$\int_0^1 (x-1)^{-1/5} dx = \lim_{t\to 1^-} \int_0^t (x-1)^{-1/5} dx = \lim_{t\to 1^-} \left[\frac{5}{4}(x-1)^{4/5}\right]_0^t = \lim_{t\to 1^-} \left[\frac{5}{4}(t-1)^{4/5} - \frac{5}{4}\right] = -\frac{5}{4}$$ and

$\int_1^{33} (x-1)^{-1/5} dx = \lim_{t \to 1^+} \int_t^{33} (x-1)^{-1/5} dx = \lim_{t \to 1^+} \left[\frac{5}{4} (x-1)^{4/5} \right]_t^{33} = \lim_{t \to 1^+} \left[\frac{5}{4} \cdot 16 - \frac{5}{4} (t-1)^{4/5} \right] = 20$. Thus,

$\int_0^{33} (x-1)^{-1/5} dx = -\frac{5}{4} + 20 = \frac{75}{4}$. Convergent

34. $f(y) = 1/(4y-1)$ has an infinite discontinuity at $y = \frac{1}{4}$.

$$\int_{1/4}^{1} \frac{1}{4y-1}\, dy = \lim_{t \to (1/4)^+} \int_t^1 \frac{1}{4y-1}\, dy = \lim_{t \to (1/4)^+} \left[\frac{1}{4} \ln |4y-1| \right]_t^1$$

$$= \lim_{t \to (1/4)^+} \left[\frac{1}{4} \ln 3 - \frac{1}{4} \ln (4t-1) \right] = \infty$$

so $\int_{1/4}^{1} \frac{1}{4y-1}\, dy$ diverges, and hence, $\int_0^1 \frac{1}{4y-1}\, dy$ diverges. Divergent

35.

$$\int_0^{\pi} \sec x\, dx = \int_0^{\pi/2} \sec x\, dx + \int_{\pi/2}^{\pi} \sec x\, dx \text{ . } \int_0^{\pi/2} \sec x\, dx = \lim_{t \to \pi/2^-} \int_0^t \sec x\, dx$$

$$= \lim_{t \to \pi/2^-} [\ln |\sec x + \tan x|]_0^t = \lim_{t \to \pi/2^-} \ln |\sec t + \tan t| = \infty$$. Divergent

36. $\int_0^4 \frac{dx}{x^2+x-6} = \int_0^4 \frac{dx}{(x+3)(x-2)} = \int_0^2 \frac{dx}{(x-2)(x+3)} + \int_2^4 \frac{dx}{(x-2)(x+3)}$, and

$$\int_0^2 \frac{dx}{(x-2)(x+3)} = \lim_{t \to 2^-} \int_0^t \left[\frac{1/5}{x-2} - \frac{1/5}{x+3} \right] dx \text{ [partial fractions} = \lim_{t \to 2^-} \left[\frac{1}{5} \ln \left| \frac{x-2}{x+3} \right| \right]_0^t$$

$$= \lim_{t \to 2^-} \frac{1}{5} \left[\ln \left| \frac{t-2}{t+3} \right| - \ln \frac{2}{3} \right] = -\infty$$ .Divergent

37. There is an infinite discontinuity at $x=0$. $\int_{-1}^{1} \frac{e^x}{e^x-1}\, dx = \int_{-1}^{0} \frac{e^x}{e^x-1}\, dx + \int_0^1 \frac{e^x}{e^x-1}\, dx$.

$$\int_{-1}^{0}\frac{e^x}{e^x-1}\,dx=\lim_{t\to 0^-}\int_{-1}^{t}\frac{e^x}{e^x-1}\,dx=\lim_{t\to 0^-}\Big[\ln\left|e^x-1\right|\Big]_{-1}^{t}=\lim_{t\to 0^-}\Big[\ln\left|e^t-1\right|-\ln\left|e^{-1}-1\right|\Big]=-\infty\ ,$$

so $\int_{-1}^{1}\frac{e^x}{e^x-1}\,dx$ is divergent. The integral $\int_{0}^{1}\frac{e^x}{e^x-1}\,dx$ also diverges since

$$\int_{0}^{1}\frac{e^x}{e^x-1}\,dx=\lim_{t\to 0^+}\int_{t}^{1}\frac{e^x}{e^x-1}\,dx=\lim_{t\to 0^+}\Big[\ln\left|e^x-1\right|\Big]_{t}^{1}=\lim_{t\to 0^+}\Big[\ln|e-1|-\ln\left|e^t-1\right|\Big]=\infty\ .$$

Divergent

38. $\int_{0}^{2}\frac{x-3}{2x-3}\,dx=\int_{0}^{3/2}\frac{x-3}{2x-3}\,dx+\int_{3/2}^{2}\frac{x-3}{2x-3}\,dx$ and

$\int\frac{x-3}{2x-3}\,dx=\frac{1}{2}\int\frac{2x-6}{2x-3}\,dx=\frac{1}{2}\int\left[1-\frac{3}{2x-3}\right]dx=\frac{1}{2}\,x-\frac{3}{4}\ln|2x-3|+C$, so

$\int_{0}^{3/2}\frac{x-3}{2x-3}\,dx=\lim_{t\to 3/2^-}\frac{1}{4}\left[2x-3\ln|2x-3|\right]_{0}^{t}=\infty$. Divergent

39.

$$I=\int_{0}^{2}z^2\ln z\,dz=\lim_{t\to 0^+}\int_{t}^{2}z^2\ln z\,dz=\lim_{t\to 0^+}\left[\frac{z^3}{3^2}(3\ln z-1)\right]_{t}^{2}$$

$$=\lim_{t\to 0^+}\left[\frac{8}{9}(3\ln 2-1)-\frac{1}{9}t^3(3\ln t-1)\right]=\frac{8}{3}\ln 2-\frac{8}{9}-\frac{1}{9}\lim_{t\to 0^+}\left[t^3(3\ln t-1)\right]=\frac{8}{3}\ln 2-\frac{8}{9}-\frac{1}{9}L\ .$$

Now $L=\lim_{t\to 0^+}\left[t^3(3\ln t-1)\right]=\lim_{t\to 0^+}\frac{3\ln t-1}{t^{-3}}=\lim_{t\to 0^+}\frac{3/t}{-3/t^4}=\lim_{t\to 0^+}\left(-t^3\right)=0$. Thus, $L=0$ and $I=\frac{8}{3}\ln 2-\frac{8}{9}$.

Convergent

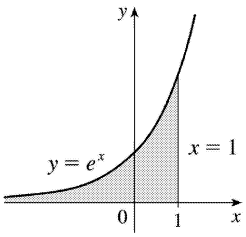
40. Integrate by parts with $u=\ln x$, $dv=dx\,/\sqrt{x}\ \Rightarrow du=dx/x$, $v=2\sqrt{x}$.

$$\int_{0}^{1}\frac{\ln x}{\sqrt{x}}\,dx=\lim_{t\to 0^+}\int_{t}^{1}\frac{\ln x}{\sqrt{x}}\,dx=\lim_{t\to 0^+}\left(\left[2\sqrt{x}\ln x\right]_{t}^{1}-2\int_{t}^{1}\frac{dx}{\sqrt{x}}\right)=\lim_{t\to 0^+}\left(-2\sqrt{t}\ln t-4\left[\sqrt{x}\,\right]_{t}^{1}\right)$$

$$=\lim_{t\to 0^+}\left(-2\sqrt{t}\ln t-4+4\sqrt{t}\,\right)=-4$$

since $\lim_{t\to 0^+} \sqrt{t}\ln t = \lim_{t\to 0^+} \frac{\ln t}{t^{-1/2}} = \lim_{t\to 0^+} \frac{1/t}{-t^{-3/2}/2} = \lim_{t\to 0^+} \left(-2\sqrt{t}\right) = 0$. Convergent

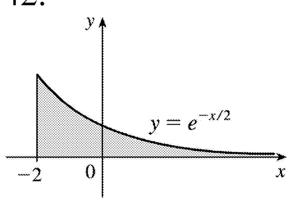
41.



$$\text{Area} = \int_{-\infty}^{1} e^{x}\,dx = \lim_{t\to -\infty} \left[e^{x}\right]_t^1$$

$$= e - \lim_{t\to -\infty} e^{t} = e$$

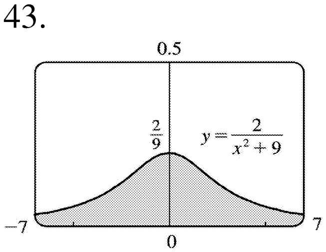
42.



$$\text{Area} = \int_{-2}^{\infty} e^{-x/2}\,dx = -2\lim_{t\to \infty} \left[e^{-x/2}\right]_{-2}^{t}$$

$$= -2\lim_{t\to \infty} e^{-t/2} + 2e = 2e$$

43.

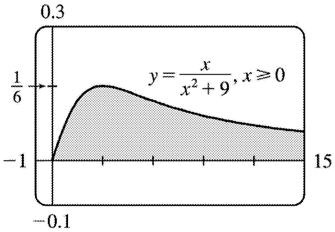


$$\text{Area} = \int_{-\infty}^{\infty} \frac{2}{x^2+9}\,dx = 2\cdot 2\int_{0}^{\infty} \frac{1}{x^2+9}\,dx$$

$$=4\lim_{t\to\infty}\int_0^t \frac{1}{x^2+9}\,dx=4\lim_{t\to\infty}\left[\frac{1}{3}\tan^{-1}\frac{x}{3}\right]_0^t$$

$$=\frac{4}{3}\lim_{t\to\infty}\left[\tan^{-1}\frac{t}{3}-0\right]=\frac{4}{3}\cdot\frac{\pi}{2}=\frac{2\pi}{3}$$

44.



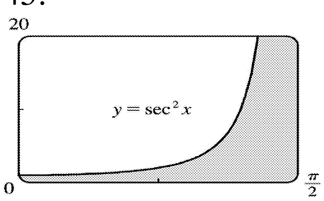
$$\text{Area }=\int_0^\infty \frac{x}{x^2+9}\,dx=\lim_{t\to\infty}\int_0^t \frac{x}{x^2+9}\,dx$$

$$=\lim_{t\to\infty}\left[\frac{1}{2}\ln\left(x^2+9\right)\right]_0^t$$

$$=\frac{1}{2}\lim_{t\to\infty}\left[\ln\left(t^2+9\right)-\ln 9\right]=\infty$$

Infinite area

45.



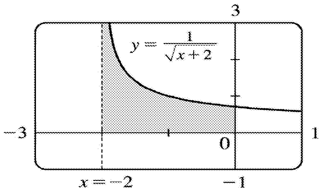
$$\text{Area }=\int_0^{\pi/2}\sec^2 x\,dx=\lim_{t\to(\pi/2)^-}\int_0^t\sec^2 x\,dx$$

$$=\lim_{t\to(\pi/2)^-}\left[\tan x\right]_0^t=\lim_{t\to(\pi/2)^-}(\tan t-0)$$

$$=\infty$$

Infinite area

46.



$$\text{Area} = \int_{-2}^{0} \frac{1}{\sqrt{x+2}}\,dx = \lim_{t\to -2^+} \int_{t}^{0} \frac{1}{\sqrt{x+2}}\,dx$$

$$= \lim_{t\to -2^+} \left[2\sqrt{x+2}\,\right]_t^0 = \lim_{t\to -2^+} \left(2\sqrt{2} - 2\sqrt{t+2}\,\right)$$

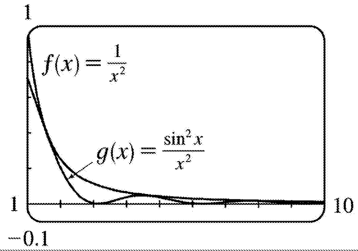
$$= 2\sqrt{2} - 0 = 2\sqrt{2}$$

47. **(a)**

t	$\int_1^t g(x)\,dx$
2	0.447453
5	0.577101
10	0.621306
100	0.668479
1000	0.672957
10 , 000	0.673407

$g(x) = \dfrac{\sin^2 x}{x^2}$. It appears that the integral is convergent.

(b) $-1 \le \sin x \le 1 \Rightarrow 0 \le \sin^2 x \le 1 \Rightarrow 0 \le \dfrac{\sin^2 x}{x^2} \le \dfrac{1}{x^2}$. Since $\int_1^\infty \dfrac{1}{x^2}\,dx$ is convergent (Equation 2 with $p = 2 > 1$), $\int_1^\infty \dfrac{\sin^2 x}{x^2}\,dx$ is convergent by the Comparison Theorem.



(c)

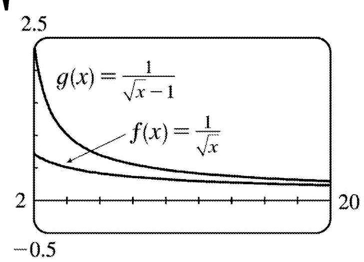
Since $\int_1^\infty f(x)\,dx$ is finite and the area under $g(x)$ is less than the area under $f(x)$ on any interval $[1,t]$, $\int_1^\infty g(x)\,dx$ must be finite; that is, the integral is convergent.

48. **(a)**

t	$\int_2^t g(x)dx$
5	3.830327
10	6.801200
100	23.328769
1000	69.023361
10 , 000	208.124560

$g(x)=\dfrac{1}{\sqrt{x}-1}$. It appears that the integral is divergent.

(b) For $x\geq 2$, $\sqrt{x}>\sqrt{x}-1\Rightarrow \dfrac{1}{\sqrt{x}}<\dfrac{1}{\sqrt{x}-1}$. Since $\int_2^\infty \dfrac{1}{\sqrt{x}}\,dx$ is divergent (Equation 2 with $p=\dfrac{1}{2}\leq 1$), $\int_2^\infty \dfrac{1}{\sqrt{x}-1}\,dx$ is divergent by the Comparison Theorem.



(c)

Since $\int_2^\infty f(x)\,dx$ is infinite and the area under $g(x)$ is greater than the area under $f(x)$ on any interval $[2,t]$, $\int_2^\infty g(x)\,dx$ must be infinite; that is, the integral is divergent.

49. For $x\geq 1$, $\dfrac{\cos^2 x}{1+x^2}\leq\dfrac{1}{1+x^2}<\dfrac{1}{x^2}$. $\int_1^\infty \dfrac{1}{x^2}\,dx$ is convergent by Equation 2 with $p=2>1$, so

$\int_1^\infty \frac{\cos^2 x}{1+x^2}\,dx$ is convergent by the Comparison Theorem.

50. For $x \geq 1$, $\frac{2+e^{-x}}{x} > \frac{2}{x}$ [since $e^{-x} > 0$] $> \frac{1}{x}$. $\int_1^\infty \frac{1}{x}\,dx$ is divergent by Equation 2 with $p=1 \leq 1$, so $\int_1^\infty \frac{2+e^{-x}}{x}\,dx$ is divergent by the Comparison Theorem.

51. For $x \geq 1$, $x+e^{2x} > e^{2x} > 0 \Rightarrow \frac{1}{x+e^{2x}} \leq \frac{1}{e^{2x}} = e^{-2x}$ on $[1,\infty)$.

$\int_1^\infty e^{-2x}\,dx = \lim_{t\to\infty}\left[-\frac{1}{2}e^{-2x}\right]_1^t = \lim_{t\to\infty}\left[-\frac{1}{2}e^{-2t}+\frac{1}{2}e^{-2}\right] = \frac{1}{2}e^{-2}$. Therefore, $\int_1^\infty e^{-2x}\,dx$ is convergent, and by the Comparison Theorem, $\int_1^\infty \frac{}{dx}\, x+e^{2x}$ is also convergent.

52. For $x \geq 1$, $0 < \frac{x}{\sqrt{1+x^6}} < \frac{x}{\sqrt{x^6}} = \frac{x}{x^3} = \frac{1}{x^2}$. $\int_1^\infty \frac{1}{x^2}\,dx$ is convergent by Equation 2 with $p=2>1$, so

$$\int_1^\infty \frac{x}{\sqrt{1+x^6}}\,dx$$ is convergent by the Comparison Theorem.

53. $\frac{1}{x\sin x} \geq \frac{1}{x}$ on $\left(0, \frac{\pi}{2}\right]$ since $0 \leq \sin x \leq 1$. $\int_0^{\pi/2} \frac{dx}{x} = \lim_{t\to 0^+}\int_t^{\pi/2}\frac{dx}{x} = \lim_{t\to 0^+}[\ln x]_t^{\pi/2}$.

But $\ln t \to -\infty$ as $t \to 0^+$, so $\int_0^{\pi/2}\frac{dx}{x}$ is divergent, and by the Comparison Theorem, $\int_0^{\pi/2}\frac{dx}{x\sin x}$ is also divergent.

54. For $0 \leq x \leq 1$, $e^{-x} \leq 1 \Rightarrow \frac{e^{-x}}{\sqrt{x}} \leq \frac{1}{\sqrt{x}}$.

$\int_0^1 \frac{1}{\sqrt{x}}\,dx = \lim_{t\to 0^+}\int_t^1 \frac{1}{\sqrt{x}}\,dx = \lim_{t\to 0^+}\left[2\sqrt{x}\,\right]_t^1 = \lim_{t\to 0^+}\left(2-2\sqrt{t}\,\right) = 2$ is convergent. Therefore, $\int_0^1 \frac{e^{-x}}{\sqrt{x}}\,dx$ is convergent by the Comparison Theorem.

55. $\int_0^\infty \frac{dx}{\sqrt{x}\,(1+x)} = \int_0^1 \frac{dx}{\sqrt{x}\,(1+x)} + \int_1^\infty \frac{dx}{\sqrt{x}\,(1+x)} = \lim_{t\to 0^+}\int_t^1 \frac{dx}{\sqrt{x}\,(1+x)} + \lim_{t\to\infty}\int_1^t \frac{dx}{\sqrt{x}\,(1+x)}$. Now

$$\int \frac{dx}{\sqrt{x}\,(1+x)} = \int \frac{2u\,du}{u\left(1+u^2\right)} \quad [\,u=\sqrt{x}, x=u^2, dx=2u\,du\,]$$

$$= 2\int \frac{du}{1+u^2} = 2\tan^{-1}u + C = 2\tan^{-1}\sqrt{x} + C\ ,$$

so

$$\int_0^\infty \frac{dx}{\sqrt{x}\,(1+x)} = \lim_{t\to 0^+}\left[2\tan^{-1}\sqrt{x}\,\right]_t^1 + \lim_{t\to\infty}\left[2\tan^{-1}\sqrt{x}\,\right]_1^t$$

$$= \lim_{t\to 0^+}\left[2\left(\frac{\pi}{4}\right) - 2\tan^{-1}\sqrt{t}\,\right] + \lim_{t\to\infty}\left[2\tan^{-1}\sqrt{t} - 2\left(\frac{\pi}{4}\right)\right] = \frac{\pi}{2} - 0 + 2\left(\frac{\pi}{2}\right) - \frac{\pi}{2} = \pi$$

.

56. $\int_2^\infty \frac{dx}{x\sqrt{x^2-4}} = \int_2^3 \frac{dx}{x\sqrt{x^2-4}} + \int_3^\infty \frac{dx}{x\sqrt{x^2-4}} = \lim_{t\to 2^+}\int_t^3 \frac{dx}{x\sqrt{x^2-4}} + \lim_{t\to\infty}\int_3^t \frac{dx}{x\sqrt{x^2-4}}$. Now

$$\int \frac{dx}{x\sqrt{x^2-4}} = \int \frac{2\sec\theta\,\tan\theta\,d\theta}{2\sec\theta\,2\tan\theta} \quad [\,x=2\sec\theta\ ,\text{ where } 0\le\theta<\pi/2 \text{ or } \pi\le\theta<3\pi/2\,]$$

$$= \frac{1}{2}\theta + C = \frac{1}{2}\sec^{-1}\left(\frac{1}{2}x\right) + C\ ,\text{ so}$$

$$\int_2^\infty \frac{dx}{x\sqrt{x^2-4}} = \lim_{t\to 2^+}\left[\frac{1}{2}\sec^{-1}\left(\frac{1}{2}x\right)\right]_t^3 + \lim_{t\to\infty}\left[\frac{1}{2}\sec^{-1}\left(\frac{1}{2}x\right)\right]_3^t$$

$$=\frac{1}{2}\sec^{-1}\left(\frac{3}{2}\right)-0+\frac{1}{2}\left(\frac{\pi}{2}\right)-\frac{1}{2}\sec^{-1}\left(\frac{3}{2}\right)=\frac{\pi}{4}$$

57. If $p=1$, then $\int_0^1 \frac{dx}{x^p}=\lim_{t\to 0^+}\int_t^1 \frac{dx}{x}=\lim_{t\to 0^+}[\ln x]_t^1=\infty$. Divergent.

If $p\neq 1$, then

$\int_0^1 \frac{dx}{x^p}=\lim_{t\to 0^+}\int_t^1 \frac{dx}{x^p}$ (note that the integral is not improper if $p<0$)

$$=\lim_{t\to 0^+}\left[\frac{x^{-p+1}}{-p+1}\right]_t^1=\lim_{t\to 0^+}\frac{1}{1-p}\left[1-\frac{1}{t^{p-1}}\right]$$

If $p>1$, then $p-1>0$, so $\frac{1}{t^{p-1}}\to\infty$ as $t\to 0^+$, and the integral diverges.

If $p<1$, then $p-1<0$, so $\frac{1}{t^{p-1}}\to 0$ as $t\to 0^+$ and $\int_0^1 \frac{dx}{x^p}=\frac{1}{1-p}\left[\lim_{t\to 0^+}\left(1-t^{1-p}\right)\right]=\frac{1}{1-p}$.

Thus, the integral converges if and only if $p<1$, and in that case its value is $\frac{1}{1-p}$.

58. Let $u=\ln x$. Then $du=dx/x\Rightarrow\int_e^\infty \frac{dx}{x(\ln x)^p}=\int_1^\infty \frac{du}{u^p}$. By Example 4, this converges to $\frac{1}{p-1}$ if $p>1$ and diverges otherwise.

59. First suppose $p=-1$. Then

$\int_0^1 x^p\ln x\,dx=\int_0^1 \frac{\ln x}{x}\,dx=\lim_{t\to 0^+}\int_t^1 \frac{\ln x}{x}\,dx=\lim_{t\to 0^+}\left[\frac{1}{2}(\ln x)^2\right]_t^1=-\frac{1}{2}\lim_{t\to 0^+}(\ln t)^2=-\infty$, so the integral diverges.

Now suppose $p\neq -1$. Then integration by parts gives

$\int x^p\ln x\,dx=\frac{x^{p+1}}{p+1}\ln x-\int\frac{x^p}{p+1}\,dx=\frac{x^{p+1}}{p+1}\ln x-\frac{x^{p+1}}{(p+1)^2}+C$. If $p<-1$, then $p+1<0$, so

$$\int_0^1 x^p\ln x\,dx=\lim_{t\to 0^+}\left[\frac{x^{p+1}}{p+1}\ln x-\frac{x^{p+1}}{(p+1)^2}\right]_t^1=\frac{-1}{(p+1)^2}-\left(\frac{1}{p+1}\right)\lim_{t\to 0^+}\left[t^{p+1}\left(\ln t-\frac{1}{p+1}\right)\right]=\infty\ .$$

If $p>-1$, then $p+1>0$ and

$$\int_0^1 x^p \ln x\,dx = \frac{-1}{(p+1)^2} - \left(\frac{1}{p+1}\right)\lim_{t\to 0^+}\frac{\ln t - 1/(p+1)}{t^{-(p+1)}} = \frac{-1}{(p+1)^2} - \left(\frac{1}{p+1}\right)\lim_{t\to 0^+}\frac{1/t}{-(p+1)t^{-(p+2)}}$$

$$= \frac{-1}{(p+1)^2} + \frac{1}{(p+1)^2}\lim_{t\to 0^+} t^{p+1} = \frac{-1}{(p+1)^2}$$

Thus, the integral converges to $-\dfrac{1}{(p+1)^2}$ if $p>-1$ and diverges otherwise.

60. **(a)**

$n=0$:

$$\int_0^\infty x^n e^{-x}dx = \lim_{t\to\infty}\int_0^t e^{-x}dx = \lim_{t\to\infty}\left[-e^{-x}\right]_0^t$$

$$= \lim_{t\to\infty}\left[-e^{-t}+1\right] = 0+1 = 1$$

$n=1$:

$\int_0^\infty x^n e^{-x}dx = \lim_{t\to\infty}\int_0^t xe^{-x}dx$. To evaluate $\int xe^{-x}dx$, we'll use integration by parts

with $u=x$, $dv=e^{-x}dx \Rightarrow du=dx$, $v=-e^{-x}$.

So $\int xe^{-x}dx = -xe^{-x} - \int -e^{-x}dx = -xe^{-x} - e^{-x} + C = (-x-1)e^{-x} + C$ and

$$\lim_{t\to\infty}\int_0^t xe^{-x}dx = \lim_{t\to\infty}\left[(-x-1)e^{-x}\right]_0^t$$

$$= \lim_{t\to\infty}\left[(-t-1)e^{-t}+1\right] = \lim_{t\to\infty}\left[-te^{-t}-e^{-t}+1\right]$$

$= 0-0+1$ [use l'Hospital's Rule] $=1$

$n=2$:

$\int_0^\infty x^n e^{-x}dx = \lim_{t\to\infty}\int_0^t x^2 e^{-x}dx$. To evaluate $\int x^2 e^{-x}dx$, we could use integration by parts

again or Formula 97. Thus,

$$\lim_{t\to\infty}\int_0^t x^2 e^{-x}dx = \lim_{t\to\infty}\left[-x^2 e^{-x}\right]_0^t + 2\lim_{t\to\infty}\int_0^t xe^{-x}dx$$

$= 0+0+2(1)$ [use l'Hospital's Rule and the result for $n=1$] $=2$

$n=3$:

$$\int_0^\infty x^n e^{-x}dx = \lim_{t\to\infty}\int_0^t x^3e^{-x}dx = \lim_{t\to\infty}\left[-x^3e^{-x}\right]_0^t + 3\lim_{t\to\infty}\int_0^t x^2e^{-x}dx$$

$=0+0+3(2)$ [use l'Hospital's Rule and the result for $n=2$] $=6$

(b) For $n=1$, 2 , and 3 , we have $\int_0^\infty x^n e^{-x}dx=1$, 2 , and 6 . The values for the integral are equal to the factorials for n , so we guess $\int_0^\infty x^n e^{-x}dx=\mathrm{n}!$.

(c) Suppose that $\int_0^\infty x^k e^{-x}dx=\mathrm{k}!$ for some positive integer k . Then $\int_0^\infty x^{k+1}e^{-x}dx=\lim_{t\to\infty}\int_0^t x^{k+1}e^{-x}dx$. To evaluate $\int x^{k+1}e^{-x}dx$, we use parts with $u=x^{k+1}$, $dv=e^{-x}dx \Rightarrow du=(k+1)x^k dx$, $v=-e^{-x}$. So $\int x^{k+1}e^{-x}dx=-x^{k+1}e^{-x}-\int -(k+1)x^k e^{-x}dx=-x^{k+1}e^{-x}+(k+1)\int x^k e^{-x}dx$ and

$$\lim_{t\to\infty}\int_0^t x^{k+1}e^{-x}dx = \lim_{t\to\infty}\left[-x^{k+1}e^{-x}\right]_0^t + (k+1)\lim_{t\to\infty}\int_0^t x^k e^{-x}dx$$

$$=\lim_{t\to\infty}\left[-t^{k+1}e^{-t}+0\right]+(k+1)\mathrm{k}!=0+0+(k+1)!=(k+1)!\ ,$$

so the formula holds for $k+1$. By induction, the formula holds for all positive integers. (Since $0!=1$, the formula holds for $n=0$, too.)

61. **(a)** $I=\int_{-\infty}^\infty x\,dx=\int_{-\infty}^0 x\,dx+\int_0^\infty x\,dx$, and $\int_0^\infty x\,dx=\lim_{t\to\infty}\int_0^t x\,dx=\lim_{t\to\infty}\left[\frac{1}{2}x^2\right]_0^t=\lim_{t\to\infty}\left[\frac{1}{2}t^2-0\right]=\infty$, so I is divergent.

(b) $\int_{-t}^t x\,dx=\left[\frac{1}{2}x^2\right]_{-t}^t=\frac{1}{2}t^2-\frac{1}{2}t^2=0$, so $\lim_{t\to\infty}\int_{-t}^t x\,dx=0$. Therefore, $\int_{-\infty}^\infty x\,dx\neq\lim_{t\to\infty}\int_{-t}^t x\,dx$.

62. Let $k=\dfrac{M}{2RT}$ so that $\bar{\mathrm{v}}=\dfrac{4}{\sqrt{\pi}}k^{3/2}\int_0^\infty v^3e^{-kv^2}dv$. Let I denote the integral and use parts to integrate I . Let $\alpha=v^2$, $d\beta=ve^{-kv^2}dv \Rightarrow d\alpha=2v\,dv$, $\beta=-\dfrac{1}{2k}e^{-kv^2}$:

$$I = \lim_{t\to\infty}\left[-\frac{1}{2k}v^2e^{-kv^2}\right]_0^t + \frac{1}{k}\int_0^\infty ve^{-kv^2}dv = -\frac{1}{2k}\lim_{t\to\infty}\left(t^2e^{-kt^2}\right) + \frac{1}{k}\lim_{t\to\infty}\left[-\frac{1}{2k}e^{-kv^2}\right]_0^t$$
$$= -\frac{1}{2k}\cdot 0 - \frac{1}{2k^2}(0-1) = \frac{1}{2k^2}$$

Thus, $\bar{v} = \frac{4}{\sqrt{\pi}}k^{3/2}\cdot\frac{1}{2k^2} = \frac{2}{(k\pi)^{1/2}} = \frac{2}{[\pi M/(2RT)]^{1/2}} = \frac{2\sqrt{2}\sqrt{RT}}{\sqrt{\pi M}} = \sqrt{\frac{8RT}{\pi M}}$.

63. Volume $= \int_1^\infty \pi\left(\frac{1}{x}\right)^2 dx = \pi\lim_{t\to\infty}\int_1^t \frac{}{dx}\,x^2 = \pi\lim_{t\to\infty}\left[-\frac{1}{x}\right]_1^t = \pi\lim_{t\to\infty}\left(1-\frac{1}{t}\right) = \pi < \infty$.

64. Work $= \int_R^\infty \frac{GMm}{r^2}dr = \lim_{t\to\infty}\int_R^t \frac{GMm}{r^2}dr = \lim_{t\to\infty} GMm\left[\frac{-1}{r}\right]_R^t = GMm_{t\to\infty}\left(\frac{-1}{t}+\frac{1}{R}\right) = \frac{GMm}{R}$, where M= mass of Earth $=5.98\times 10^{24}$ kg, m= mass of satellite $=10^3$ kg, R= radius of Earth $=6.37\times 10^6$ m, and G= gravitational constant $=6.67\times 10^{-11}$ N · m ^{2}/ kg.

Therefore, Work $= \frac{6.67\times 10^{-11}\cdot 5.98\times 10^{24}\cdot 10^3}{6.37\times 10^6} \approx 6.26\times 10^{10}$ J.

65. Work $= \int_R^\infty F\,dr = \lim_{t\to\infty}\int_R^t \frac{GmM}{r^2}dr = \lim_{t\to\infty} GmM\left(\frac{1}{R}-\frac{1}{t}\right) = \frac{GmM}{R}$. The initial kinetic energy provides the work, so $\frac{1}{2}mv_0^2 = \frac{GmM}{R} \Rightarrow v_0 = \sqrt{\frac{2GM}{R}}$.

66. $y(s) = \int_s^R \frac{2r}{\sqrt{r^2-s^2}}x(r)\,dr$ and $x(r) = \frac{1}{2}(R-r)^2 \Rightarrow$

$$y(s) = \lim_{t\to s^+}\int_t^R \frac{r(R-r)^2}{\sqrt{r^2-s^2}}\,dr = \lim_{t\to s^+}\int_t^R \frac{r^3-2Rr^2+R^2 r}{\sqrt{r^2-s^2}}\,dr$$

$$= \lim_{t\to s^+}\left[\int_t^R \frac{r^3}{dr}\sqrt{r^2-s^2} - 2R\int_t^R \frac{r^2}{dr}\sqrt{r^2-s^2} + R^2\int_t^R \frac{r}{dr}\sqrt{r^2-s^2}\right]$$

$$= \lim_{t\to s^+}\left(I_1 - 2RI_2 + R^2 I_3\right) = L$$

For I_1 : Let $u=\sqrt{r^2-s^2} \Rightarrow u^2=r^2-s^2$, $r^2=u^2+s^2$, $2r\,dr=2u\,du$, so, omitting limits and constant of integration,

$$I_1 = \int \frac{\left(u^2+s^2\right)u}{u}\,du = \int\left(u^2+s^2\right)du = \frac{1}{3}u^3+s^2u = \frac{1}{3}u\left(u^2+3s^2\right)$$

$$= \frac{1}{3}\sqrt{r^2-s^2}\left(r^2-s^2+3s^2\right) = \frac{1}{3}\sqrt{r^2-s^2}\left(r^2+2s^2\right)$$

For I_2 : Using Formula 44, $I_2=\frac{r}{2}\sqrt{r^2-s^2}+\frac{s^2}{2}\ln\left|r+\sqrt{r^2-s^2}\right|$.

For I_3 : Let $u=r^2-s^2 \Rightarrow du=2r\,dr$. Then $I_3=\frac{1}{2}\int\frac{du}{\sqrt{u}}=\frac{1}{2}\cdot 2\sqrt{u}=\sqrt{r^2-s^2}$.

Thus,

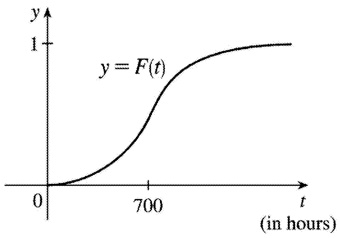
$$L = \lim_{t\to s^+}\left[\frac{1}{3}\sqrt{r^2-s^2}\left(r^2+2s^2\right)-2R\left(\frac{r}{2}\sqrt{r^2-s^2}+\frac{s^2}{2}\ln\left|r+\sqrt{r^2-s^2}\right|\right)+R^2\sqrt{r^2-s^2}\right]_t^R$$

$$= \lim_{t\to s^+}\left[\frac{1}{3}\sqrt{R^2-s^2}\left(R^2+2s^2\right)-2R\left(\frac{R}{2}\sqrt{R^2-s^2}+\frac{s^2}{2}\ln\left|R+\sqrt{R^2-s^2}\right|\right)+R^2\sqrt{R^2-s^2}\right]$$

$$-\lim_{t\to s^+}\left[\frac{1}{3}\sqrt{t^2-s^2}\left(t^2+2s^2\right)-2R\left(\frac{t}{2}\sqrt{t^2-s^2}+\frac{s^2}{2}\ln\left|t+\sqrt{t^2-s^2}\right|\right)+R^2\sqrt{t^2-s^2}\right]$$

$$= \left[\frac{1}{3}\sqrt{R^2-s^2}\left(R^2+2s^2\right)-Rs^2\ln\left|R+\sqrt{R^2-s^2}\right|\right]-\left[-Rs^2\ln|s|\right]$$

$$= \frac{1}{3}\sqrt{R^2-s^2}\left(R^2+2s^2\right)-Rs^2\ln\left(\frac{R+\sqrt{R^2-s^2}}{s}\right)$$

67. **(a)** 

(b) $r(t)=F'(t)$ is the rate at which the fraction $F(t)$ of burnt–out bulbs increases as t increases. This could be interpreted as a fractional burnout rate.

(c) $\int_0^\infty r(t)\,dt=\lim_{x\to\infty} F(x)=1$, since all of the bulbs will eventually burn out.

68.

$$I=\int_0^\infty te^{kt}\,dt=\lim_{s\to\infty}\left[\frac{1}{k^2}(kt-1)e^{kt}\right]_0^s \quad \text{[Formula 96, or parts]}$$

$$=\lim_{s\to\infty}\left[\left(\frac{1}{k}se^{ks}-\frac{1}{k^2}e^{ks}\right)-\left(-\frac{1}{k^2}\right)\right].$$

Since $k<0$ the first two terms approach 0 (you can verify that the first term does so with l'Hospital's Rule), so the limit is equal to $1/k^2$. Thus, $M=-kI=-k\left(1/k^2\right)=-1/k=-1/(-0.000121)\approx 8264.5$ years.

69. $I=\int_a^\infty \frac{1}{x^2+1}\,dx=\lim_{t\to\infty}\int_a^t \frac{1}{x^2+1}\,dx=\lim_{t\to\infty}\left[\tan^{-1}x\right]_a^t=\lim_{t\to\infty}\left(\tan^{-1}t-\tan^{-1}a\right)=\frac{\pi}{2}-\tan^{-1}a$. $I<0.001\Rightarrow$ $\frac{\pi}{2}-\tan^{-1}a<0.001\Rightarrow \tan^{-1}a>\frac{\pi}{2}-0.001\Rightarrow a>\tan\left(\frac{\pi}{2}-0.001\right)\approx 1000$.

70. $f(x)=e^{-x^2}$ and $\Delta x=\frac{4-0}{8}=\frac{1}{2}$.

$$\int_0^4 f(x)\,dx \approx S_8 = \frac{1}{2\cdot 3}[f(0)+4f(0.5)+2f(1)+\cdots+2f(3)+4f(3.5)+f(4)]$$

$$\approx \frac{1}{6}(5.31717808)\approx 0.8862$$

Now $x>4\Rightarrow -x\cdot x<-x\cdot 4\Rightarrow e^{-x^2}<e^{-4x}\Rightarrow \int_4^\infty e^{-x^2}\,dx<\int_4^\infty e^{-4x}\,dx$.

$\int_4^\infty e^{-4x}dx = \lim_{t\to\infty}\left[-\frac{1}{4}e^{-4x}\right]_4^t = -\frac{1}{4}\left(0-e^{-16}\right) = 1/\left(4e^{16}\right) \approx 0.0000000281 < 0.0000001$, as desired.

71. **(a)** $F(s) = \int_0^\infty f(t)e^{-st}dt = \int_0^\infty e^{-st}dt = \lim_{n\to\infty}\left[-\frac{e^{-st}}{s}\right]_0^n = \lim_{n\to\infty}\left(\frac{e^{-sn}}{-s}+\frac{1}{s}\right)$. This converges to $\frac{1}{s}$ only if $s>0$. Therefore $F(s)=\frac{1}{s}$ with domain $\{s \mid s>0\}$.

(b)

$$F(s) = \int_0^\infty f(t)e^{-st}dt = \int_0^\infty e^t e^{-st}dt = \lim_{n\to\infty}\int_0^n e^{t(1-s)}dt = \lim_{n\to\infty}\left[\frac{1}{1-s}e^{t(1-s)}\right]_0^n$$
$$= \lim_{n\to\infty}\left(\frac{e^{(1-s)n}}{1-s}-\frac{1}{1-s}\right)$$

This converges only if $1-s<0 \Rightarrow s>1$, in which case $F(s)=\frac{1}{s-1}$ with domain $\{s \mid s>1\}$.

(c) $F(s) = \int_0^\infty f(t)e^{-st}dt = \lim_{n\to\infty}\int_0^n te^{-st}dt$. Use integration by parts: let $u=t$, $dv=e^{-st}dt \Rightarrow du=dt$, $v=-\frac{e^{-st}}{s}$. Then $F(s)=\lim_{n\to\infty}\left[-\frac{t}{s}e^{-st}-\frac{1}{s^2}e^{-st}\right]_0^n = \lim_{n\to\infty}\left(\frac{-n}{se^{sn}}-\frac{1}{s^2e^{sn}}+0+\frac{1}{s^2}\right)=\frac{1}{s^2}$ only if $s>0$. Therefore, $F(s)=\frac{1}{s^2}$ and the domain of F is $\{s \mid s>0\}$.

72. $0 \le f(t) \le Me^{at} \Rightarrow 0 \le f(t)e^{-st} \le Me^{at}e^{-st}$ for $t \ge 0$. Now use the Comparison Theorem:

$$\int_0^\infty Me^{at}e^{-st}dt = \lim_{n\to\infty} M\int_0^n e^{t(a-s)}dt = M\cdot\lim_{n\to\infty}\left[\frac{1}{a-s}e^{t(a-s)}\right]_0^n = M\cdot\lim_{n\to\infty}\frac{1}{a-s}\left[e^{n(a-s)}-1\right]$$

This is convergent only when $a-s<0 \Rightarrow s>a$. Therefore, by the Comparison Theorem, $F(s)=\int_0^\infty f(t)e^{-st}dt$ is also convergent for $s>a$.

73. $G(s)=\int_0^\infty f'(t)e^{-st}dt$. Integrate by parts with $u=e^{-st}$, $dv=f'(t)dt \Rightarrow du=-se^{-st}$, $v=f(t)$:

$$G(s)=\lim_{n\to\infty}\left[f(t)e^{-st}\right]_0^n+s\int_0^\infty f(t)e^{-st}dt=\lim_{n\to\infty}f(n)e^{-sn}-f(0)+sF(s)$$

But $0\le f(t)\le Me^{at}\Rightarrow 0\le f(t)e^{-st}\le Me^{at}e^{-st}$ and $\lim_{t\to\infty}Me^{t(a-s)}=0$ for $s>a$. So by the Squeeze Theorem,

$\lim_{t\to\infty}f(t)e^{-st}=0$ for $s>a\Rightarrow G(s)=0-f(0)+sF(s)=sF(s)-f(0)$ for $s>a$ .f}(0) for $s>a$.

74. Assume without loss of generality that $a<b$. Then

$$\begin{aligned}\int_{-\infty}^{a}f(x)dx+\int_a^\infty f(x)dx &= \lim_{t\to-\infty}\int_t^a f(x)dx+\lim_{u\to\infty}\int_a^u f(x)dx\\ &= \lim_{t\to-\infty}\int_t^a f(x)dx+\lim_{u\to\infty}\left[\int_a^b f(x)dx+\int_b^u f(x)dx\right]\\ &= \lim_{t\to-\infty}\int_t^a f(x)dx+\int_a^b f(x)dx+\lim_{u\to\infty}\int_b^u f(x)dx\\ &= \lim_{t\to-\infty}\left[\int_t^a f(x)dx+\int_a^b f(x)dx\right]+\int_b^\infty f(x)dx\\ &= \lim_{t\to-\infty}\int_t^b f(x)dx+\int_b^\infty f(x)dx\\ &= \int_{-\infty}^b f(x)dx+\int_b^\infty f(x)dx\end{aligned}$$

75. We use integration by parts: let $u=x$, $dv=xe^{-x^2}dx\Rightarrow du=dx$, $v=-\frac{1}{2}e^{-x^2}$. So

$$\begin{aligned}\int_0^\infty x^2e^{-x^2}dx &= \lim_{t\to\infty}\left[-\frac{1}{2}xe^{-x^2}\right]_0^t+\frac{1}{2}\int_0^\infty e^{-x^2}dx\\ &= \lim_{t\to\infty}\left[-t\Big/\left(2e^{t^2}\right)\right]+\frac{1}{2}\int_0^\infty e^{-x^2}dx=\frac{1}{2}\int_0^\infty e^{-x^2}dx\end{aligned}$$

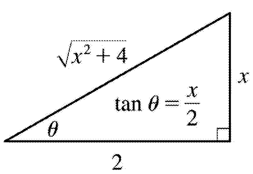
(The limit is 0 by l’Hospital’s Rule.)

76.

$\int_0^\infty e^{-x^2}\,dx$ is the area under the curve $y=e^{-x^2}$ for $0\le x<\infty$ and $0<y\le 1$. Solving $y=e^{-x^2}$ for x , we get $y=e^{-x^2}\Rightarrow \ln y=-x^2\Rightarrow -\ln y=x^2\Rightarrow x=\pm\sqrt{-\ln y}$. Since x is positive, choose $x=\sqrt{-\ln y}$, and the area is represented by $\int_0^1 \sqrt{-\ln y}\,dy$. Therefore, each integral represents the same area, so the integrals are equal.

77. For the first part of the integral, let $x=2\tan\theta\Rightarrow dx=2\sec^2\theta\,d\theta$.

$\int \frac{1}{\sqrt{x^2+4}}\,dx=\int \frac{2\sec^2\theta}{2\sec\theta}\,d\theta=\int \sec\theta\,d\theta=\ln|\sec\theta+\tan\theta|$. From the figure, $\tan\theta=\frac{x}{2}$, and $\sec\theta=\frac{\sqrt{x^2+4}}{2}$. So



$$I=\int_0^\infty\left(\frac{1}{\sqrt{x^2+4}}-\frac{C}{x+2}\right)dx=\lim_{t\to\infty}\left[\ln\left|\frac{\sqrt{x^2+4}}{2}+\frac{x}{2}\right|-C\ln|x+2|\right]_0^t$$

$$=\lim_{t\to\infty}\left[\ln\frac{\sqrt{t^2+4}+t}{2}-C\ln(t+2)-(\ln 1-C\ln 2)\right]$$

$$=\lim_{t\to\infty}\left[\ln\left(\frac{\sqrt{t^2+4}+t}{2(t+2)^C}\right)+\ln 2^C\right]$$

$$=\ln\left(\lim_{t\to\infty}\frac{t+\sqrt{t^2+4}}{(t+2)^C}\right)+\ln 2^{C-1}$$

Now $L=\lim_{t\to\infty}\frac{t+\sqrt{t^2+4}}{(t+2)^C}=\lim_{t\to\infty}\frac{1+t/\sqrt{t^2+4}}{C(t+2)^{C-1}}=\frac{2}{C\lim_{t\to\infty}(t+2)^{C-1}}$.

If $C<1$, $L=\infty$ and I diverges. If $C=1$, $L=2$ and I converges to

$\ln 2+\ln 2^0=\ln 2$. If $C>1$, $L=0$ and I diverges to $-\infty$.

78.

$$I=\int_0^\infty \left(\frac{x}{x^2+1} - \frac{C}{3x+1} \right) dx = \lim_{t\to\infty} \left[\frac{1}{2}\ln\left(x^2+1\right) - \frac{1}{3}C\ln(3x+1) \right]_0^t$$

$$=\lim_{t\to\infty}\left[\ln\left(t^2+1\right)^{1/2} - \ln(3t+1)^{C/3}\right]$$

$$=\lim_{t\to\infty}\left(\ln \frac{\left(t^2+1\right)^{1/2}}{(3t+1)^{C/3}} \right) = \ln\left(\lim_{t\to\infty} \frac{\sqrt{t^2+1}}{(3t+1)^{C/3}} \right)$$

For $C\leq 0$, the integral diverges. For $C>0$, we have

$$L=\lim_{t\to\infty}\frac{\sqrt{t^2+1}}{(3t+1)^{C/3}} = \lim_{t\to\infty}\frac{t/\sqrt{t^2+1}}{C(3t+1)^{(C/3)-1}} = \frac{1}{C}\lim_{t\to\infty}\frac{1}{(3t+1)^{(C/3)-1}} .$$

For $C/3<1 \Leftrightarrow C<3$, $L=\infty$ and I diverges. For $C=3$, $L=\frac{1}{3}$ and $I=\ln\frac{1}{3}$. For $C>3$, $L=0$ and I diverges to $-\infty$.

1. $y=2-3x \Rightarrow L=\int_{-2}^{1}\sqrt{1+(\{dy/dx)^2}\text{italic\{dx\}}=\int_{-2}^{1}\sqrt{1+(-3)^2}\,dx=\sqrt{10}\,[1-(-2)]=3\sqrt{10}$.

The arc length can be calculated using the distance formula, since the curve is a line segment, so

$$L=[\text{ distance from } (-2,8) \text{ to } (1,-1)\ ]=\sqrt{[1-(-2)]^2+[(-1)-8]^2}=\sqrt{90}=3\sqrt{10}$$

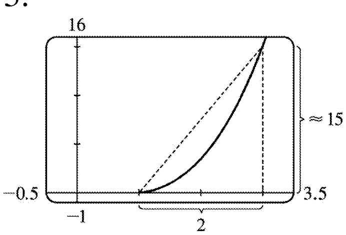
2. Using the arc length formula with $y=\sqrt{4-x^2} \Rightarrow \dfrac{dy}{dx}=-\dfrac{x}{\sqrt{4-x^2}}$, we get

$$L=\int_0^2\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx=\int_0^2\sqrt{1+\frac{x^2}{4-x^2}}\,dx=\int_0^2\frac{2\,dx}{\sqrt{4-x^2}}=2\lim_{t\to 2^-}\int_0^t\frac{dx}{\sqrt{2^2-x^2}}$$

$$=2\lim_{t\to 2^-}\left[\sin^{-1}(x/2)\right]_0^t=2\lim_{t\to 2^-}\left[\sin^{-1}(t/2)-\sin^{-1}0\right]=2\left(\frac{\pi}{2}-0\right)=\pi$$

The curve is a quarter of a circle with radius 2 , so the length of the arc is $\frac{1}{4}(2\pi\cdot 2)=\pi$, as above.

3.



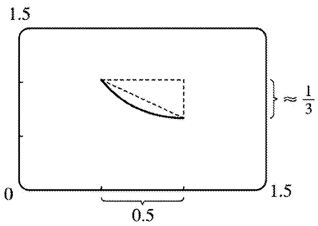
From the figure, the length of the curve is slightly larger than the hypotenuse of the triangle formed by the points $(1,0)$, $(3,0)$, and $(3,f(3))\approx(3,15)$, where $y=f(x)=\frac{2}{3}\left(x^2-1\right)^{3/2}$. This length is about $\sqrt{15^2+2^2}\approx 15$, so we might estimate the length to be 15.5 . $y=\frac{2}{3}\left(x^2-1\right)^{3/2}\Rightarrow y'=\left(x^2-1\right)^{1/2}(2x)\Rightarrow$

$1+(y')^2=1+4x^2\left(x^2-1\right)=4x^4-4x^2+1=\left(2x^2-1\right)^2$, so, using the fact that $2x^2-1>0$ for $1\le x\le 3$,

$$L=\int_1^3\sqrt{(2x^2-1)^2}\,dx=\int_1^3\left|2x^2-1\right|dx=\int_1^3(2x^2-1)\,dx=\left[\frac{2}{3}x^3-x\right]_1^3$$

$$=(18-3)-\left(\frac{2}{3}-1\right)=\frac{46}{3}=15.\overline{3}$$

4.



From the figure, the length of the curve is slightly larger than the hypotenuse of the triangle formed by the points $(0.5,f(0.5)\approx 1)$, $(1,f(0.5)\approx 1)$ and $\left(1,\frac{2}{3}\right)$, where $y=f(x)=x^3/6+1/(2x)$. This length is about $\sqrt{\left(\frac{1}{2}\right)^2+\left(\frac{1}{3}\right)^2}\approx 0.6$, so we might estimate the length to be 0.65 .

$$y=\frac{x^3}{6}+\frac{1}{2x}\Rightarrow y'=\frac{x^2}{2}-\frac{x^{-2}}{2}\Rightarrow$$
$$1+\left(y'\right)^2=1+\frac{x^4}{4}-\frac{1}{2}+\frac{x^{-4}}{4}=\frac{x^4}{4}+\frac{1}{2}+\frac{x^{-4}}{4}=\left(\frac{x^2}{2}+\frac{x^{-2}}{2}\right)^2$$

so, using the fact that the parenthetical expression is positive,

$$L=\int_{1/2}^{1}\sqrt{\left(\frac{x^2}{2}+\frac{x^{-2}}{2}\right)^2}\,dx=\int_{1/2}^{1}\left(\frac{x^2}{2}+\frac{x^{-2}}{2}\right)dx=\left[\frac{x^3}{6}-\frac{1}{2x}\right]_{1/2}^{1}$$
$$=\left(\frac{1}{6}-\frac{1}{2}\right)-\left(\frac{1}{48}-1\right)=\frac{31}{48}=0.6458\overline{3}$$

5. $y=1+6x^{3/2}\Rightarrow dy/dx=9x^{1/2}\Rightarrow 1+(dy/dx)^2=1+81x$. So

$$L=\int_0^1\sqrt{1+81x}\,dx=\int_1^{82}u^{1/2}\left(\frac{1}{81}\right)du \text{ [where } u=1+81x \text{ and } du=81\,dx \text{]}$$
$$=\frac{1}{81}\cdot\frac{2}{3}\left[u^{3/2}\right]_1^{82}=\frac{2}{243}\left(82\sqrt{82}-1\right)$$

6. $y^2=4(x+4)^3$, $y>0\Rightarrow y=2(x+4)^{3/2}\Rightarrow dy/dx=3(x+4)^{1/2}\Rightarrow$
$1+(dy/dx)^2=1+9(x+4)=9x+37$. So

$$L=\int_0^2 \sqrt{9x+37}\,dx \left[\begin{array}{l} u=9x+37, \\ du=9dx \end{array}\right]=\int_{37}^{55} u^{1/2}\left(\frac{1}{9}\right)du$$
$$=\frac{1}{9}\cdot\frac{2}{3}\left[u^{3/2}\right]_{37}^{55}=\frac{2}{27}\left(55\sqrt{55}-37\sqrt{37}\right)$$

7. $y=\frac{x^5}{6}+\frac{1}{10x^3}\Rightarrow\frac{dy}{dx}=\frac{5}{6}x^4-\frac{3}{10}x^{-4}\Rightarrow$
$1+(dy/dx)^2=1+\frac{25}{36}x^8-\frac{1}{2}+\frac{9}{100}x^{-8}=\frac{25}{36}x^8+\frac{1}{2}+\frac{9}{100}x^{-8}=\left(\frac{5}{6}x^4+\frac{3}{10}x^{-4}\right)^2$. So
$$L=\int_1^2\sqrt{\left(\frac{5}{6}x^4+\frac{3}{10}x^{-4}\right)^2}\,dx=\int_1^2\left(\frac{5}{6}x^4+\frac{3}{10}x^{-4}\right)dx=\left[\frac{1}{6}x^5-\frac{1}{10}x^{-3}\right]_1^2$$
$$=\left(\frac{32}{6}-\frac{1}{80}\right)-\left(\frac{1}{6}-\frac{1}{10}\right)=\frac{31}{6}+\frac{7}{80}=\frac{1261}{240}$$

8. $y=\frac{x^2}{2}-\frac{\ln x}{4}\Rightarrow\frac{dy}{dx}=x-\frac{1}{4x}\Rightarrow 1+\left(\frac{dy}{dx}\right)^2=x^2+\frac{1}{2}+\frac{1}{16x^2}$. So
$$L=\int_2^4\left(x+\frac{1}{4x}\right)dx=\left[\frac{x^2}{2}+\frac{\ln x}{4}\right]_2^4=\left(8+\frac{2\ln 2}{4}\right)-\left(2+\frac{\ln 2}{4}\right)=6+\frac{\ln 2}{4}\ .$$

9. $x=\frac{1}{3}\sqrt{y}\,(y-3)=\frac{1}{3}y^{3/2}-y^{1/2}\Rightarrow dx/dy=\frac{1}{2}y^{1/2}-\frac{1}{2}y^{-1/2}\Rightarrow$
$1+(dx/dy)^2=1+\frac{1}{4}y-\frac{1}{2}+\frac{1}{4}y^{-1}=\frac{1}{4}y+\frac{1}{2}+\frac{1}{4}y^{-1}=\left(\frac{1}{2}y^{1/2}+\frac{1}{2}y^{-1/2}\right)^2$. So
$$L=\int_1^9\left(\frac{1}{2}y^{1/2}+\frac{1}{2}y^{-1/2}\right)dy=\frac{1}{2}\left[\frac{2}{3}y^{3/2}+2y^{1/2}\right]_1^9=\frac{1}{2}\left[\left(\frac{2}{3}\cdot 27+2\cdot 3\right)-\left(\frac{2}{3}\cdot 1+2\cdot 1\right)\right]$$
$$=\frac{1}{2}\left(24-\frac{8}{3}\right)=\frac{1}{2}\left(\frac{64}{3}\right)=\frac{32}{3}$$

10. $y=\ln(\cos x)\Rightarrow dy/dx=-\tan x\Rightarrow 1+(dy/dx)^2=1+\tan^2 x=\sec^2 x$. So
$$L=\int_0^{\pi/3}\sqrt{\sec^2 x}\,dx=\int_0^{\pi/3}\sec x\,dx=\left[\ln|\sec x+\tan x|\right]_0^{\pi/3}=\ln\left(2+\sqrt{3}\right)-\ln(1+0)=\ln\left(2+\sqrt{3}\right)\ .$$

11. $y=\ln(\sec x)\Rightarrow$

$\frac{dy}{dx}=\frac{\sec x\tan x}{\sec x}=\tan x \Rightarrow 1+\left(\frac{dy}{dx}\right)^2=1+\tan^2 x=\sec^2 x$, so

$$L=\int_0^{\pi/4}\sqrt{\sec^2 x}\,dx=\int_0^{\pi/4}|\sec x|\,dx=\int_0^{\pi/4}\sec x\,dx=\left[\ln(\sec x+\tan x)\right]_0^{\pi/4}$$
$$=\ln\left(\sqrt{2}+1\right)-\ln(1+0)=\ln\left(\sqrt{2}+1\right)$$

12. $y=\ln x \Rightarrow \frac{dy}{dx}=\frac{1}{x} \Rightarrow \sqrt{1+\left(\frac{dy}{dx}\right)^2}=\sqrt{1+\left(\frac{1}{x}\right)^2}=\frac{\sqrt{1+x^2}}{x}$. So $L=\int_1^{\sqrt{3}}\frac{\sqrt{1+x^2}}{x}\,dx$. Now let $v=\sqrt{1+x^2}$, so $v^2=1+x^2$ and $v\,dv=x\,dx$. Thus

$$L=\int_{\sqrt{2}}^{2}\frac{v}{v^2-1}\,v\,dv=\int_{\sqrt{2}}^{2}\left(1+\frac{1/2}{v-1}-\frac{1/2}{v+1}\right)dv=\left[v+\frac{1}{2}\ln|v-1|-\frac{1}{2}\ln|v+1|\right]_{\sqrt{2}}^{2}$$
$$=\left[v-\frac{1}{2}\ln\left|\frac{v+1}{v-1}\right|\right]_{\sqrt{2}}^{2}=2-\frac{1}{2}\ln 3-\sqrt{2}+\frac{1}{2}\ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)=2-\sqrt{2}+\ln\left(\sqrt{2}+1\right)-\frac{1}{2}\ln 3$$

Or: Use Formula 23 in the table of integrals.

13. $y=\cosh x \Rightarrow y'=\sinh x \Rightarrow 1+\left(y'\right)^2=1+\sinh^2 x=\cosh^2 x$.

So $L=\int_0^1\cosh x\,dx=\left[\sinh x\right]_0^1=\sinh 1=\frac{1}{2}(e-1/e)$.

14. $y^2=4x$, $x=\frac{1}{4}y^2 \Rightarrow \frac{dx}{dy}=\frac{1}{2}y \Rightarrow 1+\left(\frac{dx}{dy}\right)^2=1+\frac{1}{4}y^2$. So

$$L=\int_0^2\sqrt{1+\frac{1}{4}y^2}\,dy=\int_0^1\sqrt{1+u^2}\cdot 2\,du$$
$$=\left[u\sqrt{1+u^2}+\ln\left|u+\sqrt{1+u^2}\right|\right]_0^1=\sqrt{2}+\ln\left(1+\sqrt{2}\right)$$

15. $y=e^x \Rightarrow y'=e^x \Rightarrow 1+\left(y'\right)^2=1+e^{2x}$. So

$$L=\int_0^1\sqrt{1+e^{2x}}\,dx=\int_1^e\sqrt{1+u^2}\,\frac{du}{u}\quad\left[u=e^x,\text{so}\,x=\ln u, dx=du/u\right]$$

$$= \int_1^e \frac{\sqrt{1+u^2}}{u^2}\, u\,du = \int_{\sqrt{2}}^{\sqrt{1+e^2}} \frac{v}{v^2-1}\, v\,dv \left[v=\sqrt{1+u^2}, \text{so } v^2=1+u^2, v\,dv=u\,du \right]$$

$$= \int_{\sqrt{2}}^{\sqrt{1+e^2}} \left(1+\frac{1/2}{v-1}-\frac{1/2}{v+1}\right)dv = \left[v+\frac{1}{2}\ln\frac{v-1}{v+1} \right]_{\sqrt{2}}^{\sqrt{1+e^2}}$$

$$= \sqrt{1+e^2}+\frac{1}{2}\ln\frac{\sqrt{1+e^2}-1}{\sqrt{1+e^2}+1}-\sqrt{2}-\frac{1}{2}\ln\frac{\sqrt{2}-1}{\sqrt{2}+1}$$

$$= \sqrt{1+e^2}-\sqrt{2}+\ln\left(\sqrt{1+e^2}-1\right)-1-\ln\left(\sqrt{2}-1\right)$$

Or: Use Formula 23 for $\int\left(\sqrt{1+u^2}/u\right)du$, or substitute $u=\tan\theta$.

16. $y=\ln\left(\frac{e^x+1}{e^x-1}\right)=\ln\left(e^x+1\right)-\ln\left(e^x-1\right) \Rightarrow y'=\frac{e^x}{e^x+1}-\frac{e^x}{e^x-1}=\frac{-2e^x}{e^{2x}-1} \Rightarrow$

$1+\left(y'\right)^2=1+\frac{4e^{2x}}{\left(e^{2x}-1\right)^2}=\frac{\left(e^{2x}+1\right)^2}{\left(e^{2x}-1\right)^2} \Rightarrow \sqrt{1+\left(y'\right)^2}=\frac{e^{2x}+1}{e^{2x}-1}=\frac{e^x+e^{-x}}{e^x-e^{-x}}=\frac{\cosh x}{\sinh x}$.

So $L=\int_a^b \frac{\cosh x}{\sinh x}\,dx=[\ln\sinh x]_a^b=\ln\sinh b-\ln\sinh a=\ln\left(\frac{\sinh b}{\sinh a}\right)=\ln\left(\frac{e^b-e^{-b}}{e^a-e^{-a}}\right)$.

17. $y=\cos x \Rightarrow dy/dx=-\sin x \Rightarrow 1+(dy/dx)^2=1+\sin^2 x$. So $L=\int_0^{2\pi}\sqrt{1+\sin^2 x}\,dx$.

18. $y=2^x \Rightarrow dy/dx=\left(2^x\right)\ln 2 \Rightarrow L=\int_0^3\sqrt{1+(\ln 2)^2 2^{2x}}\,dx$

19. $x=y+y^3 \Rightarrow dx/dy=1+3y^2 \Rightarrow 1+(dx/dy)^2=1+(1+3y^2)^2=9y^4+6y^2+2$. So $L=\int_1^4\sqrt{9y^4+6y^2+2}\,dy$.

20. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, $y=\pm b\sqrt{1-\left(x^2/a^2\right)}=\pm\frac{b}{a}\sqrt{a^2-x^2}$.

$y=\frac{b}{a}\sqrt{a^2-x^2} \Rightarrow \frac{dy}{dx}=\frac{-bx}{a\sqrt{a^2-x^2}} \Rightarrow \left(\frac{dy}{dx}\right)^2=\frac{b^2x^2}{a^2\left(a^2-x^2\right)}$.

So $L=2\int_{-a}^{a}\left[1+\frac{b^2x^2}{a^2\left(a^2-x^2\right)}\right]^{1/2}dx=\frac{4}{a}\int_{0}^{a}\left[\frac{\left(b^2-a^2\right)x^2+a^4}{a^2-x^2}\right]^{1/2}dx$.

21. $y=xe^{-x} \Rightarrow dy/dx=e^{-x}-xe^{-x}=e^{-x}(1-x) \Rightarrow 1+(dy/dx)^2=1+e^{-2x}(1-x)^2$. Let $f(x)=\sqrt{1+(dy/dx)^2}=\sqrt{1+e^{-2x}(1-x)^2}$. Then $L=\int_0^5 f(x)dx$. Since $n=10$, $\Delta x=\frac{5-0}{10}=\frac{1}{2}$. Now

$$L\approx S_{10}=\frac{1/2}{3}\left[f(0)+4f\left(\frac{1}{2}\right)+2f(1)+4f\left(\frac{3}{2}\right)+2f(2)+4f\left(\frac{5}{2}\right)+2f(3)\right.$$
$$\left.+4f\left(\frac{7}{2}\right)+2f(4)+4f\left(\frac{9}{2}\right)+f(5)\right]\approx 5.115840$$

The value of the integral produced by a calculator is 5.113568 (to six decimal places).

22. $x=y+\sqrt{y} \Rightarrow dx/dy=1+\frac{1}{2\sqrt{y}} \Rightarrow 1+(dx/dy)^2=1+\left(1+\frac{1}{2\sqrt{y}}\right)^2=2+\frac{1}{\sqrt{y}}+\frac{1}{4y}$. Let $g(y)=\sqrt{1+(dx/dy)^2}$. Then $L=\int_1^2 g(y)dy$. Since $n=10$, $\Delta y=\frac{2-1}{10}=\frac{1}{10}$. Now

$$L\approx S_{10}=\frac{1/10}{3}[g(1)+4g(1.1)+2g(1.2)+4g(1.3)+2g(1.4)+4g(1.5)$$
$$+2g(1.6)+4g(1.7)+2g(1.8)+4g(1.9)+g(2)]\approx 1.732215 ,$$

which is the same value of the integral produced by a calculator to six decimal places.

23. $y=\sec x \Rightarrow dy/dx=\sec x\tan x \Rightarrow L=\int_0^{\pi/3} f(x)dx$, where $f(x)=\sqrt{1+\sec^2 x\tan^2 x}$.

Since $n=10$, $\Delta x=\frac{\pi/3-0}{10}=\frac{\pi}{30}$. Now

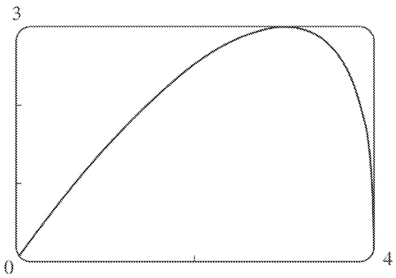
$$L\approx S_{10}=\frac{\pi/30}{3}\left[f(0)+4f\left(\frac{\pi}{30}\right)+2f\left(\frac{2\pi}{30}\right)+4f\left(\frac{3\pi}{30}\right)+2f\left(\frac{4\pi}{30}\right)+4f\left(\frac{5\pi}{30}\right)\right.$$
$$\left.+2f\left(\frac{6\pi}{30}\right)+4f\left(\frac{7\pi}{30}\right)+2f\left(\frac{8\pi}{30}\right)+4f\left(\frac{9\pi}{30}\right)+f\left(\frac{\pi}{3}\right)\right]\approx 1.569619 .$$

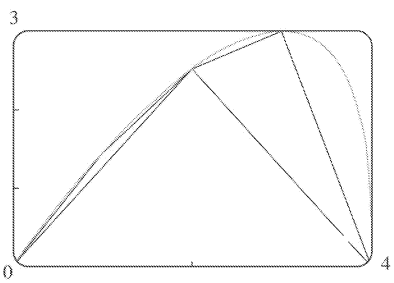
The value of the integral produced by a calculator is 1.569259 (to six decimal places).

24. $y=x\ln x \Rightarrow dy/dx=1+\ln x$. Let $f(x)=\sqrt{1+(dy/dx)^2}=\sqrt{1+(1+\ln x)^2}$.
Then $L=\int_1^3 f(x)dx$. Since $n=10$, $\Delta x=\frac{3-1}{10}=\frac{1}{5}$. Now

$$L\approx S_{10}=\frac{1/5}{3}[f(1)+4f(1.2)+2f(1.4)+4f(1.6)+2f(1.8)+4f(2)]$$
$$+2f(2.2)+4f(2.4)+2f(2.6)+4f(2.8)+f(3)]\approx 3.869618 \text{ .}$$

The value of the integral produced by a calculator is 3.869617 (to six decimal places).

25. **(a)** 

(b) 

Let $f(x)=y=x\sqrt[3]{4-x}$. The polygon with one side is just the line segment joining the points $(0,f(0))=(0,0)$ and $(4,f(4))=(4,0)$, and its length is 4 . The polygon with two sides joins the points $(0,0)$, $(2,f(2))=\left(2,2\sqrt[3]{2}\right)$ and $(4,0)$.

Its length is

$$\sqrt{(2-0)^2+\left(2\sqrt[3]{2}-0\right)^2}+\sqrt{(4-2)^2+\left(0-2\sqrt[3]{2}\right)^2}=2\sqrt{4+2^{8/3}}\approx 6.43$$

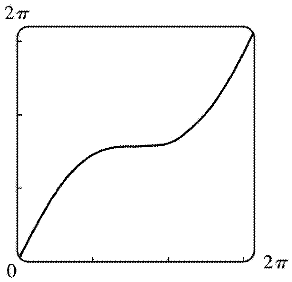
Similarly, the inscribed polygon with four sides joins the points $(0,0)$, $\left(1,\sqrt[3]{3}\right)$, $\left(2,2\sqrt[3]{2}\right)$, $(3,3)$, and $(4,0)$, so its length is

$$\sqrt{1+\left(\sqrt[3]{3}\right)^2}+\sqrt{1+\left(2\sqrt[3]{2}-\sqrt[3]{3}\right)^2}+\sqrt{1+\left(3-2\sqrt[3]{2}\right)^2}+\sqrt{1+9}\approx 7.50$$

(c) Using the arc length formula with $\frac{dy}{dx}=x\left[\frac{1}{3}(4-x)^{-2/3}(-1)\right]+\sqrt[3]{4-x}=\frac{12-4x}{3(4-x)^{2/3}}$, the length of the curve is $L=\int_0^4\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx=\int_0^4\sqrt{1+\left[\frac{12-4x}{3(4-x)^{2/3}}\right]^2}\,dx$.

(d) According to a CAS, the length of the curve is $L\approx 7.7988$. The actual value is larger than any of the approximations in part (b). This is always true, since any approximating straight line between two points on the curve is shorter than the length of the curve between the two points.

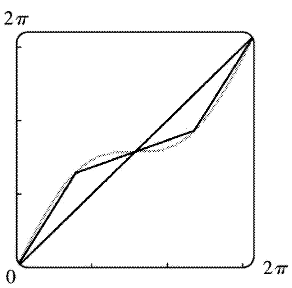
26. **(a)** Let $f(x)=y=x+\sin x$ with $0\le x\le 2\pi$.



(b) The polygon with one side is just the line segment joining the points $(0,f(0))=(0,0)$ and $(2\pi,f(2\pi))=(2\pi,2\pi)$, and its length is $\sqrt{(2\pi-0)^2+(2\pi-0)^2}=2\sqrt{2}\,\pi\approx 8.9$.
The polygon with two sides joins the points $(0,0)$, $(\pi,f(\pi))=(\pi,\pi)$, and $(2\pi,2\pi)$. Its length is

$$\sqrt{(\pi-0)^2+(\pi-0)^2}+\sqrt{(2\pi-\pi)^2+(2\pi-\pi)^2}=\sqrt{2}\,\pi+\sqrt{2}\,\pi=2\sqrt{2}\,\pi\approx 8.9$$

Note from the diagram that the two approximations are the same because the sides of the 2 –sided polygon are in fact on the same line, since $f(\pi)=\pi=\frac{1}{2}f(2\pi)$.



The four–sided polygon joins the points $(0,0)$, $\left(\frac{\pi}{2},\frac{\pi}{2}+1\right)$, (π,π) , $\left(\frac{3\pi}{2},\frac{3\pi}{2}-1\right)$, and $(2\pi,2\pi)$, so its length is

$$\sqrt{\left(\frac{\pi}{2}\right)^2+\left(\frac{\pi}{2}+1\right)^2}+\sqrt{\left(\frac{\pi}{2}\right)^2+\left(\frac{\pi}{2}-1\right)^2}+\sqrt{\left(\frac{\pi}{2}\right)^2+\left(\frac{\pi}{2}-1\right)^2}+\sqrt{\left(\frac{\pi}{2}\right)^2+\left(\frac{\pi}{2}+1\right)^2}\approx 9.4$$

(c) Using the arc length formula with $dy/dx=1+\cos x$, the length of the curve is

$$L=\int_0^{2\pi}\sqrt{1+(1+\cos x)^2}\,dx=\int_0^{2\pi}\sqrt{2+2\cos x+\cos^2 x}\,dx$$

(d) The CAS approximates the integral as 9.5076 . The actual length is larger than the approximations in part (b).

27. $x=\ln\left(1-y^2\right)\Rightarrow \frac{dx}{dy}=\frac{-2y}{1-y^2}\Rightarrow 1+\left(\frac{dx}{dy}\right)^2=1+\frac{4y^2}{\left(1-y^2\right)^2}=\frac{\left(1+y^2\right)^2}{\left(1-y^2\right)^2}$. So

$$L=\int_0^{1/2}\sqrt{\frac{\left(1+y^2\right)^2}{\left(1-y^2\right)^2}}\,dy=\int_0^{1/2}\frac{1+y^2}{1-y^2}\,dy=\ln 3-\frac{1}{2}\approx 0.599$$

28. $y=x^{4/3}\Rightarrow dy/dx=\frac{4}{3}x^{1/3}\Rightarrow 1+(dy/dx)^2=1+\frac{16}{9}x^{2/3}\Rightarrow$

$$L=\int_0^1\sqrt{1+\frac{16}{9}x^{2/3}}\,\mathrm{dx}=\int_0^{4/3}\sqrt{1+u^2}\,\frac{81}{64}\mathrm{u}^2\mathrm{du}\quad\left[u=\frac{4}{3}x^{1/3},du=\frac{4}{9}x^{-2/3}dx,dx=\frac{9}{4}x^{2/3}du=\frac{9}{4}\cdot\frac{9}{16}u^2du=\frac{81}{64}u^2du\right]$$

$$=\frac{81}{64}\left[\frac{1}{8}u\left(1+2u^2\right)\sqrt{1+u^2}-\frac{1}{8}\ln\left(u+\sqrt{1+u^2}\right)\right]_0^{4/3}$$

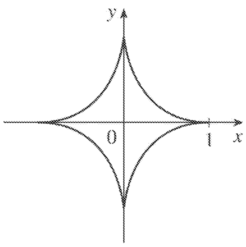
$$=\frac{81}{64}\left[\frac{1}{6}\left(1+\frac{32}{9}\right)\sqrt{\frac{25}{9}}-\frac{1}{8}\ln\left(\frac{4}{3}+\sqrt{\frac{25}{9}}\right)\right]=\frac{81}{64}\left(\frac{1}{6}\cdot\frac{41}{9}\cdot\frac{5}{3}-\frac{1}{8}\ln 3\right)$$

$$=\frac{205}{128}-\frac{81}{512}\ln 3\approx 1.4277586$$

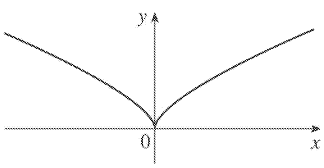
29. $y^{2/3}=1-x^{2/3}\Rightarrow y=\left(1-x^{2/3}\right)^{3/2}\Rightarrow\frac{dy}{dx}=\frac{3}{2}\left(1-x^{2/3}\right)^{1/2}\left(-\frac{2}{3}x^{-1/3}\right)=-x^{-1/3}\left(1-x^{2/3}\right)^{1/2}\Rightarrow$

$\left(\frac{dy}{dx}\right)^2 = x^{-2/3}\left(1 - x^{2/3}\right) = x^{-2/3} - 1$. Thus

$L = 4\int_0^1 \sqrt{1 + \left(x^{-2/3} - 1\right)}\, dx = 4\int_0^1 x^{-1/3} dx = 4\lim_{t\to 0^+}\left[\frac{3}{2}x^{2/3}\right]_t^1 = 6$.



30. **(a)**



(b) $y = x^{2/3} \Rightarrow 1 + \left(\frac{dy}{dx}\right)^2 = 1 + \left(\frac{2}{3}x^{-1/3}\right)^2 = 1 + \frac{4}{9}x^{-2/3}$. So $L = \int_0^1 \sqrt{1 + \frac{4}{9}x^{-2/3}}\, dx$ [an improper integral]. $x = y^{3/2} \Rightarrow 1 + \left(\frac{dx}{dy}\right)^2 = 1 + \left(\frac{3}{2}y^{1/2}\right)^2 = 1 + \frac{9}{4}y$. So $L = \int_0^1 \sqrt{1 + \frac{9}{4}y}\, dy$.

The second integral equals $\frac{4}{9}\cdot\frac{2}{3}\left[\left(1 + \frac{9}{4}y\right)^{3/2}\right]_0^1 = \frac{8}{27}\left(\frac{13\sqrt{13}}{8} - 1\right) = \frac{13\sqrt{13} - 8}{27}$. The first integral can be evaluated as follows:

$$\int_0^1 \sqrt{1 + \frac{4}{9}x^{-2/3}}\, dx = \lim_{t\to 0^+}\int_t^1 \frac{\sqrt{9x^{2/3} + 4}}{3x^{1/3}}\, dx = \lim_{t\to 0^+}\int_{9t^{2/3}}^9 \frac{\sqrt{u+4}}{18}\, du\left[u = 9x^{2/3}, du = 6x^{-1/3}dx\right]$$

$$= \int_0^9 \frac{\sqrt{u+4}}{18}\, du = \frac{1}{18}\cdot\left[\frac{2}{3}(u+4)^{3/2}\right]_0^9 = \frac{1}{27}\left(13^{3/2} - 4^{3/2}\right) = \frac{13\sqrt{13} - 8}{27}$$

(c)

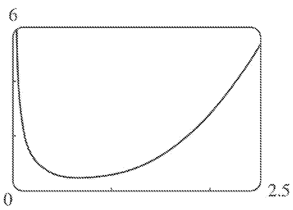
L = length of the arc of this curve from $(-1,1)$ to $(8,4)$

$$= \int_0^1 \sqrt{1 + \frac{9}{4}y}\, dy + \int_0^4 \sqrt{1 + \frac{9}{4}y}\, dy = \frac{13\sqrt{13} - 8}{27} + \frac{8}{27}\left[\left(1 + \frac{9}{4}y\right)^{3/2}\right]_0^4$$

$$=\frac{13\sqrt{13}-8}{27}+\frac{8}{27}\left(10\sqrt{10}-1\right)=\frac{13\sqrt{13}+80\sqrt{10}-16}{27}$$

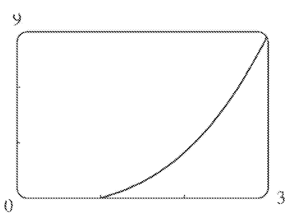
31. $y=2x^{3/2}\Rightarrow y'=3x^{1/2}\Rightarrow 1+\left(y'\right)^2=1+9x$. The arc length function with starting point $P_0(1,2)$ is

$$s(x)=\int_1^x\sqrt{1+9t}\,dt=\left[\frac{2}{27}(1+9t)^{3/2}\right]_1^x=\frac{2}{27}\left[(1+9x)^{3/2}-10\sqrt{10}\right]$$

32. **(a)** 

(b) $1+\left(\frac{dy}{dx}\right)^2=x^4+\frac{1}{2}+\frac{1}{16x^4}$,

$$\begin{aligned}s(x)&=\int_1^x\left[t^2+1/\left(4t^2\right)\right]dt\\&=\left[\frac{1}{3}t^3-1/(4t)\right]_1^x\\&=\frac{1}{3}x^3-1/(4x)-\left(\frac{1}{3}-\frac{1}{4}\right)\\&=\frac{1}{3}x^3-1/(4x)-\frac{1}{12}\quad\text{for }x\geq 1\end{aligned}$$

(c) 

33. The prey hits the ground when $y=0\Leftrightarrow 180-\frac{1}{45}x^2=0\Leftrightarrow x^2=45\cdot 180\Rightarrow x=\sqrt{8100}=90$, since x must be positive. $y'=-\frac{2}{45}x\Rightarrow 1+\left(y'\right)^2=1+\frac{4}{45^2}x^2$, so the distance traveled by the prey is

$$L=\int_0^{90}\sqrt{1+\frac{4}{45^2}x^2}\,dx=\int_0^4\sqrt{1+u^2}\left(\frac{45}{2}\,du\right)$$
$$=\frac{45}{2}\left[\frac{1}{2}u\sqrt{1+u^2}+\frac{1}{2}\ln\left(u+\sqrt{1+u^2}\right)\right]_0^4$$
$$=\frac{45}{2}\left[2\sqrt{17}+\frac{1}{2}\ln\left(4+\sqrt{17}\right)\right]=45\sqrt{17}+\frac{45}{4}\ln\left(4+\sqrt{17}\right)\approx 209.1\text{m}$$

34. $y=150-\frac{1}{40}(x-50)^2\Rightarrow y'=-\frac{1}{20}(x-50)\Rightarrow 1+\left(y'\right)^2=1+\frac{1}{20^2}(x-50)^2$, so the distance traveled by the kite is

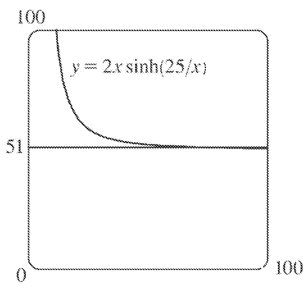
$$L=\int_0^{80}\sqrt{1+\frac{1}{20^2}(x-50)^2}\,dx=\int_{-5/2}^{3/2}\sqrt{1+u^2}\,(20du)\ [u=\frac{1}{20}(x-50),du=\frac{1}{20}\,dx]$$
$$=20\left[\frac{1}{2}u\sqrt{1+u^2}+\frac{1}{2}\ln\left(u+\sqrt{1+u^2}\right)\right]_{-5/2}^{3/2}$$
$$=10\left[\frac{3}{2}\sqrt{\frac{13}{4}}+\ln\left(\frac{3}{2}+\sqrt{\frac{13}{4}}\right)+\frac{5}{2}\sqrt{\frac{29}{4}}-\ln\left(-\frac{5}{2}+\sqrt{\frac{29}{4}}\right)\right]$$
$$=\frac{15}{2}\sqrt{13}+\frac{25}{2}\sqrt{29}+10\ln\left(\frac{3+\sqrt{13}}{-5+\sqrt{29}}\right)\approx 122.8\text{ft}$$

35. The sine wave has amplitude 1 and period 14 , since it goes through two periods in a distance of 28 in., so its equation is $y=1\sin\left(\frac{2\pi}{14}x\right)=\sin\left(\frac{\pi}{7}x\right)$. The width w of the flat metal sheet needed to make the panel is the arc length of the sine curve from $x=0$ to $x=28$. We set up the integral to evaluate w using the arc length formula with $\frac{dy}{dx}=\frac{\pi}{7}\cos\left(\frac{\pi}{7}x\right)$:
$L=\int_0^{28}\sqrt{1+\left[\frac{\pi}{7}\cos\left(\frac{\pi}{7}x\right)\right]^2}\,dx=2\int_0^{14}\sqrt{1+\left[\frac{\pi}{7}\cos\left(\frac{\pi}{7}x\right)\right]^2}\,dx$. This integral would be very difficult to evaluate exactly, so we use a CAS, and find that $L\approx 29.36$ inches.

36. **(a)** $y=c+a\cosh\left(\frac{x}{a}\right)\Rightarrow y'=\sinh\left(\frac{x}{a}\right)\Rightarrow 1+\left(y'\right)^2=1+\sinh^2\left(\frac{x}{a}\right)=\cosh^2\left(\frac{x}{a}\right)$. So

$$L=\int_{-b}^{b}\sqrt{\cosh^2\left(\frac{x}{a}\right)}\,dx=2\int_0^b\cosh\left(\frac{x}{a}\right)dx=2\left[a\sinh\left(\frac{x}{a}\right)\right]_0^b=2a\sinh\left(\frac{b}{a}\right)$$

(b) At $x=0$, $y=c+a$, so $c+a=20$. The poles are 50 ft apart, so $b=25$, and $L=51\Rightarrow 51=2a\sinh(b/a)$. From the figure, we see that $y=51$ intersects $y=2x\sinh(25/x)$ at $x\approx 72.3843$ for $x>0$. So $a\approx 72.3843$ and the wire should be attached at a distance of $y=c+a\cosh(25/a)=20-a+a\cosh(25/a)\approx 24.36$ ft above the ground.



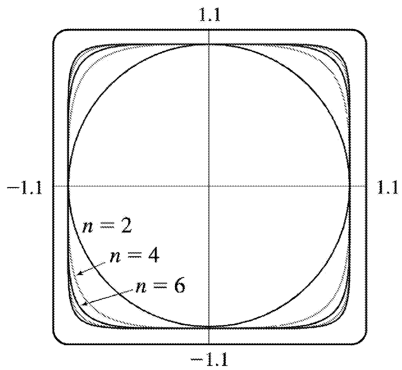
37. $y=\int_1^x\sqrt{t^3-1}\,dt\Rightarrow \frac{dy}{dx}=\sqrt{x^3-1}$ [by FTC1] $\Rightarrow 1+\left(\frac{dy}{dx}\right)^2=1+\left(\sqrt{x^3-1}\right)^2=x^3\Rightarrow$

$$L=\int_1^4\sqrt{x^3}\,dx=\int_1^4 x^{3/2}dx=\frac{2}{5}\left[x^{5/2}\right]_1^4=\frac{2}{5}(32-1)=\frac{62}{5}=12.4$$

38. By symmetry, the length of the curve in each quadrant is the same, so we'll find the length in the first quadrant and multiply by 4 . $x^{2k}+y^{2k}=1\Rightarrow y^{2k}=1-x^{2k}\Rightarrow y=\left(1-x^{2k}\right)^{1/(2k)}$ (in the first quadrant), so we use the arc length formula with

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{2k}\left(1-x^{2k}\right)^{1/(2k)-1}\left(-2kx^{2k-1}\right)\\ &= -x^{2k-1}\left(1-x^{2k}\right)^{1/(2k)-1}\end{aligned}$$

The total length is therefore



$L_{2k}=4\int_0^1 \sqrt{1+\left[-x^{2k-1}\left(1-x^{2k}\right)^{1/(2k)-1}\right]^2}\,dx=4\int_0^1 \sqrt{1+x^{2(2k-1)}\left(1-x^{2k}\right)^{1/k-2}}\,dx$ Now from the graph, we see that as k increases, the ''corners'' of these fat circles get closer to the points $(\pm 1,\pm 1)$ and $(\pm 1,\mp 1)$, and the ''edges'' of the fat circles approach the lines joining these four points. It seems plausible that as $k\to\infty$, the total length of the fat circle with $n=2k$ will approach the length of the perimeter of the square with sides of length 2 . This is supported by taking the limit as $k\to\infty$ of the equation of the fat circle in the first quadrant: $\lim_{k\to\infty}\left(1-x^{2k}\right)^{1/(2k)}=1$ for $0\leq x<1$. So we guess that $\lim_{k\to\infty} L_{2k}=4\cdot 2=8$.

1. $y=\ln x \Rightarrow ds=\sqrt{1+(dy/dx)^2}\,dx=\sqrt{1+(1/x)^2}\,dx \Rightarrow S=\int_1^3 2\pi(\ln x)\sqrt{1+(1/x)^2}\,dx$

2. $y=\sin^2 x \Rightarrow ds=\sqrt{1+(dy/dx)^2}\,dx=\sqrt{1+(2\sin x\cos x)^2}\,dx \Rightarrow S=\int_0^{\pi/2} 2\pi\sin^2 x\sqrt{1+(2\sin x\cos x)^2}\,dx$ [by (7)]

3. $y=\sec x \Rightarrow ds=\sqrt{1+(dy/dx)^2}\,dx=\sqrt{1+(\sec x\tan x)^2}\,dx \Rightarrow S=\int_0^{\pi/4} 2\pi x\sqrt{1+(\sec x\tan x)^2}\,dx$ [by (8)]

4. $y=e^x \Rightarrow ds=\sqrt{1+(dy/dx)^2}\,dx=\sqrt{1+e^{2x}}\,dx \Rightarrow S=\int_0^{\ln 2} 2\pi x\sqrt{1+e^{2x}}\,dx$ [by (8)] or $\int_1^2 2\pi(\ln y)\sqrt{1+(1/y)^2}\,dy$ [by (6)]

5. $y=x^3 \Rightarrow y'=3x^2$. So

$$S=\int_0^2 2\pi y\sqrt{1+\left(y'\right)^2}\,dx=2\pi\int_0^2 x^3\sqrt{1+9x^4}\,dx[u=1+9x^4, du=36x^3\,dx]$$

$$=\frac{2\pi}{36}\int_1^{145}\sqrt{u}\,du=\frac{\pi}{18}\left[\frac{2}{3}u^{3/2}\right]_1^{145}=\frac{\pi}{27}\left(145\sqrt{145}-1\right)$$

6. The curve $9x=y^2+18$ is symmetric about the x –axis, so we only use its top half, given by $y=3\sqrt{x-2}$. $dy/dx=\dfrac{3}{2\sqrt{x-2}}$, so $1+(dy/dx)^2=1+\dfrac{9}{4(x-2)}$. Thus,

$$S=\int_2^6 2\pi\cdot 3\sqrt{x-2}\sqrt{1+\frac{9}{4(x-2)}}\,dx=6\pi\int_2^6\sqrt{x-2+\frac{9}{4}}\,dx=6\pi\int_2^6\left(x+\frac{1}{4}\right)^{1/2}dx$$

$$=6\pi\cdot\frac{2}{3}\left[\left(x+\frac{1}{4}\right)^{3/2}\right]_2^6=4\pi\left[\left(\frac{25}{4}\right)^{3/2}-\left(\frac{9}{4}\right)^{3/2}\right]=4\pi\left(\frac{125}{8}-\frac{27}{8}\right)=4\pi\cdot\frac{98}{8}=49\pi$$

7. $y=\sqrt{x} \Rightarrow 1+(dy/dx)^2=1+\left[1/\left(2\sqrt{x}\right)\right]^2=1+1/(4x)$. So

$$S=\int_4^9 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx=\int_4^9 2\pi\sqrt{x}\sqrt{1+\frac{1}{4x}}\,dx=2\pi\int_4^9\sqrt{x+\frac{1}{4}}\,dx$$

$$= 2\pi \left[\frac{2}{3} \left(x+\frac{1}{4} \right)^{3/2} \right]_4^9 = \frac{4\pi}{3} \left[\frac{1}{8} (4x+1)^{3/2} \right]_4^9 = \frac{\pi}{6} \left(37\sqrt{37} - 17\sqrt{17} \right)$$

8. $y=\cos 2x \Rightarrow ds=\sqrt{1+(dy/dx)^2}\, dx=\sqrt{1+(-2\sin 2x)^2}\, dx \Rightarrow$

$$S=\int_0^{\pi/6} 2\pi \cos 2x \sqrt{1+4\sin^2 2x}\, dx = 2\pi \int_0^{\sqrt{3}} \sqrt{1+u^2} \left(\frac{1}{4} \right) du [u=2\sin 2x, du=4\cos 2x\, dx]$$

$$= \frac{\pi}{2} \left[\frac{1}{2} u \sqrt{1+u^2} + \frac{1}{2} \ln \left(u+\sqrt{1+u^2} \right) \right]_0^{\sqrt{3}} = \frac{\pi}{2} \left[\frac{\sqrt{3}}{2} \cdot 2 + \frac{1}{2} \ln \left(\sqrt{3}+2 \right) \right] = \frac{\pi\sqrt{3}}{2} + \frac{\pi}{4} \ln \left(2+\sqrt{3} \right)$$

9. $y=\cosh x \Rightarrow 1+(dy/dx)^2=1+\sinh^2 x=\cosh^2 x$. So

$$S = 2\pi \int_0^1 \cosh x \cosh x\, dx = 2\pi \int_0^1 \frac{1}{2} (1+\cosh 2x)\, dx = \pi \left[x+\frac{1}{2} \sinh 2x \right]_0^1$$

$$= \pi \left(1+\frac{1}{2} \sinh 2 \right) \text{ or } \pi \left[1+\frac{1}{4} \left(e^2 - e^{-2} \right) \right]$$

10. $y=\frac{x^3}{6}+\frac{1}{2x} \Rightarrow \frac{dy}{dx}=\frac{x^2}{2}-\frac{1}{2x^2} \Rightarrow$

$$\sqrt{1+(dy/dx)^2}=\sqrt{\frac{x^4}{4}+\frac{1}{2}+\frac{1}{4x^4}}=\sqrt{\left(\frac{x^2}{2}+\frac{1}{2x^2} \right)^2}=\frac{x^2}{2}+\frac{1}{2x^2} \Rightarrow$$

$$S=\int_{1/2}^1 2\pi \left(\frac{x^3}{6}+\frac{1}{2x} \right) \left(\frac{x^2}{2}+\frac{1}{2x^2} \right) dx = 2\pi \int_{1/2}^1 \left(\frac{x^5}{12}+\frac{x}{12}+\frac{x}{4}+\frac{1}{4x^3} \right) dx$$

$$=2\pi \int_{1/2}^1 \left(\frac{x^5}{12}+\frac{x}{3}+\frac{x^{-3}}{4} \right) dx = 2\pi \left[\frac{x^6}{72}+\frac{x^2}{6}-\frac{x^{-2}}{8} \right]_{1/2}^1$$

$$=2\pi \left[\left(\frac{1}{72}+\frac{1}{6}-\frac{1}{8} \right) - \left(\frac{1}{64\cdot 72}+\frac{1}{24}-\frac{1}{2} \right) \right] = 2\pi \left(\frac{263}{512} \right) = \frac{263}{256}\pi$$

11. $x=\frac{1}{3}\left(y^2+2 \right)^{3/2} \Rightarrow dx/dy=\frac{1}{2}\left(y^2+2 \right)^{1/2}(2y)=y\sqrt{y^2+2} \Rightarrow 1+(dx/dy)^2=1+y^2\left(y^2+2 \right)=\left(y^2+1 \right)^2$. So

$$S=2\pi\int_1^2 y\left(y^2+1\right)dy=2\pi\left[\frac{1}{4}y^4+\frac{1}{2}y^2\right]_1^2=2\pi\left(4+2-\frac{1}{4}-\frac{1}{2}\right)=\frac{21\pi}{2}$$

12. $x=1+2y^2 \Rightarrow 1+(dx/dy)^2=1+(4y)^2=1+16y^2$. So

$$S=2\pi\int_1^2 y\sqrt{1+16y^2}\,dy=\frac{\pi}{16}\int_1^2\left(16y^2+1\right)^{1/2}32y\,dy=\frac{\pi}{16}\left[\frac{2}{3}\left(16y^2+1\right)^{3/2}\right]_1^2$$

$$=\frac{\pi}{24}\left(65\sqrt{65}-17\sqrt{17}\right)$$

13. $y=\sqrt[3]{x}\Rightarrow x=y^3\Rightarrow 1+(dx/dy)^2=1+9y^4$. So

$$S=2\pi\int_1^2 x\sqrt{1+(dx/dy)^2}\,dy=2\pi\int_1^2 y^3\sqrt{1+9y^4}\,dy=\frac{2\pi}{36}\int_1^2\sqrt{1+9y^4}\,36y^3\,dy$$

$$=\frac{\pi}{18}\left[\frac{2}{3}\left(1+9y^4\right)^{3/2}\right]_1^2=\frac{\pi}{27}\left(145\sqrt{145}-10\sqrt{10}\right)$$

14. $y=1-x^2\Rightarrow 1+(dy/dx)^2=1+4x^2\Rightarrow$

$$S=2\pi\int_0^1 x\sqrt{1+4x^2}\,dx=\frac{\pi}{4}\int_0^1 8x\sqrt{4x^2+1}\,dx=\frac{\pi}{4}\left[\frac{2}{3}\left(4x^2+1\right)^{3/2}\right]_0^1=\frac{\pi}{6}\left(5\sqrt{5}-1\right)$$

15. $x=\sqrt{a^2-y^2}\Rightarrow dx/dy=\frac{1}{2}(a^2-y^2)^{-1/2}(-2y)=-y/\sqrt{a^2-y^2}\Rightarrow$

$$1+(dx/dy)^2=1+\frac{y^2}{a^2-y^2}=\frac{a^2-y^2}{a^2-y^2}+\frac{y^2}{a^2-y^2}=\frac{a^2}{a^2-y^2}\Rightarrow$$

$$S=\int_0^{a/2}2\pi\sqrt{a^2-y^2}\,\frac{a}{\sqrt{a^2-y^2}}\,dy=2\pi\int_0^{a/2}a\,dy=2\pi a\left[y\right]_0^{a/2}=2\pi a\left(\frac{a}{2}-0\right)=\pi a^2$$. Note that this is $\frac{1}{4}$ the surface area of a sphere of radius a , and the length of the interval $y=0$ to $y=a/2$ is $\frac{1}{4}$ the length of the interval $y=-a$ to $y=a$.

16. $x=a\cosh(y/a)\Rightarrow 1+(dx/dy)^2=1+\sinh^2(y/a)=\cosh^2(y/a)$. So

$$S = 2\pi \int_{-a}^{a} a\cosh\left(\frac{y}{a}\right)\cosh\left(\frac{y}{a}\right)dy = 4\pi a\int_{0}^{a}\cosh^{2}\left(\frac{y}{a}\right)dy = 2\pi a\int_{0}^{a}\left[1+\cosh\left(\frac{2y}{a}\right)\right]dy$$

$$= 2\pi a\left[y+\frac{a}{2}\sinh\left(\frac{2y}{a}\right)\right]_{0}^{a} = 2\pi a\left[a+\frac{a}{2}\sinh 2\right] = 2\pi a^{2}\left[1+\frac{1}{2}\sinh 2\right] \text{ or } \frac{\pi a^{2}\left(e^{2}+4-e^{-2}\right)}{2}$$

17. $y=\ln x \Rightarrow dy/dx=1/x \Rightarrow 1+(dy/dx)^2=1+1/x^2 \Rightarrow S=\int_1^3 2\pi \ln x\sqrt{1+1/x^2}\,dx$.

Let $f(x)=\ln x\sqrt{1+1/x^2}$. Since $n=10$, $\Delta x=\dfrac{3-1}{10}=\dfrac{1}{5}$. Then

$S\approx S_{10}=2\pi\cdot\dfrac{1/5}{3}\,[f(1)+4f(1.2)+2f(1.4)+\cdots+2f(2.6)+4f(2.8)+f(3)]\approx 9.023754$.

The value of the integral produced by a calculator is 9.024262 (to six decimal places).

18. $y=x+\sqrt{x} \Rightarrow dy/dx=1+\dfrac{1}{2}x^{-1/2} \Rightarrow 1+(dy/dx)^2=2+x^{-1/2}+\dfrac{1}{4}x^{-1} \Rightarrow$

$S=\displaystyle\int_1^2 2\pi\left(x+\sqrt{x}\right)\sqrt{2+\frac{1}{\sqrt{x}}+\frac{1}{4x}}\,dx$. Let $f(x)=(x+\sqrt{x})\sqrt{2+\dfrac{1}{\sqrt{x}}+\dfrac{1}{4x}}$.

Since $n=10$, $\Delta x=\dfrac{2-1}{10}=\dfrac{1}{10}$. Then

$S\approx S_{10}=2\pi\cdot\dfrac{1/10}{3}\,[f(1)+4f(1.1)+2f(1.2)+\cdots+2f(1.8)+4f(1.9)+f(2)]\approx 29.506566$.

The value of the integral produced by a calculator is 29.506568 (to six decimal places).

19. $y=\sec x \Rightarrow dy/dx=\sec x\tan x \Rightarrow 1+(dy/dx)^2=1+\sec^2 x\tan^2 x \Rightarrow$

$S=\displaystyle\int_0^{\pi/3} 2\pi\sec x\sqrt{1+\sec^2 x\tan^2 x}\,dx$. Let $f(x)=\sec x\sqrt{1+\sec^2 x\tan^2 x}$.

Since $n=10$, $\Delta x=\dfrac{\pi/3-0}{10}=\dfrac{\pi}{30}$. Then

$S\approx S_{10}=2\pi\cdot\dfrac{\pi/30}{3}\left[f(0)+4f\left(\dfrac{\pi}{30}\right)+2f\left(\dfrac{2\pi}{30}\right)+\cdots+2f\left(\dfrac{8\pi}{30}\right)+4f\left(\dfrac{9\pi}{30}\right)+f\left(\dfrac{\pi}{3}\right)\right]\approx 13.527296$

.

The value of the integral produced by a calculator is 13.516987 (to six decimal places).

20. $y=(1+e^x)^{1/2} \Rightarrow \dfrac{dy}{dx}=\dfrac{1}{2}(1+e^x)^{-1/2}\cdot e^x=\dfrac{e^x}{2(1+e^x)^{1/2}} \Rightarrow$

$$1+\left(\frac{dy}{dx}\right)^2=1+\frac{e^{2x}}{4(1+e^x)}=\frac{4+4e^x+e^{2x}}{4(1+e^x)}=\frac{(e^x+2)^2}{4(1+e^x)}\Rightarrow$$

$$S=\int_0^1 2\pi\sqrt{1+e^x}\,\frac{e^x+2}{2\sqrt{1+e^x}}\,dx=\pi\int_0^1(e^x+2)dx=\pi\Big[e^x+2x\Big]_0^1=\pi[(e+2)-(1+0)]=\pi(e+1)\ .$$

Let $f(x)=\frac{1}{2}(e^x+2)$. Since $n=10$, $\Delta x=\frac{1-0}{10}=\frac{1}{10}$. Then

$S\approx S_{10}=2\pi\cdot\frac{1/10}{3}\,[f(0)+4f(0.1)+2f(0.2)+\cdots+2f(0.8)+4f(0.9)+f(1)]\approx 11.681330$.

The value of the integral produced by a calculator is 11.681327 (to six decimal places).

21. $y=1/x\Rightarrow ds=\sqrt{1+(dy/dx)^2}\,dx=\sqrt{1+\left(-1/x^2\right)^2}\,dx=\sqrt{1+1/x^4}\,dx\Rightarrow$

$$S=\int_1^2 2\pi\cdot\frac{1}{x}\sqrt{1+\frac{1}{x^4}}\,dx=2\pi\int_1^2\frac{\sqrt{x^4+1}}{x^3}\,dx=2\pi\int_1^4\frac{\sqrt{u^2+1}}{u^2}\left(\frac{1}{2}\right)du[u=x^2,du=2x\,dx]$$

$$=\pi\int_1^4\frac{\sqrt{1+u^2}}{u^2}\,du=\pi\left[-\frac{\sqrt{1+u^2}}{u}+\ln\left(u+\sqrt{1+u^2}\right)\right]_1^4$$

$$=\pi\left[-\frac{\sqrt{17}}{4}+\ln\left(4+\sqrt{17}\right)+\frac{\sqrt{2}}{1}-\ln\left(1+\sqrt{2}\right)\right]=\pi\left[\sqrt{2}-\frac{\sqrt{17}}{4}+\ln\left(\frac{4+\sqrt{17}}{1+\sqrt{2}}\right)\right]$$

22. $y=\sqrt{x^2+1}\Rightarrow\frac{dy}{dx}=\frac{x}{\sqrt{x^2+1}}\Rightarrow ds=\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx=\sqrt{1+\frac{x^2}{x^2+1}}\,dx\Rightarrow$

$$S=\int_0^3 2\pi\sqrt{x^2+1}\sqrt{1+\frac{x^2}{x^2+1}}\,dx=2\pi\int_0^3\sqrt{2x^2+1}\,dx=2\sqrt{2}\,\pi\int_0^3\sqrt{x^2+\left(\frac{1}{\sqrt{2}}\right)^2}\,dx$$

$$=2\sqrt{2}\,\pi\left[\frac{1}{2}x\sqrt{x^2+\frac{1}{2}}+\frac{1}{4}\ln\left(x+\sqrt{x^2+\frac{1}{2}}\right)\right]_0^3$$

$$=2\sqrt{2}\pi\left[\frac{3}{2}\sqrt{9+\frac{1}{2}}+\frac{1}{4}\ln\left(3+\sqrt{9+\frac{1}{2}}\right)-\frac{1}{4}\ln\frac{1}{\sqrt{2}}\right]=2\sqrt{2}\pi\left[\frac{3}{2}\sqrt{\frac{19}{2}}+\frac{1}{4}\ln\left(3+\sqrt{\frac{19}{2}}\right)+\frac{1}{4}\right.$$

$$=2\sqrt{2}\pi\left[\frac{3}{2}\frac{\sqrt{19}}{\sqrt{2}}+\frac{1}{4}\ln\left(3\sqrt{2}+\sqrt{19}\right)\right]=3\sqrt{19}\pi+\frac{\pi}{\sqrt{2}}\ln\left(3\sqrt{2}+\sqrt{19}\right)$$

23. $y=x^3$ and $0\le y\le 1 \Rightarrow y'=3x^2$ and $0\le x\le 1$.

$$S=\int_0^1 2\pi x\sqrt{1+\left(3x^2\right)^2}\,dx=2\pi\int_0^3\sqrt{1+u^2}\,\frac{1}{6}\,du[u=3x^2,du=6x\,dx]$$

$$=\frac{\pi}{3}\int_0^3\sqrt{1+u^2}\,du=[\text{or use CAS}]\,\frac{\pi}{3}\left[\frac{1}{2}u\sqrt{1+u^2}+\frac{1}{2}\ln\left(u+\sqrt{1+u^2}\right)\right]_0^3$$

$$=\frac{\pi}{3}\left[\frac{3}{2}\sqrt{10}+\frac{1}{2}\ln\left(3+\sqrt{10}\right)\right]=\frac{\pi}{6}\left[3\sqrt{10}+\ln\left(3+\sqrt{10}\right)\right]$$

24. $y=\ln(x+1)$, $0\le x\le 1$. $ds=\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx=\sqrt{1+\left(\frac{1}{x+1}\right)^2}\,dx$, so S

$$=\int_0^1 2\pi x\sqrt{1+\frac{1}{(x+1)^2}}\,dx=\int_1^2 2\pi(u-1)\sqrt{1+\frac{1}{u^2}}\,du[u=x+1,du=dx]$$

$$=2\pi\int_1^2 u\,\frac{\sqrt{1+u^2}}{u}\,du-2\pi\int_1^2\frac{\sqrt{1+u^2}}{u}\,du=2\pi\int_1^2\sqrt{1+u^2}\,du-2\pi\int_1^2\frac{\sqrt{1+u^2}}{u}\,du$$

$$=[\text{or use CAS}]\;2\pi\left[\frac{1}{2}u\sqrt{1+u^2}+\frac{1}{2}\ln\left(u+\sqrt{1+u^2}\right)\right]_1^2-2\pi\left[\sqrt{1+u^2}-\ln\left(\frac{1+\sqrt{1+u^2}}{u}\right)\right]_1^2$$

$$=2\pi\left[\sqrt{5}+\frac{1}{2}\ln\left(2+\sqrt{5}\right)-\frac{1}{2}\sqrt{2}-\frac{1}{2}\ln\left(1+\sqrt{2}\right)\right]-2\pi\left[\sqrt{5}-\ln\left(\frac{1+\sqrt{5}}{2}\right)-\sqrt{2}+\ln\left(1+\sqrt{2}\right)\right]$$

$$=2\pi\left[\frac{1}{2}\ln\left(2+\sqrt{5}\right)+\ln\left(\frac{1+\sqrt{5}}{2}\right)+\frac{\sqrt{2}}{2}-\frac{3}{2}\ln\left(1+\sqrt{2}\right)\right]$$

25. $S=2\pi\int_1^\infty y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx=2\pi\int_1^\infty\frac{1}{x}\sqrt{1+\frac{1}{x^4}}\,dx=2\pi\int_1^\infty\frac{\sqrt{x^4+1}}{x^3}\,dx$. Rather than trying to

evaluate this integral, note that $\sqrt{x^4+1} > \sqrt{x^4} = x^2$ for $x>0$. Thus, if the area is finite,

$$S=2\pi\int_1^\infty \frac{\sqrt{x^4+1}}{x^3}\,dx > 2\pi\int_1^\infty \frac{x^2}{x^3}\,dx = 2\pi\int_1^\infty \frac{1}{x}\,dx$$

But we know that this integral diverges, so the area S is infinite.

26. $S=\int_0^\infty 2\pi y\sqrt{1+(dy/dx)^2}\,dx = 2\pi\int_0^\infty e^{-x}\sqrt{1+(-e^{-x})^2}\,dx$.

Evaluate $I=\int e^{-x}\sqrt{1+(-e^{-x})^2}\,dx$ by using the substitution $u=-e^{-x}$, $du=e^{-x}dx$.

$$I=\int\sqrt{1+u^2}\,du = \frac{1}{2}u\sqrt{1+u^2}+\frac{1}{2}\ln\left(u+\sqrt{1+u^2}\right)+C$$

$$=\frac{1}{2}(-e^{-x})\sqrt{1+e^{-2x}}+\frac{1}{2}\ln\left(-e^{-x}+\sqrt{1+e^{-2x}}\right)+C$$

Returning to the surface area integral, we have

$$S=2\pi\lim_{t\to\infty}\int_0^t e^{-x}\sqrt{1+(-e^{-x})^2}\,dx$$

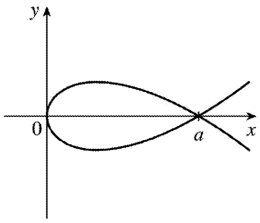
$$=2\pi\lim_{t\to\infty}\left[\frac{1}{2}(-e^{-x})\sqrt{1+e^{-2x}}+\frac{1}{2}\ln\left(-e^{-x}+\sqrt{1+e^{-2x}}\right)\right]_0^t$$

$$=2\pi\lim_{t\to\infty}\left\{\left[\frac{1}{2}(-e^{-t})\sqrt{1+e^{-2t}}+\frac{1}{2}\ln\left(-e^{-t}+\sqrt{1+e^{-2t}}\right)\right]-\left[\frac{1}{2}(-1)\sqrt{1+1}+\frac{1}{2}\ln\left(-1+\sqrt{1+1}\right)\right]\right\}$$

$$=2\pi\left\{\left[\frac{1}{2}(0)\sqrt{1}+\frac{1}{2}\ln\left(0+\sqrt{1}\right)\right]-\left[-\frac{1}{2}\sqrt{2}+\frac{1}{2}\ln\left(-1+\sqrt{2}\right)\right]\right\}$$

$$=2\pi\left\{[0]+\frac{1}{2}\left[\sqrt{2}-\ln\left(\sqrt{2}-1\right)\right]\right\}=\pi\left[\sqrt{2}-\ln\left(\sqrt{2}-1\right)\right]$$

27. Since $a>0$, the curve $3ay^2=x(a-x)^2$ only has points with $x\geq 0$. ($3ay^2\geq 0\Rightarrow x(a-x)^2\geq 0\Rightarrow x\geq 0$.) The curve is symmetric about the x –axis (since the equation is unchanged when y is replaced by $-y$). $y=0$ when $x=0$ or a , so the curve's loop extends from $x=0$ to $x=a$.



$$\frac{d}{dx}(3ay^2)=\frac{d}{dx} \Rightarrow 6ay\frac{dy}{dx}=x\cdot 2(a-x)(-1)+(a-x)^2 \Rightarrow \frac{dy}{dx}=\frac{(a-x)[-2x+a-x]}{6ay} \Rightarrow$$

$$\left(\frac{dy}{dx}\right)^2=\frac{(a-x)^2(a-3x)^2}{36a^2y^2}=\frac{(a-x)^2(a-3x)^2}{36a^2}\cdot\frac{3a}{x(a-x)^2}\left[\begin{array}{c}\text{the last fraction}\\ \text{is } 1/y^2\end{array}\right]=\frac{(a-3x)^2}{12ax} \Rightarrow$$

$$1+\left(\frac{dy}{dx}\right)^2=1+\frac{a^2-6ax+9x^2}{12ax}=\frac{12ax}{12ax}+\frac{a^2-6ax+9x^2}{12ax}=\frac{a^2+6ax+9x^2}{12ax}=\frac{(a+3x)^2}{12ax} \quad \text{for } x\neq 0\,.$$

(a)

$$S=\int_{x=0}^{a} 2\pi y\,ds=2\pi\int_0^a \frac{\sqrt{x}(a-x)}{\sqrt{3a}}\cdot\frac{a+3x}{\sqrt{12ax}}\,dx=2\pi\int_0^a \frac{(a-x)(a+3x)}{6a}\,dx$$

$$=\frac{\pi}{3a}\int_0^a (a^2+2ax-3x^2)dx=\frac{\pi}{3a}\left[a^2x+ax^2-x^3\right]_0^a=\frac{\pi}{3a}(a^3+a^3-a^3)=\frac{\pi}{3a}\cdot a^3=\frac{\pi a^2}{3}\quad.$$

Note that we have rotated the top half of the loop about the x –axis. This generates the full surface.
(b) We must rotate the full loop about the y –axis, so we get double the area obtained by rotating the top half of the loop:

$$S=2\cdot 2\pi\int_{x=0}^{a} x\,ds=4\pi\int_0^a x\frac{a+3x}{\sqrt{12ax}}\,dx=\frac{4\pi}{2\sqrt{3a}}\int_0^a x^{1/2}(a+3x)\,dx$$

$$=\frac{2\pi}{\sqrt{3a}}\int_0^a (ax^{1/2}+3x^{3/2})dx=\frac{2\pi}{\sqrt{3a}}\left[\frac{2}{3}ax^{3/2}+\frac{6}{5}x^{5/2}\right]_0^a=\frac{2\pi\sqrt{3}}{3\sqrt{a}}\left(\frac{2}{3}a^{5/2}+\frac{6}{5}a^{5/2}\right)$$

$$=\frac{2\pi\sqrt{3}}{3}\left(\frac{2}{3}+\frac{6}{5}\right)a^2=\frac{2\pi\sqrt{3}}{3}\left(\frac{28}{15}\right)a^2=\frac{56\pi\sqrt{3}\,a^2}{45}$$

28. In general, if the parabola $y=ax^2$, $-c\leq x\leq c$, is rotated about the y –axis, the surface area it generates is

$$2\pi\int_0^c x\sqrt{1+(2ax)^2}\,dx = 2\pi\int_0^{2ac}\frac{u}{2a}\sqrt{1+u^2}\,\frac{1}{2a}\,du\ [u=2ax, du=2a\,dx]$$

$$= \frac{\pi}{4a^2}\int_0^{2ac}\left(1+u^2\right)^{1/2}2u\,du = \frac{\pi}{4a^2}\left[\frac{2}{3}\left(1+u^2\right)^{3/2}\right]_0^{2ac}$$

$$= \frac{\pi}{6a^2}\left[\left(1+4a^2c^2\right)^{3/2}-1\right]$$

Here $2c=10$ ft and $ac^2=2$ ft, so $c=5$ and $a=\frac{2}{25}$. Thus, the surface area is

$$S = \frac{\pi}{6}\frac{625}{4}\left[\left(1+4\cdot\frac{4}{625}\cdot 25\right)^{3/2}-1\right] = \frac{625\pi}{24}\left[\left(1+\frac{16}{25}\right)^{3/2}-1\right] = \frac{625\pi}{24}\left(\frac{41\sqrt{41}}{125}-1\right)$$

$$= \frac{5\pi}{24}\left(41\sqrt{41}-125\right)\approx 90.01\text{ft}^2$$

29. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \Rightarrow \frac{y(dy/dx)}{b^2}=-\frac{x}{a^2} \Rightarrow \frac{dy}{dx}=-\frac{b^2x}{a^2y} \Rightarrow$

$$1+\left(\frac{dy}{dx}\right)^2 = 1+\frac{b^4x^2}{a^4y^2} = \frac{b^4x^2+a^4y^2}{a^4y^2} = \frac{b^4x^2+a^4b^2\left(1-x^2/a^2\right)}{a^4b^2\left(1-x^2/a^2\right)} = \frac{a^4b^2+b^4x^2-a^2b^2x^2}{a^4b^2-a^2b^2x^2}$$

$$= \frac{a^4+b^2x^2-a^2x^2}{a^4-a^2x^2} = \frac{a^4-\left(a^2-b^2\right)x^2}{a^2\left(a^2-x^2\right)}$$

The ellipsoid's surface area is twice the area generated by rotating the first quadrant portion of the ellipse about the x– axis. Thus,

$$S = 2\int_0^a 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = 4\pi\int_0^a \frac{b}{a}\sqrt{a^2-x^2}\,\frac{\sqrt{a^4-\left(a^2-b^2\right)x^2}}{a\sqrt{a^2-x^2}}\,dx$$

$$= \frac{4\pi b}{a^2}\int_0^a\sqrt{a^4-\left(a^2-b^2\right)x^2}\,dx = \frac{4\pi b}{a^2}\int_0^{a\sqrt{a^2-b^2}}\sqrt{a^4-u^2}\,\frac{du}{\sqrt{a^2-b^2}}\quad\left[u=\sqrt{a^2-b^2}\,x\right]$$

$$= \frac{4\pi b}{a^2\sqrt{a^2-b^2}}\left[\frac{u}{2}\sqrt{a^4-u^2}+\frac{a^4}{2}\sin^{-1}\frac{u}{a^2}\right]_0^{a\sqrt{a^2-b^2}}$$

$$= \frac{4\pi b}{a^2\sqrt{a^2-b^2}}\left[\frac{a\sqrt{a^2-b^2}}{2}\sqrt{a^4-a^2\left(a^2-b^2\right)}+\frac{a^4}{2}\sin^{-1}\frac{\sqrt{a^2-b^2}}{a}\right]=2\pi\left[b^2+\frac{a^2b\sin^{-1}\frac{\sqrt{a^2-b^2}}{a}}{\sqrt{a^2-b^2}}\right]$$

30. The upper half of the torus is generated by rotating the curve $(x-R)^2+y^2=r^2$, $y>0$, about the y–axis. $y\frac{dy}{dx}=-(x-R)\Rightarrow 1+\left(\frac{dy}{dx}\right)^2=1+\frac{(x-R)^2}{y^2}=\frac{y^2+(x-R)^2}{y^2}=\frac{r^2}{r^2-(x-R)^2}$. Thus,

$$S=2\int_{R-r}^{R+r}2\pi x\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx=4\pi\int_{R-r}^{R+r}\frac{rx}{\sqrt{r^2-(x-R)^2}}\,dx$$

$$=4\pi r\int_{-r}^{r}\frac{u+R}{\sqrt{r^2-u^2}}\,du\,[u=x-R]$$

$$=4\pi r\int_{-r}^{r}\frac{u\,du}{\sqrt{r^2-u^2}}+4\pi Rr\int_{-r}^{r}\frac{du}{\sqrt{r^2-u^2}}$$

$$=4\pi r\cdot 0+8\pi Rr\int_{0}^{r}\frac{du}{\sqrt{r^2-u^2}}$$ [since the first integrand is odd and the second is even]

$$=8\pi Rr\left[\sin^{-1}(u/r)\right]_0^r=8\pi Rr\left(\frac{\pi}{2}\right)=4\pi^2Rr$$

31. The analogue of $f\left(x_i^*\right)$ in the derivation of (4) is now $c-f\left(x_i^*\right)$, so

$$S=\lim_{n\to\infty}\sum_{i=1}^{n}2\pi\left[c-f\left(x_i^*\right)\right]\sqrt{1+\left[f'\left(x_i^*\right)\right]^2}\,\Delta x=\int_a^b 2\pi\,[c-f(x)]\sqrt{1+\left[f'(x)\right]^2}\,dx .$$

32. $y=x^{1/2}\Rightarrow y'=\frac{1}{2}x^{-1/2}\Rightarrow 1+\left(y'\right)^2=1+1/4x$, so by Exercise 31,

$S=\int_0^4 2\pi\left(4-\sqrt{x}\right)\sqrt{1+1/(4x)}\,dx$. Using a CAS, we get $S=2\pi\ln\left(\sqrt{17}+4\right)+\frac{\pi}{6}\left(31\sqrt{17}+1\right)\approx 80.6095$.

33. For the upper semicircle, $f(x)=\sqrt{r^2-x^2}$, $f'(x)=-x/\sqrt{r^2-x^2}$. The surface area generated is

$$S_1=\int_{-r}^{r}2\pi\left(r-\sqrt{r^2-x^2}\right)\sqrt{1+\frac{x^2}{r^2-x^2}}\,dx=4\pi\int_0^r\left(r-\sqrt{r^2-x^2}\right)\frac{r}{\sqrt{r^2-x^2}}\,dx$$

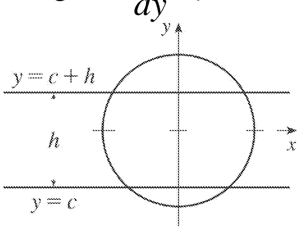
$$=4\pi\int_0^r\left(\frac{r^2}{\sqrt{r^2-x^2}}-r\right)dx$$

For the lower semicircle, $f(x)=-\sqrt{r^2-x^2}$ and $f'(x)=\frac{x}{\sqrt{r^2-x^2}}$, so $S_2=4\pi\int_0^r\left(\frac{r^2}{\sqrt{r^2-x^2}}+r\right)dx$.

Thus, the total area is

$$S=S_1+S_2=8\pi\int_0^r\left(\frac{r^2}{\sqrt{r^2-x^2}}\right)dx=8\pi\left[r^2\sin^{-1}\left(\frac{x}{r}\right)\right]_0^r=8\pi r^2\left(\frac{\pi}{2}\right)=4\pi^2r^2 .$$

34. Take the sphere $x^2+y^2+z^2=\frac{1}{4}d^2$ and let the intersecting planes be $y=c$ and $y=c+h$, where $-\frac{1}{2}d\le c\le\frac{1}{2}d-h$. The sphere intersects the $xy-$ plane in the circle $x^2+y^2=\frac{1}{4}d^2$. From this equation, we get $x\frac{dx}{dy}+y=0$, so $\frac{dx}{dy}=-\frac{y}{x}$. The desired surface area is



$$S = 2\pi \int x\,ds = 2\pi \int_c^{c+h} x\sqrt{1+(dx/dy)^2}\,dy = 2\pi \int_c^{c+h} x\sqrt{1+y^2/x^2}\,dy = 2\pi \int_c^{c+h} \sqrt{x^2+y^2}\,dy$$

$$= 2\pi \int_c^{c+h} \frac{1}{2}d\,dy = \pi d \int_c^{c+h} dy = \pi dh$$

35. In the derivation of (4), we computed a typical contribution to the surface area to be $2\pi \frac{y_{i-1}+y_i}{2} \left|P_{i-1}P_i\right|$, the area of a frustum of a cone. When $f(x)$ is not necessarily positive, the approximations $y_i = f(x_i) \approx f(x_i^*)$ and $y_{i-1} = f(x_{i-1}) \approx f(x_i^*)$ must be replaced by $y_i = \left|f(x_i)\right| \approx \left|f(x_i^*)\right|$ and $y_{i-1} = \left|f(x_{i-1})\right| \approx \left|f(x_i^*)\right|$. Thus, $2\pi \frac{y_{i-1}+y_i}{2} \left|P_{i-1}P_i\right| \approx 2\pi \left|f(x_i^*)\right| \sqrt{1+\left[f'(x_i^*)\right]^2}\,\Delta x$. Continuing with the rest of the derivation as before, we obtain $S = \int_a^b 2\pi\,|f(x)|\sqrt{1+\left[f'(x)\right]^2}\,dx$.

36. Since $g(x) = f(x)+c$, we have $g'(x) = f'(x)$. Thus,

$$S_g = \int_a^b 2\pi g(x)\sqrt{1+\left[g'(x)\right]^2}\,dx = \int_a^b 2\pi\,[f(x)+c]\sqrt{1+\left[f'(x)\right]^2}\,dx$$

$$= \int_a^b 2\pi f(x)\sqrt{1+\left[f'(x)\right]^2}\,dx + 2\pi c\int_a^b \sqrt{1+\left[f'(x)\right]^2}\,dx = S_f + 2\pi cL$$

1. The weight density of water is $\delta = 62.5\ \text{lb} / \text{ft}^3$.

(a) $P = \delta d \approx \left(62.5 \text{lb/ft}^3\right)(3\text{ft}) = 187.5\ \text{lb} / \text{ft}^2$

(b) $F = PA \approx \left(187.5 \text{lb/ft}^2\right)(5\text{ft})(2\text{ft}) = 1875$ lb. (A is the area of the bottom of the tank.)

(c) As in Example 1, the area of the i th strip is $2(\Delta x)$ and the pressure is $\delta d = \delta x_i$. Thus,

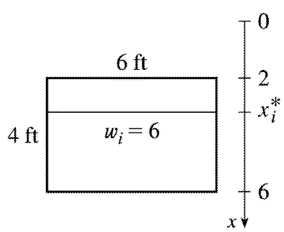
$$F = \int_0^3 \delta x \cdot 2dx \approx (62.5)(2) \int_0^3 x dx = 125 \left[\frac{1}{2} x^2 \right]_0^3 = 125 \left(\frac{9}{2} \right) = 562.5 \text{lb}$$

2. **(a)** $P = \rho g d = 1030(9.8)(2.5) = 25\,,235 \approx 2.52 \times 10^4$ Pa $= 25.2$ kPa

(b) $F = PA \approx \left(2.52 \times 10^4 \text{N/m}^2\right)\left(50\text{m}^2\right) = 1.26 \times 10^6$ N

(c) $F = \int_0^{2.5} \rho g x \cdot 5dx = (1030)(9.8)(5) \int_0^{2.5} x dx \approx 2.52 \times 10^4 \left[x^2\right]_0^{2.5} \approx 1.58 \times 10^5$ N

3. Set up a vertical x –axis as shown, with $x=0$ at the water's surface and x increasing in the downward direction. Then the area of the i th rectangular strip is $6\Delta x$ and the pressure on the strip is δx_i^* (where $\delta \approx 62.5 \text{lb/ft}^3$). Thus, the hydrostatic force on the strip is $\delta x_i^* \cdot 6\Delta x$ and the total hydrostatic force $\approx \sum_{i=1}^{n} \delta x_i^* \cdot 6\Delta x$. The total force

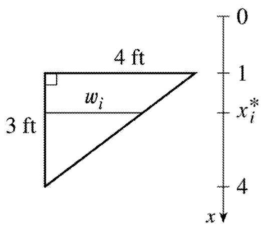


$$F = \lim_{n \to \infty} \sum_{i=1}^{n} \delta x_i^* \cdot 6\Delta x = \int_2^6 \delta x \cdot 6dx = 6\delta \int_2^6 x dx$$

$$= 6\delta \left[\frac{1}{2} x^2 \right]_2^6 = 6\delta (18-2) = 96\delta \approx 6000 \text{lb}$$

4. Set up a vertical x –axis as shown. Then the area of the i th rectangular strip is $\frac{4}{3}(4 - x_i^*)\Delta x$. The pressure on the strip is δx_i^* , so the hydrostatic force on the strip is $\delta x_i^* \cdot \frac{4}{3}(4 - x_i^*)\Delta x$ and the total force on the plate

$\approx \sum_{i=1}^{n} \delta x_i^* \cdot \frac{4}{3}(4-x_i^*)\Delta x$. The total force



$$F = \lim_{n\to\infty} \sum_{i=1}^{n} \delta x_i^* \cdot \frac{4}{3}(4-x_i^*)\Delta x = \int_1^4 \delta x \cdot \frac{4}{3}(4-x)dx = \frac{4}{3}\delta \int_1^4 (4x-x^2)dx$$

$$= \frac{4}{3}\delta \left[2x^2 - \frac{1}{3}x^3 \right]_1^4 = \frac{4}{3}\delta \left[\left(32-\frac{64}{3}\right) - \left(2-\frac{1}{3}\right) \right] = \frac{4}{3}\delta(9) = 12\delta \approx 750 \text{ lb}$$

5. Since an equation for the shape is $x^2+y^2=10^2$ ($x \geq 0$), we have $y=\sqrt{100-x^2}$. Thus, the area of the i th strip is $2\sqrt{100-(x_i^*)^2}\,\Delta x$

and the pressure on the strip is $\rho g x_i^*$, so the hydrostatic force on the strip is $\rho g x_i^* \cdot 2\sqrt{100-(x_i^*)^2}\,\Delta x$ and the total force on the

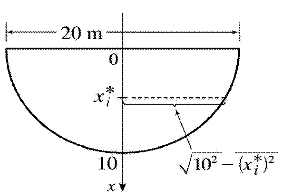


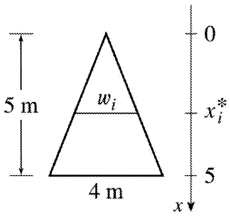
plate $\approx \sum_{i=1}^{n} \rho g x_i^* \cdot 2\sqrt{100-(x_i^*)^2}\,\Delta x$. The total force

$$F = \lim_{n\to\infty} \sum_{i=1}^{n} \rho g x_i^* \cdot 2\sqrt{100-(x_i^*)^2}\,\Delta x = \int_0^{10} 2\rho g x\sqrt{100-x^2}\,dx$$

$$= -\rho g \int_0^{10} \left(100-x^2\right)^{1/2}(-2x)dx = -\rho g \left[\frac{2}{3}(100-x^2)^{3/2} \right]_0^{10} = -\frac{2}{3}\rho g(0-1000)$$

$$= \frac{2000}{3}\rho g \approx \frac{2000}{3} \cdot 1000 \cdot 9.8 \approx 6.5\times 10^6 \text{N} \left[\rho \approx 1000 \text{kg/m}^3 \text{and} g \approx 9.8 \text{m/s}^2 . \right]$$

6. By similar triangles, $w_i/4=x_i^*/5$, so $w_i=\frac{4}{5}x_i^*$ and the area of the i th strip is $\frac{4}{5}x_i^*\Delta x$. The pressure on the strip is $\rho g x_i^*$, so the hydrostatic force on the strip is $\rho g x_i^*\cdot\frac{4}{5}x_i^*\Delta x$ and the total force on the plate $\approx\sum_{i=1}^{n}\rho g x_i^*\cdot\frac{4}{5}x_i^*\Delta x$. The total force



$$F=\lim_{n\to\infty}\sum_{i=1}^{n}\rho g x_i^*\cdot\frac{4}{5}x_i^*\Delta x=\int_0^5\rho g x\cdot\frac{4}{5}x\,dx=\frac{4}{5}\rho g\left[\frac{1}{3}x^3\right]_0^5=\frac{4}{5}\rho g\cdot\frac{125}{3}=\frac{100}{3}\rho g$$

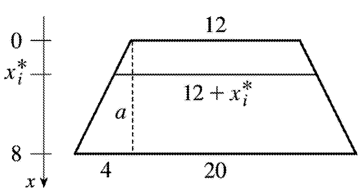
$$\approx\frac{100}{3}\cdot 1000\cdot 9.8\approx 3.3\times 10^5\ \text{N}.$$

7. Using similar triangles, $\frac{4\ \text{ft wide}}{8\ \text{ft high}}=\frac{a\ \text{ft wide}}{x_i^*\ \text{ft high}}$, so $a=\frac{1}{2}x_i^*$ and the width of the i th rectangular strip is $12+2a=12+x_i^*$. The area of the strip is $\left(12+x_i^*\right)\Delta x$. The pressure on the strip is δx_i^* .

$$F=\lim_{n\to\infty}\sum_{i=1}^{n}\delta x_i^*\left(12+x_i^*\right)\Delta x=\int_0^8\delta x\cdot(12+x)\,dx$$

$$=\delta\int_0^8\left(12x+x^2\right)dx=\delta\left[6x^2+\frac{x^3}{3}\right]_0^8=\delta\left(384+\frac{512}{3}\right)$$

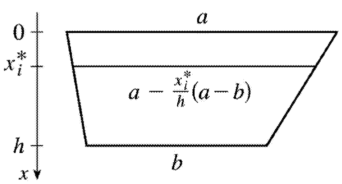
$$=(62.5)\frac{1664}{3}\approx 3.47\times 10^4\ \text{lb}$$



8. In the figure, deleting a $b\times h$ rectangle leaves a triangle with base $a-b$ and height h . By similar triangles,

$\frac{(a-b)\text{ ft wide}}{h\text{ ft high}}=\frac{d\text{ ft wide}}{\left(h-x_i^*\right)\text{ ft high}}$, so the width of the triangle is

$$d=\frac{h-x_i^*}{h}(a-b)=\left(1-\frac{x_i^*}{h}\right)(a-b)=a-b-\frac{x_i^*}{h}(a-b)$$



and the width of the trapezoid is $b+d=a-\frac{x_i^*}{h}(a-b)$. The area of the i th rectangular strip is $\left[a-\frac{x_i^*}{h}(a-b)\right]\Delta x$ and the pressure on it is $\rho g x_i^*$.

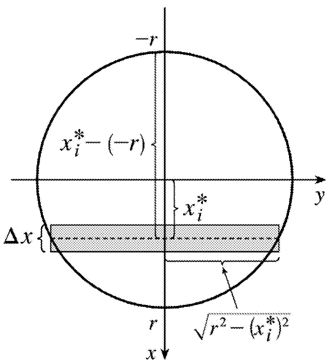
$$\begin{aligned} F &= \lim_{n\to\infty}\sum_{i=1}^{n}\rho g x_i^*\left[a-\frac{x_i^*}{h}(a-b)\right]\Delta x=\int_0^h \rho g x\left[a-\frac{x}{h}(a-b)\right]dx \\ &= \rho g a\int_0^h x\,dx+\frac{\rho g(b-a)}{h}\int_0^h x^2\,dx=\rho g a\frac{h^2}{2}+\rho g\frac{b-a}{h}\frac{h^3}{3} \\ &= \rho g h^2\left(\frac{a}{2}+\frac{b-a}{3}\right)=\rho g h^2\frac{a+2b}{6}\approx\frac{500}{3}gh^2(a+2b)\text{N} \end{aligned}$$

9. From the figure, the area of the i th rectangular strip is $2\sqrt{r^2-\left(x_i^*\right)^2}\,\Delta x$ and the pressure on it is $\rho g\left(x_i^*+r\right)$.

$$\begin{aligned} F &= \lim_{n\to\infty}\sum_{i=1}^{n}\rho g\left(x_i^*+r\right)2\sqrt{r^2-\left(x_i^*\right)^2}\,\Delta x \\ &= \int_{-r}^{r}\rho g(x+r)\cdot 2\sqrt{r^2-x^2}\,dx \end{aligned}$$

$$= \rho g\int_{-r}^{r}\sqrt{r^2-x^2}\,2x\,dx+2\rho gr\int_{-r}^{r}\sqrt{r^2-x^2}\,dx$$

The first integral is 0 because the integrand is an odd function. The second integral can be interpreted as the area of a semicircular disk with radius r , or we could make the trigonometric substitution $x=r\sin\theta$. Continuing: $F=\rho g\cdot 0+2\rho gr\cdot \frac{1}{2}\pi r^2=\rho g\pi r^3=1000g\pi r^3$ N (SI units assumed).



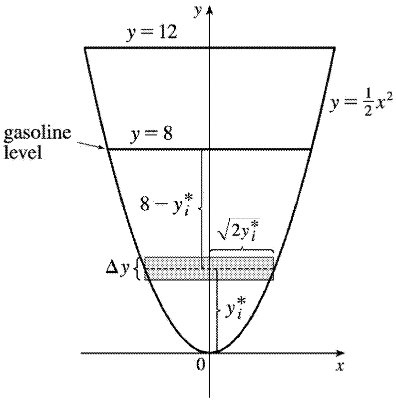
10. The area of the i th rectangular strip is $2\sqrt{2y_i^*}\,\Delta y$ and the pressure on it is $\delta d_i=\delta\left(8-y_i^*\right)$.

$$F = \int_0^8\delta(8-y)2\sqrt{2y}\,dy=42\cdot 2\cdot\sqrt{2}\int_0^8(8-y)y^{1/2}dy$$

$$= 84\sqrt{2}\int_0^8\left(8y^{1/2}-y^{3/2}\right)dy=84\sqrt{2}\left[8\cdot\frac{2}{3}y^{3/2}-\frac{2}{5}y^{5/2}\right]_0^8$$

$$= 84\sqrt{2}\left[8\cdot\frac{2}{3}\cdot 16\sqrt{2}-\frac{2}{5}\cdot 128\sqrt{2}\right]$$

$$= 84\sqrt{2}\cdot 256\sqrt{2}\left(\frac{1}{3}-\frac{1}{5}\right)=43{,}008\cdot\frac{2}{15}=5734.4\text{lb}$$

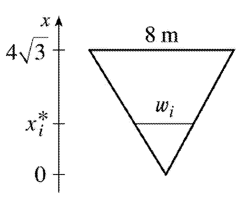


11. By similar triangles, $\dfrac{8}{4\sqrt{3}} = \dfrac{w_i}{x_i^*} \Rightarrow w_i = \dfrac{2x_i^*}{\sqrt{3}}$. The area of the i th rectangular strip is $\dfrac{2x_i^*}{\sqrt{3}}\,\Delta x$ and the pressure on it is $\rho g\left(4\sqrt{3}-x_i^*\right)$.

$$F = \int_0^{4\sqrt{3}} \rho g\left(4\sqrt{3}-x\right)\frac{2x}{\sqrt{3}}\,dx = 8\rho g\int_0^{4\sqrt{3}} x\,dx - \frac{2\rho g}{\sqrt{3}}\int_0^{4\sqrt{3}} x^2\,dx$$

$$= 4\rho g\left[x^2\right]_0^{4\sqrt{3}} - \frac{2\rho g}{3\sqrt{3}}\left[x^3\right]_0^{4\sqrt{3}} = 192\rho g - \frac{2\rho g}{3\sqrt{3}}\,64\cdot 3\sqrt{3}$$

$$= 192\rho g - 128\rho g = 64\rho g \approx 64(840)(9.8) \approx 5.27\times 10^5\ \text{N}$$



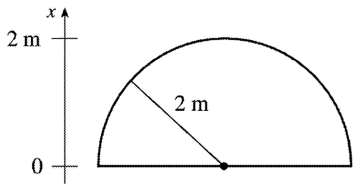
12.

$$F = \int_0^2 \rho g(10-x)2\sqrt{4-x^2}\,dx$$

$$= 20\rho g\int_0^2 \sqrt{4-x^2}\,dx - \rho g\int_0^2 \sqrt{4-x^2}\,2x\,dx$$

$$= 20\rho g\,\frac{1}{4}\pi\left(2^2\right) - \rho g\int_0^4 u^{1/2}\,du \quad [\,u = 4-x^2,\ du = -2x\,dx\,]$$

$$=20\pi\rho g-\frac{2}{3}\rho g\left[u^{3/2}\right]_0^4=20\pi\rho g-\frac{16}{3}\rho g=\rho g\left(20\pi-\frac{16}{3}\right)$$
$$=(1000)(9.8)\left(20\pi-\frac{16}{3}\right)\approx 5.63\times 10^5\text{ N}$$



13. **(a)** The top of the cube has depth $d=1$ m -20 cm $=80$ cm $=0.8$ m.

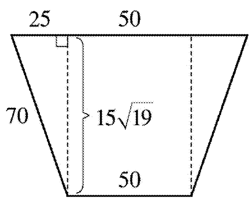
$$F=\rho gdA\approx(1000)(9.8)(0.8)(0.2)^2=313.6\approx 314\text{N}$$

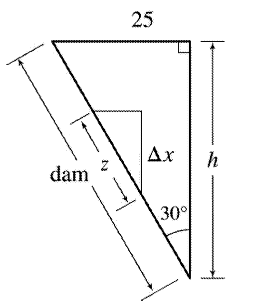
(b) The area of a strip is $0.2\,\Delta x$ and the pressure on it is $\rho g x_i^*$.

$$F=\int_{0.8}^{1}\rho gx(0.2)\,dx=0.2\rho g\left[\frac{1}{2}x^2\right]_{0.8}^{1}=(0.2\rho g)(0.18)=0.036\rho g=0.036(1000)(9.8)$$
$$=352.8\approx 353\text{N}$$

14. The height of the dam is $h=\sqrt{70^2-25^2}\cos 30^\circ=15\sqrt{19}\left(\frac{\sqrt{3}}{2}\right)$. From the solution for Exercise 8, the width of the trapezoid is $100-\frac{x}{h}(100-50)=100-\frac{50x}{h}$. From the small triangle in the second figure, $\cos 30^\circ=\frac{\Delta x}{z}\Rightarrow z=\Delta x\sec 30^\circ=2\,\Delta x/\sqrt{3}$.

$$F=\int_0^h\delta x\left(100-\frac{50x}{h}\right)\frac{2}{\sqrt{3}}\,dx=\frac{200\delta}{\sqrt{3}}\int_0^h x\,dx-\frac{100\delta}{h\sqrt{3}}\int_0^h x^2\,dx$$
$$=\frac{200\delta}{\sqrt{3}}\frac{h^2}{2}-\frac{100\delta}{h\sqrt{3}}\frac{h^3}{3}=\frac{200\delta h^2}{3\sqrt{3}}=\frac{200(62.5)}{3\sqrt{3}}\cdot\frac{12,825}{4}$$
$$\approx 7.71\times 10^6\text{lb}$$

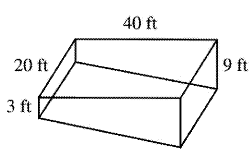




15. **(a)** The area of a strip is $20\Delta x$ and the pressure on it is δx_i .

$$F = \int_0^3 \delta x 20\, dx = 20\delta \left[\frac{1}{2} x^2 \right]_0^3 = 20\delta \cdot \frac{9}{2} = 90\delta$$

$$= 90(62.5) = 5625 \text{ lb} \approx 5.63 \times 10^3 \text{ lb}$$



(b) $F = \int_0^9 \delta x 20\, dx = 20\delta \left[\frac{1}{2} x^2 \right]_0^9 = 20\delta \cdot \frac{81}{2} = 810\delta = 810(62.5) = 50{,}625 \text{ lb} \approx 5.06 \times 10^4$ lb.

(c) For the first 3 ft, the length of the side is constant at 40 ft. For $3 < x \leq 9$, we can use similar triangles to find the length a : $\frac{a}{40} = \frac{9-x}{6} \Rightarrow a = 40 \cdot \frac{9-x}{6}$.

$$F = \int_0^3 \delta x 40\, dx + \int_3^9 \delta x (40) \frac{9-x}{6}\, dx = 40\delta \left[\frac{1}{2} x^2 \right]_0^3 + \frac{20}{3} \delta \int_3^9 \left(9x - x^2\right) dx$$

$$= 180\delta + \frac{20}{3} \delta \left[\frac{9}{2} x^2 - \frac{1}{3} x^3 \right]_3^9 = 180\delta + \frac{20}{3} \delta \left[\left(\frac{729}{2} - 243 \right) - \left(\frac{81}{2} - 9 \right) \right]$$

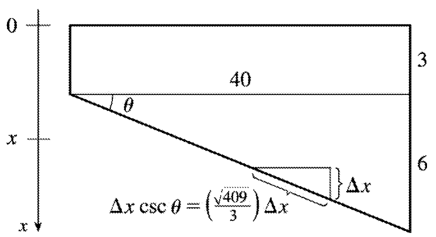
$$= 180\delta + 600\delta = 780\delta = 780(62.5) = 48{,}750 \text{ lb} \approx 4.88 \times 10^4 \text{ lb}$$

(d) For any right triangle with hypotenuse on the bottom, $\csc\theta = \dfrac{\Delta x}{\text{hypotenuse}} \Rightarrow$ hypotenuse $=\Delta x \csc\theta = \Delta x \dfrac{\sqrt{40^2+6^2}}{6} = \dfrac{\sqrt{409}}{3}\ \Delta x$.

$$F = \int_3^9 \delta x 20\,\frac{\sqrt{409}}{3}\,dx = \frac{1}{3}\left(20\sqrt{409}\right)\delta\left[\frac{1}{2}x^2\right]_3^9$$

$$= \frac{1}{3}\cdot 10\sqrt{409}\,\delta(81-9)$$

$$\approx 303{,}356\text{lb} \approx 3.03\times 10^5\text{lb}$$



16. Partition the interval $[a,b]$ by points x_i as usual and choose $x_i^* \in [x_{i-1}, x_i]$ for each i . The i th horizontal strip of the immersed plate is approximated by a rectangle of height Δx_i and width $w\left(x_i^*\right)$, so its area is $A_i \approx w\left(x_i^*\right)\Delta x_i$. For small Δx_i , the pressure P_i on the i th strip is almost constant and $P_i \approx \rho g x_i^*$ by Equation 1. The hydrostatic force F_i acting on the i th strip is $F_i = P_i A_i \approx \rho g x_i^* w\left(x_i^*\right)\Delta x_i$. Adding these forces and taking the limit as $n \to \infty$, we obtain the hydrostatic force on the immersed plate:

$$F = \lim_{n\to\infty} \sum_{i=1}^n F_i = \lim_{n\to\infty} \sum_{i=1}^n \rho g x_i^* w\left(x_i^*\right)\Delta x_i = \int_a^b \rho g x w(x)\,dx$$

17. $F = \int_2^5 \rho g x \cdot w(x)\,dx$, where $w(x)$ is the width of the plate at depth x . Since $n=6$, $\Delta x = \dfrac{5-2}{6} = \dfrac{1}{2}$, and

$$F \approx S_6 = \rho g\cdot \frac{1/2}{3}\,[2\cdot w(2) + 4\cdot 2.5\cdot w(2.5) + 2\cdot 3\cdot w(3) + 4\cdot 3.5\cdot w(3.5)]$$

$$+2\cdot 4\cdot w(4) + 4\cdot 4.5\cdot w(4.5) + 5\cdot w(5)]$$

$= \frac{1}{6}\rho g(2\cdot 0+10\cdot 0.8+6\cdot 1.7+14\cdot 2.4+8\cdot 2.9+18\cdot 3.3+5\cdot 3.6)$

$= \frac{1}{6}(1000)(9.8)(152.4)\approx 2.5\times 10^5\text{N}$

18. **(a)** From Equation 8, $\bar{x}=\frac{1}{A}\int_a^b xw(x)dx \Rightarrow A\bar{x}=\int_a^b xw(x)dx \Rightarrow \rho gA\bar{x}=\rho g\int_a^b xw(x)dx \Rightarrow$

$(\rho g\bar{x})A=\int_a^b \rho gxw(x)dx=F$ by Exercise 16.

(b) The centroid of a circle is its center. In this case, the center is at a depth of r meters, so $\bar{x}=r$. Thus, $F=(\rho g\bar{x})A=(\rho gr)(\pi r^2)=\rho g\pi r^3$.

19. The moment M of the system about the origin is $M=\sum_{i=1}^{2} m_i x_i=m_1x_1+m_2x_2=40\cdot 2+30\cdot 5=230$. The mass m of the system is $m=\sum_{i=1}^{2} m_i=m_1+m_2=40+30=70$. The center of mass of the system is $M/m=\frac{230}{70}=\frac{23}{7}$.

20. $M=m_1x_1+m_2x_2+m_3x_3=25(-2)+20(3)+10(7)=80$;
$\bar{x}=M/(m_1+m_2+m_3)=\frac{80}{55}=\frac{16}{11}$.

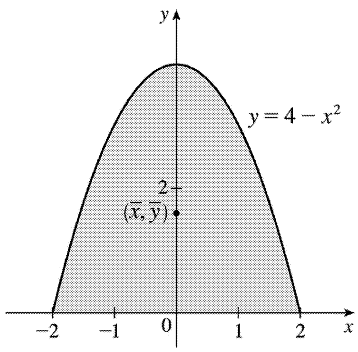
21. $m=\sum_{i=1}^{3} m_i=6+5+10=21$. $M_x=\sum_{i=1}^{3} m_iy_i=6(5)+5(-2)+10(-1)=10$;
$M_y=\sum_{i=1}^{3} m_ix_i=6(1)+5(3)+10(-2)=1$. $\bar{x}=\frac{M_y}{m}=\frac{1}{21}$ and $\bar{y}=\frac{M_x}{m}=\frac{10}{21}$, so the center of mass of the system is $\left(\frac{1}{21},\frac{10}{21}\right)$.

22. $M_x=\sum_{i=1}^{4} m_iy_i=6(-2)+5(4)+1(-7)+4(-1)=-3$, $M_y=\sum_{i=1}^{4} m_ix_i=6(1)+5(3)+1(-3)+4(6)=42$, and $m=\sum_{i=1}^{4} m_i=16$, so $\bar{x}=\frac{M_y}{m}=\frac{42}{16}=\frac{21}{8}$ and $\bar{y}=\frac{M_x}{m}=-\frac{3}{16}$; the center of mass is $(\bar{x},\bar{y})=\left(\frac{21}{8},-\frac{3}{16}\right)$.

23. Since the region in the figure is symmetric about the y –axis, we know that $\bar{x}=0$. The region is ‘‘bottom–heavy,’’ so we know that $\bar{y}<2$, and we might guess that $\bar{y}=1.5$.

$$A=\int_{-2}^{2}(4-x^2)dx=2\int_{0}^{2}(4-x^2)dx=2\left[4x-\frac{1}{3}x^3\right]_0^2$$
$$=2\left(8-\frac{8}{3}\right)=\frac{32}{3}$$

$\bar{x}=\frac{1}{A}\int_{-2}^{2}x(4-x^2)dx=0$ since $f(x)=x(4-x^2)$ is an odd function (or since the region is symmetric about the y-axis).



$$\bar{y}=\frac{1}{A}\int_{-2}^{2}\frac{1}{2}(4-x^2)^2dx=\frac{3}{32}\cdot\frac{1}{2}\cdot2\int_{0}^{2}(16-8x^2+x^4)dx=\frac{3}{32}\left[16x-\frac{8}{3}x^3+\frac{1}{5}x^5\right]_0^2$$
$$=\frac{3}{32}\left(32-\frac{64}{3}+\frac{32}{5}\right)=3\left(1-\frac{2}{3}+\frac{1}{5}\right)=3\left(\frac{8}{15}\right)=\frac{8}{5}$$

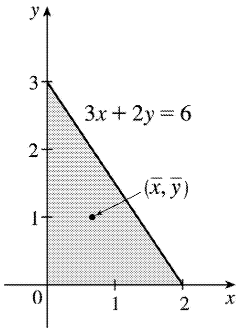
Thus, the centroid is $(\bar{x},\bar{y})=\left(0,\frac{8}{5}\right)$.

24. The region in the figure is ‘‘left–heavy’’ and ‘‘bottom–heavy,’’ so we know $\bar{x}<1$ and $\bar{y}<1.5$, and we might guess that $\bar{x}=0.7$ and $\bar{y}=1.2$.

$3x+2y=6\Leftrightarrow 2y=6-3x\Leftrightarrow y=3-\frac{3}{2}x$.

$$A=\int_0^2\left(3-\frac{3}{2}x\right)dx=\left[3x-\frac{3}{4}x^2\right]_0^2=6-3=3.$$

$$\bar{x}=\frac{1}{A}\int_0^2 x\left(3-\frac{3}{2}x\right)dx=\frac{1}{3}\int_0^2\left(3x-\frac{3}{2}x^2\right)dx$$
$$=\frac{1}{3}\left[\frac{3}{2}x^2-\frac{1}{2}x^3\right]_0^2=\frac{1}{3}(6-4)=\frac{2}{3};$$



$$\bar{y} = \frac{1}{A}\int_0^2 \frac{1}{2}\left(3 - \frac{3}{2}x\right)^2 dx = \frac{1}{3}\cdot\frac{1}{2}\int_0^2\left(9 - 9x + \frac{9}{4}x^2\right)dx = \frac{1}{6}\left[9x - \frac{9}{2}x^2 + \frac{3}{4}x^3\right]_0^2 = \frac{1}{6}(18 - 18 + 6) = 1 .$$

Thus, the centroid is $(\bar{x}, \bar{y}) = \left(\frac{2}{3}, 1\right)$.

25. The region in the figure is "right–heavy'' and "bottom–heavy,'' so we know $\bar{x} > 0.5$ and $\bar{y} < 1$, and we might guess that $\bar{x} = 0.6$ and $\bar{y} = 0.9$.

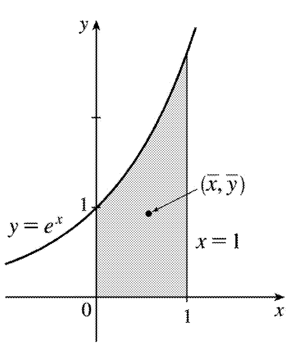
$$A = \int_0^1 e^x dx = \left[e^x\right]_0^1 = e - 1 ,$$

$$\bar{x} = \frac{1}{A}\int_0^1 xe^x dx = \frac{1}{e-1}\left[xe^x - e^x\right]_0^1 \text{ [by parts]}$$

$$= \frac{1}{e-1}\left[0 - (-1)\right] = \frac{1}{e-1} ,$$

$$\bar{y} = \frac{1}{A}\int_0^1 \frac{1}{2}\left(e^x\right)^2 dx = \frac{1}{e-1}\cdot\frac{1}{4}\left[e^{2x}\right]_0^1 = \frac{1}{4(e-1)}\left(e^2 - 1\right) = \frac{e+1}{4}$$. Thus, the centroid is

$$(\bar{x}, \bar{y}) = \left(\frac{1}{e-1}, \frac{e+1}{4}\right) \approx (0.58, 0.93) .$$



26. The region in the figure is "left–heavy'' and "bottom–heavy,'' so we know $\bar{x} < 1.5$ and $\bar{y} < 0.5$, and we might guess that

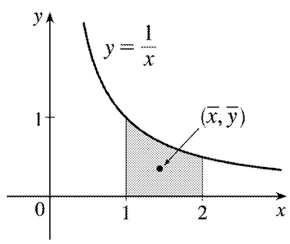
$\bar{x}=1.4$ and $\bar{y}=0.4$.

$A=\int_1^2 \frac{1}{x}\,dx=[\ln x]_1^2=\ln 2$, $\bar{x}=\frac{1}{A}\int_1^2 x\cdot\frac{1}{x}\,dx=\frac{1}{A}[x]_1^2=\frac{1}{A}=\frac{1}{\ln 2}$,

$\bar{y}=\frac{1}{A}\int_1^2 \frac{1}{2}\left(\frac{1}{x}\right)^2 dx=\frac{1}{2A}\int_1^2 x^{-2}\,dx=\frac{1}{2A}\left[-\frac{1}{x}\right]_1^2$

$=\frac{1}{2\ln 2}\left(-\frac{1}{2}+1\right)=\frac{1}{4\ln 2}$.

Thus, the centroid is $(\bar{x},\bar{y})=\left(\frac{1}{\ln 2},\frac{1}{4\ln 2}\right)\approx(1.44,0.36)$.



27. $A=\int_0^1(\sqrt{x}-x)\,dx=\left[\frac{2}{3}x^{3/2}-\frac{1}{2}x^2\right]_0^1=\frac{2}{3}-\frac{1}{2}=\frac{1}{6}$.

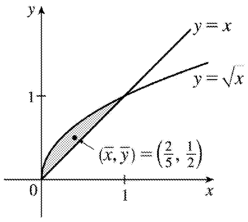
$\bar{x}=\frac{1}{A}\int_0^1 x(\sqrt{x}-x)\,dx=6\int_0^1(x^{3/2}-x^2)\,dx$

$=6\left[\frac{2}{5}x^{5/2}-\frac{1}{3}x^3\right]_0^1=6\left(\frac{2}{5}-\frac{1}{3}\right)=6\left(\frac{1}{15}\right)=\frac{2}{5}$;

$\bar{y}=\frac{1}{A}\int_0^1\frac{1}{2}\left[\left(\sqrt{x}\right)^2-x^2\right]dx=6\cdot\frac{1}{2}\int_0^1(x-x^2)\,dx$

$=3\left[\frac{1}{2}x^2-\frac{1}{3}x^3\right]_0^1=3\left(\frac{1}{2}-\frac{1}{3}\right)=\frac{1}{2}$.

Thus, the centroid is $(\bar{x},\bar{y})=\left(\frac{2}{5},\frac{1}{2}\right)$.



28.

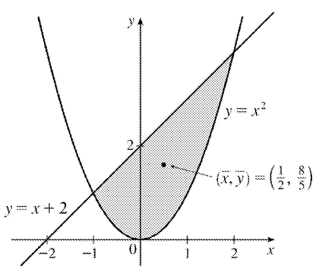
$$A=\int_{-1}^{2}(x+2-x^2)dx=\left[\frac{1}{2}x^2+2x-\frac{1}{3}x^3\right]_{-1}^{2}$$

$$=\left(2+4-\frac{8}{3}\right)-\left(\frac{1}{2}-2+\frac{1}{3}\right)=\frac{9}{2}.$$

$$\bar{x}=\frac{1}{A}\int_{-1}^{2}x(x+2-x^2)dx=\frac{2}{9}\int_{-1}^{2}\left(x^2+2x-x^3\right)dx$$

$$=\frac{2}{9}\left[\frac{1}{3}x^3+x^2-\frac{1}{4}x^4\right]_{-1}^{2}$$

$$=\frac{2}{9}\left[\left(\frac{8}{3}+4-4\right)-\left(-\frac{1}{3}+1-\frac{1}{4}\right)\right]=\frac{2}{9}\cdot\frac{9}{4}=\frac{1}{2}\;;$$



$$\bar{y}=\frac{1}{A}\int_{-1}^{2}\frac{1}{2}\left[(x+2)^2-(x^2)^2\right]dx=\frac{2}{9}\cdot\frac{1}{2}\int_{-1}^{2}\left(x^2+4x+4-x^4\right)dx=\frac{1}{9}\left[\frac{1}{3}x^3+2x^2+4x-\frac{1}{5}x^5\right]_{-1}^{2}$$

$$=\frac{1}{9}\left[\left(\frac{8}{3}+8+8-\frac{32}{5}\right)-\left(-\frac{1}{3}+2-4+\frac{1}{5}\right)\right]=\frac{1}{9}\left(18+\frac{9}{3}-\frac{33}{5}\right)=\frac{1}{9}\cdot\frac{72}{5}=\frac{8}{5}\;.$$

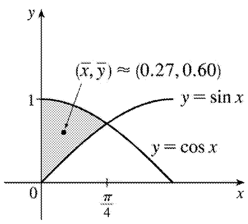
Thus, the centroid is $(\bar{x},\bar{y})=\left(\frac{1}{2},\frac{8}{5}\right)$.

29. $A=\int_{0}^{\pi/4}(\cos x-\sin x)dx=[\sin x+\cos x]_{0}^{\pi/4}=\sqrt{2}-1$,

$$\overline{x} = A^{-1}\int_0^{\pi/4} x(\cos x - \sin x)\,dx$$

$$= A^{-1}\left[x(\sin x + \cos x) + \cos x - \sin x\right]_0^{\pi/4} \text{ [integration by parts]}$$

$$= A^{-1}\left(\frac{\pi}{4}\sqrt{2} - 1\right) = \frac{\frac{1}{4}\pi\sqrt{2} - 1}{\sqrt{2} - 1}$$



$$\overline{y} = A^{-1}\int_0^{\pi/4} \frac{1}{2}\left(\cos^2 x - \sin^2 x\right)dx = \frac{1}{2A}\int_0^{\pi/4} \cos 2x\,dx = \frac{1}{4A}\left[\sin 2x\right]_0^{\pi/4} = \frac{1}{4A} = \frac{1}{4\left(\sqrt{2} - 1\right)}$$

Thus, the centroid is $\left(\overline{x}, \overline{y}\right) = \left(\frac{\pi\sqrt{2} - 4}{4\left(\sqrt{2} - 1\right)}, \frac{1}{4\left(\sqrt{2} - 1\right)}\right) \approx (0.27, 0.60)$.

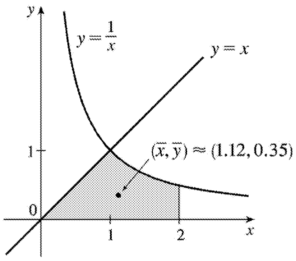
30. $A = \int_0^1 x\,dx + \int_1^2 \frac{1}{x}\,dx = \left[\frac{1}{2}x^2\right]_0^1 + \left[\ln x\right]_1^2 = \frac{1}{2} + \ln 2$,

$$\overline{x} = \frac{1}{A}\left[\int_0^1 x^2\,dx + \int_1^2 1\,dx\right] = \frac{1}{A}\left(\left[\frac{1}{3}x^3\right]_0^1 + \left[x\right]_1^2\right)$$

$$= \frac{1}{A}\left(\frac{1}{3} + 1\right) = \frac{2}{1 + 2\ln 2}\cdot\frac{4}{3} = \frac{8}{3(1 + 2\ln 2)} ,$$

$$\overline{y} = \frac{1}{A}\left[\int_0^1 \frac{1}{2}x^2\,dx + \int_1^2 \frac{1}{2x^2}\,dx\right] = \frac{1}{2A}\left(\left[\frac{1}{3}x^3\right]_0^1 + \left[-\frac{1}{x}\right]_1^2\right)$$

$$= \frac{1}{2A}\left(\frac{1}{3} + \frac{1}{2}\right) = \frac{5}{12A} = \frac{5}{6 + 12\ln 2} .$$

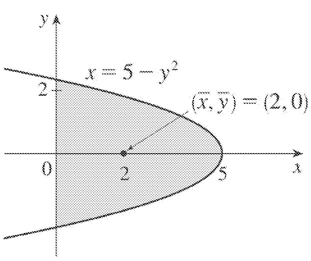


Thus, the centroid is $(\bar{x},\bar{y})=\left(\frac{8}{3(1+2\ln 2)},\frac{5}{6(1+2\ln 2)}\right)\approx(1.12,0.35)$. The principle used in this problem is stated after Example 3: the moment of the union of two nonoverlapping regions is the sum of the moments of the individual regions.

31. From the figure we see that $\bar{y}=0$. Now

$$A=\int_0^5 2\sqrt{5-x}\,dx=2\left[-\frac{2}{3}(5-x)^{3/2}\right]_0^5$$
$$=2\left(0+\frac{2}{3}\cdot 5^{3/2}\right)=\frac{20}{3}\sqrt{5}$$

so



$$\bar{x}=\frac{1}{A}\int_0^5 x\left[\sqrt{5-x}-\left(-\sqrt{5-x}\right)\right]dx=\frac{1}{A}\int_0^5 2x\sqrt{5-x}\,dx$$
$$=\frac{1}{A}\int_{\sqrt{5}}^0 2\left(5-u^2\right)u(-2u)\,du[u=\sqrt{5-x},x=5-u^2,u^2=5-x,dx=-2u\,du]$$
$$=\frac{4}{A}\int_0^{\sqrt{5}} u^2\left(5-u^2\right)du=\frac{4}{A}\left[\frac{5}{3}u^3-\frac{1}{5}u^5\right]_0^{\sqrt{5}}=\frac{3}{5\sqrt{5}}\left(\frac{25}{3}\sqrt{5}-5\sqrt{5}\right)=5-3=2$$

Thus, the centroid is $(\bar{x},\bar{y})=(2,0)$.

32. By symmetry, $M_y=0$ and $\bar{x}=0$;

$A=\frac{1}{2}\pi\cdot 1^2+4$, so $m=\rho A=5(\frac{\pi}{2}+4)=\frac{5}{2}(\pi+8)$;

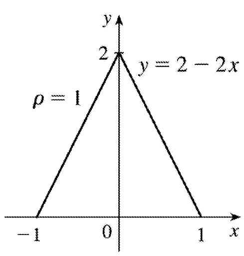
$M_x=\rho\cdot 2\int_0^1\frac{1}{2}[(\sqrt{1-x^2})^2-(-2)^2]dx=5\int_0^1\left(-x^2-3\right)dx=-5\left[\frac{1}{3}x^3+3x\right]_0^1=-5\cdot\frac{10}{3}=-\frac{50}{3}$;

$\bar{y}=\frac{1}{m}M_x=\frac{2}{5(\pi+8)}\cdot\frac{-50}{3}=-\frac{20}{3(\pi+8)}$. Thus, the centroid is $(\bar{x},\bar{y})=\left(0,\frac{-20}{3(\pi+8)}\right)$.

33. By symmetry, $M_y=0$ and $\bar{x}=0$. $A=\frac{1}{2}bh=\frac{1}{2}\cdot 2\cdot 2=2$.

$$\begin{aligned}
M_x &= \rho\int_{-1}^{1}\frac{1}{2}(2-2x)^2dx=2\rho\int_0^1\frac{1}{2}(2-2x)^2dx\\
&= \left(2\cdot 1\cdot\frac{1}{2}\cdot 2^2\right)\int_0^1(1-x)^2dx\\
&= 4\int_1^0 u^2(-du)[u=1-x,du=-dx]\\
&= -4\left[\frac{1}{3}u^3\right]_1^0=-4\left(-\frac{1}{3}\right)=\frac{4}{3}
\end{aligned}$$

$\bar{y}=\frac{1}{m}M_x=\frac{1}{\rho A}M_x=\frac{1}{1\cdot 2}\cdot\frac{4}{3}=\frac{2}{3}$. Thus, the centroid is $(\bar{x},\bar{y})=\left(0,\frac{2}{3}\right)$.



34. By symmetry about the line $y=x$, we expect that $\bar{x}=\bar{y}$. $A=\frac{1}{4}\pi r^2$, so $m=\rho A=2A=\frac{1}{2}\pi r^2$.

$M_x=\rho\int_0^r\frac{1}{2}\left(\sqrt{r^2-x^2}\right)^2dx=2\cdot\frac{1}{2}\int_0^r\left(r^2-x^2\right)dx=\left[r^2x-\frac{1}{3}x^3\right]_0^r=\frac{2}{3}r^3$.

$M_y=\rho\int_0^r x\sqrt{r^2-x^2}\,dx=\int_0^r\left(r^2-x^2\right)^{1/2}2x\,dx=\int_0^{r^2}u^{1/2}du=\left[\frac{2}{3}u^{3/2}\right]_0^{r^2}=\frac{2}{3}r^3$. $\bar{x}=\frac{1}{m}M_y=\frac{2}{\pi r^2}\left(\frac{2}{3}r^3\right)=\frac{4}{3\pi}r$

; $\bar{y}=\frac{1}{m}M_x=\frac{2}{\pi r^2}\left(\frac{2}{3}r^3\right)=\frac{4}{3\pi}r$. Thus, the centroid is $(\bar{x},\bar{y})=\left(\frac{4}{3\pi}r,\frac{4}{3\pi}r\right)$.

35.

$$A=\int_0^2\left(2^x-x^2\right)dx=\left[\frac{2^x}{\ln 2}-\frac{x^3}{3}\right]_0^2$$

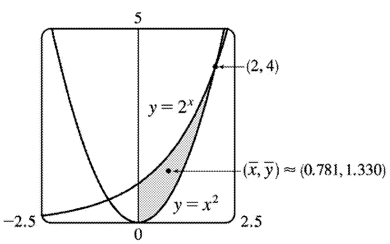
$$=\left(\frac{4}{\ln 2}-\frac{8}{3}\right)-\frac{1}{\ln 2}=\frac{3}{\ln 2}-\frac{8}{3}\approx 1.661418\ .$$

$$\overline{x}=\frac{1}{A}\int_0^2 x(2^x-x^2)dx=\frac{1}{A}\int_0^2(x2^x-x^3)dx$$

$$=\frac{1}{A}\left[\frac{x2^x}{\ln 2}-\frac{2^x}{(\ln 2)^2}-\frac{x^4}{4}\right]_0^2\text{[use parts]}$$

$$=\frac{1}{A}\left[\frac{8}{\ln 2}-\frac{4}{(\ln 2)^2}-4+\frac{1}{(\ln 2)^2}\right]$$

$$=\frac{1}{A}\left[\frac{8}{\ln 2}-\frac{3}{(\ln 2)^2}-4\right]\approx\frac{1}{A}(1.297453)\approx 0.781$$



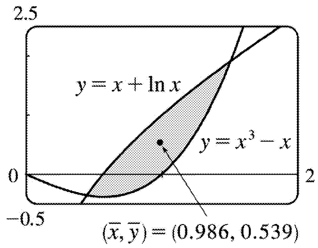
$$\overline{y}=\frac{1}{A}\int_0^2\frac{1}{2}\left[\left(2^x\right)^2-\left(x^2\right)^2\right]dx=\frac{1}{A}\int_0^2\frac{1}{2}(2^{2x}-x^4)dx=\frac{1}{A}\cdot\frac{1}{2}\left[\frac{2^{2x}}{2\ln 2}-\frac{x^5}{5}\right]_0^2$$

$$=\frac{1}{A}\cdot\frac{1}{2}\left(\frac{16}{2\ln 2}-\frac{32}{5}-\frac{1}{2\ln 2}\right)=\frac{1}{A}\left(\frac{15}{4\ln 2}-\frac{16}{5}\right)\approx\frac{1}{A}(2.210106)\approx 1.330$$

36. The curves $y=x+\ln x$ and $y=x^3-x$ intersect at $(a,c)\approx(0.447141,-0.357742)$ and $(b,d)\approx(1.507397,1.917782)$.

$$A=\int_a^b\left(x+\ln x-x^3+x\right)dx=\int_a^b\left(2x+\ln x-x^3\right)dx$$

$$=\left[x^2+x\ln x-x-\frac{1}{4}x^4\right]_a^b\approx 0.709781$$



$$\bar{x} = \frac{1}{A}\int_a^b x\left(2x+\ln x - x^3\right)dx = \frac{1}{A}\int_a^b \left(2x^2+x\ln x - x^4\right)dx$$

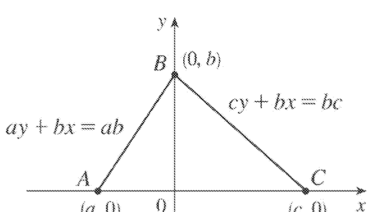
$$= \frac{1}{A}\left[\frac{2}{3}x^3+\frac{1}{4}x^2(2\ln x-1)-\frac{1}{5}x^5\right]_a^b \approx \frac{1}{A}(0.699489) \approx 0.985501$$

$$\bar{y} = \frac{1}{A}\int_a^b \frac{1}{2}\left[(x+\ln x)^2-\left(x^3-x\right)^2\right]dx = \frac{1}{2A}\int_a^b \left[2x\ln x+(\ln x)^2-x^6+2x^4\right]dx$$

$$= \frac{1}{2A}\left[x^2\ln x-\frac{1}{2}x^2+x(\ln x)^2-2x\ln x+2x-\frac{1}{7}x^7+\frac{2}{5}x^5\right]_a^b \approx \frac{1}{2A}(0.765092) \approx 0.538964$$

Thus, the centroid is $\left(\bar{x},\bar{y}\right) \approx (0.986, 0.539)$.

37. Choose x– and y– axes so that the base (one side of the triangle) lies along the x– axis with the other vertex along the positive y– axis as shown. From geometry, we know the medians intersect at a point $\frac{2}{3}$ of the way from each vertex (along the median) to the opposite side. The median from B goes to the midpoint $\left(\frac{1}{2}(a+c),0\right)$ of side AC , so the point of intersection of the medians is $\left(\frac{2}{3}\cdot\frac{1}{2}(a+c),\frac{1}{3}b\right)=\left(\frac{1}{3}(a+c),\frac{1}{3}b\right)$.



This can also be verified by finding the equations of two medians, and solving them simultaneously to find their point of intersection. Now let us compute the location of the centroid of the triangle. The area is $A=\frac{1}{2}(c-a)b$.

$$\bar{x} = \frac{1}{A}\left[\int_a^0 x\cdot\frac{b}{a}(a-x)\,dx+\int_0^c x\cdot\frac{b}{c}(c-x)dx\right] = \frac{1}{A}\left[\frac{b}{a}\int_a^0\left(ax-x^2\right)dx+\frac{b}{c}\int_0^c\left(cx-x^2\right)dx\right]$$

$$= \frac{b}{Aa}\left[\frac{1}{2}ax^2-\frac{1}{3}x^3\right]_a^0+\frac{b}{Ac}\left[\frac{1}{2}cx^2-\frac{1}{3}x^3\right]_0^c=\frac{b}{Aa}\left[-\frac{1}{2}a^3+\frac{1}{3}a^3\right]+\frac{b}{Ac}\left[\frac{1}{2}c^3-\frac{1}{3}c^3\right]$$

$$= \frac{2}{a(c-a)}\cdot\frac{-a^3}{6}+\frac{2}{c(c-a)}\cdot\frac{c^3}{6}=\frac{1}{3(c-a)}(c^2-a^2)=\frac{a+c}{3}$$

and

$$\bar{y} = \frac{1}{A}\left[\int_a^0\frac{1}{2}\left(\frac{b}{a}(a-x)\right)^2dx+\int_0^c\frac{1}{2}\left(\frac{b}{c}(c-x)\right)^2dx\right]$$

$$= \frac{1}{A}\left[\frac{b^2}{2a^2}\int_a^0\left(a^2-2ax+x^2\right)dx+\frac{b^2}{2c^2}\int_0^c\left(c^2-2cx+x^2\right)dx\right]$$

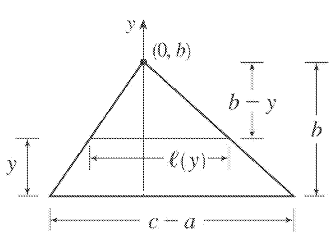
$$= \frac{1}{A}\left[\frac{b^2}{2a^2}\left[a^2x-ax^2+\frac{1}{3}x^3\right]_a^0+\frac{b^2}{2c^2}\left[c^2x-cx^2+\frac{1}{3}x^3\right]_0^c\right]$$

$$= \frac{1}{A}\left[\frac{b^2}{2a^2}\left(-a^3+a^3-\frac{1}{3}a^3\right)+\frac{b^2}{2c^2}\left(c^3-c^3+\frac{1}{3}c^3\right)\right]=\frac{1}{A}\left[\frac{b^2}{6}(-a+c)\right]=\frac{2}{(c-a)b}\cdot\frac{(c-a)b^2}{6}=\frac{b}{3}$$

Thus, the centroid is $(\bar{x},\bar{y})=\left(\frac{a+c}{3},\frac{b}{3}\right)$, as claimed.

Remarks: Actually the computation of $\bar{y}$ is all that is needed. By considering each side of the triangle in turn to be the base, we see that the centroid is $\frac{1}{3}$ of the way from each side to the opposite vertex and must therefore be the intersection of the medians.

The computation of $\bar{y}$ in this problem (and many others) can be simplified by using horizontal rather than vertical approximating rectangles. If the length of a thin rectangle at coordinate y is $\ell(y)$, then its area is $\ell(y)\Delta y$, its mass is $\rho\ell(y)\Delta y$, and its moment about the x– axis is $\Delta M_x=\rho y\ell(y)\Delta y$. Thus,



In this problem, $\ell(y)=\frac{c-a}{b}(b-y)$ by similar triangles, so

$$\bar{y}=\frac{1}{A}\int_0^b \frac{c-a}{b}\, y(b-y)dy=\frac{2}{b^2}\int_0^b\left(by-y^2\right)dy=\frac{2}{b^2}\left[\frac{1}{2}by^2-\frac{1}{3}y^3\right]_0^b=\frac{2}{b^2}\cdot\frac{b^3}{6}=\frac{b}{3}$$

Notice that only one integral is needed when this method is used.

38. Divide the lamina into three rectangles with masses 2 , 2 and 6 , with centroids $\left(-\frac{3}{2},1\right)$, $\left(0,\frac{1}{2}\right)$ and $\left(2,\frac{3}{2}\right)$, respectively. The total mass of the lamina is 10 . So, using Formulas 5, 6, and 7, we have

$\bar{x}=\frac{M_y}{m}=\frac{1}{m}\sum_{i=1}^3 m_i x_i=\frac{1}{10}\left[2\left(-\frac{3}{2}\right)+2(0)+6(2)\right]=\frac{1}{10}(9)$, and

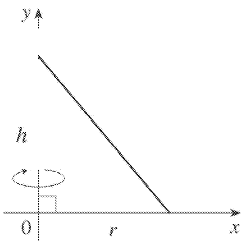
$\bar{y}=\frac{M_x}{m}=\frac{1}{m}\sum_{i=1}^3 m_i y_i=\frac{1}{10}\left[2(1)+2\left(\frac{1}{2}\right)+6\left(\frac{3}{2}\right)\right]=\frac{1}{10}(12)$.

Thus, the centroid is $(\bar{x},\bar{y})=\left(\frac{9}{10},\frac{6}{5}\right)$.

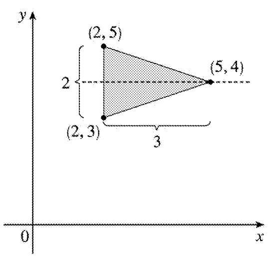
39. Divide the lamina into two triangles and one rectangle with respective masses of 2 , 2 and 4 , so that the total mass is 8 . Using the result of Exercise 37, the triangles have centroids $\left(-1,\frac{2}{3}\right)$ and $\left(1,\frac{2}{3}\right)$. The centroid of the rectangle (its center) is $\left(0,-\frac{1}{2}\right)$. So, using Formulas 5 and 7, we have $\bar{y}=\frac{M_x}{m}=\frac{1}{m}\sum_{i=1}^3 m_i y_i=\frac{1}{8}\left[2\left(\frac{2}{3}\right)+2\left(\frac{2}{3}\right)+4\left(-\frac{1}{2}\right)\right]=\frac{1}{8}\left(\frac{2}{3}\right)=\frac{1}{12}$, and $\bar{x}=0$, since the lamina is symmetric about the line $x=0$. Thus, the centroid is $(\bar{x},\bar{y})=\left(0,\frac{1}{12}\right)$.

40. A sphere can be generated by rotating a semicircle about its diameter. By Example 4, the center of mass travels a distance $2\pi\bar{y}=2\pi\left(\frac{4r}{3\pi}\right)=\frac{8r}{3}$, so by the Theorem of Pappus, the volume of the sphere is $V=Ad=\frac{\pi r^2}{2}\cdot\frac{8r}{3}=\frac{4}{3}\pi r^3$.

41. A cone of height h and radius r can be generated by rotating a right triangle about one of its legs as shown. By Exercise 37, $\bar{x}=\frac{1}{3}r$, so by the Theorem of Pappus, the volume of the cone is $V=Ad=\left(\frac{1}{2}\cdot\text{base}\cdot\text{height}\right)\cdot(2\pi\bar{x})=\frac{1}{2}rh\cdot 2\pi\left(\frac{1}{3}r\right)=\frac{1}{3}\pi r^2 h$.



42. From the symmetry in the figure, $\bar{y}=4$. So the distance traveled by the centroid when rotating the triangle about the x –axis is $d=2\pi\cdot 4=8\pi$. The area of the triangle is $A=\frac{1}{2}bh=\frac{1}{2}(2)(3)=3$. By the Theorem of Pappus, the volume of the resulting solid is $Ad=3(8\pi)=24\pi$.

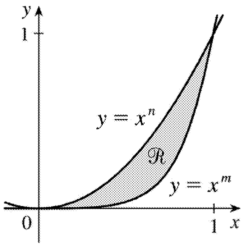


43. Suppose the region lies between two curves $y=f(x)$ and $y=g(x)$ where $f(x)\geq g(x)$, as illustrated in Figure 13. Choose points x_i with $a=x_0<x_1<\cdots<x_n=b$ and choose x_i^* to be the midpoint of the i th subinterval; that is, $x_i^*=\bar{x}_i=\frac{1}{2}\left(x_{i-1}+x_i\right)$. Then the centroid of the i th approximating rectangle R_i is its center $C_i=\left(\bar{x}_i, \frac{1}{2}\left[f\left(\bar{x}_i\right)+g\left(\bar{x}_i\right)\right]\right)$. Its area is $\left[f\left(\bar{x}_i\right)-g\left(\bar{x}_i\right)\right]\Delta x$, so its mass is $\rho\left[f\left(\bar{x}_i\right)-g\left(\bar{x}_i\right)\right]\Delta x$. Thus, $M_y\left(R_i\right)=\rho\left[f\left(\bar{x}_i\right)-g\left(\bar{x}_i\right)\right]\Delta x\cdot\bar{x}_i=\rho\bar{x}_i\left[f\left(\bar{x}_i\right)-g\left(\bar{x}_i\right)\right]\Delta x$ and $M_x\left(R_i\right)=\rho\left[f\left(\bar{x}_i\right)-g\left(\bar{x}_i\right)\right]\Delta x\cdot\frac{1}{2}\left[f\left(\bar{x}_i\right)+g\left(\bar{x}_i\right)\right]=\rho\cdot\frac{1}{2}\left[f\left(\bar{x}_i\right)^2-g\left(\bar{x}_i\right)^2\right]\Delta x$. Summing over i and taking the limit as $n\to\infty$, we get $M_y=\lim_{n\to\infty}\sum_i\rho\bar{x}_i\left[f\left(\bar{x}_i\right)-g\left(\bar{x}_i\right)\right]\Delta x=\rho\int_a^b x[f(x)-g(x)]\,dx$ and $M_x=\lim_{n\to\infty}\sum_i\rho\cdot\frac{1}{2}\left[f\left(\bar{x}_i\right)^2-g\left(\bar{x}_i\right)^2\right]\Delta x=\rho\int_a^b\frac{1}{2}\left[f(x)^2-g(x)^2\right]dx$. Thus,

$$\bar{x}=\frac{M_y}{m}=\frac{M_y}{\rho A}=\frac{1}{A}\int_a^b x[f(x)-g(x)]\,dx \text{ and } \bar{y}=\frac{M_x}{m}=\frac{M_x}{\rho A}=\frac{1}{A}\int_a^b\frac{1}{2}\left[f(x)^2-g(x)^2\right]dx$$

44. **(a)** Let $0\leq x\leq 1$. If $n<m$, then $x^n>x^m$; that is, raising x to a larger power produces a smaller

number.



(b) Using Formulas 9 and the fact that the area of is

$$A=\int_0^1 \left(x^n - x^m\right) dx = \frac{1}{n+1} - \frac{1}{m+1} = \frac{m-n}{(n+1)(m+1)}$$, we get

$$\bar{x} = \frac{(n+1)(m+1)}{m-n} \int_0^1 x\left[x^n - x^m\right] dx = \frac{(n+1)(m+1)}{m-n} \int_0^1 \left(x^{n+1} - x^{m+1}\right) dx$$

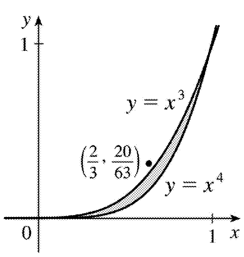
$$= \frac{(n+1)(m+1)}{m-n} \left[\frac{1}{n+2} - \frac{1}{m+2} \right] = \frac{(n+1)(m+1)}{(n+2)(m+2)} \quad and \quad \bar{y} = \frac{(n+1)(m+1)}{m-n} \int_0^1 \frac{1}{2} \left[\left(x^n\right)^2 - \left(x^m\right)^2 \right] dx = \frac{(n+1}{2(}$$

$$= \frac{(n+1)(m+1)}{2(m-n)} \left[\frac{1}{2n+1} - \frac{1}{2m+1} \right] = \frac{(n+1)(m+1)}{(2n+1)(2m+1)}$$

(c) If we take $n=3$ and $m=4$, then

$$\left(\bar{x},\bar{y}\right) = \left(\frac{4\cdot 5}{5\cdot 6}, \frac{4\cdot 5}{7\cdot 9} \right) = \left(\frac{2}{3}, \frac{20}{63} \right)$$

which lies outside R since $\left(\frac{2}{3} \right)^3 = \frac{8}{27} < \frac{20}{63}$. This is the simplest of many possibilities.



1. By the Net Change Theorem, $C(2000)-C(0)=\int_{0}^{2000} C'(x)\,dx \Rightarrow$

$$C(2000)=20,000+\int_{0}^{2000}(5-0.008x+0.000009x^2)\,dx=20,000+\Big[5x-0.004x^2+0.000003x^3\Big]_0^{2000}$$

$$=20,000+10,000-0.004(4,000,000)+0.000003(8,000,000,000)=30,000-16,000+24,000$$

$$=\$38,000$$

2. By the Net Change Theorem, $R(5000)-R(1000)=\int_{1000}^{5000} R'(x)\,dx \Rightarrow$

$$\text{R}(5000)=12{,}400+\int_{1000}^{5000}(12-0.0004\text{x})\,\text{dx}=12{,}400+\Big[12x-0.0002x^2\Big]_{1000}^{5000}$$

$$=12{,}400+(60{,}000-5{,}000)-(12{,}000-200)=\$55{,}600$$

3. If the production level is raised from 1200 units to 1600 units, then the increase in cost is

$$C(1600)-C(1200)=\int_{1200}^{1600} C'(x)\,dx=\int_{1200}^{1600}\left(74+1.1x-0.002x^2+0.00004x^3\right)dx$$

$$=\left[74x+0.55x^2-\frac{0.002}{3}x^3+0.00001x^4\right]_{1200}^{1600}$$

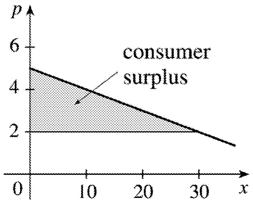
$$=64{,}331{,}733.33-20{,}464{,}800=\$43{,}866{,}933.33$$

4.

$$\text{Consumer surplus}=\int_0^{30}[p(x)-p(30)]\,dx$$

$$=\int_0^{30}\left[5-\frac{1}{10}x-\left(5-\frac{30}{10}\right)\right]dx$$

$$=\left[3x-\frac{1}{20}x^2\right]_0^{30}=90-45=\$45$$



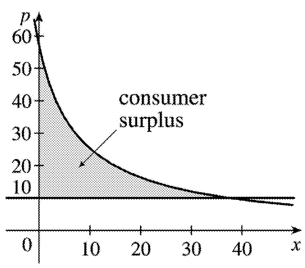
5. $p(x)=10 \Rightarrow \dfrac{450}{x+8}=10 \Rightarrow x+8=45 \Rightarrow x=37$.

$$\text{Consumer surplus} = \int_0^{37} [p(x)-10]\,dx = \int_0^{37} \left(\frac{450}{x+8} - 10 \right) dx$$

$$= \left[450 \ln (x+8) - 10x\right]_0^{37}$$

$$= (450 \ln 45 - 370) - 450 \ln 8$$

$$= 450 \ln \left(\frac{45}{8} \right) - 370 \approx \$407.25$$

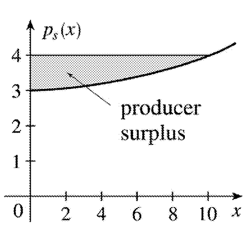


6. $p_S(x)=3+0.01x^2$. $P=p_S(10)=3+1=4$.

$$\text{Producer surplus} = \int_0^{10} [P-p_S(x)]\,dx$$

$$= \int_0^{10} \left[4-3-0.01x^2\right] dx = \left[x - \frac{0.01}{3} x^3 \right]_0^{10}$$

$$\approx 10-3.33=\$6.67$$



7. $P=p_S(x) \Rightarrow 400=200+0.2x^{3/2} \Rightarrow 200=0.2x^{3/2} \Rightarrow 1000=x^{3/2} \Rightarrow x=1000^{2/3}=100$.

$$\text{Producer surplus} = \int_0^{100} \left[P-p_S(x) \right] dx = \int_0^{100} \left[400-\left(200+0.2x^{3/2}\right) \right] dx = \int_0^{100} \left(200-\frac{1}{5}x^{3/2} \right) dx$$

$$= \left[200x-\frac{2}{25}x^{5/2} \right]_0^{100} = 20,000-8,000=\$12,000$$

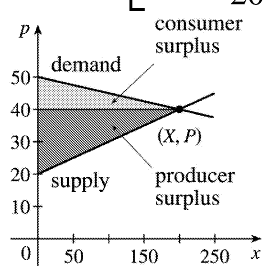
8. $p=50-\frac{1}{20}x$ and $p=20+\frac{1}{10}x$ intersect at $p=40$ and $x=200$.

$$\text{Consumer surplus} = \int_0^{200} \left(50-\frac{1}{20}x-40 \right) dx$$

$$= \left[10x-\frac{1}{40}x^2 \right]_0^{200} = \$1000$$

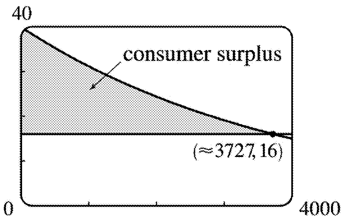
$$\text{Producer surplus} = \int_0^{200} \left(40-20-\frac{1}{10}x \right) dx$$

$$= \left[20x-\frac{1}{20}x^2 \right]_0^{200} = \$2000$$



9. $p(x)=\dfrac{800,000e^{-x/5000}}{x+20,000}=16 \Rightarrow x=x_1 \approx 3727.04$.

$$\text{Consumer surplus} = \int_0^{x_1} [p(x)-16]\, dx \approx \$37,753$$



10. The demand function is linear with slope $\frac{-0.5}{35}=-\frac{1}{70}$ and $p(400)=7.5$, so an equation is $p-7.5=-\frac{1}{70}(x-400)$ or $p=-\frac{1}{70}x+\frac{185}{14}$. A selling price of \$6 implies that $6=-\frac{1}{70}x+\frac{185}{14}\Rightarrow$ $\frac{1}{70}x=\frac{185}{14}-\frac{84}{14}=\frac{101}{14}\Rightarrow x=505$.

$$\text{Consumer surplus}=\int_0^{505}\left(-\frac{1}{70}x+\frac{185}{14}-6\right)dx=\left[-\frac{1}{140}x^2+\frac{101}{14}x\right]_0^{505}\approx\$1821.61$$

11. $f(8)-f(4)=\int_4^8 f'(t)dt=\int_4^8\sqrt{t}\,dt=\left[\frac{2}{3}t^{3/2}\right]_4^8=\frac{2}{3}\left(16\sqrt{2}-8\right)\approx\$9.75$ million

12.

$$n(9)-n(5)=\int_5^9\left(2200+10e^{0.8t}\right)dt=\left[2200t+\frac{10e^{0.8t}}{0.8}\right]_5^9=[2200t]_5^9+\frac{25}{2}\left[e^{0.8t}\right]_5^9$$

$$=2200(9-5)+12.5\left(e^{7.2}-e^{4}\right)\approx 24\text{ , }860$$

13. $F=\frac{\pi PR^4}{8\eta l}=\frac{\pi(4000)(0.008)^4}{8(0.027)(2)}\approx 1.19\times 10^{-4}\text{ cm}^3/\text{ s}$

14. If the flux remains constant, then $\frac{\pi P_0R_0^4}{8\eta l}=\frac{\pi PR^4}{8\eta l}\Rightarrow P_0R_0^4=PR^4\Rightarrow\frac{P}{P_0}=\left(\frac{R_0}{R}\right)^4$.

$R=\frac{3}{4}R_0\Rightarrow\frac{P}{P_0}=\left(\frac{R_0}{\frac{3}{4}R_0}\right)^4\Rightarrow P=P_0\left(\frac{4}{3}\right)^4\approx 3.1605P_0>3P_0$; that is, the blood pressure is more than tripled.

15.

$$\int_0^{12} c(t)\,dt = \int_0^{12} \frac{1}{4}\,t(12-t)\,dt = \int_0^{12} \left(3t - \frac{1}{4}t^2\right) dt = \left[\frac{3}{2}t^2 - \frac{1}{12}t^3\right]_0^{12} = (216-144) = 72 \text{ mg}\cdot\text{ s / L}.$$

Thus, the cardiac output is $F = \dfrac{A}{\int_0^{12} c(t)\,dt} = \dfrac{8\text{mg}}{72\text{mg}\cdot\text{ s/L}} = \dfrac{1}{9}\ \text{L / s} = \dfrac{60}{9}\ \text{L / min}.$

16. As in Example 2, we will estimate the cardiac output using Simpson's Rule with $\Delta t = 2$.

$$\begin{aligned}\int_0^{20} c(t)\,dt &\approx \frac{2}{3}\,[1(0)+4(2.4)+2(5.1)+4(7.8)+2(7.6) \\ &\quad +4(5.4)+2(3.9)+4(2.3)+2(1.6)+4(0.7)+1(0)] \\ &= \frac{2}{3}(110.8) \approx 73.87\text{mg}\cdot\text{ s/L}\end{aligned}$$

Therefore, $F \approx \dfrac{A}{73.87} = \dfrac{8}{73.87} \approx 0.1083$ L / s or 6.498 L / min.

1. **(a)** $\int_{30,000}^{40,000} f(x)dx$ is the probability that a randomly chosen tire will have a lifetime between 30 , 000 and 40 , 000 miles.

(b) $\int_{25,000}^{\infty} f(x)dx$ is the probability that a randomly chosen tire will have a lifetime of at least 25 , 000 miles.

2. **(a)** The probability that you drive to school in less than 15 minutes is $\int_0^{15} f(t)dt$.

(b) The probability that it takes you more than half an hour to get to school is $\int_{30}^{\infty} f(t)dt$.

3. **(a)** In general, we must satisfy the two conditions that are mentioned before Example 1 — namely, (1) $f(x)\geq 0$ for all x , and (2) $\int_{-\infty}^{\infty} f(x)dx=1$. For $0\leq x\leq 4$, we have $f(x)=\frac{3}{64}x\sqrt{16-x^2}\geq 0$, so $f(x)\geq 0$ for all x . Also,

$$\int_{-\infty}^{\infty} f(x)dx=\int_0^4 \frac{3}{64}x\sqrt{16-x^2}\,dx=-\frac{3}{128}\int_0^4 (16-x^2)^{1/2}(-2x)dx=-\frac{3}{128}\left[\frac{2}{3}(16-x^2)^{3/2}\right]_0^4$$

$$=-\frac{1}{64}\left[\left(16-x^2\right)^{3/2}\right]_0^4=-\frac{1}{64}(0-64)=1 .$$

Therefore, f is a probability density function.

(b)

$$P(X<2)=\int_{-\infty}^{2} f(x)dx=\int_0^2 \frac{3}{64}x\sqrt{16-x^2}\,dx=-\frac{3}{128}\int_0^2 (16-x^2)^{1/2}(-2x)dx$$

$$=-\frac{3}{128}\left[\frac{2}{3}(16-x^2)^{3/2}\right]_0^2=-\frac{1}{64}\left[(16-x^2)^{3/2}\right]_0^2=-\frac{1}{64}(12^{3/2}-16^{3/2})$$

$$=\frac{1}{64}\left(64-12\sqrt{12}\right)=\frac{1}{64}\left(64-24\sqrt{3}\right)=1-\frac{3}{8}\sqrt{3}\approx 0.350481$$

4. **(a)** For $0\leq x\leq 1$, we have $f(x)=kx^2(1-x)$, which is nonnegative if and only if $k\geq 0$. Also, $\int_{-\infty}^{\infty} f(x)dx=\int_0^1 kx^2(1-x)dx=k\int_0^1 (x^2-x^3)dx=k\left[\frac{1}{3}x^3-\frac{1}{4}x^4\right]_0^1=k/12$. Now $k/12=1\Leftrightarrow k=12$. Therefore, f is

a probability density function if and only if $k=12$.
(b) Let $k=12$.

$$P\left(X\geq \frac{1}{2}\right)=\int_{1/2}^{\infty} f(x)dx=\int_{1/2}^{1} 12x^2(1-x)dx=\int_{1/2}^{1}(12x^2-12x^3)dx=\Big[4x^3-3x^4\Big]_{1/2}^{1}$$

$$=(4-3)-\left(\frac{1}{2}-\frac{3}{16}\right)=1-\frac{5}{16}=\frac{11}{16}$$

(c) The mean

$$\mu=\int_{-\infty}^{\infty} xf(x)dx=\int_{0}^{1} x\cdot 12x^2(1-x)dx=12\int_{0}^{1}(x^3-x^4)dx=12\left[\frac{1}{4}x^4-\frac{1}{5}x^5\right]_0^1$$

$$=12\left(\frac{1}{4}-\frac{1}{5}\right)=\frac{12}{20}=\frac{3}{5}$$

5. **(a)** In general, we must satisfy the two conditions that are mentioned before Example 1 — namely, (1) $f(x)\geq 0$ for all x , and (2) $\int_{-\infty}^{\infty} f(x)dx=1$. Since $f(x)=0$ or $f(x)=0.1$, condition (1) is satisfied. For condition (2), we see that $\int_{-\infty}^{\infty} f(x)dx=\int_{0}^{10} 0.1\,dx=\left[\frac{1}{10}x\right]_0^{10}=1$. Thus, $f(x)$ is a probability density function for the spinner's values.
(b) Since all the numbers between 0 and 10 are equally likely to be selected, we expect the mean to be halfway between the endpoints of the interval; that is, $x=5$.

$\mu=\int_{-\infty}^{\infty} xf(x)dx=\int_{0}^{10} x(0.1)dx=\left[\frac{1}{20}x^2\right]_0^{10}=\frac{100}{20}=5$, as expected.

6. **(a)** As in the preceding exercise, (1) $f(x)\geq 0$ and (2) $\int_{-\infty}^{\infty} f(x)dx=\int_{0}^{10} f(x)dx=\frac{1}{2}(10)(0.2)=1$. So $f(x)$ is a probability density function.
(b)

(a) $$P(X<3)=\int_0^3 f(x)dx=\frac{1}{2}(3)(0.1)=\frac{3}{20}=0.15$$

(b) $$P(X<3)=\int_0^3 f(x)dx=\frac{1}{2}(3)(0.1)=\frac{3}{20}=0.15$$

(c) We first compute $P(X>8)$ and then subtract that value and our answer in (i) from 1 (the total probability). $P(X>8)=\int_{8}^{10} f(x)dx=\frac{1}{2}(2)(0.1)=\frac{2}{20}=0.10$. So $P(3\le X\le 8)=1-0.15-0.10=0.75$.

(d) We first compute $P(X>8)$ and then subtract that value and our answer in (i) from 1 (the total probability). $P(X>8)=\int_{8}^{10} f(x)dx=\frac{1}{2}(2)(0.1)=\frac{2}{20}=0.10$. So $P(3\le X\le 8)=1-0.15-0.10=0.75$.

(c) We find equations of the lines from $(0,0)$ to $(6,0.2)$ and from $(6,0.2)$ to $(10,0)$, and find that

$$f(x)=\begin{cases} \frac{1}{30}x & \text{if } 0\le x<6 \\ -\frac{1}{20}x+\frac{1}{2} & \text{if } 6\le x<10 \\ 0 & \text{otherwise} \end{cases}$$

$$\mu = \int_{-\infty}^{\infty} xf(x)dx=\int_{0}^{6} x\left(\frac{1}{30}x\right)dx+\int_{6}^{10} x\left(-\frac{1}{20}x+\frac{1}{2}\right)dx=\left[\frac{1}{90}x^3\right]_0^6+\left[-\frac{1}{60}x^3+\frac{1}{4}x^2\right]_6^{10}$$

$$= \frac{216}{90}+\left(-\frac{1000}{60}+\frac{100}{4}\right)-\left(-\frac{216}{60}+\frac{36}{4}\right)=\frac{16}{3}=5.\overline{3}$$

7. We need to find m so that $\int_{m}^{\infty} f(t)dt=\frac{1}{2}\Rightarrow \lim_{x\to\infty}\int_{m}^{x}\frac{1}{5}e^{-t/5}dt=\frac{1}{2}\Rightarrow \lim_{x\to\infty}\left[\frac{1}{5}(-5)e^{-t/5}\right]_m^x=\frac{1}{2}\Rightarrow$ $(-1)(0-e^{-m/5})=\frac{1}{2}\Rightarrow e^{-m/5}=\frac{1}{2}\Rightarrow -m/5=\ln\frac{1}{2}\Rightarrow m=-5\ln\frac{1}{2}=5\ln 2\approx 3.47$ min.

8. **(a)**

(a) $P(0\le X\le 200)=\int_{0}^{200}\frac{1}{1000}e^{-t/1000}dt=\left[-e^{-t/1000}\right]_0^{200}=-e^{-1/5}+1\approx 0.181$

(b) $P(0\le X\le 200)=\int_{0}^{200}\frac{1}{1000}e^{-t/1000}dt=\left[-e^{-t/1000}\right]_0^{200}=-e^{-1/5}+1\approx 0.181$

(c) $P(X>800)=\int_{800}^{\infty}\frac{1}{1000}e^{-t/1000}dt=\lim_{x\to\infty}\left[-e^{-t/1000}\right]_{800}^{x}=0+e^{-4/5}\approx 0.449$

(d) $P(X>800)=\int_{800}^{\infty}\frac{1}{1000}e^{-t/1000}dt=\lim_{x\to\infty}\left[-e^{-t/1000}\right]_{800}^{x}=0+e^{-4/5}\approx 0.449$

(b) We need to find m so that $\int_m^\infty f(t)\,dt=\frac{1}{2} \Rightarrow \lim_{x\to\infty}\int_m^x \frac{1}{1000}e^{-t/1000}dt=\frac{1}{2} \Rightarrow \lim_{x\to\infty}\left[-e^{-t/1000}\right]_m^x=\frac{1}{2} \Rightarrow$ $0+e^{-m/1000}=\frac{1}{2} \Rightarrow -m/1000=\ln\frac{1}{2} \Rightarrow m=-1000\ln\frac{1}{2}=1000\ln 2\approx 693.1$ h.

9. We use an exponential density function with $\mu=2.5$ min.

(a) $P(X>4)=\int_4^\infty f(t)\,dt=\lim_{x\to\infty}\int_4^x \frac{1}{2.5}e^{-t/2.5}dt=\lim_{x\to\infty}\left[-e^{-t/2.5}\right]_4^x=0+e^{-4/2.5}\approx 0.202$

(b) $P(0\le X\le 2)=\int_0^2 f(t)\,dt=\left[-e^{-t/2.5}\right]_0^2=-e^{-2/2.5}+1\approx 0.551$

(c) We need to find a value a so that $P(X\ge a)=0.02$, or, equivalently, $P(0\le X\le a)=0.98 \Leftrightarrow$ $\int_0^a f(t)\,dt=0.98 \Leftrightarrow \left[-e^{-t/2.5}\right]_0^a=0.98 \Leftrightarrow -e^{-a/2.5}+1=0.98 \Leftrightarrow e^{-a/2.5}=0.02 \Leftrightarrow -a/2.5=\ln 0.02 \Leftrightarrow$

$a=-2.5\ln\frac{1}{50}=2.5\ln 50\approx 9.78$ min ≈ 10 min. The ad should say that if you aren't served within 10 minutes, you get a free hamburger.

10. **(a)** With $\mu=69$ and $\sigma=2.8$, we have $P(65\le X\le 73)=\int_{65}^{73}\frac{1}{2.8\sqrt{2\pi}}\exp\left(-\frac{(x-69)^2}{2\cdot 2.8^2}\right)dx\approx 0.847$ (using a calculator or computer to estimate the integral).

(b) $P(X>6 \text{ feet })=P(X>72 \text{ inches })=1-P(0\le X\le 72)\approx 1-0.858=0.142$, so 14.2% of the adult male population is more than 6 feet tall.

11. $P(X\ge 10)=\int_{10}^\infty \frac{1}{4.2\sqrt{2\pi}}\exp\left(-\frac{(x-9.4)^2}{2\cdot 4.2^2}\right)dx$. To avoid the improper integral we approximate it by the integral from 10 to 100. Thus, $P(X\ge 10)\approx\int_{10}^{100}\frac{1}{4.2\sqrt{2\pi}}\exp\left(-\frac{(x-9.4)^2}{2\cdot 4.2^2}\right)dx\approx 0.443$ (using a calculator or computer to estimate the integral), so about 44 percent of the households throw out at least 10 lb of paper a week.

Note: We can't evaluate $1-P(0\le X\le 10)$ for this problem since a significant amount of area lies to the left of $X=0$.

12. **(a)**

$P(0\le X\le 480)=\int_0^{480} \frac{1}{12\sqrt{2\pi}} \exp\left(-\frac{(x-500)^2}{2\cdot 12^2}\right)dx\approx 0.0478$ (using a calculator or computer to estimate the integral), so there is about a 4.78% chance that a particular box contains less than 480 g of cereal.

(b) We need to find μ so that $P(0\le X<500)=0.05$. Using our calculator or computer to find $P(0\le X\le 500)$ for various values of μ , we find that if $\mu=519.73$, $P=0.05007$; and if $\mu=519.74$, $P=0.04998$. So a good target weight is at least 519.74 g.

13. $P(\mu-2\sigma\le X\le\mu+2\sigma)=\int_{\mu-2\sigma}^{\mu+2\sigma} \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)dx$. Substituting $t=\frac{x-\mu}{\sigma}$ and $dt=\frac{1}{\sigma}\,dx$ gives us

$$\int_{-2}^{2} \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\left(t^2/2\right)}(\sigma\, dt)=\frac{1}{\sqrt{2\pi}}\int_{-2}^{2} e^{-\left(t^2/2\right)}dt\approx 0.9545$$

14. Let $f(x)=\begin{cases} 0 & \text{if } x<0 \\ ce^{-cx} & \text{if } x\ge 0\end{cases}$ where $c=1/\mu$. By using parts, tables, or a CAS, we find that

(1): $\int xe^{bx}dx=\left(e^{bx}/b^2\right)(bx-1)$

(2): $\int x^2e^{bx}dx=\left(e^{bx}/b^3\right)\left(b^2x^2-2bx+2\right)$

Now

$$\begin{aligned}\sigma^2 &= \int_{-\infty}^{\infty} (x-\mu)^2 f(x)dx=\int_{-\infty}^{0} (x-\mu)^2 f(x)dx+\int_0^{\infty} (x-\mu)^2 f(x)dx \\ &= 0+\lim_{t\to\infty} c\int_0^t (x-\mu)^2 e^{-cx}dx=c\cdot\lim_{t\to\infty}\int_0^t \left(x^2e^{-cx}-2x\mu e^{-cx}+\mu^2e^{-cx}\right)dx\end{aligned}$$

Next we use (2) and (1) with $b=-c$ to get

$$\sigma^2=c\lim_{t\to\infty}\left[-\frac{e^{-cx}}{c^3}\left(c^2x^2+2cx+2\right)-2\mu\frac{e^{-cx}}{c^2}(-cx-1)+\mu^2\frac{e^{-cx}}{-c}\right]_0^t$$

Using l'Hospital's Rule several times, along with the fact that $\mu=1/c$, we get

$$\sigma^2 = c\left[0 - \left(-\frac{2}{c^3} + \frac{2}{c}\cdot\frac{1}{c^2} + \frac{1}{c^2}\cdot\frac{1}{-c}\right)\right] = c\left(\frac{1}{c^3}\right) = \frac{1}{c^2} \Rightarrow \sigma = \frac{1}{c} = \mu$$

15. **(a)** First $p(r) = \frac{4}{a_0^3} r^2 e^{-2r/a_0} \geq 0$ for $r \geq 0$. Next,

$$\int_{-\infty}^{\infty} p(r)\,dr = \int_0^{\infty} \frac{4}{a_0^3} r^2 e^{-2r/a_0}\,dr = \frac{4}{a_0^3} \lim_{t\to\infty} \int_0^t r^2 e^{-2r/a_0}\,dr$$

By using parts, tables, or a CAS , we find that $\int x^2 e^{bx}\,dx = (e^{bx}/b^3)(b^2x^2 - 2bx + 2)$. (*)

Next, we use (*) (with $b = -2/a_0$) and l'Hospital's Rule to get $\frac{4}{a_0^3}\left[\frac{a_0^3}{-8}(-2)\right] = 1$. This satisfies the second condition for a function to be a probability density function.

(b) Using l'Hospital's Rule, $\frac{4}{a_0^3} \lim_{r\to\infty} \frac{r^2}{e^{2r/a_0}} = \frac{4}{a_0^3} \lim_{r\to\infty} \frac{2r}{\left(2/a_0\right)e^{2r/a_0}} = \frac{2}{a_0^2} \lim_{r\to\infty} \frac{2}{\left(2/a_0\right)e^{2r/a_0}} = 0$.

To find the maximum of p , we differentiate:

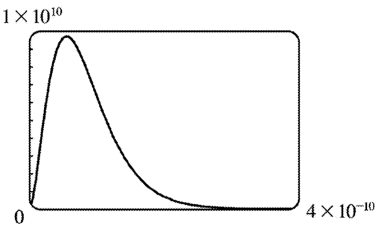
$$p'(r) = \frac{4}{a_0^3}\left[r^2 e^{-2r/a_0}\left(-\frac{2}{a_0}\right) + e^{-2r/a_0}(2r)\right] = \frac{4}{a_0^3} e^{-2r/a_0}(2r)\left(-\frac{r}{a_0} + 1\right)$$

$p'(r) = 0 \Leftrightarrow r = 0$ or $1 = \frac{r}{a_0} \Leftrightarrow r = a_0$. $p'(r)$ changes from positive to negative at $r = a_0$, so $p(r)$ has its maximum value at $r = a_0$.

(c) It is fairly difficult to find a viewing rectangle, but knowing the maximum value from part (b) helps.

$$p\left(a_0\right) = \frac{4}{a_0^3} a_0^2 e^{-2a_0/a_0} = \frac{4}{a_0} e^{-2} \approx 9{,}684{,}098{,}979$$

With a maximum of nearly 10 billion and a total area under the curve of 1 , we know that the ‘‘hump’’ in the graph must be extremely narrow.



(d) $P(r)=\int_0^r \frac{4}{a_0^3} s^2 e^{-2s/a_0} ds \Rightarrow P\left(4a_0\right)=\int_0^{4a_0} \frac{4}{a_0^3} s^2 e^{-2s/a_0} ds$. Using (*) from part (a) (with $b=-2/a_0$),

$$P\left(4a_0\right) = \frac{4}{a_0^3}\left[\frac{e^{-2s/a_0}}{-8/a_0^3}\left(\frac{4}{a_0^2}s^2+\frac{4}{a_0}s+2\right)\right]_0^{4a_0}=\frac{4}{a_0^3}\left(\frac{a_0^3}{-8}\right)\left[e^{-8}(64+16+2)-1(2)\right]$$

$$= -\frac{1}{2}\left(82e^{-8}-2\right)=1-41e^{-8}\approx 0.986$$

(e) $\mu=\int_{-\infty}^{\infty} rp(r)dr=\frac{4}{a_0^3}\lim_{t\to\infty}\int_0^t r^3 e^{-2r/a_0} dr$. Integrating by parts three times or using a CAS, we find that

$\int x^3 e^{bx} dx=\frac{e^{bx}}{b^4}\left(b^3x^3-3b^2x^2+6bx-6\right)$. So with $b=-\frac{2}{a_0}$, we use l’Hospital’s Rule, and get

$$\mu=\frac{4}{a_0^3}\left[-\frac{a_0^4}{16}(-6)\right]=\frac{3}{2}a_0 .$$

1. $y=x-x^{-1} \Rightarrow y'=1+x^{-2}$. To show that y is a solution of the differential equation, we will substitute the expressions for y and y' in the left–hand side of the equation and show that the left–hand side is equal to the right–hand side.

$$\text{LHS}=xy'+y=x\left(1+x^{-2}\right)+\left(x-x^{-1}\right)=x+x^{-1}+x-x^{-1}=2x=\text{RHS}$$

2. $y=\sin x\cos x-\cos x \Rightarrow y'=\sin x(-\sin x)+\cos x(\cos x)-(-\sin x)=\cos^2 x-\sin^2 x+\sin x$.

$$\begin{aligned}\text{LHS} &= y'+(\tan x)y=\cos^2 x-\sin^2 x+\sin x+(\tan x)(\sin x\cos x-\cos x)\\ &= \cos^2 x-\sin^2 x+\sin x+\sin^2 x-\sin x=\cos^2 x=\text{RHS},\end{aligned}$$

so y is a solution of the differential equation. Also, $y(0)=\sin 0\cos 0-\cos 0=0\cdot 1-1=-1$, so the initial condition is satisfied.

3. **(a)** $y=\sin kt \Rightarrow y'=k\cos kt \Rightarrow y''=-k^2\sin kt$. $y''+9y=0 \Rightarrow$
$-k^2\sin kt+9\sin kt=0$ for all $t \Leftrightarrow \left(9-k^2\right)\sin kt=0$ for all $t \Leftrightarrow 9-k^2=0 \Leftrightarrow k=\pm 3$

(b) $y=A\sin kt+B\cos kt \Rightarrow y'=Ak\cos kt-Bk\sin kt \Rightarrow y''=-Ak^2\sin kt-Bk^2\cos kt$.
Thus, $y''+9y=0 \Rightarrow -Ak^2\sin kt-Bk^2\cos kt+9(A\sin kt+B\cos kt)=0 \Rightarrow \left(9-k^2\right)A\sin kt+\left(9-k^2\right)B\cos kt=0$.
The last equation is true for all values of A and B if $k=\pm 3$.

4. $y=e^{rt} \Rightarrow y'=re^{rt} \Rightarrow y''=r^2e^{rt}$. $y''+y'-6y=0 \Rightarrow r^2e^{rt}+re^{rt}-6e^{rt}=0 \Rightarrow \left(r^2+r-6\right)e^{rt}=0 \Rightarrow (r+3)(r-2)=0$
$\Rightarrow r=-3$ or 2

5. **(a)** $y=e^t \Rightarrow y'=e^t \Rightarrow y''=e^t$. LHS $=y''+2y'+y=e^t+2e^t+e^t=4e^t\neq 0$, so $y=e^t$ is not a solution of the differential equation.

(b) $y=e^{-t} \Rightarrow y'=-e^{-t} \Rightarrow y''=e^{-t}$. LHS $=y''+2y'+y=e^{-t}-2e^{-t}+e^{-t}=0=$ RHS, so $y=e^{-t}$ is a solution.

(c) $y=te^{-t} \Rightarrow y'=t\left(-e^{-t}\right)+e^{-t}(1)=e^{-t}(1-t) \Rightarrow y''=e^{-t}(t-2)$.

$$\begin{aligned}\text{LHS} &= y''+2y'+y=e^{-t}(t-2)+2e^{-t}(1-t)+te^{-t}\\ &= e^{-t}[(t-2)+2(1-t)+t]=e^{-t}(0)=0=\text{RHS},\end{aligned}$$

so $y=te^{-t}$ is a solution.

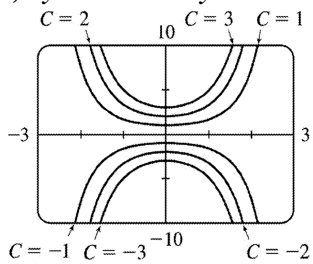
(d) $y=t^2e^{-t} \Rightarrow y'=te^{-t}(2-t) \Rightarrow y''=e^{-t}\left(t^2-4t+2\right)$.

$$\begin{aligned}\text{LHS} &= y''+2y'+y=e^{-t}\left(t^2-4t+2\right)+2te^{-t}(2-t)+t^2e^{-t}\\ &= e^{-t}\left[\left(t^2-4t+2\right)+2t(2-t)+t^2\right]=e^{-t}(2)\neq 0,\end{aligned}$$

so $y=t^2 e^{-t}$ is not a solution.

6. **(a)** $y=Ce^{x^2/2} \Rightarrow y'=Ce^{x^2/2}(2x/2)=xCe^{x^2/2}=xy$.

(b)



(c) $y(0)=5 \Rightarrow Ce^0=5 \Rightarrow C=5$, so the solution is $y=5e^{x^2/2}$.

(d) $y(1)=2 \Rightarrow Ce^{1/2}=2 \Rightarrow C=2e^{-1/2}$, so the solution is $y=2e^{-1/2}e^{x^2/2}=2e^{\left(x^2-1\right)/2}$.

7. **(a)** Since the derivative $y'=-y^2$ is always negative (or 0 if $y=0$), the function y must be decreasing (or equal to 0) on any interval on which it is defined.

(b) $y=\dfrac{1}{x+C} \Rightarrow y'=-\dfrac{1}{(x+C)^2}$. LHS $=y'=-\dfrac{1}{(x+C)^2}=-\left(\dfrac{1}{x+C}\right)^2=-y^2=$ RHS

(c) $y=0$ is a solution of $y'=-y^2$ that is not a member of the family in part (b).

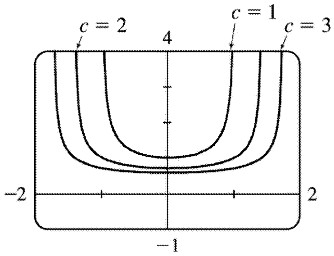
(d) If $y(x)=\dfrac{1}{x+C}$, then $y(0)=\dfrac{1}{0+C}=\dfrac{1}{C}$. Since $y(0)=0.5$, $\dfrac{1}{C}=\dfrac{1}{2} \Rightarrow C=2$, so $y=\dfrac{1}{x+2}$.

8. **(a)** If x is close to 0 , then xy^3 is close to 0 , and hence, y' is close to 0 . Thus, the graph of y must have a tangent line that is nearly horizontal. If x is large, then xy^3 is large, and the graph of y must have a tangent line that is nearly vertical. (In both cases, we assume reasonable values for y .)

(b) $y=\left(c-x^2\right)^{-1/2} \Rightarrow y'=x\left(c-x^2\right)^{-3/2}$.

RHS $=xy^3=x\left[\left(c-x^2\right)^{-1/2}\right]^3=x\left(c-x^2\right)^{-3/2}=y'=$ LHS

(c)



When x is close to 0 , y' is also close to 0 . As x gets larger, so does $\left|y'\right|$.

(d) $y(0)=(c-0)^{-1/2}=1/\sqrt{c}$ and $y(0)=2 \Rightarrow \sqrt{c}=\frac{1}{2} \Rightarrow c=\frac{1}{4}$, so $y=\left(\frac{1}{4}-x^2\right)^{-1/2}$.

9. **(a)** $\frac{dP}{dt}=1.2P\left(1-\frac{P}{4200}\right)$. Now $\frac{dP}{dt}>0 \Rightarrow 1-\frac{P}{4200}>0 \Rightarrow \frac{P}{4200}<1 \Rightarrow P<4200 \Rightarrow$ the population is increasing for $0<P<4200$.

(b) $\frac{dP}{dt}<0 \Rightarrow P>4200$

(c) $\frac{dP}{dt}=0 \Rightarrow P=4200$ or $P=0$

10. **(a)** $y=k \Rightarrow y'=0$, so $\frac{dy}{dt}=y^4-6y^3+5y^2 \Leftrightarrow 0=k^4-6k^3+5k^2 \Leftrightarrow k^2\left(k^2-6k+5\right)=0 \Leftrightarrow k^2(k-1)(k-5)=0 \Leftrightarrow k=0$, 1 , or 5

(b) y is increasing $\Leftrightarrow \frac{dy}{dt}>0 \Leftrightarrow y^2(y-1)(y-5)>0 \Leftrightarrow y \in (-\infty,0) \cup (0,1) \cup (5,\infty)$

(c) y is decreasing $\Leftrightarrow \frac{dy}{dt}<0 \Leftrightarrow y \in (1,5)$

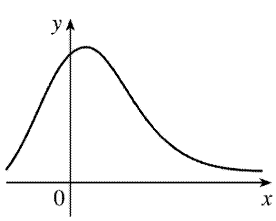
11. **(a)** This function is increasing *and* also decreasing. But $dy/dt=e^t(y-1)^2 \geq 0$ for all t , implying that the graph of the solution of the differential equation cannot be decreasing on any interval.
(b) When $y=1$, $dy/dt=0$, but the graph does not have a horizontal tangent line.

12. The graph for this exercise is shown in the figure at the right.

A. $y'=1+xy>1$ for points in the first quadrant, but we can see that $y'<0$ for some points in the first quadrant. So equation A is incorrect.

B. $y'=-2xy=0$ when $x=0$, but we can see that $y'>0$ for $x=0$. So equation B is incorrect.

C. $y'=1-2xy$ seems reasonable since:



(1) When $x=0$, y' could be 1 .

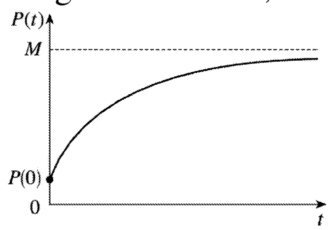
(2) When $x<0$, y' could be greater than 1 .

(3)

Solving $y'=1-2xy$ for y gives us $y=\frac{1-y'}{2x}$. If y' takes on small negative values, then as $x\to\infty$, $y\to 0^+$, as shown in the figure. Thus, the correct equation is C.

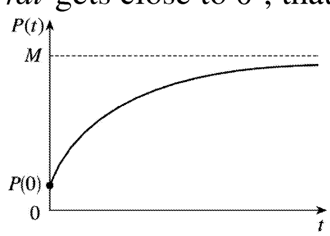
13. **(a)** P increases most rapidly at the beginning, since there are usually many simple, easily–learned sub–skills associated with learning a skill. As t increases, we would expect dP/dt to remain positive, but decrease. This is because as time progresses, the only points left to learn are the more difficult ones.

(b) $\frac{dP}{dt}=k(M-P)$ is always positive, so the level of performance P is increasing. As P gets close to M , dP/dt gets close to 0 ; that is, the performance levels off, as explained in part (a).

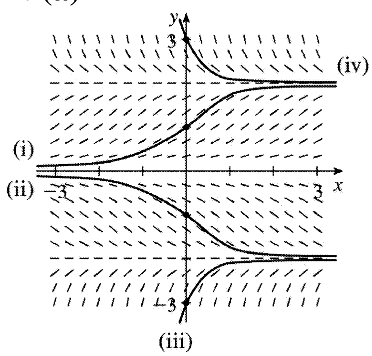
(c) 

14. **(a)** P increases most rapidly at the beginning, since there are usually many simple, easily–learned sub–skills associated with learning a skill. As t increases, we would expect dP/dt to remain positive, but decrease. This is because as time progresses, the only points left to learn are the more difficult ones.

(b) $\frac{dP}{dt}=k(M-P)$ is always positive, so the level of performance P is increasing. As P gets close to M , dP/dt gets close to 0 ; that is, the performance levels off, as explained in part (a).

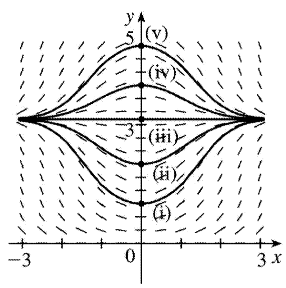
(c) 

1. **(a)**



(b) It appears that the constant functions $y=0$, $y=-2$, and $y=2$ are equilibrium solutions. Note that these three values of y satisfy the given differential equation $y'=y\left(1-\frac{1}{4}y^2\right)$.

2. **(a)**



(b) From the figure, it appears that $y=\pi$ is an equilibrium solution. From the equation $y'=x\sin y$, we see that $y=n\pi$ (n an integer) describes all the equilibrium solutions.

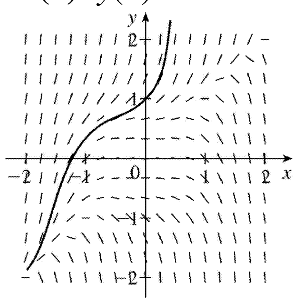
3. $y'=y-1$. The slopes at each point are independent of x , so the slopes are the same along each line parallel to the $x-$ axis. Thus, IV is the direction field for this equation. Note that for $y=1$, $y'=0$.

4. $y'=y-x=0$ on the line $y=x$, when $x=0$ the slope is y , and when $y=0$ the slope is $-x$. Direction field II satisfies these conditions.

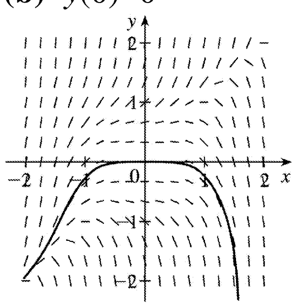
5. $y'=y^2-x^2=0 \Rightarrow y=\pm x$. There are horizontal tangents on these lines only in graph III, so this equation corresponds to direction field III.

6. $y'=y^3-x^3=0$ on the line $y=x$, when $x=0$ the slope is y^3 , and when $y=0$ the slope is $-x^3$. The graph is similar to the graph for Exercise 4, but the segments must get steeper very rapidly as they move away from the origin, because x and y are raised to the third power. This is the case in direction field I.

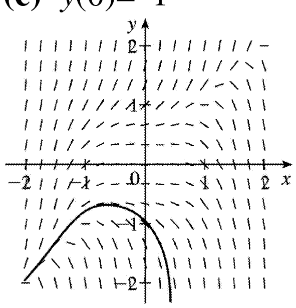
7. **(a)** $y(0)=1$



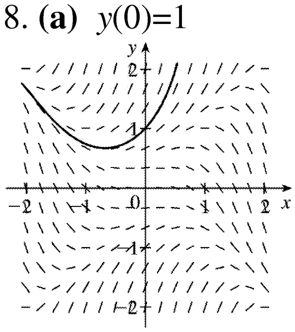
(b) $y(0)=0$



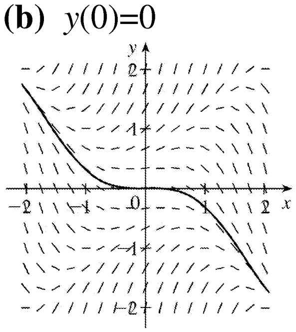
(c) $y(0)=-1$



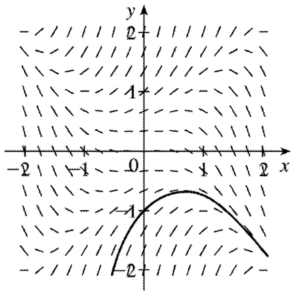
8. **(a)** $y(0)=1$



(b) $y(0)=0$



(c) $y(0)=-1$

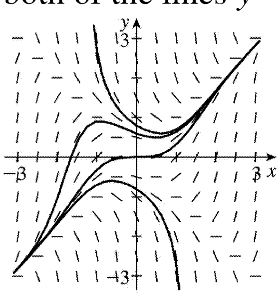


9.

10.

x	y	$y'=x^2-y^2$
± 1	± 3	-8
± 3	± 1	8
± 1	± 0.5	0.75
± 0.5	± 1	-0.75

Note that $y'=0$ for $y=\pm x$. If $|x|<|y|$, then $y'<0$; that is, the slopes are negative for all points in quadrants I and II above both of the lines $y=x$ and $y=-x$, and all points in quadrants III and IV below both of the lines $y=-x$ and $y=x$. A similar statement holds for positive slopes.

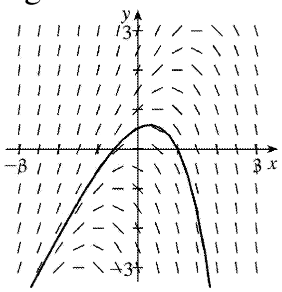


11.

x	y	$y'=y-2x$
-2	-2	2
-2	2	6
2	2	-2
2	-2	-6

Note that

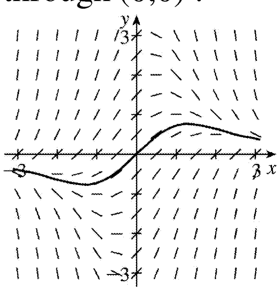
$y'=0$ for any point on the line $y=2x$. The slopes are positive to the left of the line and negative to the right of the line. The solution curve in the graph passes through $(1,0)$.



12.

x	y	$y'=1-xy$
± 1	± 1	0
± 2	± 2	-3
± 2	∓ 2	5

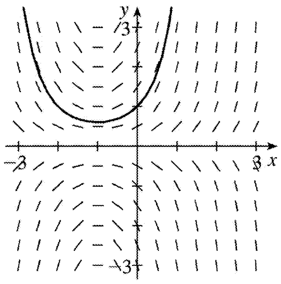
Note that $y'=0$ for any point on the hyperbola $xy=1$ (or $y=1/x$). The slopes are negative at points ''inside'' the branches and positive at points everywhere else. The solution curve in the graph passes through $(0,0)$.



13.

x	y	$y'=y+xy$
0	± 2	± 2
1	± 2	± 4
-3	± 2	∓ 4

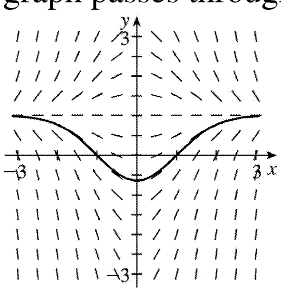
Note that $y'=y(x+1)=0$ for any point on $y=0$ or on $x=-1$. The slopes are positive when the factors y and $x+1$ have the same sign and negative when they have opposite signs. The solution curve in the graph passes through $(0,1)$.



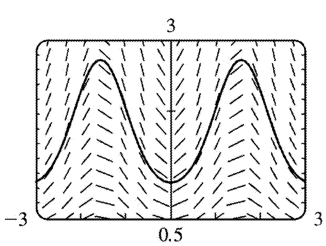
14.

x	y	$y' = x - xy$
± 2	0	± 2
± 2	3	∓ 4
± 2	-1	± 4

Note that $y' = x(1-y) = 0$ for any point on $x=0$ or on $y=1$. The slopes are positive when the factors x and $1-y$ have the same sign and negative when they have opposite signs. The solution curve in the graph passes through $(1,0)$.

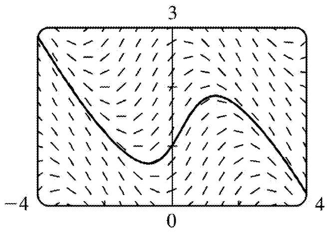


15. In Maple, we can use either directionfield (in Maple's share library) or plots [fieldplot] to plot the direction field. To plot the solution, we can either use the initial–value option in directionfield, or actually solve the equation. In Mathematica, we use PlotVectorField for the direction field, and the Plot [Evaluate[...]] construction to plot the solution, which

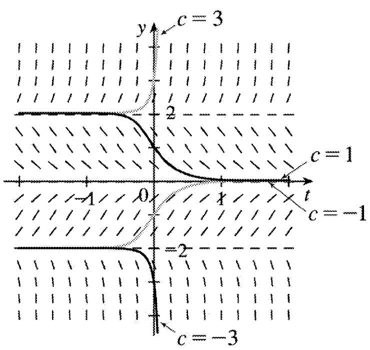


is $y = e^{(1-\cos 2x)/2}$. In Derive, use Direction_Field (in utility file ODE_APPR) to plot the direction field. Then use DSOLVE1(–y*SIN(2*x),1,x,y,0,1) (in utility file ODE1) to solve the equation. Simplify each result.

16. See Exercise 15 for specific CAS directions. The exact solution is

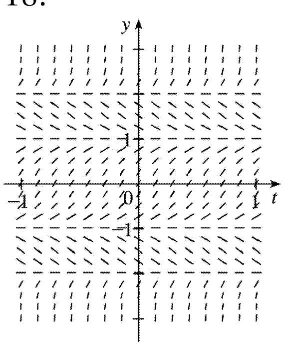


17.



$L=\lim_{t\to\infty} y(t)$ exists for $-2\le c\le 2$; $L=\pm 2$ for $c=\pm 2$ and $L=0$ for $-2<c<2$. For other values of c , L does not exist.

18.



Note that when $f(y)=0$ on the graph in the text, we have $y'=f(y)=0$; so we get horizontal segments at $y=\pm 1$, ± 2 . We get segments with negative slopes only for $1<|y|<2$. All other segments have positive slope. For the limiting behavior of solutions:

- If $y(0)>2$, then $\lim_{t\to\infty} y=\infty$ and $\lim_{t\to-\infty} y=2$.
- If $1<y(0)<2$, then $\lim_{t\to\infty} y=1$ and $\lim_{t\to-\infty} y=2$.
- If $-1<y(0)<1$, then $\lim_{t\to\infty} y=1$ and $\lim_{t\to-\infty} y=-1$.
- If $-2<y(0)<-1$, then $\lim_{t\to\infty} y=-2$ and $\lim_{t\to-\infty} y=-1$.

- If $y<-2$, then $\lim_{t\to\infty} y=-2$ and $\lim_{t\to-\infty} y=-\infty$.

19. **(a)**

(a) $h=0.4$ and $y_1=y_0+hF\left(x_0,y_0\right)\Rightarrow y_1=1+0.4\cdot 1=1.4$. $x_1=x_0+h=0+0.4=0.4$, so $y_1=y(0.4)=1.4$.

(b) $h=0.4$ and $y_1=y_0+hF\left(x_0,y_0\right)\Rightarrow y_1=1+0.4\cdot 1=1.4$. $x_1=x_0+h=0+0.4=0.4$, so $y_1=y(0.4)=1.4$.

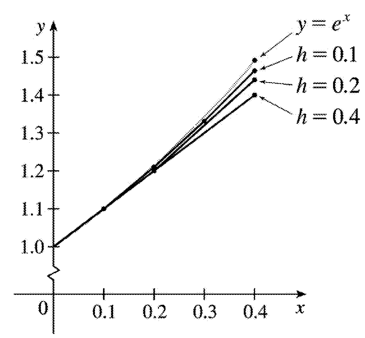
(c) $h=0.2\Rightarrow x_1=0.2$ and $x_2=0.4$, so we need to find y_2 . $y_1=y_0+hF\left(x_0,y_0\right)=1+0.2y_0=1+0.2\cdot 1=1.2$,
$y_2=y_1+hF\left(x_1,y_1\right)=1.2+0.2y_1=1.2+0.2\cdot 1.2=1.44$.

(d) $h=0.2\Rightarrow x_1=0.2$ and $x_2=0.4$, so we need to find y_2 . $y_1=y_0+hF\left(x_0,y_0\right)=1+0.2y_0=1+0.2\cdot 1=1.2$,
$y_2=y_1+hF\left(x_1,y_1\right)=1.2+0.2y_1=1.2+0.2\cdot 1.2=1.44$.

(e) $h=0.1\Rightarrow x_4=0.4$, so we need to find y_4 . $y_1=y_0+hF\left(x_0,y_0\right)=1+0.1y_0=1+0.1\cdot 1=1.1$,
$y_2=y_1+hF\left(x_1,y_1\right)=1.1+0.1y_1=1.1+0.1\cdot 1.1=1.21$,
$y_3=y_2+hF\left(x_2,y_2\right)=1.21+0.1y_2=1.21+0.1\cdot 1.21=1.331$,
$y_4=y_3+hF\left(x_3,y_3\right)=1.331+0.1y_3=1.331+0.1\cdot 1.331=1.4641$.

(f) $h=0.1\Rightarrow x_4=0.4$, so we need to find y_4 . $y_1=y_0+hF\left(x_0,y_0\right)=1+0.1y_0=1+0.1\cdot 1=1.1$,
$y_2=y_1+hF\left(x_1,y_1\right)=1.1+0.1y_1=1.1+0.1\cdot 1.1=1.21$,
$y_3=y_2+hF\left(x_2,y_2\right)=1.21+0.1y_2=1.21+0.1\cdot 1.21=1.331$,
$y_4=y_3+hF\left(x_3,y_3\right)=1.331+0.1y_3=1.331+0.1\cdot 1.331=1.4641$.

(b)



We see that the estimates are underestimates since they are all below the graph of $y=e^x$.

(c)

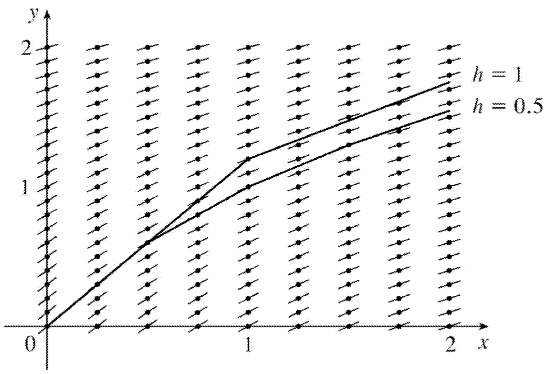
(a) For $h=0.4$: $(\text{exact value})-(\text{approximate value})=e^{0.4}-1.4\approx 0.0918$

(b) For $h=0.2$: $(\text{exact value})-(\text{approximate value})=e^{0.4}-1.44\approx 0.0518$

(c) For $h=0.1$: $(\text{exact value})-(\text{approximate value})=e^{0.4}-1.4641\approx 0.0277$

Each time the step size is halved, the error estimate also appears to be halved (approximately).

20.



As x increases, the slopes decrease and all of the estimates are above the true values. Thus, all of the estimates are overestimates.

21. $h=0.5$, $x_0=1$, $y_0=0$, and $F(x,y)=y-2x$.

Note that $x_1=x_0+h=1+0.5=1.5$, $x_2=2$, and $x_3=2.5$.

$y_1=y_0+hF\left(x_0,y_0\right)=0+0.5F(1,0)=0.5[0-2(1)]=-1$.

$y_2=y_1+hF\left(x_1,y_1\right)=-1+0.5F(1.5,-1)=-1+0.5[-1-2(1.5)]=-3$.

$y_3=y_2+hF\left(x_2,y_2\right)=-3+0.5F(2,-3)=-3+0.5[-3-2(2)]=-6.5$.

$y_4=y_3+hF\left(x_3,y_3\right)=-6.5+0.5F(2.5,-6.5)=-6.5+0.5[-6.5-2(2.5)]=-12.25$.

22. $h=0.2$, $x_0=0$, $y_0=0$, and $F(x,y)=1-xy$.

Note that $x_1=x_0+h=0+0.2=0.2$, $x_2=0.4$, $x_3=0.6$, and $x_4=0.8$.

$y_1=y_0+hF\left(x_0,y_0\right)=0+0.2F(0,0)=0.2[1-(0)(0)]=0.2$.

$y_2=y_1+hF\left(x_1,y_1\right)=0.2+0.2F(0.2,0.2)=0.2+0.2[1-(0.2)(0.2)]=0.392$.

$y_3=y_2+hF\left(x_2,y_2\right)=0.392+0.2F(0.4,0.392)=0.392+0.2[1-(0.4)(0.392)]=0.56064$.

$y_4=y_3+hF\left(x_3,y_3\right)=0.56064+0.2[1-(0.6)(0.56064)]=0.6933632$.

$y_5=y_4+hF\left(x_4,y_4\right)=0.6933632+0.2[1-(0.8)(0.6933632)]=0.782425088$.

Thus, $y(1)\approx 0.7824$.

23. $h=0.1$, $x_0=0$, $y_0=1$, and $F(x,y)=y+xy$.
Note that $x_1=x_0+h=0+0.1=0.1$, $x_2=0.2$, $x_3=0.3$, and $x_4=0.4$.
$y_1=y_0+hF(x_0,y_0)=1+0.1F(0,1)=1+0.1[1+(0)(1)]=1.1$.
$y_2=y_1+hF(x_1,y_1)=1.1+0.1F(0.1,1.1)=1.1+0.1[1.1+(0.1)(1.1)]=1.221$.
$y_3=y_2+hF(x_2,y_2)=1.221+0.1F(0.2,1.221)=1.221+0.1[1.221+(0.2)(1.221)]=1.36752$.
$y_4=y_3+hF(x_3,y_3)=1.36752+0.1F(0.3,1.36752)=1.36752+0.1[1.36752+(0.3)(1.36752)]$
$=1.5452976$.

$y_5=y_4+hF(x_4,y_4)=1.5452976+0.1F(0.4,1.5452976)$
$=1.5452976+0.1[1.5452976+(0.4)(1.5452976)]=1.761639264$.

Thus, $y(0.5)\approx 1.7616$.

24. **(a)** $h=0.2$, $x_0=1$, $y_0=0$, and $F(x,y)=x-xy$.
We need to find y_2 , because $x_1=1.2$ and $x_2=1.4$.
$y_1=y_0+hF(x_0,y_0)=0+0.2F(1,0)=0.2[1-(1)(0)]=0.2$.
$y_2=y_1+hF(x_1,y_1)=0.2+0.2F(1.2,0.2)=0.2+0.2[1.2-(1.2)(0.2)]=0.392\approx y(1.4)$.
(b) Now $h=0.1$, so we need to find y_4 .
$y_1=0+0.1[1-(1)(0)]=0.1$,
$y_2=0.1+0.1[1.1-(1.1)(0.1)]=0.199$,
$y_3=0.199+0.1[1.2-(1.2)(0.199)]=0.29512$, and
$y_4=0.29512+0.1[1.3-(1.3)(0.29512)]=0.3867544\approx y(1.4)$.

25. **(a)**
(a) H =1 , N =1$\Rightarrow y(1)=3$
(b) H =1 , N =1$\Rightarrow y(1)=3$
(c) H =0.1 , N =10$\Rightarrow y(1)\approx 2.3928$
(d) H =0.1 , N =10$\Rightarrow y(1)\approx 2.3928$
(e) H =0.01 , N =100$\Rightarrow y(1)\approx 2.3701$
(f) H =0.01 , N =100$\Rightarrow y(1)\approx 2.3701$

(g) H =0.001, N =1000 $\Rightarrow y(1)\approx 2.3681$

(h) H =0.001, N =1000 $\Rightarrow y(1)\approx 2.3681$

(b) $y=2+e^{-x^3} \Rightarrow y'=-3x^2e^{-x^3}$

$$\text{LHS}=y'+3x^2y=-3x^2e^{-x^3}+3x^2\left(2+e^{-x^3}\right)=-3x^2e^{-x^3}+6x^2+3x^2e^{-x^3}=6x^2=\text{RHS}$$

$y(0)=2+e^{-0}=2+1=3$

(c)

(a) For $h=1$: $(\text{exactvalue})-(\text{approximatevalue})=2+e^{-1}-3\approx -0.6321$

(b) For $h=1$: $(\text{exactvalue})-(\text{approximatevalue})=2+e^{-1}-3\approx -0.6321$

(c) For $h=0.1$: $(\text{exactvalue})-(\text{approximatevalue})=2+e^{-1}-2.3928\approx -0.0249$

(d) For $h=0.1$: $(\text{exactvalue})-(\text{approximatevalue})=2+e^{-1}-2.3928\approx -0.0249$

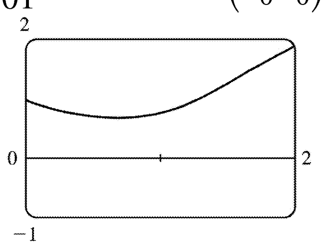
(e) For $h=0.01$: $(\text{exactvalue})-(\text{approximatevalue})=2+e^{-1}-2.3701\approx -0.0022$

(f) For $h=0.01$: $(\text{exactvalue})-(\text{approximatevalue})=2+e^{-1}-2.3701\approx -0.0022$

(g) For $h=0.001$: $(\text{exactvalue})-(\text{approximatevalue})=2+e^{-1}-2.3681\approx -0.0002$

(h) For $h=0.001$: $(\text{exactvalue})-(\text{approximatevalue})=2+e^{-1}-2.3681\approx -0.0002$

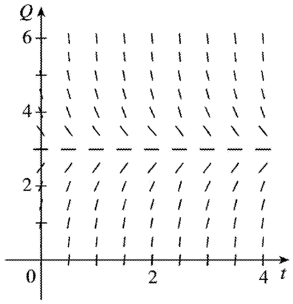
26. **(a)** We use the program from the solution to Exercise 25 with $Y_1=x^3-y^3$, H =0.01 , and N $=\dfrac{2-0}{0.01}=200$. With $\left(x_0,y_0\right)=(0,1)$, we get $y(2)\approx 1.9000$.



(b)

Notice from the graph that $y(2)\approx 1.9$, which serves as a check on our calculation in part (a).

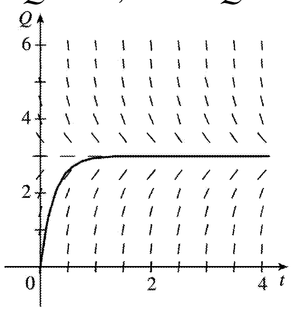
27. **(a)** $R\dfrac{dQ}{dt}+\dfrac{1}{C}Q=E(t)$ becomes $5Q'+\dfrac{1}{0.05}Q=60$ or $Q'+4Q=12$.



(b) From the graph, it appears that the limiting value of the charge Q is about 3 .

(c) If $Q'=0$, then $4Q=12 \Rightarrow Q=3$ is an equilibrium solution.

(d)



(e) $Q'+4Q=12 \Rightarrow Q'=12-4Q$. Now $Q(0)=0$, so $t_0=0$ and $Q_0=0$.

$$Q_1 = Q_0+hF\left(t_0,Q_0\right)=0+0.1(12-4\cdot 0)=1.2$$

$$Q_2 = Q_1+hF\left(t_1,Q_1\right)=1.2+0.1(12-4\cdot 1.2)=1.92$$

$$Q_3 = Q_2+hF\left(t_2,Q_2\right)=1.92+0.1(12-4\cdot 1.92)=2.352$$

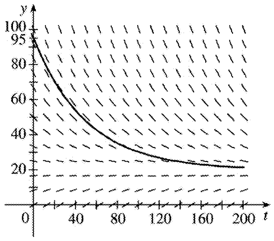
$$Q_4 = Q_3+hF\left(t_3,Q_3\right)=2.352+0.1(12-4\cdot 2.352)=2.6112$$

$$Q_5 = Q_4+hF\left(t_4,Q_4\right)=2.6112+0.1(12-4\cdot 2.6112)=2.76672$$

Thus, $Q_5=Q(0.5)\approx 2.77$ C.

28. **(a)** From Exercise .1.14, we have $dy/dt=k(y-R)$. We are given that $R=20^\circ$ C and $dy/dt=-1^\circ$ C / min when $y=70^\circ$ C. Thus, $-1=k(70-20) \Rightarrow k=-\frac{1}{50}$ and the differential equation becomes $dy/dt=-\frac{1}{50}(y-20)$.

(b)



The limiting value of the temperature is 20° C;
that is, the temperature of the room.

(c) From part (a), $dy/dt=-\frac{1}{50}(y-20)$. With $t_0=0$, $y_0=95$, and $h=2$ min, we get

$$y_1=y_0+hF\left(t_0,y_0\right)=95+2\left[-\frac{1}{50}(95-20)\right]=92$$

$$y_2=y_1+hF\left(t_1,y_1\right)=92+2\left[-\frac{1}{50}(92-20)\right]=89.12$$

$$y_3=y_2+hF\left(t_2,y_2\right)=89.12+2\left[-\frac{1}{50}(89.12-20)\right]=86.3552$$

$$y_4=y_3+hF\left(t_3,y_3\right)=86.3552+2\left[-\frac{1}{50}(86.3552-20)\right]=83.700992$$

$$y_5=y_4+hF\left(t_4,y_4\right)=83.700992+2\left[-\frac{1}{50}(83.700992-20)\right]=81.15295232$$

Thus, $y(10)\approx 81.15^{\circ}$ C.

1. $\frac{dy}{dx}=\frac{y}{x}\Rightarrow\frac{dy}{y}=\frac{dx}{x}\Rightarrow\int\frac{dy}{y}=\int\frac{dx}{x}\Rightarrow\ln|y|=\ln|x|+C\Rightarrow|y|=e^{\ln|x|+C}=e^{\ln|x|}e^{C}=e^{C}|x|\Rightarrow y=Kx$, where $K=\pm e^{C}$ is a constant. (In our derivation, K was nonzero, but we can restore the excluded case $y=0$ by allowing K to be zero.)

2. $\frac{dy}{dx}=\frac{e^{2x}}{4y^{3}}\Rightarrow 4y^{3}dy=e^{2x}dx\Rightarrow\int 4y^{3}dy=\int e^{2x}dx\Rightarrow y^{4}=\frac{1}{2}e^{2x}+C\Rightarrow\ y=\pm\sqrt[4]{\frac{1}{2}e^{2x}+C}$

3. $(x^{2}+1)y'=xy\Rightarrow\frac{dy}{dx}=\frac{xy}{x^{2}+1}\Rightarrow\frac{dy}{y}=\frac{x\,dx}{x^{2}+1}$ $[\,y\neq 0\,]\Rightarrow\int\frac{dy}{y}=\int\frac{x\,dx}{x^{2}+1}\Rightarrow\ln|y|=\frac{1}{2}\ln(x^{2}+1)+C$ [$u=x^{2}+1$, $du=2x\,dx$] $=\ln(x^{2}+1)^{1/2}+\ln e^{C}=\ln\left(e^{C}\sqrt{x^{2}+1}\right)\Rightarrow|y|=e^{C}\sqrt{x^{2}+1}\Rightarrow y=K\sqrt{x^{2}+1}$, where $K=\pm e^{C}$ is a constant. (In our derivation, K was nonzero, but we can restore the excluded case $y=0$ by allowing K to be zero.)

4. $y'=y^{2}\sin x\Rightarrow\frac{dy}{dx}=y^{2}\sin x\Rightarrow\frac{dy}{y^{2}}=\sin x\,dx\Rightarrow\int\frac{dy}{y^{2}}=\int\sin x\,dx\Rightarrow-\frac{1}{y}=-\cos x+C\Rightarrow\frac{1}{y}=\cos x-C\Rightarrow$ $y=\frac{1}{\cos x+K}$, where $K=-C$. $y=0$ is also a solution.

5. $(1+\tan y)y'=x^{2}+1\Rightarrow(1+\tan y)\frac{dy}{dx}=x^{2}+1\Rightarrow\left(1+\frac{\sin y}{\cos y}\right)dy=(x^{2}+1)dx\Rightarrow$ $\int\left(1-\frac{-\sin y}{\cos y}\right)dy=\int(x^{2}+1)dx\Rightarrow y-\ln|\cos y|=\frac{1}{3}x^{3}+x+C$. Note: The left side is equivalent to $y+\ln|\sec y|$.

6. $\frac{du}{dr}=\frac{1+\sqrt{r}}{1+\sqrt{u}}\Rightarrow(1+\sqrt{u})du=(1+\sqrt{r})dr\Rightarrow\int(1+u^{1/2})du=\int(1+r^{1/2})dr\Rightarrow\ u+\frac{2}{3}u^{3/2}=r+\frac{2}{3}r^{3/2}+C$

7. $\frac{dy}{dt}=\frac{te^{t}}{y\sqrt{1+y^{2}}}\Rightarrow y\sqrt{1+y^{2}}\,dy=te^{t}dt\Rightarrow\int y\sqrt{1+y^{2}}\,dy=\int te^{t}dt\Rightarrow\ \frac{1}{3}\left(1+y^{2}\right)^{3/2}=te^{t}-e^{t}+C\Rightarrow$ $1+y^{2}=[3(te^{t}-e^{t}+C)]^{2/3}\Rightarrow\ y=\pm\sqrt{[3(te^{t}-e^{t}+C)]^{2/3}-1}$

8. $y'=\frac{xy}{2\ln y}\Rightarrow\ \frac{2\ln y}{y}dy=x\,dx\Rightarrow\int\frac{2\ln y}{y}dy=\int x\,dx\Rightarrow(\ln y)^{2}=\frac{x^{2}}{2}+C\Rightarrow\ln y=\pm\sqrt{x^{2}/2+C}\Rightarrow$

$y=e^{\pm\sqrt{x^2/2+C}}$

9. $\frac{du}{dt}=2+2u+t+tu\Rightarrow\frac{du}{dt}=(1+u)(2+t)\Rightarrow\int\frac{du}{1+u}=\int(2+t)dt\Rightarrow\ln|1+u|=\frac{1}{2}t^2+2t+C\Rightarrow$ $|1+u|=e^{t^2/2+2t+C}=Ke^{t^2/2+2t}$, where $K=e^C\Rightarrow 1+u=\pm Ke^{t^2/2+2t}\Rightarrow u=-1\pm Ke^{t^2/2+2t}$ where $K>0$. $u=-1$ is also a solution, so $u=-1+Ae^{t^2/2+2t}$, where A is an arbitrary constant.

10. $\frac{dz}{dt}+e^{t+z}=0\Rightarrow\frac{dz}{dt}=-e^te^z\Rightarrow\int e^{-z}dz=-\int e^tdt\Rightarrow -e^{-z}=-e^t+C\Rightarrow e^{-z}=e^t-C\Rightarrow\frac{1}{e^z}=e^t-C\Rightarrow e^z=\frac{1}{e^t-C}\Rightarrow$ $z=\ln\left(\frac{1}{e^t-C}\right)\Rightarrow z=-\ln\left(e^t-C\right)$

11. $\frac{dy}{dx}=y^2+1$, $y(1)=0$. $\int\frac{dy}{y^2+1}=\int dx\Rightarrow\tan^{-1}y=x+C$. $y=0$ when $x=1$, so $1+C=\tan^{-1}0=0\Rightarrow C=-1$. Thus, $\tan^{-1}y=x-1$ and $y=\tan(x-1)$.

12. $\frac{dy}{dx}=\frac{y\cos x}{1+y^2}$, $y(0)=1$. $\left(1+y^2\right)dy=y\cos x\,dx\Rightarrow\frac{1+y^2}{y}dy=\cos x\,dx\Rightarrow\int\left(\frac{1}{y}+y\right)dy=\int\cos x\,dx\Rightarrow$ $\ln|y|+\frac{1}{2}y^2=\sin x+C$. $y(0)=1\Rightarrow\ln 1+\frac{1}{2}=\sin 0+C\Rightarrow C=\frac{1}{2}$, so $\ln|y|+\frac{1}{2}y^2=\sin x+\frac{1}{2}$. We cannot solve explicitly for y.

13. $x\cos x=(2y+e^{3y})y'\Rightarrow x\cos x\,dx=(2y+e^{3y})dy\Rightarrow\int(2y+e^{3y})dy=\int x\cos x\,dx\Rightarrow$ $y^2+\frac{1}{3}e^{3y}=x\sin x+\cos x+C$ [where the second integral is evaluated using integration by parts]. Now $y(0)=0\Rightarrow 0+\frac{1}{3}=0+1+C\Rightarrow C=-\frac{2}{3}$. Thus, a solution is $y^2+\frac{1}{3}e^{3y}=x\sin x+\cos x-\frac{2}{3}$. We cannot solve explicitly for y.

14. $\frac{dP}{dt}=\sqrt{Pt}\Rightarrow dP/\sqrt{P}=\sqrt{t}\,dt\Rightarrow\int P^{-1/2}dP=\int t^{1/2}dt\Rightarrow 2P^{1/2}=\frac{2}{3}t^{3/2}+C$. $P(1)=2\Rightarrow 2\sqrt{2}=\frac{2}{3}+C\Rightarrow C=2\sqrt{2}-\frac{2}{3}$, so $2P^{1/2}=\frac{2}{3}t^{3/2}+2\sqrt{2}-\frac{2}{3}\Rightarrow\sqrt{P}=\frac{1}{3}t^{3/2}+\sqrt{2}-\frac{1}{3}\Rightarrow$ $P=\left(\frac{1}{3}t^{3/2}+\sqrt{2}-\frac{1}{3}\right)^2$.

15. $\frac{du}{dt}=\frac{2t+\sec^2 t}{2u}$, $u(0)=-5$. $\int 2u\,du=\int\left(2t+\sec^2 t\right)dt \Rightarrow u^2=t^2+\tan t+C$, where $[u(0)]^2=0^2+\tan 0+C$ $\Rightarrow C=(-5)^2=25$. Therefore, $u^2=t^2+\tan t+25$, so $u=\pm\sqrt{t^2+\tan t+25}$. Since $u(0)=-5$, we must have $u=-\sqrt{t^2+\tan t+25}$.

16. $\frac{dy}{dt}=te^y$, $y(1)=0$. $\int e^{-y}dy=\int t\,dt \Rightarrow -e^{-y}=\frac{1}{2}t^2+C$. Since $y(1)=0$, $-e^0=\frac{1}{2}\cdot 1^2+C$. Therefore, $C=-1-\frac{1}{2}=-\frac{3}{2}$ and $-e^{-y}=\frac{1}{2}t^2-\frac{3}{2}$. So $e^{-y}=\frac{3}{2}-\frac{1}{2}t^2=\frac{3-t^2}{2} \Rightarrow e^y=\frac{2}{3-t^2} \Rightarrow y=\ln 2-\ln\left(3-t^2\right)$ for $|t|<\sqrt{3}$.

17. $y'\tan x=a+y$, $0<x<\pi/2 \Rightarrow \frac{dy}{dx}=\frac{a+y}{\tan x} \Rightarrow \frac{dy}{a+y}=\cot x\,dx$ [$a+y\neq 0$] $\Rightarrow$ $\int\frac{dy}{a+y}=\int\frac{\cos x}{\sin x}dx \Rightarrow \ln|a+y|=\ln|\sin x|+C \Rightarrow |a+y|=e^{\ln|\sin x|+C}=e^{\ln|\sin x|}\cdot e^C=e^C|\sin x| \Rightarrow$ $a+y=K\sin x$, where $K=\pm e^C$. (In our derivation, K was nonzero, but we can restore the excluded case $y=-a$ by allowing K to be zero.) $y(\pi/3)=a \Rightarrow$ $a+a=K\sin\left(\frac{\pi}{3}\right) \Rightarrow 2a=K\frac{\sqrt{3}}{2} \Rightarrow K=\frac{4a}{\sqrt{3}}$. Thus, $a+y=\frac{4a}{\sqrt{3}}\sin x$ and so $y=\frac{4a}{\sqrt{3}}\sin x-a$.

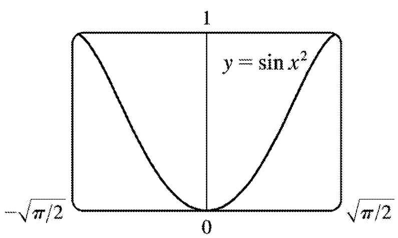
18. $xy'+y=y^2 \Rightarrow x\frac{dy}{dx}=y^2-y \Rightarrow x\,dy=(y^2-y)\,dx \Rightarrow \frac{dy}{y^2-y}=\frac{dx}{x} \Rightarrow$ $\int\frac{dy}{y(y-1)}=\int\frac{dx}{x}$ [$y\neq 0, 1$] $\Rightarrow \int\left(\frac{1}{y-1}-\frac{1}{y}\right)dy=\int\frac{dx}{x} \Rightarrow \ln|y-1|-\ln|y|=\ln|x|+C \Rightarrow$ $\ln\left|\frac{y-1}{y}\right|=\ln\left(e^C|x|\right) \Rightarrow \left|\frac{y-1}{y}\right|=e^C|x| \Rightarrow \frac{y-1}{y}=Kx$, where $K=\pm e^C \Rightarrow 1-\frac{1}{y}=Kx \Rightarrow \frac{1}{y}=1-Kx \Rightarrow$ $y=\frac{1}{1-Kx}$. Now $y(1)=-1 \Rightarrow -1=\frac{1}{1-K} \Rightarrow 1-K=-1 \Rightarrow K=2$, so $y=\frac{1}{1-2x}$.

19. $\frac{dy}{dx}=4x^3y$, $y(0)=7$. $\frac{dy}{y}=4x^3dx \Rightarrow \int\frac{dy}{y}=\int 4x^3dx \Rightarrow \ln|y|=x^4+C \Rightarrow e^{\ln|y|}=e^{x^4+C} \Rightarrow |y|=e^{x^4}e^C \Rightarrow$ $y=Ae^{x^4}$; $y(0)=7 \Rightarrow A=7 \Rightarrow y=7e^{x^4}$.

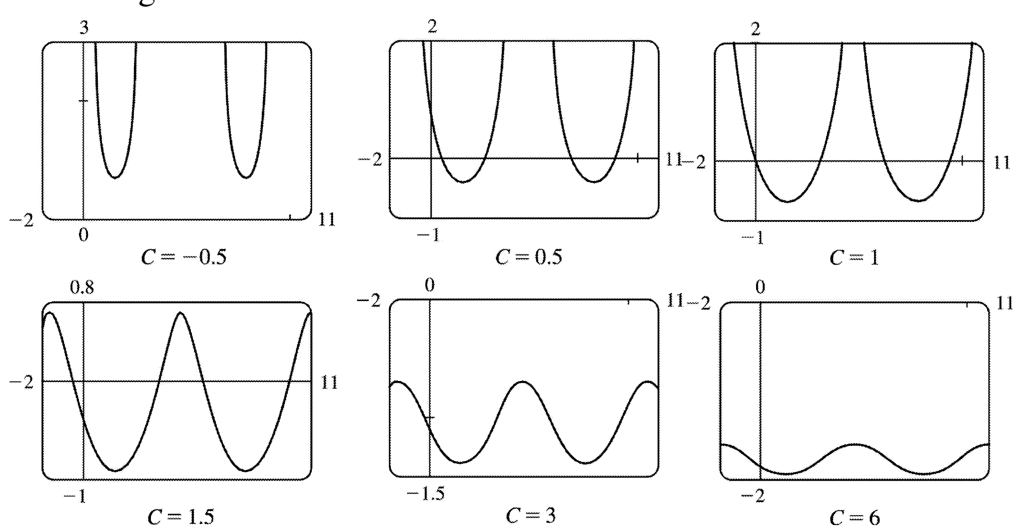
20.

$\frac{dy}{dx}=\frac{y^2}{x^3}$, $y(1)=1$. $\int\frac{dy}{y^2}=\int\frac{dx}{x^3}\Rightarrow-\frac{1}{y}=-\frac{1}{2x^2}+C$. $y(1)=1\Rightarrow-1=-\frac{1}{2}+C\Rightarrow C=-\frac{1}{2}$. So
$\frac{1}{y}=\frac{1}{2x^2}+\frac{1}{2}=\frac{2+2x^2}{2\cdot 2x^2}\Rightarrow y=\frac{2x^2}{x^2+1}$.

21. **(a)** $y'=2x\sqrt{1-y^2}\Rightarrow\frac{dy}{dx}=2x\sqrt{1-y^2}\Rightarrow\frac{dy}{\sqrt{1-y^2}}=2x\,dx\Rightarrow\int\frac{dy}{\sqrt{1-y^2}}=\int 2x\,dx\Rightarrow\sin^{-1}y=x^2+C$ for
$-\frac{\pi}{2}\le x^2+C\le\frac{\pi}{2}$.
(b) $y(0)=0\Rightarrow\sin^{-1}0=0^2+C\Rightarrow C=0$, so $\sin^{-1}y=x^2$ and $y=\sin\left(x^2\right)$ for $-\sqrt{\pi/2}\le x\le\sqrt{\pi/2}$.

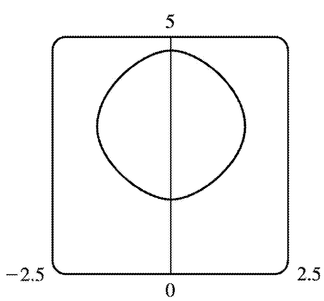


(c) For $\sqrt{1-y^2}$ to be a real number, we must have $-1\le y\le 1$; that is, $-1\le y(0)\le 1$. Thus, the initial–value problem $y'=2x\sqrt{1-y^2}$, $y(0)=2$ does *not* have a solution.
22. $e^{-y}y'+\cos x=0\Leftrightarrow\int e^{-y}dy=-\int\cos x\,dx\Leftrightarrow-e^{-y}=-\sin x+C_1\Leftrightarrow y=-\ln(\sin x+C)$. The solution is periodic, with period 2π . Note that for $C>1$, the domain of the solution is R , but for $-1<C\le 1$ it is only defined on the intervals where $\sin x+C>0$, and it is meaningless for $C\le-1$, since then $\sin x+C\le 0$, and the logarithm is undefined.

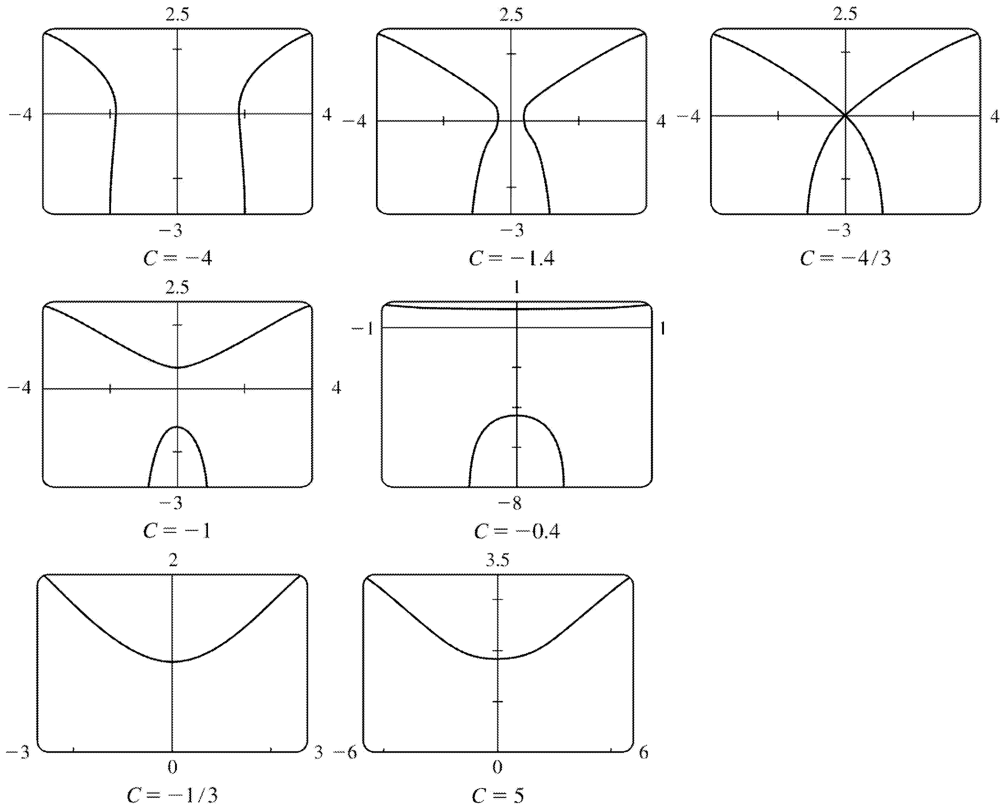


For $-1<C<1$, the solution curve consists of concave–up pieces separated by intervals on which the solution is not defined (where $\sin x+C\le 0$). For $C=1$, the solution curve consists of concave–up pieces separated by vertical asymptotes at the points where $\sin x+C=0\Leftrightarrow\sin x=-1$. For $C>1$, the curve is continuous, and as C increases, the graph moves downward, and the amplitude of the oscillations decreases.

23. $\frac{dy}{dx}=\frac{\sin x}{\sin y}$, $y(0)=\frac{\pi}{2}$. So $\int \sin y\,dy=\int \sin x\,dx\Leftrightarrow -\cos y=-\cos x+C\Leftrightarrow \cos y=\cos x-C$. From the initial condition, we need $\cos\frac{\pi}{2}=\cos 0-C\Rightarrow 0=1-C\Rightarrow C=1$, so the solution is $\cos y=\cos x-1$. Note that we cannot take $\cos^{-1}$ of both sides, since that would unnecessarily restrict the solution to the case where $-1\le\cos x-1\Leftrightarrow 0\le\cos x$, as $\cos^{-1}$ is defined only on $[-1,1]$. Instead we plot the graph using Maple’s plots [implicitplot] or Mathematica’s Plot [Evaluate[..]].



24. $\frac{dy}{dx}=\frac{x\sqrt{x^2+1}}{ye^y}\Leftrightarrow\int ye^y\,dy=\int x\sqrt{x^2+1}\,dx$. We use parts on the LHS with $u=y$, $dv=e^y\,dy$, and on the RHS we use the substitution $z=x^2+1$, so $dz=2x\,dx$. The equation becomes $ye^y-\int e^y\,dy=\frac{1}{2}\int\sqrt{z}\,dz\Leftrightarrow$ $e^y(y-1)=\frac{1}{3}\left(x^2+1\right)^{3/2}+C$, so we see that the curves are symmetric about the $y-$ axis. Every point (x,y) in the plane lies on one of the curves, namely the one for which $C=(y-1)e^y-\frac{1}{3}\left(x^2+1\right)^{3/2}$. For example,
along the $y-$ axis, $C=(y-1)e^y-\frac{1}{3}$, so the origin lies on the curve with $C=-\frac{4}{3}$. We use Maple’s plots command or Plot] in Mathematica to plot the solution curves for various values of C .



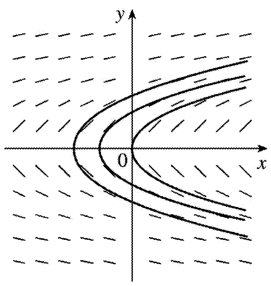
It seems that the transitional values of C are $-\dfrac{4}{3}$ and $-\dfrac{1}{3}$. For $C<-\dfrac{4}{3}$, the graph consists of left and right branches. At $C=-\dfrac{4}{3}$, the two branches become connected at the origin, and as C increases, the graph splits into top and bottom branches. At $C=-\dfrac{1}{3}$, the bottom half disappears. As C increases further, the graph moves upward, but doesn't change shape much.

25. **(a)**

x	y	$y'=1/y$
0	0.5	2
0	−0.5	−2
0	1	1
0	−1	−1
0	2	0.5

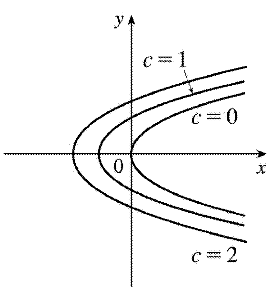
x	y	$y'=1/y$
0	−2	−0.5
0	4	0.25

0	3	$0.\overline{3}$
0	0.25	4
0	$0.\overline{3}$	3



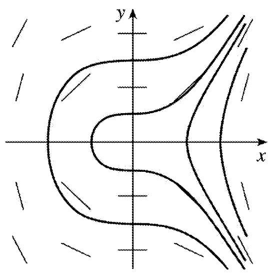
(b) $y' = 1/y \Rightarrow dy/dx = 1/y \Rightarrow$
$y\,dy = dx \Rightarrow \int y\,dy = \int dx \Rightarrow \frac{1}{2}y^2 = x + c \Rightarrow y^2 = 2(x + c)$ or $y = \pm\sqrt{2(x + c)}$.

(c)

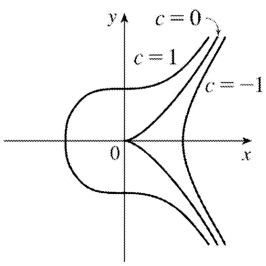


26. **(a)**

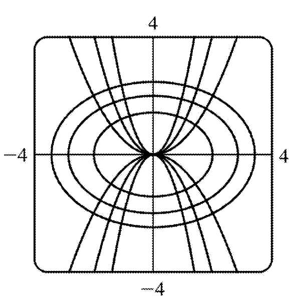
x	y	$y' = x^2/y$
1	1	1
–1	1	1
–1	–1	–1
1	–1	–1
1	2	0.5
2	1	4
2	2	2
1	0.5	2
0.5	1	0.25
2	0.5	8



(b) $y' = x^2/y \Rightarrow y\,dy = x^2\,dx$, so $\frac{1}{2}y^2 = \frac{1}{3}x^3 + c_1$, or $y = \pm\left(\frac{2}{3}x^3 + c\right)^{1/2}$.

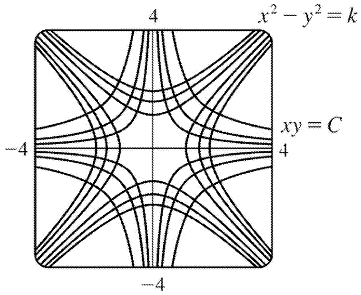
(c) 

27. The curves $y = kx^2$ form a family of parabolas with axis the $y-$ axis. Differentiating gives $y' = 2kx$, but $k = y/x^2$, so $y' = 2y/x$. Thus, the slope of the tangent line at any point (x,y) on one of the parabolas is $y' = 2y/x$, so the orthogonal trajectories must satisfy $y' = -x/(2y) \Leftrightarrow 2y\,dy = -x\,dx \Leftrightarrow y^2 = \left(-x^2\right)/2 + C_1 \Leftrightarrow$ $x^2 + 2y^2 = C$. This is a family of ellipses.

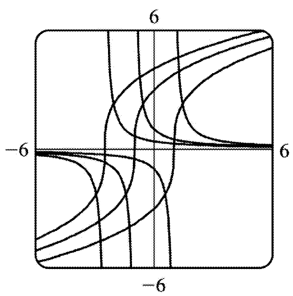


28. The curves $x^2 - y^2 = k$ form a family of hyperbolas. Differentiating gives $2x - 2y(dy/dx) = 0$ or $y' = x/y$, the slope of the tangent line at (x,y) on one of the hyperbolas. Thus, the orthogonal trajectories must satisfy $y' = -y/x \Leftrightarrow dy/y = -dx/x \Leftrightarrow \ln|y| = -\ln|x| + C_1 \Leftrightarrow \ln|x| + \ln|y| = C_1 \Leftrightarrow \ln|xy| = C_1 \Leftrightarrow$

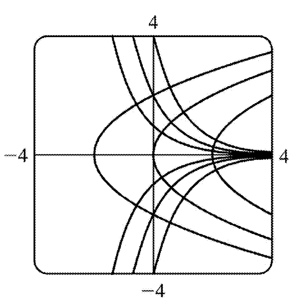
$|xy| = e^{C_1} \Leftrightarrow xy = C$. This is a family of hyperbolas.



29. Differentiating $y=(x+k)^{-1}$ gives $y'=-\frac{1}{(x+k)^2}$, but $k=\frac{1}{y}-x$, so $y'=-\frac{1}{(1/y)^2}=-y^2$. Thus, the orthogonal trajectories must satisfy $y'=-\frac{1}{-y^2}=\frac{1}{y^2}\Leftrightarrow y^2\,dy=dx\Leftrightarrow \frac{y^3}{3}=x+C$ or $y=[3(x+C)]^{1/3}$



30. Differentiating $y=ke^{-x}$ gives $y'=-ke^{-x}$, but $k=ye^{x}$, so $y'=-y$. Thus, the orthogonal trajectories must satisfy $y'=-1/(-y)=1/y\Leftrightarrow y\,dy=dx\Leftrightarrow \frac{1}{2}y^2=x+C\Leftrightarrow y=\pm[2(C+x)]^{1/2}$. This is a family of parabolas with axis the x-axis.



31. From Exercise .2.27, $\frac{dQ}{dt}=12-4Q\Leftrightarrow\int\frac{dQ}{12-4Q}=\int dt\Leftrightarrow-\frac{1}{4}\ln|12-4Q|=t+C\Leftrightarrow\ln|12-4Q|=-4t-4C$ $\Leftrightarrow|12-4Q|=e^{-4t-4C}\Leftrightarrow 12-4Q=Ke^{-4t}\Leftrightarrow 4Q=12-Ke^{-4t}\Leftrightarrow Q=3-Ae^{-4t}$. $Q(0)=0\Leftrightarrow 0=3-A\Leftrightarrow A=3\Leftrightarrow$ $Q(t)=3-3e^{-4t}$. As $t\to\infty$, $Q(t)\to 3-0=3$ (the limiting value).

32. From Exercise .2.28, $\frac{dy}{dt}=-\frac{1}{50}(y-20)\Leftrightarrow\int\frac{dy}{y-20}=\int\left(-\frac{1}{50}\right)dt\Leftrightarrow\ln|y-20|=-\frac{1}{50}t+C\Leftrightarrow$ $y-20=Ke^{-t/50}\Leftrightarrow y(t)=Ke^{-t/50}+20$. $y(0)=95\Leftrightarrow 95=K+20\Leftrightarrow K=75\Leftrightarrow y(t)=75e^{-t/50}+20$.

33. $\frac{dP}{dt}=k(M-P)\Leftrightarrow\int\frac{dP}{P-M}=\int(-k)dt\Leftrightarrow\ln|P-M|=-kt+C\Leftrightarrow|P-M|=e^{-kt+C}\Leftrightarrow P-M=Ae^{-kt}\Leftrightarrow P=M+Ae^{-kt}$. If we assume that performance is at level 0 when $t=0$, then $P(0)=0\Leftrightarrow 0=M+A\Leftrightarrow A=-M\Leftrightarrow$ $P(t)=M-Me^{-kt}$. $\lim_{t\to\infty}P(t)=M-M\cdot 0=M$.

34. **(a)** $\frac{dx}{dt}=k(a-x)(b-x)$, $a\neq b$. Using partial fractions, $\frac{1}{(a-x)(b-x)}=\frac{1/(b-a)}{a-x}-\frac{1/(b-a)}{b-x}$, so $\int\frac{dx}{(a-x)(b-x)}=\int k\,dt\Rightarrow\frac{1}{b-a}(-\ln|a-x|+\ln|b-x|)=kt+C\Rightarrow\ln\left|\frac{b-x}{a-x}\right|=(b-a)(kt+C)$. The concentrations $[\mathrm{A}]=a-x$ and $[\mathrm{B}]=b-x$ cannot be negative, so $\frac{b-x}{a-x}\geq 0$ and $\left|\frac{b-x}{a-x}\right|=\frac{b-x}{a-x}$. We now have $\ln\left(\frac{b-x}{a-x}\right)=(b-a)(kt+C)$. Since $x(0)=0$, we get $\ln\left(\frac{b}{a}\right)=(b-a)C$. Hence, $\ln\left(\frac{b-x}{a-x}\right)=(b-a)kt+\ln\left(\frac{b}{a}\right)\Rightarrow\frac{b-x}{a-x}=\frac{b}{a}e^{(b-a)kt}\Rightarrow x=\frac{b\left[e^{(b-a)kt}-1\right]}{\left[be^{(b-a)kt}/a-1\right]}=\frac{ab\left[e^{(b-a)kt}-1\right]}{be^{(b-a)kt}-a}$ moles / L.

(b) If $b=a$, then $\frac{dx}{dt}=k(a-x)^2$, so $\int\frac{dx}{(a-x)^2}=\int k\,dt$ and $\frac{1}{a-x}=kt+C$. Since $x(0)=0$, we get $C=\frac{1}{a}$.

Thus, $a-x=\frac{1}{kt+1/a}$ and $x=a-\frac{a}{akt+1}=\frac{a^2kt}{akt+1}\ \frac{\text{moles}}{\text{L}}$.

Suppose $x=[\mathrm{C}]=a/2$ when $t=20$. Then $x(20)=a/2\Rightarrow\frac{a}{2}=\frac{20a^2k}{20ak+1}\Rightarrow 40a^2k=20a^2k+a\Rightarrow 20a^2k=a\Rightarrow$ $k=\frac{1}{20a}$, so $x=\frac{a^2t/(20a)}{1+at/(20a)}=\frac{at/20}{1+t/20}=\frac{at}{t+20}\ \frac{\text{moles}}{\text{L}}$.

35. **(a)** If $a=b$, then $\frac{dx}{dt}=k(a-x)(b-x)^{1/2}$ becomes $\frac{dx}{dt}=k(a-x)^{3/2}\Rightarrow(a-x)^{-3/2}dx=k\,dt\Rightarrow$ $\int(a-x)^{-3/2}dx=\int k\,dt\Rightarrow 2(a-x)^{-1/2}=kt+C\Rightarrow\frac{2}{kt+C}=\sqrt{a-x}\Rightarrow\left(\frac{2}{kt+C}\right)^2=a-x\Rightarrow x(t)=a-\frac{4}{(kt+C)^2}$. The initial concentration of HBr is 0 , so $x(0)=0\Rightarrow 0=a-\frac{4}{C^2}\Rightarrow\frac{4}{C^2}=a\Rightarrow C^2=\frac{4}{a}\Rightarrow C=2/\sqrt{a}$ (C is positive since

$kt+C=2(a-x)^{-1/2}>0$). Thus, $x(t)=a-\dfrac{4}{\left(kt+2/\sqrt{a}\,\right)^2}$.

(b) $\dfrac{dx}{dt}=k(a-x)(b-x)^{1/2}\Rightarrow \dfrac{dx}{(a-x)\sqrt{b-x}}=k\,dt\Rightarrow \int \dfrac{dx}{(a-x)\sqrt{b-x}}=\int k\,dt$ (*) . From the hint, $u=\sqrt{b-x}\Rightarrow$ $u^2=b-x\Rightarrow 2u\,du=-dx$, so

$\int \dfrac{dx}{(a-x)\sqrt{b-x}}=\int \dfrac{-2u\,du}{\left[a-\left(b-u^2\right)\right]u}=-2\int \dfrac{du}{a-b+u^2}=-2\int \dfrac{du}{\left(\sqrt{a-b}\,\right)^2+u^2}=-2\left(\dfrac{1}{\sqrt{a-b}}\tan^{-1}\dfrac{u}{\sqrt{a-b}}\right)$. So (*) becomes $\dfrac{-2}{\sqrt{a-b}}\tan^{-1}\dfrac{\sqrt{b-x}}{\sqrt{a-b}}=kt+C$. Now $x(0)=0\Rightarrow C=\dfrac{-2}{\sqrt{a-b}}\tan^{-1}\dfrac{\sqrt{b}}{\sqrt{a-b}}$ and we have

$\dfrac{-2}{\sqrt{a-b}}\tan^{-1}\dfrac{\sqrt{b-x}}{\sqrt{a-b}}=kt-\dfrac{2}{\sqrt{a-b}}\tan^{-1}\dfrac{\sqrt{b}}{\sqrt{a-b}}\Rightarrow \dfrac{2}{\sqrt{a-b}}\left(\tan^{-1}\sqrt{\dfrac{b}{a-b}}-\tan^{-1}\sqrt{\dfrac{b-x}{a-b}}\right)=kt\Rightarrow$
$t(x)=\dfrac{2}{k\sqrt{a-b}}\left(\tan^{-1}\sqrt{\dfrac{b}{a-b}}-\tan^{-1}\sqrt{\dfrac{b-x}{a-b}}\right)$.

36. If $S=\dfrac{dT}{dr}$, then $\dfrac{dS}{dr}=\dfrac{d^2T}{dr^2}$. The differential equation $\dfrac{d^2T}{dr^2}+\dfrac{2}{r}\dfrac{dT}{dr}=0$ can be written as $\dfrac{dS}{dr}+\dfrac{2}{r}S=0$. Thus, $\dfrac{dS}{dr}=\dfrac{-2S}{r}\Rightarrow \dfrac{dS}{S}=-\dfrac{2}{r}dr\Rightarrow \int \dfrac{1}{S}dS=\int -\dfrac{2}{r}dr\Rightarrow \ln|S|=-2\ln|r|+C$. Assuming $S=dT/dr>0$ and $r>0$, we have $S=e^{-2\ln r+C}=e^{\ln r^{-2}}e^{C}=r^{-2}k\Rightarrow S=\dfrac{1}{r^2}k\Rightarrow$

$\dfrac{dT}{dr}=\dfrac{1}{r^2}k\Rightarrow dT=\dfrac{1}{r^2}k\,dr\Rightarrow \int dT=\int \dfrac{1}{r^2}k\,dr\Rightarrow T(r)=-\dfrac{k}{r}+A$.

$T(1)=15\Rightarrow 15=-k+A$ **(1)** and $T(2)=25\Rightarrow 25=-\dfrac{1}{2}k+A$ **(2)** .

Now solve for k and A : -2 **(2)** + **(1)** $\Rightarrow -35=-A$, so $A=35$ and $k=20$, and $T(r)=-20/r+35$.

37. **(a)** $\dfrac{dC}{dt}=r-kC\Rightarrow \dfrac{dC}{dt}=-(kC-r)\Rightarrow \int \dfrac{dC}{kC-r}=\int -dt\Rightarrow (1/k)\ln|kC-r|=-t+M_1\Rightarrow \ln|kC-r|=-kt+M_2$ $\Rightarrow |kC-r|=e^{-kt+M_2}\Rightarrow kC-r=M_3e^{-kt}\Rightarrow kC=M_3e^{-kt}+r\Rightarrow C(t)=M_4e^{-kt}+r/k$. $C(0)=C_0\Rightarrow C_0=M_4+r/k\Rightarrow$ $M_4=C_0-r/k\Rightarrow C(t)=\left(C_0-r/k\right)e^{-kt}+r/k$.

(b) If $C_0<r/k$, then $C_0-r/k<0$ and the formula for $C(t)$ shows that $C(t)$ increases and $\lim_{t\to\infty} C(t)=r/k$. As t increases, the formula for $C(t)$ shows how the role of C_0 steadily diminishes as that of r/k increases.

38. **(a)** Use 1 billion dollars as the x– unit and 1 day as the t– unit. Initially, there is \$10 billion of old currency in circulation, so all of the \$50 million returned to the banks is old. At time t , the amount of new currency is $x(t)$ billion dollars, so $10-x(t)$ billion dollars of currency is old. The fraction of circulating money that is old is $[10-x(t)]/10$, and the amount of old currency being returned to the banks each day is $\frac{10-x(t)}{10}\,0.05$ billion dollars. This amount of new currency per day is introduced into circulation, so $\frac{dx}{dt}=\frac{10-x}{10}\cdot 0.05=0.005(10-x)$ billion dollars per day.

(b) $\frac{dx}{10-x}=0.005\,dt \Rightarrow \frac{-dx}{10-x}=-0.005\,dt \Rightarrow \ln(10-x)=-0.005t+c \Rightarrow 10-x=Ce^{-0.005t}$, where $C=e^{c} \Rightarrow$ $x(t)=10-Ce^{-0.005t}$. From $x(0)=0$, we get $C=10$, so $x(t)=10\left(1-e^{-0.005t}\right)$.

(c) The new bills make up 90% of the circulating currency when $x(t)=0.9\cdot 10=9$ billion dollars. $9=10\left(1-e^{-0.005t}\right) \Rightarrow 0.9=1-e^{-0.005t} \Rightarrow e^{-0.005t}=0.1 \Rightarrow -0.005t=-\ln 10 \Rightarrow t=200\ln 10\approx 460.517$ days ≈ 1.26 years.

39. **(a)** Let $y(t)$ be the amount of salt (in kg) after t minutes. Then $y(0)=15$. The amount of liquid in the tank is 1000 L at all times, so the concentration at time t (in minutes) is $y(t)/1000$ kg/L and $\frac{dy}{dt}=-\left[\frac{y(t)}{1000}\,\frac{\text{kg}}{\text{L}}\right]\left(10\,\frac{\text{L}}{\text{min}}\right)=-\frac{y(t)}{100}\,\frac{\text{kg}}{\text{min}}$. $\int\frac{dy}{y}=-\frac{1}{100}\int dt \Rightarrow \ln y=-\frac{t}{100}+C$, and $y(0)=15 \Rightarrow$ $\ln 15=C$, so $\ln y=\ln 15-\frac{t}{100}$. It follows that $\ln\left(\frac{y}{15}\right)=-\frac{t}{100}$ and $\frac{y}{15}=e^{-t/100}$, so $y=15e^{-t/100}$ kg.

(b) After 20 minutes, $y=15e^{-20/100}=15e^{-0.2}\approx 12.3$ kg.

40. **(a)** If $y(t)$ is the amount of salt (in kg) after t minutes, then $y(0)=0$ and the total amount of liquid in the tank remains constant at 1000 L.

$$\frac{dy}{dt}=\left(0.05\,\frac{\text{kg}}{\text{L}}\right)\left(5\,\frac{\text{L}}{\text{min}}\right)+\left(0.04\,\frac{\text{kg}}{\text{L}}\right)\left(10\,\frac{\text{L}}{\text{min}}\right)-\left(\frac{y(t)}{1000}\,\frac{\text{kg}}{\text{L}}\right)\left(15\,\frac{\text{L}}{\text{min}}\right)$$

$$=0.25+0.40-0.015y=0.65-0.015y=\frac{130-3y}{200}\,\frac{\text{kg}}{\text{min}}$$

so $\int\frac{dy}{130-3y}=\int\frac{dt}{200}$ and $-\frac{1}{3}\ln|130-3y|=\frac{1}{200}t+C$; since $y(0)=0$, we have $-\frac{1}{3}\ln 130=C$, so $-\frac{1}{3}\ln|130-3y|=\frac{1}{200}t-\frac{1}{3}\ln 130 \Rightarrow \ln|130-3y|=-\frac{3}{200}t+\ln 130=\ln\left(130e^{-3t/200}\right)$, and $|130-3y|=130e^{-3t/200}$. Since y is continuous, $y(0)=0$, and the right–hand side is never zero, we deduce that $130-3y$ is always positive. Thus, $130-3y=130e^{-3t/200}$ and $y=\frac{130}{3}\left(1-e^{-3t/200}\right)$ kg.

(b) After one hour, $y=\frac{130}{3}\left(1-e^{-3\cdot 60/200}\right)=\frac{130}{3}\left(1-e^{-0.9}\right)\approx 25.7$ kg.

Note: As $t\to\infty$,

$y(t)\to \frac{130}{3}=43\frac{1}{3}$ kg.

41. Assume that the raindrop begins at rest, so that $v(0)=0$. $dm/dt=km$ and $(mv)'=gm\Rightarrow mv'+vm'=gm\Rightarrow mv'+v(km)=gm\Rightarrow v'+vk=g\Rightarrow dv/dt=g-kv\Rightarrow \int \frac{dv}{g-kv}=\int dt\Rightarrow -(1/k)\ln|g-kv|=t+C\Rightarrow \ln|g-kv|=-kt-kC\Rightarrow g-kv=Ae^{-kt}$. $v(0)=0\Rightarrow A=g$. So $kv=g-ge^{-kt}\Rightarrow v=(g/k)\left(1-e^{-kt}\right)$. Since $k>0$, as $t\to\infty$, $e^{-kt}\to 0$ and therefore, $\lim_{t\to\infty} v(t)=g/k$.

42. **(a)** $m\frac{dv}{dt}=-kv\Rightarrow \frac{dv}{v}=-\frac{k}{m}dt\Rightarrow \ln|v|=-\frac{k}{m}t+C$. Since $v(0)=v_0$, $\ln\left|v_0\right|=C$. Therefore, $\ln\left|\frac{v}{v_0}\right|=-\frac{k}{m}t\Rightarrow \left|\frac{v}{v_0}\right|=e^{-kt/m}\Rightarrow v(t)=\pm v_0e^{-kt/m}$. The sign is + when $t=0$, and we assume v is continuous, so that the sign is + for all t . Thus, $v(t)=v_0e^{-kt/m}$. $ds/dt=v_0e^{-kt/m}\Rightarrow s(t)=-\frac{mv_0}{k}e^{-kt/m}+C'$. From $s(0)=s_0$, we get $s_0=-\frac{mv_0}{k}+C'$, so $C'=s_0+\frac{mv_0}{k}$ and $s(t)=s_0+\frac{mv_0}{k}\left(1-e^{-kt/m}\right)$. The distance traveled from time 0 to time t is $s(t)-s_0$, so the total distance traveled is $\lim_{t\to\infty}\left[s(t)-s_0\right]=\frac{mv_0}{k}$.

Note: In finding the limit, we use the fact that $k>0$ to conclude that $\lim_{t\to\infty} e^{-kt/m}=0$.

(b) $m\frac{dv}{dt}=-kv^2\Rightarrow \frac{dv}{v^2}=-\frac{k}{m}dt\Rightarrow \frac{-1}{v}=-\frac{kt}{m}+C\Rightarrow \frac{1}{v}=\frac{kt}{m}-C$. Since $v(0)=v_0$, $C=-\frac{1}{v_0}$ and $\frac{1}{v}=\frac{kt}{m}+\frac{1}{v_0}$. Therefore, $v(t)=\frac{1}{kt/m+1/v_0}=\frac{mv_0}{kv_0t+m}$. $\frac{ds}{dt}=\frac{mv_0}{kv_0t+m}\Rightarrow s(t)=\frac{m}{k}\int\frac{kv_0dt}{kv_0t+m}=\frac{m}{k}\ln\left|kv_0t+m\right|+C'$.

Since $s(0)=s_0$, we get $s_0=\frac{m}{k}\ln m+C'\Rightarrow C'=s_0-\frac{m}{k}\ln m\Rightarrow$

$s(t)=s_0+\frac{m}{k}\left(\ln\left|kv_0t+m\right|-\ln m\right)=s_0+\frac{m}{k}\ln\left|\frac{kv_0t+m}{m}\right|$. We can rewrite the formulas for $v(t)$ and $s(t)$ as $v(t)=\frac{v_0}{1+\left(kv_0/m\right)t}$ and $s(t)=s_0+\frac{m}{k}\ln\left|1+\frac{kv_0}{m}t\right|$.

Remarks: This model of horizontal motion through a resistive medium was designed to handle the case in which

$v_0>0$. Then the term $-kv^2$ representing the resisting force causes the object to decelerate. The absolute value in the expression for $s(t)$ is unnecessary (since k , v_0 , and m are all positive), and $\lim_{t\to\infty} s(t)=\infty$. In other words, the object travels infinitely far. However, $\lim_{t\to\infty} v(t)=0$. When $v_0<0$, the term $-kv^2$ increases the magnitude of the object's negative velocity. According to the formula for $s(t)$, the position of the object approaches $-\infty$ as t approaches $m/k\left(-v_0\right)$: $\lim_{t\to -m/\left(kv_0\right)} s(t)=-\infty$. Again the object travels infinitely far, but this time the feat is accomplished in a finite amount of time. Notice also that $\lim_{t\to -m/\left(kv_0\right)} v(t)=-\infty$ when $v_0<0$, showing that the speed of the object increases without limit.

43. **(a)** The rate of growth of the area is jointly proportional to $\sqrt{A(t)}$ and $M-A(t)$; that is, the rate is proportional to the product of those two quantities. So for some constant k , $dA/dt=k\sqrt{A}\,(M-A)$. We are interested in the maximum of the function dA/dt (when the tissue grows the fastest), so we differentiate, using the Chain Rule and then substituting for dA/dt from the differential equation:

$$\frac{d}{dt}\left(\frac{dA}{dt}\right) = k\left[\sqrt{A}\,(-1)\frac{dA}{dt}+(M-A)\cdot\frac{1}{2}A^{-1/2}\frac{dA}{dt}\right]=\frac{1}{2}kA^{-1/2}\frac{dA}{dt}\left[-2A+(M-A)\right]$$
$$= \frac{1}{2}kA^{-1/2}\left[k\sqrt{A}(M-A)\right]\left[M-3A\right]=\frac{1}{2}k^2(M-A)(M-3A)$$

This is 0 when $M-A=0$ and when $M-3A=0\Leftrightarrow A(t)=M/3$. This represents a maximum by the First Derivative Test, since $\frac{d}{dt}\left(\frac{dA}{dt}\right)$ goes from positive to negative when $A(t)=M/3$.

(b) From the CAS, we get $A(t)=M\left(\dfrac{Ce^{\sqrt{M}kt}-1}{Ce^{\sqrt{M}kt}+1}\right)^2$. To get C in terms of the initial area A_0 and the maximum area M , we substitute $t=0$ and $A=A_0=A(0)$: $A_0=M\left(\dfrac{C-1}{C+1}\right)^2\Leftrightarrow(C+1)\sqrt{A_0}=(C-1)\sqrt{M}\Leftrightarrow$ $C\sqrt{A_0}+\sqrt{A_0}=C\sqrt{M}-\sqrt{M}\Leftrightarrow\sqrt{M}+\sqrt{A_0}=C\sqrt{M}-C\sqrt{A_0}\Leftrightarrow$ $\sqrt{M}+\sqrt{A_0}=C\left(\sqrt{M}-\sqrt{A_0}\right)\Leftrightarrow C=\dfrac{\sqrt{M}+\sqrt{A_0}}{\sqrt{M}-\sqrt{A_0}}$. (Notice that if $A_0=0$, then $C=1$.)

44. **(a)** According to the hint we use the Chain Rule: $m\dfrac{dv}{dt}=m\dfrac{dv}{dx}\cdot\dfrac{dx}{dt}=mv\dfrac{dv}{dx}=-\dfrac{mgR^2}{(x+R)^2}\Rightarrow$

$\int v\,dv = \int \frac{-gR^2\,dx}{(x+R)^2} \Rightarrow \frac{v^2}{2} = \frac{gR^2}{x+R} + C$. When $x=0$, $v=v_0$, so $\frac{v_0^2}{2} = \frac{gR^2}{0+R} + C \Rightarrow C = \frac{1}{2}v_0^2 - gR \Rightarrow$ $\frac{1}{2}v^2 - \frac{1}{2}v_0^2 = \frac{gR^2}{x+R} - gR$. Now at the top of its flight, the rocket's velocity will be 0 , and its height will be $x=h$. Solving for v_0 : $-\frac{1}{2}v_0^2 = \frac{gR^2}{h+R} - gR \Rightarrow \frac{v_0^2}{2} = g\left[-\frac{R^2}{R+h} + \frac{R(R+h)}{R+h}\right] = \frac{gRh}{R+h} \Rightarrow v_0 = \sqrt{\frac{2gRh}{R+h}}$.

(b) $v_e = \lim_{h\to\infty} v_0 = \lim_{h\to\infty} \sqrt{\frac{2gRh}{R+h}} = \lim_{h\to\infty} \sqrt{\frac{2gR}{(R/h)+1}} = \sqrt{2gR}$

(c) $v_e = \sqrt{2 \cdot 32\text{ft/s}^2 \cdot 3960\text{mi} \cdot 5280\text{ft/mi}} \approx 36,581\text{ ft / s} \approx 6.93\text{ mi / s}$

1. The relative growth rate is $\frac{1}{P}\frac{dP}{dt}=0.7944$, so $\frac{dP}{dt}=0.7944P$ and, by Theorem 2, $P(t)=P(0)e^{0.7944t}=2e^{0.7944t}$. Thus, $P(6)=2e^{0.7944(6)}\approx 234.99$ or about 235 members.

2. **(a)** By Theorem 2, $P(t)=P(0)e^{kt}=60e^{kt}$. In 20 minutes ($\frac{1}{3}$ hour), there are 120 cells, so $P\left(\frac{1}{3}\right)=60e^{k/3}=120\Rightarrow e^{k/3}=2\Rightarrow k/3=\ln 2\Rightarrow k=3\ln 2=\ln\left(2^3\right)=\ln 8$.

(b) $P(t)=60e^{(\ln 8)t}=60\cdot 8^t$

(c) $P(8)=60\cdot 8^8=60\cdot 2^{24}=1,006,632,960$

(d) $dP/dt=kP\Rightarrow P'(8)=kP(8)=(\ln 8)P(8)\approx 2.093$ billion cells / h

(e) $P(t)=20,000\Rightarrow 60\cdot 8^t=20,000\Rightarrow 8^t=1000/3\Rightarrow t\ln 8=\ln\left(1000/3\right)\Rightarrow t=\frac{\ln\left(1000/3\right)}{\ln 8}\approx 2.79$ h

3. **(a)** By Theorem 2, $y(t)=y(0)e^{kt}=500e^{kt}$. Now $y(3)=500e^{k(3)}=8000\Rightarrow e^{3k}=\frac{8000}{500}\Rightarrow 3k=\ln 16\Rightarrow$ $k=(\ln 16)/3$. So $y(t)=500e^{(\ln 16)t/3}=500\cdot 16^{t/3}$

(b) $y(4)=500\cdot 16^{4/3}\approx 20,159$

(c) $dy/dt=ky\Rightarrow y'(4)=ky(4)=\frac{1}{3}\ln 16\left(500\cdot 16^{4/3}\right)\approx 18,631$ cells / h

(d) $y(t)=500\cdot 16^{t/3}=30,000\Rightarrow 16^{t/3}=60\Rightarrow\frac{1}{3}t\ln 16=\ln 60\Rightarrow t=3(\ln 60)/(\ln 16)\approx 4.4$h

4. **(a)** $y(t)=y(0)e^{kt}\Rightarrow y(2)=y(0)e^{2k}=600$, $y(8)=y(0)e^{8k}=75,000$. Dividing these equations, we get $e^{8k}/e^{2k}=75,000/600\Rightarrow e^{6k}=125\Rightarrow 6k=\ln 125=\ln 5^3=3\ln 5\Rightarrow k=\frac{3}{6}\ln 5=\frac{1}{2}\ln 5$. Thus, $y(0)=600/e^{2k}=600/e^{\ln 5}=\frac{600}{5}=120$.

(b) $y(t)=y(0)e^{kt}=120e^{(\ln 5)t/2}$ or $y=120\cdot 5^{t/2}$

(c) $y(5)=120\cdot 5^{5/2}=120\cdot 25\sqrt{5}=3000\sqrt{5}\approx 6708$ bacteria.

(d) $y(t)=120\cdot 5^{t/2}\Rightarrow y'(t)=120\cdot 5^{t/2}\cdot\ln 5\cdot\frac{1}{2}=60\cdot\ln 5\cdot 5^{t/2}$.

$y'(5)=60\cdot\ln 5\cdot 5^{5/2}=60\cdot\ln 5\cdot 25\sqrt{5}\approx 5398$ bacteria / hour.

(e) $y(t)=200,000\Leftrightarrow 120e^{(\ln 5)t/2}=200,000\Leftrightarrow e^{(\ln 5)t/2}=\frac{5000}{3}\Leftrightarrow(\ln 5)t/2=\ln\frac{5000}{3}\Leftrightarrow$ $t=\left(2\ln\frac{5000}{3}\right)/\ln 5\approx 9.2$ h.

5. **(a)** Let the population (in millions) in the year t be $P(t)$. Since the initial time is the year 1750, we substitute $t-1750$ for t in Theorem 2, so the exponential model gives $P(t)=P(1750)e^{k(t-1750)}$. Then $P(1800)=980=790e^{k(1800-1750)} \Rightarrow \frac{980}{790}=e^{k(50)} \Rightarrow \ln\frac{980}{790}=50k \Rightarrow k=\frac{1}{50}\ln\frac{980}{790}\approx 0.0043104$. So with this model, we have $P(1900)=790e^{k(1900-1750)}\approx 1508$ million, and $P(1950)=790e^{k(1950-1750)}\approx 1871$ million. Both of these estimates are much too low.

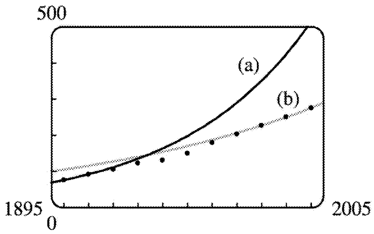
(b) In this case, the exponential model gives $P(t)=P(1850)e^{k(t-1850)} \Rightarrow$ $P(1900)=1650=1260e^{k(1900-1850)} \Rightarrow \ln\frac{1650}{1260}=k(50) \Rightarrow k=\frac{1}{50}\ln\frac{1650}{1260}\approx 0.005393$. So with this model, we estimate $P(1950)=1260e^{k(1950-1850)}\approx 2161$ million. This is still too low, but closer than the estimate of $P(1950)$ in part (a).

(c) The exponential model gives $P(t)=P(1900)e^{k(t-1900)} \Rightarrow P(1950)=2560=1650e^{k(1950-1900)} \Rightarrow$ $\ln\frac{2560}{1650}=k(50) \Rightarrow k=\frac{1}{50}\ln\frac{2560}{1650}\approx 0.008785$. With this model, we estimate $P(2000)=1650e^{k(2000-1900)}\approx 3972$ million. This is much too low. The discrepancy is explained by the fact that the world birth rate (average yearly number of births per person) is about the same as always, whereas the mortality rate (especially the infant mortality rate) is much lower, owing mostly to advances in medical science and to the wars in the first part of the twentieth century. The exponential model assumes, among other things, that the birth and mortality rates will remain constant.

6. **(a)** Let $P(t)$ be the population (in millions) in the year t . Since the initial time is the year 1900, we substitute $t-1900$ for t in Theorem 2, and find that the exponential model gives $P(t)=P(1900)e^{k(t-1900)}$ $\Rightarrow P(1910)=92=76e^{k(1910-1900)} \Rightarrow k=\frac{1}{10}\ln\frac{92}{76}\approx 0.0191$. With this model, we estimate $P(2000)=76e^{k(2000-1900)}\approx 514$ million. This estimate is much too high. The discrepancy is explained by the fact that, between the years 1900 and 1910, an enormous number of immigrants (compared to the total population) came to the United States. Since that time, immigration (as a proportion of total population) has been much lower. Also, the birth rate in the United States has declined since the turn of the century. So our calculation of the constant k was based partly on factors which no longer exist.

(b) Substituting $t-1980$ for t in Theorem 2, we find that the exponential model gives $P(t)=P(1980)e^{k(t-1980)} \Rightarrow P(1990)=250=227e^{k(1990-1980)} \Rightarrow k=\frac{1}{10}\ln\frac{250}{227}\approx 0.00965$. With this model, we estimate $P(2000)=227e^{k(2000-1980)}\approx 275.3$ million. This is quite accurate. The further estimates are $P(2010)=227e^{30k}\approx 303$ million and $P(2020)=227e^{40k}\approx 334$ million.

(c)



The model in part (a) is quite inaccurate after 1910 (off by 5 million in 1920 and 12 million in 1930). The model in part (b) is more accurate (which is not surprising, since it is based on more recent information).

7. **(a)** If $y=\left[N_2O_5\right]$ then by Theorem 2, $\frac{dy}{dt}=-0.0005y \Rightarrow y(t)=y(0)e^{-0.0005t}=Ce^{-0.0005t}$.

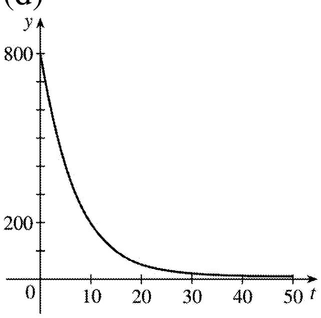
(b) $y(t)=Ce^{-0.0005t}=0.9C \Rightarrow e^{-0.0005t}=0.9 \Rightarrow -0.0005t=\ln 0.9 \Rightarrow t=-2000\ln 0.9 \approx 211$ s

8. **(a)** The mass remaining after t days is $y(t)=y(0)e^{kt}=800e^{kt}$. Since the half–life is 5.0 days,
$y(5)=800e^{5k}=400 \Rightarrow e^{5k}=\frac{1}{2} \Rightarrow$
$5k=\ln \frac{1}{2} \Rightarrow k=-(\ln 2)/5$, so $y(t)=800e^{-(\ln 2)t/5}=800\cdot 2^{-t/5}$.

(b) $y(30)=800\cdot 2^{-30/5}=12.5$ mg

(c) $800e^{-(\ln 2)t/5}=1 \Leftrightarrow -(\ln 2)\frac{t}{5}=\ln \frac{1}{800}=-\ln 800 \Leftrightarrow t=5\frac{\ln 800}{\ln 2} \approx 48$ days

(d)



9. **(a)** If $y(t)$ is the mass (in mg) remaining after t years, then $y(t)=y(0)e^{kt}=100e^{kt}$.
$y(30)=100e^{30k}=\frac{1}{2}(100) \Rightarrow e^{30k}=\frac{1}{2} \Rightarrow k=-(\ln 2)/30 \Rightarrow y(t)=100e^{-(\ln 2)t/30}=100\cdot 2^{-t/30}$

(b) $y(100)=100\cdot 2^{-100/30} \approx 9.92$ mg

(c) $100e^{-(\ln 2)t/30}=1 \Rightarrow$

$-(\ln 2)t/30=\ln \frac{1}{100} \Rightarrow t=-30 \frac{\ln 0.01}{\ln 2} \approx 199.3$ years

10. **(a)** If $y(t)$ is the mass after t days and $y(0)=A$, then $y(t)=Ae^{kt}$. $y(3)=Ae^{3k}=0.58A \Rightarrow$ $e^{3k}=0.58 \Rightarrow 3k=\ln 0.58 \Rightarrow k=\frac{1}{3}\ln 0.58$. Then $Ae^{(\ln 0.58)t/3}=\frac{1}{2}A \Leftrightarrow$ $\ln e^{(\ln 0.58)t/3}=\ln \frac{1}{2} \Leftrightarrow \frac{(\ln 0.58)t}{3}=\ln \frac{1}{2}$, so the half–life is $t=-\frac{3\ln 2}{\ln 0.58} \approx 3.82$ days.

(b) $Ae^{(\ln 0.58)t/3}=0.10A \Leftrightarrow \frac{(\ln 0.58)t}{3}=\ln \frac{1}{10} \Leftrightarrow t=-\frac{3\ln 10}{\ln 0.58} \approx 12.68$ days

11. Let $y(t)$ be the level of radioactivity. Thus, $y(t)=y(0)e^{-kt}$ and k is determined by using the half–life: $y(5730)=\frac{1}{2}y(0) \Rightarrow y(0)e^{-k(5730)}=\frac{1}{2}y(0) \Rightarrow e^{-5730k}=\frac{1}{2} \Rightarrow -5730k=\ln \frac{1}{2} \Rightarrow k=-\frac{\ln \frac{1}{2}}{5730}=\frac{\ln 2}{5730}$. If 74% of the ^{14}C remains, then we know that $y(t)=0.74y(0) \Rightarrow 0.74=e^{-t(\ln 2)/5730} \Rightarrow \ln 0.74=-\frac{t\ln 2}{5730} \Rightarrow$ $t=-\frac{5730(\ln 0.74)}{\ln 2} \approx 2489 \approx 2500$ years.

12. From the information given, we know that $\frac{dy}{dx}=2y \Rightarrow y=Ce^{2x}$ by Theorem 2. To calculate C we use the point $(0,5)$: $5=Ce^{2(0)} \Rightarrow C=5$. Thus, the equation of the curve is $y=5e^{2x}$.

13. **(a)** Using Newton's Law of Cooling, $\frac{dT}{dt}=k(T-T_s)$, we have $\frac{dT}{dt}=k(T-75)$.
Now let $y=T-75$, so $y(0)=T(0)-75=185-75=110$, so y is a solution of the initial–value problem $dy/dt=ky$ with $y(0)=110$ and by Theorem 2 we have $y(t)=y(0)e^{kt}=110e^{kt}$.
$y(30)=110e^{30k}=150-75 \Rightarrow e^{30k}=\frac{75}{110}=\frac{15}{22} \Rightarrow k=\frac{1}{30}\ln \frac{15}{22}$,
so $y(t)=110e^{\frac{1}{30}t\ln\left(\frac{15}{22}\right)}$ and $y(45)=110e^{\frac{45}{30}\ln\left(\frac{15}{22}\right)} \approx 62^\circ$ F. Thus, $T(45) \approx 62+75=137^\circ$ F.

(b) $T(t)=100 \Rightarrow y(t)=25$. $y(t)=110e^{\frac{1}{30}t\ln\left(\frac{15}{22}\right)}=25 \Rightarrow e^{\frac{1}{30}t\ln\left(\frac{15}{22}\right)}=\frac{25}{110} \Rightarrow \frac{1}{30}t\ln \frac{15}{22}=\ln \frac{25}{110} \Rightarrow$

$$t=\frac{30\ln \frac{25}{110}}{\ln \frac{15}{22}} \approx 116 \text{ min.}$$

14. **(a)** Let $T(t)$= temperature after t minutes. Newton's Law of Cooling implies that $\frac{dT}{dt}=k(T-5)$. Let $y(t)=T(t)-5$. Then $\frac{dy}{dt}=ky$, so $y(t)=y(0)e^{kt}=15e^{kt}\Rightarrow T(t)=5+15e^{kt}\Rightarrow$ $T(1)=5+15e^{k}=12\Rightarrow e^{k}=\frac{7}{15}\Rightarrow k=\ln\frac{7}{15}$, so $T(t)=5+15e^{\ln(7/15)t}$ and $T(2)=5+15e^{2\ln(7/15)}\approx 8.3^\circ$ C.

(b) $5+15e^{\ln(7/15)t}=6$ when $e^{\ln(7/15)t}=\frac{1}{15}\Rightarrow \ln\left(\frac{7}{15}\right)t=\ln\frac{1}{15}\Rightarrow t=\frac{\ln\frac{1}{15}}{\ln\frac{7}{15}}\approx 3.6$ min.

15. $\frac{dT}{dt}=k(T-20)$. Letting $y=T-20$, we get $\frac{dy}{dt}=ky$, so $y(t)=y(0)e^{kt}$. $y(0)=T(0)-20=5-20=-15$, so $y(25)=y(0)e^{25k}=-15e^{25k}$, and $y(25)=T(25)-20=10-20=-10$, so $-15e^{25k}=-10\Rightarrow e^{25k}=\frac{2}{3}$. Thus, $25k=\ln\left(\frac{2}{3}\right)$ and $k=\frac{1}{25}\ln\left(\frac{2}{3}\right)$, so $y(t)=y(0)e^{kt}=-15e^{(1/25)\ln(2/3)t}$. More simply, $e^{25k}=\frac{2}{3}\Rightarrow$ $e^{k}=\left(\frac{2}{3}\right)^{1/25}\Rightarrow e^{kt}=\left(\frac{2}{3}\right)^{t/25}\Rightarrow y(t)=-15\cdot\left(\frac{2}{3}\right)^{t/25}$.

(a) $T(50)=20+y(50)=20-15\cdot\left(\frac{2}{3}\right)^{50/25}=20-15\cdot\left(\frac{2}{3}\right)^{2}=20-\frac{20}{3}=13.\bar{3}^\circ$ C

(b) $15=T(t)=20+y(t)=20-15\cdot\left(\frac{2}{3}\right)^{t/25}\Rightarrow 15\cdot\left(\frac{2}{3}\right)^{t/25}=5\Rightarrow\left(\frac{2}{3}\right)^{t/25}=\frac{1}{3}\Rightarrow$ $(t/25)\ln\left(\frac{2}{3}\right)=\ln\left(\frac{1}{3}\right)\Rightarrow t=25\ln\left(\frac{1}{3}\right)\Big/\ln\left(\frac{2}{3}\right)\approx 67.74$ min.

16. $\frac{dT}{dt}=k(T-20)$. Let $y=T-20$. Then $\frac{dy}{dt}=ky$, so $y(t)=y(0)e^{kt}$. $y(0)=T(0)-20=95-20=75$, so $y(t)=75e^{kt}$. When $T(t)=70$, $\frac{dT}{dt}=-1^\circ$ C / min. Equivalently, $\frac{dy}{dt}=-1$ when $y(t)=50$. Thus, $-1=\frac{dy}{dt}=ky(t)=50k$ and $50=y(t)=75e^{kt}$. The first relation implies $k=-1/50$, so the second relation says $50=75e^{-t/50}$. Thus, $e^{-t/50}=\frac{2}{3}\Rightarrow -t/50=\ln\left(\frac{2}{3}\right)\Rightarrow t=-50\ln\left(\frac{2}{3}\right)\approx 20.27$ min.

17. **(a)** Let $P(h)$ be the pressure at altitude h . Then $dP/dh=kP\Rightarrow P(h)=P(0)e^{kh}=101.3e^{kh}$. $P(1000)=101.3e^{1000k}=87.14\Rightarrow 1000k=\ln\left(\frac{87.14}{101.3}\right)\Rightarrow k=\frac{1}{1000}\ln\left(\frac{87.14}{101.3}\right)\Rightarrow P(h)=101.3$ $e^{\frac{1}{1000}h\ln\left(\frac{87.14}{101.3}\right)}$, so $P(3000)=101.3e^{3\ln\left(\frac{87.14}{101.3}\right)}\approx 64.5$ kPa.

(b) $P(6187)=101.3e^{\frac{6187}{1000}\ln\left(\frac{87.14}{101.3}\right)} \approx 39.9$ kPa

18. **(a)** Using $A=A_0\left(1+\frac{r}{n}\right)^{nt}$ with $A_0=500$, $r=0.14$, and $t=2$, we have:

(i) Annually: $n=1$; $A=500\left(1+\frac{0.14}{1}\right)^{1\cdot 2}=\$649.80$

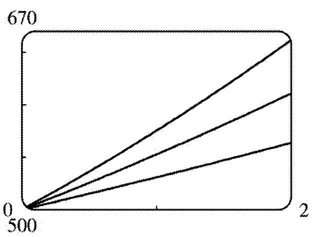
(ii) Quarterly: $n=4$; $A=500\left(1+\frac{0.14}{4}\right)^{4\cdot 2}=\$658.40$

(iii) Monthly: $n=12$; $A=500\left(1+\frac{0.14}{12}\right)^{12\cdot 2}=\$660.49$

(iv) Daily: $n=365$; $A=500\left(1+\frac{0.14}{365}\right)^{365\cdot 2}=\$661.53$

(v) Hourly: $n=365\cdot 24$; $A=500\left(1+\frac{0.14}{365\cdot 24}\right)^{365\cdot 24\cdot 2}=\$661.56$

(vi) Continuously: $A=500e^{(0.14)2}=\$661.56$

(b) 

$A_{0.14}(2)=\$661.56$, $A_{0.10}(2)=\$610.70$, and $A_{0.06}(2)=\$563.75$.

19. **(a)** Using $A=A_0\left(1+\frac{r}{n}\right)^{nt}$ with $A_0=3000$, $r=0.05$, and $t=5$, we have:

(i) Annually: $n=1$; $A=3000\left(1+\frac{0.05}{1}\right)^{1\cdot 5}=\$3828.84$

(ii) Semiannually: $n=2$; $A=3000\left(1+\frac{0.05}{2}\right)^{2\cdot 5}=\$3840.25$

(iii) Monthly: $n=12$; $A=3000\left(1+\frac{0.05}{12}\right)^{12\cdot 5}=\$3850.08$

(iv) Weekly: $n=52$; $A=3000\left(1+\frac{0.05}{52}\right)^{52\cdot 5}=\$3851.61$

(v) Daily: $n=365$; $A=3000\left(1+\frac{0.05}{365}\right)^{365\cdot 5}=\$3852.01$

(vi) Continuously: $A=3000e^{(0.05)5}=\$3852.08$

(b) $dA/dt=0.05A$ and $A(0)=3000$.

20. **(a)** $A_0e^{0.06t}=2A_0 \Leftrightarrow e^{0.06t}=2 \Leftrightarrow 0.06t=\ln 2 \Leftrightarrow t=\frac{50}{3}\ln 2\approx 11.55$, so the investment will double in about 11.55 years.

(b) The annual interest rate in $A=A_0(1+r)^t$ is r . From part (a), we have $A=A_0e^{0.06t}$. These amounts must be equal, so $(1+r)^t=e^{0.06t} \Rightarrow 1+r=e^{0.06} \Rightarrow r=e^{0.06}-1\approx 0.0618=6.18\%$, which is the equivalent annual interest rate.

21. **(a)** $\frac{dP}{dt}=kP-m=k\left(P-\frac{m}{k}\right)$. Let $y=P-\frac{m}{k}$, so $\frac{dy}{dt}=\frac{dP}{dt}$ and the differential equation becomes $\frac{dy}{dt}=ky$. The solution is $y=y_0e^{kt} \Rightarrow P-\frac{m}{k}=\left(P_0-\frac{m}{k}\right)e^{kt} \Rightarrow P(t)=\frac{m}{k}+\left(P_0-\frac{m}{k}\right)e^{kt}$.

(b) Since $k>0$, there will be an exponential expansion $\Leftrightarrow P_0-\frac{m}{k}>0 \Leftrightarrow m<kP_0$.

(c) The population will be constant if $P_0-\frac{m}{k}=0 \Leftrightarrow m=kP_0$. It will decline if $P_0-\frac{m}{k}<0 \Leftrightarrow m>kP_0$.

(d) $P_0=8,000,000$, $k=\alpha-\beta=0.016$, $m=210,000 \Rightarrow m>kP_0$ ($=128,000$), so by part (c), the population was declining.

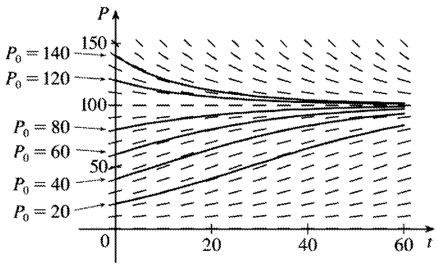
22. **(a)** $\frac{dy}{dt}=ky^{1+c} \Rightarrow y^{-1-c}dy=k\,dt \Rightarrow \frac{y^{-c}}{-c}=kt+C$. Since $y(0)=y_0$, we have $C=\frac{y_0^{-c}}{-c}$. Thus, $\frac{y^{-c}}{-c}=kt+\frac{y_0^{-c}}{-c}$, or $y^{-c}=y_0^{-c}-ckt$. So $y^c=\frac{1}{y_0^{-c}-ckt}=\frac{y_0^c}{1-cy_0^ckt}$ and $y(t)=\frac{y_0}{\left(1-cy_0^ckt\right)^{1/c}}$.

(b) $y(t)\to\infty$ as $1-cy_0^ckt\to 0$, that is, as $t\to\frac{1}{cy_0^ck}$. Define $T=\frac{1}{cy_0^ck}$. Then $\lim_{t\to T^-} y(t)=\infty$.

(c) According to the data given, we have $c=0.01$, $y(0)=2$, and $y(3)=16$, where the time t is given in months. Thus, $y_0=2$ and $16=y(3)=\frac{y_0}{\left(1-cy_0^ck\cdot 3\right)^{1/c}}$. Since $T=\frac{1}{cy_0^ck}$, we will solve for cy_0^ck .

$16=\dfrac{2}{\left(1-3cy_0^c k\right)^{100}} \Rightarrow 1-3cy_0^c k=\left(\dfrac{1}{8}\right)^{0.01}=8^{-0.01} \Rightarrow cy_0^c k=\dfrac{1}{3}\left(1-8^{-0.01}\right)$. Thus, doomsday occurs when $t=T=\dfrac{1}{cy_0^c k}=\dfrac{3}{1-8^{-0.01}}\approx 145.77$ months or 12.15 years.

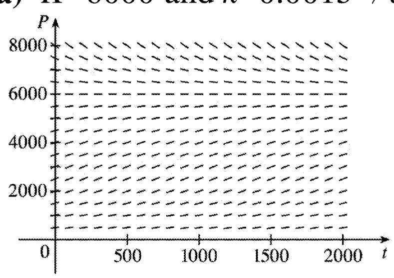
1. **(a)** $dP/dt=0.05P-0.0005P^2=0.05P(1-0.01P)=0.05P\,(1-P/100)$. Comparing to Equation 1, $dP/dt=kP\,(1-P/K)$, we see that the carrying capacity is $K=100$ and the value of k is 0.05 .
(b) The slopes close to 0 occur where P is near 0 or 100 . The largest slopes appear to be on the line $P=50$. The solutions are increasing for $0<P_0<100$ and decreasing for $P_0>100$.



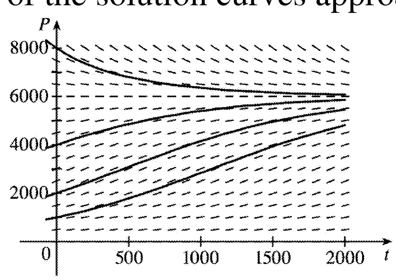
(c)
All of the solutions approach $P=100$ as t increases. As in part (b), the solutions differ since for $0<P_0<100$ they are increasing, and for $P_0>100$ they are decreasing. Also, some have an IP and some don't. It appears that the solutions which have $P_0=20$ and $P_0=40$ have inflection points at $P=50$.

(d) The equilibrium solutions are $P=0$ (trivial solution) and $P=100$. The increasing solutions move away from $P=0$ and all nonzero solutions approach $P=100$ as $t\to\infty$.

2. **(a)** $K=6000$ and $k=0.0015 \Rightarrow dP/dt=0.0015P\,(1-P/6000)$.



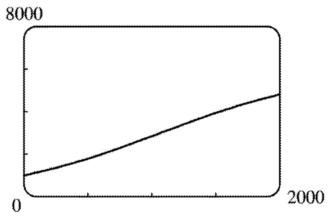
(b)
All of the solution curves approach 6000 as $t\to\infty$.



(c)
The curves with $P_0=1000$ and $P_0=2000$ appear to be concave upward at first and then concave downward. The curve with $P_0=4000$ appears to be concave downward everywhere. The curve with $P_0=8000$ appears to be concave upward everywhere. The inflection points are where the population grows the fastest.

(d) See the solution to Exercise .2.25 for a possible program to calculate $P(50)$. We find that $P(50)\approx 1064$.

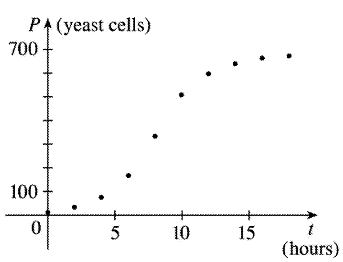
(e) Using Equation 4 with $K=6000$, $k=0.0015$, and $P_0=1000$, we have $P(t)=\dfrac{K}{1+Ae^{-kt}}=\dfrac{6000}{1+Ae^{-0.0015t}}$, where $A=\dfrac{K-P_0}{P_0}=\dfrac{6000-1000}{1000}=5$. Thus, $P(50)=\dfrac{6000}{1+5e^{-0.0015(50)}}\approx 1064.1$, which is extremely close to the estimate obtained in part (d).

(f) 

The curves are very similar.

3. **(a)** $\dfrac{dy}{dt}=ky\left(1-\dfrac{y}{K}\right)\Rightarrow y(t)=\dfrac{K}{1+Ae^{-kt}}$ with $A=\dfrac{K-y(0)}{y(0)}$. With $K=8\times 10^7$, $k=0.71$, and $y(0)=2\times 10^7$, we get the model $y(t)=\dfrac{8\times 10^7}{1+3e^{-0.71t}}$, so $y(1)=\dfrac{8\times 10^7}{1+3e^{-0.71}}\approx 3.23\times 10^7$ kg.

(b) $y(t)=4\times 10^7\Rightarrow \dfrac{8\times 10^7}{1+3e^{-0.71t}}=4\times 10^7\Rightarrow 2=1+3e^{-0.71t}\Rightarrow e^{-0.71t}=\dfrac{1}{3}\Rightarrow -0.71t=\ln\dfrac{1}{3}\Rightarrow t=\dfrac{\ln 3}{0.71}\approx 1.55$ years

4. **(a)** 

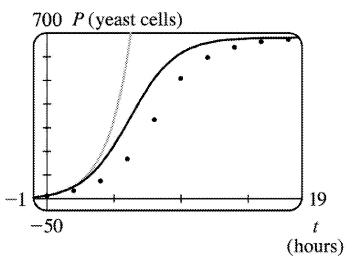
From the graph, we estimate the carrying capacity K for the yeast population to be 680 .

(b) An estimate of the initial relative growth rate is $\dfrac{1}{P_0}\dfrac{dP}{dt}=\dfrac{1}{18}\cdot\dfrac{39-18}{2-0}=\dfrac{7}{12}=0.58\overline{3}$.

(c) An exponential model is $P(t)=18e^{7t/12}$. A logistic model is $P(t)=\dfrac{680}{1+Ae^{-7t/12}}$, where $A=\dfrac{680-18}{18}=\dfrac{331}{9}$.

(d)

Time in Hours	Observed Values	Exponential Model	Logistic Model
0	18	18	18
2	39	58	55
4	80	186	149
6	171	596	322
8	336	1914	505
10	509	6147	614
12	597	19 , 739	658
14	640	63 , 389	673
16	664	203 , 558	678
18	672	653 , 679	679



The exponential model is a poor fit for anything beyond the first two observed values. The logistic model varies more for the middle values than it does for the values at either end, but provides a good general fit, as shown in the figure.

(e) $P(7)=\dfrac{680}{1+\dfrac{331}{9}e^{-7(7/12)}}\approx 420$ yeast cells

5. **(a)** We will assume that the difference in the birth and death rates is 20 million / year. Let $t=0$ correspond to the year 1990 and use a unit of 1 billion for all calculations.

$k\approx \dfrac{1}{P}\dfrac{dP}{dt}=\dfrac{1}{5.3}(0.02)=\dfrac{1}{265}$, so

$$\frac{dP}{dt}=kP\left(1-\frac{P}{K}\right)=\frac{1}{265}P\left(1-\frac{P}{100}\right), P \text{ in billions}$$

(b) $A=\dfrac{K-P_0}{P_0}=\dfrac{100-5.3}{5.3}=\dfrac{947}{53}\approx 17.8679$. $P(t)=\dfrac{K}{1+Ae^{-kt}}=\dfrac{100}{1+\dfrac{947}{53}e^{-(1/265)t}}$, so $P(10)\approx 5.49$

billion.
(c) $P(110)\approx 7.81$, and $P(510)\approx 27.72$. The predictions are 7.81 billion in the year 2100 and 27.72 billion in 2500.
(d) If $K=50$, then $P(t)=\dfrac{50}{1+\frac{447}{53}e^{-(1/265)t}}$. So $P(10)\approx 5.48$, $P(110)\approx 7.61$, and $P(510)\approx 22.41$. The predictions become 5.48 billion in the year 2000, 7.61 billion in 2100, and 22.41 billion in the year 2500.

6. **(a)** If we assume that the carrying capacity for the world population is 100 billion, it would seem reasonable that the carrying capacity for the U.S. is 3 — 5 billion by using current populations and simple proportions. We will use $K=4$ billion or 4000 million. With $t=0$ corresponding to 1980, we have

$$P(t)=\frac{4000}{1+\left(\frac{4000-250}{250}\right)e^{-kt}}=\frac{4000}{1+15e^{-kt}}$$

(b) $P(10)=275\Rightarrow \dfrac{4000}{1+15e^{-10k}}=275\Rightarrow 1+15e^{-10k}=\dfrac{4000}{275}\Rightarrow e^{-10k}=\dfrac{\frac{160}{11}-1}{15}\Rightarrow -10k=\ln\dfrac{149}{165}\Rightarrow$
$k=-\dfrac{1}{10}\ln\dfrac{149}{165}\approx 0.01019992$.
(c) $2100-1990=110$ and $P(110)\approx 680$ million.
$2200-1990=210$ and $P(210)\approx 1449$ million, or about 1.4 billion.
(d) $P(t)=300\Rightarrow \dfrac{4000}{1+15e^{-kt}}=300\Rightarrow 1+15e^{-kt}=\dfrac{40}{3}\Rightarrow e^{-kt}=\dfrac{37}{3}\cdot\dfrac{1}{15}\Rightarrow -kt=\ln\dfrac{37}{45}\Rightarrow$
$t=10\dfrac{\ln\frac{37}{45}}{\ln\frac{149}{165}}\approx 19.19\approx 19$. So we predict that the U.S. population will exceed 300 million in the year $1990+19=2009$.

7. **(a)** Our assumption is that $\dfrac{dy}{dt}=ky(1-y)$, where y is the fraction of the population that has heard the rumor.
(b) Using the logistic equation (1), $\dfrac{dP}{dt}=kP\left(1-\dfrac{P}{K}\right)$, we substitute $y=\dfrac{P}{K}$, $P=Ky$, and $\dfrac{dP}{dt}=K\dfrac{dy}{dt}$, to obtain

$K\frac{dy}{dt}=k(Ky)(1-y)\Leftrightarrow\frac{dy}{dt}=ky(1-y)$, our equation in part (a). Now the solution to (1) is $P(t)=\dfrac{K}{1+Ae^{-kt}}$,

where $A=\dfrac{K-P_0}{P_0}$. We use the same substitution to obtain $Ky=\dfrac{K}{1+\dfrac{K-Ky_0}{Ky_0}e^{-kt}}$

$\Rightarrow y=\dfrac{y_0}{y_0+\left(1-y_0\right)e^{-kt}}$.

Alternatively, we could use the same steps as outlined in ''The Analytic Solution,'' following Example 2.

(c) Let t be the number of hours since 8 A.M. Then $y_0=y(0)=\frac{80}{1000}=0.08$ and $y(4)=\frac{1}{2}$, so

$\frac{1}{2}=y(4)=\dfrac{0.08}{0.08+0.92e^{-4k}}$. Thus, $0.08+0.92e^{-4k}=0.16$, $e^{-4k}=\frac{0.08}{0.92}=\frac{2}{23}$, and $e^{-k}=\left(\frac{2}{23}\right)^{1/4}$, so

$y=\dfrac{0.08}{0.08+0.92\,(2/23)^{t/4}}=\dfrac{2}{2+23\,(2/23)^{t/4}}$. Solving this equation for t , we get

$2y+23y\left(\frac{2}{23}\right)^{t/4}=2\Rightarrow\left(\frac{2}{23}\right)^{t/4}=\frac{2-2y}{23y}\Rightarrow\left(\frac{2}{23}\right)^{t/4}=\frac{2}{23}\cdot\frac{1-y}{y}\Rightarrow\left(\frac{2}{23}\right)^{t/4-1}=\frac{1-y}{y}$. It follows

that $\frac{t}{4}-1=\dfrac{\ln\left[(1-y)/y\right]}{\ln\frac{2}{23}}$, so $t=4\left[1+\dfrac{\ln\left((1-y)/y\right)}{\ln\frac{2}{23}}\right]$.

When $y=0.9$, $\frac{1-y}{y}=\frac{1}{9}$, so $t=4\left(1-\dfrac{\ln 9}{\ln\frac{2}{23}}\right)\approx 7.6$ h or 7 h 36 min. Thus, 90% of the population will have heard the rumor by 3:36 P.M.

8. **(a)** $P(0)=P_0=400$, $P(1)=1200$ and $K=10,000$. From the solution to the logistic differential

equation $P(t)=\dfrac{P_0K}{P_0+\left(K-P_0\right)e^{-kt}}$, we get $P=\dfrac{400(10{,}000)}{400+(9600)e^{-kt}}=\dfrac{10{,}000}{1+24e^{-kt}}$. $P(1)=1200\Rightarrow$

$1+24e^{-k}=\frac{100}{12}\Rightarrow e^{k}=\frac{288}{88}\Rightarrow k=\ln\frac{36}{11}$. So $P=\dfrac{10{,}000}{1+24e^{-t\ln(36/11)}}=\dfrac{10{,}000}{1+24\cdot(11/36)^{t}}$.

(b) $5000=\dfrac{10{,}000}{1+24\,(11/36)^{t}}\Rightarrow 24\left(\frac{11}{36}\right)^{t}=1\Rightarrow t\ln\frac{11}{36}=\ln\frac{1}{24}\Rightarrow t\approx 2.68$ years.

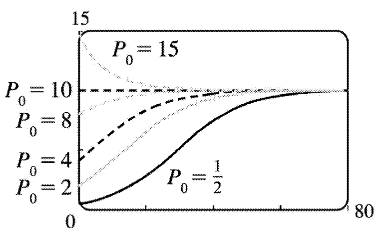
9. **(a)** $\frac{dP}{dt}=kP\left(1-\frac{P}{K}\right)\Rightarrow$

$$\frac{d^2P}{dt^2}=k\left[P\left(-\frac{1}{K}\frac{dP}{dt}\right)+\left(1-\frac{P}{K}\right)\frac{dP}{dt}\right]=k\frac{dP}{dt}\left(-\frac{P}{K}+1-\frac{P}{K}\right)$$

$$=k\left[kP\left(1-\frac{P}{K}\right)\right]\left(1-\frac{2P}{K}\right)=k^2P\left(1-\frac{P}{K}\right)\left(1-\frac{2P}{K}\right)$$

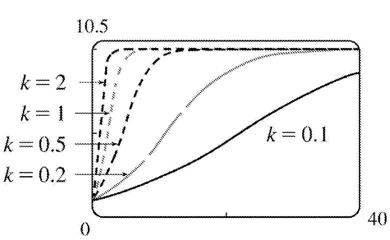
(b) P grows fastest when P' has a maximum, that is, when $P''=0$. From part (a), $P''=0\Leftrightarrow P=0$, $P=K$, or $P=K/2$. Since $0<P<K$, we see that $P''=0\Leftrightarrow P=K/2$.

10.



First we keep k constant (at 0.1 , say) and change P_0 in the function $P=\frac{10P_0}{P_0+\left(10-P_0\right)e^{-0.1t}}$. (Notice that P_0 is the $P-$ intercept.) If $P_0=0$, the function is 0 everywhere. For $0<P_0<5$, the curve has an inflection point, which moves to the right as P_0 decreases. If $5<P_0<10$, the graph is concave down everywhere. (We are

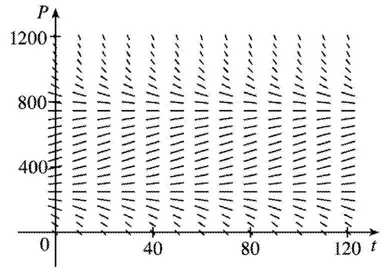
considering only $t\geq 0$.) If $P_0=10$, the function is the constant function $P=10$, and if $P_0>10$, the function decreases. For all $P_0\neq 0$, $\lim_{t\to\infty} P=10$.



Now we instead keep P_0 constant (at $P_0=1$) and change k in the function $P=\frac{10}{1+9e^{-kt}}$. It seems that as k increases, the graph approaches the line $P=10$ more and more quickly. (Note that the only difference in the shape of the curves is in the horizontal scaling; if we choose suitable x –scales, the graphs all look the same.)

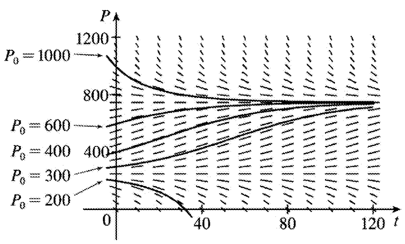
11. **(a)** The term –15 represents a harvesting of fish at a constant rate — in this case, 15 fish / week. This is the rate at which fish are caught.

(b)



(c) From the graph in part (b), it appears that $P(t)=250$ and $P(t)=750$ are the equilibrium solutions. We confirm this analytically by solving the equation $dP/dt=0$ as follows: $0.08P\,(1-P/1000)-15=0\Rightarrow$ $0.08P-0.00008P^2-15=0\Rightarrow -0.00008\left(P^2-1000P+187{,}500\right)=0\Rightarrow (P-250)(P-750)=0\Rightarrow P=250$ or 750 .

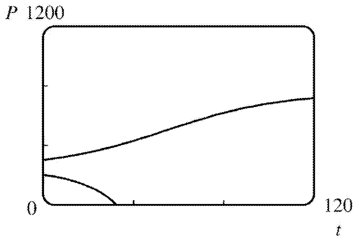
(d)

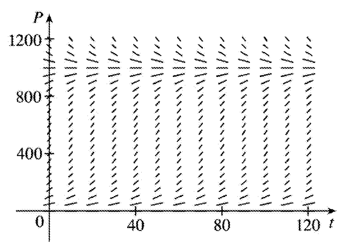


For $0<P_0<250$, $P(t)$ decreases to 0 . For $P_0=250$, $P(t)$ remains constant. For $250<P_0<750$, $P(t)$ increases and approaches 750 . For $P_0=750$, $P(t)$ remains constant. For $P_0>750$, $P(t)$ decreases and approaches 750 .

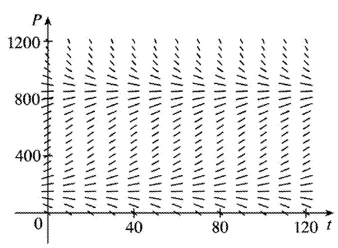
(e) $\dfrac{dP}{dt}=0.08P\left(1-\dfrac{P}{1000}\right)-15\Leftrightarrow -\dfrac{100{,}000}{8}\cdot\dfrac{dP}{dt}=\left(0.08P-0.00008P^2-15\right)\cdot\left(-\dfrac{100{,}000}{8}\right)\Leftrightarrow -12$,
$500\,\dfrac{dP}{dt}=P^2-1000P+187$, $500\Leftrightarrow \dfrac{dP}{(P-250)(P-750)}=-\dfrac{1}{12{,}500}\,dt\Leftrightarrow$
$\int\left(\dfrac{-1/500}{P-250}+\dfrac{1/500}{P-750}\right)dP=-\dfrac{1}{12{,}500}\,dt\Leftrightarrow\int\left(\dfrac{1}{P-250}-\dfrac{1}{P-750}\right)dP=\dfrac{1}{25}\,dt\Leftrightarrow$
$\ln|P-250|-\ln|P-750|=\dfrac{1}{25}\,t+C\Leftrightarrow \ln\left|\dfrac{P-250}{P-750}\right|=\dfrac{1}{25}\,t+C\Leftrightarrow\left|\dfrac{P-250}{P-750}\right|=e^{t/25+C}=ke^{t/25}\Leftrightarrow$
$\dfrac{P-250}{P-750}=ke^{t/25}\Leftrightarrow P-250=Pke^{t/25}-750ke^{t/25}\Leftrightarrow P-Pke^{t/25}=250-750ke^{t/25}\Leftrightarrow P(t)=\dfrac{250-750ke^{t/25}}{1-ke^{t/25}}$. If $t=0$ and $P=200$, then
$200=\dfrac{250-750k}{1-k}\Leftrightarrow 200-200k=250-750k\Leftrightarrow 550k=50\Leftrightarrow k=\dfrac{1}{11}$. Similarly, if $t=0$ and $P=300$, then

$k=-\frac{1}{9}$. Simplifying P with these two values of k gives us $P(t)=\frac{250\left(3e^{t/25}-11\right)}{e^{t/25}-11}$ and $P(t)=\frac{750\left(e^{t/25}+3\right)}{e^{t/25}+9}$.

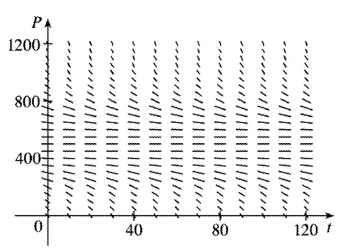


12. **(a)** 

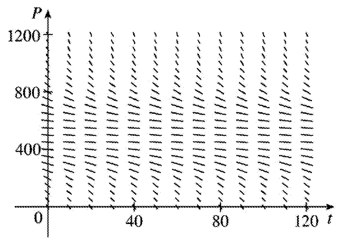
c=0



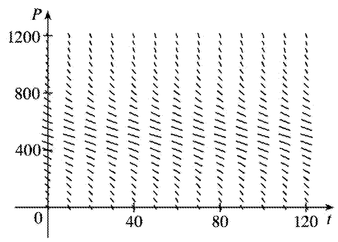
c=10



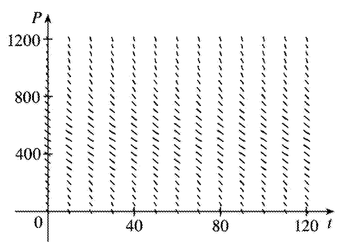
c=20



$c=21$



c=25

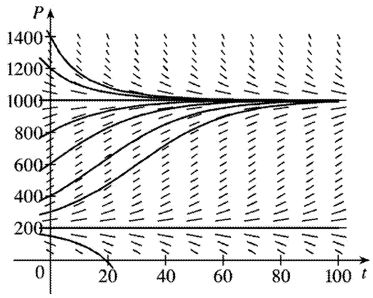


$c=30$

(b) For $0 \le c \le 20$, there is at least one equilibrium solution. For $c>20$, the population always dies out.

(c) $\frac{dP}{dt}=0.08P-0.00008P^2-c$. $\frac{dP}{dt}=0 \Leftrightarrow P=\frac{-0.08\pm\sqrt{(0.08)^2-4(-0.00008)(-c)}}{2(-0.00008)}$, which has at least one solution when the discriminant is nonnegative $\Rightarrow 0.0064-0.00032c \ge 0 \Leftrightarrow c \le 20$. For $0 \le c \le 20$, there is at least one value of P such that $dP/dt=0$ and hence, at least one equilibrium solution. For $c>20$, $dP/dt<0$ and the population always dies out.
(d) The weekly catch should be less than 20 fish per week.

13. **(a)** $\frac{dP}{dt}=(kP)\left(1-\frac{P}{K}\right)\left(1-\frac{m}{P}\right)$. If $m<P<K$, then $dP/dt=(+)(+)(+)=+ \Rightarrow P$ is increasing. If $0<P<m$, then $dP/dt=(+)(+)(-)=- \Rightarrow P$ is decreasing.
(b)



$k=0.08$, $K=1000$, and $m=200 \Rightarrow$

$$\frac{dP}{dt}=0.08P\left(1-\frac{P}{1000}\right)\left(1-\frac{200}{P}\right)$$

For $0<P_0<200$, the population dies out. For $P_0=200$, the population is steady. For $200<P_0<1000$, the population increases and approaches 1000 . For $P_0>1000$, the population decreases and approaches 1000 .

The equilibrium solutions are $P(t)=200$ and $P(t)=1000$.

(c) $\frac{dP}{dt}=kP\left(1-\frac{P}{K}\right)\left(1-\frac{m}{P}\right)=kP\left(\frac{K-P}{K}\right)\left(\frac{P-m}{P}\right)=\frac{k}{K}(K-P)(P-m) \Leftrightarrow \int \frac{dP}{(K-P)(P-m)}=\int \frac{k}{K}\,dt$.

By partial fractions, $\frac{1}{(K-P)(P-m)}=\frac{A}{K-P}+\frac{B}{P-m}$, so $A(P-m)+B(K-P)=1$.

If $P=m$, $B=\frac{1}{K-m}$; if $P=K$, $A=\frac{1}{K-m}$, so $\frac{1}{K-m}\int\left(\frac{1}{K-P}+\frac{1}{P-m}\right)dP=\int \frac{k}{K}\,dt \Rightarrow$

$\frac{1}{K-m}\left(-\ln|K-P|+\ln|P-m|\right)=\frac{k}{K}t+M \Rightarrow \frac{1}{K-m}\ln\left|\frac{P-m}{K-P}\right|=\frac{k}{K}t+M \Rightarrow \ln\left|\frac{P-m}{K-P}\right|=(K-m)\frac{k}{K}t+M_1$

$\Leftrightarrow \frac{P-m}{K-P}=De^{(K-m)(k/K)t}$.

Let $t=0$: $\frac{P_0-m}{K-P_0}=D$. So $\frac{P-m}{K-P}=\frac{P_0-m}{K-P_0}e^{(K-m)(k/K)t}$. Solving for P , we get

$$P(t)=\frac{m\left(K-P_0\right)+K\left(P_0-m\right)e^{(K-m)(k/K)t}}{K-P_0+\left(P_0-m\right)e^{(K-m)(k/K)t}} .$$

(d) If $P_0<m$, then $P_0-m<0$. Let $N(t)$ be the numerator of the expression for $P(t)$ in part (c). Then $N(0)=P_0(K-m)>0$, and $P_0-m<0 \Leftrightarrow \lim_{t\to\infty} K\left(P_0-m\right)e^{(K-m)(k/K)t}=-\infty \Rightarrow \lim_{t\to\infty} N(t)=-\infty$. Since N is continuous, there is a number t such that $N(t)=0$ and thus $P(t)=0$. So the species will become extinct.

14. **(a)**

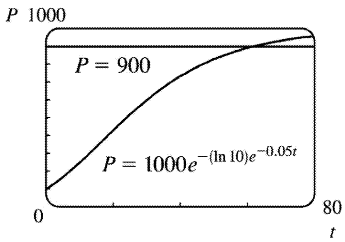
$\frac{dP}{dt}=c\ln\left(\frac{K}{P}\right)P \Rightarrow \int \frac{dP}{P\ln(K/P)}=\int c\,dt$. Let $u=\ln\left(\frac{K}{P}\right)=\ln K-\ln P \Rightarrow du=-\frac{dP}{P}$

$\Rightarrow \int -\frac{du}{u}=ct+D \Rightarrow \ln|u|=-ct-D \Rightarrow |u|=e^{-(ct+D)} \Rightarrow |\ln(K/P)|=e^{-(ct+D)} \Rightarrow$

$\ln(K/P)=\pm e^{-(ct+D)}$. Letting $t=0$, we get $\ln\left(K/P_0\right)=\pm e^{-D}$, so

$\ln(K/P)=\pm e^{-ct-D}=\pm e^{-ct}e^{-D}=\ln\left(K/P_0\right)e^{-ct} \Rightarrow K/P=e^{\ln\left(K/P_0\right)e^{-ct}} \Rightarrow P(t)=Ke^{-\ln\left(K/P_0\right)e^{-ct}}$, $c\neq 0$.

(b) $\lim_{t\to\infty} P(t)=\lim_{t\to\infty} Ke^{-\ln\left(K/P_0\right)e^{-ct}}=Ke^{-\ln\left(K/P_0\right)\cdot 0}=Ke^0=K$



(c)

The graphs look very similar. For the Gompertz function, $P(40)\approx 732$, nearly the same as the logistic function. The Gompertz function reaches $P=900$ at $t\approx 61.7$ and its value at $t=80$ is about 959 , so it doesn't increase quite as fast as the logistic curve.

(d) $\frac{dP}{dt}=c\ln\left(\frac{K}{P}\right)P=cP(\ln K-\ln P)\Rightarrow$

$$\frac{d^2P}{dt^2} = c\left[P\left(-\frac{1}{P}\frac{dP}{dt}\right)+(\ln K-\ln P)\frac{dP}{dt}\right]=c\frac{dP}{dt}\left[-1+\ln\left(\frac{K}{P}\right)\right]$$

$$= c[c\ln(K/P)P][\ln(K/P)-1]=c^2P\ln(K/P)[\ln(K/P)-1]$$

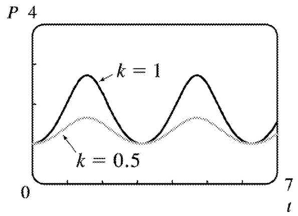
Since $0<P<K$, $P''=0 \Leftrightarrow \ln(K/P)=1 \Leftrightarrow K/P=e \Leftrightarrow P=K/e$. $P''>0$ for $0<P<K/e$ and $P''<0$ for $K/e<P<K$, so P' is a maximum (and P grows fastest) when $P=K/e$.

Note: If $P>K$, then $\ln(K/P)<0$, so $P''(t)>0$.

15. **(a)** $dP/dt=kP\cos(rt-\phi) \Rightarrow (dP)/P=k\cos(rt-\phi)\,dt \Rightarrow \int(dP)/P=k\square(rt-\phi)\,dt \Rightarrow$
$\ln P=(k/r)\sin(rt-\phi)+C$. (Since this is a growth model, $P>0$ and we can write $\ln P$ instead of $\ln|P|$.) Since $P(0)=P_0$, we obtain $\ln P_0=(k/r)\sin(-\phi)+C=-(k/r)\sin\phi+C \Rightarrow C=\ln P_0+(k/r)\sin\phi$. Thus, $\ln P=(k/r)\sin(rt-\phi)+\ln P_0+(k/r)\sin\phi$, which we can rewrite as $\ln\left(P/P_0\right)=(k/r)[\sin(rt-\phi)+\sin\phi]$ or, after exponentiation, $P(t)=P_0e^{(k/r)[\sin(rt-\phi)+\sin\phi]}$.

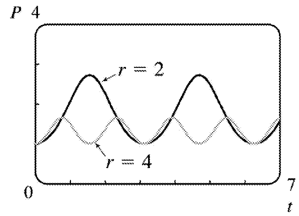
(b)

As k increases, the amplitude increases, but the minimum value stays the same.



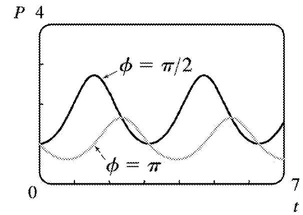
Comparing values of k with $P_0 = 1$, $r = 2$, and $\phi = \pi/2$

As r increases, the amplitude and the period decrease.



Comparing values of r with $P_0 = 1$, $k = 1$, and $\phi = \pi/2$

A change in ϕ produces slight adjustments in the phase shift and amplitude.



Comparing values of ϕ with $P_0 = 1$, $k = 1$, and $r = 2$

$P(t)$ oscillates between $P_0 e^{(k/r)(1+\sin\phi)}$ and $P_0 e^{(k/r)(-1+\sin\phi)}$ (the extreme values are attained when $rt-\phi$ is an odd multiple of $\frac{\pi}{2}$), so $\lim_{t\to\infty} P(t)$ does not exist.

16. **(a)** $dP/dt=kP\cos^2(rt-\phi)\Rightarrow (dP)/P=k\cos^2(rt-\phi)\,dt\Rightarrow \int (dP)/P=k\square^2(rt-\phi)\,dt\Rightarrow$

$\ln P=k\int \frac{1+\cos(2(rt-\phi))}{2}\,dt=\frac{k}{2}t+\frac{k}{4r}(2(rt-\phi))+C$. From $P(0)=P_0$, we get

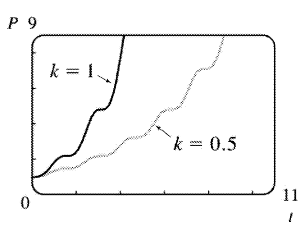
$\ln P_0=\frac{k}{4r}(-2\phi)+C=C-\frac{k}{4r}\sin 2\phi$, so $C=\ln P_0+\frac{k}{4r}\sin 2\phi$ and

$\ln P=\frac{k}{2}t+\frac{k}{4r}\sin(2(rt-\phi))+\ln P_0+\frac{k}{4r}\sin 2\phi$. Simplifying, we get

$\ln\frac{P}{P_0}=\frac{k}{2}t+\frac{k}{4r}[\sin(2(rt-\phi))+\sin 2\phi]=f(t)$, or $P(t)=P_0 e^{f(t)}$.

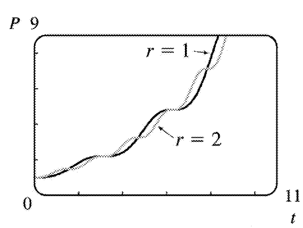
(b)

An increase in k stretches the graph of P vertically while maintaining $P(0)=P_0$.



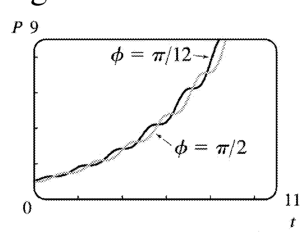
Comparing values of k with $P_0 = 1$, $r = 2$, and $\phi = \pi/2$

An increase in r compresses the graph of P horizontally — similar to changing the period in Exercise 15.



Comparing values of r with $P_0 = 1$, $k = 0.5$, and $\phi = \pi/2$

As in Exercise 15, a change in ϕ only makes slight adjustments in the growth of P , as shown in the figure.



Comparing values of ϕ with $P_0 = 1$, $k = 0.5$, and $r = 2$

$f'(t)=k/2+[k/(4r)][2r\cos(2(rt-\phi))]=(k/2)[1+\cos(2(rt-\phi))]\geq 0$. Since $P(t)=P_0 e^{f(t)}$, we have $P'(t)=P_0 f'(t)e^{f(t)}\geq 0$, with equality only when $\cos(2(rt-\phi))=-1$; that is, when $rt-\phi$ is an odd multiple of $\frac{\pi}{2}$. Therefore, $P(t)$ is an increasing function on $(0,\infty)$. P can also be written as $P(t)=P_0 e^{kt/2} e^{(k/4r)[\sin(2(rt-\phi))+\sin 2\phi]}$. The second exponential oscillates between $e^{(k/4r)(1+\sin 2\phi)}$ and $e^{(k/4r)(-1+\sin 2\phi)}$, while the first one, $e^{kt/2}$, grows without bound. So $\lim_{t\to\infty} P(t)=\infty$.

17. By Equation (4), $P(t)=\frac{K}{1+Ae^{-kt}}$. By comparison, if $c=(\ln A)/k$ and $u=\frac{1}{2}k(t-c)$, then

$1+\tanh u=1+\frac{e^u-e^{-u}}{e^u+e^{-u}}=\frac{e^u+e^{-u}}{e^u+e^{-u}}+\frac{e^u-e^{-u}}{e^u+e^{-u}}=\frac{2e^u}{e^u+e^{-u}}\cdot\frac{e^{-u}}{e^{-u}}=\frac{2}{1+e^{-2u}}$ and

$e^{-2u}=e^{-k(t-c)}=e^{kc}e^{-kt}=e^{\ln A}e^{-kt}=Ae^{-kt}$, so

$\frac{1}{2}K\left[1+\tanh\left(\frac{1}{2}k(t-c)\right)\right]=\frac{K}{2}[1+\tanh u]=\frac{K}{2}\cdot\frac{2}{1+e^{-2u}}=\frac{K}{1+e^{-2u}}=\frac{K}{1+Ae^{-kt}}=P(t)$.

1. $y'+e^{x}y=x^{2}y^{2}$ is not linear since it cannot be put into the standard linear form (1), $y'+P(x)y=Q(x)$.

2. $y+\sin x=x^{3}y' \Rightarrow x^{3}y'-y=\sin x \Rightarrow y'+\left(-\frac{1}{x^{3}}\right)y=\frac{\sin x}{x^{3}}$. This equation is in the standard linear form (1), so it is linear.

3. $xy'+\ln x-x^{2}y=0 \Rightarrow xy'-x^{2}y=-\ln x \Rightarrow y'+(-x)y=-\frac{\ln x}{x}$, which is in the standard linear form (1), so this equation is linear.

4. $y'+\cos y=\tan x$ is not linear since it cannot be put into the standard linear form (1), $y'+P(x)y=Q(x)$.

5. Comparing the given equation, $y'+2y=2e^{x}$, with the general form, $y'+P(x)y=Q(x)$, we see that $P(x)=2$ and the integrating factor is $I(x)=e^{\int P(x)\,dx}=e^{\int 2\,dx}=e^{2x}$. Multiplying the differential equation by $I(x)$ gives $e^{2x}y'+2e^{2x}y=2e^{3x} \Rightarrow \left(e^{2x}y\right)'=2e^{3x} \Rightarrow e^{2x}y=\int 2e^{3x}dx \Rightarrow e^{2x}y=\frac{2}{3}e^{3x}+C \Rightarrow y=\frac{2}{3}e^{x}+Ce^{-2x}$.

6. $y'=x+5y \Rightarrow y'-5y=x$. $I(x)=e^{\int P(x)\,dx}=e^{\int (-5)\,dx}=e^{-5x}$. Multiplying the differential equation by $I(x)$ gives $e^{-5x}y'-5e^{-5x}y=xe^{-5x} \Rightarrow \left(e^{-5x}y\right)'=xe^{-5x} \Rightarrow e^{-5x}y=\int xe^{-5x}dx=-\frac{1}{5}xe^{-5x}-\frac{1}{25}e^{-5x}+C$ [by parts] $\Rightarrow$ $y=-\frac{1}{5}x-\frac{1}{25}+Ce^{5x}$

7. $xy'-2y=x^{2}$ [divide by x] $\Rightarrow y'+\left(-\frac{2}{x}\right)y=x$ (*).
$I(x)=e^{\int P(x)dx}=e^{\int (-2/x)dx}=e^{-2\ln|x|}=e^{\ln|x|^{-2}}=e^{\ln(1/x^{2})}=1/x^{2}$. Multiplying the differential equation (*) by $I(x)$ gives $\frac{1}{x^{2}}y'-\frac{2}{x^{3}}y=\frac{1}{x} \Rightarrow \left(\frac{1}{x^{2}}y\right)'=\frac{1}{x} \Rightarrow \frac{1}{x^{2}}y=\ln|x|+C \Rightarrow y=x^{2}(\ln|x|+C)=x^{2}\ln|x|+Cx^{2}$.

8. $x^{2}y'+2xy=\cos^{2}x \Rightarrow y'+\frac{2}{x}y=\frac{\cos^{2}x}{x^{2}}$. $I(x)=e^{\int P(x)dx}=e^{\int 2/x\,dx}=e^{2\ln|x|}=e^{\ln(x^{2})}=x^{2}$. Multiplying by $I(x)$ gives us our original equation back. You may have noticed this immediately, since $P(x)$ is the derivative of the coefficient of y' . We rewrite it as $(x^{2}y)'=\cos^{2}x$. Thus,
$x^{2}y=\int\cos^{2}x\,dx=\int\frac{1}{2}(1+\cos 2x)\,dx=\frac{1}{2}x+\frac{1}{4}\sin 2x+C \Rightarrow$

$y=\frac{1}{2x}+\frac{1}{4x^2}\sin 2x+\frac{C}{x^2}$ or $y=\frac{1}{2x}+\frac{1}{2x^2}\sin x\cos x+\frac{C}{x^2}$.

9. Since $P(x)$ is the derivative of the coefficient of y', we can write the differential equation $xy'+y=\sqrt{x}$ in the easily integrable form $(xy)'=\sqrt{x}\Rightarrow xy=\frac{2}{3}x^{3/2}+C\Rightarrow y=\frac{2}{3}\sqrt{x}+C/x$.

10. $y'-y=1/x$, so $I(x)=e^{\int(-1)\,dx}=e^{-x}$. Multiplying the differential equation by $I(x)$ gives $e^{-x}y'-e^{-x}y=e^{-x}/x\Rightarrow\left(e^{-x}y\right)'=e^{-x}/x\Rightarrow y=e^{x}\left[\int\left(e^{-x}/x\right)dx+C\right]$.

11. $I(x)=e^{\int 2x\,dx}=e^{x^2}$. Multiplying the differential equation $y'+2xy=x^2$ by $I(x)$ gives $e^{x^2}y'+2xe^{x^2}y=x^2e^{x^2}\Rightarrow\left(e^{x^2}y\right)'=x^2e^{x^2}$. Thus
$y=e^{-x^2}\left[\int x^2e^{x^2}dx+C\right]=e^{-x^2}\left[\frac{1}{2}xe^{x^2}-\int\frac{1}{2}e^{x^2}dx+C\right]=\frac{1}{2}x+Ce^{-x^2}-e^{-x^2}\int\frac{1}{2}e^{x^2}dx$.

12. $I(x)=e^{\int-\tan x\,dx}=e^{\ln|\cos x|}=\cos x$ (since $-\frac{\pi}{2}<x<\frac{\pi}{2}$). Multiplying the differential equation by $I(x)$ gives $y'\cos x-y\tan x\cos x=x\cos x\sin 2x\Rightarrow(y\cos x)'=x\cos x\sin 2x$. So

$$y=\frac{1}{\cos x}\left[\int x\cos x\sin 2x\,dx+C\right]=\frac{1}{\cos x}\left[\int 2x\cos^2 x\sin x\,dx+C\right]$$
$$=\frac{1}{\cos x}\left[\frac{-2x\cos^3 x}{3}+\frac{2}{3}\left(\sin x-\frac{\sin^3 x}{3}\right)+C\right]=\frac{-2x\cos^2 x}{3}+\frac{C}{\cos x}+2\tan x\,\frac{3-\sin^2 x}{9}$$

13. $(1+t)\frac{du}{dt}+u=1+t$, $t>0$ [divide by $1+t$] $\Rightarrow\frac{du}{dt}+\frac{1}{1+t}u=1$ (*), which has the form $u'+P(t)u=Q(t)$. The integrating factor is $I(t)=e^{\int P(t)dt}=e^{\int[1/(1+t)]\,dt}=e^{\ln(1+t)}=1+t$. Multiplying (*) by $I(t)$ gives us our original equation back. We rewrite it as $[(1+t)u]'=1+t$. Thus,
$(1+t)u=\int(1+t)\,dt=t+\frac{1}{2}t^2+C\Rightarrow u=\frac{t+\frac{1}{2}t^2+C}{1+t}$ or $u=\frac{t^2+2t+2C}{2(t+1)}$.

14. $t\ln t\,\frac{dr}{dt}+r=te^t\Rightarrow\frac{dr}{dt}+\frac{1}{t\ln t}r=\frac{e^t}{\ln t}$. $I(t)=e^{\int dt/(t\ln t)}=e^{\ln(\ln t)}=\ln t$. Multiplying by $\ln t$ gives

$\ln t \dfrac{dr}{dt} + \dfrac{1}{t} r = e^t \Rightarrow [(\ln t)r]' = e^t \Rightarrow (\ln t)r = e^t + C \Rightarrow r = \dfrac{e^t + C}{\ln t}$.

15. $y' = x + y \Rightarrow y' + (-1)y = x$. $I(x) = e^{\int(-1)dx} = e^{-x}$. Multiplying by e^{-x} gives $e^{-x}y' - e^{-x}y = xe^{-x} \Rightarrow$ $(e^{-x}y)' = xe^{-x} \Rightarrow e^{-x}y = \int xe^{-x}dx = -xe^{-x} - e^{-x} + C$ [integration by parts with $u = x$, $dv = e^{-x}dx$] $\Rightarrow y = -x - 1 + Ce^x$. $y(0) = 2 \Rightarrow -1 + C = 2 \Rightarrow C = 3$, so $y = -x - 1 + 3e^x$.

16. $t \dfrac{dy}{dt} + 2y = t^3$, $t > 0$, $y(1) = 0$. Divide by t to get $\dfrac{dy}{dt} + \dfrac{2}{t} y = t^2$, which is linear. $I(t) = e^{\int(2/t)dt} = e^{2\ln t} = t^2$. Multiplying by t^2 gives $t^2 \dfrac{dy}{dt} + 2ty = t^4 \Rightarrow \left(t^2 y\right)' = t^4 \Rightarrow t^2 y = \dfrac{1}{5} t^5 + C \Rightarrow y = \dfrac{t^3}{5} + \dfrac{C}{t^2}$. Thus, $0 = y(1) = \dfrac{1}{5} + C$ $\Rightarrow C = -\dfrac{1}{5}$, so $y = \dfrac{t^3}{5} - \dfrac{1}{5t^2}$.

17. $\dfrac{dv}{dt} - 2tv = 3t^2 e^{t^2}$, $v(0) = 5$. $I(t) = e^{\int(-2t)dt} = e^{-t^2}$. Multiply the differential equation by $I(t)$ to get $e^{-t^2} \dfrac{dv}{dt} - 2te^{-t^2} v = 3t^2 \Rightarrow \left(e^{-t^2} v\right)' = 3t^2 \Rightarrow e^{-t^2} v = \int 3t^2 dt = t^3 + C \Rightarrow v = t^3 e^{t^2} + Ce^{t^2}$. $5 = v(0) = 0 \cdot 1 + C \cdot 1 = C$, so $v = t^3 e^{t^2} + 5e^{t^2}$.

18. $2xy' + y = 6x$, $x > 0 \Rightarrow y' + \dfrac{1}{2x} y = 3$. $I(x) = e^{\int 1/(2x)dx} = e^{(1/2)\ln x} = e^{\ln x^{1/2}} = \sqrt{x}$. Multiplying by $\sqrt{x}$ gives $\sqrt{x}\, y' + \dfrac{1}{2\sqrt{x}} y = 3\sqrt{x} \Rightarrow \left(\sqrt{x}\, y\right)' = 3\sqrt{x} \Rightarrow \sqrt{x}\, y = \int 3\sqrt{x}\, dx = 2x^{3/2} + C \Rightarrow y = 2x + \dfrac{C}{\sqrt{x}}$. $y(4) = 20 \Rightarrow$ $8 + \dfrac{C}{2} = 20 \Rightarrow C = 24$, so $y = 2x + \dfrac{24}{\sqrt{x}}$.

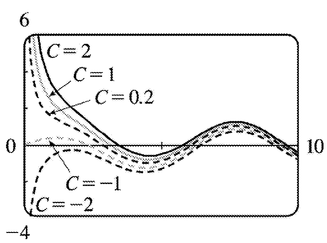
19. $xy' = y + x^2 \sin x \Rightarrow y' - \dfrac{1}{x} y = x\sin x$. $I(x) = e^{\int(-1/x)dx} = e^{-\ln x} = e^{\ln x^{-1}} = \dfrac{1}{x}$. Multiplying by $\dfrac{1}{x}$ gives $\dfrac{1}{x} y' - \dfrac{1}{x^2} y = \sin x \Rightarrow \left(\dfrac{1}{x} y\right)' = \sin x \Rightarrow \dfrac{1}{x} y = -\cos x + C \Rightarrow y = -x\cos x + Cx$. $y(\pi) = 0 \Rightarrow -\pi \cdot (-1) + C\pi = 0 \Rightarrow C = -1$, so $y = -x\cos x - x$.

20. $x \dfrac{dy}{dx} - \dfrac{y}{x+1} = x \Rightarrow y' - \dfrac{y}{x(x+1)} = 1\,(x > 0)$, so $I(x) = e^{-\int 1/[x(x+1)]dx} = e^{-(\ln|x| - \ln|x+1|)} = \dfrac{x+1}{x}$. Multiplying the differential equation by $I(x)$ gives

$\frac{x+1}{x}y' - \frac{y}{x(x+1)}\frac{x+1}{x} = \frac{x+1}{x} \Rightarrow \left(\frac{x+1}{x}y\right)' = \frac{x+1}{x}$. Then $y = \frac{x}{x+1}\left[\int\left(1+\frac{1}{x}\right)dx + C\right] = \frac{x}{x+1}(x+\ln x + C)$. But $0 = y(1) = \frac{1}{2}[1+C]$ so $C=-1$ and the solution to the initial–value problem is $y = \frac{x}{x+1}(x-1+\ln x)$.

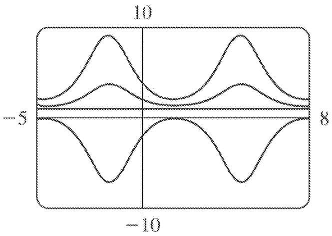
21. $y' + \frac{1}{x}y = \cos x$ ($x \neq 0$), so $I(x) = e^{\int(1/x)\,dx} = e^{\ln|x|} = x$ (for $x>0$). Multiplying the differential equation by $I(x)$ gives $xy' + y = x\cos x \Rightarrow (xy)' = x\cos x$. Thus,

$$y = \frac{1}{x}\left[\int x\cos x\,dx + C\right] = \frac{1}{x}[x\sin x + \cos x + C]$$
$$= \sin x + \frac{\cos x}{x} + \frac{C}{x}$$



The solutions are asymptotic to the $y-$ axis (except for $C=-1$). In fact, for $C>-1$, $y\to\infty$ as $x\to 0^+$, whereas for $C<-1$, $y\to-\infty$ as $x\to 0^+$. As x gets larger, the solutions approximate $y=\sin x$ more closely. The graphs for larger C lie above those for smaller C . The distance between the graphs lessens as x increases.

22. $I(x) = e^{\int\cos x\,dx} = e^{\sin x}$. Multiplying the differential equation by $I(x)$ gives $e^{\sin x}y' + \cos x\cdot e^{\sin x}y = \cos x\cdot e^{\sin x} \Rightarrow \left(e^{\sin x}y\right)' = \cos x\cdot e^{\sin x} \Rightarrow y = e^{-\sin x}\left[\int\cos x\cdot e^{\sin x}dx + C\right] = 1 + Ce^{-\sin x}$. The graphs for $C=-3$, 0 , 1 , and 3 are shown. As the values of C get further from zero the graph is stretched away from the line $y=1$, which is the value for $C=0$. The graphs are all periodic in x , with a period of 2π .



23. Setting $u=y^{1-n}$, $\frac{du}{dx}=(1-n)y^{-n}\frac{dy}{dx}$ or $\frac{dy}{dx}=\frac{y^{n}}{1-n}\frac{du}{dx}=\frac{u^{n/(1-n)}}{1-n}\frac{du}{dx}$. Then the Bernoulli differential equation becomes $\frac{u^{n/(1-n)}}{1-n}\frac{du}{dx}+P(x)u^{1/(1-n)}=Q(x)u^{n/(1-n)}$ or $\frac{du}{dx}+(1-n)P(x)u=Q(x)(1-n)$.

24. Here $y'+\frac{y}{x}=-y^{2}$, so $n=2$, $P(x)=\frac{1}{x}$ and $Q(x)=-1$. Setting $u=y^{-1}$, u satisfies $u'-\frac{1}{x}u=1$. Then $I(x)=e^{\int(-1/x)\,dx}=\frac{1}{x}$ (for $x>0$) and $u=x\left(\int\frac{1}{x}\,dx+C\right)=x(\ln|x|+C)$. Thus, $y=\frac{1}{x(C+\ln|x|)}$.

25. $y'+\frac{2}{x}y=\frac{y^{3}}{x^{2}}$. Here $n=3$, $P(x)=\frac{2}{x}$, $Q(x)=\frac{1}{x^{2}}$ and setting $u=y^{-2}$, u satisfies $u'-\frac{4u}{x}=-\frac{2}{x^{2}}$.
Then $I(x)=e^{\int(-4/x)\,dx}=x^{-4}$ and $u=x^{4}\left(\int-\frac{2}{x^{6}}\,dx+C\right)=x^{4}\left(\frac{2}{5x^{5}}+C\right)=Cx^{4}+\frac{2}{5x}$.
Thus, $y=\pm\left(Cx^{4}+\frac{2}{5x}\right)^{-1/2}$.

26. Here $n=3$, $P(x)=1$, $Q(x)=x$ and setting $u=y^{-2}$, u satisfies $u'-2u=-2x$. Then $I(x)=e^{\int(-2)\,dx}=e^{-2x}$ and $u=e^{2x}\left[\int-2xe^{-2x}dx+C\right]=e^{2x}\left(xe^{-2x}+\frac{1}{2}e^{-2x}+C\right)=x+\frac{1}{2}+Ce^{2x}$.
So $y^{-2}=x+\frac{1}{2}+Ce^{2x}\Rightarrow y=\pm\left[x+\frac{1}{2}+Ce^{2x}\right]^{-1/2}$.

27. **(a)** $2\frac{dI}{dt}+10I=40$ or $\frac{dI}{dt}+5I=20$. Then the integrating factor is $e^{\int 5\,dt}=e^{5t}$. Multiplying the differential equation by the integrating factor gives $e^{5t}\frac{dI}{dt}+5Ie^{5t}=20e^{5t}\Rightarrow\left(e^{5t}I\right)'=20e^{5t}\Rightarrow$ $I(t)=e^{-5t}\left[\int 20e^{5t}\,dt+C\right]=4+Ce^{-5t}$. But $0=I(0)=4+C$, so $I(t)=4-4e^{-5t}$.
(b) $I(0.1)=4-4e^{-0.5}\approx 1.57$ A

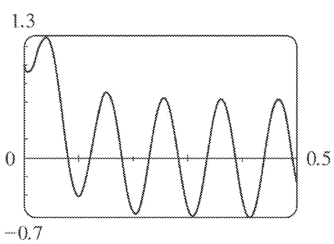
28. **(a)** $\frac{dI}{dt}+20I=40\sin 60t$, so the integrating factor is e^{20t} . Multiplying the differential equation by the integrating factor gives $e^{20t}\frac{dI}{dt}+20Ie^{20t}=40e^{20t}\sin 60t\Rightarrow \left(e^{20t}I\right)'=40e^{20t}\sin 60t\Rightarrow$

$$\begin{aligned}I(t)&=e^{-20t}\left[\int 40e^{20t}\sin 60t\,dt+C\right]\\&=e^{-20t}\left[40e^{20t}\left(\frac{1}{4000}\right)(20\sin 60t-60\cos 60t)\right]+Ce^{-20t}\\&=\frac{\sin 60t-3\cos 60t}{5}+Ce^{-20t}\end{aligned}$$

But $1=I(0)=-\frac{3}{5}+C$, so $I(t)=\frac{\sin 60t-3\cos 60t+8e^{-20t}}{5}$.

(b)

$$\begin{aligned}I(0.1)&=\frac{\sin 6-3\cos 6+8e^{-2}}{5}\\&\approx -0.42\text{ A}\end{aligned}$$

(c) 

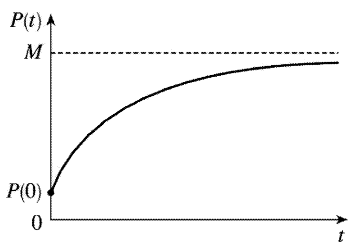
29. $5\frac{dQ}{dt}+20Q=60$ with $Q(0)=0$ C. Then the integrating factor is $e^{\int 4dt}=e^{4t}$, and multiplying the differential equation by the integrating factor gives $e^{4t}\frac{dQ}{dt}+4e^{4t}Q=12e^{4t}\Rightarrow\left(e^{4t}Q\right)'=12e^{4t}\Rightarrow$ $Q(t)=e^{-4t}\left[\int 12e^{4t}dt+C\right]=3+Ce^{-4t}$. But $0=Q(0)=3+C$ so $Q(t)=3\left(1-e^{-4t}\right)$ is the charge at time t and $I=dQ/dt=12e^{-4t}$ is the current at time t .

30. $2\frac{dQ}{dt}+100Q=10\sin 60t$ or $\frac{dQ}{dt}+50Q=5\sin 60t$. Then the integrating factor is $e^{\int 50dt}=e^{50t}$, and multiplying the differential equation by the integrating factor gives $e^{50t}\frac{dQ}{dt}+50e^{50t}Q=5e^{50t}\sin 60t\Rightarrow$ $\left(e^{50t}Q\right)'=5e^{50t}\sin 60t\Rightarrow$

$$Q(t)=e^{-50t}\left[\int 5e^{50t}\sin 60t\,dt+C\right]=e^{-50t}\left[5e^{50t}\left(\frac{1}{6100}\right)(50\sin 60t-60\cos 60t)\right]+Ce^{-50t}$$
$$=\frac{1}{122}(5\sin 60t-6\cos 60t)+Ce^{-50t}$$

But $0=Q(0)=-\frac{6}{122}+C$ so $C=\frac{3}{61}$ and $Q(t)=\frac{5\sin 60t-6\cos 60t}{122}+\frac{3e^{-50t}}{61}$ is the charge at time t , while the current is $I(t)=\frac{dQ}{dt}=\frac{150\cos 60t+180\sin 60t-150e^{-50t}}{61}$.

31. $\frac{dP}{dt}+kP=kM$, so $I(t)=e^{\int k\,dt}=e^{kt}$. Multiplying the differential equation by $I(t)$ gives $e^{kt}\frac{dP}{dt}+kPe^{kt}=kMe^{kt}\Rightarrow\left(e^{kt}P\right)'=kMe^{kt}\Rightarrow P(t)=e^{-kt}\left(\int kMe^{kt}dt+C\right)=M+Ce^{-kt}$, $k>0$. Furthermore, it is reasonable to assume that $0\le P(0)\le M$, so $-M\le C\le 0$.



32. Since $P(0)=0$, we have $P(t)=M(1-e^{-kt})$. If $P_1(t)$ is Jim's learning curve, then $P_1(1)=25$ and $P_1(2)=45$. Hence, $25=M_1(1-e^{-k})$ and $45=M_1(1-e^{-2k})$, so $1-25/M_1=e^{-k}$ or $k=-\ln\left(1-\frac{25}{M_1}\right)=\ln\left(\frac{M_1}{M_1-25}\right)$. But $45=M_1(1-e^{-2k})$ so $45=M_1\left[1-\left(\frac{M_1-25}{M_1}\right)^2\right]$ or $45=\frac{50M_1-625}{M_1}$. Thus, $M_1=125$ is the maximum number of units per hour Jim is capable of processing. Similarly, if $P_2(t)$ is Mark's learning curve, then $P_2(1)=35$ and $P_2(2)=50$. So $k=\ln\left(\frac{M_2}{M_2-35}\right)$ and $50=M_2\left[1-\left(\frac{M_2-35}{M_2}\right)^2\right]$ or $M_2=61.25$. Hence the maximum number of units per hour for Mark is approximately 61 . Another approach would be to use the midpoints of the intervals so that $P_1(0.5)=25$ and $P_1(1.5)=45$. Doing so gives us $M_1\approx 52.6$ and $M_2\approx 51.8$.

33. $y(0)=0$ kg. Salt is added at a rate of $\left(0.4\,\frac{\text{kg}}{\text{L}}\right)\left(5\,\frac{\text{L}}{\text{min}}\right)=2\,\frac{\text{kg}}{\text{min}}$. Since solution is drained from the tank at a rate of 3 L / min, but salt solution is added at a rate of 5 L / min, the tank, which starts out with 100 L of water, contains $(100+2t)$ L of liquid after t min. Thus, the salt concentration at time t is $\frac{y(t)}{100+2t}\,\frac{\text{kg}}{\text{L}}$. Salt therefore leaves the tank at a rate of $\left(\frac{y(t)}{100+2t}\,\frac{\text{kg}}{\text{L}}\right)\left(3\,\frac{\text{L}}{\text{min}}\right)=\frac{3y}{100+2t}\,\frac{\text{kg}}{\text{min}}$. Combining the rates at which salt enters and leaves the tank, we get $\frac{dy}{dt}=2-\frac{3y}{100+2t}$. Rewriting this equation as $\frac{dy}{dt}+\left(\frac{3}{100+2t}\right)y=2$, we see that it is linear. $I(t)=\exp\left(\int\frac{3\,dt}{100+2t}\right)=\exp\left(\frac{3}{2}\ln(100+2t)\right)=(100+2t)^{3/2}$. Multiplying the differential equation by $I(t)$ gives $(100+2t)^{3/2}\frac{dy}{dt}+3(100+2t)^{1/2}y=2(100+2t)^{3/2}\Rightarrow\left[(100+2t)^{3/2}y\right]'=2(100+2t)^{3/2}\Rightarrow$ $(100+2t)^{3/2}y=\frac{2}{5}(100+2t)^{5/2}+C\Rightarrow y=\frac{2}{5}(100+2t)+C(100+2t)^{-3/2}$. Now $0=y(0)=\frac{2}{5}(100)+C\cdot100^{-3/2}=40+\frac{1}{1000}C\Rightarrow C=-40,000$, so $y=\left[\frac{2}{5}(100+2t)-40{,}000(100+2t)^{-3/2}\right]$ kg. From this solution (no pun intended), we calculate the salt concentration at time t to be $C(t)=\frac{y(t)}{100+2t}=\left[\frac{-40{,}000}{(100+2t)^{5/2}}+\frac{2}{5}\right]\frac{\text{kg}}{\text{L}}$. In particular, $C(20)=\frac{-40{,}000}{140^{5/2}}+\frac{2}{5}\approx0.2275\,\frac{\text{kg}}{\text{L}}$ and $y(20)=\frac{2}{5}(140)-40,000(140)^{-3/2}\approx31.85$ kg.

34. Let $y(t)$ denote the amount of chlorine in the tank at time t (in seconds). $y(0)=(0.05\text{g/L})(400\text{L})=20$ g. The amount of liquid in the tank at time t is $(400-6t)$ L since 4 L of water enters the tank each second and 10 L of liquid leaves the tank each second. Thus, the concentration of chlorine at time t is $\frac{y(t)}{400-6t}\,\frac{\text{g}}{\text{L}}$. Chlorine doesn't enter the tank, but it leaves at a rate of $\left[\frac{y(t)}{400-6t}\,\frac{\text{g}}{\text{L}}\right]\left[10\,\frac{\text{L}}{\text{s}}\right]=\frac{10\,y(t)}{400-6t}\,\frac{\text{g}}{\text{s}}=\frac{5\,y(t)}{200-3t}\,\frac{\text{g}}{\text{s}}$. Therefore, $\frac{dy}{dt}=-\frac{5y}{200-3t}\Rightarrow\int\frac{dy}{y}=\int\frac{-5\,dt}{200-3t}\Rightarrow\ln y=\frac{5}{3}\ln(200-3t)+C\Rightarrow$ $y=\exp\left(\frac{5}{3}\ln(200-3t)+C\right)=e^{C}(200-3t)^{5/3}$. Now $20=y(0)=e^{C}\cdot200^{5/3}\Rightarrow e^{C}=\frac{20}{200^{5/3}}$, so $y(t)=20\,\frac{(200-3t)^{5/3}}{200^{5/3}}=20(1-0.015t)^{5/3}$ g for $0\le t\le 66\,\frac{2}{3}$ s, at which time the tank is empty.

35. **(a)**

$\frac{dv}{dt}+\frac{c}{m}\,v=g$ and $I(t)=e^{\int (c/m)\,dt}=e^{(c/m)t}$, and multiplying the differential equation by $I(t)$ gives $e^{(c/m)t}\,\frac{dv}{dt}+\frac{vce^{(c/m)t}}{m}=ge^{(c/m)t}\Rightarrow\left[e^{(c/m)t}v\right]'=ge^{(c/m)t}$. Hence, $v(t)=e^{-(c/m)t}\left[\int ge^{(c/m)t}\,dt+K\right]=mg/c+Ke^{-(c/m)t}$. But the object is dropped from rest, so $v(0)=0$ and $K=-mg/c$. Thus, the velocity at time t is $v(t)=(mg/c)\left[1-e^{-(c/m)t}\right]$.

(b) $\lim_{t\to\infty} v(t)=mg/c$

(c) $s(t)=\int v(t)\,dt=(mg/c)\left[t+(m/c)e^{-(c/m)t}\right]+c_1$ where $c_1=s(0)-m^2g/c^2$. $s(0)$ is the initial position, so $s(0)=0$ and $s(t)=(mg/c)\left[t+(m/c)e^{-(c/m)t}\right]-m^2g/c^2$.

36. $v=(mg/c)(1-e^{-ct/m})\Rightarrow$
$\frac{dv}{dm}=\frac{mg}{c}\left(0-e^{-ct/m}\cdot\frac{ct}{m^2}\right)+\frac{g}{c}\left(1-e^{-ct/m}\right)\cdot 1=-\frac{gt}{m}\,e^{-ct/m}+\frac{g}{c}-\frac{g}{c}\,e^{-ct/m}=\frac{g}{c}\left(1-e^{-ct/m}-\frac{ct}{m}\,e^{-ct/m}\right)\Rightarrow$
$\frac{c}{g}\,\frac{dv}{dm}=1-\left(1+\frac{ct}{m}\right)e^{-ct/m}=1-\frac{1+ct/m}{e^{ct/m}}=1-\frac{1+Q}{e^{Q}}$, where $Q=\frac{ct}{m}\geq 0$. Since $e^{Q}>1+Q$ for all $Q>0$, it follows that $dv/dm>0$ for $t>0$. In other words, for all $t>0$, v increases as m increases.

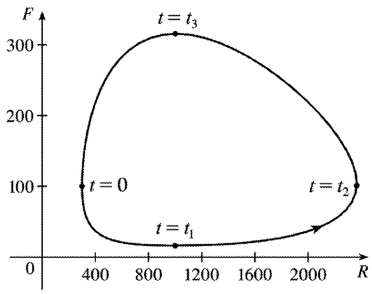
1. **(a)** $dx/dt=-0.05x+0.0001xy$. If $y=0$, we have $dx/dt=-0.05x$, which indicates that in the absence of y , x declines at a rate proportional to itself. So x represents the predator population and y represents the prey population. The growth of the prey population, $0.1y$ (from $dy/dt=0.1y-0.005xy$), is restricted only by encounters with predators (the term $-0.005xy$). The predator population increases only through the term $0.0001xy$; that is, by encounters with the prey and not through additional food sources.

(b) $dy/dt=-0.015y+0.00008xy$. If $x=0$, we have $dy/dt=-0.015y$, which indicates that in the absence of x , y would decline at a rate proportional to itself. So y represents the predator population and x represents the prey population. The growth of the prey population, $0.2x$ (from $dx/dt=0.2x-0.0002x^2-0.006xy=0.2x(1-0.001x)-0.006xy$), is restricted by a carrying capacity of 1000 and by encounters with predators (the term $-0.006xy$). The predator population increases only through the term $0.00008xy$; that is, by encounters with the prey and not through additional food sources.

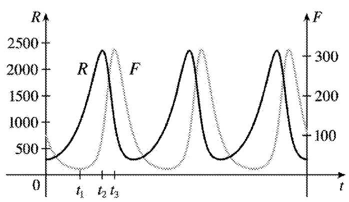
2. **(a)** $dx/dt=0.12x-0.0006x^2+0.00001xy$. $dy/dt=0.08y+0.00004xy$.
The xy terms represent encounters between the two species x and y . An increase in y makes dx/dt (the growth rate of x) larger due to the positive term $0.00001xy$. An increase in x makes dy/dt (the growth rate of y) larger due to the positive term $0.00004xy$. Hence, the system describes a cooperation model.

(b) $dx/dt=0.15x-0.0002x^2-0.0006xy=0.15x\,(1-x/750)-0.0006xy$.
$dy/dt=0.2y-0.00008y^2-0.0002xy=0.2y\,(1-y/2500)-0.0002xy$.
The system shows that x and y have carrying capacities of 750 and 2500 . An increase in x reduces the growth rate of y due to the negative term $-0.0002xy$. An increase in y reduces the growth rate of x due to the negative term $-0.0006xy$. Hence, the system describes a competition model.

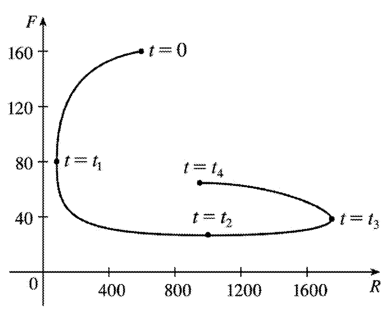
3. **(a)** At $t=0$, there are about 300 rabbits and 100 foxes. At $t=t_1$, the number of foxes reaches a minimum of about 20 while the number of rabbits is about 1000 . At $t=t_2$, the number of rabbits reaches a maximum of about 2400 , while the number of foxes rebounds to 100 . At $t=t_3$, the number of rabbits decreases to about 1000 and the number of foxes reaches a maximum of about 315 . As t increases, the number of foxes decreases greatly to 100 , and the number of rabbits decreases to 300 (the initial populations), and the cycle starts again.



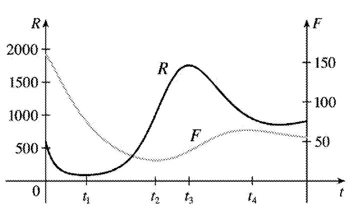
(b)



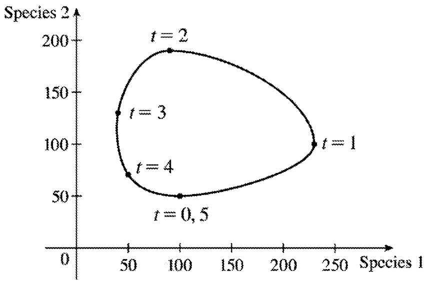
4. **(a)** At $t=0$, there are about 600 rabbits and 160 foxes. At $t=t_1$, the number of rabbits reaches a minimum of about 80 and the number of foxes is also 80 . At $t=t_2$, the number of foxes reaches a minimum of about 25 while the number of rabbits rebounds to 1000 . At $t=t_3$, the number of foxes has increased to 40 and the rabbit population has reached a maximum of about 1750 . The curve ends at $t=t_4$, where the number of foxes has increased to 65 and the number of rabbits has decreased to about 950 .



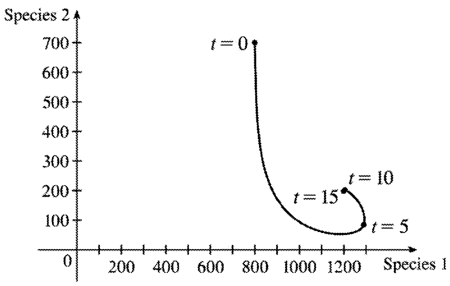
(b)



5.



6.



7. $\dfrac{dW}{dR} = \dfrac{-0.02W+0.00002RW}{0.08R-0.001RW} \Leftrightarrow (0.08-0.001W)R\,dW=(-0.02+0.00002R)W\,dR \Leftrightarrow$
$\dfrac{0.08-0.001W}{W}\,dW=\dfrac{-0.02+0.00002R}{R}\,dR \Leftrightarrow \int\left(\dfrac{0.08}{W}-0.001\right)dW=\int\left(-\dfrac{0.02}{R}+0.00002\right)dR \Leftrightarrow$
$0.08\ln|W|-0.001W=-0.02\ln|R|+0.00002R+K \Leftrightarrow 0.08\ln W+0.02\ln R=0.001W+0.00002R+K \Leftrightarrow$
$\ln\left(W^{0.08}R^{0.02}\right)=0.00002R+0.001W+K \Leftrightarrow W^{0.08}R^{0.02}=e^{0.00002R+0.001W+K} \Leftrightarrow$
$R^{0.02}W^{0.08}=Ce^{0.00002R}e^{0.001W} \Leftrightarrow \dfrac{R^{0.02}W^{0.08}}{e^{0.00002R}e^{0.001W}}=C$.

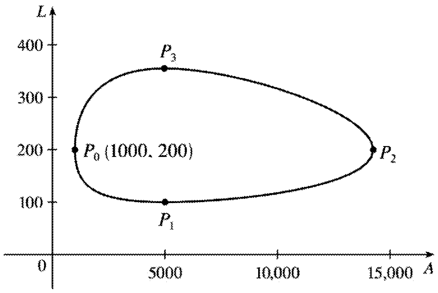
In general, if $\dfrac{dy}{dx}=\dfrac{-ry+bxy}{kx-axy}$, then $C=\dfrac{x^r y^k}{e^{bx}e^{ay}}$.

8. **(a)** A and L are constant $\Rightarrow A'=0$ and $L'=0 \Rightarrow \left\{\begin{array}{l} 0=2A-0.01AL \\ 0=-0.5L+0.0001AL \end{array}\right\} \Rightarrow$
$\left\{\begin{array}{l} 0=A(2-0.01L) \\ 0=L(-0.5+0.0001A) \end{array}\right.$

So either $A=L=0$ or $L=\dfrac{2}{0.01}=200$ and $A=\dfrac{0.5}{0.0001}=5000$. The trivial solution $A=L=0$ just says that if there aren't any aphids or ladybugs, then the populations will not change. The non–trivial solution, $L=200$ and $A=5000$, indicates the population sizes needed so that there are no changes in either the number of aphids or the number of ladybugs.

(b) $\dfrac{dL}{dA}=\dfrac{dL/dt}{dA/dt}=\dfrac{-0.5L+0.0001AL}{2A-0.01AL}$

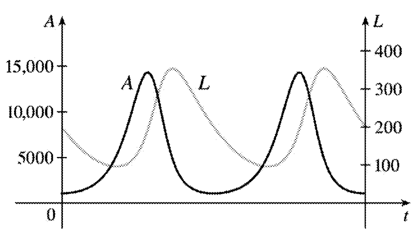
(c) C100708cC100708c.texC100708c.tex



(d)
At $P_0(1000,200)$, $dA/dt=0$ and $dL/dt=-80<0$, so the number of ladybugs is decreasing and hence, we are proceeding in a counterclockwise direction. At P_0 , there aren't enough aphids to support the ladybug population, so the number of ladybugs decreases and the number of aphids begins to increase. The ladybug population reaches a minimum at $P_1(5000,100)$ while the aphid population increases in a dramatic way, reaching its maximum at $P_2(14{,}250,200)$.

Meanwhile, the ladybug population is increasing from P_1 to $P_3(5000,355)$, and as we pass through P_2 , the increasing number of ladybugs starts to deplete the aphid population. At P_3 the ladybugs reach a maximum population, and start to decrease due to the reduced aphid population. Both populations then decrease until P_0 , where the cycle starts over again.

(e) Both graphs have the same period and the graph of L peaks about a quarter of a cycle after the graph of A .



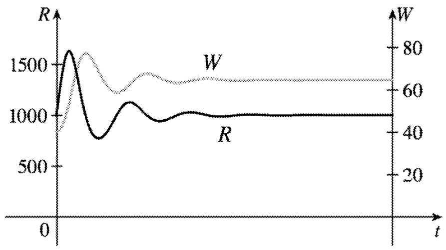
9. **(a)** Letting $W=0$ gives us $dR/dt=0.08R(1-0.0002R)$. $dR/dt=0 \Leftrightarrow R=0$ or 5000 . Since $dR/dt>0$ for $0<R<5000$, we would expect the rabbit population to *increase* to 5000 for these values of R . Since $dR/dt<0$ for $R>5000$, we would expect the rabbit population to *decrease* to 5000 for these values of R . Hence, in the absence of wolves, we would expect the rabbit population to stabilize at 5000 .

(b) R and W are constant $\Rightarrow R'=0$ and $W'=0 \Rightarrow$

$$\left\{ \begin{array}{l} 0=0.08R(1-0.0002R)-0.001RW \\ 0=-0.02W+0.00002RW \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} 0=R\left[0.08(1-0.0002R)-0.001W\right] \\ 0=W(-0.02+0.00002R) \end{array} \right.$$

The second equation is true if $W=0$ or

$R=\dfrac{0.02}{0.00002}=1000$. If $W=0$ in the first equation, then either $R=0$ or $R=\dfrac{1}{0.0002}=5000$. If $R=1000$, then $0=1000\left[0.08(1-0.0002\cdot 1000)-0.001W\right]\Leftrightarrow 0=80(1-0.2)-W\Leftrightarrow W=64$.
Case (i): $W=0$, $R=0$: both populations are zero
Case (ii): $W=0$, $R=5000$: see part (a)
Case (iii): $R=1000$, $W=64$: the predator/prey interaction balances and the populations are stable.
(c) The populations of wolves and rabbits fluctuate around 64 and 1000 , respectively, and eventually stabilize at those values.

(d) 

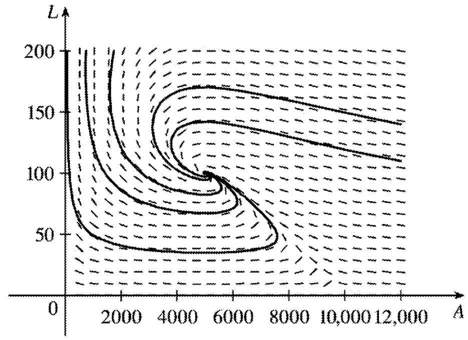
10. **(a)** If $L=0$, $dA/dt=2A(1-0.0001A)$, so $dA/dt=0\Leftrightarrow A=0$ or $A=\dfrac{1}{0.0001}=10,000$. Since $dA/dt>0$ for $0<A<10,000$, we expect the aphid population to *increase* to 10 , 000 for these values of A . Since $dA/dt<0$ for $A>10,000$, we expect the aphid population to *decrease* to 10 , 000 for these values of A . Hence, in the absence of ladybugs we expect the aphid population to stabilize at 10 , 000 .

(b) A and L are constant $\Rightarrow A'=0$ and $L'=0\Rightarrow$

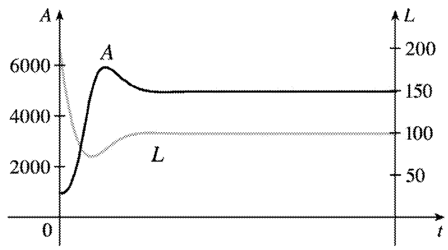
$$\left\{\begin{array}{l}0=2A(1-0.0001A)-0.01AL\\0=-0.5L+0.0001AL\end{array}\right\}\Rightarrow\left\{\begin{array}{l}0=A\left[2(1-0.0001A)-0.01L\right]\\0=L(-0.5+0.0001A)\end{array}\right.$$

The second equation is true if $L=0$ or $A=\dfrac{0.5}{0.0001}=5000$. If $L=0$ in the first equation, then either $A=0$ or $A=\dfrac{1}{0.0001}=10,000$. If $A=5000$, then $0=5000\left[2(1-0.0001\cdot 5000)-0.01L\right]\Leftrightarrow 0=10,000(1-0.5)-50L\Leftrightarrow 50L=5000\Leftrightarrow L=100$. The equilibrium solutions are: (i) $L=0,A=0$ (ii) $L=0,A=10,000$ (iii) $A=5000,L=100$

(c) $\dfrac{dL}{dA}=\dfrac{dL/dt}{dA/dt}=\dfrac{-0.5L+0.0001AL}{2A(1-0.0001A)-0.01AL}$

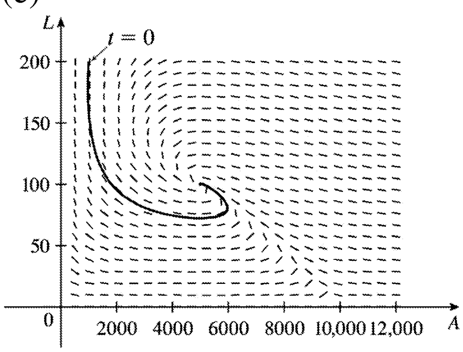


(d) All of the phase trajectories spiral tightly around the equilibrium solution $(5000,100)$.



The graph of A peaks just after the graph of L has a minimum.

(e)



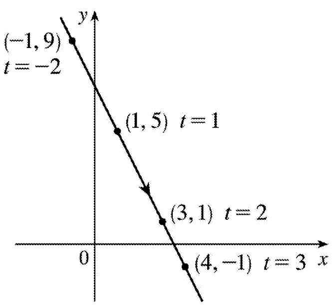
At t=0 , the ladybug population decreases rapidly and the aphid population decreases slightly before beginning to increase. As the aphid population continues to increase, the ladybug population reaches a minimum at about $(5000,75)$. The ladybug population starts to increase and quickly stabilizes at 100 , while the aphid population stabilizes at 5000 .

1. $x = 1 + t$, $y = 5 - 2t$, $-2 \le t \le 3$

(a)

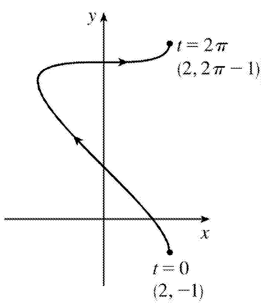
t	-2	-1	0	1	2	3
x	-1	0	1	2	3	4
y	9	7	5	3	1	-1

(b) $x = 1 + t \Rightarrow t = x - 1 \Rightarrow y = 5 - 2(x - 1)$, so $y = -2x + 7$, $-1 \le x \le 4$.



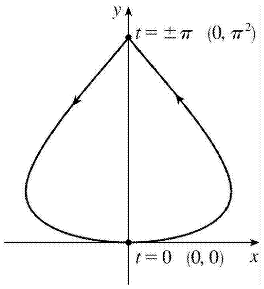
2. $x = 2\cos t$, $y = t - \cos t$, $0 \le t \le 2\pi$

t	0	$\pi/2$	π	$3\pi/2$	2π
x	2	0	-2	0	2
y	-1	$\pi/2$ 1.57	$\pi+1$ 4.14	$3\pi/2$ 4.71	$2\pi-1$ 5.28



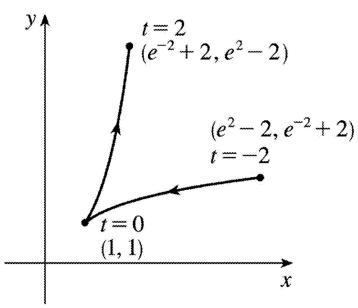
3. $x = 5\sin t$, $y = t^2$, $-\pi \le t \le \pi$

t	$-\pi$	$-\pi/2$	0	$\pi/2$	π
x	0	-5	0	5	0
y	π^2 9.87	$\pi^2/4$ 2.47	0	$\pi^2/4$ 2.47	π^2 9.87



4. $x=e^{-t}+t$, $y=e^{t}-t$, $-2\leq t\leq 2$

t	-2	-1	0	1	2
x	$e^{2}-2$ 5.39	$e-1$ 1.72	1	$e^{-1}+1$ 1.37	$e^{-2}+2$ 2.14
y	$e^{-2}+2$ 2.14	$e^{-1}+1$ 1.37	1	$e-1$ 1.72	$e^{2}-2$ 5.39

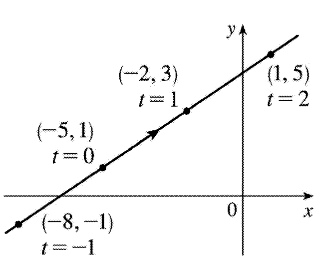


5. $x=3t-5$, $y=2t+1$

(a)

t	-2	-1	0	1	2	3	4
x	-11	-8	-5	-2	1	4	7
y	-3	-1	1	3	5	7	9

(b) $x=3t-5\Rightarrow 3t=x+5\Rightarrow t=\frac{1}{3}(x+5)\Rightarrow y=2\cdot\frac{1}{3}(x+5)+1$, so $y=\frac{2}{3}x+\frac{13}{3}$.

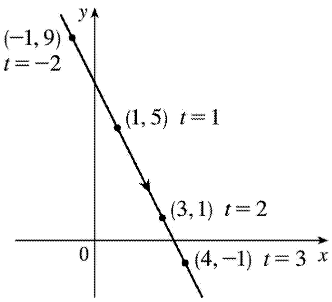


6. $x=1+t$, $y=5-2t$, $-2\leq t\leq 3$

(a)

t	-2	-1	0	1	2	3
x	-1	0	1	2	3	4
y	9	7	5	3	1	-1

(b) $x=1+t \Rightarrow t=x-1 \Rightarrow y=5-2(x-1)$, so $y=-2x+7$, $-1 \leq x \leq 4$.

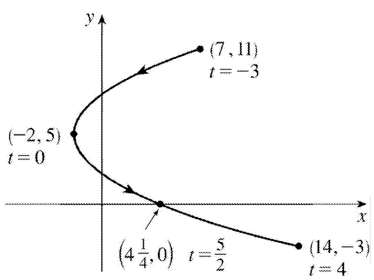


7. $x=t^2-2$, $y=5-2t$, $-3 \leq t \leq 4$

(a)

t	-3	-2	-1	0	1	2	3	4
x	7	2	-1	-2	-1	2	7	14
y	11	9	7	5	3	1	-1	-3

(b) $y=5-2t \Rightarrow 2t=5-y \Rightarrow t=\frac{1}{2}(5-y) \Rightarrow x=\left[\frac{1}{2}(5-y)\right]^2-2$, so $x=\frac{1}{4}(5-y)^2-2$, $-3 \leq y \leq 11$.

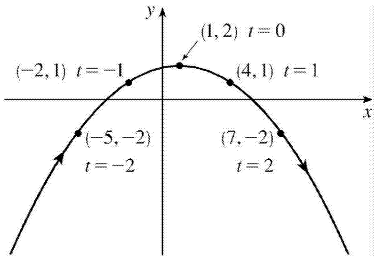


8. $x=1+3t$, $y=2-t^2$

(a)

t	-3	-2	-1	0	1	2	3
x	-8	-5	-2	1	4	7	10
y	-7	-2	1	2	1	-2	-7

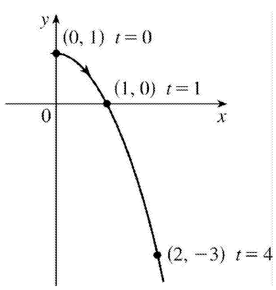
(b) $x=1+3t \Rightarrow t=\frac{1}{3}(x-1) \Rightarrow y=2-\left[\frac{1}{3}(x-1)\right]^2$, so $y=-\frac{1}{9}(x-1)^2+2$.



9. **(a)** $x=\sqrt{t}$, $y=1-t$

t	0	1	2	3	4
x	0	1	1.414	1.732	2
y	1	0	–1	–2	–3

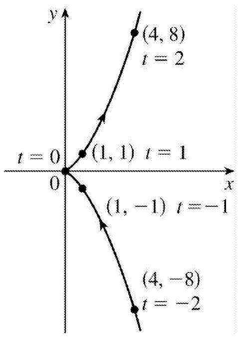
(b) $x=\sqrt{t}$, $\Rightarrow t=x^2 \Rightarrow y=1-t=1-x^2$.
Since $t\geq 0$, $x\geq 0$



10. **(a)** $x=t^2$, $y=t^3$

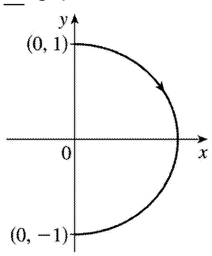
t	–2	–1	0	1	2
x	4	1	0	1	4
y	–8	–1	0	1	8

(b) $y=t^3 \Rightarrow t=\sqrt[3]{y} \Rightarrow x=t^2=\left(\sqrt[3]{y}\right)^2=y^{2/3}$.
$t\in R$, $y\in R$, $x\geq 0$.



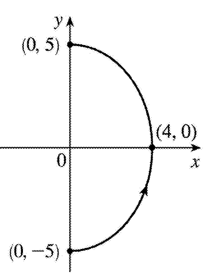
11. **(a)** $x = \sin\theta$, $y = \cos\theta$, $0 \le \theta \le \pi$. $x^2 + y^2 = \sin^2\theta + \cos^2\theta = 1$. Since $0 \le \theta \le \pi$, we have $\sin\theta \ge 0$, so $x \ge 0$.

(b)



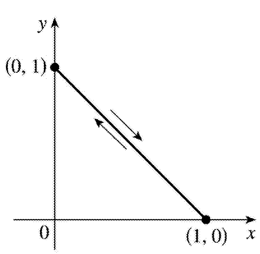
12. **(a)** $x = 4\cos\theta$, $y = 5\sin\theta$, $-\pi/2 \le \theta \le \pi/2$.
$\left(\frac{x}{4}\right)^2 + \left(\frac{y}{5}\right)^2 = \cos^2\theta + \sin^2\theta = 1$, which is an ellipse with x –intercepts $(\pm 4, 0)$ and y –intercepts $(0, \pm 5)$. We obtain the portion of the ellipse with $x \ge 0$ since $4\cos\theta \ge 0$ for $-\pi/2 \le \theta \le \pi/2$.

(b)



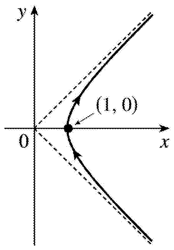
13. **(a)** $x = \sin^2\theta$, $y = \cos^2\theta$. $x + y = \sin^2\theta + \cos^2\theta = 1$, $0 \le x \le 1$. Note that the curve is at $(0,1)$ whenever $\theta = \pi n$ and is at $(1,0)$ whenever $\theta = \frac{\pi}{2}\, n$ for every integer n .

(b)



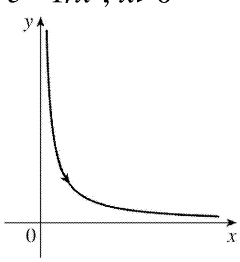
14. **(a)** $x=\sec\theta$, $y=\tan\theta$, $-\frac{\pi}{2}<\theta<\frac{\pi}{2}$. $x^2-y^2=\sec^2\theta-\tan^2\theta=1$, $x\geq 1$, or $x=\sqrt{y^2+1}$.

(b)



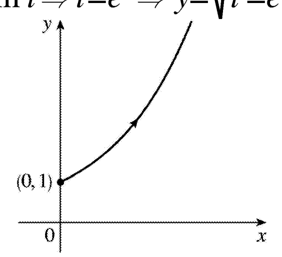
15. **(a)** $x=e^t$, $y=e^{-t}$.
$y=1/e^t=1/x$, $x>0$

(b)



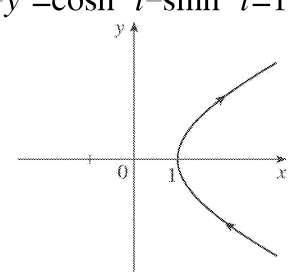
16. **(a)** $x=\ln t$, $y=\sqrt{t}$, $t\geq 1$.
$x=\ln t\Rightarrow t=e^x\Rightarrow y=\sqrt{t}=e^{x/2}$, $x\geq 0$.

(b)

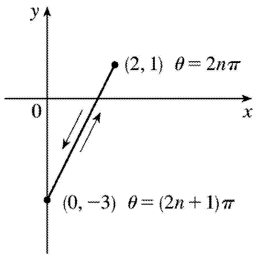


17. **(a)** $x=\cosh t$, $y=\sinh$ t,
$x^2-y^2=\cosh^2 t-\sinh^2 t=1$, $x\geq 1$

(b)



18. **(a)** $x=1+\cos\theta \Rightarrow \cos\theta = x-1$.
$y=2\cos\theta - 1=2(x-1)-1=2x-3$, $0\leq x\leq 2$.

(b) 

19. $x^2+y^2=\cos^2\pi t+\sin^2\pi t=1$, $1\leq t\leq 2$, so the particle moves counterclockwise along the circle $x^2+y^2=1$ from $(-1,0)$ to $(1,0)$, along the lower half of the circle.

20. $(x-2)^2+(y-3)^2=\cos^2 t+\sin^2 t=1$, so the motion takes place on a unit circle centered at $(2,3)$. As t goes from 0 to 2π , the particle makes one complete counterclockwise rotation around the circle, starting and ending at $(3,3)$.

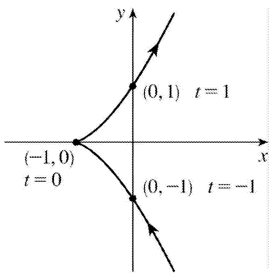
21. $\left(\frac{1}{2}x\right)^2+\left(\frac{1}{3}y\right)^2=\sin^2 t+\cos^2 t=1$, so the particle moves once clockwise along the ellipse $\frac{1}{4}x^2+\frac{1}{9}y^2=1$, starting and ending at $(0,3)$.

22. $x=\cos^2 t=y^2$, so the particle moves along the parabola $x=y^2$. As t goes from 0 to 4π , the particle moves from $(1,1)$ down to $(1,-1)$ (at $t=\pi$), back up to $(1,1)$ again (at $t=2\pi$), and then repeats this entire cycle between $t=2\pi$ and $t=4\pi$.

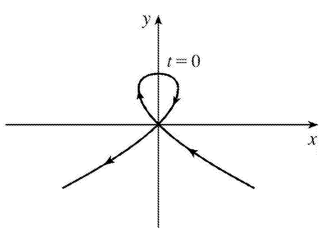
23. We must have $1\leq x\leq 4$ and $2\leq y\leq 3$. So the graph of the curve must be contained in the rectangle [1,4] by [2,3] .

24. **(a)** From the first graph, we have $1\leq x\leq 2$. From the second graph, we have $-1\leq y\leq 1$. The only choice that satisfies either of those conditions is III.
(b) From the first graph, the values of x cycle through the values from -2 to 2 four times. From the second graph, the values of y cycle through the values from -2 to 2 six times. Choice I satisfies these conditions.
(c) From the first graph, the values of x cycle through the values from -2 to 2 three times. From the second graph, we have $0\leq y\leq 2$. Choice IV satisfies these conditions.
(d) From the first graph, the values of x cycle through the values from -2 to 2 two times. From the second graph, the values of y do the same thing. Choice II satisfies these conditions.

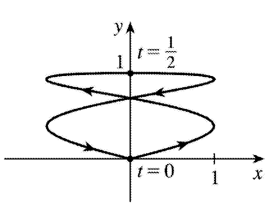
25. When $t=-1$, $(x,y)=(0,-1)$. As t increases to 0 , x decreases to -1 and y increases to 0 . As t increases from 0 to 1 , x increases to 0 and y increases to 1 . As t increases beyond 1 , both x and y increase. For $t<-1$, x is positive and decreasing and y is negative and increasing. We could achieve greater accuracy by estimating x – and y –values for selected values of t from the given graphs and plotting the corresponding points.



26. For $t<-1$, x is positive and decreasing, while y is negative and increasing (these points are in Quadrant IV). When $t=-1$, $(x,y)=(0,0)$ and, as t increases from -1 to 0 , x becomes negative and y increases from 0 to 1 .
At $t=0$, $(x,y)=(0,1)$ and, as t increases from 0 to 1 , y decreases from 1 to 0 and x is positive. At $t=1,(x,y)=(0,0)$ again, so the loop is completed. For $t>1$, x and y both become large negative. This enables us to draw a rough sketch. We could achieve greater accuracy by estimating x – and y –values for selected values of t from the given graphs and plotting the corresponding points.



27. When $t=0$ we see that $x=0$ and $y=0$, so the curve starts at the origin. As t increases from 0 to $\frac{1}{2}$, the graphs show that y increases from 0 to 1 while x increases from 0 to 1 , decreases to 0 and to -1 , then increases back to 0 , so we arrive at the point $(0,1)$. Similarly, as t increases from $\frac{1}{2}$ to 1 , y decreases from 1 to 0 while x repeats its

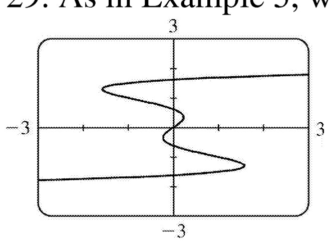


pattern, and we arrive back at the origin. We could achieve greater accuracy by estimating x – and y –

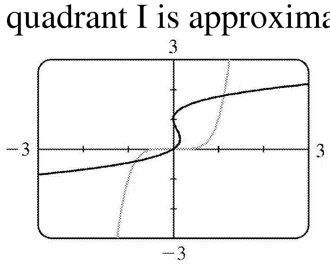
values for selected values of t from the given graphs and plotting the corresponding points.

28. **(a)** Note that as $t\to-\infty$, we have $x\to-\infty$ and $y\to\infty$, whereas when $t\to\infty$, both x and $y\to\infty$. This description fits only IV.
(b) Note that as $t\to\pm\infty$, $y\to-\infty$. This is only the case with VI.
(c) If $t=0$, then $(x,y)=(\sin 0,\sin 0)=(0,0)$. Also, $|x|=|\sin 3t|\leq 1$ for all t , and $|y|=|\sin 4t|\leq 1$ for all t . The only graph which includes the point $(0,0)$ and which has $|x|\leq 1$ and $|y|\leq 1$, is V.
(d) Note that as $t\to-\infty$, both x and $y\to-\infty$, and as $t\to\infty$, both x and $y\to\infty$. This description fits only III. (Also note that, since $\sin 2t$ and $\sin 3t$ lie between -1 and 1 , the curve never strays very far from the line $y=x$.)
(e) Note that both $x(t)$ and $y(t)$ are periodic with period 2π and satisfy $|x|\leq 1$ and $|y|\leq 1$. Now the only $y-$ intercepts occur when $x=\sin(t+\sin t)=0\Leftrightarrow t=0$ or π . So there should be two $y-$ intercepts: $y(0)=\cos 1\approx 0.54$ and $y(\pi)=\cos(\pi-1)\approx-0.54$. Similarly, there should be two $x-$ intercepts: $x\left(\frac{\pi}{2}\right)=\sin\left(\frac{\pi}{2}+1\right)\approx 0.54$ and $x\left(\frac{3\pi}{2}\right)=\sin\left(\frac{3\pi}{2}-1\right)\approx-0.54$. The only curve with these $x-$ and $y-$ intercepts is I.
(f) Note that $x(t)$ is periodic with period 2π , so the only $y-$ intercepts occur when $x=\cos t=0\Leftrightarrow$ $t=\frac{\pi}{2}$ or $\frac{3\pi}{2}$. Also, the graph is symmetric about the $x-$ axis, since $y(-t)=\sin(-t+\sin 5(-t))=\sin(-t-\sin 5t)=-\sin(t+\sin 5t)=-y(t)$, and $x(-t)=\cos(-t)=\cos t=x(t)$. The only graph which has only two $y-$ intercepts, and is symmetric about the $x-$ axis, is II.

29. As in Example 5, we let $y=t$ and $x=t-3t^3+t^5$ and use a t –interval of $[-2\pi,2\pi]$.



30. We use $x_1=t$, $y_1=t^5$ and $x_2=t(t-1)^2$, $y_2=t$ with $-2\pi\leq t\leq 2\pi$. There are 3 points of intersection; $(0,0)$ is fairly obvious. The point in quadrant III is approximately $(-0.8,-0.4)$ and the point in quadrant I is approximately $(1.1,1.8)$.



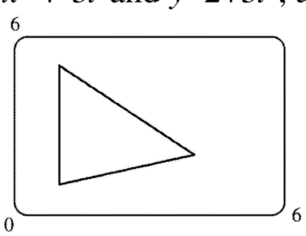
31. **(a)** $x=x_1+(x_2-x_1)t$, $y=y_1+(y_2-y_1)t$, $0\leq t\leq 1$. Clearly the curve passes through $P_1\left(x_1,y_1\right)$ when

$t=0$ and through $P_2\left(x_2,y_2\right)$ when $t=1$. For $0<t<1$, x is strictly between x_1 and x_2 and y is strictly between y_1 and y_2 . For every value of t , x and y satisfy the relation $y-y_1=\frac{y_2-y_1}{x_2-x_1}\left(x-x_1\right)$, which is the equation of the line through $P_1\left(x_1,y_1\right)$ and $P_2\left(x_2,y_2\right)$.

Finally, any point (x,y) on that line satisfies $\frac{y-y_1}{y_2-y_1}=\frac{x-x_1}{x_2-x_1}$; if we call that common value t , then the given parametric equations yield the point (x,y) ; and any (x,y) on the line between $P_1\left(x_1,y_1\right)$ and $P_2\left(x_2,y_2\right)$ yields a value of t in $[0,1]$. So the given parametric equations exactly specify the line segment from $P_1\left(x_1,y_1\right)$ to $P_2\left(x_2,y_2\right)$.

(b) $x=-2+[3-(-2)]t=-2+5t$ and $y=7+(-1-7)t=7-8t$ for $0\leq t\leq 1$.

32. For the side of the triangle from A to B , use $\left(x_1,y_1\right)=(1,1)$ and $\left(x_2,y_2\right)=(4,2)$. Hence, the equations are $x=x_1+\left(x_2-x_1\right)t=1+(4-1)t=1+3t$, $y=y_1+\left(y_2-y_1\right)t=1+(2-1)t=1+t$. Graphing $x=1+3t$ and $y=1+t$ with $0\leq t\leq 1$ gives us the side of the triangle from A to B . Similarly, for the side BC we use $x=4-3t$ and $y=2+3t$, and for the side AC we use $x=1$ and $y=1+4t$.



33. The circle $x^2+y^2=4$ can be represented parametrically by $x=2\cos t$, $y=2\sin t$; $0\leq t\leq 2\pi$. The circle $x^2+(y-1)^2=4$ can be represented by $x=2\cos t$, $y=1+2\sin t$; $0\leq t\leq 2\pi$. This representation gives us the circle with a counterclockwise orientation starting at $(2,1)$.

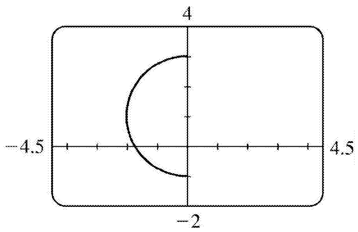
(a) To get a clockwise orientation, we could change the equations to $x=2\cos t$, $y=1-2\sin t$, $0\leq t\leq 2\pi$.

(b) To get three times around in the counterclockwise direction, we use the original equations $x=2\cos t$, $y=1+2\sin t$ with the domain expanded to $0\leq t\leq 6\pi$.

(c) To start at $(0,3)$ using the original equations, we must have $x_1=0$; that is, $2\cos t=0$. Hence, $t=\frac{\pi}{2}$. So we use $x=2\cos t$, $y=1+2\sin t$; $\frac{\pi}{2}\leq t\leq \frac{3\pi}{2}$.

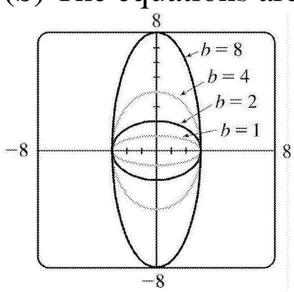
Alternatively, if we want t to start at 0 , we could change the equations of the curve. For example, we could use $x=-2\sin t$, $y=1+2\cos t$, $0\leq t\leq \pi$.

34.



35. **(a)** Let $x^2/a^2 = \sin^2 t$ and $y^2/b^2 = \cos^2 t$ to obtain $x = a\sin t$ and $y = b\cos t$ with $0 \le t \le 2\pi$ as possible parametric equations for the ellipse $x^2/a^2 + y^2/b^2 = 1$.

(b) The equations are $x = 3\sin t$ and $y = b\cos t$ for $b \in \{1,2,4,8\}$.



(c) As b increases, the ellipse stretches vertically.

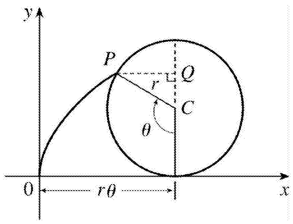
36. The possible parametrizations of the curve $y = x^3$ include

(1) $x = t$, $y = t^3$, $t \in R$

(2) $x = -t$, $y = -t^3$, $t \in R$

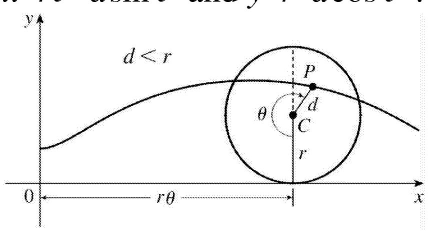
(3) $x = t + 1$, $y = (t+1)^3$, $t \in R$

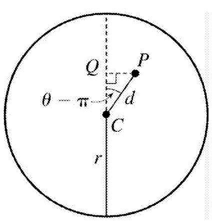
37. The case $\frac{\pi}{2} < \theta < \pi$ is illustrated. C has coordinates $(r\theta, r)$ as in Example 6, and Q has coordinates $(r\theta, r + r\cos(\pi - \theta)) = (r\theta, r(1 - \cos\theta))$, so P has coordinates $(r\theta - r\sin(\pi - \theta), r(1 - \cos\theta)) = (r(\theta - \sin\theta), r(1 - \cos\theta))$. Again we have the parametric equations $x = r(\theta - \sin\theta)$, $y = r(1 - \cos\theta)$.

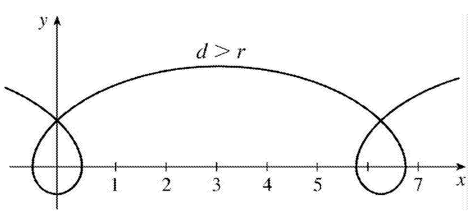


38. The first two diagrams depict the case $\pi < \theta < \frac{3\pi}{2}$, $d<r$. As in Example 6, C has coordinates $(r\theta, r)$.

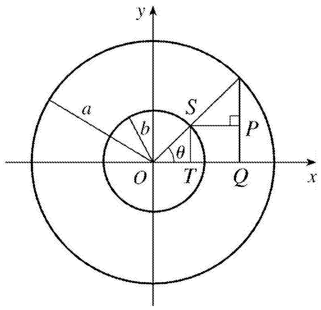
Now Q (in the second diagram) has coordinates $(r\theta, r+d\cos(\theta-\pi))=(r\theta, r-d\cos\theta)$, so a typical point P of the trochoid has coordinates $(r\theta+d\sin(\theta-\pi), r-d\cos\theta)$. That is, P has coordinates (x,y) , where $x=r\theta-d\sin\theta$ and $y=r-d\cos\theta$. When $d=r$, these equations agree with those of the cycloid.





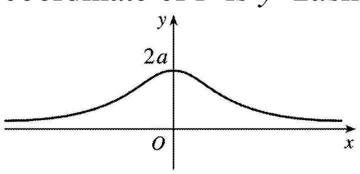


39. It is apparent that $x=|OQ|$ and $y=|QP|=|ST|$. From the diagram, $x=|OQ|=a\cos\theta$ and $y=|ST|=b\sin\theta$. Thus, the parametric equations are $x=a\cos\theta$ and $y=b\sin\theta$. To eliminate θ we rearrange: $\sin\theta=y/b \Rightarrow \sin^2\theta=(y/b)^2$ and $\cos\theta=x/a \Rightarrow \cos^2\theta=(x/a)^2$. Adding the two equations: $\sin^2\theta+\cos^2\theta=1=x^2/a^2+y^2/b^2$. Thus, we have an ellipse.



40. A has coordinates $(a\cos\theta, a\sin\theta)$. Since OA is perpendicular to AB, ΔOAB is a right triangle and B has coordinates $(a\sec\theta, 0)$. It follows that P has coordinates $(a\sec\theta, b\sin\theta)$. Thus, the parametric equations are $x=a\sec\theta$, $y=b\sin\theta$.

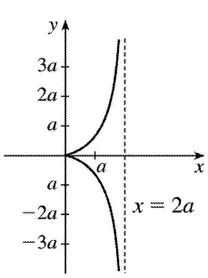
41. $C=(2a\cot\theta, 2a)$, so the x– coordinate of P is $x=2a\cot\theta$. Let $B=(0,2a)$. Then $\angle OAB$ is a right angle and $\angle OBA=\theta$, so $|OA|=2a\sin\theta$ and $A=((2a\sin\theta)\cos\theta, (2a\sin\theta)\sin\theta)$. Thus, the y– coordinate of P is $y=2a\sin^2\theta$.



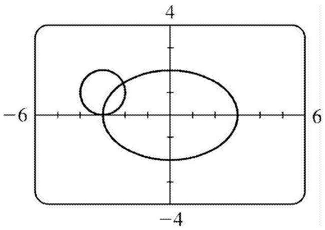
42. Let θ be the angle of inclination of segment OP. Then $|OB|=\dfrac{2a}{\cos\theta}$. Let $C=(2a,0)$. Then by use of right triangle OAC we see that $|OA|=2a\cos\theta$. Now

$$
\begin{aligned}
|OP| &= |AB|=|OB|-|OA|=2a\left(\frac{1}{\cos\theta}-\cos\theta\right) \\
&= 2a\,\frac{1-\cos^2\theta}{\cos\theta}=2a\,\frac{\sin^2\theta}{\cos\theta}=2a\sin\theta\tan\theta
\end{aligned}
$$

So P has coordinates $x=2a\sin\theta\tan\theta\cdot\cos\theta=2a\sin^2\theta$ and $y=2a\sin\theta\tan\theta\cdot\sin\theta=2a\sin^2\theta\tan\theta$.



43. **(a)**



There are 2 points of intersection:
$(-3,0)$ and approximately $(-2.1,1.4)$.

(b) A collision point occurs when $x_1=x_2$ and $y_1=y_2$ for the same t . So solve the equations:

$$\begin{aligned} 3\sin t &= -3+\cos t \quad \mathbf{(1)} \\ 2\cos t &= 1+\sin t \quad \mathbf{(2)} \end{aligned}$$

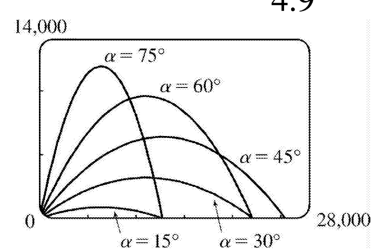
From **(2)** , $\sin t=2\cos t-1$. Substituting into **(1)** , we get $3(2\cos t-1)=-3+\cos t \Rightarrow$ $5\cos t=0$ (*) $\Rightarrow \cos t=0 \Rightarrow t=\frac{\pi}{2}$ or $\frac{3\pi}{2}$. We check that $t=\frac{3\pi}{2}$ satisfies **(1)** and **(2)** but $t=\frac{\pi}{2}$ does not. So the only collision point occurs when $t=\frac{3\pi}{2}$, and this gives the point $(-3,0)$.
(c) The circle is centered at $(3,1)$ instead of $(-3,1)$. There are still 2 intersection points: $(3,0)$ and $(2.1,1.4)$, but there are no collision points, since (*) in part (b) becomes $5\cos t=6 \Rightarrow \cos t=\frac{6}{5}>1$.

44. **(a)** If $\alpha=30^\circ$ and $v_0=500$ m / s, then the equations become $x=(500\cos 30^\circ)t=250\sqrt{3}t$ and $y=\left(500\sin 30^\circ\right)t-\frac{1}{2}(9.8)t^2=250t-4.9t^2$. $y=0$ when $t=0$ (when the gun is fired) and again when $t=\frac{250}{4.9}\approx 51$ s. Then $x=\left(250\sqrt{3}\right)\left(\frac{250}{4.9}\right)\approx 22{,}092$ m, so the bullet hits the ground about 22 km from the gun.
The formula for y is quadratic in t . To find the maximum y –value, we will complete the square:

$$y=-4.9\left(t^2-\frac{250}{4.9}t\right)=-4.9\left[t^2-\frac{250}{4.9}t+\left(\frac{125}{4.9}\right)^2\right]+\frac{125^2}{4.9}=-4.9\left(t-\frac{125}{4.9}\right)^2+\frac{125^2}{4.9}\le\frac{125^2}{4.9}$$

with equality when $t=\frac{125}{4.9}$ s, so the maximum height attained is $\frac{125^2}{4.9}\approx 3189$ m.

(b)

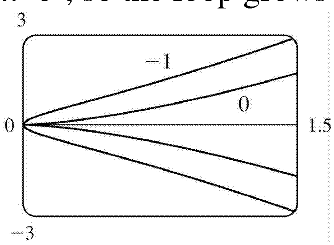


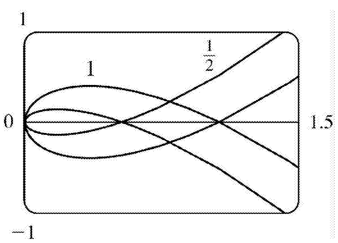
As α ($0^{\circ} < \alpha < 90^{\circ}$) increases up to 45° , the projectile attains a greater height and a greater range. As α increases past 45° , the projectile attains a greater height, but its range decreases.

(c) $x = (v_0 \cos\alpha)t \Rightarrow t = \dfrac{x}{v_0 \cos\alpha}$.

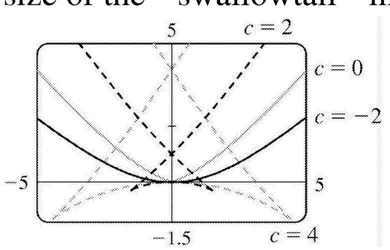
$y = (v_0 \sin\alpha)t - \dfrac{1}{2}gt^2 \Rightarrow y = (v_0 \sin\alpha)\dfrac{x}{v_0\cos\alpha} - \dfrac{g}{2}\left(\dfrac{x}{v_0\cos\alpha}\right)^2 = (\tan\alpha)x - \left(\dfrac{g}{2v_0^2\cos^2\alpha}\right)x^2$, which is the equation of a parabola (quadratic in x).

45. $x = t^2, y = t^3 - ct$. We use a graphing device to produce the graphs for various values of c with $-\pi \le t \le \pi$. Note that all the members of the family are symmetric about the $x-$ axis. For $c<0$, the graph does not cross itself, but for $c=0$ it has a cusp at $(0,0)$ and for $c>0$ the graph crosses itself at $x=c$, so the loop grows larger as c increases.

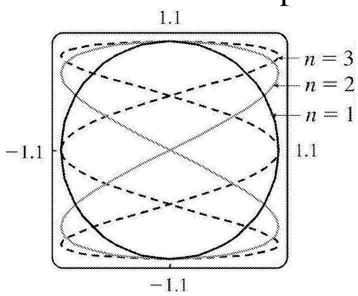




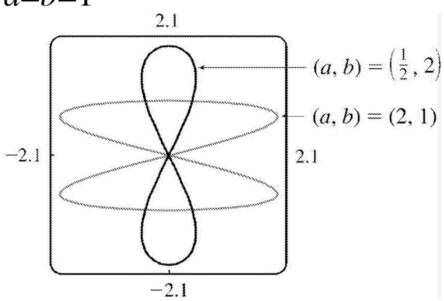
46. $x = 2ct - 4t^3, y = -ct^2 + 3t^4$. We use a graphing device to produce the graphs for various values of c with $-\pi \le t \le \pi$. Note that all the members of the family are symmetric about the $y-$ axis. When $c<0$, the graph resembles that of a polynomial of even degree, but when $c=0$ there is a corner at the origin, and when $c>0$, the graph crosses itself at the origin, and has two cusps below the $x-$ axis. The size of the ''swallowtail'' increases as c increases.



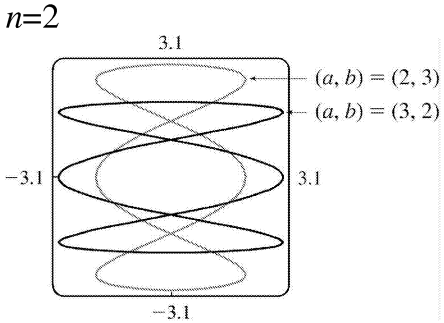
47. Note that all the Lissajous figures are symmetric about the $x-$ axis. The parameters a and b simply stretch the graph in the $x-$ and $y-$ directions respectively. For $a=b=n=1$ the graph is simply a circle with radius 1 . For $n=2$ the graph crosses itself at the origin and there are loops above and below the $x-$ axis. In general, the figures have $n-1$ points of intersection, all of which are on the $y-$ axis, and a total of n closed loops.



$a=b=1$

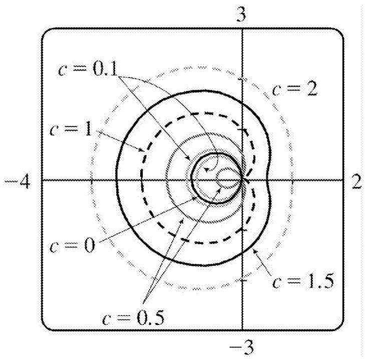


$n=2$



$n=3$

48. We use $-\pi \le t \le \pi$ in the viewing rectangle $[-4,2]\times[-3,3]$. We first observe that for $c=0$, we obtain a circle with center $\left(-\frac{1}{2},0\right)$ and radius $\frac{1}{2}$. As the value of c increases, there is a larger outer loop and a smaller inner loop until $c=1$, when we obtain a curve with a dent (called a **cardioid**). As c increases, we get a curve with a dimple (called a **limacon**) until $c=2$. For $c>2$, we have convex limacons. For negative values of c , we obtain the same graphs as for positive c , but with different values of t corresponding to the points on the curve.


3
-4
2
-3
c = 0.1
c = 2
c = 1
c = 0
c = 1.5
c = 0.5

1. $x=t-t^3$, $y=2-5t \Rightarrow \frac{dy}{dt}=-5$, $\frac{dx}{dt}=1-3t^2$, and $\frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{-5}{1-3t^2}$ or $\frac{5}{3t^2-1}$.

2. $x=te^t$, $y=t+e^t \Rightarrow \frac{dy}{dt}=1+e^t$, $\frac{dx}{dt}=te^t+e^t$, and $\frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{1+e^t}{te^t+e^t}$.

3. $x=t^4+1$, $y=t^3+t$; $t=-1$. $\frac{dy}{dt}=3t^2+1$, $\frac{dx}{dt}=4t^3$, and $\frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{3t^2+1}{4t^3}$.

When $t=-1$, (x,y)=(2,−2) and $dy/dx=\frac{4}{-4}=-1$, so an equation of the tangent to the curve at the point corresponding to $t=-1$ is $y-(-2)=(-1)(x-2)$, or $y=-x$.

4. $x=2t^2+1$, $y=\frac{1}{3}t^3-t$; $t=3$. $\frac{dy}{dt}=t^2-1$, $\frac{dx}{dt}=4t$, and $\frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{t^2-1}{4t}$. When $t=3$, (x,y)=(19,6) and $dy/dx=\frac{8}{12}=\frac{2}{3}$, so an equation of the tangent line is $y-6=\frac{2}{3}(x-19)$, or $y=\frac{2}{3}x-\frac{20}{3}$.

5. $x=e^{\sqrt{t}}$, $y=t-\ln t^2$; $t=1$. $\frac{dy}{dt}=1-\frac{2t}{t^2}=1-\frac{2}{t}$, $\frac{dx}{dt}=\frac{e^{\sqrt{t}}}{2\sqrt{t}}$, and $\frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{1-2/t}{e^{\sqrt{t}}/(2\sqrt{t})}\cdot\frac{2t}{2t}=\frac{2t-4}{\sqrt{t}\,e^{\sqrt{t}}}$. When $t=1$, (x,y)=(e,1) and $\frac{dy}{dx}=-\frac{2}{e}$, so an equation of the tangent line is $y-1=-\frac{2}{e}(x-e)$, or $y=-\frac{2}{e}x+3$.

6. $x=\cos\theta+\sin 2\theta$, $y=\sin\theta+\cos 2\theta$; $\theta=0$. $\frac{dy}{dx}=\frac{dy/d\theta}{dx/d\theta}=\frac{\cos\theta-2\sin 2\theta}{-\sin\theta+2\cos 2\theta}$. When $\theta=0$, (x,y)=(1,1) and $dy/dx=\frac{1}{2}$, so an equation of the tangent to the curve is $y-1=\frac{1}{2}(x-1)$, or $y=\frac{1}{2}x+\frac{1}{2}$.

7. **(a)** $x=e^t$, $y=(t-1)^2$; (1,1) . $\frac{dy}{dt}=2(t-1)$, $\frac{dx}{dt}=e^t$, and $\frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{2(t-1)}{e^t}$.

At (1,1) , $t=0$ and $\frac{dy}{dx}=-2$, so an equation of the tangent is $y-1=-2(x-1)$, or $y=-2x+3$.

(b) $x=e^t \Rightarrow t=\ln x$, so $y=(t-1)^2=(\ln x-1)^2$ and $\frac{dy}{dx}=2(\ln x-1)\left(\frac{1}{x}\right)$. When $x=1$, $\frac{dy}{dx}=2(-1)(1)=-2$, so an equation of the tangent is $y=-2x+3$, as in part (a).

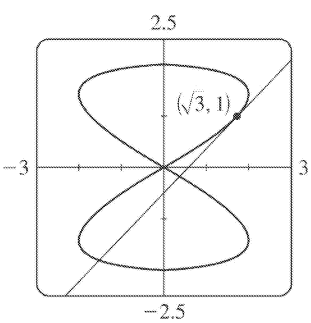
8. **(a)** $x=\tan\theta$, $y=\sec\theta$; $\left(1,\sqrt{2}\right)$. $\dfrac{dy}{dx}=\dfrac{dy/d\theta}{dx/d\theta}=\dfrac{\sec\theta\tan\theta}{\sec^2\theta}=\dfrac{\tan\theta}{\sec\theta}=\sin\theta$.

When (x,y)=$\left(1,\sqrt{2}\right)$, $\theta=\dfrac{\pi}{4}$ (or $\dfrac{\pi}{4}+2\pi n$ for some integer n), so $dy/dx=\sin\dfrac{\pi}{4}=\sqrt{2}/2$. Thus, an equation of the tangent to the curve is $y-\sqrt{2}=\left(\sqrt{2}/2\right)(x-1)$, or $y=\left(\sqrt{2}/2\right)x+\left(\sqrt{2}/2\right)$.

(b) $\tan^2\theta+1=\sec^2\theta\Rightarrow x^2+1=y^2$, so $\dfrac{d}{dx}(x^2+1)=\dfrac{d}{dx}(y^2)\Rightarrow 2x=2y\dfrac{dy}{dx}$.

When (x,y)=$\left(1,\sqrt{2}\right)$, $\dfrac{dy}{dx}=\dfrac{x}{y}=\dfrac{1}{\sqrt{2}}=\dfrac{\sqrt{2}}{2}$, so an equation of the tangent is $y-\sqrt{2}=\left(\sqrt{2}/2\right)(x-1)$, as in part (a).

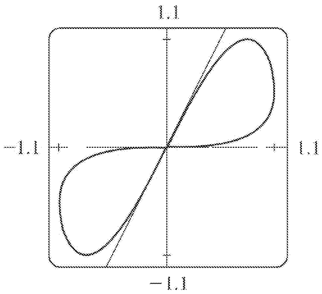
9. $x=2\sin 2t$, $y=2\sin t$; $\left(\sqrt{3},1\right)$. $\dfrac{dy}{dx}=\dfrac{dy/dt}{dx/dt}=\dfrac{2\cos t}{2\cdot 2\cos 2t}=\dfrac{\cos t}{2\cos 2t}$. The point $\left(\sqrt{3},1\right)$ corresponds to $t=\dfrac{\pi}{6}$, so the slope of the tangent at that point is $\dfrac{\cos\frac{\pi}{6}}{2\cos\frac{\pi}{3}}=\dfrac{\frac{\sqrt{3}}{2}}{2\cdot\frac{1}{2}}=\dfrac{\sqrt{3}}{2}$. An equation of the tangent is therefore $(y-1)=\dfrac{\sqrt{3}}{2}\left(x-\sqrt{3}\right)$, or $y=\dfrac{\sqrt{3}}{2}x-\dfrac{1}{2}$.



10. $x=\sin t$, $y=\sin(t+\sin t)$; $(0,0)$.

$$\frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{\cos(t+\sin t)(1+\cos t)}{\cos t}=(\sec t+1)\cos(t+\sin t)$$

Note that there are two tangents at the point $(0,0)$, since both $t=0$ and $t=\pi$ correspond to the origin. The tangent corresponding to $t=0$ has slope $(\sec 0+1)\cos(0+\sin 0)=2\cos 0=2$, and its equation is $y=2x$. The tangent corresponding to $t=\pi$ has slope $(\sec\pi+1)\cos(\pi+\sin\pi)=0$, so it is the $x-$ axis.



11. $x=4+t^2$, $y=t^2+t^3 \Rightarrow \dfrac{dy}{dx}=\dfrac{dy/dt}{dx/dt}=\dfrac{2t+3t^2}{2t}=1+\dfrac{3}{2}t \Rightarrow$

$\dfrac{d^2y}{dx^2}=\dfrac{d}{dx}\left(\dfrac{dy}{dx}\right)=\dfrac{d(dy/dx)/dt}{dx/dt}=\dfrac{(d/dt)\left(1+\frac{3}{2}t\right)}{2t}=\dfrac{3/2}{2t}=\dfrac{3}{4t}$. The curve is CU when $\dfrac{d^2y}{dx^2}>0$, that is, when $t>0$.

12. $x=t^3-12t$, $y=t^2-1 \Rightarrow \dfrac{dy}{dx}=\dfrac{dy/dt}{dx/dt}=\dfrac{2t}{3t^2-12} \Rightarrow$

$\dfrac{d^2y}{dx^2}=\dfrac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{dx/dt}=\dfrac{\frac{(3t^2-12)\cdot 2-2t(6t)}{(3t^2-12)^2}}{3t^2-12}=\dfrac{-6t^2-24}{(3t^2-12)^3}=\dfrac{-6(t^2+4)}{3^3(t^2-4)^3}=\dfrac{-2(t^2+4)}{9(t^2-4)^3}$. Thus, the curve is CU when $t^2-4<0 \Rightarrow |t|<2 \Rightarrow -2<t<2$.

13. $x=t-e^t$, $y=t+e^{-t} \Rightarrow$

$\dfrac{dy}{dx}=\dfrac{dy/dt}{dx/dt}=\dfrac{1-e^{-t}}{1-e^t}=\dfrac{1-\frac{1}{e^t}}{1-e^t}=\dfrac{\frac{e^t-1}{e^t}}{1-e^t}=-e^{-t} \Rightarrow \dfrac{d^2y}{dx^2}=\dfrac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{dx/dt}=\dfrac{\frac{d}{dt}(-e^{-t})}{dx/dt}=\dfrac{e^{-t}}{1-e^t}$. The curve is CU when $e^t<1 \Rightarrow t<0$.

14. $x=t+\ln t$, $y=t-\ln t \Rightarrow \dfrac{dy}{dx}=\dfrac{dy/dt}{dx/dt}=\dfrac{1-1/t}{1+1/t}=\dfrac{t-1}{t+1}=1-\dfrac{2}{t+1} \Rightarrow$

$\dfrac{d^2y}{dx^2}=\dfrac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{dx/dt}=\dfrac{\frac{d}{dt}\left(1-\frac{2}{t+1}\right)}{1+1/t}=\dfrac{2/(t+1)^2}{(t+1)/t}=\dfrac{2t}{(t+1)^3}$, so the curve is CU for all t in its domain, that is, $t>0$.

15. $x=2\sin t$, $y=3\cos t$, $0<t<2\pi$.

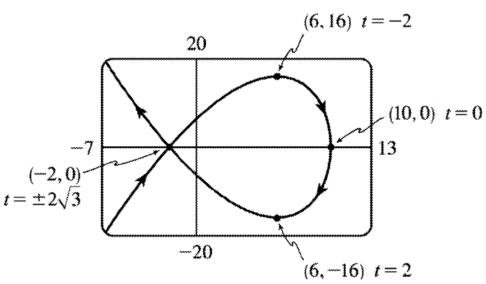
$\frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{-3\sin t}{2\cos t}=-\frac{3}{2}\tan t$, so $\frac{d^2y}{dx^2}=\frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{dx/dt}=\frac{-\frac{3}{2}\sec^2 t}{2\cos t}=-\frac{3}{4}\sec^3 t$.

The curve is CU when $\sec^3 t<0\Rightarrow\sec t<0\Rightarrow\cos t<0\Rightarrow\frac{\pi}{2}<t<\frac{3\pi}{2}$.

16. $x=\cos 2t$, $y=\cos t$, $0<t<\pi$. $\frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{-\sin t}{-2\sin 2t}=\frac{\sin t}{2\cdot 2\sin t\cos t}=\frac{1}{4\cos t}=\frac{1}{4}\sec t$, so $\frac{d^2y}{dx^2}=\frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{dx/dt}=\frac{\frac{1}{4}\sec t\tan t}{-4\sin t\cos t}=-\frac{1}{16}\sec^3 t$. The curve is CU when $\sec^3 t<0\Rightarrow\sec t<0\Rightarrow\cos t<0\Rightarrow$ $\frac{\pi}{2}<t<\pi$.

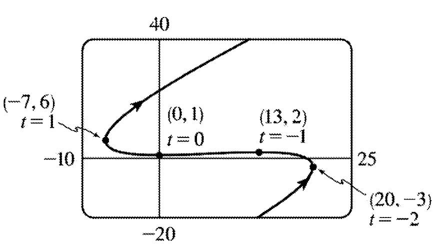
17. $x=10-t^2$, $y=t^3-12t$.

$dy/dt=3t^2-12=3(t+2)(t-2)$, so $dy/dt=0\Leftrightarrow t=\pm 2\Leftrightarrow(x,y)=(6,\mp 16)$. $dx/dt=-2t$, so $dx/dt=0\Leftrightarrow t=0\Leftrightarrow$ $(x,y)=(10,0)$. The curve has horizontal tangents at $(6,\pm 16)$ and a vertical tangent at $(10,0)$.



18. $x=2t^3+3t^2-12t$, $y=2t^3+3t^2+1$.

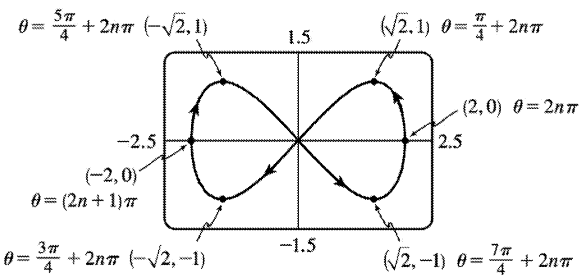
$dy/dt=6t^2+6t=6t(t+1)$, so $dy/dt=0\Leftrightarrow t=0$ or $-1\Leftrightarrow(x,y)=(0,1)$ or $(13,2)$. $dx/dt=6t^2+6t-12=6(t+2)(t-1)$, so $dx/dt=0\Leftrightarrow t=-2$ or $1\Leftrightarrow$



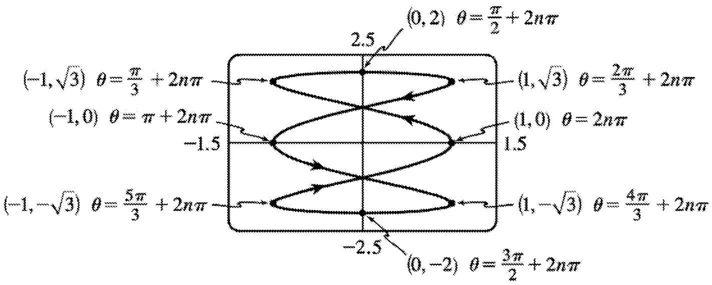
$(x,y)=(20,-3)$ or $(-7,6)$. The curve has horizontal tangents at $(0,1)$ and $(13,2)$, and vertical tangents at $(20,-3)$ and $(-7,6)$.

19. $x=2\cos\theta$, $y=\sin 2\theta$.

$dy/d\theta=2\cos 2\theta$, so $dy/d\theta=0 \Leftrightarrow 2\theta=\frac{\pi}{2}+n\pi$ (n an integer) $\Leftrightarrow \theta=\frac{\pi}{4}+\frac{\pi}{2}n \Leftrightarrow (x,y)=\left(\pm\sqrt{2},\pm 1\right)$. Also, $dx/d\theta=-2\sin\theta$, so $dx/d\theta=0 \Leftrightarrow \theta=n\pi \Leftrightarrow (x,y)=(\pm 2,0)$. The curve has horizontal tangents at $\left(\pm\sqrt{2},\pm 1\right)$ (four points), and vertical tangents at $(\pm 2,0)$.

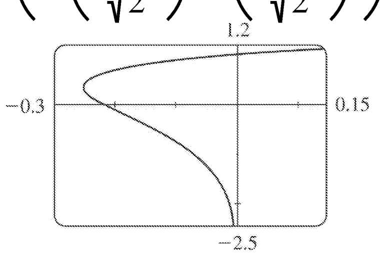


20. $x=\cos 3\theta$, $y=2\sin\theta$. $dy/d\theta=2\cos\theta$, so $dy/d\theta=0 \Leftrightarrow \theta=\frac{\pi}{2}+n\pi$ (n an integer) $\Leftrightarrow (x,y)=(0,\pm 2)$. Also, $dx/d\theta=-3\sin 3\theta$, so $dx/d\theta=0 \Leftrightarrow 3\theta=n\pi \Leftrightarrow \theta=\frac{\pi}{3}n \Leftrightarrow (x,y)=(\pm 1,0)$ or $\left(\pm 1,\pm\sqrt{3}\right)$. The curve has horizontal tangents at $(0,\pm 2)$, and vertical tangents at $(\pm 1,0)$, $\left(\pm 1,-\sqrt{3}\right)$ and $\left(\pm 1,\sqrt{3}\right)$.

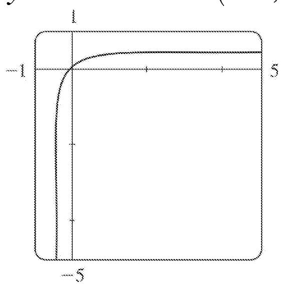


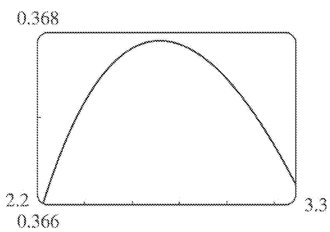
21. From the graph, it appears that the leftmost point on the curve $x=t^4-t^2$, $y=t+\ln t$ is about $(-0.25,0.36)$. To find the exact coordinates, we find the value of t for which the graph has a vertical tangent, that is, $0=dx/dt=4t^3-2t \Leftrightarrow 2t\left(2t^2-1\right)=0 \Leftrightarrow 2t\left(\sqrt{2}\,t+1\right)\left(\sqrt{2}\,t-1\right)=0 \Leftrightarrow t=0$ or $\pm\frac{1}{\sqrt{2}}$. The negative and 0 roots are inadmissible since $y(t)$ is only defined for $t>0$, so the leftmost point must be

$$\left(x\left(\frac{1}{\sqrt{2}}\right),y\left(\frac{1}{\sqrt{2}}\right)\right)=\left(\left(\frac{1}{\sqrt{2}}\right)^4-\left(\frac{1}{\sqrt{2}}\right)^2,\frac{1}{\sqrt{2}}+\ln\frac{1}{\sqrt{2}}\right)=\left(-\frac{1}{4},\frac{1}{\sqrt{2}}-\frac{1}{2}\ln 2\right)$$



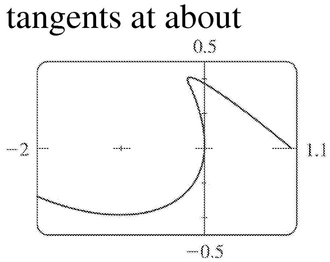
22. The curve is symmetric about the line $y=-x$ since replacing t with $-t$ has the effect of replacing (x,y) with $(-y,-x)$, so if we can find the highest point (x_h,y_h), then the leftmost point is $(x_l,y_l)=(-y_h,-x_h)$. After carefully zooming in, we estimate that the highest point on the curve $x=te^t$, $y=te^{-t}$ is about $(2.7,0.37)$.

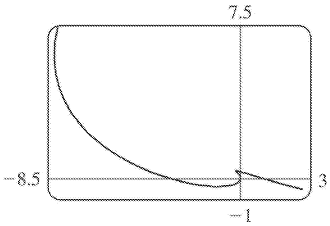




To find the exact coordinates of the highest point, we find the value of t for which the curve has a horizontal tangent, that is, $dy/dt=0\Leftrightarrow t(-e^{-t})+e^{-t}=0\Leftrightarrow(1-t)e^{-t}=0\Leftrightarrow t=1$. This corresponds to the point $(x(1),y(1))=(e,1/e)$. To find the leftmost point, we find the value of t for which $0=dx/dt=te^t+e^t\Leftrightarrow$ $(1+t)e^t=0\Leftrightarrow t=-1$. This corresponds to the point $(x(-1),y(-1))=(-1/e,-e)$. As $t\to-\infty$, $x(t)=te^t\to 0^-$ by l'Hospital's Rule and $y(t)=te^{-t}\to-\infty$, so the $y-$ axis is an asymptote. As $t\to\infty$, $x(t)\to\infty$ and $y(t)\to 0^+$, so the $x-$ axis is the other asymptote. The asymptotes can also be determined from the graph, if we use a larger t –interval.

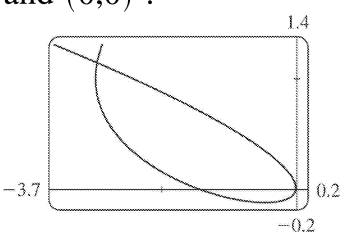
23. We graph the curve $x=t^4-2t^3-2t^2$, $y=t^3-t$ in the viewing rectangle $[-2,1.1]$ by $[-0.5,0.5]$. This rectangle corresponds approximately to $t\in[-1,0.8]$. We estimate that the curve has horizontal tangents at about

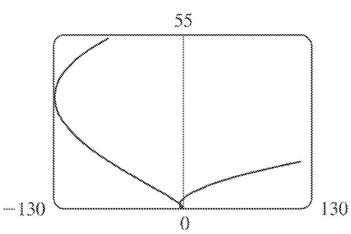




$(-1,-0.4)$ and $(-0.17,0.39)$ and vertical tangents at about $(0,0)$ and $(-0.19,0.37)$. We calculate $\frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{3t^2-1}{4t^3-6t^2-4t}$. The horizontal tangents occur when $dy/dt=3t^2-1=0 \Leftrightarrow t=\pm\frac{1}{\sqrt{3}}$, so both horizontal tangents are shown in our graph. The vertical tangents occur when $dx/dt=2t(2t^2-3t-2)=0 \Leftrightarrow 2t(2t+1)(t-2)=0 \Leftrightarrow t=0$, $-\frac{1}{2}$ or 2 . It seems that we have missed one vertical tangent, and indeed if we plot the curve on the $t-$ interval $[-1.2,2.2]$ we see that there is another vertical tangent at $(-8,6)$.

24. We graph the curve $x=t^4+4t^3-8t^2$, $y=2t^2-t$ in the viewing rectangle $[-3.7,0.2]$ by $[-0.2,1.4]$. It appears that there is a horizontal tangent at about $(-0.4,-0.1)$, and vertical tangents at about $(-3,1)$ and $(0,0)$.

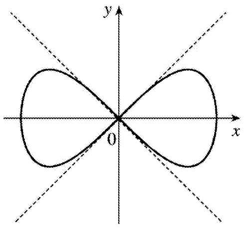




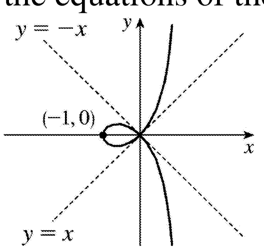
We calculate $\frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{4t-1}{4t^3+12t^2-16t}$, so there is a horizontal tangent where $dy/dt=4t-1=0 \Leftrightarrow t=\frac{1}{4}$. This point (the lowest point) is shown in the first graph. There are vertical tangents where $dx/dt=4t^3+12t^2-16t=0 \Leftrightarrow 4t(t^2+3t-4)=0 \Leftrightarrow 4t(t+4)(t-1)=0$. We have missed one vertical tangent corresponding to $t=-4$, and if we plot the graph for $t\in[-5,3]$, we see that the curve has another vertical tangent line at approximately $(-128,36)$.

25. $x=\cos t$, $y=\sin t\cos t$. $\frac{dx}{dt}=-\sin t$, $\frac{dy}{dt}=-\sin^2 t+\cos^2 t=\cos 2t$. $(x,y)=(0,0) \Leftrightarrow \cos t=0 \Leftrightarrow t$ is an odd multiple of

$\frac{\pi}{2}$. When $t=\frac{\pi}{2}$, $\frac{dx}{dt}=-1$ and $\frac{dy}{dt}=-1$, so $\frac{dy}{dx}=1$. When $t=\frac{3\pi}{2}$, $\frac{dx}{dt}=1$ and $\frac{dy}{dt}=-1$. So $\frac{dy}{dx}=-1$. Thus, $y=x$ and $y=-x$ are both tangent to the curve at $(0,0)$.



26. $x=1-2\cos^2 t=-\cos 2t$, $y=(\tan t)(1-2\cos^2 t)=-(\tan t)\cos 2t$. To find a point where the curve crosses itself, we look for two values of t that give the same point (x,y) . Call these values t_1 and t_2 . Then $\cos^2 t_1=\cos^2 t_2$ (from the equation for x) and either $\tan t_1=\tan t_2$ or $\cos^2 t_1=\cos^2 t_2=\frac{1}{2}$ (from the equation for y). We can satisfy $\cos^2 t_1=\cos^2 t_2$ and $\tan t_1=\tan t_2$ by choosing t_1 arbitrarily and taking $t_2=t_1+\pi$, so evidently the whole curve is retraced every time t traverses an interval of length π . Thus, we can restrict our attention to the interval $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$. If $t_2=-t_1$, then $\cos^2 t_2=\cos^2 t_1$, but $\tan t_2=-\tan t_1$. This suggests that we try to satisfy the condition $\cos^2 t_1=\cos^2 t_2=\frac{1}{2}$. Taking $t_1=\frac{\pi}{4}$ and $t_2=-\frac{\pi}{4}$ gives $(x,y)=(0,0)$ for both values of t . $\frac{dx}{dt}=2\sin 2t$,and $\frac{dy}{dt}=2\sin 2t\tan t-\cos 2t\sec^2 t$. When $t=\frac{\pi}{4}$, $\frac{dx}{dt}=2$ and $\frac{dy}{dt}=2$, so $\frac{dy}{dx}=1$. When $t=-\frac{\pi}{4}$, $\frac{dx}{dt}=-2$ and $\frac{dy}{dt}=2$, so $\frac{dy}{dx}=-1$. Thus, the equations of the two tangents at $(0,0)$ are $y=x$ and $y=-x$.



27. **(a)** $x=r\theta-d\sin\theta$, $y=r-d\cos\theta$; $\frac{dx}{d\theta}=r-d\cos\theta$, $\frac{dy}{d\theta}=d\sin\theta$. So $\frac{dy}{dx}=\frac{d\sin\theta}{r-d\cos\theta}$.
(b) If $0<d<r$, then $|d\cos\theta|\leq d<r$, so $r-d\cos\theta\geq r-d>0$. This shows that $dx/d\theta$ never vanishes, so the trochoid can have no vertical tangent if $d<r$.

28. $x=a\cos^3\theta$, $y=a\sin^3\theta$.
(a)

$\frac{dx}{d\theta}=-3a\cos^2\theta\sin\theta$, $\frac{dy}{d\theta}=3a\sin^2\theta\cos\theta$, so $\frac{dy}{dx}=-\frac{\sin\theta}{\cos\theta}=-\tan\theta$.

(b) The tangent is horizontal $\Leftrightarrow dy/dx=0\Leftrightarrow\tan\theta=0\Leftrightarrow\theta=n\pi\Leftrightarrow(x,y)=(\pm a,0)$. The tangent is vertical $\Leftrightarrow\cos\theta=0\Leftrightarrow\theta$ is an odd multiple of $\frac{\pi}{2}\Leftrightarrow(x,y)=(0,\pm a)$.

(c) $dy/dx=\pm1\Leftrightarrow\tan\theta=\pm1\Leftrightarrow\theta$ is an odd multiple of $\frac{\pi}{4}\Leftrightarrow(x,y)=\left(\pm\frac{\sqrt{2}}{4}a,\pm\frac{\sqrt{2}}{4}a\right)$ (All sign choices are valid.)

29. The line with parametric equations $x=-7t$, $y=12t-5$ is $y=12\left(-\frac{1}{7}x\right)-5$, which has slope $-\frac{12}{7}$. The curve $x=t^3+4t$, $y=6t^2$ has slope $\frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{12t}{3t^2+4}$. This equals $-\frac{12}{7}\Leftrightarrow3t^2+4=-7t\Leftrightarrow$ $(3t+4)(t+1)=0\Leftrightarrow t=-1$ or $t=-\frac{4}{3}\Leftrightarrow(x,y)=(-5,6)$ or $\left(-\frac{208}{27},\frac{32}{3}\right)$.

30. $x=3t^2+1$, $y=2t^3+1$, $\frac{dx}{dt}=6t$, $\frac{dy}{dt}=6t^2$, so $\frac{dy}{dx}=\frac{6t^2}{6t}=t$ (even where $t=0$).
So at the point corresponding to parameter value t , an equation of the tangent line is $y-(2t^3+1)=t\left[x-(3t^2+1)\right]$. If this line is to pass through $(4,3)$, we must have $3-(2t^3+1)=t\left[4-(3t^2+1)\right]$ $\Leftrightarrow2t^3-2=3t^3-3t\Leftrightarrow t^3-3t+2=0\Leftrightarrow(t-1)^2(t+2)=0\Leftrightarrow t=1$ or -2 . Hence, the desired equations are $y-3=x-4$, or $y=x-1$, tangent to the curve at $(4,3)$, and $y-(-15)=-2(x-13)$, or $y=-2x+11$, tangent to the curve at $(13,-15)$.

31. By symmetry of the ellipse about the $x-$ and $y-$ axes,

$$A=4\int_0^a y\,dx=4\int_{\pi/2}^0 b\sin\theta(-a\sin\theta)d\theta=4ab\int_0^{\pi/2}\sin^2\theta\,d\theta=4ab\int_0^{\pi/2}\frac{1}{2}(1-\cos2\theta)d\theta$$

$$=2ab\left[\theta-\frac{1}{2}\sin2\theta\right]_0^{\pi/2}=2ab\left(\frac{\pi}{2}\right)=\pi ab$$

32. $t+1/t=2.5\Leftrightarrow t=\frac{1}{2}$ or 2 , and for $\frac{1}{2}<t<2$, we have $t+1/t<2.5$. $x=-\frac{3}{2}$ when $t=\frac{1}{2}$ and $x=\frac{3}{2}$ when $t=2$. \

$$A=\int_{-3/2}^{3/2}(2.5-y)dx=\int_{1/2}^{2}\left(\frac{5}{2}-t-1/t\right)(1+1/t^2)dt\ [x=t-1/t,dx=(1+1/t^2)dt]$$

$$=\int_{1/2}^{2}\left(-t+\frac{5}{2}-2t^{-1}+\frac{5}{2}t^{-2}-t^{-3}\right)dt=\left[\frac{-t^2}{2}+\frac{5t}{2}-2\ln|t|-\frac{5}{2t}+\frac{1}{2t^2}\right]_{1/2}^{2}$$

$$= \left(-2+5-2\ln 2-\frac{5}{4}+\frac{1}{8} \right)-\left(-\frac{1}{8}+\frac{5}{4}+2\ln 2-5+2 \right)=\frac{15}{4}-4\ln 2$$

33.

$$A=\int_0^1 (y-1)\,dx=\int_{\pi/2}^0 (e^t-1)(-\sin t)\,dt=\int_0^{\pi/2} (e^t\sin t-\sin t)\,dt=\left[\frac{1}{2}e^t(\sin t-\cos t)+\cos t \right]_0^{\pi/2}$$

$$=\frac{1}{2}(e^{\pi/2}-1)$$

34. By symmetry, $A=4\int_0^a y\,dx=4\int_{\pi/2}^0 a\sin^3\theta(-3a\cos^2\theta\sin\theta)\,d\theta=12a^2\int_0^{\pi/2}\sin^4\theta\cos^2\theta\,d\theta$. Now

$$\int\sin^4\theta\cos^2\theta\,d\theta = \int\sin^2\theta\left(\frac{1}{4}\sin^2 2\theta \right)d\theta=\frac{1}{8}\int(1-\cos 2\theta)\sin^2 2\theta\,d\theta$$

$$= \frac{1}{8}\int\left[\frac{1}{2}(1-\cos 4\theta)-\sin^2 2\theta\cos 2\theta \right]d\theta=\frac{1}{16}\theta-\frac{1}{64}\sin 4\theta-\frac{1}{48}\sin^3 2\theta+C$$

so $\int_0^{\pi/2}\sin^4\theta\cos^2\theta\,d\theta=\left[\frac{1}{16}\theta-\frac{1}{64}\sin 4\theta-\frac{1}{48}\sin^3 2\theta \right]_0^{\pi/2}=\frac{\pi}{32}$. Thus, $A=12a^2\left(\frac{\pi}{32} \right)=\frac{3}{8}\pi a^2$.

35.

$$A=\int_0^{2\pi r} y\,dx=\int_0^{2\pi}(r-d\cos\theta)(r-d\cos\theta)\,d\theta=\int_0^{2\pi}(r^2-2dr\cos\theta+d^2\cos^2\theta)\,d\theta$$

$$=\left[r^2\theta-2dr\sin\theta+\frac{1}{2}d^2\left(\theta+\frac{1}{2}\sin 2\theta \right) \right]_0^{2\pi}=2\pi r^2+\pi d^2$$

36. **(a)** By symmetry, the area of is twice the area inside above the x– axis. The top half of the loop is described by $x=t^2$, $y=t^3-3t$, $-\sqrt{3}\le t\le 0$, so, using the Substitution Rule with $y=t^3-3t$ and $dx=2t\,dt$, we find that

$$\text{area} = 2\int_0^3 y\,dx=2\int_0^{-\sqrt{3}}(t^3-3t)2t\,dt=2\int_0^{-\sqrt{3}}(2t^4-6t^2)\,dt=2\left[\frac{2}{5}t^5-2t^3 \right]_0^{-\sqrt{3}}$$

$$= 2\left[\frac{2}{5}(-3^{1/2})^5-2(-3^{1/2})^3 \right]=2\left[\frac{2}{5}\left(-9\sqrt{3}\right)-2\left(-3\sqrt{3}\right) \right]=\frac{24}{5}\sqrt{3}$$

(b) Here we use the formula for disks and use the Substitution Rule as in part (a):

$$\text{volume} = \pi\int_0^3 y^2\,dx=\pi\int_0^{-\sqrt{3}}(t^3-3t)^2 2t\,dt=2\pi\int_0^{-\sqrt{3}}(t^6-6t^4+9t^2)t\,dt$$

$$= 2\pi\left[\frac{1}{8}t^8-t^6+\frac{9}{4}t^4 \right]_0^{-\sqrt{3}}=2\pi\left[\frac{1}{8}(-3^{1/2})^8-(-3^{1/2})^6+\frac{9}{4}(-3^{1/2})^4 \right]$$

$$= 2\pi\left[\frac{81}{8} - 27 + \frac{81}{4}\right] = \frac{27}{4}\pi$$

(c) By symmetry, the $y-$ coordinate of the centroid is 0 . To find the $x-$ coordinate, we note that it is the same as the $x-$ coordinate of the centroid of the top half of , the area of which is $\frac{1}{2}\cdot\frac{24}{5}\sqrt{3}=\frac{12}{5}\sqrt{3}$. So, using Formula .3.8 with $A=\frac{12}{5}\sqrt{3}$, we get

$$\bar{x} = \frac{5}{12\sqrt{3}}\int_0^3 xy\,dx = \frac{5}{12\sqrt{3}}\int_0^{-\sqrt{3}} t^2(t^3-3t)2t\,dt = \frac{5}{6\sqrt{3}}\left[\frac{1}{7}t^7 - \frac{3}{5}t^5\right]_0^{-\sqrt{3}}$$

$$= \frac{5}{6\sqrt{3}}\left[\frac{1}{7}(-3^{1/2})^7 - \frac{3}{5}(-3^{1/2})^5\right] = \frac{5}{6\sqrt{3}}\left[-\frac{27}{7}\sqrt{3}+\frac{27}{5}\sqrt{3}\right] = \frac{9}{7}$$

So the coordinates of the centroid of are $(\mathrm{x},\mathrm{y})=\left(\frac{9}{7},0\right)$.

37. $x=t-t^2$, $y=\frac{4}{3}t^{3/2}$, $1\le t\le 2$. $dx/dt=1-2t$ and $dy/dt=2t^{1/2}$, so $(dx/dt)^2+(dy/dt)^2=(1-2t)^2+(2t^{1/2})^2=1-4t+4t^2+4t=1+4t^2$. Thus, $L=\int_a^b\sqrt{(dx/dt)^2+(dy/dt)^2}\,dt=\int_1^2\sqrt{1+4t^2}\,dt$.

38. $x=1+e^t$, $y=t^2$, $-3\le t\le 3$. $dx/dt=e^t$ and $dy/dt=2t$, so $(dx/dt)^2+(dy/dt)^2=e^{2t}+4t^2$. Thus, $L=\int_a^b\sqrt{(dx/dt)^2+(dy/dt)^2}\,dt=\int_{-3}^3\sqrt{e^{2t}+4t^2}\,dt$.

39. $x=t+\cos t$, $y=t-\sin t$, $0\le t\le 2\pi$. $dx/dt=1-\sin t$ and $dy/dt=1-\cos t$, so

$$(dx/dt)^2+(dy/dt)^2 = (1-\sin t)^2+(1-\cos t)^2=(1-2\sin t+\sin^2 t)+(1-2\cos t+\cos^2 t)$$
$$= 3-2\sin t-2\cos t$$

Thus, $L=\int_a^b\sqrt{(dx/dt)^2+(dy/dt)^2}\,dt=\int_0^{2\pi}\sqrt{3-2\sin t-2\cos t}\,dt$.

40. $x=\ln t$, $y=\sqrt{t+1}$, $1\le t\le 5$. $\frac{dx}{dt}=\frac{1}{t}$ and $\frac{dy}{dt}=\frac{1}{2\sqrt{t+1}}$, so $\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2=\frac{1}{t^2}+\frac{1}{4(t+1)}=\frac{t^2+4t+4}{4t^2(t+1)}$. Thus,

$$L=\int_a^b\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt=\int_1^5\sqrt{\frac{t^2+4t+4}{4t^2(t+1)}}\,dt=\int_1^5\sqrt{\frac{(t+2)^2}{(2t)^2(t+1)}}\,dt=\int_1^5\frac{t+2}{2t\sqrt{t+1}}\,dt\ .$$

41. $x=1+3t^2$, $y=4+2t^3$, $0\le t\le 1$. $dx/dt=6t$ and $dy/dt=6t^2$, so $(dx/dt)^2+(dy/dt)^2=36t^2+36t^4$. Thus,

$$L=\int_0^1\sqrt{36t^2+36t^4}\,dt=\int_0^1 6t\sqrt{1+t^2}\,dt=6\int_1^2\sqrt{u}\left(\frac{1}{2}du\right)\ [u=1+t^2,\ du=2t\,dt]$$

$$=3\left[\frac{2}{3}u^{3/2}\right]_1^2=2(2^{3/2}-1)=2\left(2\sqrt{2}-1\right)$$

42. $x=a(\cos\theta+\theta\sin\theta)$, $y=a(\sin\theta-\theta\cos\theta)$, $0\le\theta\le\pi$.

$$\begin{aligned}(dx/d\theta)^2+(dy/d\theta)^2 &= a^2\left[(-\sin\theta+\theta\cos\theta+\sin\theta)^2+(\cos\theta+\theta\sin\theta-\cos\theta)^2\right]\\ &= a^2\theta^2\left(\cos^2\theta+\sin^2\theta\right)=(a\theta)^2\end{aligned}$$

Thus, $L=\int_0^\pi a\theta\,d\theta=a\left[\frac{1}{2}\theta^2\right]_0^\pi=\frac{1}{2}\pi^2a$.

43. $x=\dfrac{t}{1+t}$, $y=\ln(1+t)$, $0\le t\le 2$. $\dfrac{dx}{dt}=\dfrac{(1+t)\cdot 1-t\cdot 1}{(1+t)^2}=\dfrac{1}{(1+t)^2}$ and $\dfrac{dy}{dt}=\dfrac{1}{1+t}$, so

$$\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2=\frac{1}{(1+t)^4}+\frac{1}{(1+t)^2}=\frac{1}{(1+t)^4}\left[1+(1+t)^2\right]=\frac{t^2+2t+2}{(1+t)^4}$$. Thus,

$$L=\int_0^2\frac{\sqrt{t^2+2t+2}}{(1+t)^2}\,dt=\int_1^3\frac{\sqrt{u^2+1}}{u^2}\,du[u=t+1,du=dt]=\left[-\frac{\sqrt{u^2+1}}{u}+\ln\left(u+\sqrt{u^2+1}\right)\right]_1^3$$

$$=-\frac{\sqrt{10}}{3}+\ln\left(3+\sqrt{10}\right)+\sqrt{2}-\ln\left(1+\sqrt{2}\right)$$

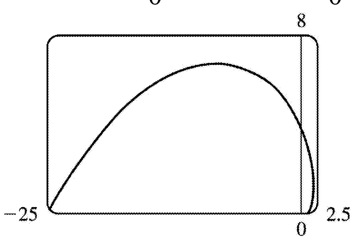
44. $x=e^t+e^{-t}$, $y=5-2t$, $0\le t\le 3$. $dx/dt=e^t-e^{-t}$ and $dy/dt=-2$, so $(dx/dt)^2+(dy/dt)^2=e^{2t}-2+e^{-2t}+4=e^{2t}+2+e^{-2t}=(e^t+e^{-t})^2$. Thus, $L=\int_0^3(e^t+e^{-t})dt=\left[e^t-e^{-t}\right]_0^3=e^3-e^{-3}-(1-1)=e^3-e^{-3}$.

45. $x=e^t\cos t$, $y=e^t\sin t$, $0\le t\le\pi$.

$$\begin{aligned}\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2 &= \left[e^t(\cos t-\sin t)\right]^2+\left[e^t(\sin t+\cos t)\right]^2\\ &= \left(e^t\right)^2\left(\cos^2t-2\cos t\sin t+\sin^2t\right)\end{aligned}$$

$$+\left(e^{t}\right)^{2}\left(\sin^{2}t+2\sin t\cos t+\cos^{2}t\right)$$
$$=\ e^{2t}\left(2\cos^{2}t+2\sin^{2}t\right)=2e^{2t}$$

Thus, $L=\int_{0}^{\pi}\sqrt{2e^{2t}}\,dt=\int_{0}^{\pi}\sqrt{2}\,e^{t}\,dt=\sqrt{2}\left[e^{t}\right]_{0}^{\pi}=\sqrt{2}\left(e^{\pi}-1\right)$.

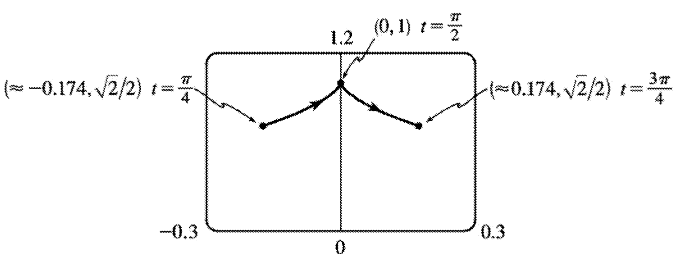


46. $x=\cos t+\ln\left(\tan\frac{1}{2}t\right)$, $y=\sin t$, $\pi/4\le t\le 3\pi/4$.

$$\frac{dx}{dt}=-\sin t+\frac{\frac{1}{2}\sec^{2}(t/2)}{\tan(t/2)}=-\sin t+\frac{1}{2\sin(t/2)\cos(t/2)}=-\sin t+\frac{1}{\sin t}$$

and $\frac{dy}{dt}=\cos t$, so $\left(\frac{dx}{dt}\right)^{2}+\left(\frac{dy}{dt}\right)^{2}=\sin^{2}t-2+\frac{1}{\sin^{2}t}+\cos^{2}t=1-2+\csc^{2}t=\cot^{2}t$. Thus,

$$L=\int_{\pi/4}^{3\pi/4}|\cot t|\,dt=2\int_{\pi/4}^{\pi/2}\cot t\,dt=2[\ln|\sin t|]_{\pi/4}^{\pi/2}=2\left(\ln 1-\ln\frac{1}{\sqrt{2}}\right)=2\left(0+\ln\sqrt{2}\right)=2\left(\frac{1}{2}\ln 2\right)=\ln 2.$$



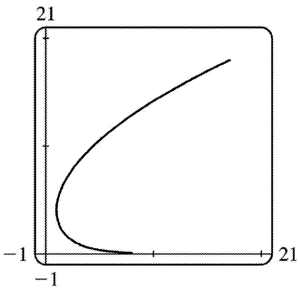
47. $x=e^{t}-t$, $y=4e^{t/2}$, $-8\le t\le 3$.

$$(dx/dt)^{2}+(dy/dt)^{2}=(e^{t}-1)^{2}+\left(2e^{t/2}\right)^{2}=e^{2t}-2e^{t}+1+4e^{t}=e^{2t}+2e^{t}+1=(e^{t}+1)^{2}$$

Thus,

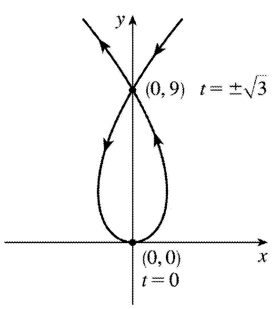
$$L=$$

$$\int_{-8}^{3}\sqrt{(e^t+1)^2}\,dt=\int_{-8}^{3}(e^t+1)\,dt=\Big[e^t+t\Big]_{-8}^{3}=(e^3+3)-(e^{-8}-8)=e^3-e^{-8}+11.$$



48. $x=3t-t^3$, $y=3t^2$. $dx/dt=3-3t^2$ and $dy/dt=6t$, so $\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2=(3-3t^2)^2+(6t)^2=(3+3t^2)^2$ and the length of the loop is given by

$$L=\int_{-\sqrt{3}}^{\sqrt{3}}(3+3t^2)\,dt=2\int_0^{\sqrt{3}}(3+3t^2)\,dt=2[3t+t^3]_0^{\sqrt{3}}$$
$$=2\left(3\sqrt{3}+3\sqrt{3}\right)=12\sqrt{3}.$$



49. $x=t-e^t$, $y=t+e^t$, $-6\le t\le 6$.
$\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2=\left(1-e^t\right)^2+\left(1+e^t\right)^2=\left(1-2e^t+e^{2t}\right)+\left(1+2e^t+e^{2t}\right)=2+2e^{2t}$, so $L=\int_{-6}^{6}\sqrt{2+2e^{2t}}\,dt$. Set $f(t)=\sqrt{2+2e^{2t}}$. Then by Simpson's Rule with $n=6$ and $\Delta t=\frac{6-(-6)}{6}=2$, we get
$L\approx\frac{2}{3}[f(-6)+4f(-4)+2f(-2)+4f(0)+2f(2)+4f(4)+f(6)]\approx 612.3053$.

50. $x=2a\cot\theta\Rightarrow dx/dt=-2a\csc^2\theta$ and $y=2a\sin^2\theta\Rightarrow dy/dt=4a\sin\theta\cos\theta=2a\sin 2\theta$.
So $L=\int_{\pi/4}^{\pi/2}\sqrt{4a^2\csc^4\theta+4a^2\sin^2 2\theta}\,d\theta=2a\int_{\pi/4}^{\pi/2}\sqrt{\csc^4\theta+\sin^2 2\theta}\,d\theta$. Using Simpson's Rule with $n=4$,

$\Delta\theta=\dfrac{\pi/2-\pi/4}{4}=\dfrac{\pi}{16}$, and $f(\theta)=\sqrt{\csc^4\theta+\sin^2 2\theta}$, we get

$L\approx 2a\cdot S_4=(2a)\,\dfrac{\pi}{16\cdot 3}\left[f\left(\dfrac{\pi}{4}\right)+4f\left(\dfrac{5\pi}{16}\right)+2f\left(\dfrac{3\pi}{8}\right)+4f\left(\dfrac{7\pi}{16}\right)+f\left(\dfrac{\pi}{2}\right)\right]\approx 2.2605a$.

51. $x=\sin^2 t$, $y=\cos^2 t$, $0\le t\le 3\pi$.

$(dx/dt)^2+(dy/dt)^2=(2\sin t\cos t)^2+(-2\cos t\sin t)^2=8\sin^2 t\cos^2 t=2\sin^2 2t\Rightarrow$

Distance $=\int_0^{3\pi}\sqrt{2}\,|\sin 2t|\,dt=6\sqrt{2}\int_0^{\pi/2}\sin 2t\,dt$ [bysymmetry]$=-3\sqrt{2}\,[\cos 2t]_0^{\pi/2}$

$=-3\sqrt{2}\,(-1-1)=6\sqrt{2}$

The full curve is traversed as t goes from 0 to $\dfrac{\pi}{2}$, because the curve is the segment of $x+y=1$ that lies in the first quadrant (since x , $y\ge 0$), and this segment is completely traversed as t goes from 0 to $\dfrac{\pi}{2}$. Thus, $L=\int_0^{\pi/2}\sin 2t\,dt=\sqrt{2}$, as above.

52. $x=\cos^2 t$, $y=\cos t$, $0\le t\le 4\pi$. $\left(\dfrac{dx}{dt}\right)^2+\left(\dfrac{dy}{dt}\right)^2=(-2\cos t\sin t)^2+(-\sin t)^2=\sin^2 t(4\cos^2 t+1)$

Distance $=\int_0^{4\pi}|\sin t|\sqrt{4\cos^2 t+1}\,dt=4\int_0^{\pi}\sin t\sqrt{4\cos^2 t+1}\,dt$

$=-4\int_1^{-1}\sqrt{4u^2+1}\,du[u=\cos t,du=-\sin t\,dt]=4\int_{-1}^{1}\sqrt{4u^2+1}\,du=8\int_0^1\sqrt{4u^2+1}\,du$

$=8\int_0^{\tan^{-1}2}\sec\theta\cdot\dfrac{1}{2}\sec^2\theta\,d\theta\;\; 2u=\tan\theta,2du=\sec^2\theta\,d\theta\,]$

$=4\int_0^{\tan^{-1}2}\sec^3\theta\,d\theta=[2\sec\theta\tan\theta+2\ln|\sec\theta+\tan\theta|]_0^{\tan^{-1}2}=4\sqrt{5}+2\ln\left(\sqrt{5}+2\right)$

Thus, $L=\int_0^{\pi}|\sin t|\sqrt{4\cos^2 t+1}\,dt=\sqrt{5}+\dfrac{1}{2}\ln\left(\sqrt{5}+2\right)$.

53. $x=a\sin\theta$, $y=b\cos\theta$, $0\le\theta\le 2\pi$.

$\left(\dfrac{dx}{d\theta}\right)^2+\left(\dfrac{dy}{d\theta}\right)^2=(a\cos\theta)^2+(-b\sin\theta)^2=a^2\cos^2\theta+b^2\sin^2\theta=a^2\left(1-\sin^2\theta\right)+b^2\sin^2\theta$

$=a^2-\left(a^2-b^2\right)\sin^2\theta=a^2-c^2\sin^2\theta=a^2\left(1-\dfrac{c^2}{a^2}\sin^2\theta\right)=a^2\left(1-e^2\sin^2\theta\right)$

So $L=4\int_0^{\pi/2}\sqrt{a^2\left(1-e^2\sin^2\theta\right)}\,d\theta=4a\int_0^{\pi/2}\sqrt{1-e^2\sin^2\theta}\,d\theta$.

54. $x=a\cos^3\theta$, $y=a\sin^3\theta$.

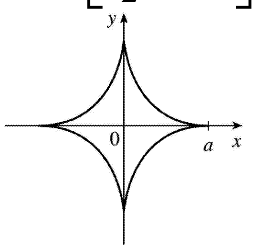
$(dx/d\theta)^2+(dy/d\theta)^2=\left(-3a\cos^2\theta\sin\theta\right)^2+\left(3a\sin^2\theta\cos\theta\right)^2=9a^2\cos^4\theta\sin^2\theta+9a^2\sin^4\theta\cos^2\theta$

$=9a^2\sin^2\theta\cos^2\theta\left(\cos^2\theta+\sin^2\theta\right)=9a^2\sin^2\theta\cos^2\theta$.

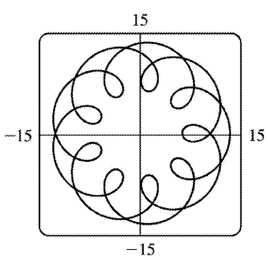
The graph has four–fold symmetry and the curve in the first quadrant corresponds to $0\le\theta\le\pi/2$. Thus,

$L=4\int_0^{\pi/2}3a\sin\theta\cos\theta\,d\theta$ [since $a>0$ and $\sin\theta$ and $\cos\theta$ are positive for $0\le\theta\le\pi/2$]

$=12a\left[\frac{1}{2}\sin^2\theta\right]_0^{\pi/2}=12a\left(\frac{1}{2}-0\right)=6a$.



55. **(a)** Notice that $0\le t\le 2\pi$ does not give the complete curve because $x(0)\ne x(2\pi)$. In fact, we must take $t\in[0,4\pi]$ in order to obtain the complete curve, since the first term in each of the parametric equations has period 2π and the second has period $\frac{2\pi}{11/2}=\frac{4\pi}{11}$, and the least common integer multiple of these two numbers is 4π .

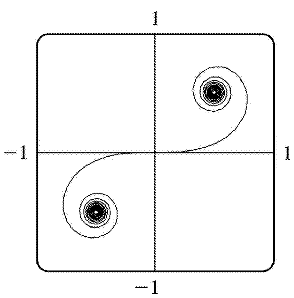


(b) We use the CAS to find the derivatives dx/dt and dy/dt , and then use Formula 1 to find the arc length. Recent versions of Maple express the integral $\int_0^{4\pi}\sqrt{(dx/dt)^2+(dy/dt)^2}\,dt$ as $88E\left(2\sqrt{2}\,i\right)$, where $E(x)$ is the elliptic integral $\int_0^1\frac{\sqrt{1-x^2t^2}}{\sqrt{1-t^2}}\,dt$ and i is the imaginary number $\sqrt{-1}$. Some earlier versions of Maple (as well as Mathematica) cannot do the integral exactly, so we use the command `evalf(Int(sqrt(diff(x,t)^2+diff(y,t)^2),t=0..4*Pi));` to estimate the length, and find that the arc length is approximately 294.03 . Derive's `Para_arc_length` function in the utility file `Int_apps` simplifies the integral to

$11\int_0^{4\pi}\sqrt{-4\cos t\cos\left(\frac{11t}{2}\right)-4\sin t\sin\left(\frac{11t}{2}\right)+5}\;dt$.

56. **(a)** It appears that as $t\to\infty$, $(x,y)\to\left(\frac{1}{2},\frac{1}{2}\right)$, and as $t\to-\infty$, $(x,y)\to\left(-\frac{1}{2},-\frac{1}{2}\right)$.

(b) By the Fundamental Theorem of Calculus, $dx/dt=\cos\left(\frac{\pi}{2}t^2\right)$ and $dy/dt=\sin\left(\frac{\pi}{2}t^2\right)$, so by Formula 6, the length of the curve from the origin to the point with parameter value t is



$$L=\int_0^t\sqrt{(dx/du)^2+(dy/du)^2}\;du=\int_0^t\sqrt{\cos^2\left(\frac{\pi}{2}u^2\right)+\sin^2\left(\frac{\pi}{2}u^2\right)}\;du$$

$$=\int_0^t 1\,du=t\ [\text{or } -t \text{ if } t<0]$$

We have used u as the dummy variable so as not to confuse it with the upper limit of integration.

57. $x=t-t^2$, $y=\frac{4}{3}t^{3/2}$, $1\le t\le 2$. $\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2=(1-2t)^2+(2t^{1/2})^2=1-4t+4t^2+4t=1+4t^2$, so $S=\int_1^2 2\pi y\,ds=\int_1^2 2\pi\cdot\frac{4}{3}t^{3/2}\sqrt{1+4t^2}\;dt=\int_1^2\frac{8\pi}{3}t^{3/2}\sqrt{1+4t^2}\;dt$.

58. $x=\sin^2 t$, $y=\sin 3t$, $0\le t\le\frac{\pi}{3}$. $dx/dt=2\sin t\cos t=\sin 2t$ and $dy/dt=3\cos 3t$, so $(dx/dt)^2+(dy/dt)^2=\sin^2 2t+9\cos^2 3t$ and $S=\int 2\pi y\,ds=\int_0^{\pi/3}2\pi\sin 3t\sqrt{\sin^2 2t+9\cos^2 3t}\;dt$.

59. $x=t^3$, $y=t^2$, $0\le t\le 1$. $\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2=\left(3t^2\right)^2+(2t)^2=9t^4+4t^2$.

$$S=\int_0^1 2\pi y\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\;dt=\int_0^1 2\pi t^2\sqrt{9t^4+4t^2}\;dt=2\pi\int_0^1 t^2\sqrt{t^2(9t^2+4)}\;dt$$

$$= 2\pi\int_4^{13}\left(\frac{u-4}{9}\right)\sqrt{u}\left(\frac{1}{18}du\right)\left[\begin{array}{ll} u=9t^2+4 & t^2=(u-4)/9 \\ du=18t\,dt & \text{so, } t\,dt=\frac{1}{18}\,du\end{array}\right] = \frac{2\pi}{9\cdot 18}\int_4^{13}(u^{3/2}-4u^{1/2})du$$

$$= \frac{\pi}{81}\left[\frac{2}{5}u^{5/2}-\frac{8}{3}u^{3/2}\right]_4^{13} = \frac{\pi}{81}\cdot\frac{2}{15}[3u^{5/2}-20u^{3/2}]_4^{13}$$

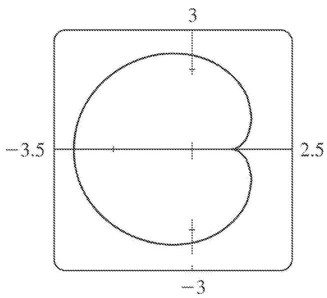
$$= \frac{2\pi}{1215}\left[\left(3\cdot 13^2\sqrt{13}-20\cdot 13\sqrt{13}\right)-(3\cdot 32-20\cdot 8)\right]$$

$$= \frac{2\pi}{1215}\left(247\sqrt{13}+64\right)$$

60. $x=3t-t^3$, $y=3t^2$, $0\le t\le 1$. $\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2=\left(3-3t^2\right)^2+(6t)^2=9(1+2t^2+t^4)=\left[3(1+t^2)\right]^2$.
$S=\int_0^1 2\pi\cdot 3t^2\cdot 3(1+t^2)dt=18\pi\int_0^1\left(t^2+t^4\right)dt=18\pi\left[\frac{1}{3}t^3+\frac{1}{5}t^5\right]_0^1=\frac{48}{5}\pi$

61. $x=a\cos^3\theta$, $y=a\sin^3\theta$, $0\le\theta\le\frac{\pi}{2}$.
$\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2=\left(-3a\cos^2\theta\sin\theta\right)^2+\left(3a\sin^2\theta\cos\theta\right)^2=9a^2\sin^2\theta\cos^2\theta$.
$S=\int_0^{\pi/2}2\pi\cdot a\sin^3\theta\cdot 3a\sin\theta\cos\theta\,d\theta=6\pi a^2\int_0^{\pi/2}\sin^4\theta\cos\theta\,d\theta=\frac{6}{5}\pi a^2\left[\sin^5\theta\right]_0^{\pi/2}=\frac{6}{5}\pi a^2$

62. $\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2=(-2\sin\theta+2\sin 2\theta)^2+(2\cos\theta-2\cos 2\theta)^2$
$=4\left[\left(\sin^2\theta-2\sin\theta\sin 2\theta+\sin^2 2\theta\right)+\left(\cos^2\theta-2\cos\theta\cos 2\theta+\cos^2 2\theta\right)\right]$
$=4[1+1-2(\cos 2\theta\cos\theta+\sin 2\theta\sin\theta)]=8[1-\cos(2\theta-\theta)]=8(1-\cos\theta)$
We plot the graph with parameter interval $[0,2\pi]$, and see that we should only integrate between 0 and π . (If the interval $[0,2\pi]$ were taken, the surface of revolution would be generated twice.) Also note that $y=2\sin\theta-\sin 2\theta=2\sin\theta\,(1-\cos\theta)$. So $S=\int_0^{\pi}2\pi\cdot 2\sin\theta\,(1-\cos\theta)2\sqrt{2}\sqrt{1-\cos\theta}\,d\theta$ =
$8\sqrt{2}\pi\int_0^{\pi}(1-\cos\theta)^{3/2}\sin\theta\,d\theta=8\sqrt{2}\pi\int_0^2\sqrt{u^3}\,du$ [where $u=1-\cos\theta$, $du=\sin\theta\,d\theta$] =
$8\sqrt{2}\pi\left[\left(\frac{2}{5}\right)u^{5/2}\right]_0^2=\frac{16}{5}\sqrt{2}\pi(2)^{\frac{5}{2}}=\frac{128}{5}\pi$



63. $x=t+t^3$, $y=t-\frac{1}{t^2}$, $1\le t\le 2$. $\frac{dx}{dt}=1+3t^2$ and $\frac{dy}{dt}=1+\frac{2}{t^3}$, so

$\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2=\left(1+3t^2\right)^2+\left(1+\frac{2}{t^3}\right)^2$ and

$S=\int 2\pi y\,ds=\int_1^2 2\pi\left(t-\frac{1}{t^2}\right)\sqrt{\left(1+3t^2\right)^2+\left(1+\frac{2}{t^3}\right)^2}\,dt\approx 59.101$.

64. $S=\int_{\pi/4}^{\pi/2}2\pi\cdot 2a\sin^2\theta\sqrt{\csc^4\theta+\sin^2 2\theta}\,dt=4\pi a\int_{\pi/4}^{\pi/2}\sin^2\theta\sqrt{\csc^4\theta+\sin^2 2\theta}\,d\theta$.

Using Simpson's Rule with $n=4$, $\Delta\theta=\frac{\pi/2-\pi/4}{4}=\frac{\pi}{16}$, and $f(\theta)=\sin^2\theta\sqrt{\csc^4\theta+\sin^2 2\theta}$, we get

$S\approx(4\pi a)\frac{\pi}{16\cdot 3}\left[f\left(\frac{\pi}{4}\right)+4f\left(\frac{5\pi}{16}\right)+2f\left(\frac{3\pi}{8}\right)+4f\left(\frac{7\pi}{16}\right)+f\left(\frac{\pi}{2}\right)\right]\approx 11.0893a$.

65. $x=3t^2$, $y=2t^3$, $0\le t\le 5\Rightarrow\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2=(6t)^2+(6t^2)^2=36t^2(1+t^2)\Rightarrow$

$S=\int_0^5 2\pi x\sqrt{(dx/dt)^2+(dy/dt)^2}\,dt=\int_0^5 2\pi(3t^2)6t\sqrt{1+t^2}\,dt=18\pi\int_0^5 t^2\sqrt{1+t^2}\,2t\,dt$

$=18\pi\int_1^{26}(u-1)\sqrt{u}\,du[\text{where}u=1+t^2,du=2t\,dt]=18\pi\int_1^{26}\left(u^{3/2}-u^{1/2}\right)du$

$=18\pi\left[\frac{2}{5}u^{5/2}-\frac{2}{3}u^{3/2}\right]_1^{26}=18\pi\left[\left(\frac{2}{5}\cdot 676\sqrt{26}-\frac{2}{3}\cdot 26\sqrt{26}\right)-\left(\frac{2}{5}-\frac{2}{3}\right)\right]$

$=\frac{24}{5}\pi\left(949\sqrt{26}+1\right)$

66. $x=e^t-t$, $y=4e^{t/2}$, $0\le t\le 1$. $\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2=\left(e^t-1\right)^2+\left(2e^{t/2}\right)^2=e^{2t}+2e^t+1=\left(e^t+1\right)^2$.

$S=\int_0^1 2\pi\left(e^t-t\right)\sqrt{\left(e^t-1\right)^2+\left(2e^{t/2}\right)^2}\,dt=\int_0^1 2\pi\left(e^t-t\right)\left(e^t+1\right)dt$

$$= 2\pi\left[\frac{1}{2}e^{2t}+e^{t}-(t-1)e^{t}-\frac{1}{2}t^{2}\right]_0^1=\pi\left(e^{2}+2e-6\right)$$

67. If f' is continuous and $f'(t)\neq 0$ for $a\leq t\leq b$, then either $f'(t)>0$ for all t in $[a,b]$ or $f'(t)<0$ for all t in $[a,b]$. Thus, f is monotonic (in fact, strictly increasing or strictly decreasing) on $[a,b]$. It follows that f has an inverse. Set $F=g\circ f^{-1}$, that is, define F by $F(x)=g(f^{-1}(x))$. Then $x=f(t)\Rightarrow f^{-1}(x)=t$, so $y=g(t)=g(f^{-1}(x))=F(x)$.

68. By Formula .2.5 with $y=F(x)$, $S=\int_a^b 2\pi F(x)\sqrt{1+[F'(x)]^2}\,dx$. But by Formula .2.2,

$1+[F'(x)]^2=1+\left(\frac{dy}{dx}\right)^2=1+\left(\frac{dy/dt}{dx/dt}\right)^2=\frac{(dx/dt)^2+(dy/dt)^2}{(dx/dt)^2}$. Using the Substitution Rule with

$x=x(t)$, where $a=x(\alpha)$ and $b=x(\beta)$, we have (since $dx=\frac{dx}{dt}\,dt$)

$$S=\int_\alpha^\beta 2\pi F(x(t))\sqrt{\frac{(dx/dt)^2+(dy/dt)^2}{(dx/dt)^2}}\,\frac{dx}{dt}\,dt=\int_\alpha^\beta 2\pi y\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$$

which is Formula .2.7.

69. **(a)** $\phi=\tan^{-1}\left(\frac{dy}{dx}\right)\Rightarrow\frac{d\phi}{dt}=\frac{d}{dt}\tan^{-1}\left(\frac{dy}{dx}\right)=\frac{1}{1+(dy/dx)^2}\left[\frac{d}{dt}\left(\frac{dy}{dx}\right)\right]$. But

$\frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{\dot{y}}{\dot{x}}\Rightarrow\frac{d}{dt}\left(\frac{dy}{dx}\right)=\frac{d}{dt}\left(\frac{\dot{y}}{\dot{x}}\right)=\frac{\ddot{y}\dot{x}-\ddot{x}\dot{y}}{\dot{x}^2}\Rightarrow\frac{d\phi}{dt}=\frac{1}{1+\left(\dot{y}/\dot{x}\right)^2}\left(\frac{\ddot{y}\dot{x}-\ddot{x}\dot{y}}{\dot{x}^2}\right)=\frac{\dot{x}\ddot{y}-\ddot{x}\dot{y}}{\dot{x}^2+\dot{y}^2}$

.

Using the Chain Rule, and the fact that $s=\int_0^t\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt\Rightarrow$

$\frac{ds}{dt}=\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}=\left(\dot{x}^2+\dot{y}^2\right)^{1/2}$, we have that

$$\frac{d\phi}{ds}=\frac{d\phi/dt}{ds/dt}=\left(\frac{\dot{x}\ddot{y}-\ddot{x}\dot{y}}{\dot{x}^2+\dot{y}^2}\right)\frac{1}{\left(\dot{x}^2+\dot{y}^2\right)^{1/2}}=\frac{\dot{x}\ddot{y}-\ddot{x}\dot{y}}{\left(\dot{x}^2+\dot{y}^2\right)^{3/2}}$$. So

$$\kappa=\left|\frac{d\phi}{ds}\right|=\left|\frac{\dot{x}\ddot{y}-\ddot{x}\dot{y}}{\left(\dot{x}^2+\dot{y}^2\right)^{3/2}}\right|=\frac{\left|\dot{x}\ddot{y}-\ddot{x}\dot{y}\right|}{\left(\dot{x}^2+\dot{y}^2\right)^{3/2}}$$.

(b) $x=x$ and $y=f(x)\Rightarrow\dot{x}=1$, $\ddot{x}=0$ and $\dot{y}=\frac{dy}{dx}$, $\ddot{y}=\frac{d^2y}{dx^2}$. So

$$\kappa=\frac{\left|1\cdot\left(d^2y/dx^2\right)-0\cdot(dy/dx)\right|}{\left[1+(dy/dx)^2\right]^{3/2}}=\frac{\left|d^2y/dx^2\right|}{\left[1+(dy/dx)^2\right]^{3/2}}$$.

70. **(a)** $y=x^2\Rightarrow\frac{dy}{dx}=2x\Rightarrow\frac{d^2y}{dx^2}=2$. So $\kappa=\frac{\left|d^2y/dx^2\right|}{\left[1+(dy/dx)^2\right]^{3/2}}=\frac{2}{\left(1+4x^2\right)^{3/2}}$, and at $(1,1)$,

$$\kappa=\frac{2}{5^{3/2}}=\frac{2}{5\sqrt{5}}$$.

(b) $\kappa'=\frac{d\kappa}{dx}=-3(1+4x^2)^{-5/2}(8x)=0\Leftrightarrow x=0\Rightarrow y=0$. This is a maximum since $\kappa'>0$ for $x<0$ and $\kappa'<0$ for $x>0$. So the parabola $y=x^2$ has maximum curvature at the origin.

71. $x=\theta-\sin\theta\Rightarrow\dot{x}=1-\cos\theta\Rightarrow\ddot{x}=\sin\theta$, and $y=1-\cos\theta\Rightarrow\dot{y}=\sin\theta\Rightarrow\ddot{y}=\cos\theta$. Therefore,

$$\kappa=\frac{\left|\cos\theta-\cos^2\theta-\sin^2\theta\right|}{\left[(1-\cos\theta)^2+\sin^2\theta\right]^{3/2}}=\frac{\left|\cos\theta-\left(\cos^2\theta+\sin^2\theta\right)\right|}{\left(1-2\cos\theta+\cos^2\theta+\sin^2\theta\right)^{3/2}}=\frac{\left|\cos\theta-1\right|}{(2-2\cos\theta)^{3/2}}$$. The top of the arch is characterized by a horizontal tangent, and from Example 2(b) in Section .2, the tangent is horizontal when $\theta=(2n-1)\pi$, so take $n=1$ and substitute $\theta=\pi$ into the expression for κ :

$$\kappa=\frac{\left|\cos\pi-1\right|}{(2-2\cos\pi)^{3/2}}=\frac{\left|-1-1\right|}{\left[2-2(-1)\right]^{3/2}}=\frac{1}{4}$$.

72. **(a)** Every straight line has parametrizations of the form $x=a+vt$, $y=b+wt$, where a , b are arbitrary and v , $w\neq 0$. For example, a straight line passing through distinct points (a,b) and (c,d)

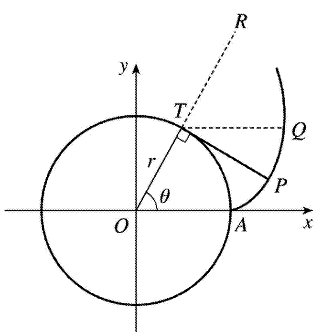
can be described as the parametrized curve $x=a+(c-a)t$, $y=b+(d-b)t$. Starting with $x=a+vt$, $y=b+wt$, we compute $\dot{x}=v$, $\dot{y}=w$, $\ddot{x}=\ddot{y}=0$, and $\kappa=\dfrac{|v\cdot 0-w\cdot 0|}{\left(v^2+w^2\right)^{3/2}}=0$.

(b) Parametric equations for a circle of radius r are $x=r\cos\theta$ and $y=r\sin\theta$. We can take the center to be the origin. So $\dot{x}=-r\sin\theta \Rightarrow \ddot{x}=-r\cos\theta$ and $\dot{y}=r\cos\theta \Rightarrow \ddot{y}=-r\sin\theta$. Therefore,

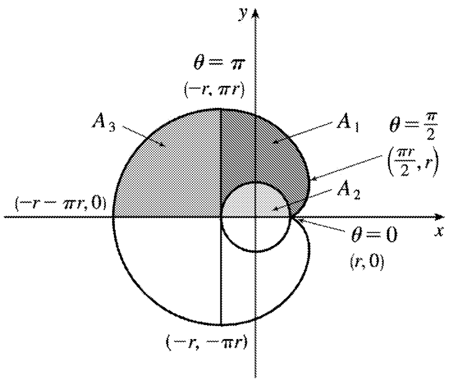
$$\kappa=\frac{\left|r^2\sin^2\theta+r^2\cos^2\theta\right|}{\left(r^2\sin^2\theta+r^2\cos^2\theta\right)^{3/2}}=\frac{r^2}{r^3}=\frac{1}{r}$$. And so for any θ (and thus any point), $\kappa=\dfrac{1}{r}$.

73. The coordinates of T are $(r\cos\theta, r\sin\theta)$. Since TP was unwound from arc TA , TP has length $r\theta$. Also $\angle PTQ=\angle PTR-\angle QTR=\frac{1}{2}\pi-\theta$, so P has coordinates

$x=r\cos\theta+r\theta\cos\left(\frac{1}{2}\pi-\theta\right)=r(\cos\theta+\theta\sin\theta)$, $y=r\sin\theta-r\theta\sin\left(\frac{1}{2}\pi-\theta\right)=r(\sin\theta-\theta\cos\theta)$.



74. If the cow walks with the rope taut, it traces out the portion of the involute in Exercise 73 corresponding to the range $0\le\theta\le\pi$, arriving at the point $(-r,\pi r)$ when $\theta=\pi$. With the rope now fully extended, the cow walks in a semicircle of radius πr , arriving at $(-r,-\pi r)$. Finally, the cow traces out another portion of the involute, namely the reflection about the $x-$ axis of the initial involute path. (This corresponds to the range $-\pi\le\theta\le 0$.) Referring to the figure, we see that the total grazing area is $2(A_1+A_3)$. A_3 is one–quarter of the area of a circle of radius πr , so $A_3=\frac{1}{4}\pi(\pi r)^2=\frac{1}{4}\pi^3 r^2$. We will compute A_1+A_2 and then subtract $A_2=\frac{1}{2}\pi r^2$ to obtain A_1 .



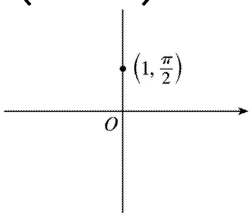
To find A_1+A_2 , first note that the rightmost point of the involute is $\left(\frac{\pi}{2}\text{ r,r}\right)$. The leftmost point of the involute is $(-\text{r},\pi r)$. Thus, $A_1+A_2=\int_{\theta=\pi}^{\pi/2} y\,dx-\int_{\theta=0}^{\pi/2} y\,dx=\int_{\theta=\pi}^{0} y\,dx$. Now $y\,dx=r(\sin\theta-\theta\cos\theta)r\theta\cos\theta\,d\theta=r^2(\theta\sin\theta\cos\theta-\theta^2\cos^2\theta)d\theta$. Integrate: $\left(1/r^2\right)\int y\,dx=-\theta\cos^2\theta-\frac{1}{2}\left(\theta^2-1\right)\sin\theta\cos\theta-\frac{1}{6}\theta^3+\frac{1}{2}\theta+C$. This enables us to compute

$$A_1+A_2 = r^2\left[-\theta\cos^2\theta-\frac{1}{2}\left(\theta^2-1\right)\sin\theta\cos\theta-\frac{1}{6}\theta^3+\frac{1}{2}\theta\right]_{\pi}^{0}=r^2\left[0-\left(-\pi-\frac{\pi^3}{6}+\frac{\pi}{2}\right)\right]$$

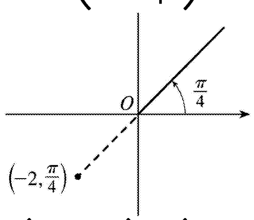
$$= r^2\left(\frac{\pi}{2}+\frac{\pi^3}{6}\right)$$

Therefore, $A_1=\left(A_1+A_2\right)-A_2=\frac{1}{6}\pi^3r^2$, so the grazing area is $2\left(A_1+A_3\right)=2\left(\frac{1}{6}\pi^3r^2+\frac{1}{4}\pi^3r^2\right)=\frac{5}{6}\pi^3r^2$.

1. **(a)** By adding 2π to $\frac{\pi}{2}$, we obtain the point $\left(1, \frac{5\pi}{2}\right)$. The direction opposite $\frac{\pi}{2}$ is $\frac{3\pi}{2}$, so $\left(-1, \frac{3\pi}{2}\right)$ is a point that satisfies the $r<0$ requirement.

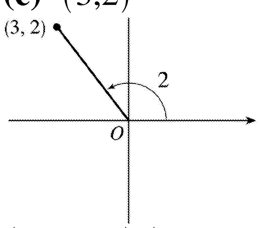


(b) $\left(-2, \frac{\pi}{4}\right)$



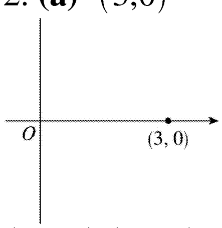
$\left(2, \frac{5\pi}{4}\right), \left(-2, \frac{9\pi}{4}\right)$

(c) $(3,2)$



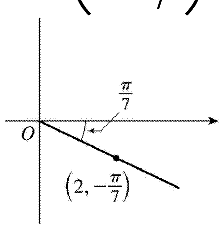
$(3,2+2\pi),(-3,2+\pi)$

2. **(a)** $(3,0)$



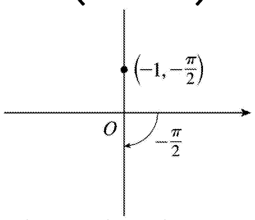
$(3,2\pi),(-3,\pi)$

(b) $\left(2, -\frac{\pi}{7}\right)$

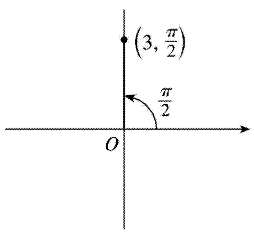


$\left(2,\frac{13\pi}{7}\right),\left(-2,\frac{6\pi}{7}\right)$

(c) $\left(-1,-\frac{\pi}{2}\right)$



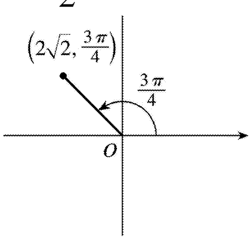
$\left(1,\frac{\pi}{2}\right),\left(-1,\frac{3\pi}{2}\right)$



3. **(a)**

$x=3\cos\frac{\pi}{2}=3(0)=0$ and

$y=3\sin\frac{\pi}{2}=3(1)=3$ give us the Cartesian coordinates $(0,3)$.

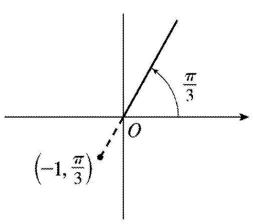


(b)

$$x=2\sqrt{2}\cos\frac{3\pi}{4}$$

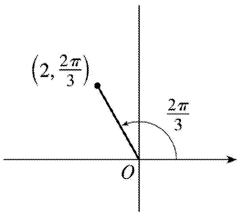
$$=2\sqrt{2}\left(-\frac{1}{\sqrt{2}}\right)=-2 \text{ and}$$

$y=2\sqrt{2}\sin\frac{3\pi}{4}=2\sqrt{2}\left(\frac{1}{\sqrt{2}}\right)=2$ give us $(-2,2)$.



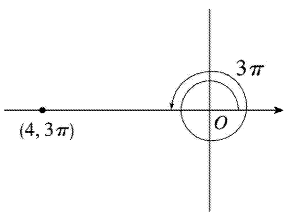
(c)

$x = -1\cos\frac{\pi}{3} = -\frac{1}{2}$ and
$y = -1\sin\frac{\pi}{3} = -\frac{\sqrt{3}}{2}$ give
us $\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$.



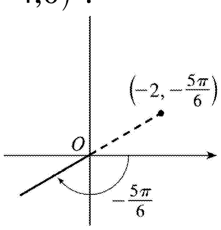
4. **(a)**

$x = 2\cos\frac{2\pi}{3} = -1$ and $y = 2\sin\frac{2\pi}{3} = \sqrt{3}$ give us $(-1, \sqrt{3})$.



(b)

$x = 4\cos 3\pi = -4$ and $y = 4\sin 3\pi = 0$ give
us $(-4, 0)$.



(c)

$x = -2\cos\left(-\frac{5\pi}{6}\right) = \sqrt{3}$ and $y = -2\sin\left(-\frac{5\pi}{6}\right) = 1$ give us $(\sqrt{3}, 1)$.

5. **(a)** $x = 1$ and $y = 1 \Rightarrow r = \sqrt{1^2 + 1^2} = \sqrt{2}$ and $\theta = \tan^{-1}\left(\frac{1}{1}\right) = \frac{\pi}{4}$. Since $(1, 1)$ is in the first quadrant, the polar coordinates are (i) $\left(\sqrt{2}, \frac{\pi}{4}\right)$ and (ii) $\left(-\sqrt{2}, \frac{5\pi}{4}\right)$.

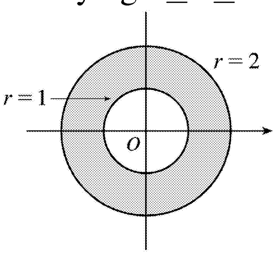
(b) $x = 2\sqrt{3}$ and $y = -2 \Rightarrow r = \sqrt{(2\sqrt{3})^2 + (-2)^2} = \sqrt{12 + 4} = \sqrt{16} = 4$ and
$\theta = \tan^{-1}\left(-\frac{2}{2\sqrt{3}}\right) = \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) = -\frac{\pi}{6}$. Since $(2\sqrt{3}, -2)$ is in the fourth quadrant and

$0 \le \theta \le 2\pi$, the polar coordinates are (i) $\left(4, \frac{11\pi}{6}\right)$ and (ii) $\left(-4, \frac{5\pi}{6}\right)$.

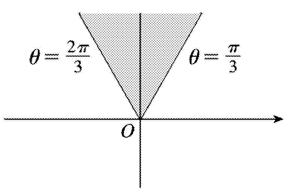
6. **(a)** $(x,y)=(-1,-\sqrt{3})$, $r=\sqrt{1+3}=2$, $\tan\theta = y/x = \sqrt{3}$ and (x,y) is in the third quadrant, so $\theta = \frac{4\pi}{3}$. The polar coordinates are (i) $\left(2, \frac{4\pi}{3}\right)$ and (ii) $\left(-2, \frac{\pi}{3}\right)$.

(b) $(x,y)=(-2,3)$, $r=\sqrt{4+9}=\sqrt{13}$, $\tan\theta = y/x = -\frac{3}{2}$ and (x,y) is in the second quadrant, so $\theta = \tan^{-1}\left(-\frac{3}{2}\right)+\pi$. The polar coordinates are (i) $(\sqrt{13}, \theta)$ and (ii) $(-\sqrt{13}, \theta+\pi)$.

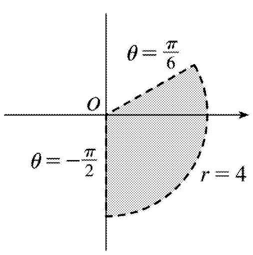
7. The curves $r=1$ and $r=2$ represent circles with center O and radii 1 and 2. The region in the plane satisfying $1 \le r \le 2$ consists of both circles and the shaded region between them in the figure.



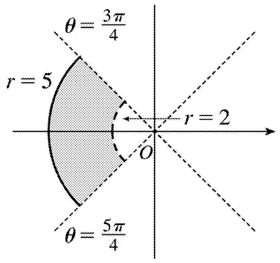
8. $r \ge 0$, $\pi/3 \le \theta \le 2\pi/3$



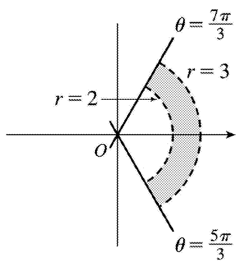
9. The region satisfying $0 \le r < 4$ and $-\pi/2 \le \theta < \pi/6$ does not include the circle $r=4$ nor the line $\theta = \frac{\pi}{6}$.



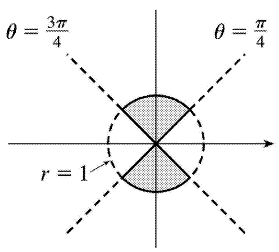
10. $2 < r \le 5$, $3\pi/4 < \theta < 5\pi/4$



11. $2<r<3$, $\frac{5\pi}{3} \le \theta \le \frac{7\pi}{3}$



12. $-1\le r\le 1$, $\frac{\pi}{4} \le \theta \le \frac{3\pi}{4}$



13. $(r,\theta)=\left(1, \frac{\pi}{6}\right) \Rightarrow x=1\cos\frac{\pi}{6}=\frac{\sqrt{3}}{2}$ and $y=1\sin\frac{\pi}{6}=\frac{1}{2}$.
$(r,\theta)=\left(3, \frac{3\pi}{4}\right) \Rightarrow x=3\cos\frac{3\pi}{4}=-\frac{3\sqrt{2}}{2}$ and $y=3\sin\frac{3\pi}{4}=\frac{3\sqrt{2}}{2}$. The distance between them is

$$\sqrt{\left[\frac{\sqrt{3}}{2}-\left(-\frac{3\sqrt{2}}{2}\right)\right]^2+\left(\frac{1}{2}-\frac{3\sqrt{2}}{2}\right)^2}=\sqrt{\frac{1}{4}\left(\sqrt{3}+3\sqrt{2}\right)^2+\frac{1}{4}\left(1-3\sqrt{2}\right)^2}$$

$$=\sqrt{\frac{1}{4}}\sqrt{(3+6\sqrt{6}+18)+(1-6\sqrt{2}+18)}=\frac{1}{2}\sqrt{40+6\sqrt{6}-6\sqrt{2}}$$

14. The points $\left(r_1,\theta_1\right)$ and $\left(r_2,\theta_2\right)$ in Cartesian coordinates are $\left(r_1\cos\theta_1, r_1\sin\theta_1\right)$ and $\left(r_2\cos\theta_2, r_2\sin\theta_2\right)$, respectively.The *square* of the distance between them is

$$\left(r_2\cos\theta_2 - r_1\cos\theta_1\right)^2 + \left(r_2\sin\theta_2 - r_1\sin\theta_1\right)^2 =$$
$$\left(r_2^2\cos^2\theta_2 - 2r_1r_2\cos\theta_1\cos\theta_2 + r_1^2\cos^2\theta_1\right) + \left(r_2^2\sin^2\theta_2 - 2r_1r_2\sin\theta_1\sin\theta_2 + r_1^2\sin^2\theta_1\right) = r_1^2\left(\sin^2\theta_1 + \cos^2\theta_1\right) + r_2^2$$
$$= r_1^2 - 2r_1r_2\cos\left(\theta_1 - \theta_2\right) + r_2^2,$$

so the distance between them is $\sqrt{r_1^2 - 2r_1r_2\cos(\theta_1 - \theta_2) + r_2^2}$.

15. $r=2 \Leftrightarrow \sqrt{x^2+y^2} = 2 \Leftrightarrow x^2+y^2=4$, a circle of radius 2 centered at the origin.

16. $r\cos\theta = 1 \Leftrightarrow x=1$, a vertical line.

17. $r=3\sin\theta \Rightarrow r^2=3r\sin\theta \Leftrightarrow x^2+y^2=3y \Leftrightarrow x^2+\left(y-\frac{3}{2}\right)^2=\left(\frac{3}{2}\right)^2$, a circle of radius $\frac{3}{2}$ centered at $\left(0,\frac{3}{2}\right)$. The first two equations are actually equivalent since $r^2=3r\sin\theta \Rightarrow r(r-3\sin\theta)=0 \Rightarrow r=0$ or $r=3\sin\theta$. But $r=3\sin\theta$ gives the point $r=0$ (the pole) when $\theta=0$. Thus, the single equation $r=3\sin\theta$ is equivalent to the compound condition ($r=0$ or $r=3\sin\theta$).

18. $r=2\sin\theta+2\cos\theta \Rightarrow r^2=2r\sin\theta+2r\cos\theta \Leftrightarrow x^2+y^2=2y+2x \Leftrightarrow \left(x^2-2x+1\right)+\left(y^2-2y+1\right)=2 \Leftrightarrow$ $(x-1)^2+(y-1)^2=2$. The first implication is reversible since $r^2=2r\sin\theta+2r\cos\theta \Rightarrow r=0$ or $r=2\sin\theta+2\cos\theta$, but the curve $r=2\sin\theta+2\cos\theta$ passes through the pole $(r=0)$ when $\theta=-\frac{\pi}{4}$, so $r=2\sin\theta+2\cos\theta$ includes the single point of $r=0$. The curve is a circle of radius $\sqrt{2}$, centered at $(1,1)$.

19. $r=\csc\theta \Leftrightarrow r=\frac{1}{\sin\theta} \Leftrightarrow r\sin\theta=1 \Leftrightarrow y=1$, a horizontal line 1 unit above the x –axis.

20. $r=\tan\theta\sec\theta=\frac{\sin\theta}{\cos^2\theta} \Rightarrow r\cos^2\theta=\sin\theta \Leftrightarrow (r\cos\theta)^2=r\sin\theta \Leftrightarrow x^2=y$, a parabola with vertex at the origin opening upward. The first implication is reversible since $\cos\theta=0$ would imply $\sin\theta=r\cos^2\theta=0$, contradicting the fact that $\cos^2\theta+\sin^2\theta=1$.

21. $x=3 \Leftrightarrow r\cos\theta=3 \Leftrightarrow r=3/\cos\theta \Leftrightarrow r=3\sec\theta$.

22. $x^2+y^2=9 \Leftrightarrow r^2=9 \Leftrightarrow r=3$. [$r=-3$ gives the same curve.]

23. $x=-y^2 \Leftrightarrow r\cos\theta=-r^2\sin^2\theta \Leftrightarrow \cos\theta=-r\sin^2\theta \Leftrightarrow r=-\dfrac{\cos\theta}{\sin^2\theta}=-\cot\theta\csc\theta$.

24. $x+y=9 \Leftrightarrow r\cos\theta+r\sin\theta=9 \Leftrightarrow r=9/(\cos\theta+\sin\theta)$.

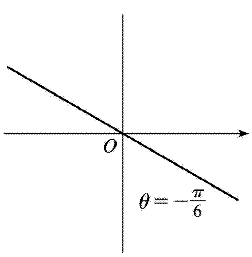
25. $x^2+y^2=2cx \Leftrightarrow r^2=2cr\cos\theta \Leftrightarrow r^2-2cr\cos\theta=0 \Leftrightarrow r(r-2c\cos\theta)=0 \Leftrightarrow r=0$ or $r=2c\cos\theta$. $r=0$ is included in $r=2c\cos\theta$ when $\theta=\dfrac{\pi}{2}+n\pi$, so the curve is represented by the single equation $r=2c\cos\theta$.

26. $x^2-y^2=1 \Leftrightarrow (r\cos\theta)^2-(r\sin\theta)^2=1 \Leftrightarrow r^2\left(\cos^2\theta-\sin^2\theta\right)=1 \Leftrightarrow r^2\cos 2\theta=1 \Rightarrow r^2=\sec 2\theta$

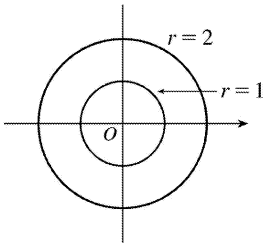
27. **(a)** The description leads immediately to the polar equation $\theta=\dfrac{\pi}{6}$, and the Cartesian equation $\tan\theta=y/x \Rightarrow y=\left(\tan\dfrac{\pi}{6}\right)x=\dfrac{1}{\sqrt{3}}x$ is slightly more difficult to derive.
(b) The easier description here is the Cartesian equation $x=3$.

28. **(a)** Because its center is not at the origin, it is more easily described by its Cartesian equation, $(x-2)^2+(y-3)^2=5^2$.
(b) This circle is more easily given in polar coordinates: $r=4$. The Cartesian equation is also simple: $x^2+y^2=16$.

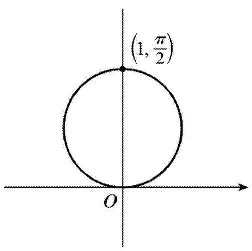
29. $\theta=-\pi/6$



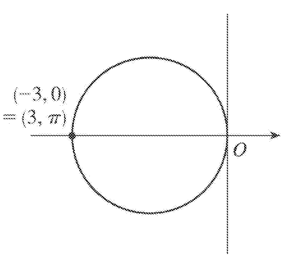
30. $r^2-3r+2=0 \Leftrightarrow (r-1)(r-2)=0 \Leftrightarrow r=1$ or $r=2$



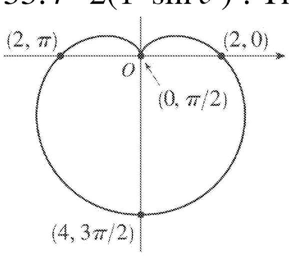
31. $r=\sin\theta \Leftrightarrow r^2=r\sin\theta \Leftrightarrow x^2+y^2=y \Leftrightarrow x^2+\left(y-\frac{1}{2}\right)^2=\left(\frac{1}{2}\right)^2$. The reasoning here is the same as in Exercise 17. This is a circle of radius $\frac{1}{2}$ centered at $\left(0,\frac{1}{2}\right)$.



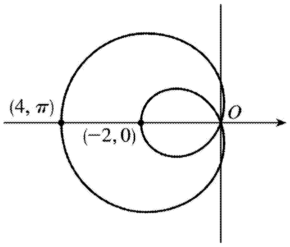
32. $r=-3\cos\theta \Leftrightarrow r^2=-3r\cos\theta \Leftrightarrow x^2+y^2=-3x \Leftrightarrow \left(x+\frac{3}{2}\right)^2+y^2=\left(\frac{3}{2}\right)^2$. This curve is a circle of radius $\frac{3}{2}$ centered at $\left(-\frac{3}{2},0\right)$.



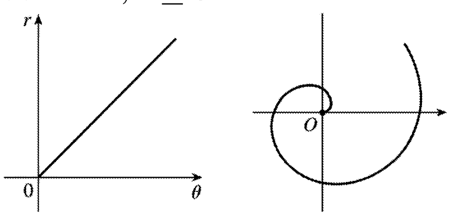
33. $r=2(1-\sin\theta)$. This curve is a cardioid.



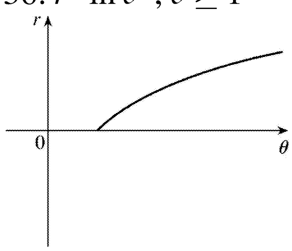
34. $r=1-3\cos\theta$. This is a limacon.

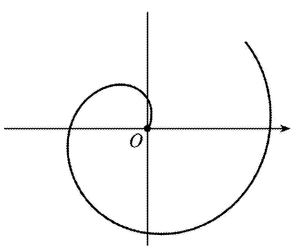


35. $r=\theta$, $\theta \geq 0$

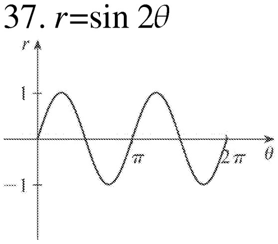


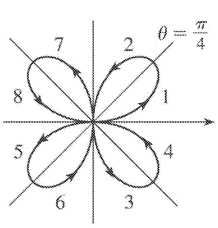
36. $r=\ln\theta$, $\theta \geq 1$



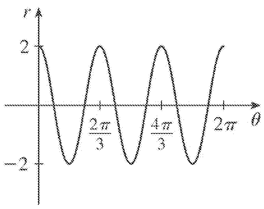


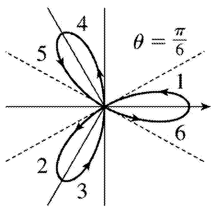
37. $r=\sin 2\theta$



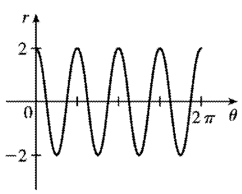


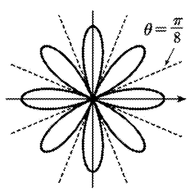
38. $r = 2\cos 3\theta$



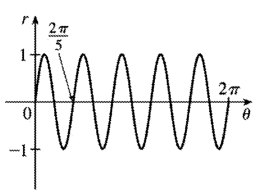


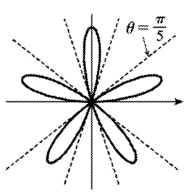
39. $r = 2\cos 4\theta$



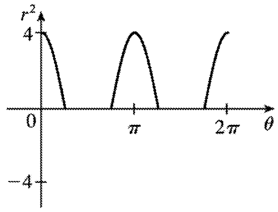


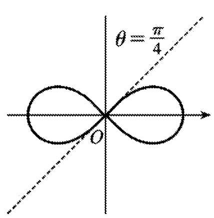
40. $r = \sin 5\theta$



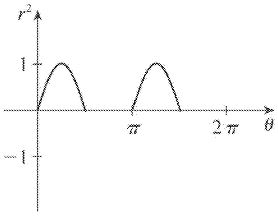


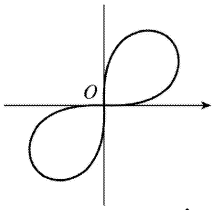
41. $r^2 = 4\cos 2\theta$



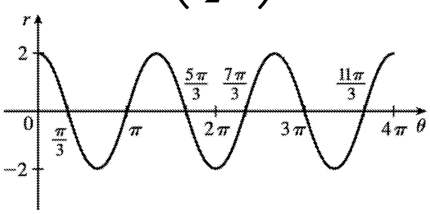


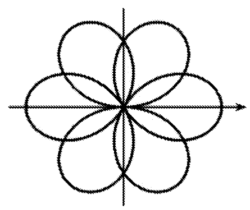
42. $r^2 = \sin 2\theta$



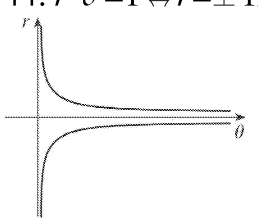


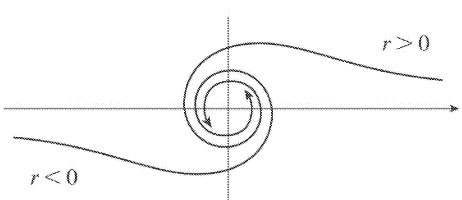
43. $r = 2\cos\left(\frac{3}{2}\theta\right)$



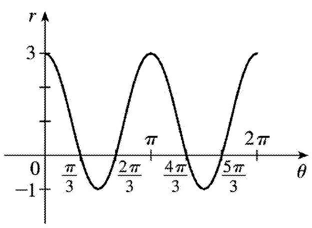


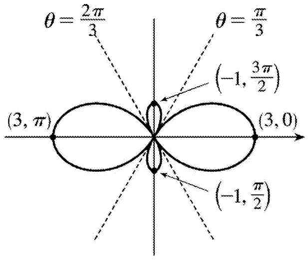
44. $r^2\theta = 1 \Leftrightarrow r = \pm 1/\sqrt{\theta}$ for $\theta > 0$



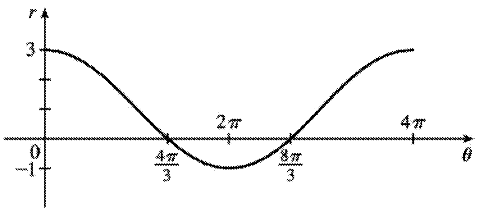


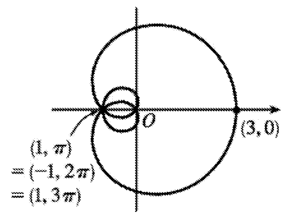
45. $r = 1 + 2\cos 2\theta$



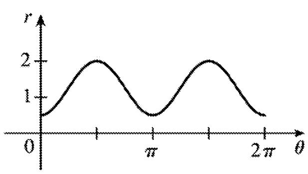


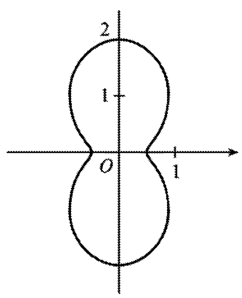
46. r=1+2cos (θ/2)



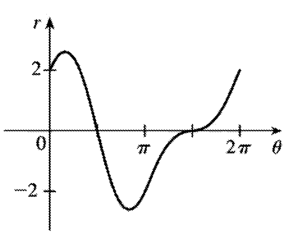


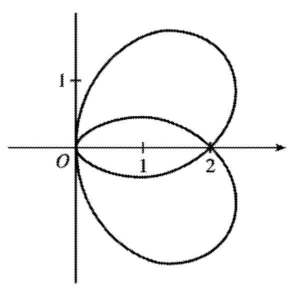
47. For θ=0 , π , and 2π , r has its minimum value of about 0.5 . For $\theta=\frac{\pi}{2}$ and $\frac{3\pi}{2}$, r attains its maximum value of 2 . We see that the graph has a similar shape for $0\leq\theta\leq\pi$ and $\pi\leq\theta\leq 2\pi$.





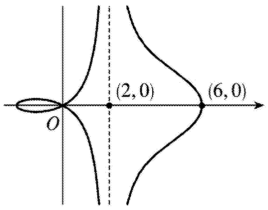
48.





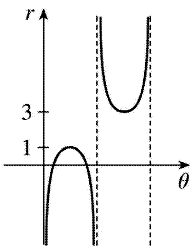
49. $x=r\cos\theta=(4+2\sec\theta)\cos\theta=4\cos\theta+2$. Now, $r\to\infty\Rightarrow(4+2\sec\theta)\to\infty\Rightarrow\theta\to\left(\frac{\pi}{2}\right)^-$ or $\theta\to\left(\frac{3\pi}{2}\right)^+$ (since we need only consider $0\leq\theta<2\pi$), so $\lim_{r\to\infty} x=\lim_{\theta\to\pi/2^-}(4\cos\theta+2)=2$. Also, $r\to-\infty\Rightarrow(4+2\sec\theta)\to-\infty\Rightarrow\theta\to\left(\frac{\pi}{2}\right)^+$ or $\theta\to\left(\frac{3\pi}{2}\right)^-$, so $\lim_{r\to-\infty} x=\lim_{\theta\to\pi/2^+}(4\cos\theta+2)=2$.

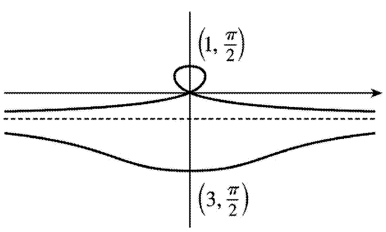
Therefore, $\lim_{r\to\pm\infty} x=2 \Rightarrow x=2$ is a vertical asymptote.



50. $y=r\sin\theta=2\sin\theta-\csc\theta\sin\theta=2\sin\theta-1$. $r\to\infty\Rightarrow(2-\csc\theta)\to\infty\Rightarrow\csc\theta\to-\infty\Rightarrow\theta\to\pi^+$ (since we need only consider $0\le\theta\le2\pi$) and so $\lim_{r\to\infty} y=\lim_{\theta\to\pi^+} 2\sin\theta-1=-1$. Also $r\to-\infty\Rightarrow(2-\csc\theta)\to-\infty\Rightarrow$

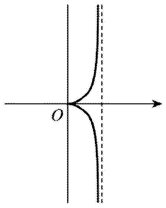
$\csc\theta\to\infty\Rightarrow$





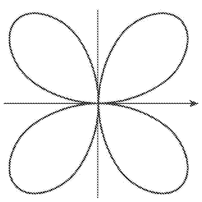
$\theta\to\pi^-$ and so $\lim_{r\to-\infty} x=\lim_{\theta\to\pi^-} 2\sin\theta-1=-1$. Therefore $\lim_{r\to\pm\infty} y=-1\Rightarrow y=-1$ is a horizontal asymptote.

51. To show that $x=1$ is an asymptote we must prove $\lim_{r\to\pm\infty} x=1$. $x=r\cos\theta=(\sin\theta\tan\theta)\cos\theta=\sin^2\theta$. Now, $r\to\infty\Rightarrow\sin\theta\tan\theta\to\infty\Rightarrow\theta\to\left(\frac{\pi}{2}\right)^-$, so $\lim_{r\to\infty} x=\lim_{\theta\to\pi/2^-}\sin^2\theta=1$. Also, $r\to-\infty\Rightarrow$ $\sin\theta\tan\theta\to-\infty\Rightarrow\theta\to\left(\frac{\pi}{2}\right)^+$, so $\lim_{r\to-\infty} x=\lim_{\theta\to\pi/2^+}\sin^2\theta=1$.



Therefore, $\lim_{r \to \pm\infty} x = 1 \Rightarrow x = 1$ is a vertical asymptote. Also notice that $x = \sin^2\theta \geq 0$ for all θ, and $x = \sin^2\theta \leq 1$ for all θ. And $x \neq 1$, since the curve is not defined at odd multiples of $\frac{\pi}{2}$. Therefore, the curve lies entirely within the vertical strip $0 \leq x < 1$.

52. The equation is $\left(x^2+y^2\right)^3 = 4x^2y^2$, but using polar coordinates we know that $x^2+y^2=r^2$ and $x = r\cos\theta$ and $y = r\sin\theta$. Substituting into the given equation: $r^6 = 4r^2\cos^2\theta\, r^2\sin^2\theta \Rightarrow$ $r^2 = 4\cos^2\theta\sin^2\theta \Rightarrow r = \pm 2\cos\theta\sin\theta = \pm\sin 2\theta$. $r = \pm\sin 2\theta$ is sketched at right.



53. **(a)** We see that the curve crosses itself at the origin, where $r=0$ (in fact the inner loop corresponds to negative r– values), so we solve the equation of the limacon for $r=0 \Leftrightarrow c\sin\theta = -1 \Leftrightarrow \sin\theta = -1/c$. Now if $|c| < 1$, then this equation has no solution and hence there is no inner loop. But if $c < -1$, then on the interval $(0, 2\pi)$ the equation has the two solutions $\theta = \sin^{-1}(-1/c)$ and $\theta = \pi - \sin^{-1}(-1/c)$, and if $c > 1$, the solutions are $\theta = \pi + \sin^{-1}(1/c)$ and $\theta = 2\pi - \sin^{-1}(1/c)$. In each case, $r < 0$ for θ between the two solutions, indicating a loop.
(b) For $0 < c < 1$, the dimple (if it exists) is characterized by the fact that y has a local maximum at $\theta = \frac{3\pi}{2}$. So we determine for what c– values $\frac{d^2y}{d\theta^2}$ is negative at $\theta = \frac{3\pi}{2}$, since by the Second Derivative Test this indicates a maximum: $y = r\sin\theta = \sin\theta + c\sin^2\theta \Rightarrow$ $\frac{dy}{d\theta} = \cos\theta + 2c\sin\theta\cos\theta = \cos\theta + c\sin 2\theta \Rightarrow \frac{d^2y}{d\theta^2} = -\sin\theta + 2c\cos 2\theta$. At $\theta = \frac{3\pi}{2}$, this is equal to $-(-1) + 2c(-1) = 1 - 2c$, which is negative only for $c > \frac{1}{2}$. A similar argument shows that for $-1 < c < 0$, y only has a local minimum at

$\theta = \frac{\pi}{2}$ (indicating a dimple) for $c < -\frac{1}{2}$.

54. **(a)** $r = \sin\ (\theta/2)$. This equation must correspond to one of II, III or VI, since these are the only graphs which are bounded. In fact it must be VI, since this is the only graph which is completed after a rotation of exactly 4π .
(b) $r = \sin\ (\theta/2)$. This equation must correspond to one of II, III or VI, since these are the only graphs which are bounded. In fact it must be VI, since this is the only graph which is completed after a rotation of exactly 4π .
(c) $r = \sin\ (\theta/4)$. This equation must correspond to III, since this is the only graph which is completed after a rotation of exactly 8π .
(d) $r = \sin\ (\theta/4)$. This equation must correspond to III, since this is the only graph which is completed after a rotation of exactly 8π .
(e) $r = \sec\ (3\theta)$. This must correspond to IV, since the graph is unbounded at $\theta = \frac{\pi}{6}$, $\frac{\pi}{2}$, $\frac{2\pi}{3}$, and so on.
(f) $r = \sec\ (3\theta)$. This must correspond to IV, since the graph is unbounded at $\theta = \frac{\pi}{6}$, $\frac{\pi}{2}$, $\frac{2\pi}{3}$, and so on.
(g) $r = \theta \sin \theta$. This must correspond to V. Note that $r = 0$ whenever θ is a multiple of π . This graph is unbounded, and each time θ moves through an interval of 2π , the same basic shape is repeated (because of the periodic $\sin \theta$ factor) but it gets larger each time (since θ increases each time we go around.)
(h) $r = \theta \sin \theta$. This must correspond to V. Note that $r = 0$ whenever θ is a multiple of π . This graph is unbounded, and each time θ moves through an interval of 2π , the same basic shape is repeated (because of the periodic $\sin \theta$ factor) but it gets larger each time (since θ increases each time we go around.)
(i) $r = 1 + 4\cos 5\theta$. This corresponds to II, since it is bounded, has fivefold rotational symmetry, and takes only one takes only one rotation through 2π to be complete.
(j) $r = 1 + 4\cos 5\theta$. This corresponds to II, since it is bounded, has fivefold rotational symmetry, and takes only one takes only one rotation through 2π to be complete.
(k) $r = 1/\sqrt{\theta}$. This corresponds to I, since it is unbounded at $\theta = 0$, and r decreases as θ increases; in fact $r \to 0$ as $\theta \to \infty$.
(l) $r = 1/\sqrt{\theta}$. This corresponds to I, since it is unbounded at $\theta = 0$, and r decreases as θ increases; in fact $r \to 0$ as $\theta \to \infty$.

55. $r = 2\sin \theta \Rightarrow x = r\cos \theta = 2\sin \theta \cos \theta = \sin 2\theta$, $y = r\sin \theta = 2\sin^2 \theta \Rightarrow$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{2 \cdot 2\sin \theta \cos \theta}{\cos 2\theta \cdot 2} = \frac{\sin 2\theta}{\cos 2\theta} = \tan 2\theta$$

When $\theta=\frac{\pi}{6}$, $\frac{dy}{dx}=\tan\left(2\cdot\frac{\pi}{6}\right)=\tan\frac{\pi}{3}=\sqrt{3}$.

56. $r=2-\sin\theta \Rightarrow x=r\cos\theta=(2-\sin\theta)\cos\theta$, $y=r\sin\theta=(2-\sin\theta)\sin\theta \Rightarrow$

$$\frac{dy}{dx}=\frac{dy/d\theta}{dx/d\theta}=\frac{(2-\sin\theta)\cos\theta+\sin\theta(-\cos\theta)}{(2-\sin\theta)(-\sin\theta)+\cos\theta(-\cos\theta)}=\frac{2\cos\theta-2\sin\theta\cos\theta}{-2\sin\theta+\sin^2\theta-\cos^2\theta}=\frac{2\cos\theta-\sin 2\theta}{-2\sin\theta-\cos 2\theta}$$

When $\theta=\frac{\pi}{3}$, $\frac{dy}{dx}=\frac{2(1/2)-\left(\sqrt{3}/2\right)}{-2\left(\sqrt{3}/2\right)-(-1/2)}=\frac{1-\sqrt{3}/2}{-\sqrt{3}+1/2}\cdot\frac{2}{2}=\frac{2-\sqrt{3}}{1-2\sqrt{3}}$.

57. $r=1/\theta \Rightarrow x=r\cos\theta=(\cos\theta)/\theta$, $y=r\sin\theta=(\sin\theta)/\theta \Rightarrow$

$$\frac{dy}{dx}=\frac{dy/d\theta}{dx/d\theta}=\frac{\sin\theta(-1/\theta^2)+(1/\theta)\cos\theta}{\cos\theta(-1/\theta^2)-(1/\theta)\sin\theta}\cdot\frac{\theta^2}{\theta^2}=\frac{-\sin\theta+\theta\cos\theta}{-\cos\theta-\theta\sin\theta}$$

When $\theta=\pi$, $\frac{dy}{dx}=\frac{-0+\pi(-1)}{-(-1)-\pi(0)}=\frac{-\pi}{1}=-\pi$.

58. $r=\ln\theta \Rightarrow x=r\cos\theta=\ln\theta\cos\theta$, $y=r\sin\theta=\ln\theta\sin\theta \Rightarrow$

$$\frac{dy}{dx}=\frac{dy/d\theta}{dx/d\theta}=\frac{\sin\theta\ (1/\theta)+\ln\theta\cos\theta}{\cos\theta\ (1/\theta)-\ln\theta\sin\theta}\cdot\frac{\theta}{\theta}=\frac{\sin\theta+\theta\ln\theta\cos\theta}{\cos\theta-\theta\ln\theta\sin\theta}$$

When $\theta=e$, $\frac{dy}{dx}=\frac{\sin e+e\ln e\cos e}{\cos e-e\ln e\sin e}=\frac{\sin e+e\cos e}{\cos e-e\sin e}$.

59. $r=1+\cos\theta \Rightarrow x=r\cos\theta=\cos\theta+\cos^2\theta$, $y=r\sin\theta=\sin\theta+\sin\theta\cos\theta \Rightarrow$

$$\frac{dy}{dx}=\frac{dy/d\theta}{dx/d\theta}=\frac{\cos\theta+\cos^2\theta-\sin^2\theta}{-\sin\theta-2\cos\theta\sin\theta}=\frac{\cos\theta+\cos 2\theta}{-\sin\theta-\sin 2\theta}$$

When $\theta=\frac{\pi}{6}$, $\frac{dy}{dx}=\frac{\frac{\sqrt{3}}{2}+\frac{1}{2}}{-\frac{1}{2}-\frac{\sqrt{3}}{2}}=\frac{\frac{\sqrt{3}}{2}+\frac{1}{2}}{-\left(\frac{1}{2}+\frac{\sqrt{3}}{2}\right)}=-1$.

60. $r=\sin 3\theta \Rightarrow x=r\cos\theta=\sin 3\theta\cos\theta$, $y=r\sin\theta=\sin 3\theta\sin\theta \Rightarrow$

$$\frac{dy}{dx}=\frac{dy/d\theta}{dx/d\theta}=\frac{3\cos 3\theta\sin\theta+\sin 3\theta\cos\theta}{3\cos 3\theta\cos\theta-\sin 3\theta\sin\theta}$$

When $\theta=\frac{\pi}{6}$, $\frac{dy}{dx}=\frac{3(0)(1/2)+1\left(\sqrt{3}/2\right)}{3(0)\left(\sqrt{3}/2\right)-1(1/2)}=\frac{\sqrt{3}/2}{-1/2}=-\sqrt{3}$.

61. $r=3\cos\theta \Rightarrow x=r\cos\theta=3\cos\theta\cos\theta$, $y=r\sin\theta=3\cos\theta\sin\theta \Rightarrow dy/d\theta=-3\sin^2\theta+3\cos^2\theta=3\cos 2\theta=0$ $\Rightarrow 2\theta=\frac{\pi}{2}$ or $\frac{3\pi}{2} \Leftrightarrow \theta=\frac{\pi}{4}$ or $\frac{3\pi}{4}$. So the tangent is horizontal at $\left(\frac{3}{\sqrt{2}},\frac{\pi}{4}\right)$ and $\left(-\frac{3}{\sqrt{2}},\frac{3\pi}{4}\right)$ $\left[\text{sameas}\left(\frac{3}{\sqrt{2}},-\frac{\pi}{4}\right)\right]$. $dx/d\theta=-6\sin\theta\cos\theta=-3\sin 2\theta=0 \Rightarrow 2\theta=0$ or $\pi \Leftrightarrow \theta=0$ or $\frac{\pi}{2}$. So the tangent is vertical at $(3,0)$ and $\left(0,\frac{\pi}{2}\right)$.

62. $y=r\sin\theta=\cos\theta\sin\theta+\sin^2\theta=\frac{1}{2}\sin 2\theta+\sin^2\theta \Rightarrow dy/d\theta=\cos 2\theta+\sin 2\theta=0 \Rightarrow \tan 2\theta=-1 \Rightarrow 2\theta=\frac{3\pi}{4}$ or $\frac{7\pi}{4} \Leftrightarrow \theta=\frac{3\pi}{8}$ or $\frac{7\pi}{8} \Rightarrow$ horizontal tangents at $\left(\cos\frac{3\pi}{8}+\sin\frac{3\pi}{8},\frac{3\pi}{8}\right)$ and $\left(\cos\frac{7\pi}{8}+\sin\frac{7\pi}{8},\frac{7\pi}{8}\right)$. $x=r\cos\theta=\cos^2\theta+\cos\theta\sin\theta \Rightarrow dx/d\theta=-\sin 2\theta+\cos 2\theta=0 \Rightarrow \tan 2\theta=1 \Rightarrow$ $2\theta=\frac{\pi}{4}$ or $\frac{5\pi}{4} \Leftrightarrow \theta=\frac{\pi}{8}$ or $\frac{5\pi}{8} \Rightarrow$ vertical tangents at $\left(\cos\frac{\pi}{8}+\sin\frac{\pi}{8},\frac{\pi}{8}\right)$ and $\left(\cos\frac{5\pi}{8}+\sin\frac{5\pi}{8},\frac{5\pi}{8}\right)$.

Note: These expressions can be simplified using trigonometric identities. For example, $\cos\frac{\pi}{8}+\sin\frac{\pi}{8}=\frac{1}{2}\sqrt{4+2\sqrt{2}}$.

63. $r=1+\cos\theta \Rightarrow x=r\cos\theta=\cos\theta(1+\cos\theta)$, $y=r\sin\theta=\sin\theta(1+\cos\theta) \Rightarrow$ $dy/d\theta=(1+\cos\theta)\cos\theta-\sin^2\theta=2\cos^2\theta+\cos\theta-1=(2\cos\theta-1)(\cos\theta+1)=0 \Rightarrow \cos\theta=\frac{1}{2}$ or $-1 \Rightarrow \theta=\frac{\pi}{3}$, π , or $\frac{5\pi}{3} \Rightarrow$ horizontal tangent at $\left(\frac{3}{2},\frac{\pi}{3}\right)$, $(0,\pi)$[the pole] , and $\left(\frac{3}{2},\frac{5\pi}{3}\right)$. $dx/d\theta=-(1+\cos\theta)\sin\theta-\cos\theta\sin\theta=-\sin\theta(1+2\cos\theta)=0 \Rightarrow \sin\theta=0$ or $\cos\theta=-\frac{1}{2} \Rightarrow \theta=0$, π , $\frac{2\pi}{3}$, or $\frac{4\pi}{3} \Rightarrow$ vertical tangent at $(2,0)$, $\left(\frac{1}{2},\frac{2\pi}{3}\right)$, and $\left(\frac{1}{2},\frac{4\pi}{3}\right)$. Note that the tangent is horizontal, not vertical when $\theta=\pi$, since $\lim_{\theta\to\pi}\frac{dy/d\theta}{dx/d\theta}=0$.

64. $\frac{dy}{d\theta}=e^{\theta}\sin\theta+e^{\theta}\cos\theta=e^{\theta}(\sin\theta+\cos\theta)=0 \Rightarrow \sin\theta=-\cos\theta \Rightarrow \tan\theta=-1 \Rightarrow \theta=-\frac{1}{4}\pi+n\pi$ (n any integer) $\Rightarrow$ horizontal tangents at $\left(e^{\pi(n-1/4)},\pi\left(n-\frac{1}{4}\right)\right)$. $\frac{dx}{d\theta}=e^{\theta}\cos\theta-e^{\theta}\sin\theta=e^{\theta}(\cos\theta-\sin\theta)=0 \Rightarrow \sin\theta=\cos\theta \Rightarrow \tan\theta=1 \Rightarrow \theta=\frac{1}{4}\pi+n\pi$ (n any integer) $\Rightarrow$

vertical tangents at $\left(e^{\pi(n+1/4)}, \pi\left(n+\frac{1}{4}\right)\right)$.

65. $r=\cos 2\theta \Rightarrow x=r\cos\theta=\cos 2\theta\cos\theta$, $y=r\sin\theta=\cos 2\theta\sin\theta \Rightarrow$

$$dy/d\theta=-2\sin 2\theta\sin\theta+\cos 2\theta\cos\theta=-4\sin^2\theta\cos\theta+\left(\cos^3\theta-\sin^2\theta\cos\theta\right)=$$
$$\cos\theta\left(\cos^2\theta-5\sin^2\theta\right)=\cos\theta\left(1-6\sin^2\theta\right)=0\Rightarrow$$

$\cos\theta=0$ or $\sin\theta=\pm\frac{1}{\sqrt{6}} \Rightarrow \theta=\frac{\pi}{2}$, $\frac{3\pi}{2}$, α , $\pi-\alpha$, $\pi+\alpha$, or $2\pi-\alpha$ (where $\alpha=\sin^{-1}\frac{1}{\sqrt{6}}$).
So the tangent is horizontal at $\left(-1,\frac{\pi}{2}\right)$, $\left(-1,\frac{3\pi}{2}\right)$, $\left(\frac{2}{3},\alpha\right)$, $\left(\frac{2}{3},\pi-\alpha\right)$, $\left(\frac{2}{3},\pi+\alpha\right)$, and $\left(\frac{2}{3},2\pi-\alpha\right)$.

$$dx/d\theta=-2\sin 2\theta\cos\theta-\cos 2\theta\sin\theta=-4\sin\theta\cos^2\theta-(2\cos^2\theta-1)\sin\theta=\sin\theta(1-6\cos^2\theta)=0\Rightarrow$$

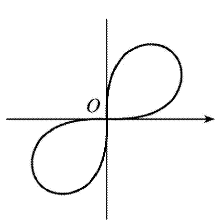
$\sin\theta=0$ or $\cos\theta=\pm\frac{1}{\sqrt{6}} \Rightarrow \theta=0$, π , β , $\pi-\beta$, $\pi+\beta$, or $2\pi-\beta$ (where $\beta=\cos^{-1}\frac{1}{\sqrt{6}}$).
So the tangent is vertical at $(1,0)$, $(1,\pi)$, $\left(-\frac{2}{3},\beta\right)$, $\left(-\frac{2}{3},\pi-\beta\right)$, $\left(-\frac{2}{3},\pi+\beta\right)$, and $\left(-\frac{2}{3},2\pi-\beta\right)$.

66.

By differentiating implicitly, $r^2=\sin 2\theta \Rightarrow$ $2r(dr/d\theta)=2\cos 2\theta \Rightarrow dr/d\theta=(1/r)\cos 2\theta$, so $\frac{dy}{d\theta}$

$$=\frac{1}{r}\cos 2\theta\sin\theta+r\cos\theta=\frac{1}{r}\left(\cos 2\theta\sin\theta+r^2\cos\theta\right)$$

$$=\frac{1}{r}(\cos 2\theta\sin\theta+\sin 2\theta\cos\theta)=\frac{1}{r}\sin 3\theta$$



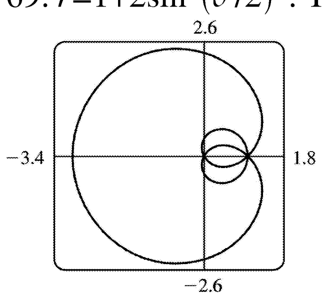
This is 0 when $\sin 3\theta=0\Rightarrow\theta=0$,

$\frac{\pi}{3}$ or $\frac{4\pi}{3}$ (restricting θ to the domain of the lemniscate), so there are horizontal tangents at $\left(\sqrt[4]{\frac{3}{4}},\frac{\pi}{3}\right)$, $\left(\sqrt[4]{\frac{3}{4}},\frac{4\pi}{3}\right)$ and $(0,0)$. Similarly, $dx/d\theta=(1/r)\cos 3\theta=0$ when $\theta=\frac{\pi}{6}$ or $\frac{7\pi}{6}$, so there are vertical tangents at $\left(\sqrt[4]{\frac{3}{4}},\frac{\pi}{6}\right)$ and $\left(\sqrt[4]{\frac{3}{4}},\frac{7\pi}{6}\right)$.

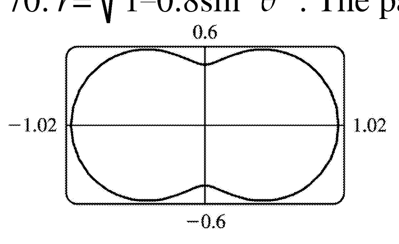
67. $r=a\sin\theta+b\cos\theta\Rightarrow r^2=ar\sin\theta+br\cos\theta\Rightarrow x^2+y^2=ay+bx\Rightarrow$
$x^2-bx+\left(\frac{1}{2}b\right)^2+y^2-ay+\left(\frac{1}{2}a\right)^2=\left(\frac{1}{2}b\right)^2+\left(\frac{1}{2}a\right)^2\Rightarrow\left(x-\frac{1}{2}b\right)^2+\left(y-\frac{1}{2}a\right)^2=\frac{1}{4}\left(a^2+b^2\right)$, and this is a circle with center $\left(\frac{1}{2}b,\frac{1}{2}a\right)$ and radius $\frac{1}{2}\sqrt{a^2+b^2}$.

68. These curves are circles which intersect at the origin and at $\left(\frac{1}{\sqrt{2}}a,\frac{\pi}{4}\right)$. At the origin, the first circle has a horizontal tangent and the second a vertical one, so the tangents are perpendicular here. For the first circle ($r=a\sin\theta$), $dy/d\theta=a\cos\theta\sin\theta+a\sin\theta\cos\theta=a\sin 2\theta=a$ at $\theta=\frac{\pi}{4}$ and $dx/d\theta=a\cos^2\theta-a\sin^2\theta=a\cos 2\theta=0$ at $\theta=\frac{\pi}{4}$, so the tangent here is vertical. Similarly, for the second circle ($r=a\cos\theta$), $dy/d\theta=a\cos 2\theta=0$ and $dx/d\theta=-a\sin 2\theta=-a$ at $\theta=\frac{\pi}{4}$, so the tangent is horizontal, and again the tangents are perpendicular.

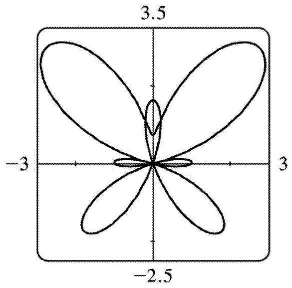
69. $r=1+2\sin(\theta/2)$. The parameter interval is $[0,4\pi]$.



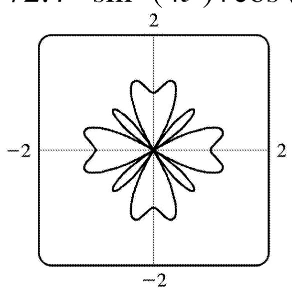
70. $r=\sqrt{1-0.8\sin^2\theta}$. The parameter interval is $[0,2\pi]$.



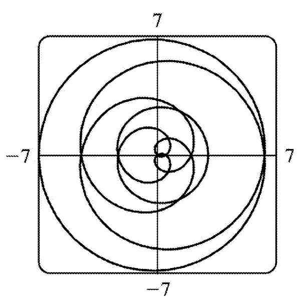
71. $r=e^{\sin\theta}-2\cos(4\theta)$. The parameter interval is $[0,2\pi]$.



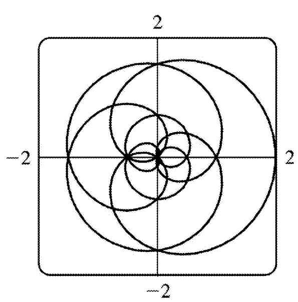
72. $r=\sin^2(4\theta)+\cos(4\theta)$. The parameter interval is $[0,2\pi]$.



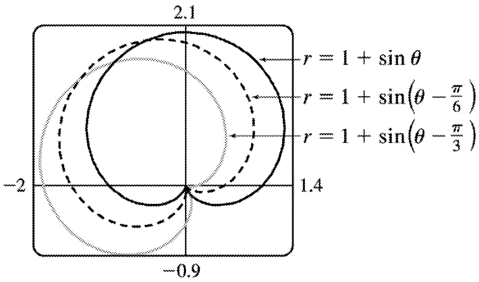
73. $r=2-5\sin(\theta/6)$. The parameter interval is $[-6\pi,6\pi]$.



74. $r=\cos(\theta/2)+\cos(\theta/3)$. The parameter interval is $[-6\pi,6\pi]$.

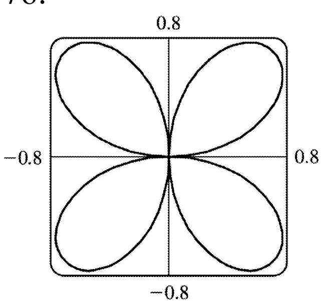


75.



It appears that the graph of $r=1+\sin\left(\theta-\frac{\pi}{6}\right)$ is the same shape as the graph of $r=1+\sin\theta$, but rotated counterclockwise about the origin by $\frac{\pi}{6}$. Similarly, the graph of $r=1+\sin\left(\theta-\frac{\pi}{3}\right)$ is rotated by $\frac{\pi}{3}$. In general, the
graph of $r=f(\theta-\alpha)$ is the same shape as that of $r=f(\theta)$, but rotated counterclockwise through α about the origin. That is, for any point $\left(r_0,\theta_0\right)$ on the curve $r=f(\theta)$, the point $\left(r_0,\theta_0+\alpha\right)$ is on the curve $r=f(\theta-\alpha)$, since $r_0=f\left(\theta_0\right)=f\left(\left(\theta_0+\alpha\right)-\alpha\right)$.

76.



From the graph, the highest points seem to have $y\approx 0.77$. To find the exact value, we solve $dy/d\theta=0$.
$y=r\sin\theta=\sin\theta\sin 2\theta \Rightarrow$

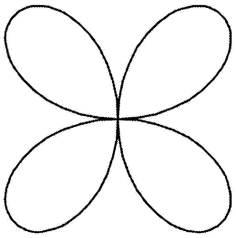
$$\begin{aligned} dy/d\theta &= 2\sin\theta\cos 2\theta+\cos\theta\sin 2\theta \\ &= 2\sin\theta\left(2\cos^2\theta-1\right)+\cos\theta\left(2\sin\theta\cos\theta\right) \\ &= 2\sin\theta\left(3\cos^2\theta-1\right) \end{aligned}$$

In the first quadrant, this is 0 when $\cos\theta=\frac{1}{\sqrt{3}} \Leftrightarrow \sin\theta=\sqrt{\frac{2}{3}} \Leftrightarrow$

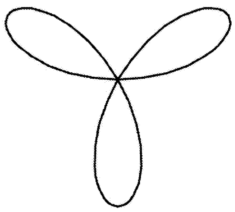
$y=2\sin^2\theta\cos\theta=2\cdot\frac{2}{3}\cdot\frac{1}{\sqrt{3}}=\frac{4\sqrt{3}}{9}\approx 0.77$.

77. **(a)** $r=\sin n\theta$. From the graphs, it seems that when n is even, the number of loops in the curve (called a rose) is $2n$, and when n is odd, the number of loops is simply n .
This is because in the case of n odd, every point on the graph is traversed twice, due to the fact that

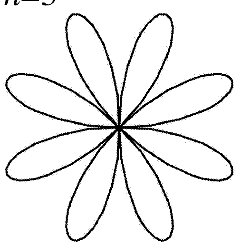
$$r(\theta+\pi)=\sin\left[n(\theta+\pi)\right]=\sin n\theta\cos n\pi+\cos n\theta\sin n\pi=\begin{cases}\sin n\theta & \text{if } n \text{ is even}\\ -\sin n\theta & \text{if } n \text{ is odd}\end{cases}$$



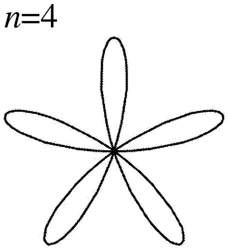
n=2



n=3

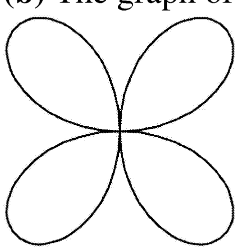


n=4

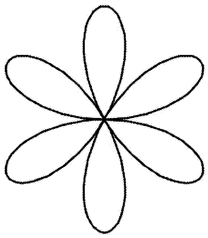


n=5

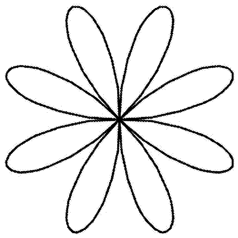
(b) The graph of $r=|\sin n\theta|$ has $2n$ loops whether n is odd or even, since $r(\theta+\pi)=r(\theta)$.



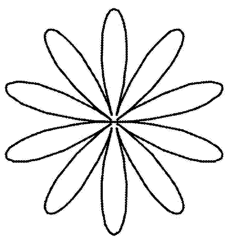
n=2



$n=3$



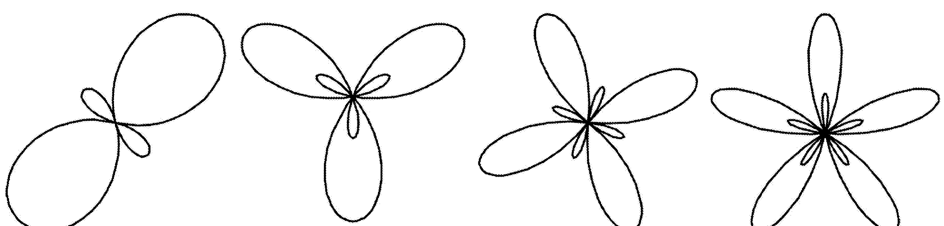
$n=4$



$n=5$

78. $r=1+c\sin n\theta$. We vary n while keeping c constant at 2 . As n changes, the curves change in the same way as those in Exercise 77: the number of loops increases. Note that if n is even, the smaller loops are outside the larger ones; if n is odd, they are inside.

$c=2$

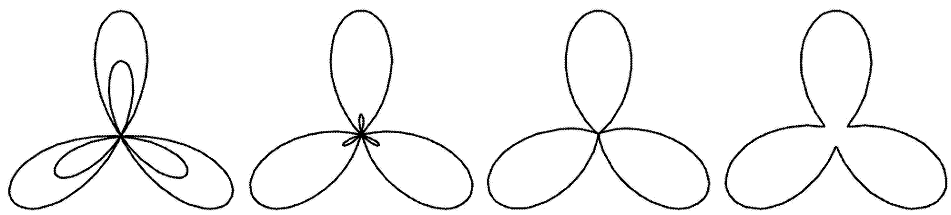


$n=2$ $n=3$ $n=4$ $n=5$

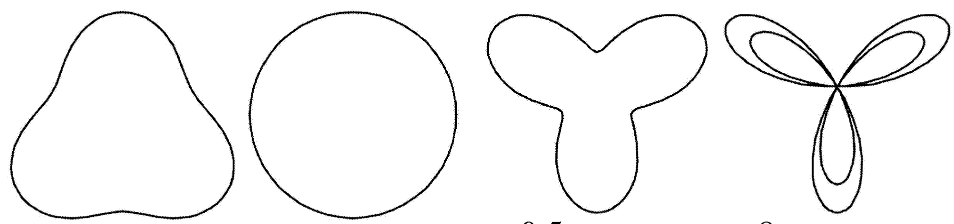
Now we vary c while keeping $n=3$. As c increases toward 0 , the entire graph gets smaller (the graphs below are not to scale) and the smaller loops shrink in relation to the large ones. At $c=-1$, the small loops disappear entirely, and for $-1<c<1$, the graph is a simple, closed curve (at $c=0$ it is a circle). As c continues to increase, the same changes are seen, but in reverse order, since $1+(-c)\sin n\theta=1+c\sin n(\theta+\pi)$, so the graph for $c=c_0$ is the same as that for $c=-c_0$, with a rotation through π . As $c\to\infty$, the smaller loops get relatively closer in size to the large ones. Note that the distance between the outermost points of corresponding inner and outer loops is always 2 .

Maple's `animate` command (or Mathematica's `Animate`) is very useful for seeing the changes that occur as c varies.

$n=3$

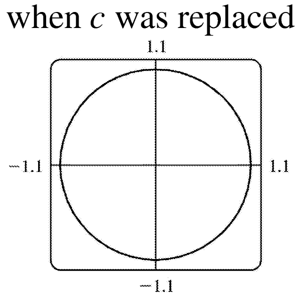


$c=-4$ $c=-1.4$ $c=-1$ $c=-0.8$

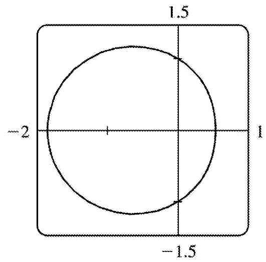


$c=-0.2$ $c=0$ $c=0.5$ $c=8$

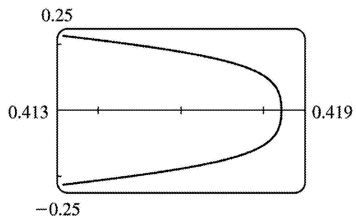
79. $r=\dfrac{1-a\cos\theta}{1+a\cos\theta}$. We start with $a=0$, since in this case the curve is simply the circle $r=1$. As a increases, the graph moves to the left, and its right side becomes flattened. As a increases through about 0.4 , the right side seems to grow a dimple, which upon closer investigation (with narrower $\theta-$ ranges) seems to appear at $a\approx 0.42$ (the actual value is $\sqrt{2}-1$). As $a\to 1$, this dimple becomes more pronounced, and the curve begins to stretch out horizontally, until at $a=1$ the denominator vanishes at $\theta=\pi$, and the dimple becomes an actual cusp. For $a>1$ we must choose our parameter interval carefully, since $r\to\infty$ as $1+a\cos\theta\to 0\Leftrightarrow\theta\to\pm\cos^{-1}(-1/a)$. As a increases from 1 , the curve splits into two parts. The left part has a loop, which grows larger as a increases, and the right part grows broader vertically, and its left tip develops a dimple when $a\approx 2.42$ (actually, $\sqrt{2}+1$). As a increases, the dimple grows more and more pronounced. If $a<0$, we get the same graph as we do for the corresponding positive $a-$ value, but with a rotation through π about the pole, as happened when c was replaced with $-c$ in Exercise 78.



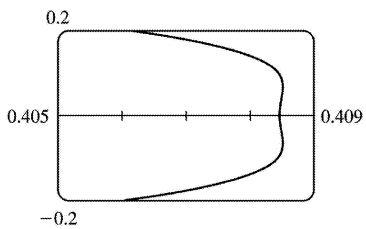
$a=0$



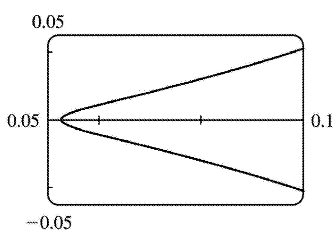
a=0.3



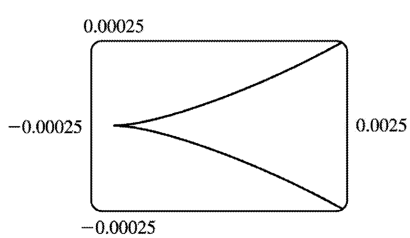
a=0.41,$|\theta| \leq 0.5$



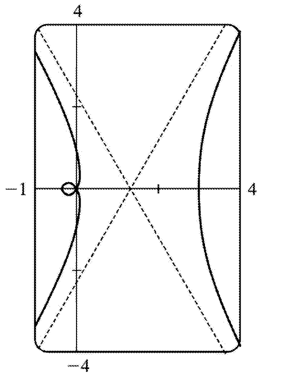
a=0.42,$|\theta| \leq 0.5$



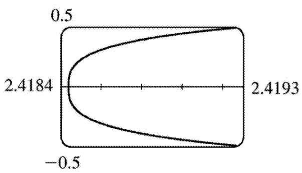
a=0.9,$|\theta| \leq 0.5$



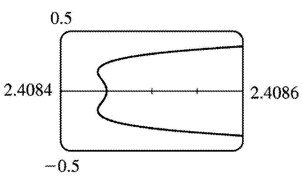
$a=1, |\theta| \leq 0.1$



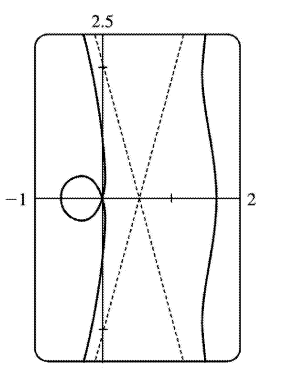
$a=2$



$a=2.41, |\theta-\pi| \leq 0.2$



$a=2.42, |\theta-\pi| \leq 0.2$



$a=4$

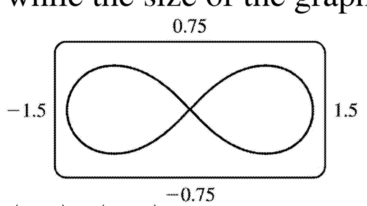
80. Most graphing devices cannot plot implicit polar equations, so we must first find an explicit expression (or expressions) for r in terms of θ , a , and c . We note that the given equation is a quadratic in r^2 , so we use the quadratic formula and find that

$$r^2=\frac{2c^2\cos 2\theta \pm\sqrt{4c^4\cos^2 2\theta-4\left(c^4-a^4\right)}}{2}=c^2\cos 2\theta \pm\sqrt{a^4-c^4\sin^2 2\theta}\ \text{ so}$$

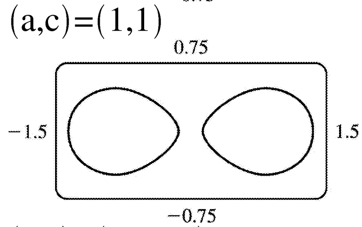
$$r=\pm\sqrt{c^2\cos 2\theta\pm\sqrt{a^4-c^4\sin^2 2\theta}}\ .$$

So for each graph, we must plot four curves to be sure of plotting all the points which satisfy the given equation. Note that all four functions have period π .

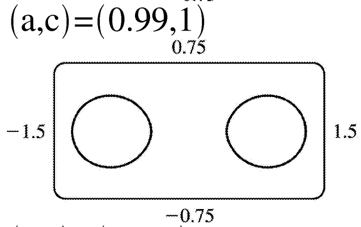
We start with the case $a=c=1$, and the resulting curve resembles the symbol for infinity. If we let a decrease, the curve splits into two symmetric parts, and as a decreases further, the parts become smaller, further apart, and rounder. If instead we let a increase from 1 , the two lobes of the curve join together, and as a increases further they continue to merge, until at $a\approx 1.4$, the graph no longer has dimples, and has an oval shape. As $a\to\infty$, the oval becomes larger and rounder, since the c^2 and c^4 terms lose their significance. Note that the shape of the graph seems to depend only on the ratio c/a , while the size of the graph varies as c and a jointly increase.



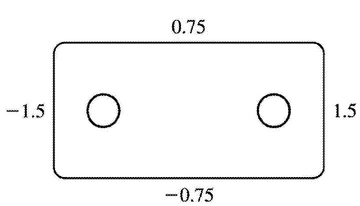
(a,c)=(1,1)



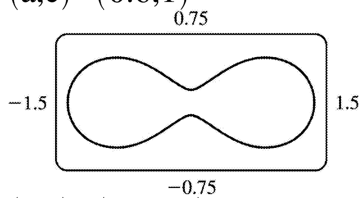
(a,c)=(0.99,1)



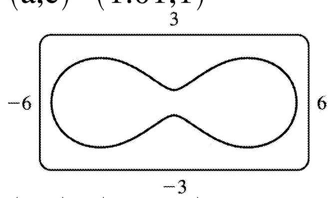
(a,c)=(0.9,1)



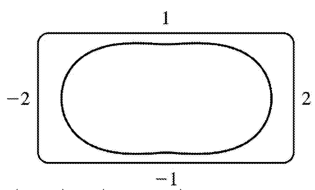
(a,c)=(0.6,1)



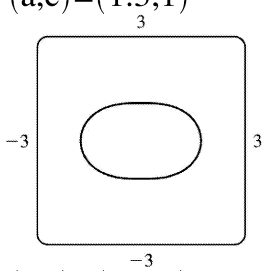
(a,c)=(1.01,1)



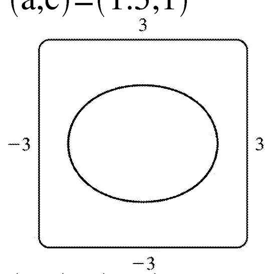
(a,c)=(4.04,4)



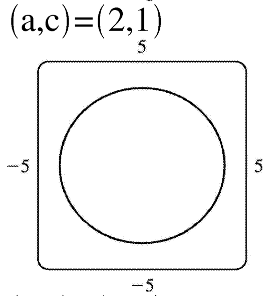
(a,c)=(1.3,1)



(a,c)=(1.5,1)



(a,c)=(2,1)



(a,c)=(4,1)

81.

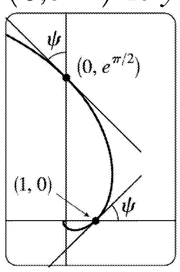
$$\tan\psi=\tan(\phi-\theta)=\frac{\tan\phi-\tan\theta}{1+\tan\phi\,\tan\theta}=\frac{\dfrac{dy}{dx}-\tan\theta}{1+\dfrac{dy}{dx}\tan\theta}=\frac{\dfrac{dy/d\theta}{dx/d\theta}-\tan\theta}{1+\dfrac{dy/d\theta}{dx/d\theta}\tan\theta}$$

$$=\frac{\dfrac{dy}{d\theta}-\dfrac{dx}{d\theta}\tan\theta}{\dfrac{dx}{d\theta}+\dfrac{dy}{d\theta}\tan\theta}=\frac{\left(\dfrac{dr}{d\theta}\sin\theta+r\cos\theta\right)-\tan\theta\left(\dfrac{dr}{d\theta}\cos\theta-r\sin\theta\right)}{\left(\dfrac{dr}{d\theta}\cos\theta-r\sin\theta\right)+\tan\theta\left(\dfrac{dr}{d\theta}\sin\theta+r\cos\theta\right)}$$

$$=\frac{r\cos\theta+r\cdot\dfrac{\sin^2\theta}{\cos\theta}}{\dfrac{dr}{d\theta}\cos\theta+\dfrac{dr}{d\theta}\cdot\dfrac{\sin^2\theta}{\cos\theta}}=\frac{r\cos^2\theta+r\sin^2\theta}{\dfrac{dr}{d\theta}\cos^2\theta+\dfrac{dr}{d\theta}\sin^2\theta}=\frac{r}{dr/d\theta}$$

82. **(a)** $r=e^{\theta}\Rightarrow dr/d\theta=e^{\theta}$, so by Exercise 81, $\tan\psi=r/e^{\theta}=1\Rightarrow\psi=\arctan 1=\dfrac{\pi}{4}$.

(b) The Cartesian equation of the tangent line at $(1,0)$ is $y=x-1$, and that of the tangent line at $\left(0,e^{\pi/2}\right)$ is $y=e^{\pi/2}-x$.



(c) Let a be the tangent of the angle between the tangent and radial lines, that is, $a=\tan\psi$. Then, by Exercise 81 , $a=\dfrac{r}{dr/d\theta}\Rightarrow\dfrac{dr}{d\theta}=\dfrac{1}{a}r\Rightarrow r=Ce^{\theta/a}$ (by Theorem 10.4.2).

1. $r=\sqrt{\theta}$, $0\le\theta\le\frac{\pi}{4}$. $A=\int_0^{\pi/4}\frac{1}{2}r^2d\theta=\int_0^{\pi/4}\frac{1}{2}\left(\sqrt{\theta}\right)^2d\theta=\int_0^{\pi/4}\frac{1}{2}\theta\,d\theta=\left[\frac{1}{4}\theta^2\right]_0^{\pi/4}=\frac{1}{64}\pi^2$

2. $r=e^{\theta/2}$, $\pi\le\theta\le2\pi$. $A=\int_\pi^{2\pi}\frac{1}{2}\left(e^{\theta/2}\right)^2d\theta=\int_\pi^{2\pi}\frac{1}{2}e^{\theta}d\theta=\frac{1}{2}\left[e^{\theta}\right]_\pi^{2\pi}=\frac{1}{2}\left(e^{2\pi}-e^{\pi}\right)$

3. $r=\sin\theta$, $\frac{\pi}{3}\le\theta\le\frac{2\pi}{3}$.

$$A=\int_{\pi/3}^{2\pi/3}\frac{1}{2}\sin^2\theta\,d\theta=\frac{1}{4}\int_{\pi/3}^{2\pi/3}(1-\cos2\theta)d\theta=\frac{1}{4}\left[\theta-\frac{1}{2}\sin2\theta\right]_{\pi/3}^{2\pi/3}$$

$$=\frac{1}{4}\left[\frac{2\pi}{3}-\frac{1}{2}\sin\frac{4\pi}{3}-\frac{\pi}{3}+\frac{1}{2}\sin\frac{2\pi}{3}\right]=\frac{1}{4}\left[\frac{2\pi}{3}-\frac{1}{2}\left(-\frac{\sqrt{3}}{2}\right)-\frac{\pi}{3}+\frac{1}{2}\left(\frac{\sqrt{3}}{2}\right)\right]=\frac{1}{4}\left(\frac{\pi}{3}+\frac{\sqrt{3}}{2}\right)$$

4. $r=\sqrt{\sin\theta}$, $0\le\theta\le\pi$. $A=\int_0^{\pi}\frac{1}{2}\left(\sqrt{\sin\theta}\right)^2d\theta=\int_0^{\pi}\frac{1}{2}\sin\theta\,d\theta=\left[-\frac{1}{2}\cos\theta\right]_0^{\pi}=\frac{1}{2}+\frac{1}{2}=1$

5. $r=\theta$, $0\le\theta\le\pi$. $A=\int_0^{\pi}\frac{1}{2}\theta^2d\theta=\left[\frac{1}{6}\theta^3\right]_0^{\pi}=\frac{1}{6}\pi^3$

6. $r=1+\sin\theta$, $\frac{\pi}{2}\le\theta\le\pi$.

$$A=\int_{\pi/2}^{\pi}\frac{1}{2}(1+\sin\theta)^2d\theta=\frac{1}{2}\int_{\pi/2}^{\pi}\left(1+2\sin\theta+\sin^2\theta\right)d\theta=\frac{1}{2}\int_{\pi/2}^{\pi}\left[1+2\sin\theta+\frac{1}{2}(1-\cos2\theta)\right]d\theta$$

$$=\frac{1}{2}\left[\theta-2\cos\theta+\frac{1}{2}\theta-\frac{1}{4}\sin2\theta\right]_{\pi/2}^{\pi}=\frac{1}{2}\left[\pi+2+\frac{\pi}{2}-0-\left(\frac{\pi}{2}-0+\frac{\pi}{4}-0\right)\right]=\frac{1}{2}\left(\frac{3\pi}{4}+2\right)=\frac{3\pi}{8}+1$$

7. $r=4+3\sin\theta$, $-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$.

$$A=\int_{-\pi/2}^{\pi/2}\frac{1}{2}(4+3\sin\theta)^2d\theta=\frac{1}{2}\int_{-\pi/2}^{\pi/2}\left(16+24\sin\theta+9\sin^2\theta\right)d\theta$$

$$=\frac{1}{2}\int_{-\pi/2}^{\pi/2}\left(16+9\sin^2\theta\right)d\theta\ \text{[byTheorem5.5.(b)]}$$

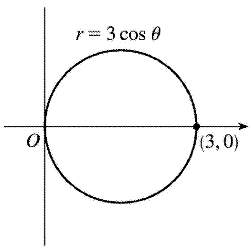
$$=\frac{1}{2}\cdot2\int_0^{\pi/2}\left[16+9\cdot\frac{1}{2}(1-\cos2\theta)\right]d\theta\ \text{[byTheorem5.5.(a)]}$$

$$=\int_0^{\pi/2}\left(\frac{41}{2}-\frac{9}{2}\cos2\theta\right)d\theta=\left[\frac{41}{2}\theta-\frac{9}{4}\sin2\theta\right]_0^{\pi/2}=\left(\frac{41\pi}{4}-0\right)-(0-0)=\frac{41\pi}{4}$$

8. $r=\sin4\theta$, $0\le\theta\le\frac{\pi}{4}$. $A=\int_0^{\pi/4}\frac{1}{2}\sin^24\theta\,d\theta=\int_0^{\pi/4}\frac{1}{4}(1-\cos8\theta)d\theta=\left[\frac{1}{4}\theta-\frac{1}{32}\sin8\theta\right]_0^{\pi/4}=\frac{\pi}{16}$

9. The area above the polar axis is bounded by $r=3\cos\theta$ for $\theta=0$ to $\theta=\pi/2$ (*not* π) . By symmetry,

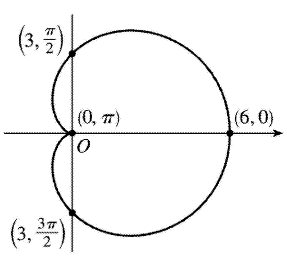
$$A = 2\int_0^{\pi/2} \frac{1}{2} r^2 d\theta = \int_0^{\pi/2} (3\cos\theta)^2 d\theta$$
$$= 3^2 \int_0^{\pi/2} \cos^2\theta \, d\theta = 9\int_0^{\pi/2} \frac{1}{2}(1+\cos 2\theta) d\theta$$
$$= \frac{9}{2}\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\pi/2} = \frac{9}{2}\left[\left(\frac{\pi}{2}+0\right)-(0+0)\right] = \frac{9\pi}{4} .$$



Also, note that this is a circle with radius $\frac{3}{2}$, so its area is $\pi\left(\frac{3}{2}\right)^2 = \frac{9\pi}{4}$.

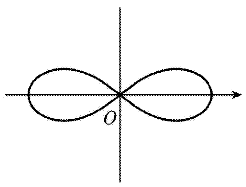
10.

$$A = \int_0^{2\pi} \frac{1}{2} r^2 d\theta = \int_0^{2\pi} \frac{1}{2} [3(1+\cos\theta)]^2 d\theta$$
$$= \frac{9}{2}\int_0^{2\pi} \left(1+2\cos\theta+\cos^2\theta\right) d\theta$$
$$= \frac{9}{2}\int_0^{2\pi} \left[1+2\cos\theta + \frac{1}{2}(1+\cos 2\theta)\right] d\theta$$
$$= \frac{9}{2}\left[\frac{3}{2}\theta + 2\sin\theta + \frac{1}{4}\sin 2\theta\right]_0^{2\pi} = \frac{27}{2}\pi$$



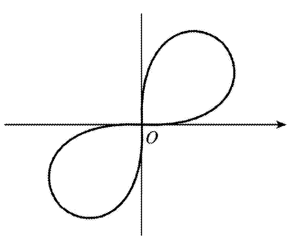
11. The curve $r^2 = 4\cos 2\theta$ goes through the pole when $\theta = \pi/4$, so we'll find the area for $0 \le \theta \le \pi/4$ and multiply it by 4 .

$$A=4\int_0^{\pi/4}\frac{1}{2}r^2\,d\theta=2\int_0^{\pi/4}(4\cos 2\theta)\,d\theta$$
$$=8\int_0^{\pi/4}\cos 2\theta\,d\theta=4\big[\sin 2\theta\big]_0^{\pi/4}=4(1-0)=4$$



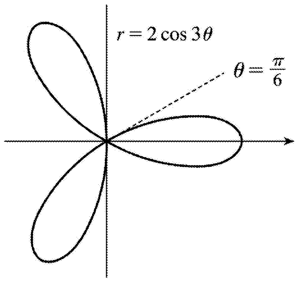
12. The curve $r^2=\sin 2\theta$ goes through the pole when $\theta=\pi/2$, so we'll find the area for $0\le\theta\le\pi/2$ and multiply it by 2 .

$$A=2\int_0^{\pi/2}\frac{1}{2}r^2\,d\theta=\int_0^{\pi/2}\sin 2\theta\,d\theta=-\frac{1}{2}[\cos 2\theta]_0^{\pi/2}$$
$$=-\frac{1}{2}(-1-1)=-\frac{1}{2}(-2)=1$$



13. One–sixth of the area lies above the polar axis and is bounded by the curve $r=2\cos 3\theta$ for $\theta=0$ to $\theta=\pi/6$.

$$A=6\int_0^{\pi/6}\frac{1}{2}(2\cos 3\theta)^2\,d\theta=12\int_0^{\pi/6}\cos^2 3\theta\,d\theta$$
$$=\frac{12}{2}\int_0^{\pi/6}(1+\cos 6\theta)\,d\theta$$
$$=6\left[\theta+\frac{1}{6}\sin 6\theta\right]_0^{\pi/6}=6\left(\frac{\pi}{6}\right)=\pi$$



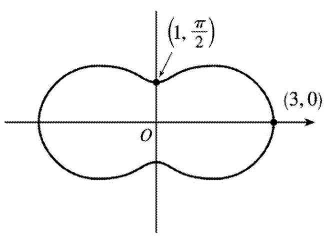
14.

$$A = \int_0^{2\pi} \frac{1}{2}(2+\cos 2\theta)^2 d\theta = \frac{1}{2}\int_0^{2\pi}(4+4\cos 2\theta + \cos^2 2\theta)d\theta$$

$$= \frac{1}{2}\int_0^{2\pi}\left(4+4\cos 2\theta + \frac{1}{2} + \frac{1}{2}\cos 4\theta\right)d\theta$$

$$= \frac{1}{2}\left[\frac{9}{2}\theta + 2\sin 2\theta + \frac{1}{8}\sin 4\theta\right]_0^{2\pi}$$

$$= \frac{1}{2}(9\pi) = \frac{9\pi}{2}$$



15.

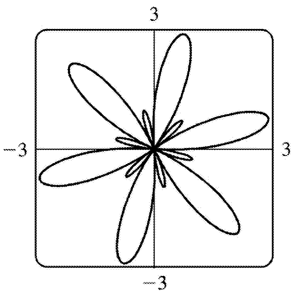
$$A = \int_0^{2\pi} \frac{1}{2}(1+2\sin 6\theta)^2 d\theta = \frac{1}{2}\int_0^{2\pi}(1+4\sin 6\theta + 4\sin^2 6\theta)d\theta$$

$$= \frac{1}{2}\int_0^{2\pi}\left[1+4\sin 6\theta + 4\cdot\frac{1}{2}(1-\cos 12\theta)\right]d\theta$$

$$= \frac{1}{2}\int_0^{2\pi}(3+4\sin 6\theta - 2\cos 12\theta)d\theta$$

$$= \frac{1}{2}\left[3\theta - \frac{2}{3}\cos 6\theta - \frac{1}{6}\sin 12\theta\right]_0^{2\pi}$$

$$= \frac{1}{2}\left[(6\pi - \frac{2}{3} - 0) - \left(0 - \frac{2}{3} - 0\right)\right] = 3\pi\ .$$



16.

$$A=\int_0^{\pi}\frac{1}{2}(2\sin\theta+3\sin 9\theta)^2\,d\theta=2\int_0^{\pi/2}\frac{1}{2}(2\sin\theta+3\sin 9\theta)^2\,d\theta$$

$$=\int_0^{\pi/2}(4\sin^2\theta+12\sin\theta\sin 9\theta+9\sin^2 9\theta)\,d\theta$$

$$=\int_0^{\pi/2}\left[2(1-\cos 2\theta)+12\cdot\frac{1}{2}(\cos(\theta-9\theta)-\cos(\theta+9\theta))+\frac{9}{2}(1-\cos 18\theta)\right]d\theta$$

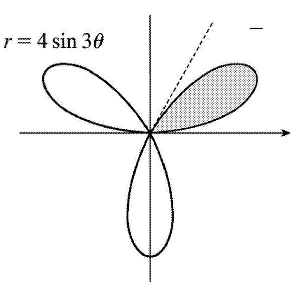
$$=\int_0^{\pi/2}(2-2\cos 2\theta+6\cos 8\theta-6\cos 10\theta+\frac{9}{2}-\frac{9}{2}\cos 18\theta)\,d\theta$$

$$=\left[\frac{13}{2}\theta-\sin 2\theta+\frac{3}{4}\sin 8\theta-\frac{3}{5}\sin 10\theta-\frac{1}{4}\sin 18\theta\right]_0^{\pi/2}=\frac{13}{4}\pi$$

17. The shaded loop is traced out from $\theta=0$ to $\theta=\pi/2$. $A=\int_0^{\pi/2}\frac{1}{2}r^2\,d\theta=\frac{1}{2}\int_0^{\pi/2}\sin^2 2\theta\,d\theta$

$$=\frac{1}{2}\int_0^{\pi/2}\frac{1}{2}(1-\cos 4\theta)\,d\theta=\frac{1}{4}\left[\theta-\frac{1}{4}\sin 4\theta\right]_0^{\pi/2}=\frac{1}{4}\left(\frac{\pi}{2}\right)=\frac{\pi}{8}$$

18. $A=\int_0^{\pi/3}\frac{1}{2}(4\sin 3\theta)^2\,d\theta=8\int_0^{\pi/3}\sin^2 3\theta\,d\theta=4\int_0^{\pi/3}(1-\cos 6\theta)\,d\theta=4\left[\theta-\frac{1}{6}\sin 6\theta\right]_0^{\pi/3}=\frac{4\pi}{3}$

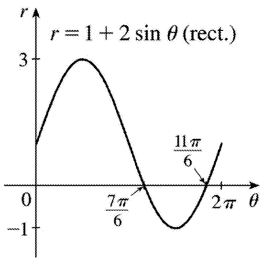


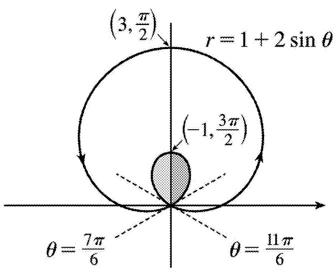
19. $r=0\Rightarrow 3\cos 5\theta=0\Rightarrow 5\theta=\frac{\pi}{2}\Rightarrow\theta=\frac{\pi}{10}$.

$$A=\int_{-\pi/10}^{\pi/10}\frac{1}{2}(3\cos 5\theta)^2\,d\theta=\int_0^{\pi/10}9\cos^2 5\theta\,d\theta=\frac{9}{2}\int_0^{\pi/10}(1+\cos 10\theta)\,d\theta=\frac{9}{2}\left[\theta+\frac{1}{10}\sin 10\theta\right]_0^{\pi/10}=\frac{9\pi}{20}$$

20. $A=2\int_0^{\pi/8}\frac{1}{2}(2\cos 4\theta)^2\,d\theta=2\int_0^{\pi/8}(1+\cos 8\theta)\,d\theta=2\left[\theta+\frac{1}{8}\sin 8\theta\right]_0^{\pi/8}=\frac{\pi}{4}$

21.





This is a limacon, with inner loop traced out between $\theta=\frac{7\pi}{6}$ and $\frac{11\pi}{6}$
.

$$A=2\int_{7\pi/6}^{3\pi/2}\frac{1}{2}(1+2\sin\theta)^2\,d\theta=\int_{7\pi/6}^{3\pi/2}\left(1+4\sin\theta+4\sin^2\theta\right)d\theta$$

$$=\int_{7\pi/6}^{3\pi/2}\left[1+4\sin\theta+4\cdot\frac{1}{2}(1-\cos 2\theta)\right]d\theta=[\theta-4\cos\theta+2\theta-\sin 2\theta]_{7\pi/6}^{3\pi/2}$$

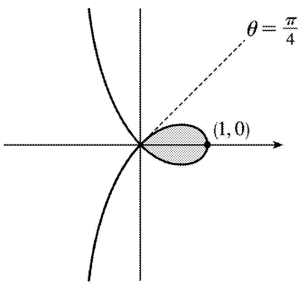
$$=\left(\frac{9\pi}{2}\right)-\left(\frac{7\pi}{2}+2\sqrt{3}-\frac{\sqrt{3}}{2}\right)=\pi-\frac{3\sqrt{3}}{2}$$

22. To determine when the strophoid $r=2\cos\theta-\sec\theta$ passes through the pole, we solve $r=0\Rightarrow 2\cos\theta-\frac{1}{\cos\theta}=0\Rightarrow$
$2\cos^2\theta-1=0\Rightarrow\cos^2\theta=\frac{1}{2}\Rightarrow\cos\theta=\pm\frac{1}{\sqrt{2}}\Rightarrow$
$\theta=\frac{\pi}{4}$ or $\theta=\frac{3\pi}{4}$ for $0\le\theta\le\pi$ with $\theta\ne\frac{\pi}{2}$.

$$A=2\int_0^{\pi/4}\frac{1}{2}(2\cos\theta-\sec\theta)^2\,d\theta=\int_0^{\pi/4}(4\cos^2\theta-4+\sec^2\theta)\,d\theta$$

$$=\int_0^{\pi/4}\left[4\cdot\frac{1}{2}(1+\cos 2\theta)-4+\sec^2\theta\right]d\theta=\int_0^{\pi/4}(-2+2\cos 2\theta+\sec^2\theta)\,d\theta$$

$$= \left[-2\theta+\sin 2\theta+\tan\theta\right]_0^{\pi/4}=\left(-\frac{\pi}{2}+1+1\right)-0=2-\frac{\pi}{2}$$



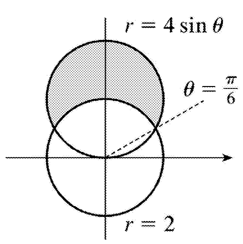
23. $4\sin\theta=2\Leftrightarrow\sin\theta=\frac{1}{2}\Leftrightarrow\theta=\frac{\pi}{6}$ or $\frac{5\pi}{6}$ (for $0\le\theta\le 2\pi$). We'll subtract the unshaded area from the shaded area for $\pi/6\le\theta\le\pi/2$ and double that value.

$$A = 2\int_{\pi/6}^{\pi/2}\frac{1}{2}(4\sin\theta)^2 d\theta-2\int_{\pi/6}^{\pi/2}\frac{1}{2}(2)^2 d\theta=2\int_{\pi/6}^{\pi/2}\frac{1}{2}\left[(4\sin\theta)^2-2^2\right]d\theta$$

$$= \int_{\pi/6}^{\pi/2}\left(16\sin^2\theta-4\right)d\theta=\int_{\pi/6}^{\pi/2}[8(1-\cos 2\theta)-4]d\theta$$

$$= \int_{\pi/6}^{\pi/2}(4-8\cos 2\theta)d\theta=\left[4\theta-4\sin 2\theta\right]_{\pi/6}^{\pi/2}$$

$$= (2\pi-0)-\left(\frac{2\pi}{3}-4\cdot\frac{\sqrt{3}}{2}\right)=\frac{4}{3}\pi+2\sqrt{3}$$

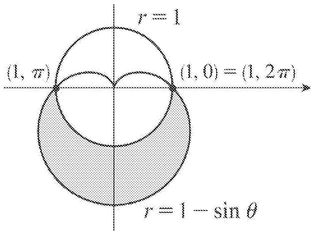


24. $1-\sin\theta=1\Rightarrow\sin\theta=0\Rightarrow\theta=0$ or $\pi\Rightarrow$

$$A = \int_{\pi}^{2\pi}\frac{1}{2}\left[(1-\sin\theta)^2-1\right]d\theta=\frac{1}{2}\int_{\pi}^{2\pi}\left(\sin^2\theta-2\sin\theta\right)d\theta$$

$$= \frac{1}{4}\int_{\pi}^{2\pi}(1-\cos 2\theta-4\sin\theta)d\theta=\frac{1}{4}\left[\theta-\frac{1}{2}\sin 2\theta+4\cos\theta\right]_{\pi}^{2\pi}$$

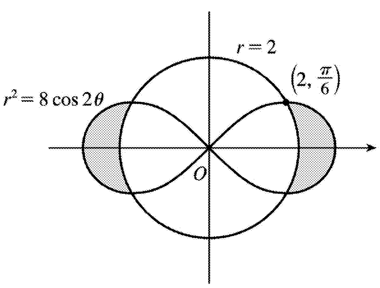
$$= \frac{1}{4}\pi+2$$



25. To find the area inside the leminiscate $r^2=8\cos 2\theta$ and outside the circle $r=2$, we first note that the two curves intersect when $r^2=8\cos 2\theta$ and $r=2$; i.e., when $\cos 2\theta=\frac{1}{2}$. For $-\pi<\theta\leq\pi$, $\cos 2\theta=\frac{1}{2}$ $\Leftrightarrow 2\theta=\pm\pi/3$ or $\pm 5\pi/3 \Leftrightarrow \theta=\pm\pi/6$ or $\pm 5\pi/6$. The figure shows that the desired area is 4 times the area between the curves from 0 to $\pi/6$. Thus,

$$A = 4\int_0^{\pi/6}\left[\frac{1}{2}(8\cos 2\theta)-\frac{1}{2}(2)^2\right]d\theta=8\int_0^{\pi/6}(2\cos 2\theta-1)\,d\theta$$

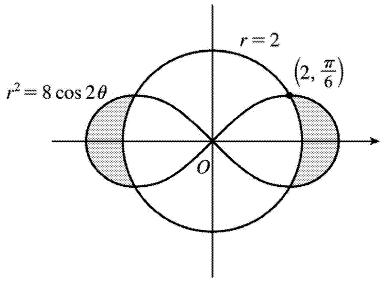
$$= 8\left[\sin 2\theta-\theta\right]_0^{\pi/6}=8\left(\sqrt{3}/2-\pi/6\right)=4\sqrt{3}-4\pi/3$$



26. To find the area inside the leminiscate $r^2=8\cos 2\theta$ and outside the circle $r=2$, we first note that the two curves intersect when $r^2=8\cos 2\theta$ and $r=2$; i.e., when $\cos 2\theta=\frac{1}{2}$. For $-\pi<\theta\leq\pi$, $\cos 2\theta=\frac{1}{2}$ $\Leftrightarrow 2\theta=\pm\pi/3$ or $\pm 5\pi/3 \Leftrightarrow \theta=\pm\pi/6$ or $\pm 5\pi/6$. The figure shows that the desired area is 4 times the area between the curves from 0 to $\pi/6$. Thus,

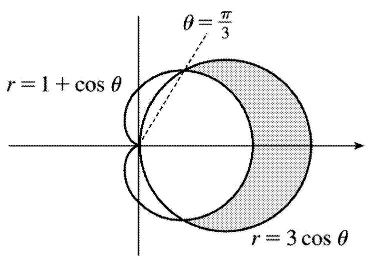
$$A = 4\int_0^{\pi/6}\left[\frac{1}{2}(8\cos 2\theta)-\frac{1}{2}(2)^2\right]d\theta=8\int_0^{\pi/6}(2\cos 2\theta-1)\,d\theta$$

$$= 8\left[\sin 2\theta-\theta\right]_0^{\pi/6}=8\left(\sqrt{3}/2-\pi/6\right)=4\sqrt{3}-4\pi/3$$



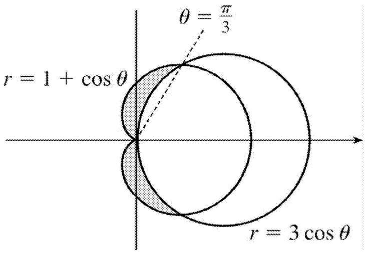
27. $3\cos\theta = 1+\cos\theta \Leftrightarrow \cos\theta = \dfrac{1}{2} \Rightarrow \theta = \dfrac{\pi}{3}$ or $-\dfrac{\pi}{3}$.

$$A = 2\int_0^{\pi/3} \frac{1}{2}\left[(3\cos\theta)^2 - (1+\cos\theta)^2\right] d\theta$$
$$= \int_0^{\pi/3} \left(8\cos^2\theta - 2\cos\theta - 1\right) d\theta$$
$$= \int_0^{\pi/3} [4(1+\cos 2\theta) - 2\cos\theta - 1]\, d\theta$$
$$= \int_0^{\pi/3} (3+4\cos 2\theta - 2\cos\theta)\, d\theta$$
$$= [3\theta + 2\sin 2\theta - 2\sin\theta]_0^{\pi/3}$$
$$= \pi + \sqrt{3} - \sqrt{3} = \pi$$



28. Note that $r = 1+\cos\theta$ goes through the pole when $\theta = \pi$, but $r = 3\cos\theta$ goes through the pole when $\theta = \pi/2$.

$$A = 2\int_{\pi/3}^{\pi} \frac{1}{2}(1+\cos\theta)^2 d\theta - 2\int_{\pi/3}^{\pi/2} \frac{1}{2}(3\cos\theta)^2 d\theta$$
$$= \int_{\pi/3}^{\pi} \left[1 + 2\cos\theta + \frac{1}{2}(1+\cos 2\theta)\right] d\theta - \frac{9}{2}\int_{\pi/3}^{\pi/2} (1+\cos 2\theta)\, d\theta$$
$$= \left[\theta + 2\sin\theta + \frac{1}{2}\left(\theta + \frac{1}{2}\sin 2\theta\right)\right]_{\pi/3}^{\pi} - \frac{9}{2}\left[\theta + \frac{1}{2}\sin 2\theta\right]_{\pi/3}^{\pi/2}$$
$$= \left(\pi - \frac{9}{8}\sqrt{3}\right) - \frac{9}{2}\left(\frac{\pi}{6} - \frac{1}{4}\sqrt{3}\right) = \frac{\pi}{4}$$

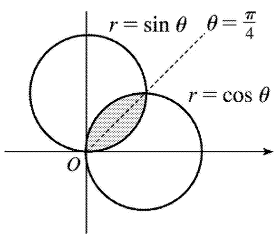


29.

$$A = 2\int_0^{\pi/4} \frac{1}{2}\sin^2\theta\, d\theta = \int_0^{\pi/4} \frac{1}{2}(1 - \cos 2\theta)\, d\theta$$

$$= \frac{1}{2}\left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{\pi/4} = \frac{1}{2}\left[\left(\frac{\pi}{4} - \frac{1}{2}\cdot 1\right) - (0 - 0)\right]$$

$$= \frac{1}{8}\pi - \frac{1}{4}$$

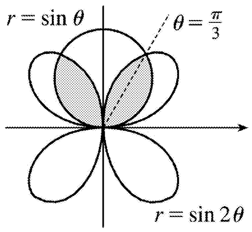


30. $r = \sin 2\theta$ takes on both positive and negative values. $\sin\theta = \square\, 2\theta = \pm 2\sin\theta\cos\theta \Rightarrow$ $\sin\theta\,(1 \pm 2\cos\theta) = 0$. From the figure we can see that the intersections occur where $\cos\theta = \pm\frac{1}{2}$, or $\theta = \frac{\pi}{3}$ and $\frac{2\pi}{3}$.

$$A = 2\left[\int_0^{\pi/3} \frac{1}{2}\sin^2\theta\, d\theta + \int_{\pi/3}^{\pi/2} \frac{1}{2}\sin^2 2\theta\, d\theta\right]$$

$$= \int_0^{\pi/3} \frac{1}{2}(1 - \cos 2\theta)\, d\theta + \int_{\pi/3}^{\pi/2} \frac{1}{2}(1 - \cos 4\theta)\, d\theta$$

$$= \frac{1}{2}\left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{\pi/3} + \frac{1}{2}\left[\theta - \frac{1}{4}\sin 4\theta\right]_{\pi/3}^{\pi/2} = \frac{4\pi - 3\sqrt{3}}{16}$$

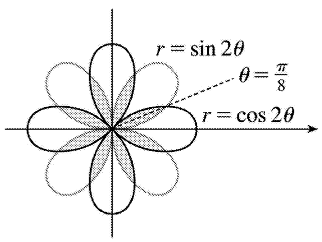


31. $\sin 2\theta = \cos 2\theta \Rightarrow \dfrac{\sin 2\theta}{\cos 2\theta} = 1 \Rightarrow \tan 2\theta = 1 \Rightarrow$

$2\theta = \dfrac{\pi}{4} \Rightarrow \theta = \dfrac{\pi}{8} \Rightarrow$

$$A = 8 \cdot 2\int_0^{\pi/8} \frac{1}{2} \sin^2 2\theta \, d\theta = 8\int_0^{\pi/8} \frac{1}{2} (1 - \cos 4\theta) d\theta$$

$$= 4\left[\theta - \frac{1}{4}\sin 4\theta\right]_0^{\pi/8} = 4\left(\frac{\pi}{8} - \frac{1}{4} \cdot 1\right) = \frac{1}{2}\pi - 1$$



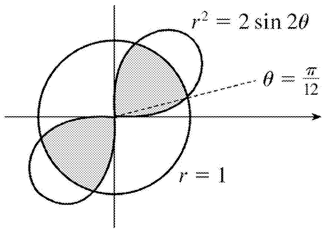
32. $2\sin 2\theta = 1^2 \Rightarrow \sin 2\theta = \dfrac{1}{2} \Rightarrow 2\theta = \dfrac{\pi}{6}$ or $\dfrac{5\pi}{6} \Rightarrow \theta = \dfrac{\pi}{12}$ or $\dfrac{5\pi}{12}$.

$$A = 4\left[\int_0^{\pi/12} \frac{1}{2} \cdot 2\sin 2\theta \, d\theta + \int_{\pi/12}^{\pi/4} \frac{1}{2} \left(1^2\right) d\theta\right]$$

$$= \left[-2\cos 2\theta\right]_0^{\pi/12} + \left[2\theta\right]_{\pi/12}^{\pi/4}$$

$$= -2\left(\frac{\sqrt{3}}{2} - 1\right) + 2\left(\frac{1}{4}\pi - \frac{1}{12}\pi\right)$$

$$= 2 - \sqrt{3} + \frac{\pi}{3}$$

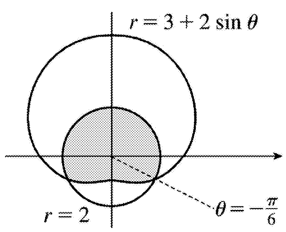


33.

$$A = 2\left[\int_{-\pi/2}^{-\pi/6} \frac{1}{2}(3+2\sin\theta)^2 d\theta + \int_{-\pi/6}^{\pi/2} \frac{1}{2}2^2 d\theta\right]$$

$$= \int_{-\pi/2}^{-\pi/6} \left(9 + 12\sin\theta + 4\sin^2\theta\right) d\theta + [4\theta]_{-\pi/6}^{\pi/2}$$

$$= [9\theta - 12\cos\theta + 2\theta - \sin 2\theta]_{-\pi/2}^{-\pi/6} + \frac{8\pi}{3} = \frac{19\pi}{3} - \frac{11\sqrt{3}}{2}$$



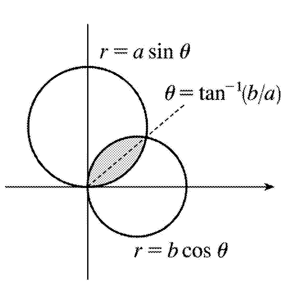
34. Let $\alpha = \tan^{-1}(b/a)$. Then

$$A = \int_0^{\alpha} \frac{1}{2}(a\sin\theta)^2 d\theta + \int_{\alpha}^{\pi/2} \frac{1}{2}(b\cos\theta)^2 d\theta$$

$$= \frac{1}{4}a^2\left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{\alpha} + \frac{1}{4}b^2\left[\theta + \frac{1}{2}\sin 2\theta\right]_{\alpha}^{\pi/2}$$

$$= \frac{1}{4}\alpha\left(a^2 - b^2\right) + \frac{1}{8}\pi b^2 - \frac{1}{4}\left(a^2 + b^2\right)(\sin\alpha\cos\alpha)$$

$$= \frac{1}{4}\left(a^2 - b^2\right)\tan^{-1}(b/a) + \frac{1}{8}\pi b^2 - \frac{1}{4}ab$$



35. The darker shaded region (from $\theta = 0$ to $\theta = 2\pi/3$) represents $\frac{1}{2}$ of the desired area plus $\frac{1}{2}$ of the area of the inner loop. From this area, we'll subtract $\frac{1}{2}$ of the area of the inner loop (the lighter shaded region from $\theta = 2\pi/3$
to $\theta = \pi$), and then double that difference to obtain the desired area.

$$A = 2\left[\int_0^{2\pi/3} \frac{1}{2} \left(\frac{1}{2} + \cos\theta \right)^2 d\theta - \int_{2\pi/3}^{\pi} \frac{1}{2} \left(\frac{1}{2} + \cos\theta \right)^2 d\theta \right]$$

$$= \int_0^{2\pi/3} \left(\frac{1}{4} + \cos\theta + \cos^2\theta \right) d\theta - \int_{2\pi/3}^{\pi} \left(\frac{1}{4} + \cos\theta + \cos^2\theta \right) d\theta$$

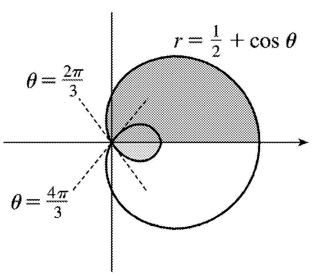
$$= \int_0^{2\pi/3} \left[\frac{1}{4} + \cos\theta + \frac{1}{2}(1 + \cos 2\theta) \right] d\theta$$

$$- \int_{2\pi/3}^{\pi} \left[\frac{1}{4} + \cos\theta + \frac{1}{2}(1 + \cos 2\theta) \right] d\theta$$

$$= \left[\frac{\theta}{4} + \sin\theta + \frac{\theta}{2} + \frac{\sin 2\theta}{4} \right]_0^{2\pi/3} - \left[\frac{\theta}{4} + \sin\theta + \frac{\theta}{2} + \frac{\sin 2\theta}{4} \right]_{2\pi/3}^{\pi}$$

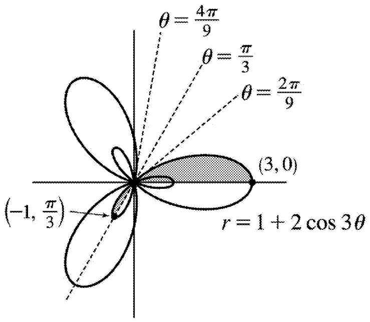
$$= \left(\frac{\pi}{6} + \frac{\sqrt{3}}{2} + \frac{\pi}{3} - \frac{\sqrt{3}}{8} \right) - \left(\frac{\pi}{4} + \frac{\pi}{2} \right) + \left(\frac{\pi}{6} + \frac{\sqrt{3}}{2} + \frac{\pi}{3} - \frac{\sqrt{3}}{8} \right)$$

$$= \frac{\pi}{4} + \frac{3}{4}\sqrt{3} = \frac{1}{4}\left(\pi + 3\sqrt{3} \right)$$



36. $r = 0 \Rightarrow 1 + 2\cos 3\theta = 0 \Rightarrow \cos 3\theta = -\frac{1}{2} \Rightarrow 3\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$ (for $0 \le 3\theta \le 2\pi$) $\Rightarrow \theta = \frac{2\pi}{9}, \frac{4\pi}{9}$. The darker shaded region (from $\theta = 0$ to $\theta = 2\pi/9$) represents $\frac{1}{2}$ of the desired area plus $\frac{1}{2}$ of the area of the inner loop. From this area, we'll subtract $\frac{1}{2}$ of the area of the inner loop (the lighter shaded region from $\theta = 2\pi/9$ to $\theta = \pi/3$), and then double that difference to obtain the desired area.

$$A = 2\left[\int_0^{2\pi/9} \frac{1}{2} (1 + 2\cos 3\theta)^2 d\theta - \int_{2\pi/9}^{\pi/3} \frac{1}{2} (1 + 2\cos 3\theta)^2 d\theta \right]$$



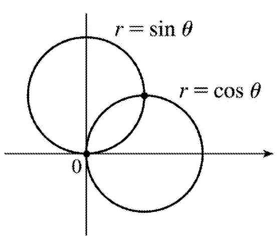
Now

$$r^2 = (1+2\cos 3\theta)^2 = 1+4\cos 3\theta + 4\cos^2 3\theta = 1+4\cos 3\theta + 4\cdot\frac{1}{2}(1+\cos 6\theta)$$
$$= 1+4\cos 3\theta + 2 + 2\cos 6\theta = 3+4\cos 3\theta + 2\cos 6\theta$$

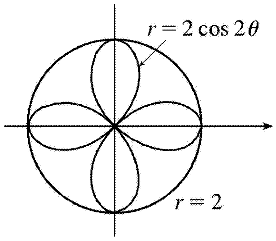
and $\int r^2 d\theta = 3\theta + \frac{4}{3}\sin 3\theta + \frac{1}{3}\sin 6\theta + C$, so

$$A = \left[3\theta + \frac{4}{3}\sin 3\theta + \frac{1}{3}\sin 6\theta\right]_0^{2\pi/9} - \left[3\theta + \frac{4}{3}\sin 3\theta + \frac{1}{3}\sin 6\theta\right]_{2\pi/9}^{\pi/3}$$
$$= \left[\left(\frac{2\pi}{3} + \frac{4}{3}\cdot\frac{\sqrt{3}}{2} + \frac{1}{3}\cdot\frac{-\sqrt{3}}{2}\right) - 0\right] - \left[(\pi+0+0) - \left(\frac{2\pi}{3} + \frac{4}{3}\cdot\frac{\sqrt{3}}{2} + \frac{1}{3}\cdot\frac{-\sqrt{3}}{2}\right)\right]$$
$$= \frac{4\pi}{3} + \frac{4}{3}\sqrt{3} - \frac{1}{3}\sqrt{3} - \pi = \frac{\pi}{3} + \sqrt{3}$$

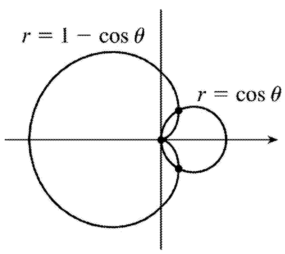
37. The two circles intersect at the pole since $(0,0)$ satisfies the first equation and $\left(0, \frac{\pi}{2}\right)$ the second. The other intersection point $\left(\frac{1}{\sqrt{2}}, \frac{\pi}{4}\right)$ occurs where $\sin\theta = \cos\theta$.



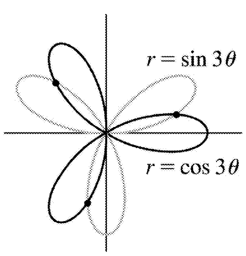
38. $2\cos 2\theta = \pm 2 \Rightarrow \cos 2\theta = \pm 1 \Rightarrow \theta = 0$, $\frac{\pi}{2}$, π , or $\frac{3\pi}{2}$, so the points are $(2,0)$, $\left(2, \frac{\pi}{2}\right)$, $(2,\pi)$, and $\left(2, \frac{3\pi}{2}\right)$.



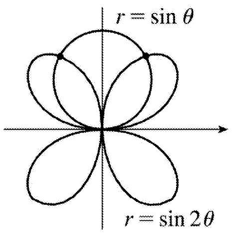
39. The curves intersect at the pole since $\left(0, \frac{\pi}{2}\right)$ satisfies $r=\cos\theta$ and $(0,0)$ satisfies $r=1-\cos\theta$. Now $\cos\theta=1-\cos\theta \Rightarrow 2\cos\theta=1 \Rightarrow \cos\theta=\frac{1}{2} \Rightarrow \theta=\frac{\pi}{3}$ or $\frac{5\pi}{3} \Rightarrow$ the other intersection points are $\left(\frac{1}{2}, \frac{\pi}{3}\right)$ and $\left(\frac{1}{2}, \frac{5\pi}{3}\right)$.



40. Clearly the pole lies on both curves. $\sin 3\theta=\cos 3\theta \Rightarrow \tan 3\theta=1 \Rightarrow 3\theta=\frac{\pi}{4}+n\pi$ (n any integer) $\Rightarrow$ $\theta=\frac{\pi}{12}+\frac{\pi}{3}n \Rightarrow \theta=\frac{\pi}{12}$, $\frac{5\pi}{12}$, or $\frac{3\pi}{4}$, so the three remaining intersection points are $\left(\frac{1}{\sqrt{2}}, \frac{\pi}{12}\right)$, $\left(-\frac{1}{\sqrt{2}}, \frac{5\pi}{12}\right)$, and $\left(\frac{1}{\sqrt{2}}, \frac{3\pi}{4}\right)$.

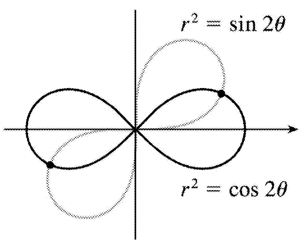


41. The pole is a point of intersection. $\sin\theta=\sin 2\theta=2\sin\theta\cos\theta \Leftrightarrow \sin\theta(1-2\cos\theta)=0 \Leftrightarrow \sin\theta=0$ or $\cos\theta=\frac{1}{2} \Rightarrow \theta=0, \pi, \frac{\pi}{3}, -\frac{\pi}{3} \Rightarrow \left(\frac{\sqrt{3}}{2}, \frac{\pi}{3}\right)$ and $\left(\frac{\sqrt{3}}{2}, \frac{2\pi}{3}\right)$ (by symmetry) are the other intersection points.

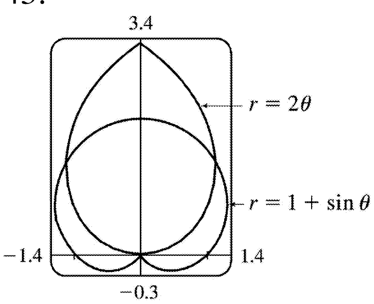


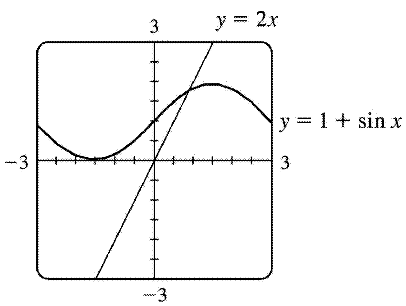
42. Clearly the pole is a point of intersection. $\sin 2\theta = \cos 2\theta \Rightarrow \tan 2\theta = 1 \Rightarrow 2\theta = \frac{\pi}{4} + 2n\pi$ (since $\sin 2\theta$ and $\cos 2\theta$ must be positive in the equations) $\Rightarrow \theta = \frac{\pi}{8} + n\pi \Rightarrow \theta = \frac{\pi}{8}$ or $\frac{9\pi}{8}$. So the curves also intersect at

$\left(\frac{1}{\sqrt[4]{2}}, \frac{\pi}{8}\right)$ and $\left(\frac{1}{\sqrt[4]{2}}, \frac{9\pi}{8}\right)$.



43.





From the first graph, we see that the pole is one point of intersection. By zooming in or using the cursor, we find the $\theta-$ values of the intersection points to be $\alpha \approx 0.88786 \approx 0.89$ and $\pi - \alpha \approx 2.25$. (The first of these values may be more easily estimated by plotting $y=1+\sin x$ and $y=2x$ in rectangular coordinates; see the second graph.)

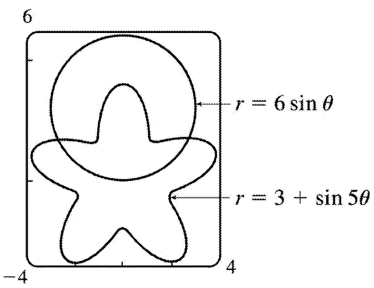
By symmetry, the total area contained is twice the area contained in the first quadrant, that is,

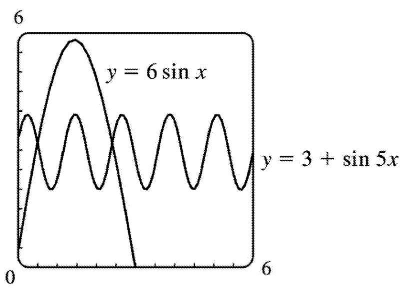
$$A = 2\int_0^{\alpha} \frac{1}{2}(2\theta)^2 d\theta + 2\int_{\alpha}^{\pi/2} \frac{1}{2}(1+\sin\theta)^2 d\theta = \int_0^{\alpha} 4\theta^2 d\theta + \int_{\alpha}^{\pi/2}\left[1+2\sin\theta+\frac{1}{2}(1-\cos 2\theta)\right]d\theta$$

$$= \left[\frac{4}{3}\theta^3\right]_0^{\alpha} + \left[\theta - 2\cos\theta + \left(\frac{1}{2}\theta - \frac{1}{4}\sin 2\theta\right)\right]_{\alpha}^{\pi/2}$$

$$= \frac{4}{3}\alpha^3 + \left[\left(\frac{\pi}{2}+\frac{\pi}{4}\right) - \left(\alpha - 2\cos\alpha + \frac{1}{2}\alpha - \frac{1}{4}\sin 2\alpha\right)\right] \approx 3.4645$$

44.





From the first graph, it appears that the $\theta-$ values of the points of intersection are $\alpha \approx 0.57504 \approx 0.58$ and $\pi - \alpha \approx 2.57$. (These values may be more easily estimated by plotting $y=3+\sin 5x$ and $y=6\sin x$ in rectangular coordinates; see the second graph.) By symmetry, the total area enclosed in both curves is

$$A = 2\int_0^{\alpha} \frac{1}{2}(6\sin\theta)^2 d\theta + 2\int_{\alpha}^{\pi/2} \frac{1}{2}(3+\sin 5\theta)^2 d\theta = \int_0^{\alpha} 36\sin^2\theta\, d\theta + \int_{\alpha}^{\pi/2}\left(9+6\sin 5\theta + \sin^2 5\theta\right)d\theta$$

$$= \int_0^{\alpha} 36\cdot\frac{1}{2}(1-\cos 2\theta)d\theta + \int_{\alpha}^{\pi/2}\left[9+6\sin 5\theta + \frac{1}{2}(1-\cos 10\theta)\right]d\theta$$

$$= \left[36\left(\frac{1}{2}\theta - \frac{1}{4}\sin 2\theta\right)\right]_0^{\alpha} + \left[9\theta - \frac{6}{5}\cos 5\theta + \left(\frac{1}{2}\theta - \frac{1}{20}\sin 10\theta\right)\right]_{\alpha}^{\pi/2} \approx 10.41$$

45.

$$L = \int_a^b \sqrt{r^2 + (dr/d\theta)^2}\, d\theta = \int_0^{\pi/3}\sqrt{(3\sin\theta)^2 + (3\cos\theta)^2}\, d\theta = \int_0^{\pi/3}\sqrt{9(\sin^2\theta + \cos^2\theta)}\, d\theta$$

$$= 3\int_0^{\pi/3} d\theta = 3[\theta]_0^{\pi/3} = 3\left(\frac{\pi}{3}\right) = \pi\ .$$

As a check, note that the circumference of a circle with radius $\frac{3}{2}$ is $2\pi\left(\frac{3}{2}\right)=3\pi$, and since $\theta=0$ to $\pi=\frac{\pi}{3}$ traces out $\frac{1}{3}$ of the circle (from $\theta=0$ to $\theta=\pi$), $\frac{1}{3}(3\pi)=\pi$.

46.

$$L=\int_a^b\sqrt{r^2+(dr/d\theta)^2}\,d\theta=\int_0^{2\pi}\sqrt{\left(e^{2\theta}\right)^2+\left(2e^{2\theta}\right)^2}\,d\theta=\int_0^{2\pi}\sqrt{e^{4\theta}+4e^{4\theta}}\,d\theta=\int_0^{2\pi}\sqrt{5e^{4\theta}}\,d\theta$$
$$=\sqrt{5}\int_0^{2\pi}e^{2\theta}\,d\theta=\frac{\sqrt{5}}{2}\left[e^{2\theta}\right]_0^{2\pi}=\frac{\sqrt{5}}{2}\left(e^{4\pi}-1\right)$$

47.

$$L=\int_a^b\sqrt{r^2+(dr/d\theta)^2}\,d\theta=\int_0^{2\pi}\sqrt{\left(\theta^2\right)^2+(2\theta)^2}\,d\theta=\int_0^{2\pi}\sqrt{\theta^4+4\theta^2}\,d\theta$$
$$=\int_0^{2\pi}\sqrt{\theta^2\left(\theta^2+4\right)}\,d\theta=\int_0^{2\pi}\theta\sqrt{\theta^2+4}\,d\theta$$

Now let $u=\theta^2+4$, so that $du=2\theta\,d\theta$ $\left[\theta\,d\theta=\frac{1}{2}\,du\right]$ and

$$\int_0^{2\pi}\theta\sqrt{\theta^2+4}\,\mathrm{d}\theta=\int_4^{4\pi^2+4}\frac{1}{2}\sqrt{u}\,\mathrm{du}=\frac{1}{2}\cdot\frac{2}{3}\left[u^{3/2}\right]_4^{4\left(\pi^2+1\right)}=\frac{1}{3}\left[4^{3/2}\left(\pi^2+1\right)^{3/2}-4^{3/2}\right]$$
$$=\frac{8}{3}\left[\left(\pi^2+1\right)^{3/2}-1\right]$$

48.

$$L=\int_a^b\sqrt{r^2+(dr/d\theta)^2}\,d\theta=\int_0^{2\pi}\sqrt{\theta^2+1}\,d\theta=\left[\frac{\theta}{2}\sqrt{\theta^2+1}+\frac{1}{2}\ln\left(\theta+\sqrt{\theta^2+1}\right)\right]_0^{2\pi}$$
$$=\pi\sqrt{4\pi^2+1}+\frac{1}{2}\ln\left(2\pi+\sqrt{4\pi^2+1}\right)$$

49. $L=\int_a^b\sqrt{r^2+(dr/d\theta)^2}\,d\theta=\int_0^{2\pi}\sqrt{\left(\theta^2\right)^2+(2\theta)^2}\,d\theta=\int_0^{2\pi}\sqrt{\theta^4+4\theta^2}\,d\theta$
$=\int_0^{2\pi}\sqrt{\theta^2\left(\theta^2+4\right)}\,d\theta=\int_0^{2\pi}\theta\sqrt{\theta^2+4}\,d\theta$

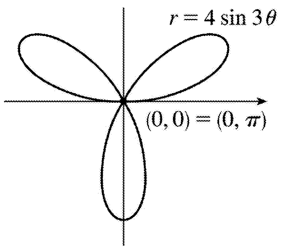
Now let $u=\theta^2+4$, so that $du=2\theta\,d\theta$ $\left[\theta\,d\theta=\frac{1}{2}\,du\right]$ and

$$\int_0^{2\pi} \theta\sqrt{\theta^2+4}\, d\theta = \int_4^{4\pi^2+4} \frac{1}{2}\sqrt{u}\, du = \frac{1}{2}\cdot\frac{2}{3}\left[u^{3/2}\right]_4^{4(\pi^2+1)} = \frac{1}{3}\left[4^{3/2}\left(\pi^2+1\right)^{3/2} - 4^{3/2}\right] = \frac{8}{3}\left[\left(\pi^2+1\right)^{3/2} - 1\right]$$

50. The curve $r=4\sin 3\theta$ is completely traced with $0\le\theta\le\pi$.

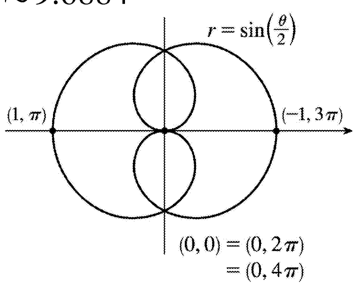
$$r^2+\left(\frac{dr}{d\theta}\right)^2=(4\sin 3\theta)^2+(12\cos 3\theta)^2 \Rightarrow$$

$$L=\int_0^{\pi}\sqrt{16\sin^2 3\theta+144\cos^2 3\theta}\, d\theta\approx 26.7298$$



51. The curve $r=\sin\left(\frac{\theta}{2}\right)$ is completely traced with $0\le\theta\le 4\pi$.

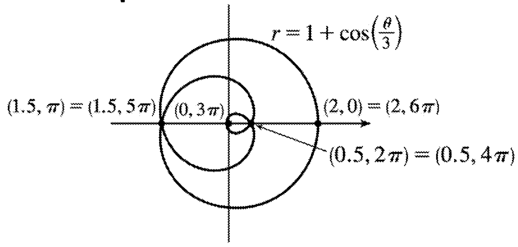
$$r^2+\left(\frac{dr}{d\theta}\right)^2=\sin^2\left(\frac{\theta}{2}\right)+\left[\frac{1}{2}\cos\left(\frac{\theta}{2}\right)\right]^2 \Rightarrow L=\int_0^{4\pi}\sqrt{\sin^2\left(\frac{\theta}{2}\right)+\frac{1}{4}\cos^2\left(\frac{\theta}{2}\right)}\, d\theta$$
$$\approx 9.6884$$



52. The curve $r=1+\cos\left(\frac{\theta}{3}\right)$ is completely traced with $0\le\theta\le 6\pi$.

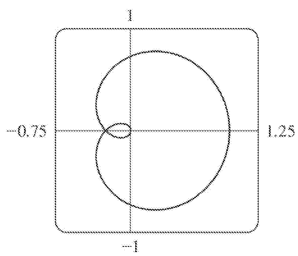
$$r^2+\left(\frac{dr}{d\theta}\right)^2=\left[1+\cos\left(\frac{\theta}{3}\right)\right]^2+\left[-\frac{1}{3}\sin\left(\frac{\theta}{3}\right)\right]^2 \Rightarrow$$

$$L=\int_0^{6\pi}\sqrt{\left[1+\cos\left(\frac{\theta}{3}\right)\right]^2+\frac{1}{9}\sin^2\left(\frac{\theta}{3}\right)}\, d\theta\approx 19.6676$$



53. The curve $r=\cos^4(\theta/4)$ is completely traced with $0\le\theta\le 4\pi$.

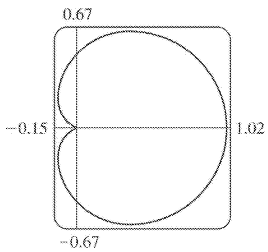
$$r^2+(dr/d\theta)^2=[\cos^4(\theta/4)]^2+\left[4\cos^3(\theta/4)\cdot(-\sin(\theta/4))\cdot\frac{1}{4}\right]^2$$
$$=\cos^8(\theta/4)+\cos^6(\theta/4)\sin^2(\theta/4)$$
$$=\cos^6(\theta/4)[\cos^2(\theta/4)+\sin^2(\theta/4)]$$
$$=\cos^6(\theta/4)$$



$$L=\int_0^{4\pi}\sqrt{\cos^6(\theta/4)}\,d\theta=\int_0^{4\pi}\left|\cos^3(\theta/4)\right|d\theta$$
$$=2\int_0^{2\pi}\cos^3(\theta/4)d\theta\ \text{[since }\cos^3(\theta/4)\ge 0\text{ for }0\le\theta\le 2\pi\text{]}=8\int_0^{\pi/2}\cos^3 u\,du\left[u=\frac{1}{4}\theta\right]$$
$$=8\left[\frac{1}{3}(2+\cos^2 u)\sin u\right]_0^{\pi/2}=\frac{8}{3}[(2\cdot 1)-(3\cdot 0)]=\frac{16}{3}$$

54. The curve $r=\cos^2(\theta/2)$ is completely traced with $0\le\theta\le 2\pi$.

$$r^2+(dr/d\theta)^2=[\cos^2(\theta/2)]^2+\left[2\cos(\theta/2)\cdot(-\sin(\theta/2))\cdot\frac{1}{2}\right]^2$$
$$=\cos^4(\theta/2)+\cos^2(\theta/2)\sin^2(\theta/2)$$
$$=\cos^2(\theta/2)[\cos^2(\theta/2)+\sin^2(\theta/2)]$$
$$=\cos^2(\theta/2)$$



$$L=\int_0^{2\pi}\sqrt{\cos^2(\theta/2)}\,d\theta=\int_0^{2\pi}|\cos(\theta/2)|\,d\theta=2\int_0^{\pi}\cos(\theta/2)\,d\theta\ \left[\text{ since }\cos(\theta/2)\geq 0\text{ for }0\leq\theta\leq\pi\ \right]$$

$$=4\int_0^{\pi/2}\cos u\,du\left[u=\frac{1}{2}\theta\right]=4\left[\sin u\right]_0^{\pi/2}=4(1-0)=4$$

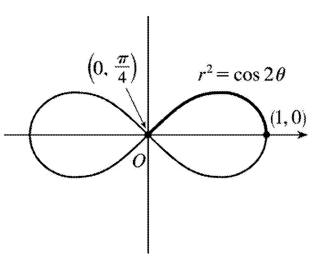
55. **(a)** From (.2.7),

$$S=\int_a^b 2\pi y\sqrt{(dx/d\theta)^2+(dy/d\theta)^2}\,d\theta$$

$$=\int_a^b 2\pi y\sqrt{r^2+(dr/d\theta)^2}\,d\theta\ \text{[fromthederivationofEquation.4.5}$$

$$=\int_a^b 2\pi r\sin\theta\sqrt{r^2+(dr/d\theta)^2}\,d\theta$$

(b) The curve $r^2=\cos 2\theta$ goes through the pole when $\cos 2\theta=0\Rightarrow 2\theta=\frac{\pi}{2}\Rightarrow\theta=\frac{\pi}{4}$. We'll rotate the curve from $\theta=0$ to $\theta=\frac{\pi}{4}$ and double this value to obtain the total surface area generated. $r^2=\cos 2\theta$ $\Rightarrow 2r\frac{dr}{d\theta}=-2\sin 2\theta\Rightarrow\left(\frac{dr}{d\theta}\right)^2=\frac{\sin^2 2\theta}{r^2}=\frac{\sin^2 2\theta}{\cos 2\theta}$.



$$S=2\int_0^{\pi/4}2\pi\sqrt{\cos 2\theta}\sin\theta\sqrt{\cos 2\theta+\left(\sin^2 2\theta\right)/\cos 2\theta}\,d\theta$$

$$=4\pi\int_0^{\pi/4}\sqrt{\cos 2\theta}\sin\theta\sqrt{\frac{\cos^2 2\theta+\sin^2 2\theta}{\cos 2\theta}}\,d\theta=4\pi\int_0^{\pi/4}\sqrt{\cos 2\theta}\sin\theta\,\frac{1}{\sqrt{\cos 2\theta}}\,d\theta$$

$$= 4\pi\int_0^{\pi/4}\sin\theta\, d\theta = 4\pi\left[-\cos\theta\right]_0^{\pi/4} = -4\pi\left(\frac{\sqrt{2}}{2} - 1\right) = 2\pi\left(2 - \sqrt{2}\right)$$

56. **(a)** Rotation around $\theta = \frac{\pi}{2}$ is the same as rotation around the $y-$ axis, that is, $S = \int_a^b 2\pi x\, ds$ where $ds = \sqrt{(dx/dt)^2 + (dy/dt)^2}\, dt$ for a parametric equation, and for the special case of a polar equation, $x = r\cos\theta$ and $ds = \sqrt{(dx/d\theta)^2 + (dy/d\theta)^2}\, d\theta = \sqrt{r^2 + (dr/d\theta)^2}\, d\theta$. Therefore, for a polar equation rotated around $\theta = \frac{\pi}{2}$, $S = \int_a^b 2\pi r\cos\theta \sqrt{r^2 + (dr/d\theta)^2}\, d\theta$.

(b) As in the solution for Exercise 55(b), we can double the surface area generated by rotating the curve from $\theta = 0$ to $\theta = \frac{\pi}{4}$ to obtain the total surface area.

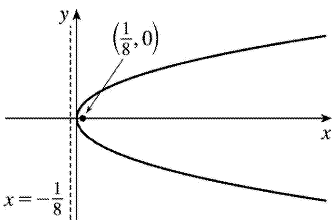
$$S = 2\int_0^{\pi/4} 2\pi\sqrt{\cos 2\theta}\cos\theta\sqrt{\cos 2\theta + (\sin^2 2\theta)/\cos 2\theta}\, d\theta$$

$$= 4\pi\int_0^{\pi/4}\sqrt{\cos 2\theta}\cos\theta\sqrt{\frac{\cos^2 2\theta + \sin^2 2\theta}{\cos 2\theta}}\, d\theta$$

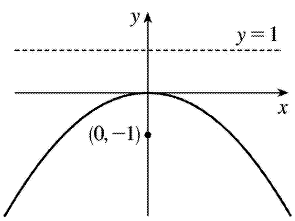
$$= 4\pi\int_0^{\pi/4}\sqrt{\cos 2\theta}\cos\theta\,\frac{1}{\sqrt{\cos 2\theta}}\, d\theta$$

$$= 4\pi\int_0^{\pi/4}\cos\theta\, d\theta = 4\pi\left[\sin\theta\right]_0^{\pi/4} = 4\pi\left(\frac{\sqrt{2}}{2} - 0\right) = 2\sqrt{2}\,\pi$$

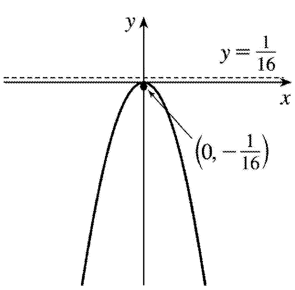
1. $x=2y^2 \Rightarrow y^2=\frac{1}{2}x$. $4p=\frac{1}{2}$, so $p=\frac{1}{8}$. The vertex is $(0,0)$, the focus is $\left(\frac{1}{8},0\right)$, and the directrix is $x=-\frac{1}{8}$.



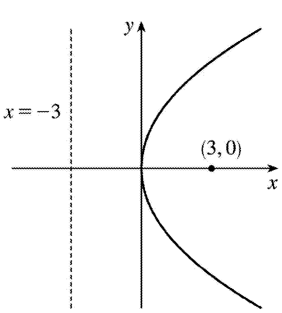
2. $4y+x^2=0 \Rightarrow x^2=-4y$. $4p=-4$, so $p=-1$. The vertex is $(0,0)$, the focus is $(0,-1)$, and the directrix is $y=1$.



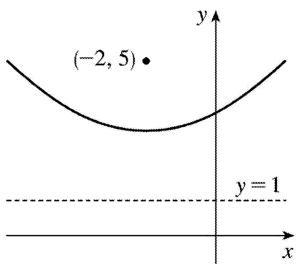
3. $4x^2=-y \Rightarrow x^2=-\frac{1}{4}y$. $4p=-\frac{1}{4}$, so $p=-\frac{1}{16}$. The vertex is $(0,0)$, the focus is $\left(0,-\frac{1}{16}\right)$, and the directrix is $y=\frac{1}{16}$.



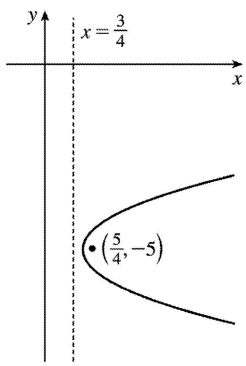
4. $y^2=12x$. $4p=12$, so $p=3$. The vertex is $(0,0)$, the focus is $(3,0)$, and the directrix is $x=-3$.



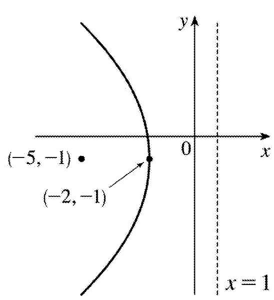
5. $(x+2)^2=8(y-3)$. $4p=8$, so $p=2$. The vertex is $(-2,3)$, the focus is $(-2,5)$, and the directrix is $y=1$.



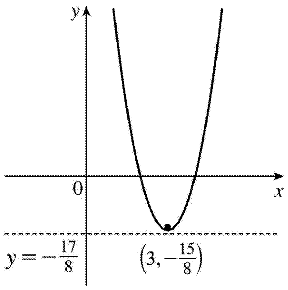
6. $x-1=(y+5)^2$. $4p=1$, so $p=\frac{1}{4}$. The vertex is $(1,-5)$, the focus is $\left(\frac{5}{4},-5\right)$, and the directrix is $x=\frac{3}{4}$.



7. $y^2+2y+12x+25=0 \Rightarrow y^2+2y+1=-12x-24 \Rightarrow (y+1)^2=-12(x+2)$. $4p=-12$, so $p=-3$. The vertex is $(-2,-1)$, the focus is $(-5,-1)$, and the directrix is $x=1$.



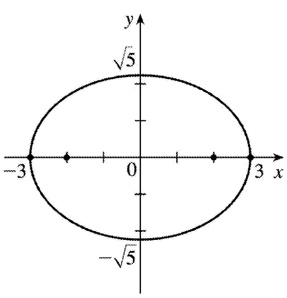
8. $y+12x-2x^2=16 \Rightarrow 2x^2-12x=y-16 \Rightarrow 2(x^2-6x+9)=y-16+18 \Rightarrow 2(x-3)^2=y+2 \Rightarrow (x-3)^2=\frac{1}{2}(y+2)$. $4p=\frac{1}{2}$, so $p=\frac{1}{8}$. The vertex is $(3,-2)$, the focus is $\left(3,-\frac{15}{8}\right)$, and the directrix is $y=-\frac{17}{8}$.



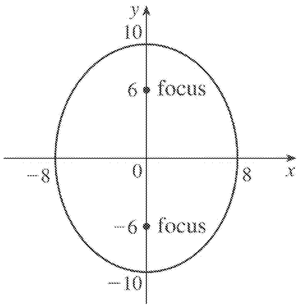
9. The equation has the form $y^2=4px$, where $p<0$. Since the parabola passes through $(-1,1)$, we have $1^2=4p(-1)$, so $4p=-1$ and an equation is $y^2=-x$ or $x=-y^2$. $4p=-1$, so $p=-\frac{1}{4}$ and the focus is $\left(-\frac{1}{4},0\right)$ while the directrix is $x=\frac{1}{4}$.

10. The vertex is $(2,-2)$, so the equation is of the form $(x-2)^2=4p(y+2)$, where $p>0$. The point $(0,0)$ is on the parabola, so $4=4p(2)$ and $4p=2$. Thus, an equation is $(x-2)^2=2(y+2)$. $4p=2$, so $p=\frac{1}{2}$ and the focus is $\left(2,-\frac{3}{2}\right)$ while the directrix is $y=-\frac{5}{2}$.

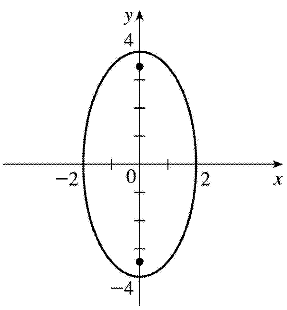
11. $\frac{x^2}{9}+\frac{y^2}{5}=1 \Rightarrow a=\sqrt{9}=3$, $b=\sqrt{5}$, $c=\sqrt{a^2-b^2}=\sqrt{9-5}=2$. The ellipse is centered at $(0,0)$, with vertices at $(\pm 3,0)$. The foci are $(\pm 2,0)$.



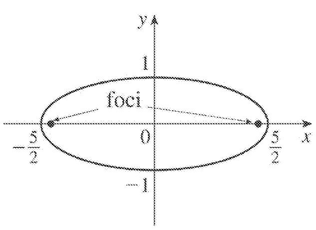
12. $\frac{x^2}{64}+\frac{y^2}{100}=1 \Rightarrow a=\sqrt{100}=10$, $b=\sqrt{64}=8$, $c=\sqrt{a^2-b^2}=\sqrt{100-64}=6$. The ellipse is centered at $(0,0)$, with vertices at $(0,\pm 10)$. The foci are $(0,\pm 6)$.



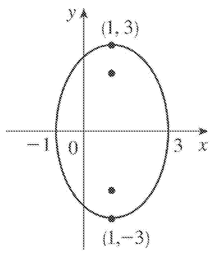
13. $4x^2+y^2=16 \Rightarrow \frac{x^2}{4}+\frac{y^2}{16}=1 \Rightarrow a=\sqrt{16}=4$, $b=\sqrt{4}=2$, $c=\sqrt{a^2-b^2}=\sqrt{16-4}=2\sqrt{3}$. The ellipse is centered at $(0,0)$, with vertices at $(0,\pm 4)$. The foci are $\left(0,\pm 2\sqrt{3}\right)$.



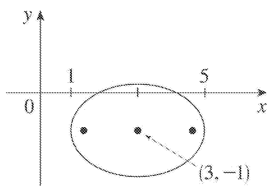
14. $4x^2+25y^2=25 \Rightarrow \frac{x^2}{25/4}+\frac{y^2}{1}=1 \Rightarrow a=\sqrt{\frac{25}{4}}=\frac{5}{2}$, $b=\sqrt{1}=1$, $c=\sqrt{a^2-b^2}=\sqrt{\frac{25}{4}-1}=\sqrt{\frac{21}{4}}=\frac{\sqrt{21}}{2}$. The ellipse is centered at $(0,0)$, with vertices at $\left(\pm\frac{5}{2},0\right)$. The foci are $\left(\pm\frac{\sqrt{21}}{2},0\right)$.



15. $9x^2-18x+4y^2=27 \Leftrightarrow 9(x^2-2x+1)+4y^2=27+9 \Leftrightarrow 9(x-1)^2+4y^2=36 \Leftrightarrow \frac{(x-1)^2}{4}+\frac{y^2}{9}=1 \Rightarrow a=3$, $b=2$, $c=\sqrt{5} \Rightarrow$ center $(1,0)$, vertices $(1,\pm 3)$, foci $\left(1,\pm\sqrt{5}\right)$



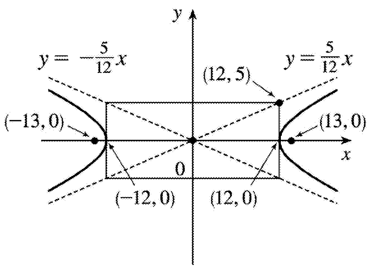
16. $x^2-6x+2y^2+4y=-7 \Leftrightarrow$
$x^2-6x+9+2(y^2+2y+1)=-7+9+2 \Leftrightarrow (x-3)^2+2(y+1)^2=4 \Leftrightarrow$
$\frac{(x-3)^2}{4}+\frac{(y+1)^2}{2}=1 \Rightarrow a=2$, $b=\sqrt{2}=c \Rightarrow$ center $(3,-1)$, vertices $(1,-1)$ and $(5,-1)$, foci $\left(3\pm\sqrt{2},-1\right)$



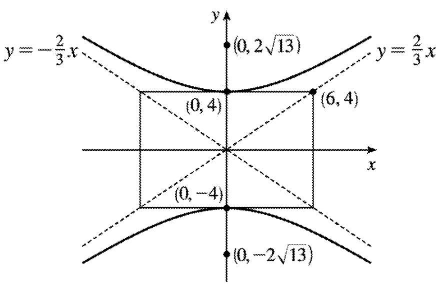
17. The center is $(0,0)$, $a=3$, and $b=2$, so an equation is $\frac{x^2}{4}+\frac{y^2}{9}=1$. $c=\sqrt{a^2-b^2}=\sqrt{5}$, so the foci are $\left(0,\pm\sqrt{5}\right)$.

18. The ellipse is centered at $(2,1)$, with $a=3$ and $b=2$. An equation is $\frac{(x-2)^2}{9}+\frac{(y-1)^2}{4}=1$.
$c=\sqrt{a^2-b^2}=\sqrt{5}$, so the foci are $\left(2\pm\sqrt{5},1\right)$.

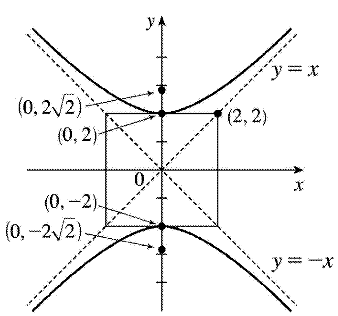
19. $\frac{x^2}{144}-\frac{y^2}{25}=1 \Rightarrow a=12$, $b=5$, $c=\sqrt{144+25}=13 \Rightarrow$ center $(0,0)$, vertices $(\pm 12,0)$, foci $(\pm 13,0)$, asymptotes $y=\pm\frac{5}{12}x$.
Note: It is helpful to draw a $2a$ –by– $2b$ rectangle whose center is the center of the hyperbola. The asymptotes are the extended diagonals of the rectangle.



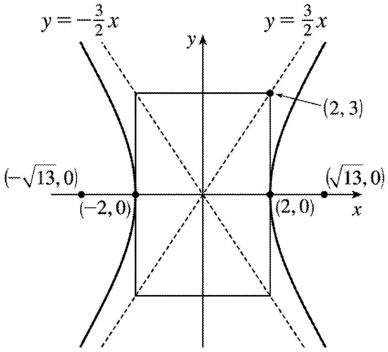
20. $\frac{y^2}{16} - \frac{x^2}{36} = 1 \Rightarrow a=4$, $b=6$, $c=\sqrt{a^2+b^2}=\sqrt{16+36}=\sqrt{52}=2\sqrt{13}$. The center is $(0,0)$, the vertices are $(0,\pm 4)$, the foci are $\left(0,\pm 2\sqrt{13}\right)$, and the asymptotes are the lines $y=\pm \frac{a}{b}x=\pm \frac{2}{3}x$.



21. $y^2 - x^2 = 4 \Leftrightarrow \frac{y^2}{4} - \frac{x^2}{4} = 1 \Rightarrow a=\sqrt{4}=2=b$, $c=\sqrt{4+4}=2\sqrt{2} \Rightarrow$ center $(0,0)$, vertices $(0,\pm 2)$, foci $\left(0,\pm 2\sqrt{2}\right)$, asymptotes $y=\pm x$

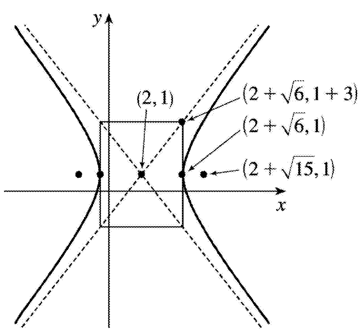


22. $9x^2 - 4y^2 = 36 \Leftrightarrow \frac{x^2}{4} - \frac{y^2}{9} = 1 \Rightarrow a=\sqrt{4}=2$, $b=\sqrt{9}=3$, $c=\sqrt{4+9}=\sqrt{13} \Rightarrow$ center $(0,0)$, vertices $(\pm 2,0)$, foci $\left(\pm\sqrt{13},0\right)$, asymptotes $y=\pm \frac{3}{2}x$



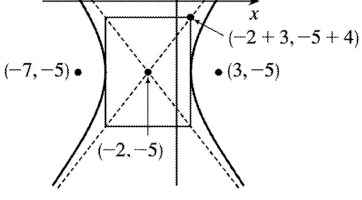
23. $2y^2-4y-3x^2+12x=-8 \Leftrightarrow$

$2(y^2-2y+1)-3(x^2-4x+4)=-8+2-12 \Leftrightarrow 2(y-1)^2-3(x-2)^2=-18 \Leftrightarrow \frac{(x-2)^2}{6}-\frac{(y-1)^2}{9}=1 \Rightarrow a=\sqrt{6}$, $b=3$, $c=\sqrt{15} \Rightarrow$ center $(2,1)$, vertices $\left(2\pm\sqrt{6},1\right)$, foci $\left(2\pm\sqrt{15},1\right)$, asymptotes $y-1=\pm\frac{3}{\sqrt{6}}(x-2)$ or $y-1=\pm\frac{\sqrt{6}}{2}(x-2)$



24. $16x^2+64x-9y^2-90y=305 \Leftrightarrow$

$16(x^2+4x+4)-9(y^2+10y+25)=305+64-225 \Leftrightarrow 16(x+2)^2-9(y+5)^2=144 \Leftrightarrow \frac{(x+2)^2}{9}-\frac{(y+5)^2}{16}=1 \Rightarrow a=3$, $b=4$, $c=5 \Rightarrow$ center $(-2,-5)$, vertices $(-5,-5)$ and $(1,-5)$, foci $(-7,-5)$ and $(3,-5)$, asymptotes $y+5=\pm\frac{4}{3}(x+2)$



25. $x^2=y+1 \Leftrightarrow x^2=1(y+1)$. This is an equation of a *parabola* with $4p=1$, so $p=\frac{1}{4}$. The vertex is $(0,-1)$ and the focus is $\left(0,-\frac{3}{4}\right)$.

26. $x^2=y^2+1 \Leftrightarrow x^2-y^2=1$. This is an equation of a *hyperbola* with vertices $(\pm 1,0)$. The foci are at $(\pm\sqrt{1+1},0)=(\pm\sqrt{2},0)$.

27. $x^2=4y-2y^2 \Leftrightarrow x^2+2y^2-4y=0 \Leftrightarrow x^2+2(y^2-2y+1)=2 \Leftrightarrow x^2+2(y-1)^2=2 \Leftrightarrow \frac{x^2}{2}+\frac{(y-1)^2}{1}=1$. This is an equation of an *ellipse* with vertices at $(\pm\sqrt{2},1)$. The foci are at $(\pm\sqrt{2-1},1)=(\pm 1,1)$.

28. $y^2-8y=6x-16 \Leftrightarrow y^2-8y+16=6x \Leftrightarrow (y-4)^2=6x$. This is an equation of a *parabola* with $4p=6$, so $p=\frac{3}{2}$. The vertex is $(0,4)$ and the focus is $\left(\frac{3}{2},4\right)$.

29. $y^2+2y=4x^2+3 \Leftrightarrow y^2+2y+1=4x^2+4 \Leftrightarrow (y+1)^2-4x^2=4 \Leftrightarrow \frac{(y+1)^2}{4}-x^2=1$. This is an equation of a *hyperbola* with vertices $(0,-1\pm 2)=(0,1)$ and $(0,-3)$. The foci are at $\left(0,-1\pm\sqrt{4+1}\right)=\left(0,-1\pm\sqrt{5}\right)$.

30. $4x^2+4x+y^2=0 \Leftrightarrow 4\left(x^2+x+\frac{1}{4}\right)+y^2=1 \Leftrightarrow 4\left(x+\frac{1}{2}\right)^2+y^2=1 \Leftrightarrow \frac{\left(x+\frac{1}{2}\right)^2}{1/4}+y^2=1$. This is an equation of an *ellipse* with vertices $\left(-\frac{1}{2},0\pm 1\right)=\left(-\frac{1}{2},\pm 1\right)$. The foci are at $\left(-\frac{1}{2},0\pm\sqrt{1-\frac{1}{4}}\right)=\left(-\frac{1}{2},\pm\sqrt{3}/2\right)$.

31. The parabola with vertex $(0,0)$ and focus $(0,-2)$ opens downward and has $p=-2$, so its equation is $x^2=4py=-8y$.

32. The parabola with vertex $(1,0)$ and directrix $x=-5$ opens to the right and has $p=6$, so its equation is $y^2=4p(x-1)=24(x-1)$.

33. The distance from the focus $(-4,0)$ to the directrix $x=2$ is $2-(-4)=6$, so the distance from the focus to the vertex is $\frac{1}{2}(6)=3$ and the vertex is $(-1,0)$. Since the focus is to the left of the vertex, $p=-3$. An equation is $y^2=4p(x+1) \Rightarrow y^2=-12(x+1)$.

34. The distance from the focus (3,6) to the vertex $(3,2)$ is $6-2=4$. Since the focus is above the vertex, $p=4$. An equation is $(x-3)^2=4p(y-2) \Rightarrow (x-3)^2=16(y-2)$.

35. The parabola must have equation $y^2=4px$, so $(-4)^2=4p(1) \Rightarrow p=4 \Rightarrow y^2=16x$.

36. Vertical axis $\Rightarrow (x-h)^2=4p(y-k)$. Substituting $(-2,3)$ and $(0,3)$ gives $(-2-h)^2=4p(3-k)$ and $(-h)^2=4p(3-k) \Rightarrow (-2-h)^2=(-h)^2 \Rightarrow 4+4h+h^2=h^2 \Rightarrow h=-1 \Rightarrow 1=4p(3-k)$. Substituting $(1,9)$ gives $[1-(-1)]^2=4p(9-k) \Rightarrow 4=4p(9-k)$. Solving for p from these equations gives $p=\frac{1}{4(3-k)}=\frac{1}{9-k} \Rightarrow$ $4(3-k)=9-k \Rightarrow k=1 \Rightarrow p=\frac{1}{8} \Rightarrow (x+1)^2=\frac{1}{2}(y-1) \Rightarrow 2x^2+4x-y+3=0$.

37. The ellipse with foci $(\pm 2,0)$ and vertices $(\pm 5,0)$ has center $(0,0)$ and a horizontal major axis, with $a=5$ and $c=2$, so $b=\sqrt{a^2-c^2}=\sqrt{21}$. An equation is $\frac{x^2}{25}+\frac{y^2}{21}=1$.

38. The ellipse with foci $(0,\pm 5)$ and vertices $(0,\pm 13)$ has center $(0,0)$ and a vertical major axis, with $c=5$ and $a=13$, so $b=\sqrt{a^2-c^2}=12$. An equation is $\frac{x^2}{144}+\frac{y^2}{169}=1$.

39. Since the vertices are $(0,0)$ and $(0,8)$, the ellipse has center $(0,4)$ with a vertical axis and $a=4$. The foci at $(0,2)$ and $(0,6)$ are 2 units from the center, so $c=2$ and $b=\sqrt{a^2-c^2}=\sqrt{4^2-2^2}=\sqrt{12}$. An equation is $\frac{(x-0)^2}{b^2}+\frac{(y-4)^2}{a^2}=1 \Rightarrow \frac{x^2}{12}+\frac{(y-4)^2}{16}=1$.

40. Since the foci are $(0,-1)$ and $(8,-1)$, the ellipse has center $(4,-1)$ with a horizontal axis and $c=4$. The vertex $(9,-1)$ is 5 units from the center, so $a=5$ and $b=\sqrt{a^2-c^2}=\sqrt{5^2-4^2}=\sqrt{9}$. An equation is $\frac{(x-4)^2}{a^2}+\frac{(y+1)^2}{b^2}=1 \Rightarrow \frac{(x-4)^2}{25}+\frac{(y+1)^2}{9}=1$.

41. Center $(2,2)$, $c=2$, $a=3 \Rightarrow b=\sqrt{5} \Rightarrow \frac{1}{9}(x-2)^2+\frac{1}{5}(y-2)^2=1$

42. Center $(0,0)$, $c=2$, major axis horizontal $\Rightarrow$

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and $b^2=a^2-c^2=a^2-4$. Since the ellipse passes through $(2,1)$, we have

$2a=\left|PF_1\right|+\left|PF_2\right|=\sqrt{17}+1 \Rightarrow a^2=\frac{9+\sqrt{17}}{2}$ and $b^2=\frac{1+\sqrt{17}}{2}$, so the ellipse has equation

$\frac{2x^2}{9+\sqrt{17}}+\frac{2y^2}{1+\sqrt{17}}=1$.

43. Center $(0,0)$, vertical axis, $c=3$, $a=1 \Rightarrow b=\sqrt{8}=2\sqrt{2} \Rightarrow y^2-\frac{1}{8}x^2=1$

44. Center $(0,0)$, horizontal axis, $c=6$, $a=4 \Rightarrow b=2\sqrt{5} \Rightarrow \frac{1}{16}x^2-\frac{1}{20}y^2=1$

45. Center $(4,3)$, horizontal axis, $c=3$, $a=2 \Rightarrow b=\sqrt{5} \Rightarrow \frac{1}{4}(x-4)^2-\frac{1}{5}(y-3)^2=1$

46. Center $(2,3)$, vertical axis, $c=5$, $a=3 \Rightarrow b=4 \Rightarrow \frac{1}{9}(y-3)^2-\frac{1}{16}(x-2)^2=1$

47. Center $(0,0)$, horizontal axis, $a=3$, $\frac{b}{a}=2 \Rightarrow b=6 \Rightarrow \frac{1}{9}x^2-\frac{1}{36}y^2=1$

48. Center $(4,2)$, horizontal axis, asymptotes $y-2=\pm(x-4) \Rightarrow c=2$, $b/a=1 \Rightarrow a=b \Rightarrow c^2=4=a^2+b^2=2a^2 \Rightarrow$ $a^2=2 \Rightarrow \frac{1}{2}(x-4)^2-\frac{1}{2}(y-2)^2=1$

49. In Figure 8, we see that the point on the ellipse closest to a focus is the closer vertex (which is a distance $a-c$ from it) while the farthest point is the other vertex (at a distance of $a+c$). So for this lunar orbit, $(a-c)+(a+c)=2a=(1728+110)+(1728+314)$, or $a=1940$; and $(a+c)-(a-c)=2c=314-110$, or $c=102$. Thus, $b^2=a^2-c^2=3,753,196$, and the equation is $\frac{x^2}{3,763,600}+\frac{y^2}{3,753,196}=1$.

50. **(a)** Choose V to be the origin, with $x-$ axis through V and F . \ Then F is $(p,0)$, A is $(p,5)$, so substituting A into the equation $y^2=4px$ gives $25=4p^2$ so $p=\frac{5}{2}$ and $y^2=10x$.

(b) $x=11 \Rightarrow y=\sqrt{110} \Rightarrow |CD|=2\sqrt{110}$

51. **(a)** Set up the coordinate system so that A is $(-200,0)$ and B is $(200,0)$.
$|PA|-|PB|=(1200)\,(980)=1,176,000$ ft

$=\frac{2450}{11}$ mi $=2a \Rightarrow a=\frac{1225}{11}$, and $c=200$ so $b^2=c^2-a^2=\frac{3,339,375}{121} \Rightarrow \frac{121x^2}{1,500,625}-\frac{121y^2}{3,339,375}=1$.

(b) Due north of $B \Rightarrow x=200 \Rightarrow \frac{(121)(200)^2}{1,500,625}-\frac{121y^2}{3,339,375}=1 \Rightarrow y=\frac{133,575}{539} \approx 248$ mi

52. $\left|PF_1\right|-\left|PF_2\right|=\pm 2a \Leftrightarrow \sqrt{(x+c)^2+y^2}-\sqrt{(x-c)^2+y^2}=\pm 2a \Leftrightarrow \sqrt{(x+c)^2+y^2}=\sqrt{(x-c)^2+y^2}\pm 2a \Leftrightarrow$ $(x+c)^2+y^2=(x-c)^2+y^2+4a^2\pm 4a\sqrt{(x-c)^2+y^2} \Leftrightarrow 4cx-4a^2=\pm 4a\sqrt{(x-c)^2+y^2} \Leftrightarrow$ $c^2x^2-2a^2cx+a^4=a^2\left(x^2-2cx+c^2+y^2\right) \Leftrightarrow \left(c^2-a^2\right)x^2-a^2y^2=a^2\left(c^2-a^2\right) \Leftrightarrow b^2x^2-a^2y^2=a^2b^2$ (where $b^2=c^2-a^2$) $\Leftrightarrow \frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

53. The function whose graph is the upper branch of this hyperbola is concave upward. The function is $y=f(x)=a\sqrt{1+\frac{x^2}{b^2}}=\frac{a}{b}\sqrt{b^2+x^2}$, so $y'=\frac{a}{b}x\left(b^2+x^2\right)^{-1/2}$ and $y''=\frac{a}{b}\left[\left(b^2+x^2\right)^{-1/2}-x^2\left(b^2+x^2\right)^{-3/2}\right]=ab\left(b^2+x^2\right)^{-3/2}>0$ for all x , and so f is concave upward.

54. We can follow exactly the same sequence of steps as in the derivation of Formula 4, except we use the points $(1,1)$ and $(-1,-1)$ in the distance formula (first equation of that derivation) so $\sqrt{(x-1)^2+(y-1)^2}+\sqrt{(x+1)^2+(y+1)^2}=4$ will lead (after moving the second term to the right, squaring, and simplifying) to $2\sqrt{(x+1)^2+(y+1)^2}=x+y+4$, which, after squaring and simplifying again, leads to $3x^2-2xy+3y^2=8$.

55. **(a)** If $k>16$, then $k-16>0$, and $\frac{x^2}{k}+\frac{y^2}{k-16}=1$ is an *ellipse* since it is the sum of two squares on the left side.

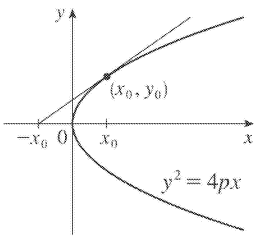
(b) If $0<k<16$, then $k-16<0$, and $\frac{x^2}{k}+\frac{y^2}{k-16}=1$ is a *hyperbola* since it is the difference of two squares on the left side.

(c) If $k<0$, then $k-16<0$, and there is *no curve* since the left side is the sum of two negative terms, which cannot equal 1.

(d) In case (a), $a^2=k$, $b^2=k-16$, and $c^2=a^2-b^2=16$, so the foci are at $(\pm 4,0)$. In case (b), $k-16<0$, so

$a^2=k$, $b^2=16-k$, and $c^2=a^2+b^2=16$, and so again the foci are at $(\pm 4,0)$.

56. **(a)** $y^2=4px \Rightarrow 2yy'=4p \Rightarrow y'=\frac{2p}{y}$, so the tangent line is $y-y_0=\frac{2p}{y_0}(x-x_0) \Rightarrow yy_0-y_0^2=2p(x-x_0) \Leftrightarrow$ $yy_0-4px_0=2px-2px_0 \Rightarrow yy_0=2p(x+x_0)$.



(b) The $x-$ intercept is $-x_0$.

57. Use the parametrization $x=2\cos t$, $y=\sin t$, $0\leq t\leq 2\pi$ to get

$$L=4\int_0^{\pi/2}\sqrt{(dx/dt)^2+(dy/dt)^2}\,dt=4\int_0^{\pi/2}\sqrt{4\sin^2 t+\cos^2 t}\,dt=4\int_0^{\pi/2}\sqrt{3\sin^2 t+1}\,dt$$

Using Simpson's Rule with $n=10$, $\Delta t=\frac{\pi/2-0}{10}=\frac{\pi}{20}$, and $f(t)=\sqrt{3\sin^2 t+1}$, we get

$$L\approx \frac{4}{3}\left(\frac{\pi}{20}\right)\left[f(0)+4f\left(\frac{\pi}{20}\right)+2f\left(\frac{2\pi}{20}\right)+\cdots+2f\left(\frac{8\pi}{20}\right)+4f\left(\frac{9\pi}{20}\right)+f\left(\frac{\pi}{2}\right)\right]\approx 9.69$$

58. The length of the major axis is $2a$, so $a=\frac{1}{2}(1.18\times 10^{10})=5.9\times 10^9$. The length of the minor axis is $2b$, so $b=\frac{1}{2}(1.14\times 10^{10})=5.7\times 10^9$. An equation of the ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, or converting into parametric equations, $x=a\cos\theta$ and $y=b\sin\theta$. So

$$L=4\int_0^{\pi/2}\sqrt{(dx/d\theta)^2+(dy/d\theta)^2}\,d\theta=4\int_0^{\pi/2}\sqrt{a^2\sin^2\theta+b^2\cos^2\theta}\,d\theta$$

Using Simpson's Rule with $n=10$, $\Delta\theta=\frac{\pi/2-0}{10}=\frac{\pi}{20}$, and $f(\theta)=\sqrt{a^2\sin^2\theta+b^2\cos^2\theta}$, we get

$L\approx 4\cdot S_{10}$

$$= 4\cdot\frac{\pi}{20\cdot 3}\left[f(0)+4f\left(\frac{\pi}{20}\right)+2f\left(\frac{2\pi}{20}\right)+\cdots+2f\left(\frac{8\pi}{20}\right)+4f\left(\frac{9\pi}{20}\right)+f\left(\frac{\pi}{2}\right)\right]$$
$$\approx 3.64\times 10^{10}\text{km}$$

59. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\Rightarrow \frac{2x}{a^2}+\frac{2yy'}{b^2}=0\Rightarrow y'=-\frac{b^2x}{a^2y}$ ($y\neq 0$). Thus, the slope of the tangent line at P is $-\frac{b^2x_1}{a^2y_1}$. The slope of F_1P is $\frac{y_1}{x_1+c}$ and of F_2P is $\frac{y_1}{x_1-c}$. By the formula from Problems Plus, we have

$$\tan\alpha=\frac{\frac{y_1}{x_1+c}+\frac{b^2x_1}{a^2y_1}}{1-\frac{b^2x_1y_1}{a^2y_1(x_1+c)}}=\frac{a^2y_1^2+b^2x_1(x_1+c)}{a^2y_1(x_1+c)-b^2x_1y_1}=\frac{a^2b^2+b^2cx_1}{c^2x_1y_1+a^2cy_1}\left[\begin{array}{r}\text{using } b^2x_1^2+a^2y_1^2=a^2b^2\\ \text{and } a^2-b^2=c^2\end{array}\right]=$$

$$\frac{b^2(cx_1+a^2)}{cy_1(cx_1+a^2)}=\frac{b^2}{cy_1}\text{ and}$$

$$\tan\beta=\frac{-\frac{y_1}{x_1-c}-\frac{b^2x_1}{a^2y_1}}{1-\frac{b^2x_1y_1}{a^2y_1(x_1-c)}}=\frac{-a^2y_1^2-b^2x_1\left(x_1-c\right)}{a^2y_1(x_1-c)-b^2x_1y_1}=\frac{-a^2b^2+b^2cx_1}{c^2x_1y_1-a^2cy_1}=\frac{b^2(cx_1-a^2)}{cy_1(cx_1-a^2)}=\frac{b^2}{cy_1}\text{ So }\alpha=\beta$$

60. The slopes of the line segments F_1P and F_2P are $\frac{y_1}{x_1+c}$ and $\frac{y_1}{x_1-c}$, where P is $\left(x_1,y_1\right)$.

Differentiating implicitly, $\frac{2x}{a^2}-\frac{2yy'}{b^2}=0\Rightarrow y'=\frac{b^2x}{a^2y}\Rightarrow$ the slope of the tangent at P is $\frac{b^2x_1}{a^2y_1}$, so by the formula from Problems Plus,

$$\tan\alpha = \frac{\dfrac{b^2x_1}{a^2y_1} - \dfrac{y_1}{x_1+c}}{1+\dfrac{b^2x_1y_1}{a^2y_1(x_1+c)}} = \frac{b^2x_1(x_1+c)-a^2y_1^2}{a^2y_1(x_1+c)+b^2x_1y_1} = \frac{b^2(cx_1+a^2)}{cy_1(cx_1+a^2)} \left[\begin{array}{r}\text{using } x_1^2/a^2-y_1^2/b^2=1 \\ \text{and } a^2+b^2=c^2\end{array}\right] = \frac{b^2}{cy_1}$$

and

$$\tan\beta = \frac{-\dfrac{b^2x_1}{a^2y_1} + \dfrac{y_1}{x_1-c}}{1+\dfrac{b^2x_1y_1}{a^2y_1(x_1-c)}} = \frac{-b^2x_1(x_1-c)+a^2y_1^2}{a^2y_1(x_1-c)+b^2x_1y_1} = \frac{b^2(cx_1-a^2)}{cy_1(cx_1-a^2)} = \frac{b^2}{cy_1}$$

So $\alpha=\beta$.

1. The directrix $y=6$ is above the focus at the origin, so we use the form with " $+e\sin\theta$ " in the denominator. (See Theorem 6 and Figure 2.) $r=\dfrac{ed}{1+e\sin\theta}=\dfrac{\frac{7}{4}\cdot 6}{1+\frac{7}{4}\sin\theta}=\dfrac{42}{4+7\sin\theta}$

2. The directrix $x=4$ is to the right of the focus at the origin, so we use the form with " $+e\cos\theta$ " in the denominator. $e=1$ for a parabola, so an equation is $r=\dfrac{ed}{1+e\cos\theta}=\dfrac{1\cdot 4}{1+1\cos\theta}=\dfrac{4}{1+\cos\theta}$

3. The directrix $x=-5$ is to the left of the focus at the origin, so we use the form with " $-e\cos\theta$ " in the denominator. $r=\dfrac{ed}{1-e\cos\theta}=\dfrac{\frac{3}{4}\cdot 5}{1-\frac{3}{4}\cos\theta}=\dfrac{15}{4-3\cos\theta}$

4. The directrix $y=-2$ is below the focus at the origin, so we use the form with " $-e\sin\theta$ " in the denominator. $r=\dfrac{ed}{1-e\sin\theta}=\dfrac{2\cdot 2}{1-2\sin\theta}=\dfrac{4}{1-2\sin\theta}$

5. The vertex $(4,3\pi/2)$ is 4 units below the focus at the origin, so the directrix is 8 units below the focus $(d=8)$, and we use the form with " $-e\sin\theta$ " in the denominator. $e=1$ for a parabola, so an equation is $r=\dfrac{ed}{1-e\sin\theta}=\dfrac{1(8)}{1-1\sin\theta}=\dfrac{8}{1-\sin\theta}$.

6. The vertex $P(1,\pi/2)$ is 1 unit above the focus F at the origin, so $|PF|=1$ and we use the form with " $+e\sin\theta$ " in the denominator. The distance from the focus to the directrix l is d , so $e=\dfrac{|PF|}{|Pl|}\Rightarrow 0.8=\dfrac{1}{d-1}\Rightarrow 0.8d-0.8=1\Rightarrow 0.8d=1.8\Rightarrow d=2.25$.

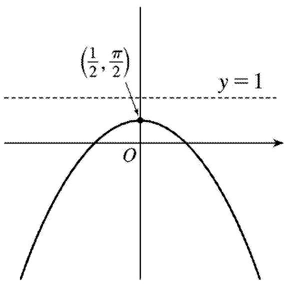
An equation is $r=\dfrac{ed}{1+e\sin\theta}=\dfrac{0.8(2.25)}{1+0.8\sin\theta}\cdot\dfrac{5}{5}=\dfrac{9}{5+4\sin\theta}$.

7. The directrix $r=4\sec\theta$ (equivalent to $r\cos\theta=4$ or $x=4$) is to the right of the focus at the origin, so we will use the form with " $+e\cos\theta$ " in the denominator. The distance from the focus to the directrix is $d=4$, so an equation is $r=\dfrac{ed}{1+e\cos\theta}=\dfrac{0.5(4)}{1+0.5\cos\theta}\cdot\dfrac{2}{2}=\dfrac{4}{2+\cos\theta}$.

8. The directrix $r=-6\csc\theta$ (equivalent to $r\sin\theta=-6$ or $y=-6$) is below the focus at the origin, so we will use the form with " $-e\sin\theta$ " in the denominator. The distance from the focus to the directrix is $d=6$, so an equation is $r=\dfrac{ed}{1-e\sin\theta}=\dfrac{3(6)}{1-3\sin\theta}=\dfrac{18}{1-3\sin\theta}$.

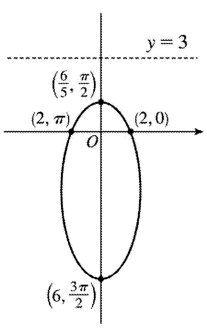
9. $r=\dfrac{1}{1+\sin\theta}=\dfrac{ed}{1+e\sin\theta}$, where $d=e=1$.

(a) Eccentricity $=e=1$
(b) Since $e=1$, the conic is a parabola.
(c) Since " $+e\sin\theta$ '' appears in the denominator, the directrix is above the focus at the origin. $d=|Fl|=1$, so an equation of the directrix is $y=1$.
(d) The vertex is at $\left(\frac{1}{2},\frac{\pi}{2}\right)$, midway between the focus and the directrix.



10. $r=\dfrac{6}{3+2\sin\theta}=\dfrac{2}{1+\frac{2}{3}\sin\theta}=\dfrac{\frac{2}{3}\cdot 3}{1+\frac{2}{3}\sin\theta}$

(a) $e=\dfrac{2}{3}$
(b) Ellipse
(c) $y=3$
(d) Vertices $\left(\frac{6}{5},\frac{\pi}{2}\right)$ and $\left(6,\frac{3\pi}{2}\right)$; center $\left(\frac{12}{5},\frac{3\pi}{2}\right)$



11. $r=\dfrac{12}{4-\sin\theta}\cdot\dfrac{1/4}{1/4}=\dfrac{3}{1-\frac{1}{4}\sin\theta}$, where $e=\dfrac{1}{4}$ and $ed=3\Rightarrow d=12$.

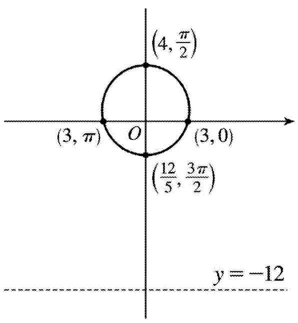
(a) Eccentricity $=e=\dfrac{1}{4}$

(b) Since

$e=\frac{1}{4}<1$, the conic is an ellipse.

(c) Since " $-e\sin\theta$ " appears in the denominator, the directrix is below the focus at the origin. $d=|Fl|=12$, so an equation of the directrix is $y=-12$.

(d) The vertices are $\left(4,\frac{\pi}{2}\right)$ and $\left(\frac{12}{5},\frac{3\pi}{2}\right)$, so the center is midway between them, that is, $\left(\frac{4}{5},\frac{\pi}{2}\right)$.



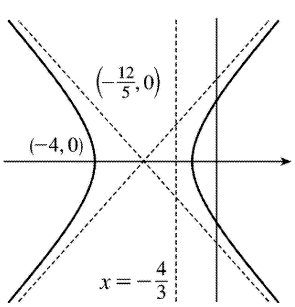
12. $r=\frac{4}{2-3\cos\theta}=\frac{2}{1-\frac{3}{2}\cos\theta}=\frac{\frac{3}{2}\cdot\frac{4}{3}}{1-\frac{3}{2}\cos\theta}$

(a) $e=\frac{3}{2}$

(b) Hyperbola

(c) $x=-\frac{4}{3}$

(d) The vertices are $(-4,0)$ and $\left(\frac{4}{5},\pi\right)=\left(-\frac{4}{5},0\right)$, so the center is $\left(-\frac{12}{5},0\right)$. The asymptotes are parallel to $\theta=\pm\cos^{-1}\frac{2}{3}$. [Their slopes are $\pm\tan\left(\cos^{-1}\frac{2}{3}\right)=\pm\frac{\sqrt{5}}{2}$]



13.

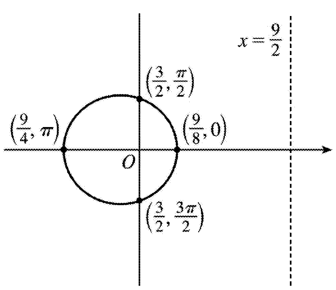
$r=\dfrac{9}{6+2\cos\theta}\cdot\dfrac{1/6}{1/6}=\dfrac{3/2}{1+\frac{1}{3}\cos\theta}$, where $e=\dfrac{1}{3}$ and $ed=\dfrac{3}{2}\Rightarrow d=\dfrac{9}{2}$.

(a) Eccentricity $=e=\dfrac{1}{3}$

(b) Since $e=\dfrac{1}{3}<1$, the conic is an ellipse.

(c) Since " $+e\cos\theta$ '' appears in the denominator, the directrix is to the right of the focus at the origin. $d=|Fl|=\dfrac{9}{2}$, so an equation of the directrix is $x=\dfrac{9}{2}$.

(d) The vertices are $\left(\dfrac{9}{8},0\right)$ and $\left(\dfrac{9}{4},\pi\right)$, so the center is midway between them, that is, $\left(\dfrac{9}{16},\pi\right)$.



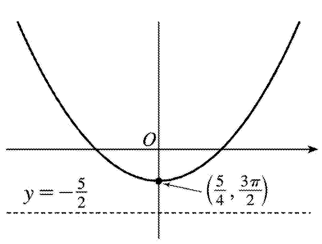
14. $r=\dfrac{5}{2-2\sin\theta}=\dfrac{\frac{5}{2}}{1-\sin\theta}$

(a) $e=1$

(b) Parabola

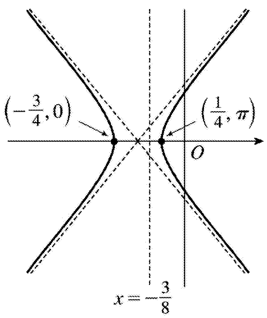
(c) $y=-\dfrac{5}{2}$

(d) The focus is $(0,0)$, so the vertex is $\left(\dfrac{5}{4},\dfrac{3\pi}{2}\right)$ and the parabola opens up.



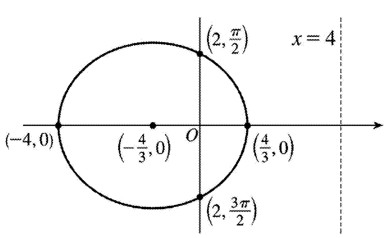
15. $r=\dfrac{3}{4-8\cos\theta}\cdot\dfrac{1/4}{1/4}=\dfrac{3/4}{1-2\cos\theta}$, where $e=2$ and $ed=\dfrac{3}{4}\Rightarrow d=\dfrac{3}{8}$.

(a) Eccentricity $=e=2$
(b) Since $e=2>1$, the conic is a hyperbola.
(c) Since " $-e\cos\theta$ '' appears in the denominator, the directrix is to the left of the focus at the origin. $d=|Fl|=\frac{3}{8}$, so an equation of the directrix is $x=-\frac{3}{8}$.
(d) The vertices are $\left(-\frac{3}{4},0\right)$ and $\left(\frac{1}{4},\pi\right)$, so the center is midway between them, that is, $\left(\frac{1}{2},\pi\right)$.



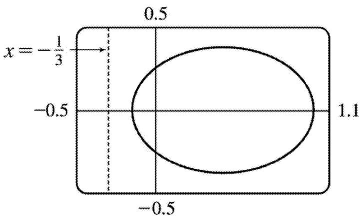
16. $r=\dfrac{4}{2+\cos\theta}=\dfrac{2}{1+\frac{1}{2}\cos\theta}=\dfrac{\frac{1}{2}\cdot 4}{1+\frac{1}{2}\cos\theta}$

(a) $e=\frac{1}{2}$
(b) Ellipse
(c) $x=4$
(d) The vertices are $\left(\frac{4}{3},0\right)$ and $(4,\pi)=(-4,0)$, so the center is $\left(-\frac{4}{3},0\right)$.

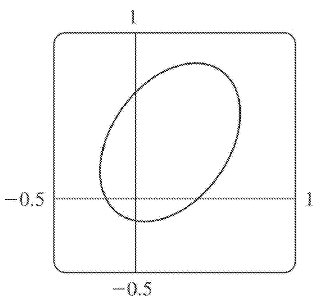


17. **(a)** The equation is $r=\dfrac{1}{4-3\cos\theta}=\dfrac{1/4}{1-\frac{3}{4}\cos\theta}$, so $e=\frac{3}{4}$ and $ed=\frac{1}{4}\Rightarrow d=\frac{1}{3}$. The conic is an ellipse, and the equation of its directrix is $x=r\cos\theta=-\frac{1}{3}\Rightarrow r=-\dfrac{1}{3\cos\theta}$. We must be careful in our

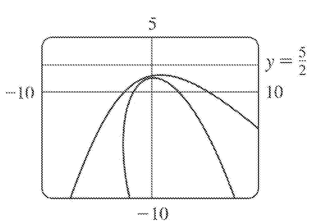
choice of parameter values in this equation ($-1 \leq \theta \leq 1$ works well).



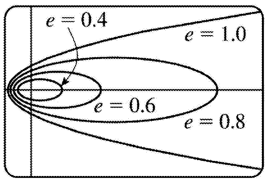
(b) The equation is obtained by replacing θ with $\theta - \frac{\pi}{3}$ in the equation of the original conic (see Example 4), so $r = \dfrac{1}{4-3\cos\left(\theta - \frac{\pi}{3}\right)}$.



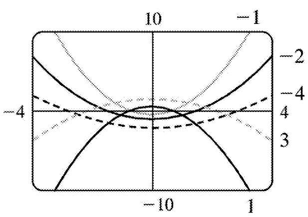
18. $r = \dfrac{5}{2+2\sin\theta} = \dfrac{5/2}{1+\sin\theta}$, so $e=1$ and $d = \frac{5}{2}$. The equation of the directrix is $y = r\sin\theta = \frac{5}{2} \Rightarrow r = \dfrac{5}{2\sin\theta}$. If the parabola is rotated about its focus (the origin) through $\frac{\pi}{6}$, its equation is the same as that of the original, with θ replaced by $\theta - \frac{\pi}{6}$ (see Example 4), so $r = \dfrac{5}{2+2\sin\left(\theta - \pi/6\right)}$. In graphing each of these curves, we must be careful to select parameter ranges which prevent the denominator from vanishing while still showing enough of the curve.



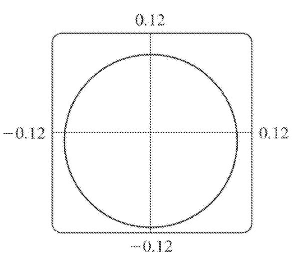
19. For $e<1$ the curve is an ellipse. It is nearly circular when e is close to 0 . As e increases, the graph is stretched out to the right, and grows larger (that is, its right–hand focus moves to the right while its left–hand focus remains at the origin.) At $e=1$, the curve becomes a parabola with focus at the origin.



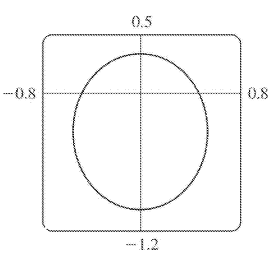
20. **(a)** The value of d does not seem to affect the shape of the conic (a parabola) at all, just its size, position, and orientation (for $d<0$ it opens upward, for $d>0$ it opens downward).



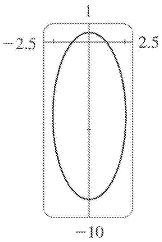
(b) We consider only positive values of e . When $0<e<1$, the conic is an ellipse. As $e\to 0^{+}$, the graph approaches perfect roundness and zero size. As e increases, the ellipse becomes more elongated, until at $e=1$ it turns into a parabola. For $e>1$, the conic is a hyperbola, which moves downward and gets broader as e continues to increase.



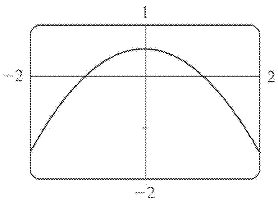
$e=0.1$



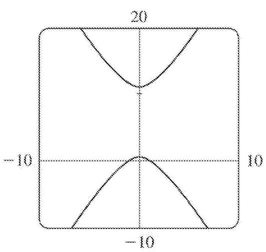
$e=0.5$



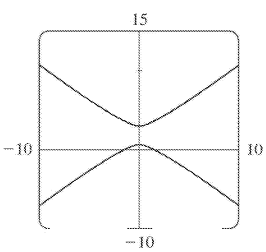
e=0.9



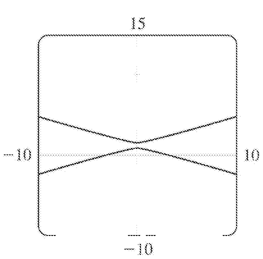
e=1



e=1.1

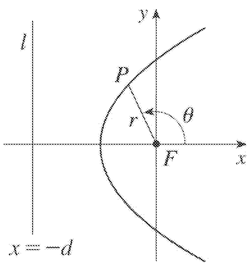


e=1.5

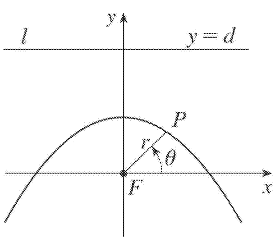


$e=10$

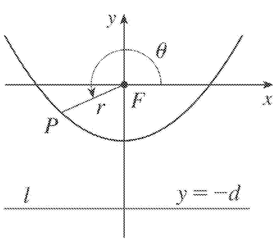
21. $|PF|=e|Pl| \Rightarrow r=e[d-r\cos(\pi-\theta)]=e(d+r\cos\theta) \Rightarrow r(1-e\cos\theta)=ed \Rightarrow r=\dfrac{ed}{1-e\cos\theta}$



22. $|PF|=e|Pl| \Rightarrow r=e[d-r\sin\theta] \Rightarrow r(1+e\sin\theta)=ed \Rightarrow r=\dfrac{ed}{1+e\sin\theta}$



23. $|PF|=e|Pl| \Rightarrow r=e[d-r\sin(\theta-\pi)]=e(d+r\sin\theta) \Rightarrow r(1-e\sin\theta)=ed \Rightarrow r=\dfrac{ed}{1-e\sin\theta}$



24. The parabolas intersect at the two points where $\dfrac{c}{1+\cos\theta}=\dfrac{d}{1-\cos\theta} \Rightarrow \cos\theta=\dfrac{c-d}{c+d} \Rightarrow r=\dfrac{c+d}{2}$.

For the first parabola, $\dfrac{dr}{d\theta}=\dfrac{c\sin\theta}{(1+\cos\theta)^2}$, so

$$\frac{dy}{dx}=\frac{(dr/d\theta)\sin\theta+r\cos\theta}{(dr/d\theta)\cos\theta-r\sin\theta}=\frac{c\sin^2\theta+c\cos\theta(1+\cos\theta)}{c\sin\theta\cos\theta-c\sin\theta(1+\cos\theta)}=\frac{1+\cos\theta}{-\sin\theta}$$

and similarly for the second,

$\frac{dy}{dx}=\frac{1-\cos\theta}{\sin\theta}=\frac{\sin\theta}{1+\cos\theta}$. Since the product of these slopes is -1 , the parabolas intersect at right angles.

25. **(a)** If the directrix is $x=-d$, then $r=\frac{ed}{1-e\cos\theta}$, and, from (4), $a^2=\frac{e^2d^2}{\left(1-e^2\right)^2}\Rightarrow ed=a(1-e^2)$.

Therefore, $r=\frac{a(1-e^2)}{1-e\cos\theta}$.

(b) $e=0.017$ and the major axis $=2a=2.99\times10^8\Rightarrow a=1.495\times10^8$. Therefore

$r=\frac{1.495\times10^8\left[1-(0.017)^2\right]}{1-0.017\cos\theta}\approx\frac{1.49\times10^8}{1-0.017\cos\theta}$.

26. **(a)** The Sun is at point F in Figure 1 so that perihelion is in the positive x –direction and aphelion is in the negative x –direction. At perihelion, $\theta=0$, so $r=\frac{a\left(1-e^2\right)}{1+e\cos 0}=\frac{a(1-e)(1+e)}{1+e}=a(1-e)$.

At aphelion, $\theta=\pi$, so $r=\frac{a\left(1-e^2\right)}{1+e\cos\pi}=\frac{a(1-e)(1+e)}{1-e}=a(1+e)$.

(b) At perihelion, $r=a(1-e)\approx(1.495\times10^8)(1-0.017)\approx1.47\times10^8$ km.

At aphelion, $r=a(1+e)\approx\left(1.495\times10^8\right)(1+0.017)\approx1.52\times10^8$ km.

27. Here $2a=$ length of major axis $=36.18$ AU $\Rightarrow a=18.09$ AU and $e=0.97$. By Exercise 25(a), the equation of the orbit is $r=\frac{18.09\left[1-(0.97)^2\right]}{1-0.97\cos\theta}\approx\frac{1.07}{1-0.97\cos\theta}$. By Exercise 26(a), the maximum distance from the comet to the sun is $18.09(1+0.97)\approx35.64$ AU or about 3.314 billion miles.

28. Here $2a=$ length of major axis $=356.5$ AU $\Rightarrow a=178.25$ AU and $e=0.9951$. By Exercise 25(a), the equation of the orbit is $r=\frac{178.25\left[1-(0.9951)^2\right]}{1-0.9951\cos\theta}\approx\frac{1.7426}{1-0.9951\cos\theta}$. By Exercise 26(a), the minimum distance from the comet to the sun is $178.25\,(1-0.9951)\approx0.8734$ AU or about 81 million miles.

29. The minimum distance is at perihelion, where

$4.6\times10^7=r=a(1-e)=a(1-0.206)=a(0.794)\Rightarrow a=4.6\times10^7/0.794$. So the maximum distance, which is at aphelion, is $r=a(1+e)=\left(4.6\times10^7/0.794\right)(1.206)\approx7.0\times10^7$ km.

30. At perihelion, $r=a(1-e)=4.43\times10^9$, and at aphelion, $r=a(1+e)=7.37\times10^9$. Adding, we get

$2a=11.80\times 10^9$, so $a=5.90\times 10^9$ km. Therefore $1+e=a(1+e)/a=\dfrac{7.37}{5.90}\approx 1.249$ and $e\approx 0.249$.

31. From Exercise 29, we have $e=0.206$ and $a(1-e)=4.6\times 10^7$ km. Thus, $a=4.6\times 10^7/0.794$. From Exercise 25, we can write the equation of Mercury's orbit as $r=a\dfrac{1-e^2}{1-e\cos\theta}$. So since

$$\frac{dr}{d\theta}=\frac{-a(1-e^2)e\sin\theta}{(1-e\cos\theta)^2}\Rightarrow$$

$$r^2+\left(\frac{dr}{d\theta}\right)^2=\frac{a^2\left(1-e^2\right)^2}{(1-e\cos\theta)^2}+\frac{a^2\left(1-e^2\right)^2e^2\sin^2\theta}{(1-e\cos\theta)^4}=\frac{a^2\left(1-e^2\right)^2}{(1-e\cos\theta)^4}\left(1-2e\cos\theta+e^2\right)$$

the length of the orbit is

$$L=\int_0^{2\pi}\sqrt{r^2+(dr/d\theta)^2}\,d\theta=a\left(1-e^2\right)\int_0^{2\pi}\frac{\sqrt{1+e^2-2e\cos\theta}}{(1-e\cos\theta)^2}\,d\theta\approx 3.6\times 10^8\text{km}$$

This seems reasonable, since Mercury's orbit is nearly circular, and the circumference of a circle of radius a is $2\pi a\approx 3.6\times 10^8$ km.

1. **(a)** A sequence is an ordered list of numbers. It can also be defined as a function whose domain is the set of positive integers.
(b) The terms a_n approach 8 as n becomes large. In fact, we can make a_n as close to 8 as we like by taking n sufficiently large.
(c) The terms a_n become large as n becomes large. In fact, we can make a_n as large as we like by taking n sufficiently large.

2. **(a)** From Definition 1, a convergent sequence is a sequence for which $\lim_{n\to\infty} a_n$ exists. Examples: $\{1/n\}$, $\left\{1/2^n\right\}$
(b) A divergent sequence is a sequence for which $\lim_{n\to\infty} a_n$ *does not* exist. Examples: $\{n\}$, $\{\sin n\}$

3. $a_n=1-(0.2)^n$, so the sequence is $\{0.8,0.96,0.992,0.9984,0.99968,\dots\}$.

4. $a_n=\dfrac{n+1}{3n-1}$, so the sequence is $\left\{\dfrac{2}{2},\dfrac{3}{5},\dfrac{4}{8},\dfrac{5}{11},\dfrac{6}{14},\dots\right\}=\left\{1,\dfrac{3}{5},\dfrac{1}{2},\dfrac{5}{11},\dfrac{3}{7},\dots\right\}$.

5. $a_n=\dfrac{3(-1)^n}{n!}$, so the sequence is $\left\{\dfrac{-3}{1},\dfrac{3}{2},\dfrac{-3}{6},\dfrac{3}{24},\dfrac{-3}{120},\dots\right\}=\left\{-3,\dfrac{3}{2},-\dfrac{1}{2},\dfrac{1}{8},-\dfrac{1}{40},\dots\right\}$.

6. $a_n=2\cdot 4\cdot 6\cdot \;\cdot\;\cdot\;\cdot\;\cdot\;(2n)$, so the sequence is
$\{2,2\cdot 4,2\cdot 4\cdot 6,2\cdot 4\cdot 6\cdot 8,2\cdot 4\cdot 6\cdot 8\cdot 10,\dots\}=\{2,8,48,384,3840,\dots\}$.

7. $a_1=3$, $a_{n+1}=2a_n-1$. Each term is defined in terms of the preceding term.
$a_2=2a_1-1=2(3)-1=5$. $a_3=2a_2-1=2(5)-1=9$. $a_4=2a_3-1=2(9)-1=17$. $a_5=2a_4-1=2(17)-1=33$. The sequence is $\{3,5,9,17,33,\dots\}$.

8. $a_1=4$, $a_{n+1}=\dfrac{a_n}{a_n-1}$. Each term is defined in terms of the preceding term.

$a_2=\dfrac{a_1}{a_1-1}=\dfrac{4}{4-1}=\dfrac{4}{3}$. $a_3=\dfrac{a_2}{a_2-1}=\dfrac{4/3}{\frac{4}{3}-1}=\dfrac{4/3}{1/3}=4$. Since $a_3=a_1$, we can see that the terms of the sequence will alternately equal 4 and 4/3 , so the sequence is $\left\{4,\dfrac{4}{3},4,\dfrac{4}{3},4,\dots\right\}$.

9. The numerators are all 1 and the denominators are powers of 2 , so $a_n = \frac{1}{2^n}$.

10. The numerators are all 1 and the denominators are multiples of 2 , so $a_n = \frac{1}{2n}$.

11. $\{2,7,12,17,\ldots\}$. Each term is larger than the preceding one by 5 , so $a_n = a_1 + d(n-1) = 2 + 5(n-1) = 5n-3$.

12. $\left\{ -\frac{1}{4}, \frac{2}{9}, -\frac{3}{16}, \frac{4}{25}, \ldots \right\}$. The numerator of the n th term is n and its denominator is $(n+1)^2$. Including the alternating signs, we get $a_n = (-1)^n \frac{n}{(n+1)^2}$.

13. $\left\{ 1, -\frac{2}{3}, \frac{4}{9}, -\frac{8}{27}, \ldots \right\}$. Each term is $-\frac{2}{3}$ times the preceding one, so $a_n = \left(-\frac{2}{3} \right)^{n-1}$.

14. $\{5,1,5,1,5,1,\ldots\}$. The average of 5 and 1 is 3 , so we can think of the sequence as alternately adding 2 and -2 to 3 . Thus, $a_n = 3 + (-1)^{n+1} \cdot 2$.

15. $a_n = n(n-1)$. $a_n \to \infty$ as $n \to \infty$, so the sequence diverges.

16. $a_n = \frac{n+1}{3n-1} = \frac{1+1/n}{3-1/n}$, so $a_n \to \frac{1+0}{3-0} = \frac{1}{3}$ as $n \to \infty$. Converges

17. $a_n = \frac{3+5n^2}{n+n^2} = \frac{\left(3+5n^2\right)/n^2}{\left(n+n^2\right)/n^2} = \frac{5+3/n^2}{1+1/n}$, so $a_n \to \frac{5+0}{1+0} = 5$ as $n \to \infty$. Converges

18. $a_n = \frac{\sqrt{n}}{1+\sqrt{n}} = \frac{1}{1/\sqrt{n}+1}$, so $a_n \to \frac{1}{0+1} = 1$ as $n \to \infty$. Converges

19. $a_n = \frac{2^n}{3^{n+1}} = \frac{1}{3} \left(\frac{2}{3} \right)^n$, so $\lim_{n\to\infty} a_n = \frac{1}{3} \lim_{n\to\infty} \left(\frac{2}{3} \right)^n = \frac{1}{3} \cdot 0 = 0$ by (8) with $r = \frac{2}{3}$. Converges

20. $a_n = \frac{n}{1+\sqrt{n}} = \frac{\sqrt{n}}{1/\sqrt{n}+1}$. The numerator approaches ∞ and the denominator approaches $0+1=1$ as

$n\to\infty$, so $a_n\to\infty$ as $n\to\infty$ and the sequence diverges.

21. $a_n=\dfrac{(-1)^{n-1}n}{n^2+1}=\dfrac{(-1)^{n-1}}{n+1/n}$, so $0\le\left|a_n\right|=\dfrac{1}{n+1/n}\le\dfrac{1}{n}\to0$ as $n\to\infty$, so $a_n\to0$ by the Squeeze Theorem and Theorem 6. Converges

22. $a_n=\dfrac{(-1)^n n^3}{n^3+2n^2+1}$. Now $\left|a_n\right|=\dfrac{n^3}{n^3+2n^2+1}=\dfrac{1}{1+\dfrac{2}{n}+\dfrac{1}{n^3}}\to1$ as $n\to\infty$, but the terms of the sequence $\{a_n\}$ alternate in sign, so the sequence $a_1, a_3, a_5, \ldots$ converges to -1 and the sequence $a_2, a_4, a_6, \ldots$ converges to $+1$. This shows that the given sequence diverges since its terms don’t approach a single real number.

23. $a_n=\cos(n/2)$. This sequence diverges since the terms don’t approach any particular real number as $n\to\infty$. The terms take on values between -1 and 1 .

24. $a_n=\cos(2/n)$. As $n\to\infty$, $2/n\to0$, so $\cos(2/n)\to\cos0=1$. Converges

25. $a_n=\dfrac{(2n-1)!}{(2n+1)!}=\dfrac{(2n-1)!}{(2n+1)(2n)(2n-1)!}=\dfrac{1}{(2n+1)(2n)}\to0$ as $n\to\infty$. Converges

26. $2n\to\infty$ as $n\to\infty$, so since $\lim\limits_{x\to\infty}\arctan x=\dfrac{\pi}{2}$, we have $\lim\limits_{n\to\infty}\arctan 2n=\dfrac{\pi}{2}$. Converges

27. $a_n=\dfrac{e^n+e^{-n}}{e^{2n}-1}\cdot\dfrac{e^{-n}}{e^{-n}}=\dfrac{1+e^{-2n}}{e^n-e^{-n}}\to\dfrac{1+0}{e^n-0}\to0$ as $n\to\infty$. Converges

28. $a_n=\dfrac{\ln n}{\ln 2n}=\dfrac{\ln n}{\ln 2+\ln n}=\dfrac{1}{\dfrac{\ln 2}{\ln n}+1}\to\dfrac{1}{0+1}\to1$ as $n\to\infty$. Converges

29. $a_n=n^2e^{-n}=\dfrac{n^2}{e^n}$. Since $\lim\limits_{x\to\infty}\dfrac{x^2}{e^x}=\lim\limits_{x\to\infty}\dfrac{2x}{e^x}=\lim\limits_{x\to\infty}\dfrac{2}{e^x}=0$, it follows from Theorem 3 that $\lim\limits_{n\to\infty}a_n=0$. Converges

30. $a_n = n\cos n\pi = n(-1)^n$. Since $\left|a_n\right| = n \to \infty$ as $n \to \infty$, the given sequence diverges.

31. $0 \le \frac{\cos^2 n}{2^n} \le \frac{1}{2^n}$, so since $\lim_{n\to\infty} \frac{1}{2^n} = 0$, $\left\{ \frac{\cos^2 n}{2^n} \right\}$ converges to 0 by the Squeeze Theorem.

32. $a_n = \ln(n+1) - \ln n = \ln\left(\frac{n+1}{n}\right) = \ln\left(1+\frac{1}{n}\right) \to \ln(1) = 0$ as $n \to \infty$. Converges

33. $a_n = n\sin(1/n) = \frac{\sin(1/n)}{1/n}$. Since $\lim_{x\to\infty} \frac{\sin(1/x)}{1/x} = \lim_{t\to 0^+} \frac{\sin t}{t}$ [] where $t = 1/x$]=1 , it follows from Theorem 3 that $\left\{a_n\right\}$ converges to 1.

34. $a_n = \sqrt{n} - \sqrt{n^2 - 1} = \sqrt{n^2 \cdot \frac{1}{n}} - \sqrt{n^2\left(1-\frac{1}{n^2}\right)} = n\left(\frac{1}{\sqrt{n}} - \sqrt{1-\frac{1}{n^2}}\right) \to n(0-1) \to -n$ as $n \to \infty$, so $a_n \to -\infty$ as $n \to \infty$. Diverges

35. $a_n = \left(1+\frac{2}{n}\right)^{1/n} \Rightarrow \ln a_n = \frac{1}{n}\ln\left(1+\frac{2}{n}\right)$. As $n \to \infty$, $\frac{1}{n} \to 0$ and $\ln\left(1+\frac{2}{n}\right) \to 0$, so $\ln a_n \to 0$. Thus, $a_n \to e^0 = 1$ as $n \to \infty$. Converges

36. $a_n = \frac{\sin 2n}{1+\sqrt{n}}$. $\left|a_n\right| \le \frac{1}{1+\sqrt{n}}$ and $\lim_{n\to\infty} \frac{1}{1+\sqrt{n}} = 0$, so $\frac{-1}{1+\sqrt{n}} \le a_n \le \frac{1}{1+\sqrt{n}} \Rightarrow \lim_{n\to\infty} a_n = 0$ by the Squeeze Theorem. Converges

37. $\{0,1,0,0,1,0,0,0,1,\dots\}$ diverges since the sequence takes on only two values, 0 and 1, and never stays arbitrarily close to either one (or any other value) for n sufficiently large.

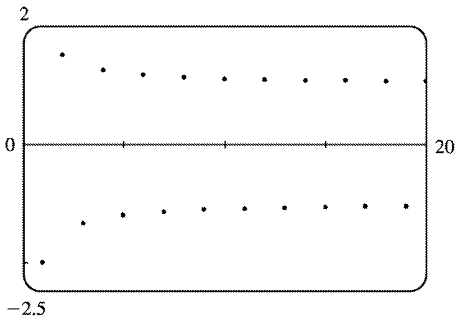
38. $\left\{ \frac{1}{1}, \frac{1}{3}, \frac{1}{2}, \frac{1}{4}, \frac{1}{3}, \frac{1}{5}, \frac{1}{4}, \frac{1}{6}, \dots \right\}$. $a_{2n-1} = \frac{1}{n}$ and $a_{2n} = \frac{1}{n+2}$ for all positive integers n . $\lim_{n\to\infty} a_n = 0$ since $\lim_{n\to\infty} a_{2n-1} = \lim_{n\to\infty} \frac{1}{n} = 0$ and $\lim_{n\to\infty} a_{2n} = \lim_{n\to\infty} \frac{1}{n+2} = 0$. For n sufficiently large, a_n can be made as close to 0 as we like. Converges

39. $a_n = \frac{n!}{2^n} = \frac{1}{2} \cdot \frac{2}{2} \cdot \frac{3}{2} \cdot \cdots \cdot \frac{(n-1)}{2} \cdot \frac{n}{2} \ge \frac{1}{2} \cdot \frac{n}{2} = \frac{n}{4} \to \infty$ as $n \to \infty$, so $\left\{a_n\right\}$ diverges.

40.

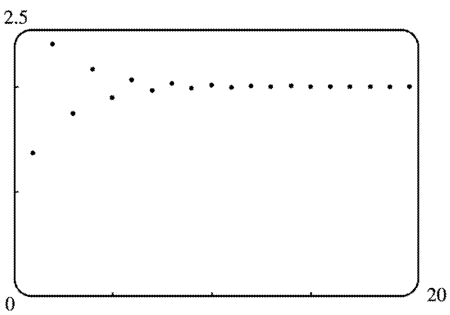
$0<\left|a_n\right|=\frac{3^n}{n!}=\frac{3}{1}\cdot\frac{3}{2}\cdot\frac{3}{3}\cdot\ \cdots\ \cdot\frac{3}{(n-1)}\cdot\frac{3}{n}\le\frac{3}{1}\cdot\frac{3}{2}\cdot\frac{3}{n}=\frac{27}{2n}\rightarrow 0$ as $n\rightarrow\infty$, so by the Squeeze Theorem and Theorem 6, $\left\{(-3)^n/n\right\}$ converges to 0.

41.



From the graph, we see that the sequence $\left\{(-1)^n\frac{n+1}{n}\right\}$ is divergent, since it oscillates between 1 and -1 (approximately).

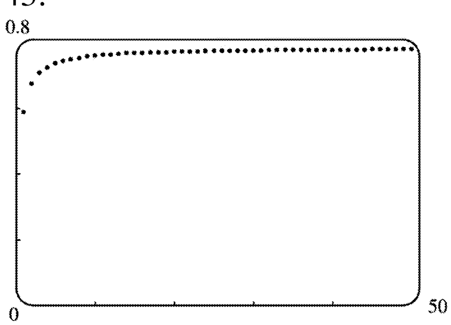
42.



From the graph, it appears that the sequence converges to 2.

$\left\{\left(-\frac{2}{\pi}\right)^n\right\}$ converges to 0 by (6), and hence $\left\{2+\left(-\frac{2}{\pi}\right)^n\right\}$ converges to $2+0=2$.

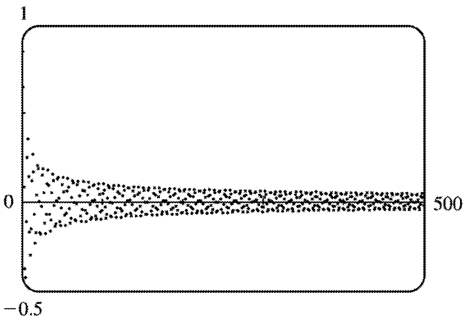
43.



From the graph, it appears that the sequence converges to about 0.78.

$\lim_{n\rightarrow\infty}\frac{2n}{2n+1}=\lim_{n\rightarrow\infty}\frac{2}{2+1/n}=1$, so $\lim_{n\rightarrow\infty}\arctan\left(\frac{2n}{2n+1}\right)=\arctan 1=\frac{\pi}{4}$.

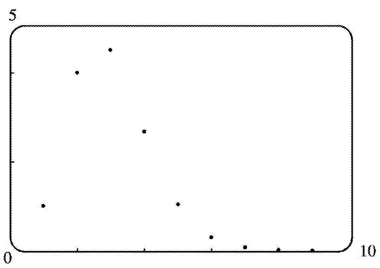
44.



From the graph, it appears that the sequence converges (slowly) to 0 .

$0 \le \frac{|\sin n|}{\sqrt{n}} \le \frac{1}{\sqrt{n}} \to 0$ as $n \to \infty$, so by the Squeeze Theorem and Theorem 6, $\left\{ \frac{\sin n}{\sqrt{n}} \right\}$ converges to 0 .

45.



From the graph, it appears that the sequence converges to 0 .

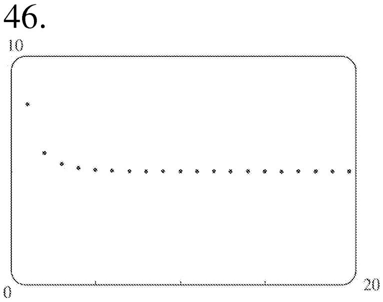
$$0 < a_n = \frac{n^3}{n!} = \frac{n}{n} \cdot \frac{n}{(n-1)} \cdot \frac{n}{(n-2)} \cdot \frac{1}{(n-3)} \cdot \cdot \cdot \cdot \cdot \frac{1}{3} \cdot \frac{1}{2} \cdot \frac{1}{1}$$

$$\le \frac{n^2}{(n-1)(n-2)(n-3)} \quad [\text{for } n \ge 4]$$

$$= \frac{1/n}{(1-1/n)\,(1-2/n)\,(1-3/n)} \to 0 \text{ as } n \to \infty$$

So by the Squeeze Theorem, $\left\{ n^3/n! \right\}$ converges to 0 .

46.



From the graph, it appears that the sequence converges to 5.

$$5=\sqrt[n]{5^n} \le \sqrt[n]{3^n+5^n} \le \sqrt[n]{5^n+5^n} = \sqrt[n]{2}\,\sqrt[n]{5^n}$$
$$= \sqrt[n]{2}\cdot 5 \to 5 \text{ as } n\to\infty \quad \lim_{n\to\infty} 2^{1/n}=2^0=1]$$

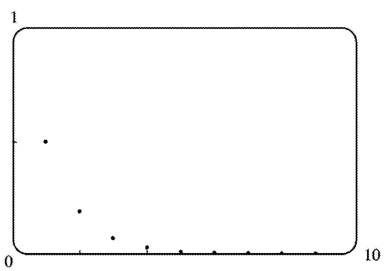
Hence, $a_n \to 5$ by the Squeeze Theorem.

Alternate Solution: Let $y=\left(3^x+5^x\right)^{1/x}$. Then

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\ln\left(3^x+5^x\right)}{x} = \lim_{x\to\infty} \frac{3^x \ln 3+5^x \ln 5}{3^x+5^x}$$
$$= \lim_{x\to\infty} \frac{\left(\frac{3}{5}\right)^x \ln 3+\ln 5}{\left(\frac{3}{5}\right)^x+1} = \ln 5$$

so $\lim_{x\to\infty} y=e^{\ln 5}=5$, and so $\left\{\sqrt[n]{3^n+5^n}\right\}$ converges to 5 .

47.

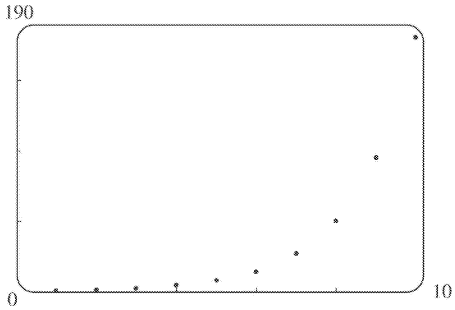


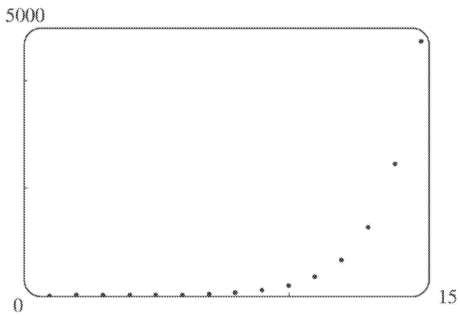
From the graph, it appears that the sequence approaches 0 .

$$0<a_n = \frac{1\cdot 3\cdot 5\cdot \cdots \cdot(2n-1)}{(2n)^n} = \frac{1}{2n}\cdot\frac{3}{2n}\cdot\frac{5}{2n}\cdot \cdots \cdot \frac{2n-1}{2n}$$
$$\le \frac{1}{2n}\cdot(1)\cdot(1)\cdot \cdots \cdot(1) = \frac{1}{2n} \to 0 \quad \text{as} \quad n\to\infty$$

So by the Squeeze Theorem, $\left\{\frac{1\cdot 3\cdot 5\cdot \cdots \cdot(2n-1)}{(2n)^n}\right\}$ converges to 0 .

48.





From the graphs, it seems that the sequence diverges. $a_n = \frac{1 \cdot 3 \cdot 5 \cdot \cdot \cdot \cdot \cdot (2n-1)}{n!}$. We first prove by induction that $a_n \geq \left(\frac{3}{2} \right)^{n-1}$ for all n . This is clearly true for $n=1$, so let $P(n)$ be the statement that the above is true for n . We must show it is then true for $n+1$.
$a_{n+1} = a_n \cdot \frac{2n+1}{n+1} \geq \left(\frac{3}{2} \right)^{n-1} \cdot \frac{2n+1}{n+1}$ (induction hypothesis). But $\frac{2n+1}{n+1} \geq \frac{3}{2}$, and so we get that $a_{n+1} \geq \left(\frac{3}{2} \right)^{n-1} \cdot \frac{3}{2} = \left(\frac{3}{2} \right)^{n}$ which is $P(n+1)$. Thus, we have proved our first assertion, so since $\left\{ \left(\frac{3}{2} \right)^{n-1} \right\}$ diverges (by (8)), so does the given sequence $\left\{ a_n \right\}$.

49. **(a)** $a_n = 1000(1.06)^n \Rightarrow a_1 = 1060$, $a_2 = 1123.60$, $a_3 = 1191.02$, $a_4 = 1262.48$, and $a_5 = 1338.23$.

(b) $\lim_{n \to \infty} a_n = 1000 \lim_{n \to \infty} (1.06)^n$, so the sequence diverges by (8) with $r = 1.06 > 1$.

50. $a_{n+1} = \begin{cases} \frac{1}{2} a_n & \text{if } a_n \text{ is an even number} \\ 3a_n + 1 & \text{if } a_n \text{ is an odd number} \end{cases}$ When $a_1 = 11$, the first 40 terms are 11 , 34 , 17 , 52 , 26 , 13 , 40 , 20 , 10 , 5 , 16 , 8 , 4 , 2 , 1 , 4 , 2 , 1 , 4 , 2 , 1 , 4 , 2 , 1 , 4 , 2 , 1 , 4 , 2 , 1 , 4 , 2 , 1 , 4 , 2 , 1 , 4 , 2 , 1 , 4 . When $a_1 = 25$, the first 40 terms are 25 , 76 , 38 , 19 , 58 , 29 , 88 , 44 , 22 , 11 , 34 , 17 , 52 , 26 , 13 , 40 , 20 , 10 , 5 , 16 , 8 , 4 , 2 , 1 , 4 , 2 , 1 , 4 , 2 , 1 , 4 , 2 , 1 , 4 , 2 , 1 , 4 , 2 , 1 , 4 . The famous Collatz conjecture is that this sequence always reaches 1 , regardless of the

starting point a_1 .

51. If $|r|\geq 1$, then $\{r^n\}$ diverges by (8), so $\{nr^n\}$ diverges also, since $|nr^n|=n|r^n|\geq|r^n|$. If $|r|<1$ then $\lim_{x\to\infty} xr^x=\lim_{x\to\infty}\frac{x}{r^{-x}}=\lim_{x\to\infty}\frac{1}{(-\ln r)r^{-x}}=\lim_{x\to\infty}\frac{r^x}{-\ln r}=0$, so $\lim_{n\to\infty} nr^n=0$, and hence $\{nr^n\}$ converges whenever $|r|<1$.

52. **(a)** Let $\lim_{n\to\infty} a_n=L$. By Definition 1, this means that for every $\varepsilon>0$ there is an integer N such that $|a_n-L|<\varepsilon$ whenever $n>N$. Thus, $|a_{n+1}-L|<\varepsilon$ whenever $n+1>N\Leftrightarrow n>N-1$. It follows that $\lim_{n\to\infty} a_{n+1}=L$ and so $\lim_{n\to\infty} a_n=\lim_{n\to\infty} a_{n+1}$.

(b) If $L=\lim_{n\to\infty} a_n$ then $\lim_{n\to\infty} a_{n+1}=L$ also, so L must satisfy $L=1/(1+L)\Rightarrow L^2+L-1=0\Rightarrow L=\frac{-1+\sqrt{5}}{2}$ (since L has to be non–negative if it exists).

53. Since $\{a_n\}$ is a decreasing sequence, $a_n>a_{n+1}$ for all $n\geq 1$. Because all of its terms lie between 5 and 8 , $\{a_n\}$ is a bounded sequence. By the Monotonic Sequence Theorem, $\{a_n\}$ is convergent; that is, $\{a_n\}$ has a limit L . L must be less than 8 since $\{a_n\}$ is decreasing, so $5\leq L<8$.

54. $a_n=1/5^n$ defines a decreasing geometric sequence since $a_{n+1}=\frac{1}{5}a_n<a_n$ for each $n\geq 1$. The sequence is bounded since $0<a_n\leq\frac{1}{5}$ for all $n\geq 1$.

55. $a_n=\frac{1}{2n+3}$ is decreasing since $a_{n+1}=\frac{1}{2(n+1)+3}=\frac{1}{2n+5}<\frac{1}{2n+3}=a_n$ for each $n\geq 1$. The sequence is bounded since $0<a_n\leq\frac{1}{5}$ for all $n\geq 1$. Note that $a_1=\frac{1}{5}$.

56. $a_n=\frac{2n-3}{3n+4}$ defines an increasing sequence since for $f(x)=\frac{2x-3}{3x+4}$, $f'(x)=\frac{(3x+4)(2)-(2x-3)(3)}{(3x+4)^2}=\frac{17}{(3x+4)^2}>0$. The sequence is bounded since $a_n\geq a_1=-\frac{1}{7}$ for $n\geq 1$, and $a_n<\frac{2n-3}{3n}<\frac{2n}{3n}=\frac{2}{3}$ for $n\geq 1$.

57. $a_n = \cos\left(n\pi/2\right)$ is not monotonic. The first few terms are $0, -1, 0, 1, 0, -1, 0, 1, \ldots$. In fact, the sequence consists of the terms $0, -1, 0, 1$ repeated over and over again in that order. The sequence is bounded since $\left|a_n\right| \le 1$ for all $n \ge 1$.

58. $a_n = ne^{-n}$ defines a positive decreasing sequence since the function $f(x) = xe^{-x}$ is decreasing for $x > 1$. $[f'(x) = e^{-x} - xe^{-x} = e^{-x}(1-x) < 0]$ for $x > 1$.] The sequence is bounded above by $a_1 = \frac{1}{e}$ and below by 0.

59. $a_n = \frac{n}{n^2+1}$ defines a decreasing sequence since for $f(x) = \frac{x}{x^2+1}$, $f'(x) = \frac{\left(x^2+1\right)(1) - x(2x)}{\left(x^2+1\right)^2} = \frac{1-x^2}{\left(x^2+1\right)^2} \le 0$ for $x \ge 1$. The sequence is bounded since $0 < a_n \le \frac{1}{2}$ for all $n \ge 1$.

60. $a_n = n + \frac{1}{n}$ defines an increasing sequence since the function $g(x) = x + \frac{1}{x}$ is increasing for $x > 1$. $[g'(x) = 1 - 1/x^2 > 0]$ for $x > 1$.] The sequence is unbounded since $a_n \to \infty$ as $n \to \infty$. (It is, however, bounded below by $a_1 = 2$.)

61. $a_1 = 2^{1/2}$, $a_2 = 2^{3/4}$, $a_3 = 2^{7/8}$, $\ldots$, so $a_n = 2^{\left(2^n-1\right)/2^n} = 2^{1-\left(1/2^n\right)}$. $\lim_{n\to\infty} a_n = \lim_{n\to\infty} 2^{1-\left(1/2^n\right)} = 2^1 = 2$.
Alternate solution : Let $L = \lim_{n\to\infty} a_n$. (We could show the limit exists by showing that $\left\{a_n\right\}$ is bounded and increasing.) Then L must satisfy $L = \sqrt{2 \cdot L} \Rightarrow L^2 = 2L \Rightarrow L(L-2) = 0$. $L \ne 0$ since the sequence increases, so $L = 2$.

62. **(a)** Let P_n be the statement that $a_{n+1} \ge a_n$ and $a_n \le 3$. P_1 is obviously true. We will assume that P_n is true and then show that as a consequence P_{n+1} must also be true. $a_{n+2} \ge a_{n+1} \Leftrightarrow \sqrt{2+a_{n+1}} \ge \sqrt{2+a_n} \Leftrightarrow 2+a_{n+1} \ge 2+a_n \Leftrightarrow a_{n+1} \ge a_n$, which is the induction hypothesis. $a_{n+1} \le 3 \Leftrightarrow \sqrt{2+a_n} \le 3 \Leftrightarrow 2+a_n \le 9 \Leftrightarrow a_n \le 7$, which is certainly true because we are assuming that $a_n \le 3$. So P_n is true for all n , and so $a_1 \le a_n \le 3$ (showing that the sequence is bounded), and hence by the Monotonic Sequence Theorem, $\lim_{n\to\infty} a_n$ exists.
(b) If

$L=\lim_{n\to\infty} a_n$, then $\lim_{n\to\infty} a_{n+1}=L$ also, so $L=\sqrt{2+L} \Rightarrow L^2=2+L \Leftrightarrow L^2-L-2=0 \Leftrightarrow (L+1)(L-2)=0 \Leftrightarrow L=2$ (since L can't be negative).

63. We show by induction that $\{a_n\}$ is increasing and bounded above by 3 .
Let P_n be the proposition that $a_{n+1}>a_n$ and $0<a_n<3$. Clearly P_1 is true. Assume that P_n is true. Then $a_{n+1}>a_n \Rightarrow \frac{1}{a_{n+1}} < \frac{1}{a_n} \Rightarrow -\frac{1}{a_{n+1}} > -\frac{1}{a_n}$.
Now $a_{n+2}=3-\frac{1}{a_{n+1}} >3-\frac{1}{a_n} =a_{n+1} \Leftrightarrow P_{n+1}$. This proves that $\{a_n\}$ is increasing and bounded above by 3 , so $1=a_1<a_n<3$, that is, $\{a_n\}$ is bounded, and hence convergent by the Monotonic Sequence Theorem.
If $L=\lim_{n\to\infty} a_n$, then $\lim_{n\to\infty} a_{n+1}=L$ also, so L must satisfy $L=3-1/L \Rightarrow L^2-3L+1=0 \Rightarrow L=\frac{3\pm\sqrt{5}}{2}$. But $L>1$, so $L=\frac{3+\sqrt{5}}{2}$.

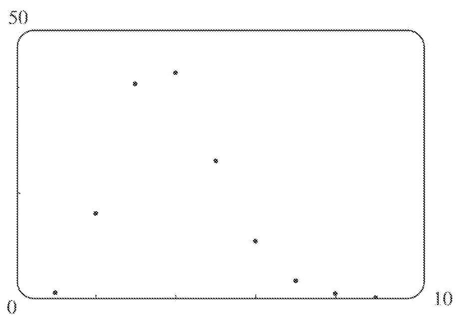
64. We use induction. Let P_n be the statement that $0<a_{n+1}\le a_n\le 2$. Clearly P_1 is true, since $a_2=1/(3-2)=1$. Now assume that P_n is true. Then $a_{n+1}\le a_n \Rightarrow -a_{n+1}\ge -a_n \Rightarrow 3-a_{n+1}\ge 3-a_n \Rightarrow$
$a_{n+2}=\frac{1}{3-a_{n+1}} \le \frac{1}{3-a_n} =a_{n+1}$. Also $a_{n+2}>0$ (since $3-a_{n+1}$ is positive) and $a_{n+1}\le 2$ by the induction hypothesis, so P_{n+1} is true.
To find the limit, we use the fact that $\lim_{n\to\infty} a_n=\lim_{n\to\infty} a_{n+1} \Rightarrow L=\frac{1}{3-L} \Rightarrow L^2-3L+1=0 \Rightarrow L=\frac{3\pm\sqrt{5}}{2}$. But $L\le 2$, so we must have $L=\frac{3-\sqrt{5}}{2}$.

65. **(a)** Let a_n be the number of rabbit pairs in the n th month. Clearly $a_1=1=a_2$. In the n th month, each pair that is 2 or more months old (that is, a_{n-2} pairs) will produce a new pair to add to the a_{n-1} pairs already present. Thus, $a_n=a_{n-1}+a_{n-2}$, so that $\{a_n\} = \{f_n\}$, the Fibonacci sequence.
(b) $a_n=\frac{f_{n+1}}{f_n} \Rightarrow a_{n-1}=\frac{f_n}{f_{n-1}} = \frac{f_{n-1}+f_{n-2}}{f_{n-1}} =1+\frac{f_{n-2}}{f_{n-1}} =1+\frac{1}{f_{n-1}/f_{n-2}} =1+\frac{1}{a_{n-2}}$. If $L=\lim_{n\to\infty} a_n$, then

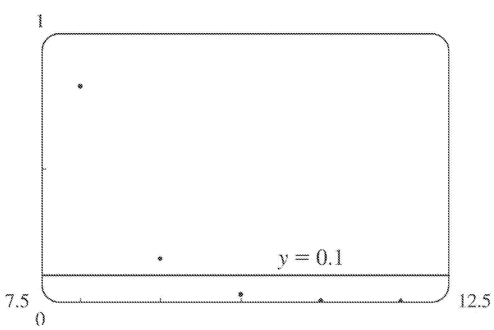
$L=\lim_{n\to\infty} a_{n-1}$ and $L=\lim_{n\to\infty} a_{n-2}$, so L must satisfy $L=1+\frac{1}{L} \Rightarrow L^2-L-1=0 \Rightarrow L=\frac{1+\sqrt{5}}{2}$ (since L must be positive).

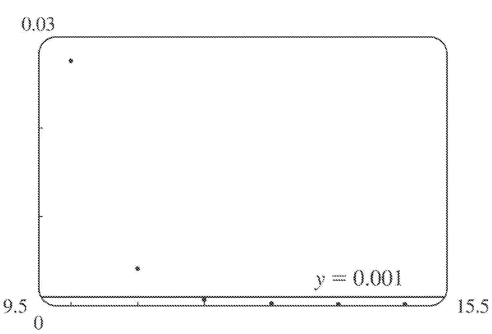
66. **(a)** If f is continuous, then $f(L)=f\left(\lim_{n\to\infty} a_n\right)=\lim_{n\to\infty} f\left(a_n\right)=\lim_{n\to\infty} a_{n+1}=L$ by Exercise 52(a).

(b) By repeatedly pressing the cosine key on the calculator (that is, taking cosine of the previous answer) until the displayed value stabilizes, we see that $L\approx 0.73909$.

67. **(a)** 

From the graph, it appears that the sequence $\left\{\frac{n^5}{n!}\right\}$ converges to 0 , that is, $\lim_{n\to\infty} \frac{n^5}{n!}=0$.

(b) 



From the first graph, it seems that the smallest possible value of N corresponding to $\varepsilon=0.1$ is 9 , since $n^5/n!<0.1$ whenever $n\geq 10$, but $9^5/9!>0.1$. From the second graph, it seems that for $\varepsilon=0.001$, the smallest possible value for N is 11 .

68. Let $\varepsilon>0$ and let N be any positive integer larger than $\ln(\varepsilon)/\ln|r|$. If $n>N$ then $n>\ln(\varepsilon)/\ln|r|$ $\Rightarrow n\ln|r|<\ln\varepsilon \Rightarrow \ln\left(|r|^n\right)<\ln\varepsilon \Rightarrow |r|^n<\varepsilon \Rightarrow \left|r^n-0\right|<\varepsilon$, and so by Definition 1, $\lim_{n\to\infty} r^n=0$.

69. If $\lim_{n\to\infty}\left|a_n\right|=0$ then $\lim_{n\to\infty}-\left|a_n\right|=0$, and since $-\left|a_n\right|\le a_n\le\left|a_n\right|$, we have that $\lim_{n\to\infty} a_n=0$ by the Squeeze Theorem.

70. **(a)**

$$\frac{b^{n+1}-a^{n+1}}{b-a}=b^n+b^{n-1}a+b^{n-2}a^2+b^{n-3}a^3+\cdot\cdot\cdot+ba^{n-1}+a^n$$
$$<b^n+b^{n-1}b+b^{n-2}b^2+b^{n-3}b^3+\cdot\cdot\cdot+bb^{n-1}+b^n=(n+1)b^n$$

(b) Since $b-a>0$, we have $b^{n+1}-a^{n+1}<(n+1)b^n(b-a)\Rightarrow b^{n+1}-(n+1)b^n(b-a)<a^{n+1}\Rightarrow b^n\left[(n+1)a-nb\right]<a^{n+1}$.

(c) With this substitution, $(n+1)a-nb=1$, and so $b^n=\left(1+\frac{1}{n}\right)^n<a^{n+1}=\left(1+\frac{1}{n+1}\right)^{n+1}$.

(d) With this substitution, we get $\left(1+\frac{1}{2n}\right)^n\left(\frac{1}{2}\right)<1\Rightarrow\left(1+\frac{1}{2n}\right)^n<2\Rightarrow\left(1+\frac{1}{2n}\right)^{2n}<4$.

(e) $a_n<a_{2n}$ since $\left\{a_n\right\}$ is increasing, so $a_n<a_{2n}<4$.

(f) Since $\left\{a_n\right\}$ is increasing and bounded above by 4 , $a_1\le a_n\le 4$, and so $\left\{a_n\right\}$ is bounded and monotonic, and hence has a limit by Theorem 11.

71. **(a)** First we show that $a>a_1>b_1>b$.

$a_1-b_1=\frac{a+b}{2}-\sqrt{ab}=\frac{1}{2}\left(a-2\sqrt{ab}+b\right)=\frac{1}{2}\left(\sqrt{a}-\sqrt{b}\right)^2>0$ (since $a>b$) $\Rightarrow a_1>b_1$. Also $a-a_1=a-\frac{1}{2}(a+b)=\frac{1}{2}(a-b)>0$ and $b-b_1=b-\sqrt{ab}=\sqrt{b}\left(\sqrt{b}-\sqrt{a}\right)<0$, so $a>a_1>b_1>b$. In the same way we can show that $a_1>a_2>b_2>b_1$ and so the given assertion is true for $n=1$. Suppose it is true for $n=k$, that is, $a_k>a_{k+1}>b_{k+1}>b_k$. Then

$$\begin{aligned}a_{k+2}-b_{k+2} &= \frac{1}{2}\left(a_{k+1}+b_{k+1}\right)-\sqrt{a_{k+1}b_{k+1}}=\frac{1}{2}\left(a_{k+1}-2\sqrt{a_{k+1}b_{k+1}}+b_{k+1}\right)\\ &= \frac{1}{2}\left(\sqrt{a_{k+1}}-\sqrt{b_{k+1}}\right)^2>0\end{aligned}$$

$$a_{k+1}-a_{k+2} = a_{k+1}-\frac{1}{2}\left(a_{k+1}+b_{k+1}\right)=\frac{1}{2}\left(a_{k+1}-b_{k+1}\right)>0$$

and $b_{k+1}-b_{k+2}=b_{k+1}-\sqrt{a_{k+1}b_{k+1}}=\sqrt{b_{k+1}}\left(\sqrt{b_{k+1}}-\sqrt{a_{k+1}}\right)<0\Rightarrow a_{k+1}>a_{k+2}>b_{k+2}>b_{k+1}$, so the assertion is true for $n=k+1$. Thus, it is true for all n by mathematical induction.

(b) From part (a) we have $a>a_n>a_{n+1}>b_{n+1}>b_n>b$, which shows that both sequences, $\left\{a_n\right\}$ and $\left\{b_n\right\}$, are monotonic and bounded. So they are both convergent by the Monotonic Sequence Theorem.

(c) Let $\lim_{n\to\infty} a_n=\alpha$ and $\lim_{n\to\infty} b_n=\beta$. Then $\lim_{n\to\infty} a_{n+1}=\lim_{n\to\infty}\frac{a_n+b_n}{2}\Rightarrow\alpha=\frac{\alpha+\beta}{2}\Rightarrow$ $2\alpha=\alpha+\beta\Rightarrow\alpha=\beta$.

72. **(a)** Let $\varepsilon>0$. Since $\lim_{n\to\infty} a_{2n}=L$, there exists N_1 such that $\left|a_{2n}-L\right|<\varepsilon$ for $n>N_1$. Since $\lim_{n\to\infty} a_{2n+1}=L$, there exists N_2 such that $\left|a_{2n+1}-L\right|<\varepsilon$ for $n>N_2$. Let $N=\max\left\{2N_1,2N_2+1\right\}$ and let $n>N$. If n is even, then $n=2m$ where $m>N_1$, so $\left|a_n-L\right|=\left|a_{2m}-L\right|<\varepsilon$. If n is odd, then $n=2m+1$, where $m>N_2$, so $\left|a_n-L\right|=\left|a_{2m+1}-L\right|<\varepsilon$. Therefore $\lim_{n\to\infty} a_n=L$.

(b) $a_1=1$, $a_2=1+\frac{1}{1+1}=\frac{3}{2}=1.5$, $a_3=1+\frac{1}{5/2}=\frac{7}{5}=1.4$, $a_4=1+\frac{1}{12/5}=\frac{17}{12}=1.41\overline{6}$, $a_5=1+\frac{1}{29/12}=\frac{41}{29}\approx1.413793$, $a_6=1+\frac{1}{70/29}=\frac{99}{70}\approx1.414286$, $a_7=1+\frac{1}{169/70}=\frac{239}{169}\approx1.414201$, $a_8=1+\frac{1}{408/169}=\frac{577}{408}\approx1.414216$. Notice that $a_1<a_3<a_5<a_7$ and $a_2>a_4>a_6>a_8$. It appears that the odd terms are increasing and the even terms are decreasing. Let's prove that $a_{2n-2}>a_{2n}$ and $a_{2n-1}<a_{2n+1}$ by mathematical induction. Suppose that $a_{2k-2}>a_{2k}$. Then $1+a_{2k-2}>1+a_{2k}\Rightarrow\frac{1}{1+a_{2k-2}}<\frac{1}{1+a_{2k}}\Rightarrow$

$1+\frac{1}{1+a_{2k-2}}<1+\frac{1}{1+a_{2k}}\Rightarrow a_{2k-1}<a_{2k+1}\Rightarrow1+a_{2k-1}<1+a_{2k+1}\Rightarrow\frac{1}{1+a_{2k-1}}>\frac{1}{1+a_{2k+1}}\Rightarrow$

$1+\frac{1}{1+a_{2k-1}}>1+\frac{1}{1+a_{2k+1}}\Rightarrow a_{2k}>a_{2k+2}$. We have thus shown, by induction, that the odd terms are increasing and the even terms are decreasing. Also all terms lie between 1 and 2 , so both $\left\{a_n\right\}$ and $\left\{b_n\right\}$ are bounded monotonic sequences and are therefore convergent by Theorem 11. Let $\lim_{n\to\infty} a_{2n}=L$. Then $\lim_{n\to\infty} a_{2n+2}=L$ also. We have $a_{n+2}=1+\frac{1}{1+1+1/(1+a_n)}=1+\frac{1}{(3+2a_n)/(1+a_n)}=\frac{4+3a_n}{3+2a_n}$, so

$a_{2n+2}=\dfrac{4+3a_{2n}}{3+2a_{2n}}$. Taking limits of both sides, we get $L=\dfrac{4+3L}{3+2L}\Rightarrow 3L+2L^2=4+3L\Rightarrow L^2=2\Rightarrow L=\sqrt{2}$ (since $L>0$). Thus, $\lim_{n\to\infty} a_{2n}=\sqrt{2}$. Similarly we find that $\lim_{n\to\infty} a_{2n+1}=\sqrt{2}$. So, by part (a), $\lim_{n\to\infty} a_n=\sqrt{2}$.

73. **(a)** Suppose $\{p_n\}$ converges to p . Then

$$p_{n+1}=\frac{bp_n}{a+p_n}\Rightarrow \lim_{n\to\infty} p_{n+1}=\frac{b\lim_{n\to\infty} p_n}{a+\lim_{n\to\infty} p_n}\Rightarrow p=\frac{bp}{a+p}\Rightarrow p^2+ap=bp\Rightarrow p(p+a-b)=0\Rightarrow p=0 \text{ or } p=b-a .$$

(b) $p_{n+1}=\dfrac{bp_n}{a+p_n}=\dfrac{\frac{b}{a}p_n}{1+\frac{p_n}{a}}<\dfrac{b}{a}p_n$ since $1+\dfrac{p_n}{a}>1$.

(c) By part (b), $p_1<\left(\frac{b}{a}\right)p_0$, $p_2<\left(\frac{b}{a}\right)p_1<\left(\frac{b}{a}\right)^2p_0$, $p_3<\left(\frac{b}{a}\right)p_2<\left(\frac{b}{a}\right)^3p_0$, etc. In general, $p_n<\left(\frac{b}{a}\right)^np_0$, so $\lim_{n\to\infty} p_n\le\lim_{n\to\infty}\left(\frac{b}{a}\right)^n\cdot p_0=0$ since $b<a$.

(d) Let $a<b$. We first show, by induction, that if $p_0<b-a$, then $p_n<b-a$ and $p_{n+1}>p_n$.

For $n=0$, we have $p_1-p_0=\dfrac{bp_0}{a+p_0}-p_0=\dfrac{p_0(b-a-p_0)}{a+p_0}>0$ since $p_0<b-a$. So $p_1>p_0$.

Now we suppose the assertion is true for $n=k$, that is, $p_k<b-a$ and $p_{k+1}>p_k$. Then

$b-a-p_{k+1}=b-a-\dfrac{bp_k}{a+p_k}=\dfrac{a(b-a)+bp_k-ap_k-bp_k}{a+p_k}=\dfrac{a(b-a-p_k)}{a+p_k}>0$ because $p_k<b-a$. So $p_{k+1}<b-a$. And $p_{k+2}-p_{k+1}=\dfrac{bp_{k+1}}{a+p_{k+1}}-p_{k+1}=\dfrac{p_{k+1}(b-a-p_{k+1})}{a+p_{k+1}}>0$ since $p_{k+1}<b-a$. Therefore, $p_{k+2}>p_{k+1}$. Thus, the assertion is true for $n=k+1$. It is therefore true for all n by mathematical induction. A similar proof by induction shows that if $p_0>b-a$, then $p_n>b-a$ and $\{p_n\}$ is decreasing. In either case the sequence $\{p_n\}$ is bounded and monotonic, so it is convergent by the Monotonic Sequence Theorem. It then follows from part (a) that $\lim_{n\to\infty} p_n=b-a$.

1. Using Theorem 5 with $\sum_{n=0}^{\infty} b_n (x-5)^n$, $b_n = \dfrac{f^{(n)}(a)}{n!}$, so $b_8 = \dfrac{f^{(8)}(5)}{8!}$.

2. **(a)** Using Formula 6, a power series expansion of f at 1 must have the form $f(1)+f'(1)(x-1)+\cdots$. Comparing to the given series, $1.6-0.8(x-1)+\cdots$, we must have $f'(1)=-0.8$. But from the graph, $f'(1)$ is positive. Hence, the given series is *not* the Taylor series of f centered at 1 .

(b) A power series expansion of f at 2 must have the form $f(2)+f'(2)(x-2)+\frac{1}{2}f''(2)(x-2)^2+\cdots$. Comparing to the given series, $2.8+0.5(x-2)+1.5(x-2)^2-0.1(x-2)^3+\cdots$, we must have $\frac{1}{2}f''(2)=1.5$; that is, $f''(2)$ is positive. But from the graph, f is concave downward near $x=2$, so $f''(2)$ must be negative. Hence, the given series is *not* the Taylor series of f centered at 2 .

3.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\cos x$	1
1	$-\sin x$	0
2	$-\cos x$	-1
3	$\sin x$	0
4	$\cos x$	1
⋮	⋮	⋮

We use Equation 7 with $f(x)=\cos x$.

$$\cos x = f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\frac{f^{(3)}(0)}{3!}x^3+\frac{f^{(4)}(0)}{4!}x^4+\cdots$$

$$=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\cdots=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}$$

If $a_n=\dfrac{(-1)^n x^{2n}}{(2n)!}$, then

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{x^{2n+2}}{(2n+2)!}\cdot\frac{(2n)!}{x^{2n}}\right|=x^2\lim_{n\to\infty}\frac{1}{(2n+2)(2n+1)}=0<1 \text{ for all } x .$$

So $R=\infty$ (Ratio Test).

4.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\sin 2x$	0
1	$2\cos 2x$	2
2	$-2^2\sin 2x$	0
3	$-2^3\cos 2x$	-2^3
4	$2^4\sin 2x$	0
⋮	⋮	⋮

$f^{(n)}(0)=0$ if n is even and $f^{(2n+1)}(0)=(-1)^n 2^{2n+1}$, so

$$\sin 2x = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{f^{(2n+1)}(0)}{(2n+1)!} x^{2n+1}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1} x^{2n+1}}{(2n+1)!}$$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{2^2 |x|^2}{(2n+3)(2n+2)} = 0<1 \text{ for all } x,$$

so $R=\infty$ (Ratio Test).

5.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$(1+x)^{-3}$	1
1	$-3(1+x)^{-4}$	-3
2	$12(1+x)^{-5}$	12
3	$-60(1+x)^{-6}$	-60
4	$360(1+x)^{-7}$	360
⋮	⋮	⋮

$$(1+x)^{-3} = f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3+\frac{f^{(4)}(0)}{4!}x^4+\cdots$$

$$=1-3x+\frac{4\cdot 3}{2!}x^2-\frac{5\cdot 4\cdot 3}{3!}x^3+\frac{6\cdot 5\cdot 4\cdot 3}{4!}x^4-\cdots$$

$$=1-3x+\frac{4\cdot 3\cdot 2}{2\cdot 2!}x^2-\frac{5\cdot 4\cdot 3\cdot 2}{2\cdot 3!}x^3+\frac{6\cdot 5\cdot 4\cdot 3\cdot 2}{2\cdot 4!}x^4-\cdots$$

$$=\sum_{n=0}^{\infty}\frac{(-1)^n(n+2)!\,x^n}{2(\mathrm{n}!)}=\sum_{n=0}^{\infty}\frac{(-1)^n(n+2)(n+1)x^n}{2}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{(n+3)(n+2)x^{n+1}}{2}\cdot\frac{2}{(n+2)(n+1)x^n}\right|=|x|\lim_{n\to\infty}\frac{n+3}{n+1}=|x|<1$$ for convergence,

so $R=1$ (Ratio Test).

6.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\ln(1+x)$	0
1	$(1+x)^{-1}$	1
2	$-(1+x)^{-2}$	-1
3	$2(1+x)^{-3}$	2
4	$-6(1+x)^{-4}$	-6
5	$24(1+x)^{-5}$	24
⋮	⋮	⋮

$$\ln(1+x) = f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3$$

$$+\frac{f^{(4)}(0)}{4!}x^4+\frac{f^{(5)}(0)}{5!}x^5+\cdots$$

$$=x-\frac{1}{2}x^2+\frac{2}{6}x^3-\frac{6}{24}x^4+\frac{24}{120}x^5-\cdots$$

$$=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\frac{x^5}{5}-\cdots=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}x^n$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{n+1}\cdot\frac{n}{x^n}\right| = \lim_{n\to\infty}\frac{|x|}{1+1/n} = |x| < 1 \text{ for convergence, so } R=1 .$$

7.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	e^{5x}	1
1	$5e^{5x}$	5
2	5^2e^{5x}	25
3	5^3e^{5x}	125
4	5^4e^{5x}	625
⋮	⋮	⋮

$$e^{5x} = \sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n = \sum_{n=0}^{\infty}\frac{5^n}{n!}x^n .$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left[\frac{5^{n+1}|x|^{n+1}}{(n+1)!}\cdot\frac{n!}{5^n|x|^n}\right]$$

$$= \lim_{n\to\infty}\frac{5|x|}{n+1} = 0 < 1 \text{ for all } x, \text{ so } R=\infty .$$

8.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	xe^x	0
1	$(x+1)e^x$	1
2	$(x+2)e^x$	2
3	$(x+3)e^x$	3
⋮	⋮	⋮

$$xe^x = \sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n = \sum_{n=0}^{\infty}\frac{n}{n!}x^n = \sum_{n=1}^{\infty}\frac{n}{n!}x^n = \sum_{n=1}^{\infty}\frac{x^n}{(n-1)!} .$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left[\frac{|x|^{n+1}}{n!}\cdot\frac{(n-1)!}{|x|^n}\right]=\lim_{n\to\infty}\frac{|x|}{n}=0<1 \text{ for all } x \text{ , so } R=\infty \text{ .}$$

9.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	sinh x	0
1	cosh x	1
2	sinh x	0
3	cosh x	1
4	sinh x	0
. . .	. . .	. . .

$$f^{(n)}(0)=\begin{cases} 0 & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd}\end{cases} \qquad \text{so } \sinh x=\sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!} \text{ .}$$

Use the Ratio Test to find R . If $a_n=\dfrac{x^{2n+1}}{(2n+1)!}$, then

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{2n+3}}{(2n+3)!}\cdot\frac{(2n+1)!}{x^{2n+1}}\right|$$

$$= x^2\cdot\lim_{n\to\infty}\frac{1}{(2n+3)(2n+2)}=0<1$$

for all x , so $R=\infty$.

10.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	cosh x	1
1	sinh x	0
2	cosh x	1
3	sinh x	0
. . .	. . .	. . .

$$f^{(n)}(0)=\begin{cases} 1 & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd}\end{cases} \qquad \text{so } \cosh x=\sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!} \text{ .}$$

Use the Ratio Test to find R . If

$a_n = \dfrac{x^{2n}}{(2n)!}$, then

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{x^{2n+2}}{(2n+2)!} \cdot \frac{(2n)!}{x^{2n}} \right|$$

$$= x^2 \cdot \lim_{n\to\infty} \frac{1}{(2n+2)(2n+1)} = 0 < 1$$

for all x , so $R=\infty$.

11.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
1	$1+2x$	5
2	2	2
3	0	0
4	0	0
. . .	. . .	. . .

$$f(x) = 7+5(x-2)+\frac{2}{2!}(x-2)^2+\sum_{n=3}^{\infty}\frac{0}{n!}(x-2)^n$$

$$= 7+5(x-2)+(x-2)^2$$

Since $a_n=0$ for large n , $R=\infty$.

12.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	x^3	-1
1	$3x^2$	3
2	$6x$	-6
3	6	6
4	0	0
5	0	0
. . .	. . .	. . .

$$f(x) = -1+3(x+1) - \frac{6}{2!}(x+1)^2 + \frac{6}{3!}(x+1)^3$$

$$= -1+3(x+1) - 3(x+1)^2 + (x+1)^3$$

Since $a_n = 0$ for large n , $R=\infty$.

13. Clearly, $f^{(n)}(x)=e^x$, so $f^{(n)}(3)=e^3$ and $e^x = \sum_{n=0}^{\infty} \frac{e^3}{n!}(x-3)^n$. If $a_n = \frac{e^3}{n!}(x-3)^n$, then

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{e^3(x-3)^{n+1}}{(n+1)!}\cdot\frac{n!}{e^3(x-3)^n}\right| = \lim_{n\to\infty}\frac{|x-3|}{n+1} = 0<1$$ for all x , so $R=\infty$.

14.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\ln x$	$\ln 2$
1	x^{-1}	$\frac{1}{2}$
2	$-x^{-2}$	$-\frac{1}{4}$
3	$2x^{-3}$	$\frac{2}{8}$
4	$-3\cdot 2x^{-4}$	$-\frac{3\cdot 2}{16}$
⋮	⋮	⋮

$f^{(n)}(2) = \frac{(-1)^{n-1}(n-1)!}{2^n}$ for $n \geq 1$, so $\ln x = \ln 2 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(x-2)^n}{n\cdot 2^n}$.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{|x-2|}{2}\lim_{n\to\infty}\frac{n}{n+1} = \frac{|x-2|}{2} < 1$$ for convergence, so $|x-2|<2 \Rightarrow R=2$.

15.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\cos x$	-1

1	$-\sin x$	0
2	$-\cos x$	1
3	$\sin x$	0
4	$\cos x$	-1
. . .	. . .	. . .

$$\cos x=\sum_{k=0}^{\infty}\frac{f^{(k)}(\pi)}{k!}(x-\pi)^k=-1+\frac{(x-\pi)^2}{2!}-\frac{(x-\pi)^4}{4!}+\frac{(x-\pi)^6}{6!}-\cdots=\sum_{n=0}^{\infty}(-1)^{n+1}\frac{(x-\pi)^{2n}}{(2n)!}\ .$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left[\frac{|x-\pi|^{2n+2}}{(2n+2)!}\cdot\frac{(2n)!}{|x-\pi|^{2n}}\right]=\lim_{n\to\infty}\frac{|x-\pi|^2}{(2n+2)(2n+1)}=0<1 \text{ for all } x \text{ , so } R=\infty\ .$$

16.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\sin x$	1
1	$\cos x$	0
2	$-\sin x$	-1
3	$-\cos x$	0
4	$\sin x$	1
. . .	. . .	. . .

$$\sin x=\sum_{k=0}^{\infty}\frac{f^{(k)}(\pi/2)}{k!}\left(x-\frac{\pi}{2}\right)^k$$

$$=1-\frac{(x-\pi/2)^2}{2!}+\frac{(x-\pi/2)^4}{4!}-\frac{(x-\pi/2)^6}{6!}+\cdots$$

$$=\sum_{n=0}^{\infty}(-1)^n\frac{(x-\pi/2)^{2n}}{(2n)!}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left[\frac{|x-\pi/2|^{2n+2}}{(2n+2)!}\cdot\frac{(2n)!}{|x-\pi/2|^{2n}}\right]=\lim_{n\to\infty}\frac{|x-\pi/2|^2}{(2n+2)(2n+1)}=0<1 \text{ for all } x \text{ , so } R=\infty\ .$$

17.

n	$f^{(n)}(x)$	$f^{(n)}(0)$

0	$x^{-1/2}$	$\frac{1}{3}$
1	$-\frac{1}{2}x^{-3/2}$	$-\frac{1}{2}\cdot\frac{1}{3^3}$
2	$\frac{3}{4}x^{-5/2}$	$-\frac{1}{2}\cdot\left(-\frac{3}{2}\right)\cdot\frac{1}{3^5}$
3	$-\frac{15}{8}x^{-7/2}$	$-\frac{1}{2}\cdot\left(-\frac{3}{2}\right)\cdot\left(-\frac{5}{2}\right)\cdot\frac{1}{3^7}$
⋮	⋮	⋮

$$\frac{1}{\sqrt{x}}=\frac{1}{3}-\frac{1}{2\cdot 3^3}(x-9)+\frac{3}{2^2\cdot 3^5}\frac{(x-9)^2}{2!}-\frac{3\cdot 5}{2^3\cdot 3^7}\frac{(x-9)^3}{3!}+\cdots$$

$$=\sum_{n=0}^{\infty}(-1)^n\frac{1\cdot 3\cdot 5\cdot\,\cdots\,\cdot(2n-1)}{2^n\cdot 3^{2n+1}\cdot n!}(x-9)^n\ .$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left[\frac{1\cdot 3\cdot 5\cdot\,\cdots\,\cdot(2n-1)[2(n+1)-1]\,|x-9|^{n+1}}{2^{n+1}\cdot 3^{[2(n+1)+1]}\cdot(n+1)!}\cdot\frac{2^n\cdot 3^{2n+1}\square!}{1\cdot 3\cdot 5\cdot\,\cdots\,\cdot(2n-1)\,|x-9|^n}\right]$$

$$=\lim_{n\to\infty}\left[\frac{(2n+1)|x-9|}{2\cdot 3^2(n+1)}\right]=\frac{1}{9}\,|x-9|<1$$

for convergence, so $|x-9|<9$ and $R=9$.

18.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	x^{-2}	1
1	$-2x^{-3}$	-2
2	$6x^{-4}$	6

3	$-24x^{-5}$	-24
4	$120x^{-6}$	120
. . .	. . .	. . .

$$x^{-2}=1-2(x-1)+6\cdot\frac{(x-1)^2}{2!}-24\cdot\frac{(x-1)^3}{3!}+120\cdot\frac{(x-1)^4}{4!}-\cdots$$
$$=1-2(x-1)+3(x-1)^2-4(x-1)^3+5(x-1)^4-\cdots$$
$$=\sum_{n=0}^{\infty}(-1)^n(n+1)(x-1)^n .$$

$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\frac{(n+2)|x-1|^{n+1}}{(n+1)|x-1|^n}=\lim_{n\to\infty}\left[\frac{n+2}{n+1}\cdot|x-1|\right]=|x-1|<1$ for convergence, so $R=1$.

19. If $f(x)=\cos x$, then $f^{(n+1)}(x)=\pm\sin x$ or $\pm\cos x$. In each case, $\left|f^{(n+1)}(x)\right|\le 1$, so by Formula 9 with $a=0$ and $M=1$, $\left|R_n(x)\right|\le\frac{1}{(n+1)!}|x|^{n+1}$. Thus, $\left|R_n(x)\right|\to 0$ as $n\to\infty$ by Equation 10. So $\lim_{n\to\infty}R_n(x)=0$ and, by Theorem 8, the series in Exercise 3 represents $\cos x$ for all x .

20. If $f(x)=\sin x$, then $f^{(n+1)}(x)=\pm\sin x$ or $\pm\cos x$. In each case, $\left|f^{(n+1)}(x)\right|\le 1$, so by Formula 9 with $a=0$ and $M=1$, $\left|R_n(x)\right|\le\frac{1}{(n+1)!}\left|x-\frac{\pi}{2}\right|^{n+1}$. Thus, $\left|R_n(x)\right|\to 0$ as $n\to\infty$ by Equation 10. So $\lim_{n\to\infty}R_n(x)=0$ and, by Theorem 8, the series in Exercise 16 represents $\sin x$ for all x .

21. If $f(x)=\sinh x$, then for all n , $f^{(n+1)}(x)=\cosh x$ or $\sinh x$. Since $|\sinh x|<|\cosh x|=\cosh x$ for all x , we have $\left|f^{(n+1)}(x)\right|\le\cosh x$ for all n . If d is any positive number and $|x|\le d$, then $\left|f^{(n+1)}(x)\right|\le\cosh x\le\cosh d$, so by Formula 9 with $a=0$ and $M=\cosh d$, we have $\left|R_n(x)\right|\le\frac{\cosh d}{(n+1)!}|x|^{n+1}$. It follows that $\left|R_n(x)\right|\to 0$ as $n\to\infty$ for $|x|\le d$ (by Equation 10). But d was an arbitrary positive number. So by Theorem 8, the series represents $\sinh x$ for all x .

22. If $f(x)=\cosh x$, then for all n , $f^{(n+1)}(x)=\cosh x$ or $\sinh x$. Since $|\sinh x|<|\cosh x|=\cosh x$ for all x , we have $\left|f^{(n+1)}(x)\right|\le\cosh x$ for all n . If d is any positive number and $|x|\le d$, then

$\left|f^{(n+1)}(x)\right| \leq \cosh x \leq \cosh d$, so by Formula 9 with $a=0$ and $M=\cosh d$, we have $\left|R_n(x)\right| \leq \frac{\cosh d}{(n+1)!}\ |x|^{n+1}$. It follows that $\left|R_n(x)\right| \to 0$ as $n \to \infty$ for $|x| \leq d$ (by Equation 10). But d was an arbitrary positive number. So by Theorem 8, the series represents $\cosh x$ for all x .

23. $\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \Rightarrow f(x) = \cos(\pi x) = \sum_{n=0}^{\infty} \frac{(-1)^n (\pi x)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n} x^{2n}}{(2n)!}$, $R=\infty$

24. $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \Rightarrow f(x) = e^{-x/2} = \sum_{n=0}^{\infty} \frac{(-x/2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n!} x^n$, $R=\infty$

25. $\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \Rightarrow f(x) = x \tan^{-1} x = x \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+2}}{2n+1}$, $R=1$

26. $\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \Rightarrow f(x) = \sin\left(x^4\right) = \sum_{n=0}^{\infty} (-1)^n \frac{\left(x^4\right)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{8n+4}$, $R=\infty$

27. $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \Rightarrow f(x) = x^2 e^{-x} = x^2 \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+2}}{n!}$, $R=\infty$

28. $\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \Rightarrow \cos 2x = \sum_{n=0}^{\infty} (-1)^n \frac{(2x)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n}}{(2n)!} x^{2n} \Rightarrow$
$f(x) = x \cos 2x = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n}}{(2n)!} x^{2n+1}$, $R=\infty$

29.

$$\sin^2 x = \frac{1}{2}(1-\cos 2x) = \frac{1}{2}\left[1 - \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n}}{(2n)!}\right] = \frac{1}{2}\left[1 - 1 - \sum_{n=1}^{\infty} \frac{(-1)^n (2x)^{2n}}{(2n)!}\right]$$
$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 2^{2n-1} x^{2n}}{(2n)!}, \quad R=\infty$$

30.

$$\cos^2 x = \frac{1}{2}(1+\cos 2x) = \frac{1}{2}\left[1 + \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n}}{(2n)!}\right] = \frac{1}{2}\left[1 + 1 + \sum_{n=1}^{\infty} \frac{(-1)^n 2^{2n} x^{2n}}{(2n)!}\right]$$

$$=1+\sum_{n=1}^{\infty}\frac{(-1)^n 2^{2n-1}x^{2n}}{(2n)!}\quad, R=\infty$$

Another method: Use $\cos^2 x=1-\sin^2 x$ and Exercise 29.

31. $\frac{\sin x}{x}=\frac{1}{x}\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n+1)!}$ and this series also gives the required value at $x=0$ (namely 1); $R=\infty$.

32.
$\frac{x-\sin x}{x^3}=\frac{1}{x^3}\left[x-\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}\right]=\frac{1}{x^3}\left[x-x-\sum_{n=1}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}\right]=\frac{1}{x^3}\left[-\sum_{n=0}^{\infty}\frac{(-1)^{n+1} x^{2n+3}}{(2n+3)!}\right]$
$=\frac{1}{x^3}\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+3}}{(2n+3)!}=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n+3)!}$ and this series also gives the required value at $x=0$ (namely 1/6); $R=\infty$.

33.

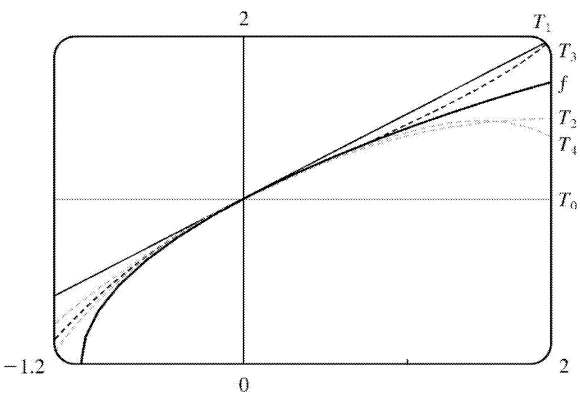
n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$(1+x)^{1/2}$	1
1	$\frac{1}{2}(1+x)^{-1/2}$	$\frac{1}{2}$
2	$-\frac{1}{4}(1+x)^{-3/2}$	$-\frac{1}{4}$
3	$\frac{3}{8}(1+x)^{-5/2}$	$\frac{3}{8}$
4	$-\frac{15}{16}(1+x)^{-7/2}$	$-\frac{15}{16}$
⋮	⋮	⋮

So $f^{(n)}(0)=\frac{(-1)^{n-1}1\cdot 3\cdot 5\cdot\cdots\cdot(2n-3)}{2^n}$ for $n\geq 2$, and

$\sqrt{1+x}=1+\frac{x}{2}+\sum_{n=2}^{\infty}\frac{(-1)^{n-1}1\cdot 3\cdot 5\cdot\cdots\cdot(2n-3)}{2^n n!}x^n$. If $a_n=\frac{(-1)^{n-1}1\cdot 3\cdot 5\cdot\cdots\cdot(2n-3)}{2^n n!}x^n$,

then $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-3)(2n-1)x^{n+1}}{2^{n+1}(n+1)!}\cdot\frac{2^n n!}{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-3)x^n}\right|$

$$=\frac{|x|}{2}\lim_{n\to\infty}\frac{2n-1}{n+1}=\frac{|x|}{2}\cdot 2=|x|<1 \text{ for convergence, so } R=1 .$$

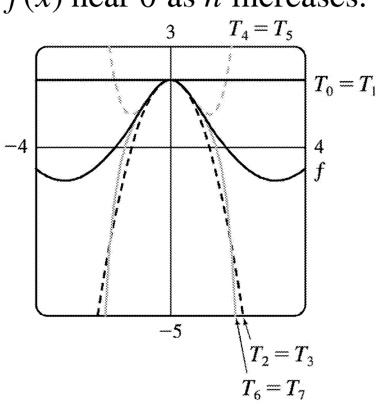


Notice that, as n increases, $T_n(x)$ becomes a better approximation to $f(x)$ for $-1<x<1$.

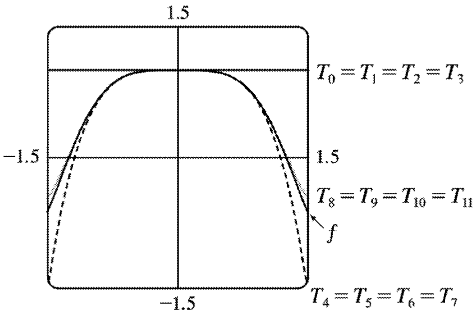
34. $e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}$, so $e^{-x^2}=\sum_{n=0}^{\infty}\frac{(-x^2)^n}{n!}=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{n!}$. Also, $\cos x=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}$, so

$f(x)=e^{-x^2}+\cos x=\sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{n!}+\frac{1}{(2n)!}\right)x^{2n}=2-\frac{3}{2}x^2+\frac{13}{24}x^4-\frac{121}{720}x^6+\cdots$.

The series for e^x and $\cos x$ converge for all x , so the same is true of the series for $f(x)$; that is, $R=\infty$. From the graphs of f and the first few Taylor polynomials, we see that $T_n(x)$ provides a closer fit to $f(x)$ near 0 as n increases.



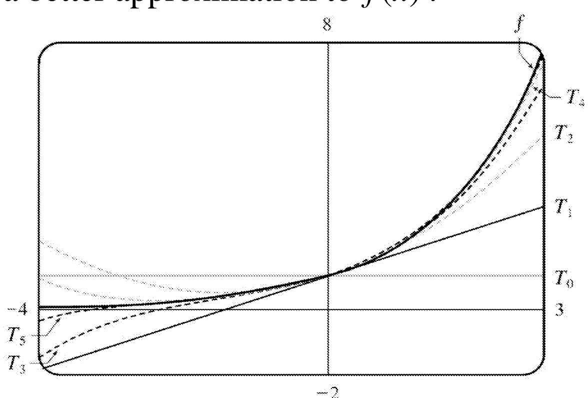
35. $\cos x = \sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n}}{(2n)!} \Rightarrow f(x) = \cos\left(x^2\right) = \sum_{n=0}^{\infty} \dfrac{(-1)^n \left(x^2\right)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \dfrac{(-1)^n x^{4n}}{(2n)!}$, $R=\infty$



Notice that, as n increases, $T_n(x)$ becomes a better approximation to $f(x)$.

36. $2^x = \left(e^{\ln 2}\right)^x = e^{x\ln 2} = \sum_{n=0}^{\infty} \dfrac{(x\ln 2)^n}{n!} = \sum_{n=0}^{\infty} \dfrac{(\ln 2)^n x^n}{n!}$, $R=\infty$.

Notice that, as n increases, $T_n(x)$ becomes
a better approximation to $f(x)$.



37. $e^x = \sum_{n=0}^{\infty} \dfrac{x^n}{n!}$, so

$e^{-0.2} = \sum_{n=0}^{\infty} \dfrac{(-0.2)^n}{n!} = 1 - 0.2 + \dfrac{1}{2!}(0.2)^2 - \dfrac{1}{3!}(0.2)^3 + \dfrac{1}{4!}(0.2)^4 - \dfrac{1}{5!}(0.2)^5 + \dfrac{1}{6!}(0.2)^6 - \cdots$. But

$\dfrac{1}{6!}(0.2)^6 = 8.\overline{8} \times 10^{-8}$, so by the Alternating Series Estimation Theorem, $e^{-0.2} \approx \sum_{n=0}^{5} \dfrac{(-0.2)^n}{n!} \approx 0.81873$
, correct to five decimal places.

38. $3^\circ = \dfrac{\pi}{60}$ radians and $\sin x = \sum_{n=0}^{\infty} \dfrac{(-1)^n x^{2n+1}}{(2n+1)!}$, so

$\sin\frac{\pi}{60}=\frac{\pi}{60}-\frac{\left(\frac{\pi}{60}\right)^3}{3!}+\frac{\left(\frac{\pi}{60}\right)^5}{5!}-\cdots=\frac{\pi}{60}-\frac{\pi^3}{1{,}296{,}000}+\frac{\pi^5}{93{,}312{,}000{,}000}-\cdots$. But $\frac{\pi^5}{93{,}312{,}000{,}000}<10^{-8}$, so by the Alternating Series Estimation Theorem,

$\sin\frac{\pi}{60}\approx\frac{\pi}{60}-\frac{\pi^3}{1{,}296{,}000}\approx 0.05234$.

39. $\cos x=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}\Rightarrow\cos(x^3)=\sum_{n=0}^{\infty}(-1)^n\frac{(x^3)^{2n}}{(2n)!}=\sum_{n=0}^{\infty}(-1)^n\frac{x^{6n}}{(2n)!}\Rightarrow$

$x\cos(x^3)=\sum_{n=0}^{\infty}(-1)^n\frac{x^{6n+1}}{(2n)!}\Rightarrow\int x\cos(x^3)\,dx=C+\sum_{n=0}^{\infty}(-1)^n\frac{x^{6n+2}}{(6n+2)(2n)!}$, with $R=\infty$.

40. $\frac{\sin x}{x}=\frac{1}{x}\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n+1)!}$, so

$\int\frac{\sin x}{x}\,dx=\int\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n+1)!}\,dx=C+\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)(2n+1)!}$

41. Using the series from Exercise 33 and substituting x^3 for x , we get

$$\int\sqrt{x^3+1}\,dx=\int\left[1+\frac{x^3}{2}+\sum_{n=2}^{\infty}\frac{(-1)^{n-1}1\cdot3\cdot5\cdot\cdots\cdot(2n-3)}{2^n n!}x^{3n}\right]dx$$

$$=C+x+\frac{x^4}{8}+\sum_{n=2}^{\infty}\frac{(-1)^{n-1}1\cdot3\cdot5\cdot\cdots\cdot(2n-3)}{2^n n!(3n+1)}x^{3n+1}$$

42. $e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}\Rightarrow e^x-1=\sum_{n=1}^{\infty}\frac{x^n}{n!}\Rightarrow\frac{e^x-1}{x}=\sum_{n=1}^{\infty}\frac{x^{n-1}}{n!}\Rightarrow$

$\int\frac{e^x-1}{x}\,dx=C+\sum_{n=1}^{\infty}\frac{x^n}{n\cdot n!}$, with $R=\infty$.

43. By Exercise 39, $\int x\cos(x^3)\,dx=C+\sum_{n=0}^{\infty}(-1)^n\frac{x^{6n+2}}{(6n+2)(2n)!}$, so $\int_0^1 x\cos(x^3)\,dx$

$=\left[\sum_{n=0}^{\infty}(-1)^n\frac{x^{6n+2}}{(6n+2)(2n)!}\right]_0^1=\sum_{n=0}^{\infty}\frac{(-1)^n}{(6n+2)(2n)!}=\frac{1}{2}-\frac{1}{8\cdot2!}+\frac{1}{14\cdot4!}-\frac{1}{20\cdot6!}+\cdots$, but

$\frac{1}{20\cdot6!}=\frac{1}{14{,}400}\approx0.000069$, so

$\int_0^1 x\cos(x^3)\,dx \approx \frac{1}{2} - \frac{1}{16} + \frac{1}{336} \approx 0.440$ (correct to three decimal places) by the Alternating Series Estimation Theorem.

44. From the table of Maclaurin series in Section .10, we see that

$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$ for x in $[-1,1]$ and $\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$ for all real numbers x, so

$\tan^{-1}(x^3) + \sin(x^3) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{6n+3}}{2n+1} + \sum_{n=0}^{\infty} (-1)^n \frac{x^{6n+3}}{(2n+1)!}$ for x^3 in $[-1,1] \Leftrightarrow x$ in $[-1,1]$. Thus,

$$I = \int_0^{0.2} dx = \int_0^{0.2} \sum_{n=0}^{\infty} (-1)^n x^{6n+3} \left(\frac{1}{2n+1} + \frac{1}{(2n+1)!} \right) dx$$

$$= \left[\sum_{n=0}^{\infty} (-1)^n \frac{x^{6n+4}}{6n+4} \left(\frac{1}{2n+1} + \frac{1}{(2n+1)!} \right) \right]_0^{0.2}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{(0.2)^{6n+4}}{6n+4} \left(\frac{1}{2n+1} + \frac{1}{(2n+1)!} \right) = \frac{(0.2)^4}{4}(1+1) - \frac{(0.2)^{10}}{10} \left(\frac{1}{3} + \frac{1}{3!} \right) + \cdots .$$

But $\frac{(0.2)^{10}}{10} \left(\frac{1}{3} + \frac{1}{3!} \right) = \frac{(0.2)^{10}}{20} = 5.12 \times 10^{-9}$, so by the Alternating Series Estimation Theorem,

$I \approx \frac{(0.2)^4}{2} = 0.000\,80$ (correct to five decimal places).

45. We first find a series representation for $f(x) = (1+x)^{-1/2}$, and then substitute.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$(1+x)^{-1/2}$	1
1	$-\frac{1}{2}(1+x)^{-3/2}$	$-\frac{1}{2}$
2	$\frac{3}{4}(1+x)^{-5/2}$	$\frac{3}{4}$
3	$-\frac{15}{8}(1+x)^{-7/2}$	$-\frac{15}{8}$
⋮	⋮	⋮

$\frac{1}{\sqrt{1+x}}=1-\frac{x}{2}+\frac{3}{4}\left(\frac{x^2}{2!}\right)-\frac{15}{8}\left(\frac{x^3}{3!}\right)+\cdots \Rightarrow \frac{1}{\sqrt{1+x^3}}=1-\frac{1}{2}x^3+\frac{3}{8}x^6-\frac{5}{16}x^9+\cdots \Rightarrow$

$\int_0^{0.1}\frac{dx}{\sqrt{1+x^3}}=\left[x-\frac{1}{8}x^4+\frac{3}{56}x^7-\frac{1}{32}x^{10}+\cdots\right]_0^{0.1}\approx(0.1)-\frac{1}{8}(0.1)^4$, by the Alternating Series Estimation Theorem, since $\frac{3}{56}(0.1)^7\approx 0.0000000054<10^{-8}$, which is the maximum desired error.

Therefore, $\int_0^{0.1}\frac{dx}{\sqrt{1+x^3}}\approx 0.09998750$.

46. $\int_0^{0.5} x^2 e^{-x^2}dx=\int_0^{0.5}\Sigma_{n=0}^{\infty}\frac{(-1)^n x^{2n+2}}{n!}dx=\Sigma_{n=0}^{\infty}\left[\frac{(-1)^n x^{2n+3}}{n!(2n+3)}\right]_0^{1/2}=\Sigma_{n=0}^{\infty}\frac{(-1)^n}{n!(2n+3)2^{2n+3}}$ and since the term with $n=2$ is $\frac{1}{1792}<0.001$, we use $\Sigma_{n=0}^{1}\frac{(-1)^n}{n!(2n+3)2^{2n+3}}=\frac{1}{24}-\frac{1}{160}\approx 0.0354$.

47.

$$\lim_{x\to 0}\frac{x-\tan^{-1}x}{x^3}=\lim_{x\to 0}\frac{x-\left(x-\frac{1}{3}x^3+\frac{1}{5}x^5-\frac{1}{7}x^7+\cdots\right)}{x^3}=\lim_{x\to 0}\frac{\frac{1}{3}x^3-\frac{1}{5}x^5+\frac{1}{7}x^7-\cdots}{x^3}$$

$$=\lim_{x\to 0}\left(\frac{1}{3}-\frac{1}{5}x^2+\frac{1}{7}x^4-\cdots\right)=\frac{1}{3}$$

since power series are continuous functions.

48.

$$\lim_{x\to 0}\frac{1-\cos x}{1+x-e^x}=\lim_{x\to 0}\frac{1-\left(1-\frac{1}{2!}x^2+\frac{1}{4!}x^4-\frac{1}{6!}x^6+\cdots\right)}{1+x-\left(1+x+\frac{1}{2!}x^2+\frac{1}{3!}x^3+\frac{1}{4!}x^4+\frac{1}{5!}x^5+\frac{1}{6!}x^6+\cdots\right)}$$

$$=\lim_{x\to 0}\frac{\frac{1}{2!}x^2-\frac{1}{4!}x^4+\frac{1}{6!}x^6-\cdots}{-\frac{1}{2!}x^2-\frac{1}{3!}x^3-\frac{1}{4!}x^4-\frac{1}{5!}x^5-\frac{1}{6!}x^6-\cdots}$$

$$=\lim_{x\to 0}\frac{\frac{1}{2!}-\frac{1}{4!}x^2+\frac{1}{6!}x^4-\cdots}{-\frac{1}{2!}-\frac{1}{3!}x-\frac{1}{4!}x^2-\frac{1}{5!}x^3-\frac{1}{6!}x^4-\cdots}=\frac{\frac{1}{2}-0}{-\frac{1}{2}-0}=-1$$

since power series are continuous functions.

49.

$$\lim_{x\to 0}\frac{\sin x-x+\frac{1}{6}x^3}{x^5}=\lim_{x\to 0}\frac{\left(x-\frac{1}{3!}x^3+\frac{1}{5!}x^5-\frac{1}{7!}x^7+\cdots\right)-x+\frac{1}{6}x^3}{x^5}$$

$$=\lim_{x\to 0}\frac{\frac{1}{5!}x^5-\frac{1}{7!}x^7+\cdots}{x^5}=\lim_{x\to 0}\left(\frac{1}{5!}-\frac{x^2}{7!}+\frac{x^4}{9!}-\cdots\right)=\frac{1}{5!}=\frac{1}{120}$$

since power series are continuous functions.

50.

$$\lim_{x\to 0}\frac{\tan x-x}{x^3}=\lim_{x\to 0}\frac{\left(x+\frac{1}{3}x^3+\frac{2}{15}x^5+\cdots\right)-x}{x^3}=\lim_{x\to 0}\frac{\frac{1}{3}x^3+\frac{2}{15}x^5+\cdots}{x^3}=\lim_{x\to 0}\left(\frac{1}{3}+\frac{2}{15}x^2+\cdots\right)=\frac{1}{3}$$

since power series are continuous functions.

51. As in Example 8(a), we have $e^{-x^2}=1-\frac{x^2}{1!}+\frac{x^4}{2!}-\frac{x^6}{3!}+\cdots$ and we know that $\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\cdots$ from Equation 16. Therefore, $e^{-x^2}\cos x=\left(1-x^2+\frac{1}{2}x^4-\cdots\right)\left(1-\frac{1}{2}x^2+\frac{1}{24}x^4-\cdots\right)$. Writing only the terms with degree ≤ 4, we get

$$e^{-x^2}\cos x=1-\frac{1}{2}x^2+\frac{1}{24}x^4-x^2+\frac{1}{2}x^4+\frac{1}{2}x^4+\cdots=1-\frac{3}{2}x^2+\frac{25}{24}x^4+\cdots .$$

52.

$$
\begin{array}{r|l}
 & 1+\frac{1}{2}x^2+\frac{1}{24}x^4-\cdots \\ \hline
1-\frac{1}{2}x^2+\frac{5}{24}x^4-\cdots & 1 \\
 & 1-\frac{1}{2}x^2+\frac{1}{24}x^4-\cdots \\ \hline
 & \frac{1}{2}x^2-\frac{1}{24}x^4+\cdots \\
 & \frac{1}{2}x^2-\frac{1}{4}x^4+\cdots \\ \hline
 & \frac{5}{24}x^4+\cdots \\
 & \frac{5}{24}x^4+\cdots \\ \hline
 & \cdots
\end{array}
$$

$\sec x=\dfrac{1}{\cos x}=\dfrac{1}{1-\frac{1}{2}x^2+\frac{1}{24}x^4-\cdots}$. From the long division above, $\sec x=1+\frac{1}{2}x^2+\frac{5}{24}x^4+\cdots$.

53.

$$
\begin{array}{r|l}
 & 1+\frac{1}{6}x^2+\frac{7}{360}x^4+\cdots \\ \hline
x-\frac{1}{6}x^3+\frac{1}{120}x^5-\cdots & x \\
 & x-\frac{1}{6}x^3+\frac{1}{120}x^5-\cdots \\ \hline
 & \frac{1}{6}x^3-\frac{1}{120}x^5+\cdots \\
 & \frac{1}{6}x^3-\frac{1}{36}x^5+\cdots \\ \hline
 & \frac{7}{360}x^5+\cdots \\
 & \frac{7}{360}x^5+\cdots \\ \hline
 & \cdots
\end{array}
$$

$\frac{x}{\sin x}=\frac{x}{x-\frac{1}{6}x^3+\frac{1}{120}x^5-\cdots}$. From the long division above, $\frac{x}{\sin x}=1+\frac{1}{6}x^2+\frac{7}{360}x^4+\cdots$.

54. From Example 6 in Section .9, we have $\ln(1-x)=-x-\frac{1}{2}x^2-\frac{1}{3}x^3-\cdots$, $|x|<1$. Therefore,

$$\begin{aligned} e^x\ln(1-x) &= \left(1+x+\frac{1}{2}x^2+\cdots\right)\left(-x-\frac{1}{2}x^2-\frac{1}{3}x^3-\cdots\right) \\ &= -x-\frac{1}{2}x^2-\frac{1}{3}x^3-x^2-\frac{1}{2}x^3-\frac{1}{2}x^3-\cdots \\ &= -x-\frac{3}{2}x^2-\frac{4}{3}x^3-\cdots, |x|<1 \end{aligned}$$

55. $\sum_{n=0}^{\infty}(-1)^n\frac{x^{4n}}{n!}=\sum_{n=0}^{\infty}\frac{\left(-x^4\right)^n}{n!}=e^{-x^4}$, by (11).

56. $\sum_{n=0}^{\infty}\frac{(-1)^n\pi^{2n}}{6^{2n}(2n)!}=\sum_{n=0}^{\infty}(-1)^n\frac{\left(\frac{\pi}{6}\right)^{2n}}{(2n)!}=\cos\frac{\pi}{6}=\frac{\sqrt{3}}{2}$, by (16).

57. $\sum_{n=0}^{\infty}\frac{(-1)^n\pi^{2n+1}}{4^{2n+1}(2n+1)!}=\sum_{n=0}^{\infty}\frac{(-1)^n\left(\frac{\pi}{4}\right)^{2n+1}}{(2n+1)!}=\sin\frac{\pi}{4}=\frac{1}{\sqrt{2}}$, by (15).

58. $\sum_{n=0}^{\infty}\frac{3^n}{5^n n!}=\sum_{n=0}^{\infty}\frac{(3/5)^n}{n!}=e^{3/5}$, by (11).

59. $3+\frac{9}{2!}+\frac{27}{3!}+\frac{81}{4!}+\cdots=\frac{3^1}{1!}+\frac{3^2}{2!}+\frac{3^3}{3!}+\frac{3^4}{4!}+\cdots=\sum_{n=1}^{\infty}\frac{3^n}{n!}=\sum_{n=0}^{\infty}\frac{3^n}{n!}-1=e^3-1$, by (11).

60. $1-\ln 2+\frac{(\ln 2)^2}{2!}-\frac{(\ln 2)^3}{3!}+\cdots=\sum_{n=0}^{\infty}\frac{(-\ln 2)^n}{n!}=e^{-\ln 2}=\left(e^{\ln 2}\right)^{-1}=2^{-1}=\frac{1}{2}$, by (11).

61. Assume that $|f'''(x)|\le M$, so $f'''(x)\le M$ for $a\le x\le a+d$. Now $\int_a^x f'''(t)\,dt\le\int_a^x M\,dt\Rightarrow$ $f''(x)-f''(a)\le M(x-a)\Rightarrow f''(x)\le f''(a)+M(x-a)$. Thus, $\int_a^x f''(t)\,dt\le\int_a^x\left[f''(a)+M(t-a)\right]dt$ $\Rightarrow$

$f'(x)-f'(a)\le f''(a)(x-a)+\frac{1}{2}M(x-a)^2 \Rightarrow f'(x)\le f'(a)+f''(a)(x-a)+\frac{1}{2}M(x-a)^2 \Rightarrow$

$\int_a^x f'(t)\,dt \le \int_a^x \left[f'(a)+f''(a)(t-a)+\frac{1}{2}M(t-a)^2 \right] dt \Rightarrow$

$f(x)-f(a)\le f'(a)(x-a)+\frac{1}{2}f''(a)(x-a)^2+\frac{1}{6}M(x-a)^3$. So

$f(x)-f(a)-f'(a)(x-a)-\frac{1}{2}f''(a)(x-a)^2\le \frac{1}{6}M(x-a)^3$. But

$R_2(x)=f(x)-T_2(x)=f(x)-f(a)-f'(a)(x-a)-\frac{1}{2}f''(a)(x-a)^2$, so $R_2(x)\le \frac{1}{6}M(x-a)^3$.

A similar argument using $f'''(x)\ge -M$ shows that $R_2(x)\ge -\frac{1}{6}M(x-a)^3$. So

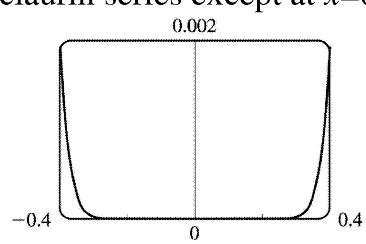
$\left|R_2\left(x_2\right)\right| \le \frac{1}{6}M|x-a|^3$.

Although we have assumed that $x>a$, a similar calculation shows that this inequality is also true if $x<a$.

62. **(a)** $f(x)=\begin{cases} e^{-1/x^2} & \text{if } x\ne 0 \\ 0 & \text{if } x=0 \end{cases}$ so

$f'(0)=\lim_{x\to 0}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0}\frac{e^{-1/x^2}}{x}=\lim_{x\to 0}\frac{1/x}{e^{1/x^2}}=\lim_{x\to 0}\frac{x}{2e^{1/x^2}}=0$ (using l'Hospital's Rule and simplifying in the penultimate step). Similarly, we can use the definition of the derivative and l'Hospital's Rule to show that $f''(0)=0$, $f^{(3)}(0)=0$, … , $f^{(n)}(0)=0$, so that the Maclaurin series for f consists entirely of zero terms. But since $f(x)\ne 0$ except for $x=0$, we see that f cannot equal its Maclaurin series except at $x=0$.

(b)



From the graph, it seems that the function is extremely flat at the origin. In fact, it could be said to be ‘‘infinitely flat’’ at $x=0$, since all of its derivatives are 0 there.

1. The general binomial series in (2) is

$$(1+x)^k=\sum_{n=0}^{\infty}\binom{k}{n}x^n=1+kx+\frac{k(k-1)}{2!}x^2+\frac{k(k-1)(k-2)}{3!}x^3+\cdots\ .$$

$$(1+x)^{1/2} = \sum_{n=0}^{\infty}\binom{\frac{1}{2}}{n}x^n=1+\left(\frac{1}{2}\right)x+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}x^2+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}x^3+\cdots$$

$$= 1+\frac{x}{2}-\frac{x^2}{2^2\cdot 2!}+\frac{1\cdot 3\cdot x^3}{2^3\cdot 3!}-\frac{1\cdot 3\cdot 5\cdot x^4}{2^4\cdot 4!}+\cdots$$

$$= 1+\frac{x}{2}+\sum_{n=2}^{\infty}\frac{(-1)^{n-1}1\cdot 3\cdot 5\cdot\cdots\cdot(2n-3)x^n}{2^n\cdot n!}\ \text{for}\ |x|<1,\text{so}\ R=1$$

2. $\frac{1}{(1+x)^4}=(1+x)^{-4}=\sum_{n=0}^{\infty}\binom{-4}{n}x^n$. The binomial coefficient is

$$\binom{-4}{n} = \frac{(-4)(-5)(-6)\cdot\cdots\cdot(-4-n+1)}{n!}=\frac{(-4)(-5)(-6)\cdot\cdots\cdot[-(n+3)]}{n!}$$

$$= \frac{(-1)^n\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot\cdots\cdot(n+1)(n+2)(n+3)}{2\cdot 3\cdot n!}=\frac{(-1)^n(n+1)(n+2)(n+3)}{6}$$

Thus, $\frac{1}{(1+x)^4}=\sum_{n=0}^{\infty}\frac{(-1)^n(n+1)(n+2)(n+3)}{6}x^n$ for $|x|<1$, so $R=1$.

3. $\frac{1}{(2+x)^3}=\frac{1}{[2(1+x/2)]^3}=\frac{1}{8}\left(1+\frac{x}{2}\right)^{-3}=\frac{1}{8}\sum_{n=0}^{\infty}\binom{-3}{n}\left(\frac{x}{2}\right)^n$. The binomial coefficient is

$$\binom{-3}{n} = \frac{(-3)(-4)(-5)\cdot\cdots\cdot(-3-n+1)}{n!}=\frac{(-3)(-4)(-5)\cdot\cdots\cdot[-(n+2)]}{n!}$$

$$= \frac{(-1)^n\cdot 2\cdot 3\cdot 4\cdot 5\cdot\cdots\cdot(n+1)(n+2)}{2\cdot n!}=\frac{(-1)^n(n+1)(n+2)}{2}$$

Thus, $\frac{1}{(2+x)^3}=\frac{1}{8}\sum_{n=0}^{\infty}\frac{(-1)^n(n+1)(n+2)}{2}\frac{x^n}{2^n}=\sum_{n=0}^{\infty}\frac{(-1)^n(n+1)(n+2)x^n}{2^{n+4}}$ for $\left|\frac{x}{2}\right|<1\Leftrightarrow$ $|x|<2$, so $R=2$.

4.

$$(1-x)^{2/3}=\sum_{n=0}^{\infty}\binom{\frac{2}{3}}{n}(-x)^n$$

$$=1+\frac{2}{3}(-x)+\frac{\frac{2}{3}\left(-\frac{1}{3}\right)}{2!}(-x)^2+\frac{\frac{2}{3}\left(-\frac{1}{3}\right)\left(-\frac{4}{3}\right)}{3!}(-x)^3+\cdot\cdot\cdot$$

$$=1-\frac{2}{3}x+\sum_{n=2}^{\infty}\frac{(-1)^{n-1}(-1)^n\cdot 2\cdot[1\cdot 4\cdot 7\cdot\cdot\cdot\cdot\cdot(3n-5)]}{3^n\cdot n!}x^n$$

$$=1-\frac{2}{3}x-2\sum_{n=2}^{\infty}\frac{1\cdot 4\cdot 7\cdot\cdot\cdot\cdot\cdot(3n-5)}{3^n\cdot n!}x^n$$

and $|-x|<1\Leftrightarrow|x|<1$, so $R=1$.

5.

$$\sqrt[4]{1-8x}=(1-8x)^{1/4}=\sum_{n=0}^{\infty}\binom{\frac{1}{4}}{n}(-8x)^n$$

$$=1+\frac{1}{4}(-8x)+\frac{\frac{1}{4}\left(-\frac{3}{4}\right)}{2!}(-8x)^2+\frac{\left(\frac{1}{4}\right)\left(-\frac{3}{4}\right)\left(-\frac{7}{4}\right)}{3!}(-8x)^3+\cdot\cdot\cdot$$

$$=1-2x+\sum_{n=2}^{\infty}\frac{(-1)^n(-1)^{n-1}\cdot 3\cdot 7\cdot\cdot\cdot\cdot\cdot(4n-5)8^n}{4^n\cdot n!}x^n$$

$$=1-2x-\sum_{n=2}^{\infty}\frac{3\cdot 7\cdot\cdot\cdot\cdot\cdot(4n-5)2^n}{n!}x^n$$

and $|-8x|<1\Leftrightarrow|x|<\frac{1}{8}$, so $R=\frac{1}{8}$.

6.

$$\frac{1}{\sqrt[5]{32-x}}=\frac{1}{2\sqrt[5]{1-x/32}}=\frac{1}{2}\left(1-\frac{x}{32}\right)^{-1/5}=\frac{1}{2}\sum_{n=0}^{\infty}\binom{-\frac{1}{5}}{n}\left(-\frac{x}{32}\right)^n=\frac{1}{2}\sum_{n=0}^{\infty}\binom{-\frac{1}{5}}{n}\frac{(-1)^n x^n}{2^{5n}}$$

$$=\frac{1}{2}\left[1+\left(-\frac{1}{5}\right)\left(-\frac{x}{2^5}\right)+\frac{\left(-\frac{1}{5}\right)\left(-\frac{6}{5}\right)}{2!}\frac{x^2}{2^{10}}+\frac{\left(-\frac{1}{5}\right)\left(-\frac{6}{5}\right)\left(-\frac{11}{5}\right)}{3!}\left(-\frac{x^3}{2^{15}}\right)+\cdot\cdot\cdot\right.$$

$$= \frac{1}{2} + \frac{1}{5\cdot 2^6}x + \frac{1\cdot 6}{5^2\cdot 2!\cdot 2^{11}}x^2 + \frac{1\cdot 6\cdot 11}{5^3\cdot 3!\cdot 2^{16}}x^3 + \cdots = \frac{1}{2} + \sum_{n=1}^{\infty}\frac{1\cdot 6\cdot\cdots\cdot(5n-4)}{5^n 2^{5n+1} n!}x^n$$

The radius of convergence is 32 .

7. We must write the binomial in the form (1+ expression), so we'll factor out a 4 .

$$\frac{x}{\sqrt{4+x^2}} = \frac{x}{\sqrt{4\left(1+x^2/4\right)}} = \frac{x}{2\sqrt{1+x^2/4}} = \frac{x}{2}\left(1+\frac{x^2}{4}\right)^{-1/2} = \frac{x}{2}\sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}\left(\frac{x^2}{4}\right)^n$$

$$= \frac{x}{2}\left[1+\left(-\frac{1}{2}\right)\frac{x^2}{4} + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}\left(\frac{x^2}{4}\right)^2 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!}\left(\frac{x^2}{4}\right)^3 + \cdots\right]$$

$$= \frac{x}{2} + \frac{x}{2}\sum_{n=1}^{\infty}(-1)^n\frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{2^n\cdot 4^n\cdot n!}x^{2n}$$

$$= \frac{x}{2} + \sum_{n=1}^{\infty}(-1)^n\frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{n!\,2^{3n+1}}x^{2n+1} \text{ and } \frac{x^2}{4}<1 \Leftrightarrow \frac{|x|}{2}<1 \Leftrightarrow$$

$|x|<2$, so $R=2$.

8.

$$\frac{x^2}{\sqrt{2+x}} = \frac{x^2}{\sqrt{2(1+x/2)}} = \frac{x^2}{\sqrt{2}}\left(1+\frac{x}{2}\right)^{-1/2} = \frac{x^2}{\sqrt{2}}\sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}\left(\frac{x}{2}\right)^n$$

$$= \frac{x^2}{\sqrt{2}}\left[1+\left(-\frac{1}{2}\right)\left(\frac{x}{2}\right) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}\left(\frac{x}{2}\right)^2 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!}\left(\frac{x}{2}\right)^3 + \cdots\right]$$

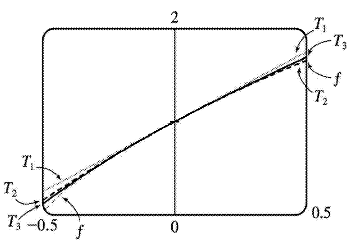
$$= \frac{x^2}{\sqrt{2}} + \frac{x^2}{\sqrt{2}}\sum_{n=1}^{\infty}(-1)^n\frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{n!\,2^{2n}}x^n$$

$$= \frac{x^2}{\sqrt{2}} + \sum_{n=1}^{\infty}(-1)^n\frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{n!\,2^{2n+1/2}}x^{n+2} \text{ and } \left|\frac{x}{2}\right|<1 \Leftrightarrow |x|<2 \text{ , so } R=2 \text{ .}$$

9.

$$(1+2x)^{3/4} = 1+\frac{3}{4}(2x)+\frac{\left(\frac{3}{4}\right)\left(-\frac{1}{4}\right)}{2!}(2x)^2+\frac{\left(\frac{3}{4}\right)\left(-\frac{1}{4}\right)\left(-\frac{5}{4}\right)}{3!}(2x)^3+\cdots$$
$$=1+\frac{3}{2}x+3\sum_{n=2}^{\infty}(-1)^{n+1}\frac{1\cdot 5\cdot 9\cdot\cdots\cdot(4n-7)}{4^n\cdot n!}\cdot 2^n x^n$$
$$=1+\frac{3}{2}x+3\sum_{n=2}^{\infty}(-1)^{n+1}\frac{1\cdot 5\cdot 9\cdot\cdots\cdot(4n-7)}{2^n\cdot n!}x^n \text{ and } |2x|<1\Leftrightarrow|x|<\frac{1}{2}\text{ , so } R=\frac{1}{2}\text{ .}$$

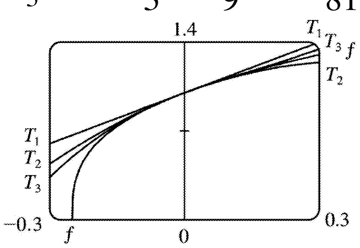
The three Taylor polynomials are $T_1(x)=1+\frac{3}{2}x$, $T_2(x)=1+\frac{3}{2}x-\frac{3}{8}x^2$, and $T_3(x)=1+\frac{3}{2}x-\frac{3}{8}x^2+\frac{5}{16}x^3$.



10.

$$\sqrt[3]{1+4x}=(1+4x)^{1/3}$$
$$=1+\frac{1}{3}(4x)+\frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)}{2!}(4x)^2+\frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)}{3!}(4x)^3+\cdots$$
$$=1+\frac{4}{3}x+\sum_{n=2}^{\infty}(-1)^{n+1}\frac{2\cdot 5\cdot 8\cdot\cdots\cdot(3n-4)}{3^n\cdot n!}(4x)^n \text{ and } |4x|<1\Leftrightarrow|x|<\frac{1}{4}\text{ , so } R=\frac{1}{4}\text{ .}$$

The three Taylor polynomials are $T_1(x)=1+\frac{4}{3}x$, $T_2(x)=1+\frac{4}{3}x-\frac{16}{9}x^2$, and $T_3(x)=1+\frac{4}{3}x-\frac{16}{9}x^2+\frac{320}{81}x^3$.



11. **(a)**

$$1/\sqrt{1-x^2}=\left[1+\left(-x^2\right)\right]^{-1/2}$$
$$=1+\left(-\frac{1}{2}\right)\left(-x^2\right)+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}\left(-x^2\right)^2+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!}\left(-x^2\right)^3+\cdots$$
$$=1+\sum_{n=1}^{\infty}\frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{2^n\cdot n!}x^{2n}$$

(b)

$$\sin^{-1}x=\int\frac{1}{\sqrt{1-x^2}}\,dx=C+x+\sum_{n=1}^{\infty}\frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{(2n+1)2^n\cdot n!}x^{2n+1}$$
$$=x+\sum_{n=1}^{\infty}\frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{(2n+1)2^n\cdot n!}x^{2n+1}\text{ since }0=\sin^{-1}0=C\,.$$

12. **(a)** $\left(1+x^2\right)^{-1/2}=\sum_{n=0}^{\infty}\binom{-1/2}{n}x^{2n}=1+\sum_{n=1}^{\infty}\frac{(-1)^n\cdot 1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)x^{2n}}{2^n\cdot n!}$

(b) $\sinh^{-1}x=\int\frac{dx}{\sqrt{1+x^2}}=C+x+\sum_{n=1}^{\infty}\frac{(-1)^n\cdot 1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)x^{2n+1}}{2^n\cdot n!\,(2n+1)}$, but $C=0$ since $\sinh^{-1}0=0$, so

$\sinh^{-1}x=x+\sum_{n=1}^{\infty}\frac{(-1)^n\cdot 1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)x^{2n+1}}{2^n\cdot n!\,(2n+1)}$, $R=1$.

13. **(a)**

$$\sqrt[3]{1+x}=(1+x)^{1/3}=\sum_{n=0}^{\infty}\binom{\frac{1}{3}}{n}x^n$$
$$=1+\frac{1}{3}x+\frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)}{2!}x^2+\frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)}{3!}x^3+\cdots$$
$$=1+\frac{x}{3}+\sum_{n=2}^{\infty}(-1)^{n+1}\frac{2\cdot 5\cdot 8\cdot\cdots\cdot(3n-4)}{3^n\cdot n!}x^n$$

(b) $\sqrt[3]{1+x}=1+\frac{1}{3}x-\frac{1}{9}x^2+\frac{5}{81}x^3-\cdots$. $\sqrt[3]{1.01}=\sqrt[3]{1+0.01}$, so let $x=0.01$. The sum of the first two terms is then

$1+\frac{1}{3}(0.01)\approx 1.0033$. The third term is $\frac{1}{9}(0.01)^2\approx 0.00001$, which does not affect the fourth decimal place of the sum, so we have $\sqrt[3]{1.01}\approx 1.0033$.

14. **(a)**

$$1/\sqrt[4]{1+x} = (1+x)^{-1/4} = \sum_{n=0}^{\infty}\binom{-\frac{1}{4}}{n}x^n$$

$$= 1-\frac{1}{4}x+\frac{\left(-\frac{1}{4}\right)\left(-\frac{5}{4}\right)}{2!}x^2+\frac{\left(-\frac{1}{4}\right)\left(-\frac{5}{4}\right)\left(-\frac{9}{4}\right)}{3!}x^3+\cdots$$

$$= 1-\frac{1}{4}x+\sum_{n=2}^{\infty}(-1)^n\frac{1\cdot 5\cdot 9\cdot\cdots\cdot(4n-3)}{4^n\cdot n!}x^n$$

(b) $1/\sqrt[4]{1+x}=1-\frac{1}{4}x+\frac{5}{32}x^2-\frac{15}{128}x^3+\frac{195}{2048}x^4-\cdots$. $1/\sqrt[4]{1.1}=1/\sqrt[4]{1+0.1}$, so let $x=0.1$. The sum of the first four terms is then $1-\frac{1}{4}(0.1)+\frac{5}{32}(0.1)^2-\frac{15}{128}(0.1)^3\approx 0.976$. The fifth term is $\frac{195}{2048}(0.1)^4\approx 0.0000095$, which does not affect the third decimal place of the sum, so we have $1/\sqrt[4]{1.1}\approx 0.976$. (Note that the third decimal place of the sum of the first three terms is affected by the fourth term, so we need to use more than three terms for the sum.)

15. **(a)**

$$[1+(-x)]^{-2} = 1+(-2)(-x)+\frac{(-2)(-3)}{2!}(-x)^2+\frac{(-2)(-3)(-4)}{3!}(-x)^3+\cdots$$

$$= 1+2x+3x^2+4x^3+\cdots = \sum_{n=0}^{\infty}(n+1)x^n ,$$

so $\frac{x}{(1-x)^2} = x\sum_{n=0}^{\infty}(n+1)x^n = \sum_{n=0}^{\infty}(n+1)x^{n+1} = \sum_{n=1}^{\infty}nx^n$.

(b) With $x=\frac{1}{2}$ in part (a), we have $\sum_{n=1}^{\infty}n\left(\frac{1}{2}\right)^n=\sum_{n=1}^{\infty}\frac{n}{2^n}=\frac{\frac{1}{2}}{\left(1-\frac{1}{2}\right)^2}=\frac{\frac{1}{2}}{\frac{1}{4}}=2$.

16. **(a)**

$$[1+(-x)]^{-3}=\sum_{n=0}^{\infty}\binom{-3}{n}(-x)^n$$
$$=1+(-3)(-x)+\frac{(-3)(-4)}{2!}(-x)^2+\frac{(-3)(-4)(-5)}{3!}(-x)^3+\cdots$$
$$=1+\sum_{n=1}^{\infty}\frac{3\cdot 4\cdot 5\cdot\cdots\cdot(n+2)}{n!}x^n=\sum_{n=0}^{\infty}\frac{2\cdot 3\cdot 4\cdot 5\cdot\cdots\cdot(n+2)}{2\cdot n!}x^n$$
$$=\sum_{n=0}^{\infty}\frac{(n+1)(n+2)}{2}x^n\Rightarrow$$

$$\left(x+x^2\right)[1+(-x)]^{-3}=x[1+(-x)]^{-3}+x^2[1+(-x)]^{-3}$$
$$=\sum_{n=0}^{\infty}\frac{(n+1)(n+2)}{2}x^{n+1}+\sum_{n=0}^{\infty}\frac{(n+1)(n+2)}{2}x^{n+2}$$
$$=\sum_{n=1}^{\infty}\frac{n(n+1)}{2}x^n+\sum_{n=1}^{\infty}\frac{n(n+1)}{2}x^{n+1}$$
$$=x+\sum_{n=2}^{\infty}\frac{n(n+1)}{2}x^n+\sum_{n=2}^{\infty}\frac{(n-1)n}{2}x^n=x+\sum_{n=2}^{\infty}\left[\frac{n(n+1)}{2}+\frac{(n-1)n}{2}\right]x^n$$
$$=x+\sum_{n=2}^{\infty}n^2x^n=\sum_{n=1}^{\infty}n^2x^n\ ,\ -1<x<1$$

(b) Setting $x=\frac{1}{2}$ in the last series above gives the required series, so $\sum_{n=1}^{\infty}\frac{n^2}{2^n}=\frac{\frac{1}{2}+\left(\frac{1}{2}\right)^2}{\left(1-\frac{1}{2}\right)^3}=\frac{\frac{3}{4}}{\frac{1}{8}}=6$

.

17. **(a)**

$$\left(1+x^2\right)^{1/2}=1+\left(\frac{1}{2}\right)x^2+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}\left(x^2\right)^2+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}\left(x^2\right)^3+\cdots$$
$$=1+\frac{x^2}{2}+\sum_{n=2}^{\infty}\frac{(-1)^{n-1}1\cdot 3\cdot 5\cdot\cdots\cdot(2n-3)}{2^n\cdot n!}x^{2n}$$

(b) The coefficient of x^{10} (corresponding to $n=5$) in the above Maclaurin series is $\frac{f^{(10)}(0)}{10!}$, so

$$\frac{f^{(10)}(0)}{10!}=\frac{(-1)^4\cdot 1\cdot 3\cdot 5\cdot 7}{2^5\cdot 5!}\Rightarrow f^{(10)}(0)=10!\left(\frac{1\cdot 3\cdot 5\cdot 7}{2^5\cdot 5!}\right)=99\text{ , }225\text{ .}$$

18. **(a)**

$$\left(1+x^3\right)^{-1/2}=\Sigma_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}\left(x^3\right)^n$$

$$=1+\left(-\frac{1}{2}\right)\left(x^3\right)+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}\left(x^3\right)^2+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!}\left(x^3\right)^3+\cdot\cdot\cdot$$

$$=1+\Sigma_{n=1}^{\infty}\frac{(-1)^n 1\cdot 3\cdot 5\cdot\cdot\cdot\cdot\cdot(2n-1)x^{3n}}{2^n\cdot n!}$$

(b) The coefficient of x^9 (corresponding to n=3) in the preceding series is

$\frac{f^{(9)}(0)}{9!}$, so $\frac{f^{(9)}(0)}{9!}=\frac{(-1)^3 1\cdot 3\cdot 5}{2^3\cdot 3!}\Rightarrow f^{(9)}(0)=-\frac{9!\cdot 5}{8\cdot 2}=-113\text{ , }400$.

19. **(a)** $g(x)=\Sigma_{n=0}^{\infty}\binom{k}{n}x^n\Rightarrow g^{/}(x)=\Sigma_{n=1}^{\infty}\binom{k}{n}nx^{n-1}$, so

$$(1+x)g^{/}(x)=(1+x)\Sigma_{n=1}^{\infty}\binom{k}{n}nx^{n-1}=\Sigma_{n=1}^{\infty}\binom{k}{n}nx^{n-1}+\Sigma_{n=1}^{\infty}\binom{k}{n}nx^{n}$$

$$=\Sigma_{n=0}^{\infty}\binom{k}{n}(n+1)x^n+\Sigma_{n=0}^{\infty}\binom{k}{n+1}nx^n\left[\begin{array}{c}\text{Replace } n \text{ with } n+1\\ \text{in the first series}\end{array}\right]$$

$$=\Sigma_{n=0}^{\infty}(n+1)\frac{k(k-1)(k-2)\cdot\cdot\cdot(k-n+1)(k-n)}{(n+1)!}x^n$$

$$+\Sigma_{n=0}^{\infty}\left[(n)\frac{k(k-1)(k-2)\cdot\cdot\cdot(k-n+1)}{n!}\right]x^n$$

$$=\Sigma_{n=0}^{\infty}\frac{(n+1)k(k-1)(k-2)\cdot\cdot\cdot(k-n+1)}{(n+1)!}\left[(k-n)+n\right]x^n$$

$$=k\Sigma_{n=0}^{\infty}\frac{k(k-1)(k-2)\cdot\cdot\cdot(k-n+1)}{n!}x^n=k\Sigma_{n=0}^{\infty}\binom{k}{n}x^n=kg(x)$$

Thus, $g^{/}(x)=\frac{kg(x)}{1+x}$.

(b)

$h(x)=(1+x)^{-k}g(x)\Rightarrow$

$h'(x)=-k(1+x)^{-k-1}g(x)+(1+x)^{-k}g'(x)$ [ProductRule

$=-k(1+x)^{-k-1}g(x)+(1+x)^{-k}\dfrac{kg(x)}{1+x}$

$=-k(1+x)^{-k-1}g(x)+k(1+x)^{-k-1}g(x)=0$

(c) From part (b) we see that $h(x)$ must be constant for $x\in(-1,1)$, so $h(x)=h(0)=1$ for $x\in(-1,1)$. Thus, $h(x)=1=(1+x)^{-k}g(x)\Leftrightarrow g(x)=(1+x)^{k}$ for $x\in(-1,1)$.

20. By Exercise .11.1, $\sqrt{1+x}=1+\dfrac{x}{2}+\sum_{n=2}^{\infty}\dfrac{(-1)^{n-1}1\cdot 3\cdot 5\cdot\cdots\cdot(2n-3)x^{n}}{2^{n}\cdot n!}$, so

$\left(1-x^{2}\right)^{1/2}=1-\dfrac{1}{2}x^{2}-\sum_{n=2}^{\infty}\dfrac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-3)}{2^{n}\cdot n!}x^{2n}$ and

$\sqrt{1-e^{2}\sin^{2}\theta}=1-\dfrac{1}{2}e^{2}\sin^{2}\theta-\sum_{n=2}^{\infty}\dfrac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-3)}{2^{n}\cdot n!}e^{2n}\sin^{2n}\theta$. Thus,

$$L=4a\int_{0}^{\pi/2}\sqrt{1-e^{2}\sin^{2}\theta}\,d\theta=4a\int_{0}^{\pi/2}\left(1-\frac{1}{2}e^{2}\sin^{2}\theta-\sum_{n=2}^{\infty}\frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-3)}{2^{n}\cdot n!}e^{2n}\sin^{2n}\theta\right)d\theta$$

$$=4a\left[\frac{\pi}{2}-\frac{e^{2}}{2}S_{1}-\sum_{n=2}^{\infty}\frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-3)}{n!}\left(\frac{e^{2}}{2}\right)^{n}S_{n}\right]$$

where $S_{n}=\int_{0}^{\pi/2}\sin^{2n}\theta\,d\theta=\dfrac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{2\cdot 4\cdot 6\cdot\cdots\cdot 2n}\dfrac{\pi}{2}$ by Exercise 44 of 8.1.

$$L=4a\left(\frac{\pi}{2}\right)\left[1-\frac{e^{2}}{2}\cdot\frac{1}{2}-\sum_{n=2}^{\infty}\frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-3)}{n!}\left(\frac{e^{2}}{2}\right)^{n}\frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{2\cdot 4\cdot 6\cdot\cdots\cdot 2n}\right]$$

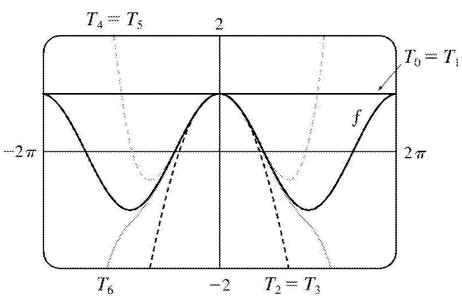
$$=2\pi a\left[1-\frac{e^{2}}{4}-\sum_{n=2}^{\infty}\frac{e^{2n}}{2^{n}}\cdot\frac{1^{2}\cdot 3^{2}\cdot 5^{2}\cdot\cdots\cdot(2n-3)^{2}(2n-1)}{n!\cdot 2^{n}\cdot n!}\right]$$

$$=2\pi a\left[1-\frac{e^{2}}{4}-\sum_{n=2}^{\infty}\frac{e^{2n}}{4^{n}}\left(\frac{1\cdot 3\cdot\cdots\cdot(2n-3)}{n!}\right)^{2}(2n-1)\right]$$

$$=2\pi a\left[1-\frac{e^{2}}{4}-\frac{3e^{4}}{64}-\frac{5e^{6}}{256}-\cdots\right]=\frac{\pi a}{128}(256-64e^{2}-12e^{4}-5e^{6}-\cdots)$$

1. **(a)**

n	$f^{(n)}(x)$	$f^{(n)}(0)$	$T_n(x)$
0	$\cos x$	1	1
1	$-\sin x$	0	1
2	$-\cos x$	-1	$1-\frac{1}{2}x^2$
3	$\sin x$	0	$1-\frac{1}{2}x^2$
4	$\cos x$	1	$1-\frac{1}{2}x^2+\frac{1}{24}x^4$
5	$-\sin x$	0	$1-\frac{1}{2}x^2+\frac{1}{24}x^4$
6	$-\cos x$	-1	$1-\frac{1}{2}x^2+\frac{1}{24}x^4-\frac{1}{720}x^6$



(b)

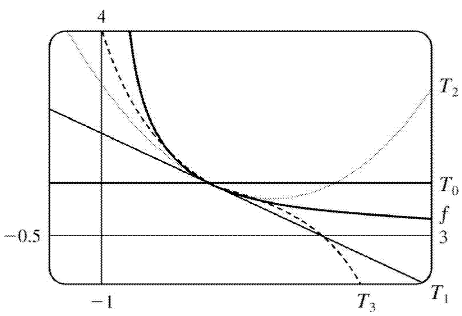
x	f	$T_0=T_1$	$T_2=T_3$	$T_4=T_5$	T_6
$\frac{\pi}{4}$	0.7071	1	0.6916	0.7074	0.7071
$\frac{\pi}{2}$	0	1	-0.2337	0.0200	-0.0009
π	-1	1	-3.9348	0.1239	-1.2114

(c) As n increases, $T_n(x)$ is a good approximation to $f(x)$ on a larger and larger interval.

2. **(a)**

n	$f^{(n)}(x)$	$f^{(n)}(0)$	$T_n(x)$

0	x^{-1}	1	1
1	$-x^{-2}$	-1	$1-(x-1)=2-x$
2	$2x^{-3}$	2	$1-(x-1)+(x-1)^2=x^2-3x+3$
3	$-6x^{-4}$	-6	$1-(x-1)+(x-1)^2-(x-1)^3=-x^3+4x^2-6x+4$



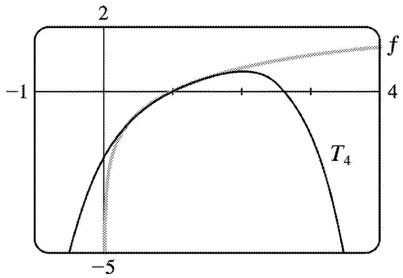
(b)

x	f	T_0	T_1	T_2	T_3
0.9	$1.\overline{1}$	1	1.1	1.11	1.111
1.3	0.7692	1	0.7	0.79	0.763

(c) As n increases, $T_n(x)$ is a good approximation to $f(x)$ on a larger and larger interval.

3.

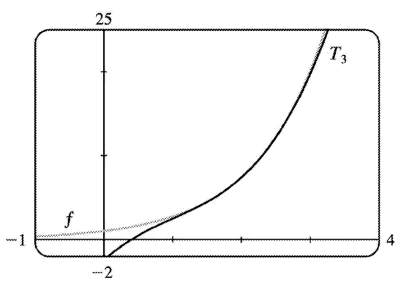
n	$f^{(n)}(x)$	$f^{(n)}(1)$
0	$\ln x$	0
1	$1/x$	1
2	$-1/x^2$	-1
3	$2/x^3$	2
4	$-6/x^4$	-6



$$T_4(x)=\sum_{n=0}^{4}\frac{f^{(n)}(1)}{n!}(x-1)^n=0+(x-1)-\frac{1}{2}(x-1)^2+\frac{1}{3}(x-1)^3-\frac{1}{4}(x-1)^4$$

4.

n	$f^{(n)}(x)$	$f^{(n)}(2)$
0	e^x	e^2
1	e^x	e^2
2	e^x	e^2
3	e^x	e^2

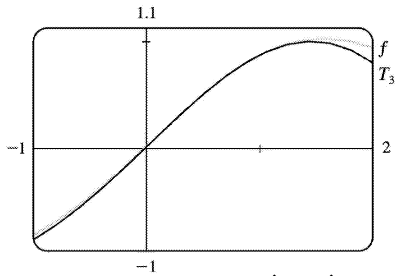


$$T_3(x)=\sum_{n=0}^{3}\frac{f^{(n)}(2)}{n!}(x-2)^n=e^2+e^2(x-2)+\frac{e^2}{2}(x-2)^2+\frac{e^2}{6}(x-2)^3$$

5.

n	$f^{(n)}(x)$	$f^{(n)}\left(\frac{\pi}{6}\right)$
0	$\sin x$	$\frac{1}{2}$
1	$\cos x$	$\frac{\sqrt{3}}{2}$

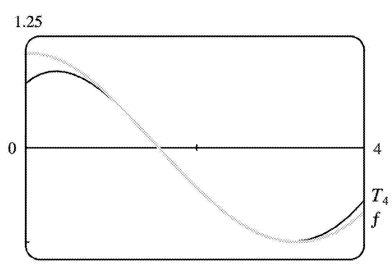
2	$-\sin x$	$-\frac{1}{2}$
3	$-\cos x$	$-\frac{\sqrt{3}}{2}$



$$T_3(x)=\sum_{n=0}^{3}\frac{f^{(n)}\left(\frac{\pi}{6}\right)}{n!}\left(x-\frac{\pi}{6}\right)^n=\frac{1}{2}+\frac{\sqrt{3}}{2}\left(x-\frac{\pi}{6}\right)-\frac{1}{4}\left(x-\frac{\pi}{6}\right)^2-\frac{\sqrt{3}}{12}\left(x-\frac{\pi}{6}\right)^3$$

6.

n	$f^{(n)}(x)$	$f^{(n)}\left(\frac{2\pi}{3}\right)$
0	$\cos x$	$-\frac{1}{2}$
1	$-\sin x$	$-\frac{\sqrt{3}}{2}$
2	$-\cos x$	$\frac{1}{2}$
3	$\sin x$	$\frac{\sqrt{3}}{2}$
4	$\cos x$	$-\frac{1}{2}$

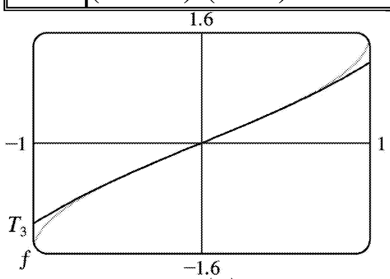


$$T_4(x)=\sum_{n=0}^{4}\frac{f^{(n)}\left(\frac{2\pi}{3}\right)}{n!}\left(x-\frac{2\pi}{3}\right)^n$$

$$=-\frac{1}{2}-\frac{\sqrt{3}}{2}\left(x-\frac{2\pi}{3}\right)+\frac{1}{4}\left(x-\frac{2\pi}{3}\right)^2+\frac{\sqrt{3}}{12}\left(x-\frac{2\pi}{3}\right)^3-\frac{1}{48}\left(x-\frac{2\pi}{3}\right)^4$$

7.

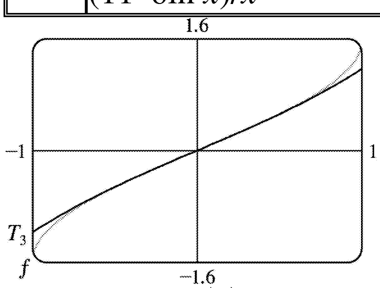
n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\arcsin x$	0
1	$1/\sqrt{1-x^2}$	1
2	$x/(1-x^2)^{3/2}$	0
3	$(2x^2+1)/(1-x^2)^{5/2}$	1



$$T_3(x)=\sum_{n=0}^{3}\frac{f^{(n)}(0)}{n!}x^n=x+\frac{x^3}{6}$$

8.

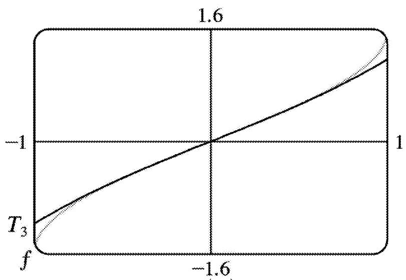
n	$f^{(n)}(x)$	$f^{(n)}(1)$
0	$(\ln x)/x$	0
1	$(1-\ln x)/x^2$	1
2	$(-3+2\ln x)/x^3$	−3
3	$(11-6\ln x)/x^4$	11



$$T_3(x)=\sum_{n=0}^{3}\frac{f^{(n)}(1)}{n!}(x-1)^n=(x-1)-\frac{3}{2}(x-1)^2+\frac{11}{6}(x-1)^3$$

9.

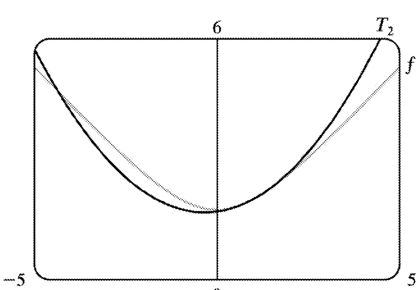
n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	xe^{-2x}	0
1	$(1-2x)e^{-2x}$	1
2	$4(x-1)e^{-2x}$	-4
3	$4(3-2x)e^{-2x}$	12



$$T_3(x)=\sum_{n=0}^{3}\frac{f^{(n)}(0)}{n!}x^n=\frac{0}{1}\cdot 1+\frac{1}{1}x^1+\frac{-4}{2}x^2+\frac{12}{6}x^3=x-2x^2+2x^3$$

10.

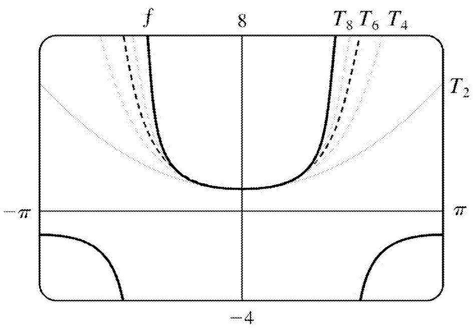
n	$f^{(n)}(x)$	$f^{(n)}(1)$
0	$\left(3+x^2\right)^{1/2}$	2
1	$x\left(3+x^2\right)^{-1/2}$	$\frac{1}{2}$
2	$3\left(3+x^2\right)^{-3/2}$	$\frac{3}{8}$



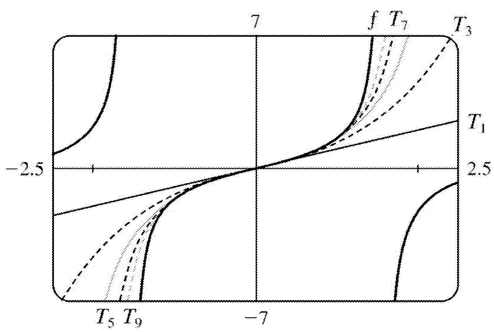
$$T_2(x)=\sum_{n=0}^{2}\frac{f^{(n)}(1)}{n!}(x-1)^n=2+\frac{1}{2}(x-1)+\frac{3/8}{2}(x-1)^2=2+\frac{1}{2}(x-1)+\frac{3}{16}(x-1)^2$$

11. In Maple, we can find the Taylor polynomials by the following method: first define `f:=sec(x);` and then set

T2:=convert(taylor(f,x=0,3),polynom);, T4:=convert(taylor(f,x=0,5),polynom);, etc. (The third argument in the taylor function is one more than the degree of the desired polynomial). We must convert to the type polynom because the output of the taylor function contains an error term which we do not want. In Mathematica, we use Tn:=Normal[Series[f,{x,0,n}]] , with n=2,4, etc. Note that in Mathematica, the "degree" argument is the same as the degree of the desired polynomial. In Derive, author sec x , then enter Calculus ,Taylor,8,0; and then simplify the expression. The eighth Taylor polynomial is $T_8(x)=1+\frac{1}{2}x^2+\frac{5}{24}x^4+\frac{61}{720}x^6+\frac{277}{8064}x^8$.



12. See Exercise 11 for the CAS commands used to generate the Taylor polynomials. The ninth Taylor polynomial for tan x is $T_9(x)=x+\frac{1}{3}x^3+\frac{2}{15}x^5+\frac{17}{315}x^7+\frac{62}{2835}x^9$.



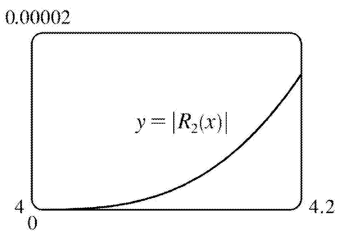
13.

n	$f^{(n)}(x)$	$f^{(n)}(4)$
0	$\sqrt{x}$	2
1	$\frac{1}{2}x^{-1/2}$	$\frac{1}{4}$
2	$-\frac{1}{4}x^{-3/2}$	$-\frac{1}{32}$
3	$\frac{3}{8}x^{-5/2}$	

(a) $f(x)=\sqrt{x}\approx T_2(x)=2+\frac{1}{4}(x-4)-\frac{1/32}{2!}(x-4)^2=2+\frac{1}{4}(x-4)-\frac{1}{64}(x-4)^2$

(b) $\left|R_2(x)\right|\le\frac{M}{3!}|x-4|^3$, where $\left|f'''(x)\right|\le M$. Now $4\le x\le 4.2\Rightarrow|x-4|\le 0.2\Rightarrow|x-4|^3\le 0.008$. Since $f'''(x)$ is decreasing on $[4,4.2]$, we can take $M=\left|f'''(4)\right|=\frac{3}{8}4^{-5/2}=\frac{3}{256}$, so $\left|R_2(x)\right|\le\frac{3/256}{6}(0.008)=\frac{0.008}{512}=0.000015625$.

(c) From the graph of $\left|R_2(x)\right|=\left|\sqrt{x}-T_2(x)\right|$, it seems that the error is less than 1.52×10^{-5} on $[4,4.2]$.



14.

n	$f^{(n)}(x)$	$f^{(n)}(1)$
0	x^{-2}	1
1	$-2x^{-3}$	-2
2	$6x^{-4}$	6
3	$-24x^{-5}$	

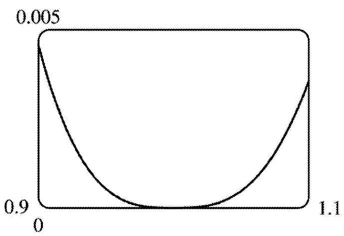
(a)

$$f(x)=x^{-2}\approx T_2(x)$$
$$=1-2(x-1)+\frac{6}{2!}(x-1)^2$$
$$=1-2(x-1)+3(x-1)^2$$

(b) $\left|R_2(x)\right|\le\frac{M}{3!}|x-1|^3$, where $\left|f'''(x)\right|\le M$. Now $0.9\le x\le 1.1\Rightarrow|x-1|\le 0.1\Rightarrow|x-1|^3\le 0.001$. Since $f'''(x)$ is decreasing on $[0.9,1.1]$, we can take

$M=\left|f^{'''}(0.9)\right|=\dfrac{24}{(0.9)^5}$, so

$$\left|R_2(x)\right| \le \frac{24/(0.9)^5}{6}(0.001)=\frac{0.004}{0.59049}$$
$$\approx 0.00677404$$

(c) 

From the graph of $\left|R_2(x)\right|=\left|x^{-2}-T_2(x)\right|$, it seems that the error is less than 0.0046 on $[0.9,1.1]$.

15.

n	$f^{(n)}(x)$	$f^{(n)}(1)$
0	$x^{2/3}$	1
1	$\frac{2}{3}x^{-1/3}$	$\frac{2}{3}$
2	$-\frac{2}{9}x^{-4/3}$	$-\frac{2}{9}$
3	$\frac{8}{27}x^{-7/3}$	$\frac{8}{27}$
4	$-\frac{56}{81}x^{-10/3}$	

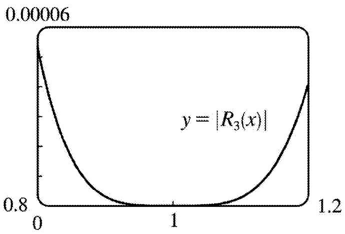
(a) $f(x)=x^{2/3}\approx T_3(x)=1+\frac{2}{3}(x-1)-\frac{2/9}{2!}(x-1)^2+\frac{8/27}{3!}(x-1)^3=1+\frac{2}{3}(x-1)-\frac{1}{9}(x-1)^2+\frac{4}{81}(x-1)^3$

(b) $\left|R_3(x)\right|\le\frac{M}{4!}|x-1|^4$, where $\left|f^{(4)}(x)\right|\le M$. Now $0.8\le x\le 1.2\Rightarrow |x-1|\le 0.2\Rightarrow |x-1|^4\le 0.0016$.

Since $\left|f^{(4)}(x)\right|$ is decreasing on $[0.8,1.2]$, we can take $M=\left|f^{(4)}(0.8)\right|=\frac{56}{81}(0.8)^{-10/3}$, so

$$\left|R_3(x)\right|\le\frac{\frac{56}{81}(0.8)^{-10/3}}{24}(0.0016)\approx 0.00009697 .$$

(c) From the graph of $\left|R_3(x)\right| = \left|x^{2/3} - T_3(x)\right|$, it seems that the error is less than 0.0000533 on $[0.8,1.2]$.



16.

n	$f^{(n)}(x)$	$f^{(n)}\left(\frac{\pi}{3}\right)$
0	$\cos x$	$\frac{1}{2}$
1	$-\sin x$	$-\frac{\sqrt{3}}{2}$
2	$-\cos x$	$-\frac{1}{2}$
3	$\sin x$	$\frac{\sqrt{3}}{2}$
4	$\cos x$	$\frac{1}{2}$
5	$-\sin x$	

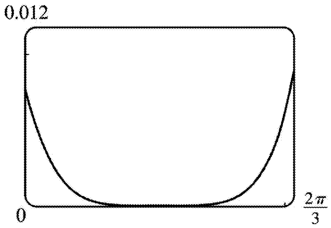
(a)

$$f(x)=\cos x \approx T_4(x)$$

$$=\frac{1}{2}-\frac{\sqrt{3}}{2}\left(x-\frac{\pi}{3}\right)-\frac{1}{4}\left(x-\frac{\pi}{3}\right)^2+\frac{\sqrt{3}}{12}\left(x-\frac{\pi}{3}\right)^3+\frac{1}{48}\left(x-\frac{\pi}{3}\right)^4$$

(b) $\left|R_4(x)\right| \le \frac{M}{5!}\left|x-\frac{\pi}{3}\right|^5$, where $\left|f^{(5)}(x)\right| \le M$. Now $0 \le x \le \frac{2\pi}{3} \Rightarrow \left(x-\frac{\pi}{3}\right)^5 \le \left(\frac{\pi}{3}\right)^5$, and letting $x=\frac{\pi}{2}$ gives $M=1$, so $\left|R_4(x)\right| \le \frac{1}{5!}\left(\frac{\pi}{3}\right)^5 \approx 0.0105$.

(c)



From the graph of $\left|R_4(x)\right| = \left|\cos x - T_4(x)\right|$, it seems that the error is less than 0.01 on $\left[0, \frac{2\pi}{3}\right]$.

17.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\tan x$	0
1	$\sec^2 x$	1
2	$2\sec^2 x \tan x$	0
3	$4\sec^2 x \tan^2 x + 2\sec^4 x$	2
4	$8\sec^2 x \tan^3 x + 16\sec^4 x \tan x$	

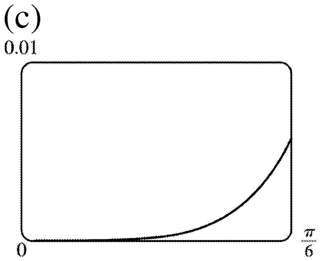
(a) $f(x) = \tan x \approx T_3(x) = x + \frac{1}{3}x^3$

(b) $\left|R_3(x)\right| \le \frac{M}{4!}|x|^4$, where $\left|f^{(4)}(x)\right| \le M$. Now $0 \le x \le \frac{\pi}{6} \Rightarrow x^4 \le \left(\frac{\pi}{6}\right)^4$, and letting $x = \frac{\pi}{6}$ $\left[\text{since} f^{(4)} \text{isincreasingon} \left(0, \frac{\pi}{6}\right)\right]$ gives

$$\left|R_3(x)\right| \le \frac{8\left(\frac{2}{\sqrt{3}}\right)^2\left(\frac{1}{\sqrt{3}}\right)^3 + 16\left(\frac{2}{\sqrt{3}}\right)^4\left(\frac{1}{\sqrt{3}}\right)}{4!}\left(\frac{\pi}{6}\right)^4$$

$$= \frac{4\sqrt{3}}{9}\left(\frac{\pi}{6}\right)^4 \approx 0.057859$$

(c)



From the graph of

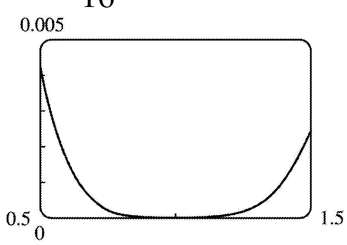
$\left|R_3(x)\right|=\left|\tan x-T_3(3)\right|$, it seems that the error is less than 0.006 on $[0,\pi/6]$.

18.

n	$f^{(n)}(x)$	$f^{(n)}(1)$
0	$\ln(1+2x)$	$\ln 3$
1	$2/(1+2x)$	$\frac{2}{3}$
2	$-4/(1+2x)^2$	$-\frac{4}{9}$
3	$16/(1+2x)^3$	$\frac{16}{27}$
4	$-96/(1+2x)^4$	

(a) $f(x)=\ln(1+2x)\approx T_3(x)=\ln 3+\frac{2}{3}(x-1)-\frac{4/9}{2!}(x-1)^2+\frac{16/27}{3!}(x-1)^3$

(b) $\left|R_3(x)\right|\leq\frac{M}{4!}|x-1|^4$, where $\left|f^{(4)}(x)\right|\leq M$. Now $0.5\leq x\leq 1.5\Rightarrow -0.5\leq x-1\leq 0.5\Rightarrow |x-1|\leq 0.5\Rightarrow$ $|x-1|^4\leq\frac{1}{16}$, and letting $x=0.5$ gives $M=6$, so $\left|R_3(x)\right|\leq\frac{6}{4!}\cdot\frac{1}{16}=\frac{1}{64}=0.015625$.



(c)
From the graph of $\left|R_3(x)\right|=\left|\ln(1+2x)-T_3(x)\right|$, it seems that the error is less than 0.005 on $[0.5,1.5]$.

19.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	e^{x^2}	1
1	$e^{x^2}(2x)$	0
2	$e^{x^2}(2+4x^2)$	2
3	$e^{x^2}(12x+8x^3)$	0
4	$e^{x^2}(12+48x^2+16x^4)$	

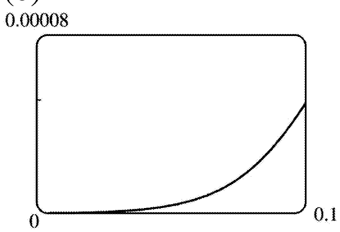
(a) $f(x)=e^{x^2}\approx T_3(x)=1+\frac{2}{2!}x^2=1+x^2$

(b) $\left|R_3(x)\right|\leq\frac{M}{4!}|x|^4$, where $\left|f^{(4)}(x)\right|\leq M$.

Now $0\leq x\leq 0.1\Rightarrow x^4\leq(0.1)^4$, and
letting $x=0.1$ gives

$\left|R_3(x)\right|\leq\frac{e^{0.01}(12+0.48+0.0016)}{24}(0.1)^4\approx 0.00006$.

(c)



From the graph of $\left|R_3(x)\right|=\left|e^{x^2}-\left(1+x^2\right)\right|$, it appears that the error is less than 0.000051 on $[0,0.1]$.

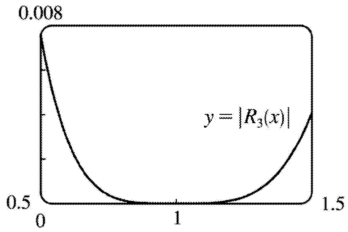
20.

n	$f^{(n)}(x)$	$f^{(n)}(1)$
0	$x\ln x$	0
1	$\ln x+1$	1
2	$1/x$	1
3	$-1/x^2$	-1
4	$2/x^3$	

(a) $f(x)=x\ln x\approx T_3(x)=(x-1)+\frac{1}{2}(x-1)^2-\frac{1}{6}(x-1)^3$

(b) $\left|R_3(x)\right|\leq\frac{M}{4!}|x-1|^4$, where $\left|f^{(4)}(x)\right|\leq M$. Now $0.5\leq x\leq 1.5\Rightarrow|x-1|\leq\frac{1}{2}\Rightarrow|x-1|^4\leq\frac{1}{16}$.
Since $\left|f^{(4)}(x)\right|$ is decreasing on $[0.5,1.5]$, we can take $M=\left|f^{(4)}(0.5)\right|=2/(0.5)^3=16$, so
$\left|R_3(x)\right|\leq\frac{16}{24}(1/16)=\frac{1}{24}=0.041\overline{6}$.

(c) From the graph of $\left|R_3(x)\right|=\left|x\ln x-T_3(x)\right|$, it seems that the error is less than 0.0076 on $[0.5,1.5]$
.



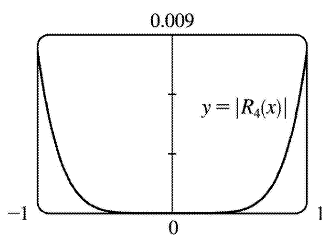
21.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$x\sin x$	0
1	$\sin x + x\cos x$	0
2	$2\cos x - x\sin x$	2
3	$-3\sin x - x\cos x$	0
4	$-4\cos x + x\sin x$	-4
5	$5\sin x + x\cos x$	

(a) $f(x) = x\sin x \approx T_4(x) = \frac{2}{2!}(x-0)^2 + \frac{-4}{4!}(x-0)^4 = x^2 - \frac{1}{6}x^4$

(b) $\left|R_4(x)\right| \le \frac{M}{5!}|x|^5$, where $\left|f^{(5)}(x)\right| \le M$. Now $-1 \le x \le 1 \Rightarrow |x| \le 1$, and a graph of $f^{(5)}(x)$ shows that $\left|f^{(5)}(x)\right| \le 5$ for $-1 \le x \le 1$. Thus, we can take $M = 5$ and get $\left|R_4(x)\right| \le \frac{5}{5!} \cdot 1^5 = \frac{1}{24} = 0.041\overline{6}$.

(c) From the graph of $\left|R_4(x)\right| = \left|x\sin x - T_4(x)\right|$, it seems that the error is less than 0.0082 on $[-1,1]$.



22.

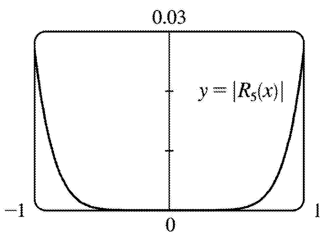
n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\sinh 2x$	0
1	$2\cosh 2x$	2
2	$4\sinh 2x$	0
3	$8\cosh 2x$	8
4	$16\sinh 2x$	0
5	$32\cosh 2x$	32

6	$64\sinh 2x$	

(a) $f(x)=\sinh 2x \approx T_5(x)=2x+\frac{8}{3!}x^3+\frac{32}{5!}x^5=2x+\frac{4}{3}x^3+\frac{4}{15}x^5$

(b) $\left|R_5(x)\right| \le \frac{M}{6!}|x|^6$, where $\left|f^{(6)}(x)\right| \le M$. For x in $[-1,1]$, we have $|x| \le 1$. Since $f^{(6)}(x)$ is an increasing odd function on $[-1,1]$, we see that $\left|f^{(6)}(x)\right| \le f^{(6)}(1)=64\sinh 2=32(e^2-e^{-2}) \approx 232.119$, so we can take $M=232.12$ and get $\left|R_5(x)\right| \le \frac{232.12}{720} \cdot 1^6 \approx 0.3224$.

(c) From the graph of $\left|R_5(x)\right|=\left|\sinh 2x-T_5(x)\right|$, it seems that the error is less than 0.027 on $[-1,1]$.



23. From Exercise 5, $\sin x=\frac{1}{2}+\frac{\sqrt{3}}{2}\left(x-\frac{\pi}{6}\right)-\frac{1}{4}\left(x-\frac{\pi}{6}\right)^2-\frac{\sqrt{3}}{12}\left(x-\frac{\pi}{6}\right)^3+R_3(x)$, where $\left|R_3(x)\right| \le \frac{M}{4!}\left|x-\frac{\pi}{6}\right|^4$ with $\left|f^{(4)}(x)\right|=|\sin x| \le M=1$. Now $x=35^\circ=(30^\circ+5^\circ)=\left(\frac{\pi}{6}+\frac{\pi}{36}\right)$ radians, so the error is

$\left|R_3\left(\frac{\pi}{36}\right)\right| \le \frac{\left(\frac{\pi}{36}\right)^4}{4!} < 0.000003$. Therefore, to five decimal places,

$\sin 35^\circ \approx \frac{1}{2}+\frac{\sqrt{3}}{2}\left(\frac{\pi}{36}\right)-\frac{1}{4}\left(\frac{\pi}{36}\right)^2-\frac{\sqrt{3}}{12}\left(\frac{\pi}{36}\right)^3 \approx 0.57358$.

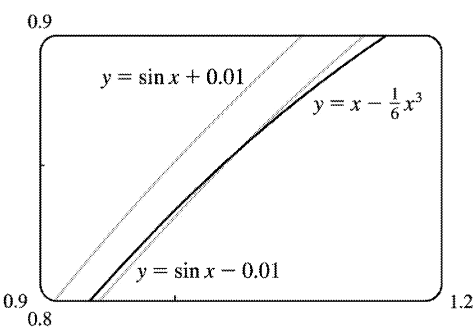
24. From Exercise 16,

$\cos x=\frac{1}{2}-\frac{\sqrt{3}}{2}\left(x-\frac{\pi}{3}\right)-\frac{1}{4}\left(x-\frac{\pi}{3}\right)^2+\frac{\sqrt{3}}{12}\left(x-\frac{\pi}{3}\right)^3+\frac{1}{48}\left(x-\frac{\pi}{3}\right)^4+R_4(x)$. Now since $x=69^\circ=(60^\circ+9^\circ)=\left(\frac{\pi}{3}+\frac{\pi}{20}\right)$ radians, the error is $\left|R_4(x)\right| \le \frac{\left(\frac{\pi}{20}\right)^5}{5!} < 8\times 10^{-7}$. Therefore, to five decimal places, $\cos 69^\circ \approx \frac{1}{2}-\frac{\sqrt{3}}{2}\left(\frac{\pi}{20}\right)-\frac{1}{4}\left(\frac{\pi}{20}\right)^2+\frac{\sqrt{3}}{12}\left(\frac{\pi}{20}\right)^3+\frac{1}{48}\left(\frac{\pi}{20}\right)^4 \approx 0.35837$.

25. All derivatives of e^x are e^x , so $\left|R_n(x)\right|\leq\dfrac{e^x}{(n+1)!}\,|x|^{n+1}$, where $0<x<0.1$. Letting $x=0.1$, $R_n(0.1)\leq\dfrac{e^{0.1}}{(n+1)!}\,(0.1)^{n+1}<0.00001$, and by trial and error we find that $n=3$ satisfies this inequality since $R_3(0.1)<0.0000046$. Thus, by adding the four terms of the Maclaurin series for e^x corresponding to $n=0$, 1 , 2 , and 3 , we can estimate $e^{0.1}$ to within 0.00001 . (In fact, this sum is $1.1051\overline{6}$ and $e^{0.1}\approx 1.10517$.)

26. Example 6 in Section .9 gives the Maclaurin series for ln $(1-x)$ as $-\sum_{n=1}^{\infty}\dfrac{x^n}{n}$ for $|x|<1$. Thus, $\ln 1.4=\ln\left[1-(-0.4)\right]=-\sum_{n=1}^{\infty}\dfrac{(-0.4)^n}{n}=\sum_{n=1}^{\infty}(-1)^{n+1}\dfrac{(0.4)^n}{n}$. Since this is an alternating series, the error is less than the first neglected term by the Alternating Series Estimation Theorem, and we find that $\left|a_6\right|=(0.4)^6/6\approx 0.0007<0.001$. So we need the first five (non–zero) terms of the Maclaurin series for the desired accuracy. (In fact, this sum is approximately 0.33698 and $\ln 1.4\approx 0.33647$.)

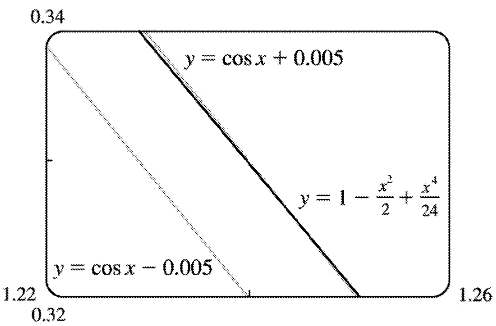
27. $\sin x=x-\dfrac{1}{3!}x^3+\dfrac{1}{5!}x^5-\cdots$. By the Alternating Series Estimation Theorem, the error in the approximation $\sin x=x-\dfrac{1}{3!}x^3$ is less than $\left|\dfrac{1}{5!}x^5\right|<0.01\Leftrightarrow\left|x^5\right|<120(0.01)\Leftrightarrow|x|<(1.2)^{1/5}\approx 1.037$. The curves $y=x-\dfrac{1}{6}x^3$ and $y=\sin x-0.01$ intersect at $x\approx 1.043$, so the graph confirms our estimate. Since both the sine function and



the given approximation are odd functions, we need to check the estimate only for $x>0$. Thus, the desired range of values for x is $-1.037<x<1.037$.

28.

$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots$. By the Alternating Series Estimation Theorem, the error is less than $\left| -\frac{1}{6!}x^6 \right| < 0.005 \Leftrightarrow x^6 < 720(0.005) \Leftrightarrow |x| < (3.6)^{1/6} \approx 1.238$. The curves $y = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4$ and $y = \cos x + 0.005$ intersect at $x \approx 1.244$, so the graph confirms our estimate. Since both the cosine function and the given approximation



are even functions, we need to check the estimate only for $x > 0$. Thus, the desired range of values for x is $-1.238 < x < 1.238$.

29. Let $s(t)$ be the position function of the car, and for convenience set $s(0) = 0$. The velocity of the car is
$v(t) = s'(t)$ and the acceleration is $a(t) = s''(t)$, so the second degree Taylor polynomial is $T_2(t) = s(0) + v(0)t + \frac{a(0)}{2}t^2 = 20t + t^2$. We estimate the distance travelled during the next second to be $s(1) \approx T_2(1) = 20 + 1 = 21$ m. The function $T_2(t)$ would not be accurate over a full minute, since the car could not possibly maintain an acceleration of 2 m / s^2 for that long (if it did, its final speed would be 140 m / s $\approx$ 313 mi / h!)

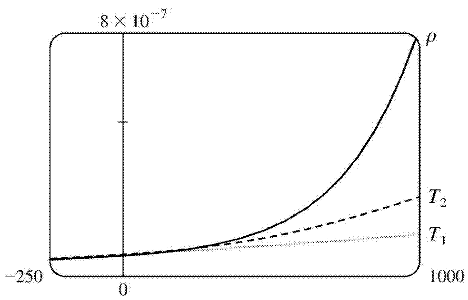
30. **(a)**

n	$\rho^{(n)}(t)$	$\rho^{(n)}(20)$
0	$\rho_{20} e^{\alpha(t-20)}$	ρ_{20}
1	$\alpha \rho_{20} e^{\alpha(t-20)}$	$\alpha \rho_{20}$
2	$\alpha^2 \rho_{20} e^{\alpha(t-20)}$	$\alpha^2 \rho_{20}$

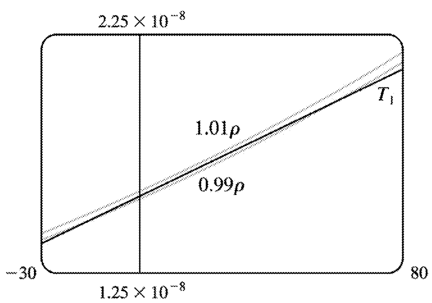
The linear approximation is $T_1(t) = \rho(20) + \rho'(20)(t-20) = \rho_{20}[1 + \alpha(t-20)]$. The quadratic approximation is

$$T_2(t)=\rho(20)+\rho'(20)(t-20)+\frac{\rho''(20)}{2}(t-20)^2=\rho_{20}\left[1+\alpha(t-20)+\frac{1}{2}\alpha^2(t-20)^2\right]$$

(b)



(c)



From the graph, it seems that $T_1(t)$ is within 1% of $\rho(t)$, that is, $0.99\rho(t)\leq T_1(t)\leq 1.01\rho(t)$, for -14° C $\leq t\leq 58^\circ$ C.

31. $E=\dfrac{q}{D^2}-\dfrac{q}{(D+d)^2}=\dfrac{q}{D^2}-\dfrac{q}{D^2(1+d/D)^2}=\dfrac{q}{D^2}\left[1-\left(1+\dfrac{d}{D}\right)^{-2}\right]$.

We use the Binomial Series to expand $(1+d/D)^{-2}$:

$$E=\frac{q}{D^2}\left[1-\left(1-2\left(\frac{d}{D}\right)+\frac{2\cdot 3}{2!}\left(\frac{d}{D}\right)^2-\frac{2\cdot 3\cdot 4}{3!}\left(\frac{d}{D}\right)^3+\cdots\right)\right]$$

$$=\frac{q}{D^2}\left[2\left(\frac{d}{D}\right)-3\left(\frac{d}{D}\right)^2+4\left(\frac{d}{D}\right)^3-\cdots\right]\approx\frac{q}{D^2}\cdot 2\left(\frac{d}{D}\right)=2qd\cdot\frac{1}{D^3}$$

when D is much larger than d; that is, when P is far away from the dipole.

32. **(a)** $\dfrac{n_1}{\ell_o}+\dfrac{n_2}{\ell_i}=\dfrac{1}{R}\left(\dfrac{n_2 s_i}{\ell_i}-\dfrac{n_1 s_o}{\ell_o}\right)$ (Equation 1) where

$$\ell_o=\sqrt{R^2+\left(s_o+R\right)^2-2R\left(s_o+R\right)\cos\phi} \text{ and } \ell_i=\sqrt{R^2+\left(s_i-R\right)^2+2R\left(s_i-R\right)\cos\phi} \quad (2)$$

Using $\cos\phi\approx 1$ gives

$$\ell_o=\sqrt{R^2+\left(s_o+R\right)^2-2R\left(s_o+R\right)}=\sqrt{R^2+s_o^2+2Rs_o+R^2-2Rs_o-2R^2}=\sqrt{s_o^2}=s_o$$

and similarly, $\ell_i=s_i$. Thus, Equation 1 becomes

$$\frac{n_1}{s_o}+\frac{n_2}{s_i}=\frac{1}{R}\left(\frac{n_2 s_i}{s_i}-\frac{n_1 s_o}{s_o}\right)\Rightarrow\frac{n_1}{s_o}+\frac{n_2}{s_i}=\frac{n_2-n_1}{R}$$

(b) Using $\cos\phi\approx 1-\frac{1}{2}\phi^2$ in (2) gives us

$$\begin{aligned}\ell_o &= \sqrt{R^2+\left(s_o+R\right)^2-2R\left(s_o+R\right)\left(1-\frac{1}{2}\phi^2\right)}\\ &= \sqrt{R^2+s_o^2+2Rs_o+R^2-2Rs_o+Rs_o\phi^2-2R^2+R^2\phi^2}=\sqrt{s_o^2+Rs_o\phi^2+R^2\phi^2}\end{aligned}$$

Anticipating that we will use the binomial series expansion $(1+x)^k\approx 1+kx$, we can write the last expression for ℓ_o as $s_o\sqrt{1+\phi^2\left(\frac{R}{s_o}+\frac{R^2}{s_o^2}\right)}$ and similarly, $\ell_i=s_i\sqrt{1-\phi^2\left(\frac{R}{s_i}-\frac{R^2}{s_i^2}\right)}$. Thus, from Equation 1, $\frac{n_1}{\ell_o}+\frac{n_2}{\ell_i}=\frac{1}{R}\left(\frac{n_2 s_i}{\ell_i}-\frac{n_1 s_o}{\ell_o}\right)\Leftrightarrow n_1\ell_o^{-1}+n_2\ell_i^{-1}=\frac{n_2}{R}\cdot\frac{s_i}{\ell_i}-\frac{n_1}{R}\cdot\frac{s_o}{\ell_o}\Leftrightarrow$

$$\frac{n_1}{s_o}\left[1+\phi^2\left(\frac{R}{s_o}+\frac{R^2}{s_o^2}\right)\right]^{-1/2}+\frac{n_2}{s_i}\left[1-\phi^2\left(\frac{R}{s_i}-\frac{R^2}{s_i^2}\right)\right]^{-1/2}$$
$$=\frac{n_2}{R}\left[1-\phi^2\left(\frac{R}{s_i}-\frac{R^2}{s_i^2}\right)\right]^{-1/2}-\frac{n_1}{R}\left[1+\phi^2\left(\frac{R}{s_o}+\frac{R^2}{s_o^2}\right)\right]^{-1/2}$$

Approximating the expressions for ℓ_o^{-1} and ℓ_i^{-1} by the first two terms in their binomial series, we get

$$\frac{n_1}{s_o}\left[1-\frac{1}{2}\phi^2\left(\frac{R}{s_o}+\frac{R^2}{s_o^2}\right)\right]+\frac{n_2}{s_i}\left[1+\frac{1}{2}\phi^2\left(\frac{R}{s_i}-\frac{R^2}{s_i^2}\right)\right]$$

$$=\frac{n_2}{R}\left[1+\frac{1}{2}\phi^2\left(\frac{R}{s_i}-\frac{R^2}{s_i^2}\right)\right]-\frac{n_1}{R}\left[1-\frac{1}{2}\phi^2\left(\frac{R}{s_o}+\frac{R^2}{s_o^2}\right)\right]\Leftrightarrow$$

$$\frac{n_1}{s_o}-\frac{n_1\phi^2}{2s_o}\left(\frac{R}{s_o}+\frac{R^2}{s_o^2}\right)+\frac{n_2}{s_i}+\frac{n_2\phi^2}{2s_i}\left(\frac{R}{s_i}-\frac{R^2}{s_i^2}\right)$$

$$=\frac{n_2}{R}+\frac{n_2\phi^2}{2R}\left(\frac{R}{s_i}-\frac{R^2}{s_i^2}\right)-\frac{n_1}{R}+\frac{n_1\phi^2}{2R}\left(\frac{R}{s_o}+\frac{R^2}{s_o^2}\right)\Leftrightarrow$$

$$\frac{n_1}{s_o}+\frac{n_2}{s_i}$$

$$=\frac{n_2}{R}-\frac{n_1}{R}+\frac{n_1\phi^2}{2s_o}\left(\frac{R}{s_o}+\frac{R^2}{s_o^2}\right)+\frac{n_1\phi^2}{2R}\left(\frac{R}{s_o}+\frac{R^2}{s_o^2}\right)+\frac{n_2\phi^2}{2R}\left(\frac{R}{s_i}-\frac{R^2}{s_i^2}\right)-\frac{n_2\phi^2}{2s_i}\left(\frac{R}{s_i}-\frac{R^2}{s_i^2}\right)$$

$$=\frac{n_2-n_1}{R}+\frac{n_1\phi^2}{2}\left(\frac{R}{s_o}+\frac{R^2}{s_o^2}\right)\left(\frac{1}{s_o}+\frac{1}{R}\right)+\frac{n_2\phi^2}{2}\left(\frac{R}{s_i}-\frac{R^2}{s_i^2}\right)\left(\frac{1}{R}-\frac{1}{s_i}\right)$$

$$=\frac{n_2-n_1}{R}+\frac{n_1\phi^2R^2}{2s_o}\left(\frac{1}{R}+\frac{1}{s_o}\right)\left(\frac{1}{R}+\frac{1}{s_o}\right)+\frac{n_2\phi^2R^2}{2s_i}\left(\frac{1}{R}-\frac{1}{s_i}\right)\left(\frac{1}{R}-\frac{1}{s_i}\right)$$

$$=\frac{n_2-n_1}{R}+\phi^2R^2\left[\frac{n_1}{2s_o}\left(\frac{1}{R}+\frac{1}{s_o}\right)^2+\frac{n_2}{2s_i}\left(\frac{1}{R}-\frac{1}{s_i}\right)^2\right]$$

From Figure 8, we see that $\sin\phi=h/R$. So if we approximate $\sin\phi$ with ϕ , we get $h=R\phi$ and $h^2=\phi^2R^2$ and hence, Equation 4, as desired.

33. **(a)** If the water is deep, then $2\pi d/L$ is large, and we know that $\tanh x\to 1$ as $x\to\infty$. So we can approximate $\tanh(2\pi d/L)\approx 1$, and so $v^2\approx gL/(2\pi)\Leftrightarrow v\approx\sqrt{gL/(2\pi)}$.
(b) From the table, the first term in the Maclaurin series of $\tanh x$ is x , so if the water is shallow, we can approximate

$\tanh \frac{2\pi d}{L} \approx \frac{2\pi d}{L}$, and so $v^2 \approx \frac{gL}{2\pi} \cdot \frac{2\pi d}{L} \Leftrightarrow v \approx \sqrt{gd}$.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\tanh x$	0
1	$\operatorname{sech}^2 x$	1
2	$-2\operatorname{sech}^2 x \tanh x$	0
3	$2\operatorname{sech}^2 x(3\tanh^2 x-1)$	-2

(c) Since $\tanh x$ is an odd function, its Maclaurin series is alternating, so the error in the approximation $\tanh \frac{2\pi d}{L} \approx \frac{2\pi d}{L}$ is less than the first neglected term, which is

$$\frac{|f'''(0)|}{3!}\left(\frac{2\pi d}{L}\right)^3 = \frac{1}{3}\left(\frac{2\pi d}{L}\right)^3 .$$

If $L > 10d$, then $\frac{1}{3}\left(\frac{2\pi d}{L}\right)^3 < \frac{1}{3}\left(2\pi \cdot \frac{1}{10}\right)^3 = \frac{\pi^3}{375}$, so the error in the approximation $v^2 = gd$ is less than $\frac{gL}{2\pi} \cdot \frac{\pi^3}{375} \approx 0.0132gL$.

34. **(a)** $4\sqrt{\frac{L}{g}}\int_0^{\pi/2} \frac{dx}{\sqrt{1-k^2\sin^2 x}} = 4\sqrt{\frac{L}{g}}\int_0^{\pi/2}\left[1+\left(-k^2\sin^2 x\right)\right]^{-1/2}dx$

$$=4\sqrt{\frac{L}{g}}\int_0^{\pi/2}\left[1-\frac{1}{2}\left(-k^2\sin^2 x\right)+\frac{\frac{1}{2}\cdot\frac{3}{2}}{2!}\left(-k^2\sin^2 x\right)^2-\frac{\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}}{3!}\left(-k^2\sin^2 x\right)^3+\cdots\right]dx$$

$$=4\sqrt{\frac{L}{g}}\int_0^{\pi/2}\left[1+\left(\frac{1}{2}\right)k^2\sin^2 x+\left(\frac{1\cdot 3}{2\cdot 4}\right)k^4\sin^4 x+\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)k^6\sin^6 x+\cdots\right]dx$$

$$=4\sqrt{\frac{L}{g}}\left[\frac{\pi}{2}+\left(\frac{1}{2}\right)\left(\frac{1}{2}\cdot\frac{\pi}{2}\right)k^2+\left(\frac{1\cdot 3}{2\cdot 4}\right)\left(\frac{1\cdot 3}{2\cdot 4}\cdot\frac{\pi}{2}\right)k^4\right.$$
$$\left.+\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\frac{\pi}{2}\right)k^6+\cdots\right]$$

[split up the integral and use the result from Exercise 8.1.44]

$$=2\pi\sqrt{\frac{L}{g}}\left[1+\frac{1^2}{2^2}k^2+\frac{1^2\cdot 3^2}{2^2\cdot 4^2}k^4+\frac{1^2\cdot 3^2\cdot 5^2}{2^2\cdot 4^2\cdot 6^2}k^6+\cdots\right]$$

(b) The first of the two inequalities is true because all of the terms in the series are positive. For the second,

$$T = 2\pi\sqrt{\frac{L}{g}}\left[1+\frac{1^2}{2^2}k^2+\frac{1^2\cdot 3^2}{2^2\cdot 4^2}k^4+\frac{1^2\cdot 3^2\cdot 5^2}{2^2\cdot 4^2\cdot 6^2}k^6+\frac{1^2\cdot 3^2\cdot 5^2\cdot 7^2}{2^2\cdot 4^2\cdot 6^2\cdot 8^2}k^8+\cdots\right]$$

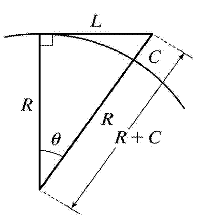
$$\leq 2\pi\sqrt{\frac{L}{g}}\left[1+\frac{1}{4}k^2+\frac{1}{4}k^4+\frac{1}{4}k^6+\frac{1}{4}k^8+\cdots\right]$$

The terms in brackets (after the first) form a geometric series with $a=\frac{1}{4}k^2$ and $r=k^2=\sin^2\left(\frac{1}{2}\theta_0\right)<1$. So $T\leq 2\pi\sqrt{\frac{L}{g}}\left[1+\frac{k^2/4}{1-k^2}\right]=2\pi\sqrt{\frac{L}{g}}\,\frac{4-3k^2}{4-4k^2}$.

(c) We substitute $L=1$, $g=9.8$, and $k=\sin\left(10^\circ/2\right)\approx 0.08716$, and the inequality from part (b) becomes $2.01090\leq T\leq 2.01093$, so $T\approx 2.0109$. The estimate $T\approx 2\pi\sqrt{L/g}\approx 2.0071$ differs by about 0.2%.

If $\theta_0=42^\circ$, then $k\approx 0.35837$ and the inequality becomes $2.07153\leq T\leq 2.08103$, so $T\approx 2.0763$. The one-term estimate is the same, and the discrepancy between the two estimates increases to about 3.4%.

35. **(a)** L is the length of the arc subtended by the angle θ, so $L=R\theta\Rightarrow\theta=L/R$. Now $\sec\theta=(R+C)/R$ $\Rightarrow R\sec\theta=R+C\Rightarrow C=R\sec\theta-R=R\sec(L/R)-R$.



(b) From Exercise 11, $\sec x\approx T_4(x)=1+\frac{1}{2}x^2+\frac{5}{24}x^4$. By part (a),

$$C\approx R\left[1+\frac{1}{2}\left(\frac{L}{R}\right)^2+\frac{5}{24}\left(\frac{L}{R}\right)^4\right]-R=R+\frac{1}{2}R\cdot\frac{L^2}{R^2}+\frac{5}{24}R\cdot\frac{L^4}{R^4}-R=\frac{L^2}{2R}+\frac{5L^4}{24R^3}.$$

(c) Taking $L=100$ km and $R=6370$ km, the formula in part (a) says that $C=R\sec(L/R)-R=6370\sec(100/6370)-6370\approx 0.78500996544$ km

The formula in part (b) says that $C\approx\frac{L^2}{2R}+\frac{5L^4}{24R^3}=\frac{100^2}{2\cdot 6370}+\frac{5\cdot 100^4}{24\cdot 6370^3}\approx 0.78500995736$ km.

The difference between these two results is only 0.000000008 08 km, or 0.000008 08 m!

36. $T_n(x)=f(a)+\frac{f'(a)}{1!}(x-a)+\frac{f''(a)}{2!}(x-a)^2+\cdots+\frac{f^{(n)}(a)}{n!}(x-a)^n$. Let $0\le m\le n$. Then

$T_n^{(m)}(x)=m!\frac{f^{(m)}(a)}{m!}(x-a)^0+(m+1)(m)\cdots(2)\frac{f^{(m+1)}(a)}{(m+1)!}(x-a)^1+\cdots$

$+n(n-1)\cdots(n-m+1)\frac{f^{(n)}(a)}{n!}(x-a)^{n-m}$

For $x=a$, all terms in this sum except the first one are 0 , so $T_n^{(m)}(a)=\frac{m!f^{(m)}(a)}{m!}=f^{(m)}(a)$.

37. Using $f(x)=T_n(x)+R_n(x)$ with $n=1$ and $x=r$, we have $f(r)=T_1(r)+R_1(r)$, where T_1 is the first-degree Taylor polynomial of f at a . Because $a=x_n$, $f(r)=f\left(x_n\right)+f'\left(x_n\right)\left(r-x_n\right)+R_1(r)$. But r is a root of f , so $f(r)=0$ and we have $0=f\left(x_n\right)+f'\left(x_n\right)\left(r-x_n\right)+R_1(r)$. Taking the first two terms to the left side gives us $f'\left(x_n\right)\left(x_n-r\right)-f\left(x_n\right)=R_1(r)$. Dividing by $f'\left(x_n\right)$, we get $x_n-r-\frac{f\left(x_n\right)}{f'\left(x_n\right)}=\frac{R_1(r)}{f'\left(x_n\right)}$. By the formula for Newton's method, the left side of the preceding equation is $x_{n+1}-r$, so $\left|x_{n+1}-r\right|=\left|\frac{R_1(r)}{f'\left(x_n\right)}\right|$. Taylor's Inequality gives us $\left|R_1(r)\right|\le\frac{\left|f''(r)\right|}{2!}\left|r-x_n\right|^2$. Combining this inequality with the facts $\left|f''(x)\right|\le M$ and $\left|f'(x)\right|\ge K$ gives us $\left|x_{n+1}-r\right|\le\frac{M}{2K}\left|x_n-r\right|^2$.

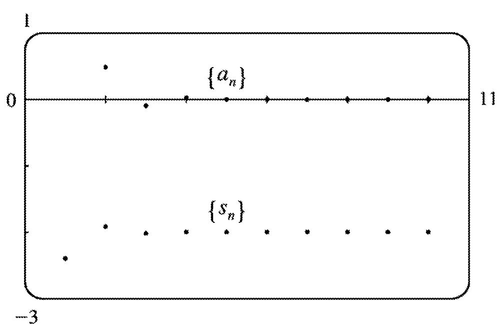
1. **(a)** A sequence is an ordered list of numbers whereas a series is the *sum* of a list of numbers.
(b) A series is convergent if the sequence of partial sums is a convergent sequence. A series is divergent if it is not convergent.

2. $\Sigma_{n=1}^{\infty} a_n = 5$ means that by adding sufficiently many terms of the series we can get as close as we like to the number 5 .

In other words, it means that $_{n\rightarrow\infty} s_n = 5$, where s_n is the n th partial sum, that is, $\Sigma_{i=1}^{n} a_i$.

3.

n	s_n
1	–2.40000
2	–1.92000
3	–2.01600
4	–1.99680
5	–2.00064
6	–1.99987
7	–2.00003
8	–1.99999
9	–2.00000
10	–2.00000



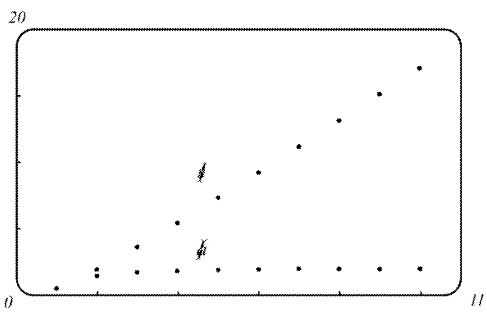
From the graph and the table, it seems that the series converges to –2 . In fact, it is a geometric series with a=–2.4 and $r=-\frac{1}{5}$, so its sum is $\Sigma_{n=1}^{\infty} \frac{12}{(-5)^n} = \frac{-2.4}{1-\left(-\frac{1}{5}\right)} = \frac{-2.4}{1.2} = -2$. Note that the dot corresponding to n=1 is part of both $\{a_n\}$ and $\{s_n\}$.

TI–86 Note: To graph $\{a_n\}$ and $\{s_n\}$, set your calculator to Param mode and DrawDot mode. (DrawDot is under GRAPH, MORE, FORMT (F3).) Now under E(t)= make the assignments: xt1=t, yt1=12/(–5)^t, xt2=t, yt2=sum seq(yt1,t,1,t,1). (sum and seq are under

LIST, OPS (F5), MORE.) Under WIND use 1,10,1,0,10,1,–3,1,1 to obtain a graph similar to the one above. Then use TRACE\;(F4) to see the values.

4.

n	s_n
1	0.50000
2	1.90000
3	3.60000
4	5.42353
5	7.30814
6	9.22706
7	11.16706
8	13.12091
9	15.08432
10	17.05462

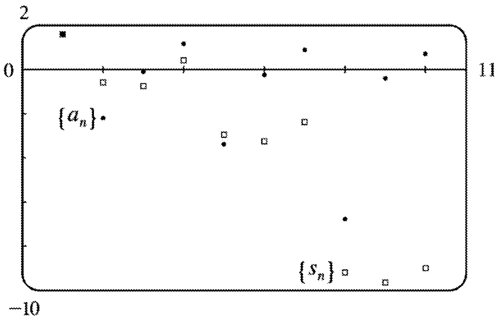


The series $\sum_{n=1}^{\infty} \frac{2n^2-1}{n^2+1}$ diverges, since its terms do not approach 0.

5.

n	s_n
1	1.55741
2	–0.62763
3	–0.77018
4	0.38764
5	–2.99287
6	–3.28388
7	–2.41243
8	–9.21214
9	–9.66446

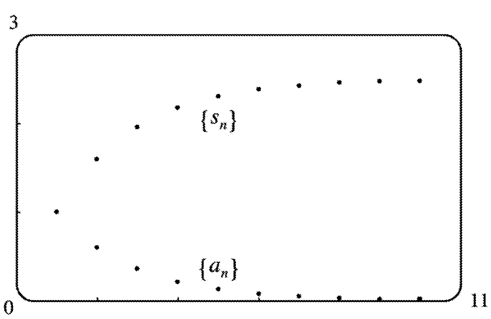
10	–9.01610



The series $\sum_{n=1}^{\infty} \tan n$ diverges, since its terms do not approach 0.

6.

n	s_n
1	1.00000
2	1.60000
3	1.96000
4	2.17600
5	2.30560
6	2.38336
7	2.43002
8	2.45801
9	2.47481
10	2.48488

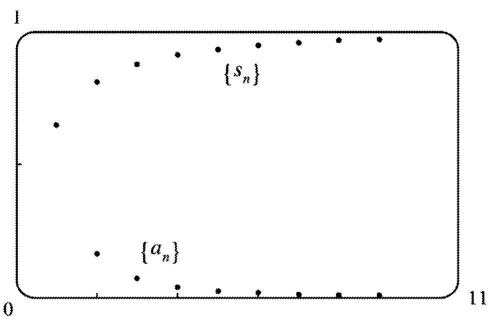


From the graph and the table, it seems that the series converges to 2.5.

In fact, it is a geometric series with $a=1$ and $r=0.6$, so its sum is $\sum_{n=1}^{\infty} (0.6)^{n-1} = \frac{1}{1-0.6} = \frac{1}{2/5} = 2.5$.

7.

n	s_n
1	0.64645
2	0.80755
3	0.87500
4	0.91056
5	0.93196
6	0.94601
7	0.95581
8	0.96296
9	0.96838
10	0.97259



From the graph, it seems that the series converges to 1 . To find the sum, we write

$$s_n = \sum_{i=1}^{n}\left(\frac{1}{i^{1.5}} - \frac{1}{(i+1)^{1.5}}\right)$$

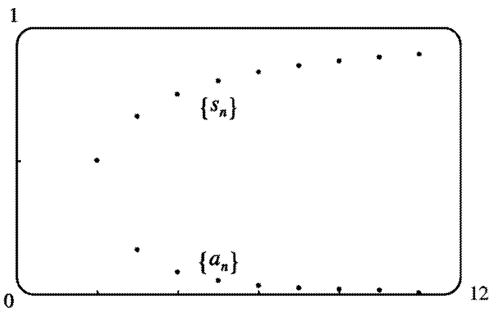
$$= \left(1 - \frac{1}{2^{1.5}}\right) + \left(\frac{1}{2^{1.5}} - \frac{1}{3^{1.5}}\right) + \left(\frac{1}{3^{1.5}} - \frac{1}{4^{1.5}}\right) + \cdots + \left(\frac{1}{n^{1.5}} - \frac{1}{(n+1)^{1.5}}\right) = 1 - \frac{1}{(n+1)^{1.5}}$$

So the sum is $\lim_{n\to\infty} s_n = 1 - 0 = 1$.

8.

n	s_n
2	0.50000
3	0.66667
4	0.75000
5	0.80000
6	0.83333

7	0.85714
8	0.87500
9	0.88889
10	0.90000
11	0.90909
100	0.99000



From the graph and the table, it seems that the series converges to 1 . To find the sum, we write

$$s_n = \sum_{i=2}^{n} \frac{1}{i(i-1)} = \sum_{i=2}^{n} \left(\frac{1}{i-1} - \frac{1}{i} \right) \quad \text{[partial fractions]}$$

$$= \left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots+\left(\frac{1}{n-1}-\frac{1}{n}\right) = 1-\frac{1}{n},$$

and so the sum is $\lim_{n\to\infty} s_n = 1-0 = 1$.

9. **(a)** $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2n}{3n+1} = \frac{2}{3}$, so the *sequence* $\{a_n\}$ is convergent by (.1.1).

(b) Since $\lim_{n\to\infty} a_n = \frac{2}{3} \neq 0$, the *series* $\sum_{n=1}^{\infty} a_n$ is divergent by the Test for Divergence (7).

10. **(a)** Both $\sum_{i=1}^{n} a_i$ and $\sum_{j=1}^{n} a_j$ represent the sum of the first n terms of the sequence $\{a_n\}$, that is, the n th partial sum.

(b) $\sum_{i=1}^{n} a_j = \underbrace{a_j + a_j + \ldots + a_j}_{n \text{ terms}} = na_j$, which, in general, is not the same as $\sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n$.

11. $3+2+\frac{4}{3}+\frac{8}{9}+\cdots$ is a geometric series with first term $a=3$ and common ratio $r=\frac{2}{3}$. Since $|r| = \frac{2}{3} < 1$, the series converges to $\frac{a}{1-r} = \frac{3}{1-2/3} = \frac{3}{1/3} = 9$.

12. $\frac{1}{8}-\frac{1}{4}+\frac{1}{2}-1+\cdots$ is a geometric series with $r=-2$. Since $|r|=2>1$, the series diverges.

13. $-2+\frac{5}{2}-\frac{25}{8}+\frac{125}{32}-\cdots$ is a geometric series with $a=-2$ and $r=\frac{5/2}{-2}=-\frac{5}{4}$. Since $|r|=\frac{5}{4}>1$, the series diverges by (4).

14. $1+0.4+0.16+0.064+\cdots$ is a geometric series with ratio 0.4 . The series converges to $\frac{a}{1-r}=\frac{1}{1-2/5}=\frac{5}{3}$ since $|r|=\frac{2}{5}<1$.

15. $\Sigma_{n=1}^{\infty}5\left(\frac{2}{3}\right)^{n-1}$ is a geometric series with $a=5$ and $r=\frac{2}{3}$. Since $|r|=\frac{2}{3}<1$, the series converges to $\frac{a}{1-r}=\frac{5}{1-2/3}=\frac{5}{1/3}=15$.

16. $\Sigma_{n=1}^{\infty}\frac{(-6)^{n-1}}{5^{n-1}}$ is a geometric series with $a=1$ and $r=-\frac{6}{5}$. The series diverges since $|r|=\frac{6}{5}>1$.

17. $\Sigma_{n=1}^{\infty}\frac{(-3)^{n-1}}{4^{n}}=\frac{1}{4}\Sigma_{n=1}^{\infty}\left(-\frac{3}{4}\right)^{n-1}$. The latter series is geometric with $a=1$ and $r=-\frac{3}{4}$. Since $|r|=\frac{3}{4}<1$, it converges to $\frac{1}{1-(-3/4)}=\frac{4}{7}$. Thus, the given series converges to $\left(\frac{1}{4}\right)\left(\frac{4}{7}\right)=\frac{1}{7}$.

18. $\Sigma_{n=0}^{\infty}\frac{1}{(\sqrt{2})^{n}}$ is a geometric series with ratio $r=\frac{1}{\sqrt{2}}$. Since $|r|=\frac{1}{\sqrt{2}}<1$, the series converges. Its sum is $\frac{1}{1-1/\sqrt{2}}=\frac{\sqrt{2}}{\sqrt{2}-1}=\frac{\sqrt{2}}{\sqrt{2}-1}\cdot\frac{\sqrt{2}+1}{\sqrt{2}+1}=\sqrt{2}(\sqrt{2}+1)=2+\sqrt{2}$.

19. $\Sigma_{n=0}^{\infty}\frac{\pi^{n}}{3^{n+1}}=\frac{1}{3}\Sigma_{n=0}^{\infty}\left(\frac{\pi}{3}\right)^{n}$ is a geometric series with ratio $r=\frac{\pi}{3}$. Since $|r|>1$, the series diverges.

20. $\Sigma_{n=1}^{\infty}\frac{e^{n}}{3^{n-1}}=3\Sigma_{n=1}^{\infty}\left(\frac{e}{3}\right)^{n}$ is a geometric series with first term $3(e/3)=e$ and ratio $r=\frac{e}{3}$. Since $|r|<1$, the series converges. Its sum is $\frac{e}{1-e/3}=\frac{3e}{3-e}$.

21. $\sum_{n=1}^{\infty} \frac{n}{n+5}$ diverges since $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{n+5} = 1 \neq 0$.

22. $\sum_{n=1}^{\infty} \frac{3}{n} = 3\sum_{n=1}^{\infty} \frac{1}{n}$ diverges since each of its partial sums is 3 times the corresponding partial sum of the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$, which diverges. In general, constant multiples of divergent series are divergent.

23. Using partial fractions, the partial sums are

$$s_n = \sum_{i=2}^{n} \frac{2}{(i-1)(i+1)} = \sum_{i=2}^{n} \left(\frac{1}{i-1} - \frac{1}{i+1} \right)$$

$$= \left(1 - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \cdots + \left(\frac{1}{n-3} - \frac{1}{n-1}\right) + \left(\frac{1}{n-2} - \frac{1}{n}\right)$$

This sum is a telescoping series and $s_n = 1 + \frac{1}{2} - \frac{1}{n-1} - \frac{1}{n}$.

Thus, $\sum_{n=2}^{\infty} \frac{2}{n^2-1} = \lim_{n\to\infty} \left(1 + \frac{1}{2} - \frac{1}{n-1} - \frac{1}{n}\right) = \frac{3}{2}$.

24. $\sum_{n=1}^{\infty} \frac{(n+1)^2}{n(n+2)}$ diverges by (7), the Test for Divergence, since

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2+2n+1}{n^2+2n} = \lim_{n\to\infty} \left(1 + \frac{1}{n^2+2n}\right) = 1 \neq 0 .$$

25. $\sum_{k=2}^{\infty} \frac{k^2}{k^2-1}$ diverges by the Test for Divergence since $\lim_{k\to\infty} a_k = \lim_{k\to\infty} \frac{k^2}{k^2-1} = 1 \neq 0$.

26. Converges. $s_n = \sum_{i=1}^{n} \frac{2}{i^2+4i+3} = \sum_{i=1}^{n} \left(\frac{1}{i+1} - \frac{1}{i+3} \right)$ (using partial fractions). The latter sum is

$$\left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{4} - \frac{1}{6}\right) + \left(\frac{1}{5} - \frac{1}{7}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+2}\right) + \left(\frac{1}{n+1} - \frac{1}{n+3}\right) = \frac{1}{2} + \frac{1}{3} - \frac{1}{n+2} - \frac{1}{n+3}$$

(telescoping series). Thus, $\sum_{n=1}^{\infty} \frac{2}{n^2+4n+3} = \lim_{n\to\infty} \left(\frac{1}{2} + \frac{1}{3} - \frac{1}{n+2} - \frac{1}{n+3} \right) = \frac{1}{2} + \frac{1}{3} = \frac{5}{6}$.

27. Converges.

$\sum_{n=1}^{\infty} \frac{3^n+2^n}{6^n} = \sum_{n=1}^{\infty} \left(\frac{3^n}{6^n} + \frac{2^n}{6^n} \right) = \sum_{n=1}^{\infty} \left[\left(\frac{1}{2} \right)^n + \left(\frac{1}{3} \right)^n \right] = \frac{1/2}{1-1/2} + \frac{1/3}{1-1/3} = 1 + \frac{1}{2} = \frac{3}{2}$

28. $\sum_{n=1}^{\infty} \left[(0.8)^{n-1} - (0.3)^n \right] = \sum_{n=1}^{\infty} (0.8)^{n-1} - \sum_{n=1}^{\infty} (0.3)^n = \frac{1}{1-0.8} - \frac{0.3}{1-0.3} = 5 - \frac{3}{7} = \frac{32}{7}$.

29. $\sum_{n=1}^{\infty} \sqrt[n]{2} = 2 + \sqrt{2} + \sqrt[3]{2} + \sqrt[4]{2} + \cdots$ diverges by the Test for Divergence since $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \sqrt[n]{2} = \lim_{n\to\infty} 2^{1/n} = 2^0 = 1 \neq 0$.

30. $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \ln \left(\frac{n}{2n+5} \right) = \lim_{n\to\infty} \ln \left(\frac{1}{2+5/n} \right) = \ln \frac{1}{2} \neq 0$, so the series diverges by the Test for Divergence.

31. $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \arctan n = \frac{\pi}{2} \neq 0$, so the series diverges by the Test for Divergence.

32. $\sum_{k=1}^{\infty} (\cos 1)^k$ is a geometric series with ratio $r = \cos 1 \approx 0.540302$. It converges because $|r| < 1$. Its sum is $\frac{\cos 1}{1-\cos 1} \approx 1.175343$.

33. The first series is a telescoping sum:

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{3}{n(n+3)} &= \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+3} \right) = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} + \frac{1}{n+1} - \frac{1}{n+2} + \frac{1}{n+2} - \frac{1}{n+3} \right) \\ &= \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right) + \sum_{n=1}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n+2} \right) + \sum_{n=1}^{\infty} \left(\frac{1}{n+2} - \frac{1}{n+3} \right) \\ &= 1 + \frac{1}{2} + \frac{1}{3} = \frac{11}{6} \end{aligned}$$

The second series is geometric with first term $\frac{5}{4}$ and ratio $\frac{1}{4}$: $\sum_{n=1}^{\infty} \frac{5}{4^n} = \frac{5/4}{1-1/4} = \frac{5}{3}$. Thus,

$\sum_{n=1}^{\infty} \left(\frac{3}{n(n+3)} + \frac{5}{4^n} \right) = \sum_{n=1}^{\infty} \frac{3}{n(n+3)} + \sum_{n=1}^{\infty} \frac{5}{4^n} = \frac{11}{6} + \frac{5}{3} = \frac{7}{2}$.

34. $\sum_{n=1}^{\infty} \left(\frac{3}{5^n} + \frac{2}{n} \right)$ diverges because $\sum_{n=1}^{\infty} \frac{2}{n} = 2\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. (If it converged, then

$\frac{1}{2}\cdot 2\sum_{n=1}^{\infty}\frac{1}{n}$ would also converge by Theorem 8(i), but we know from Example 7 that the harmonic series $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges.) If the given series converges, then the difference $\sum_{n=1}^{\infty}\left(\frac{3}{5^n}+\frac{2}{n}\right)-\sum_{n=1}^{\infty}\frac{3}{5^n}$ must converge (since $\sum_{n=1}^{\infty}\frac{3}{5^n}$ is a convergent geometric series) and equal $\sum_{n=1}^{\infty}\frac{2}{n}$, but we have just seen that $\sum_{n=1}^{\infty}\frac{2}{n}$ diverges, so the given series must also diverge.

35. $0.\overline{2}=\frac{2}{10}+\frac{2}{10^2}+\cdots$ is a geometric series with $a=\frac{2}{10}$ and $r=\frac{1}{10}$. It converges to $\frac{a}{1-r}=\frac{2/10}{1-1/10}=\frac{2}{9}$.

36. $0.\overline{73}=\frac{73}{10^2}+\frac{73}{10^4}+\cdots=\frac{73/10^2}{1-1/10^2}=\frac{73/100}{99/100}=\frac{73}{99}$

37. $3.\overline{417}=3+\frac{417}{10^3}+\frac{417}{10^6}+\cdots=3+\frac{417/10^3}{1-1/10^3}=3+\frac{417}{999}=\frac{3414}{999}=\frac{1138}{333}$

38. $6.2\overline{54}=6.2+\frac{54}{10^3}+\frac{54}{10^5}+\cdots=6.2+\frac{54/10^3}{1-1/10^2}=\frac{62}{10}+\frac{54}{990}=\frac{6192}{990}=\frac{344}{55}$

39. $0.123\overline{456}=\frac{123}{1000}+\frac{0.000456}{1-0.001}=\frac{123}{1000}+\frac{456}{999{,}000}=\frac{123{,}333}{999{,}000}=\frac{41{,}111}{333{,}000}$

40. $5.\overline{6021}=5+\frac{6021}{10^4}+\frac{6021}{10^8}+\cdots=5+\frac{6021/10^4}{1-1/10^4}=5+\frac{6021}{9999}=\frac{56{,}016}{9999}=\frac{6224}{1111}$

41. $\sum_{n=1}^{\infty}\frac{x^n}{3^n}=\sum_{n=1}^{\infty}\left(\frac{x}{3}\right)^n$ is a geometric series with $r=\frac{x}{3}$, so the series converges $\Leftrightarrow |r|<1 \Leftrightarrow \frac{|x|}{3}<1 \Leftrightarrow |x|<3$; that is, $-3<x<3$. In that case, the sum of the series is $\frac{a}{1-r}=\frac{x/3}{1-x/3}=\frac{x/3}{1-x/3}\cdot\frac{3}{3}=\frac{x}{3-x}$.

42. $\sum_{n=1}^{\infty}(x-4)^n$ is a geometric series with $r=x-4$, so the series converges $\Leftrightarrow |r|<1 \Leftrightarrow |x-4|<1 \Leftrightarrow$ $3<x<5$. In that case, the sum of the series is $\dfrac{x-4}{1-(x-4)}=\dfrac{x-4}{5-x}$.

43. $\sum_{n=0}^{\infty}4^n x^n=\sum_{n=0}^{\infty}(4x)^n$ is a geometric series with $r=4x$, so the series converges $\Leftrightarrow |r|<1 \Leftrightarrow 4|x|<1$ $\Leftrightarrow |x|<\dfrac{1}{4}$. In that case, the sum of the series is $\dfrac{1}{1-4x}$.

44. $\sum_{n=0}^{\infty}\dfrac{(x+3)^n}{2^n}$ is a geometric series with $r=\dfrac{x+3}{2}$, so the series converges $\Leftrightarrow |r|<1 \Leftrightarrow \dfrac{|x+3|}{2}<1 \Leftrightarrow$ $|x+3|<2 \Leftrightarrow -5<x<-1$. For these values of x , the sum of the series is $\dfrac{1}{1-(x+3)/2}=\dfrac{2}{2-(x+3)}=-\dfrac{2}{x+1}$.

45. $\sum_{n=0}^{\infty}\dfrac{\cos^n x}{2^n}$ is a geometric series with first term 1 and ratio $r=\dfrac{\cos x}{2}$, so it converges $\Leftrightarrow |r|<1$. But $|r|=\dfrac{|\cos x|}{2}\le\dfrac{1}{2}$ for all x . Thus, the series converges for all real values of x and the sum of the series is $\dfrac{1}{1-(\cos x)/2}=\dfrac{2}{2-\cos x}$.

46. Because $\dfrac{1}{n}\to 0$ and ln is continuous, we have $\lim\limits_{n\to\infty}\ln\left(1+\dfrac{1}{n}\right)=\ln 1=0$. We now show that the series $\sum_{n=1}^{\infty}\ln\left(1+\dfrac{1}{n}\right)=\sum_{n=1}^{\infty}\ln\left(\dfrac{n+1}{n}\right)=\sum_{n=1}^{\infty}[\ln(n+1)-\ln n]$ diverges.
$s_n=(\ln 2-\ln 1)+(\ln 3-\ln 2)+\cdots+(\ln(n+1)-\ln n)=\ln(n+1)-\ln 1=\ln(n+1)$. As $n\to\infty$, $s_n=\ln(n+1)\to\infty$, so the series diverges.

47. After defining f , We use convert(f,parfrac); in Maple, Apart in Mathematica, or Expand Rational and Simplify in Derive to find that the general term is $\dfrac{1}{(4n+1)(4n-3)}=-\dfrac{1/4}{4n+1}+\dfrac{1/4}{4n-3}$. So the n th partial sum is

$$s_n=\sum_{k=1}^{n}\left(-\frac{1/4}{4k+1}+\frac{1/4}{4k-3}\right)=\frac{1}{4}\sum_{k=1}^{n}\left(\frac{1}{4k-3}-\frac{1}{4k+1}\right)$$
$$=\frac{1}{4}\left[\left(1-\frac{1}{5}\right)+\left(\frac{1}{5}-\frac{1}{9}\right)+\left(\frac{1}{9}-\frac{1}{13}\right)+\cdots+\left(\frac{1}{4n-3}-\frac{1}{4n+1}\right)\right]=\frac{1}{4}\left(1-\frac{1}{4n+1}\right)$$

The series converges to $\lim_{n\to\infty} s_n = \frac{1}{4}$. This can be confirmed by directly computing the sum using `sum(f,1..infinity);` (in Maple), `Sum[f,{n,1,Infinity}]` (in Mathematica), or `Calculus Sum` (from 1 to ∞) and Simplify (in Derive).

48. See Exercise 47 for specific CAS commands. $\frac{n^2+3n+1}{\left(n^2+n\right)^2} = \frac{1}{n^2} + \frac{1}{n} - \frac{1}{(n+1)^2} - \frac{1}{n+1}$. So the n th partial sum is

$$s_n = \sum_{k=1}^{n} \left(\frac{1}{k^2} + \frac{1}{k} - \frac{1}{(k+1)^2} - \frac{1}{k+1} \right)$$
$$= \left(1+1-\frac{1}{2^2} - \frac{1}{2} \right) + \left(\frac{1}{2^2} + \frac{1}{2} - \frac{1}{3^2} - \frac{1}{3} \right) + \cdots + \left(\frac{1}{n^2} + \frac{1}{n} - \frac{1}{(n+1)^2} - \frac{1}{n+1} \right)$$
$$= 1+1-\frac{1}{(n+1)^2} - \frac{1}{n+1}$$

The series converges to $\lim_{n\to\infty} s_n = 2$.

49. For $n=1$, $a_1=0$ since $s_1=0$. For $n>1$,

$$a_n = s_n - s_{n-1} = \frac{n-1}{n+1} - \frac{(n-1)-1}{(n-1)+1} = \frac{(n-1)n-(n+1)(n-2)}{(n+1)n} = \frac{2}{n(n+1)}$$

Also, $\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{1-1/n}{1+1/n} = 1$.

50. $a_1 = s_1 = 3 - \frac{1}{2} = \frac{5}{2}$. For $n \neq 1$,

$$a_n = s_n - s_{n-1} = \left(3-n2^{-n}\right) - \left[3-(n-1)2^{-(n-1)}\right] = -\frac{n}{2^n} + \frac{n-1}{2^{n-1}} \cdot \frac{2}{2} = \frac{2(n-1)}{2^n} - \frac{n}{2^n} = \frac{n-2}{2^n}$$

Also, $\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \left(3 - \frac{n}{2^n} \right) = 3$ because $\lim_{x\to\infty} \frac{x}{2^x} = \lim_{x\to\infty} \frac{1}{2^x \ln 2} = 0$.

51. **(a)** The first step in the chain occurs when the local government spends D dollars. The people who receive it spend a fraction c of those D dollars, that is, Dc dollars. Those who receive the Dc

dollars spend a fraction c of it, that is, Dc^2 dollars. Continuing in this way, we see that the total spending after n transactions is $S_n = D + Dc + Dc^2 + \cdots + Dc^{n-1} = \dfrac{D(1-c^n)}{1-c}$ by (3).

(b)

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{D(1-c^n)}{1-c} = \frac{D}{1-c} \lim_{n\to\infty} (1-c^n) = \frac{D}{1-c} \quad \left(\text{since } 0<c<1 \Rightarrow \lim_{n\to\infty} c^n = 0\right)$$
$$= \frac{D}{s} \quad (\text{since } c+s=1) = kD \quad (\text{since } k=1/s)$$

If $c=0.8$, then $s=1-c=0.2$ and the multiplier is $k=1/s=5$.

52. **(a)** Initially, the ball falls a distance H, then rebounds a distance rH, falls rH, rebounds r^2H, falls r^2H, etc. The total distance it travels is

$$\begin{aligned} &H+2rH+2r^2H+2r^3H+\cdots = H\left(1+2r+2r^2+2r^3+\cdots\right) \\ &= H\left[1+2r\left(1+r+r^2+\cdots\right)\right] = H\left[1+2r\left(\frac{1}{1-r}\right)\right] = H\left(\frac{1+r}{1-r}\right) \text{meters} \end{aligned}$$

(b) From Example 3 in Section 2.1, we know that a ball falls $\frac{1}{2}gt^2$ meters in t seconds, where g is the gravitational acceleration. Thus, a ball falls h meters in $t=\sqrt{2h/g}$ seconds. The total travel time in seconds is

$$\begin{aligned} &\sqrt{\frac{2H}{g}} + 2\sqrt{\frac{2H}{g}r} + 2\sqrt{\frac{2H}{g}r^2} + 2\sqrt{\frac{2H}{g}r^3} + \cdots = \sqrt{\frac{2H}{g}}\left[1+2\sqrt{r}+2\sqrt{r}^2+2\sqrt{r}^3+\cdots\right] \\ &= \sqrt{\frac{2H}{g}}\left(1+2\sqrt{r}\left[1+\sqrt{r}+\sqrt{r}^2+\cdots\right]\right) = \sqrt{\frac{2H}{g}}\left[1+2\sqrt{r}\left(\frac{1}{1-\sqrt{r}}\right)\right] = \sqrt{\frac{2H}{g}}\,\frac{1+\sqrt{r}}{1-\sqrt{r}} \end{aligned}$$

(c) It will help to make a chart of the time for each descent and each rebound of the ball, together with the velocity just before and just after each bounce. Recall that the time in seconds needed to fall h meters is $\sqrt{2h/g}$. The ball hits the ground with velocity $-g\sqrt{2h/g} = -\sqrt{2hg}$ (taking the upward direction to be positive) and rebounds with velocity $kg\sqrt{2h/g} = k\sqrt{2hg}$, taking time $k\sqrt{2h/g}$ to reach the top of its bounce, where its velocity is 0. At that point, its height is k^2h. All these results follow from the formulas for vertical motion with gravitational acceleration $-g$:

$\dfrac{d^2y}{dt^2}=-g \Rightarrow v=\dfrac{dy}{dt}=v_0-gt \Rightarrow y=y_0+v_0t-\dfrac{1}{2}gt^2$.

number of descent	time of descent	speed before bounce	speed after bounce	time of ascent	peak height
1	$\sqrt{2H/g}$	$\sqrt{2Hg}$	$k\sqrt{2Hg}$	$k\sqrt{2H/g}$	k^2H
2	$\sqrt{2k^2H/g}$	$\sqrt{2k^2Hg}$	$k\sqrt{2k^2Hg}$	$k\sqrt{2k^2H/g}$	k^4H
3	$\sqrt{2k^4H/g}$	$\sqrt{2k^4Hg}$	$k\sqrt{2k^4Hg}$	$k\sqrt{2k^4H/g}$	k^6H
. . .	. . .	. . .	. . .	. . .	. . .

otal travel time in seconds is

$$\sqrt{\frac{2H}{g}}+\mathrm{k}\sqrt{\frac{2H}{g}}+\mathrm{k}\sqrt{\frac{2H}{g}}+\mathrm{k}^2\sqrt{\frac{2H}{g}}+\mathrm{k}^2\sqrt{\frac{2H}{g}}+\cdots=\sqrt{\frac{2H}{g}}\left(1+2k+2k^2+2k^3+\cdots\right)$$

$$=\sqrt{\frac{2H}{g}}\left[1+2k\left(1+k+k^2+\cdots\right)\right]=\sqrt{\frac{2H}{g}}\left[1+2k\left(\frac{1}{1-k}\right)\right]=\sqrt{\frac{2H}{g}}\,\frac{1+k}{1-k}$$

Another method: We could use part (b). At the top of the bounce, the height is $k^2h=rh$, so $\sqrt{r}=k$ and the result follows from part (b).

53. $\sum_{n=2}^{\infty}(1+c)^{-n}$ is a geometric series with $a=(1+c)^{-2}$ and $r=(1+c)^{-1}$, so the series converges when $\left|(1+c)^{-1}\right|<1 \Leftrightarrow |1+c|>1 \Leftrightarrow 1+c>1$ or $1+c<-1 \Leftrightarrow c>0$ or $c<-2$. We calculate the sum of the series and set it equal to 2 : $\dfrac{(1+c)^{-2}}{1-(1+c)^{-1}}=2 \Leftrightarrow \left(\dfrac{1}{1+c}\right)^2=2-2\left(\dfrac{1}{1+c}\right) \Leftrightarrow$

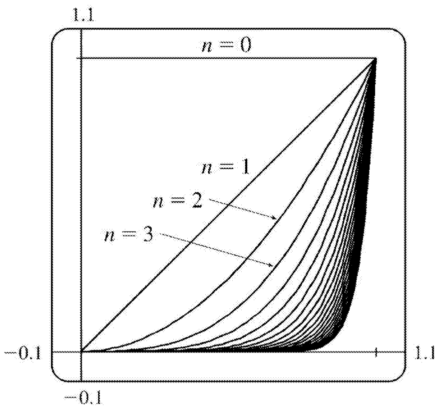
$1=2(1+c)^2-2(1+c) \Leftrightarrow 2c^2+2c-1=0 \Leftrightarrow c=\dfrac{-2\pm\sqrt{12}}{4}=\dfrac{\pm\sqrt{3}-1}{2}$. However, the negative root is inadmissible because $-2<\dfrac{-\sqrt{3}-1}{2}<0$. So $c=\dfrac{\sqrt{3}-1}{2}$.

54. The area between $y=x^{n-1}$ and $y=x^n$ for $0\le x\le 1$ is

$$\int_0^1\left(x^{n-1}-x^n\right)dx=\left[\frac{x^n}{n}-\frac{x^{n+1}}{n+1}\right]_0^1=\frac{1}{n}-\frac{1}{n+1}$$

$$=\frac{(n+1)-n}{n(n+1)}=\frac{1}{n(n+1)}$$

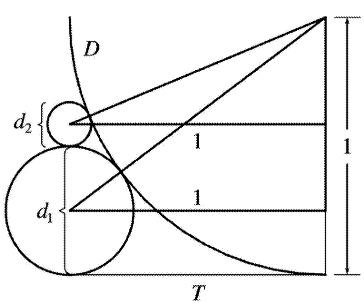
We can see from the diagram that as $n \to \infty$, the sum of the areas between the successive curves approaches the area of the unit square, that is, 1 . So $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$.



55. Let d_n be the diameter of C_n . We draw lines from the centers of the C_i to the center of D (or C), and using the Pythagorean Theorem, we can write $1^2 + \left(1 - \frac{1}{2} d_1\right)^2 = \left(1 + \frac{1}{2} d_1\right)^2 \Leftrightarrow$

$1 = \left(1 + \frac{1}{2} d_1\right)^2 - \left(1 - \frac{1}{2} d_1\right)^2 = 2d_1$ (difference of squares) $\Rightarrow d_1 = \frac{1}{2}$. Similarly,

$$1 = \left(1 + \frac{1}{2} d_2\right)^2 - \left(1 - d_1 - \frac{1}{2} d_2\right)^2 = 2d_2 + 2d_1 - d_1^2 - d_1 d_2$$
$$= (2 - d_1)(d_1 + d_2) \Leftrightarrow$$



$d_2 = \frac{1}{2 - d_1} - d_1 = \frac{\left(1 - d_1\right)^2}{2 - d_1}$, $1 = \left(1 + \frac{1}{2} d_3\right)^2 - \left(1 - d_1 - d_2 - \frac{1}{2} d_3\right)^2 \Leftrightarrow d_3 = \frac{\left[1 - \left(d_1 + d_2\right)\right]^2}{2 - \left(d_1 + d_2\right)}$, and in general, $d_{n+1} = \frac{\left(1 - \sum_{i=1}^{n} d_i\right)^2}{2 - \sum_{i=1}^{n} d_i}$. If we actually calculate d_2 and d_3 from the formulas above, we find that they are $\frac{1}{6}=\frac{1}{2\cdot 3}$ and $\frac{1}{12}=\frac{1}{3\cdot 4}$ respectively, so we suspect that in general, $d_n=\frac{1}{n(n+1)}$. To prove this, we use induction: Assume that for all $k\le n$, $d_k=\frac{1}{k(k+1)}=\frac{1}{k}-\frac{1}{k+1}$. Then $\sum_{i=1}^{n}d_i=1-\frac{1}{n+1}=\frac{n}{n+1}$ (telescoping sum). Substituting this into our formula for d_{n+1} , we get

$$d_{n+1}=\frac{\left[1-\frac{n}{n+1}\right]^2}{2-\left(\frac{n}{n+1}\right)}=\frac{\frac{1}{(n+1)^2}}{\frac{n+2}{n+1}}=\frac{1}{(n+1)(n+2)}$$, and the induction is complete.

Now, we observe that the partial sums $\sum_{i=1}^{n}d_i$ of the diameters of the circles approach 1 as $n\to\infty$; that is, $\sum_{n=1}^{\infty}a_n=\sum_{n=1}^{\infty}\frac{1}{n(n+1)}=1$, which is what we wanted to prove.

56. $|CD|=b\sin\theta$, $|DE|=|CD|\sin\theta=b\sin^2\theta$, $|EF|=|DE|\sin\theta=b\sin^3\theta$, ... Therefore, $|CD|+|DE|+|EF|+|FG|+\cdots=b\sum_{n=1}^{\infty}\sin^n\theta=b\left(\frac{\sin\theta}{1-\sin\theta}\right)$ since this is a geometric series with $r=\sin\theta$ and $|\sin\theta|<1$ (because $0<\theta<\frac{\pi}{2}$).

57. The series $1-1+1-1+1-1+\cdots$ diverges (geometric series with $r=-1$) so we cannot say that $0=1-1+1-1+1-1+\cdots$.

58. If $\sum_{n=1}^{\infty}a_n$ is convergent, then $\lim_{n\to\infty}a_n=0$ by Theorem 6, so $\lim_{n\to\infty}\frac{1}{a_n}\neq 0$, and so $\sum_{n=1}^{\infty}\frac{1}{a_n}$ is divergent by the Test for Divergence.

59. $\sum_{n=1}^{\infty}ca_n=\lim_{n\to\infty}\sum_{i=1}^{n}ca_i=\lim_{n\to\infty}c\sum_{i=1}^{n}a_i=c\lim_{n\to\infty}\sum_{i=1}^{n}a_i=c\sum_{n=1}^{\infty}a_n$, which exists by hypothesis.

60. If $\sum ca_n$ were convergent, then $\sum(1/c)(ca_n)=\sum a_n$ would be also, by Theorem 8. But this is not the case, so $\sum ca_n$ must diverge.

61. Suppose on the contrary that $\sum\left(a_n+b_n\right)$ converges. Then $\sum\left(a_n+b_n\right)$ and $\sum a_n$ are convergent series. So by Theorem 8, $\sum\left[\left(a_n+b_n\right)-a_n\right]$ would also be convergent. But $\sum\left[\left(a_n+b_n\right)-a_n\right]=\sum b_n$, a contradiction, since $\sum b_n$ is given to be divergent.

62. No. For example, take

$\Sigma a_n = \Sigma n$ and $\Sigma b_n = \Sigma(-n)$, which both diverge, yet $\Sigma\left(a_n + b_n\right) = \Sigma 0$, which converges with sum 0 .

63. The partial sums $\left\{s_n\right\}$ form an increasing sequence, since $s_n - s_{n-1} = a_n > 0$ for all n . Also, the sequence $\left\{s_n\right\}$ is bounded since $s_n \le 1000$ for all n . So by Theorem .1.11, the sequence of partial sums converges, that is, the series Σa_n is convergent.

64. **(a)**

$$\text{RHS} = \frac{1}{f_{n-1} f_n} - \frac{1}{f_n f_{n+1}} = \frac{f_n f_{n+1} - f_n f_{n-1}}{f_n^2 f_{n-1} f_{n+1}} = \frac{f_{n+1} - f_{n-1}}{f_n f_{n-1} f_{n+1}} = \frac{\left(f_{n-1} + f_n\right) - f_{n-1}}{f_n f_{n-1} f_{n+1}}$$

$$= \frac{1}{f_{n-1} f_{n+1}} = \text{LHS}$$

(b) $\Sigma_{n=2}^{\infty} \frac{1}{f_{n-1} f_{n+1}} = \Sigma_{n=2}^{\infty}\left(\frac{1}{f_{n-1} f_n} - \frac{1}{f_n f_{n+1}}\right)$

$= \lim_{n\to\infty}\left[\left(\frac{1}{f_1 f_2} - \frac{1}{f_2 f_3}\right) + \left(\frac{1}{f_2 f_3} - \frac{1}{f_3 f_4}\right) + \left(\frac{1}{f_3 f_4} - \frac{1}{f_4 f_5}\right) + \cdots + \left(\frac{1}{f_{n-1} f_n} - \frac{1}{f_n f_{n+1}}\right)\right.$

$= \lim_{n\to\infty}\left(\frac{1}{f_1 f_2} - \frac{1}{f_n f_{n+1}}\right) = \frac{1}{f_1 f_2} - 0 = \frac{1}{1\cdot 1} = 1$ because $f_n \to \infty$ as $n \to \infty$.

(c) $\Sigma_{n=2}^{\infty} \frac{f_n}{f_{n-1} f_{n+1}} = \Sigma_{n=2}^{\infty}\left(\frac{f_n}{f_{n-1} f_n} - \frac{f_n}{f_n f_{n+1}}\right)$ (as above)

$= \Sigma_{n=2}^{\infty}\left(\frac{1}{f_{n-1}} - \frac{1}{f_{n+1}}\right)$

$= \lim_{n\to\infty}\left[\left(\frac{1}{f_1} - \frac{1}{f_3}\right) + \left(\frac{1}{f_2} - \frac{1}{f_4}\right) + \left(\frac{1}{f_3} - \frac{1}{f_5}\right) + \left(\frac{1}{f_4} - \frac{1}{f_6}\right) + \cdots + \left(\frac{1}{f_{n-1}} - \frac{1}{f_{n+1}}\right)\right.$

$= \lim_{n\to\infty}\left(\frac{1}{f_1} + \frac{1}{f_2} - \frac{1}{f_n} - \frac{1}{f_{n+1}}\right) = 1 + 1 - 0 - 0 = 2$ because $f_n \to \infty$ as $n \to \infty$.

65. **(a)** At the first step, only the interval $\left(\frac{1}{3}, \frac{2}{3}\right)$ (length $\frac{1}{3}$) is removed. At the second step, we remove the intervals $\left(\frac{1}{9}, \frac{2}{9}\right)$ and $\left(\frac{7}{9}, \frac{8}{9}\right)$, which have a total length of $2\cdot\left(\frac{1}{3}\right)^2$. At the third

step, we remove 2^2 intervals, each of length $\left(\frac{1}{3}\right)^3$. In general, at the n th step we remove 2^{n-1} intervals, each of length $\left(\frac{1}{3}\right)^n$, for a length of $2^{n-1}\cdot\left(\frac{1}{3}\right)^n=\frac{1}{3}\left(\frac{2}{3}\right)^{n-1}$. Thus, the total length of all removed intervals is $\sum_{n=1}^{\infty}\frac{1}{3}\left(\frac{2}{3}\right)^{n-1}=\frac{1/3}{1-2/3}=1$ (geometric series with $a=\frac{1}{3}$ and $r=\frac{2}{3}$). Notice that at the n th step, the leftmost interval that is removed is $\left(\left(\frac{1}{3}\right)^n,\left(\frac{2}{3}\right)^n\right)$, so we never remove 0 , and 0 is in the Cantor set. Also, the rightmost interval removed is $\left(1-\left(\frac{2}{3}\right)^n,1-\left(\frac{1}{3}\right)^n\right)$, so 1 is never removed. Some other numbers in the Cantor set are $\frac{1}{3}$, $\frac{2}{3}$, $\frac{1}{9}$, $\frac{2}{9}$, $\frac{7}{9}$, and $\frac{8}{9}$.

(b) The area removed at the first step is $\frac{1}{9}$; at the second step, $8\cdot\left(\frac{1}{9}\right)^2$; at the third step, $(8)^2\cdot\left(\frac{1}{9}\right)^3$. In general, the area removed at the n th step is $(8)^{n-1}\left(\frac{1}{9}\right)^n=\frac{1}{9}\left(\frac{8}{9}\right)^{n-1}$, so the total area of all removed squares is $\sum_{n=1}^{\infty}\frac{1}{9}\left(\frac{8}{9}\right)^{n-1}=\frac{1/9}{1-8/9}=1$.

66. **(a)**

a_1	1	2	4	1	1	1000
a_2	2	3	1	4	1000	1
a_3	1.5	2.5	2.5	2.5	500.5	500.5
a_4	1.75	2.75	1.75	3.25	750.25	250.75
a_5	1.625	2.625	2.125	2.875	625.375	375.625
a_6	1.6875	2.6875	1.9375	3.0625	687.813	313.188
a_7	1.65625	2.65625	2.03125	2.96875	656.594	344.406
a_8	1.67188	2.67188	1.98438	3.01563	672.203	328.797
a_9	1.66406	2.66406	2.00781	2.99219	664.398	336.602
a_{10}	1.66797	2.66797	1.99609	3.00391	668.301	332.699
a_{11}	1.66602	2.66602	2.00195	2.99805	666.350	334.650

a_{12}	1.66699	2.66699	1.99902	3.00098	667.325	333.675

The limits seem to be $\frac{5}{3}$, $\frac{8}{3}$, 2 , 3 , 667 , and 334 . Note that the limits appear to be "weighted'' more toward a_2 . In general, we guess that the limit is $\frac{a_1+2a_2}{3}$.

(b)

$$a_{n+1}-a_n = \frac{1}{2}\left(a_n+a_{n-1}\right)-a_n = -\frac{1}{2}\left(a_n-a_{n-1}\right) = -\frac{1}{2}\left[\frac{1}{2}\left(a_{n-1}+a_{n-2}\right)-a_{n-1}\right]$$
$$= -\frac{1}{2}\left[-\frac{1}{2}\left(a_{n-1}-a_{n-2}\right)\right] = \cdots = \left(-\frac{1}{2}\right)^{n-1}\left(a_2-a_1\right)$$

Note that we have used the formula $a_k=\frac{1}{2}\left(a_{k-1}+a_{k-2}\right)$ a total of $n-1$ times in this calculation, once for each k between 3 and $n+1$. Now we can write

$$a_n = a_1+\left(a_2-a_1\right)+\left(a_3-a_2\right)+\cdots+\left(a_{n-1}-a_{n-2}\right)+\left(a_n-a_{n-1}\right)$$
$$= a_1+\Sigma_{k=1}^{n-1}\left(a_{k+1}-a_k\right)=a_1+\Sigma_{k=1}^{n-1}\left(-\frac{1}{2}\right)^{k-1}\left(a_2-a_1\right)$$

and so

$$\lim_{n\to\infty} a_n = a_1+\left(a_2-a_1\right)\Sigma_{k=1}^{\infty}\left(-\frac{1}{2}\right)^{k-1}=a_1+\left(a_2-a_1\right)\left[\frac{1}{1-(-1/2)}\right]$$
$$= a_1+\frac{2}{3}\left(a_2-a_1\right)=\frac{a_1+2a_2}{3}$$

67. **(a)** For $\Sigma_{n=1}^{\infty}\frac{n}{(n+1)!}$, $s_1=\frac{1}{1\cdot 2}=\frac{1}{2}$, $s_2=\frac{1}{2}+\frac{2}{1\cdot 2\cdot 3}=\frac{5}{6}$, $s_3=\frac{5}{6}+\frac{3}{1\cdot 2\cdot 3\cdot 4}=\frac{23}{24}$, $s_4=\frac{23}{24}+\frac{4}{1\cdot 2\cdot 3\cdot 4\cdot 5}=\frac{119}{120}$. The denominators are $(n+1)!$, so a guess would be $s_n=\frac{(n+1)!-1}{(n+1)!}$.

(b) For $n=1$, $s_1=\frac{1}{2}=\frac{2!-1}{2!}$, so the formula holds for $n=1$. Assume $s_k=\frac{(k+1)!-1}{(k+1)!}$. Then

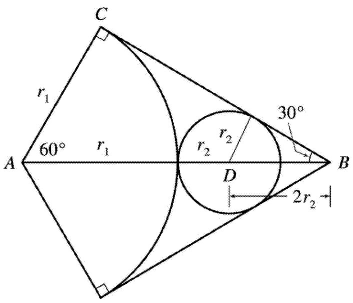
$$s_{k+1} = \frac{(k+1)!-1}{(k+1)!}+\frac{k+1}{(k+2)!}=\frac{(k+1)!-1}{(k+1)!}+\frac{k+1}{(k+1)!(k+2)}$$
$$= \frac{(k+2)!-(k+2)+k+1}{(k+2)!}=\frac{(k+2)!-1}{(k+2)!}$$

Thus, the formula is true for $n=k+1$. So by induction, the guess is correct.

(c)

$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{(n+1)!-1}{(n+1)!} = \lim_{n\to\infty} \left[1 - \frac{1}{(n+1)!} \right] = 1$ and so $\sum_{n=1}^{\infty} \frac{n}{(n+1)!} = 1$.

68.

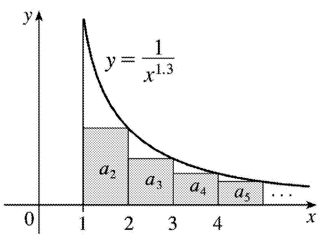


Let r_1= radius of the large circle, r_2= radius of next circle, and so on. From the figure we have $\angle BAC = 60^\circ$ and $\cos 60^\circ = r_1 / |AB|$, so $|AB| = 2r_1$ and $|DB| = 2r_2$. Therefore, $2r_1 = r_1 + r_2 + 2r_2 = r_1 + 3r_2 \Rightarrow r_1 = 3r_2$. In general, we have $r_{n+1} = \frac{1}{3} r_n$, so the total area is

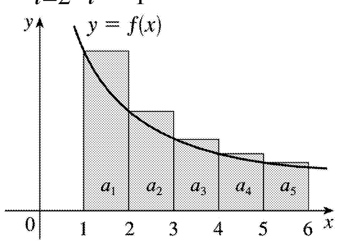
$$\begin{aligned} A &= \pi r_1^2 + 3\pi r_2^2 + 3\pi r_3^2 + \cdots \\ &= \pi r_1^2 + 3\pi r_2^2 \left(1 + \frac{1}{3^2} + \frac{1}{3^4} + \frac{1}{3^6} + \cdots \right) \\ &= \pi r_1^2 + 3\pi r_2^2 \cdot \frac{1}{1-1/9} = \pi r_1^2 + \frac{27}{8} \pi r_2^2 \end{aligned}$$

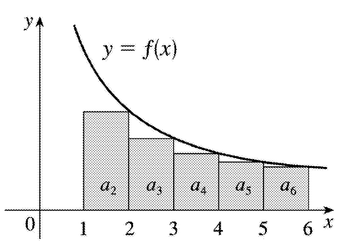
Since the sides of the triangle have length 1 , $|BC| = \frac{1}{2}$ and $\tan 30^\circ = \frac{r_1}{1/2}$. Thus, $r_1 = \frac{\tan 30^\circ}{2} = \frac{1}{2\sqrt{3}}$ $\Rightarrow r_2 = \frac{1}{6\sqrt{3}}$, so $A = \pi \left(\frac{1}{2\sqrt{3}} \right)^2 + \frac{27\pi}{8} \left(\frac{1}{6\sqrt{3}} \right)^2 = \frac{\pi}{12} + \frac{\pi}{32} = \frac{11\pi}{96}$. The area of the triangle is $\frac{\sqrt{3}}{4}$, so the circles occupy about 83.1% of the area of the triangle.

1. The picture shows that $a_2 = \frac{1}{2^{1.3}} < \int_1^2 \frac{1}{x^{1.3}}\,dx$, $a_3 = \frac{1}{3^{1.3}} < \int_2^3 \frac{1}{x^{1.3}}\,dx$, and so on, so $\sum_{n=2}^{\infty} \frac{1}{n^{1.3}} < \int_1^{\infty} \frac{1}{x^{1.3}}\,dx$. The integral converges by (.2) with $p=1.3>1$, so the series converges.



2. From the first figure, we see that $\int_1^6 f(x)dx < \sum_{i=1}^{5} a_i$. From the second figure, we see that $\sum_{i=2}^{6} a_i < \int_1^6 f(x)dx$. Thus, we have $\sum_{i=2}^{6} a_i < \int_1^6 f(x)dx < \sum_{i=1}^{5} a_i$.





3. The function $f(x)=1/x^4$ is continuous, positive, and decreasing on $[1,\infty)$, so the Integral Test applies. $\int_1^{\infty} \frac{1}{x^4}\,dx = \lim_{t\to\infty} \int_1^t x^{-4}\,dx = \lim_{t\to\infty} \left[\frac{x^{-3}}{-3} \right]_1^t = \lim_{t\to\infty} \left(-\frac{1}{3t^3} + \frac{1}{3} \right) = \frac{1}{3}$. Since this improper integral is convergent, the series $\sum_{n=1}^{\infty} \frac{1}{n^4}$ is also convergent by the Integral Test.

4. The function $f(x)=1/\sqrt[4]{x}=x^{-1/4}$ is continuous, positive, and decreasing on $[1,\infty)$, so the Integral

Test applies. $\int_1^\infty x^{-1/4}\,dx = \lim_{t\to\infty}\int_1^t x^{-1/4}\,dx = \lim_{t\to\infty}\left[\frac{4}{3}x^{3/4}\right]_1^t = \lim_{t\to\infty}\left(\frac{4}{3}t^{3/4}-\frac{4}{3}\right)=\infty$, so $\Sigma_{n=1}^{\infty} 1/\sqrt[4]{n}$ diverges.

5. The function $f(x)=1/(3x+1)$ is continuous, positive, and decreasing on $[1,\infty)$, so the Integral Test applies.

$$\int_1^\infty \frac{dx}{3x+1} = \lim_{b\to\infty}\int_1^b \frac{dx}{3x+1} = \lim_{b\to\infty}\left[\frac{1}{3}\ln(3x+1)\right]_1^b = \lim_{b\to\infty}\left[\frac{1}{3}\ln(3b+1)-\frac{1}{3}\ln 4\right]=\infty$$

so the improper integral diverges, and so does the series $\Sigma_{n=1}^{\infty} 1/(3n+1)$.

6. The function $f(x)=e^{-x}$ is continuous, positive, and decreasing on $[1,\infty)$, so the Integral Test applies. $\int_1^\infty e^{-x}\,dx = \lim_{b\to\infty}\int_1^b e^{-x}\,dx = \lim_{b\to\infty}\left[-e^{-x}\right]_1^b = \lim_{b\to\infty}\left(-e^{-b}+e^{-1}\right)=e^{-1}$, so $\Sigma_{n=1}^{\infty} e^{-n}$ converges. *Note:* This is a geometric series, with first term $a=e^{-1}$ and ratio $r=e^{-1}$. Since $|r|<1$, the series converges to $e^{-1}/(1-e^{-1})=1/(e-1)$.

7. $f(x)=xe^{-x}$ is continuous and positive on $[1,\infty)$. $f'(x)=-xe^{-x}+e^{-x}=e^{-x}(1-x)<0$ for $x>1$, so f is decreasing on $[1,\infty)$. Thus, the Integral Test applies.

$$\int_1^\infty xe^{-x}\,dx = \lim_{b\to\infty}\int_1^b xe^{-x}\,dx = \lim_{b\to\infty}\left[-xe^{-x}-e^{-x}\right]_1^b \text{(byparts)}$$

$$= \lim_{b\to\infty}\left[-be^{-b}-e^{-b}+e^{-1}+e^{-1}\right]=2/e$$

since $\lim_{b\to\infty} be^{-b} = \lim_{b\to\infty}\left(b/e^b\right) = \lim_{b\to\infty}\left(1/e^b\right)=0$ and $\lim_{b\to\infty} e^{-b}=0$. Thus, $\Sigma_{n=1}^{\infty} ne^{-n}$ converges.

8. The function $f(x)=\frac{x+2}{x+1}=1+\frac{1}{x+1}$ is continuous, positive, and decreasing on $[1,\infty)]$, so the Integral Test applies. $\int_1^\infty f(x)\,dx = \lim_{t\to\infty}\int_1^t \left(1+\frac{1}{x+1}\right)dx = \lim_{t\to\infty}\left[x+\ln(x+1)\right]_1^t = \lim_{t\to\infty}\left(t+\ln(t+1)-1-\ln 2\right)=\infty$, so $\int_1^\infty \frac{x+2}{x+1}\,dx$ is divergent and the series $\Sigma_{n=1}^{\infty}\frac{n+2}{n+1}$ is divergent. NOTE: $\lim_{n\to\infty}\frac{n+2}{n+1}=1$, so the given series diverges by the Test for Divergence.

9. The series $\sum_{n=1}^{\infty} \frac{1}{n^{0.85}}$ is a p –series with p=0.85≤ 1 , so it diverges by (1). Therefore, the series $\sum_{n=1}^{\infty} \frac{2}{n^{0.85}}$ must also diverge, for if it converged, then $\sum_{n=1}^{\infty} \frac{1}{n^{0.85}}$ would have to converge (by Theorem 8(i) in Section 11.2).

10. $\sum_{n=1}^{\infty} n^{-1.4}$ and $\sum_{n=1}^{\infty} n^{-1.2}$ are p –series with p>1 , so they converge by (1). Thus, $\sum_{n=1}^{\infty} 3n^{-1.2}$ converges by Theorem 8(i) in Section 11.2. It follows from Theorem 8(ii) that the given series $\sum_{n=1}^{\infty} (n^{-1.4}+3n^{-1.2})$ also converges.

11. $1+\frac{1}{8}+\frac{1}{27}+\frac{1}{64}+\frac{1}{125}+\cdot\cdot\cdot=\sum_{n=1}^{\infty} \frac{1}{n^3}$. This is a p –series with p=3>1 , so it converges by (1).

12. $1+\frac{1}{2\sqrt{2}}+\frac{1}{3\sqrt{3}}+\frac{1}{4\sqrt{4}}+\frac{1}{5\sqrt{5}}+\cdot\cdot\cdot=\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}}=\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$. This is a p –series with $p=\frac{3}{2}>1$, so it converges by (1).

13. $\sum_{n=1}^{\infty} \frac{5-2\sqrt{n}}{n^3}=5\sum_{n=1}^{\infty} \frac{1}{n^3}-2\sum_{n=1}^{\infty} \frac{1}{n^{5/2}}$ by Theorem .2.8, since $\sum_{n=1}^{\infty} \frac{1}{n^3}$ and $\sum_{n=1}^{\infty} \frac{1}{n^{5/2}}$ both converge by (1) (with p=3>1 and $p=\frac{5}{2}>1$). Thus, $\sum_{n=1}^{\infty} \frac{5-2\sqrt{n}}{n^3}$ converges.

14. The function $f(x)=\frac{5}{x-2}$ is continuous, positive, and decreasing on [3,∞)] , so we can apply the Integral Test. $\int_3^{\infty} \frac{5}{x-2}\, dx=\lim_{t\to\infty} \int_3^t \frac{5}{x-2}\, dx=\lim_{t\to\infty} [5\ln (x-2)]_3^t=\lim_{t\to\infty} [5\ln (t-2)-0]=\infty$, so the series $\sum_{n=3}^{\infty} \frac{5}{n-2}$ diverges.

15. The function $f(x)=\frac{1}{x^2+4}$ is continuous, positive, and decreasing on [1,∞)] , so we can apply the Integral Test.

$$\int_1^{\infty} \frac{1}{x^2+4}\, dx = \lim_{t\to\infty} \int_1^t \frac{1}{x^2+4}\, dx=\lim_{t\to\infty} \left[\frac{1}{2} \tan^{-1} \frac{x}{2} \right]_1^t=\frac{1}{2} \lim_{t\to\infty} \left[\tan^{-1} \left(\frac{t}{2} \right)-\tan^{-1} \left(\frac{1}{2} \right) \right]$$

$$= \frac{1}{2}\left[\frac{\pi}{2} - \tan^{-1}\left(\frac{1}{2}\right)\right]$$

Therefore, the series $\sum_{n=1}^{\infty} \frac{1}{n^2+4}$ converges.

16. The function $f(x)=\frac{3x+2}{x(x+1)}=\frac{2}{x}+\frac{1}{x+1}$ is continuous, positive, and decreasing on $[1,\infty)]$ since it is the sum of two such functions. Thus, we can apply the Integral Test.

$$\int_1^{\infty} \frac{3x+2}{x(x+1)}\,dx = \lim_{t\to\infty}\int_1^t \left[\frac{2}{x}+\frac{1}{x+1}\right]dx = \lim_{t\to\infty}[2\ln x+\ln(x+1)]_1^t$$
$$= \lim_{t\to\infty}[2\ln t+\ln(t+1)-\ln 2]=\infty$$

Thus, the series $\sum_{n=1}^{\infty} \frac{3n+2}{n(n+1)}$ diverges.

17. $f(x)=\frac{x}{x^2+1}$ is continuous and positive on $[1,\infty)$, and since $f'(x)=\frac{1-x^2}{\left(x^2+1\right)^2}<0$ for $x>1$, f is also decreasing. Using the Integral Test,

$\int_1^{\infty} \frac{x}{x^2+1}\,dx=\lim_{t\to\infty}\int_1^t \frac{x}{x^2+1}\,dx=\lim_{t\to\infty}\left[\frac{\ln(x^2+1)}{2}\right]_1^t=\frac{1}{2}\lim_{t\to\infty}[\ln(t^2+1)-\ln 2]=\infty$, so the series diverges.

18. The function $f(x)=\frac{1}{x^2-4x+5}=\frac{1}{(x-2)^2+1}$ is continuous, positive, and decreasing on $[2,\infty)]$, so the Integral Test applies.

$$\int_2^{\infty} f(x)dx=\lim_{t\to\infty}\int_2^t f(x)dx=\lim_{t\to\infty}\int_2^t \frac{1}{(x-2)^2+1}\,dx=\lim_{t\to\infty}[\tan^{-1}(x-2)]_2^t=\lim_{t\to\infty}[\tan^{-1}(t-2)-\tan^{-1}0]=\frac{\pi}{2}-0=\frac{\pi}{2},$$

so the series $\sum_{n=2}^{\infty} \frac{1}{n^2-4n+5}$ converges. Of course this means that $\sum_{n=1}^{\infty} \frac{1}{n^2-4n+5}$ converges too.

19. $f(x)=xe^{-x^2}$ is continuous and positive on $[1,\infty)$, and since $f'(x)=e^{-x^2}\left(1-2x^2\right)<0$ for $x>1$, f is decreasing as well. Thus, we can use the Integral Test.

$\int_1^{\infty} xe^{-x^2}\,dx=\lim_{t\to\infty}\left[-\frac{1}{2}e^{-x^2}\right]_1^t=0-\left(-\frac{1}{2}e^{-1}\right)=1/(2e)$. Since the integral converges, the series converges.

20. $f(x)=\frac{\ln x}{x^2}$ is continuous and positive for $x\geq 2$, and $f'(x)=\frac{1-2\ln x}{x^3}<0$ for $x\geq 2$, so f is decreasing. $\int_2^\infty \frac{\ln x}{x^2}\,dx=\lim_{t\to\infty}\left[-\frac{\ln x}{x}-\frac{1}{x}\right]_2^t=1$. Thus, $\sum_{n=1}^\infty \frac{\ln n}{n^2}=\sum_{n=2}^\infty \frac{\ln n}{n^2}$ converges by the Integral Test.

21. $f(x)=\frac{1}{x\ln x}$ is continuous and positive on $[2,\infty)$, and also decreasing since $f'(x)=-\frac{1+\ln x}{x^2(\ln x)^2}<0$ for $x>2$, so we can use the Integral Test. $\int_2^\infty \frac{1}{x\ln x}\,dx=\lim_{t\to\infty}[\ln(\ln x)]_2^t=\lim_{t\to\infty}[\ln(\ln t)-\ln(\ln 2)]=\infty$, so the series diverges.

22. The function $f(x)=\frac{x}{x^4+1}$ is positive, continuous, and decreasing on $[1,\infty)]$. Thus, we can apply the Integral Test.

$$\int_1^\infty \frac{x}{x^4+1}\,dx = \lim_{t\to\infty}\int_1^t \frac{\frac{1}{2}(2x)}{1+(x^2)^2}\,dx=\lim_{t\to\infty}\left[\frac{1}{2}\tan^{-1}(x^2)\right]_1^t=\frac{1}{2}\lim_{t\to\infty}[\tan^{-1}(t^2)-\tan^{-1}1]$$
$$=\frac{1}{2}\left(\frac{\pi}{2}-\frac{\pi}{4}\right)=\frac{\pi}{8}$$

so the series $\sum_{n=1}^\infty \frac{n}{n^4+1}$ converges.

23. The function $f(x)=\frac{1}{x^3+x}$ is continuous, positive, and decreasing on $[1,\infty)]$, so the Integral Test applies. We use partial fractions to evaluate the integral:

$$\int_1^\infty \frac{1}{x^3+x}\,dx = \lim_{t\to\infty}\int_1^t\left[\frac{1}{x}-\frac{x}{1+x^2}\right]dx=\lim_{t\to\infty}\left[\ln x-\frac{1}{2}\ln(1+x^2)\right]_1^t$$
$$=\lim_{t\to\infty}\left[\ln\frac{x}{\sqrt{1+x^2}}\right]_1^t=\lim_{t\to\infty}\left(\ln\frac{t}{\sqrt{1+t^2}}-\ln\frac{1}{\sqrt{2}}\right)$$

$$= \lim_{t\to\infty}\left(\ln \frac{1}{\sqrt{1+1/t^2}} + \frac{1}{2}\ln 2\right) = \frac{1}{2}\ln 2$$

so the series $\sum_{n=1}^{\infty} \frac{1}{n^3+n}$ converges.

24. $f(x)=\frac{1}{x\ln x\ln(\ln x)}$ is positive and continuous on $[3,\infty)$, and is decreasing since x, $\ln x$, and $\ln(\ln x)$ are all increasing; so we can apply the Integral Test. $\int_3^{\infty}\frac{dx}{x\ln x\ln(\ln x)} = \lim_{t\to\infty}\left[\ln(\ln(\ln x))\right]_3^t = \infty$. The integral diverges, so $\sum_{n=3}^{\infty}\frac{1}{n\ln n\ln(\ln n)}$ diverges.

25. We have already shown (in Exercise 21) that when $p=1$ the series $\sum_{n=2}^{\infty}\frac{1}{n(\ln n)^p}$ diverges, so assume that $p\neq 1$. $f(x)=\frac{1}{x(\ln x)^p}$ is continuous and positive on $[2,\infty)$, and $f'(x)=-\frac{p+\ln x}{x^2(\ln x)^{p+1}}<0$ if $x>e^{-p}$, so that f is eventually decreasing and we can use the Integral Test.

$$\int_2^{\infty}\frac{1}{x(\ln x)^p}\,dx = \lim_{t\to\infty}\left[\frac{(\ln x)^{1-p}}{1-p}\right]_2^t (\text{for } p\neq 1) = \lim_{t\to\infty}\left[\frac{(\ln t)^{1-p}}{1-p}\right] - \frac{(\ln 2)^{1-p}}{1-p}$$

This limit exists whenever $1-p<0 \Leftrightarrow p>1$, so the series converges for $p>1$.

26. As in Exercise 24, we can apply the Integral Test. $\int_3^{\infty}\frac{dx}{x\ln x(\ln\ln x)^p} = \lim_{t\to\infty}\left[\frac{(\ln\ln x)^{-p+1}}{-p+1}\right]_3^t$ (for $p\neq 1$; if $p=1$ see Exercise 24) and $\lim_{t\to\infty}\frac{(\ln\ln t)^{-p+1}}{-p+1}$ exists whenever $-p+1<0 \Leftrightarrow p>1$, so the series converges for $p>1$.

27. Clearly the series cannot converge if $p\geq -\frac{1}{2}$, because then $\lim_{n\to\infty} n(1+n^2)^p \neq 0$. Also, if $p=-1$ the series diverges (see Exercise 17). So assume $p<-\frac{1}{2}$, $p\neq -1$. Then $f(x)=x(1+x^2)^p$ is continuous,

positive, and eventually decreasing on $[1,\infty)$, and we can use the Integral Test.
$\int_1^\infty x(1+x^2)^p dx=\lim_{t\to\infty}\left[\frac{1}{2}\cdot\frac{(1+x^2)^{p+1}}{p+1}\right]_1^t=\lim_{t\to\infty}\frac{1}{2}\cdot\frac{(1+t^2)^{p+1}}{p+1}-\frac{2^p}{p+1}$. This limit exists and is finite $\Leftrightarrow$ $p+1<0\Leftrightarrow p<-1$, so the series converges whenever $p<-1$.

28. If $p\leq 0$, $\lim_{n\to\infty}\frac{\ln n}{n^p}=\infty$ and the series diverges, so assume $p>0$. $f(x)=\frac{\ln x}{x^p}$ is positive and continuous and $f'(x)<0$ for $x>e^{1/p}$, so f is eventually decreasing and we can use the Integral Test.
Integration by parts gives $\int_1^\infty\frac{\ln x}{x^p}dx=\lim_{t\to\infty}\left[\frac{x^{1-p}[(1-p)\ln x-1]}{(1-p)^2}\right]_1^t$ (for $p\neq 1$)
$=\frac{1}{(1-p)^2}\left[\lim_{t\to\infty}t^{1-p}[(1-p)\ln t-1]+1\right]$, which exists whenever $1-p<0\Leftrightarrow p>1$. Since we have already done the case $p=1$ in Exercise 25 (set $p=-1$ in that exercise), $\sum_{n=1}^\infty\frac{\ln n}{n^p}$ converges $\Leftrightarrow p>1$.

29. Since this is a $p-$ series with $p=x$, $\zeta(x)$ is defined when $x>1$. Unless specified otherwise, the domain of a function f is the set of numbers x such that the expression for $f(x)$ makes sense and defines a real number. So, in the case of a series, it's the set of numbers x such that the series is convergent.

30. **(a)** $f(x)=1/x^4$ is positive and continuous and $f'(x)=-4/x^5$ is negative for $x>0$, and so the Integral Test applies. $\sum_{n=1}^\infty\frac{1}{n^4}\approx s_{10}=\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\cdots+\frac{1}{10^4}\approx 1.082037$.

$R_{10}\leq\int_{10}^\infty\frac{1}{x^4}dx=\lim_{t\to\infty}\left[\frac{1}{-3x^3}\right]_{10}^t=\lim_{t\to\infty}\left(-\frac{1}{3t^3}+\frac{1}{3(10)^3}\right)=\frac{1}{3000}$, so the error is at most $0.000\overline{3}$.

(b) $s_{10}+\int_{11}^\infty\frac{1}{x^4}dx\leq s\leq s_{10}+\int_{10}^\infty\frac{1}{x^4}dx\Rightarrow s_{10}+\frac{1}{3(11)^3}\leq s\leq s_{10}+\frac{1}{3(10)^3}\Rightarrow$
$1.082037+0.000250=1.082287\leq s\leq 1.082037+0.000333=1.082370$, so we get $s\approx 1.08233$ with error ≤ 0.00005 .

(c) $R_n\leq\int_n^\infty\frac{1}{x^4}dx=\frac{1}{3n^3}$. So $R_n<0.00001\Rightarrow\frac{1}{3n^3}<\frac{1}{10^5}\Rightarrow 3n^3>10^5\Rightarrow n>\sqrt[3]{(10)^5/3}\approx 32.2$, that is, for

$n>32$.

31. **(a)** $f(x)=\frac{1}{x^2}$ is positive and continuous and $f'(x)=-\frac{2}{x^3}$ is negative for $x>0$, and so the Integral Test applies. $\sum_{n=1}^{\infty}\frac{1}{n^2}\approx s_{10}=\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots+\frac{1}{10^2}\approx 1.549768$.

$R_{10}\le\int_{10}^{\infty}\frac{1}{x^2}\,dx=\lim_{t\to\infty}\left[\frac{-1}{x}\right]_{10}^{t}=\lim_{t\to\infty}\left(-\frac{1}{t}+\frac{1}{10}\right)=\frac{1}{10}$, so the error is at most 0.1 .

(b) $s_{10}+\int_{11}^{\infty}\frac{1}{x^2}\,dx\le s\le s_{10}+\int_{10}^{\infty}\frac{1}{x^2}\,dx\Rightarrow s_{10}+\frac{1}{11}\le s\le s_{10}+\frac{1}{10}\Rightarrow$
$1.549768+0.090909=1.640677\le s\le 1.549768+0.1=1.649768$, so we get $s\approx 1.64522$ (the average of 1.640677 and 1.649768) with error ≤ 0.005 (the maximum of $1.649768-1.64522$ and $1.64522-1.640677$, rounded up).

(c) $R_n\le\int_n^{\infty}\frac{1}{x^2}\,dx=\frac{1}{n}$. So $R_n<0.001$ if $\frac{1}{n}<\frac{1}{1000}\Leftrightarrow n>1000$.

32. $f(x)=1/x^5$ is positive and continuous and $f'(x)=-5/x^6$ is negative for $x>0$, and so the Integral Test applies. Using (3), $R_n\le\int_n^{\infty}x^{-5}dx=\lim_{t\to\infty}\left[\frac{-1}{4x^4}\right]_n^t=\frac{1}{4n^4}$. If we take $n=5$, then $s_5\approx 1.036662$ and $R_5\le 0.0004$. So $s\approx s_5\approx 1.037$.

33. $f(x)=x^{-3/2}$ is positive and continuous and $f'(x)=-\frac{3}{2}x^{-5/2}$ is negative for $x>0$, so the Integral Test applies. From the end of Example 6, we see that the error is at most half the length of the interval. From (3), the interval is $\left(s_n+\int_{n+1}^{\infty}f(x)\mathrm{d}x, s_n+\int_n^{\infty}f(x)\,dx\right)$, so its length is □ . Thus, we need n such that

$$0.01>\frac{1}{2}\int_n^{n+1}x^{-3/2}dx=\frac{1}{2}\left[\frac{-2}{\sqrt{x}}\right]_n^{n+1}=\frac{1}{\sqrt{n}}-\frac{1}{\sqrt{n+1}}$$

$\Leftrightarrow n>13.08$ (use a graphing calculator to solve $1/\sqrt{x}-1/\sqrt{x+1}<0.01$). Again from the end of Example 6, we approximate s by the midpoint of this interval. In general, the midpoint is $\frac{1}{2}\left[\left(s_n+\int_{n+1}^{\infty}f(x)dx\right)+\left(s_n+\int_n^{\infty}f(x)dx\right)\right]=s_n+\frac{1}{2}\left(\int_{n+1}^{\infty}f(x)dx+\int_n^{\infty}f(x)dx\right)$. So using $n=14$, we have $s \approx s_{14} + \frac{1}{2}\left(\int_{14}^{\infty} x^{-3/2}\,dx + \int_{15}^{\infty} x^{-3/2}\,dx\right) \approx 2.0872 + \frac{1}{\sqrt{14}} + \frac{1}{\sqrt{15}} \approx 2.6127 \approx 2.61$. Any larger value of n will also work. For instance, $s \approx s_{30} + \frac{1}{\sqrt{30}} + \frac{1}{\sqrt{31}} \approx 2.6124$.

34. $f(x) = \frac{1}{x(\ln x)^2}$ is positive and continuous and $f'(x) = -\frac{\ln x + 2}{x^2(\ln x)^3}$ is negative for $x > 1$, so the Integral Test applies. Using (2), we need $0.01 > \int_n^{\infty} \frac{dx}{x(\ln x)^2} = \lim_{t\to\infty}\left[\frac{-1}{\ln x}\right]_n^t = \frac{1}{\ln n}$. This is true for $n > e^{100}$, so we would have to take this many terms, which would be problematic because $e^{100} \approx 2.7 \times 10^{43}$.

35. $\sum_{n=1}^{\infty} n^{-1.001} = \sum_{n=1}^{\infty} \frac{1}{n^{1.001}}$ is a convergent p –series with $p = 1.001 > 1$. Using (2), we get $R_n \leq \int_n^{\infty} x^{-1.001}\,dx = \lim_{t\to\infty}\left[\frac{x^{-0.001}}{-0.001}\right]_n^t = -1000 \lim_{t\to\infty}\left[\frac{1}{x^{0.001}}\right]_n^t = -1000\left(-\frac{1}{n^{0.001}}\right) = \frac{1000}{n^{0.001}}$. We want $R_n < 0.000000005 \Leftrightarrow \frac{1000}{n^{0.001}} < 5 \times 10^{-9} \Leftrightarrow n^{0.001} > \frac{1000}{5 \times 10^{-9}} \Leftrightarrow$
$n > \left(2 \times 10^{11}\right)^{1000} = 2^{1000} \times 10^{11{,}000} \approx 1.07 \times 10^{301} \times 10^{11{,}000} = 1.07 \times 10^{11{,}301}$.

36. **(a)** $f(x) = \left(\frac{\ln x}{x}\right)^2$ is continuous and positive for $x > 1$, and since $f'(x) = \frac{2\ln x(1 - \ln x)}{x^3} < 0$ for $x > e$, we can apply the Integral Test. Using a CAS, we get $\int_1^{\infty}\left(\frac{\ln x}{x}\right)^2 dx = 2$, so the series also converges.

(b) Since the Integral Test applies, the error in $s \approx s_n$ is $R_n \leq \int_n^{\infty}\left(\frac{\ln x}{x}\right)^2 dx = \frac{(\ln n)^2 + 2\ln n + 2}{n}$.

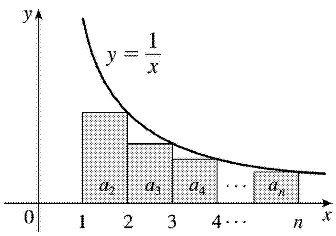
(c) By graphing the functions $y_1 = \frac{(\ln x)^2 + 2\ln x + 2}{x}$ and $y_2 = 0.05$, we see that $y_1 < y_2$ for $n \geq 1373$.

(d) Using the CAS to sum the first 1373 terms, we get $s_{1373} \approx 1.94$.

37. **(a)** From the figure,

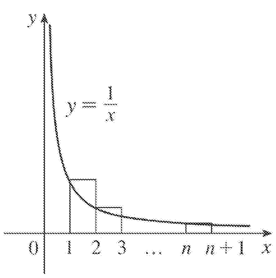
$a_2+a_3+\cdots+a_n \le \int_1^n f(x)\,dx$, so with $f(x)=\frac{1}{x}$, $\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{n} \le \int_1^n \frac{1}{x}\,dx=\ln n$. Thus,

$s_n=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{n} \le 1+\ln n$.

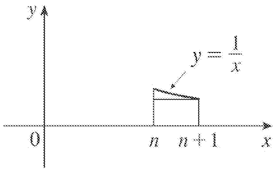


(b) By part (a), $s_{10^6} \le 1+\ln 10^6 \approx 14.82<15$ and $s_{10^9} \le 1+\ln 10^9 \approx 21.72<22$.

38. **(a)** The sum of the areas of the n rectangles in the graph to the right is $1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}$. Now $\int_1^{n+1} \frac{dx}{x}$ is less than this sum because the rectangles extend above the curve $y=1/x$, so $\int_1^{n+1} \frac{1}{x}\,dx=\ln(n+1)<1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}$, and since $\ln n<\ln(n+1)$, $0<1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-\ln n=t_n$.



(b) The area under $f(x)=1/x$ between $x=n$ and $x=n+1$ is $\int_n^{n+1} \frac{dx}{x}=\ln(n+1)-\ln n$, and this is clearly greater than the area of the inscribed rectangle in the figure to the right $\left[\text{which is } \frac{1}{n+1}\right]$, so $t_n-t_{n+1}=[\ln(n+1)-\ln n]-\frac{1}{n+1}>0$, and so $t_n>t_{n+1}$, so $\{t_n\}$ is a decreasing sequence.



(c) We have shown that $\{t_n\}$ is decreasing and that $t_n > 0$ for all n . Thus, $0 < t_n \le t_1 = 1$, so $\{t_n\}$ is a bounded monotonic sequence, and hence converges by Theorem .1.11.

39. $b^{\ln n} = \left(e^{\ln b}\right)^{\ln n} = \left(e^{\ln n}\right)^{\ln b} = n^{\ln b} = \frac{1}{n^{-\ln b}}$. This is a p– series, which converges for all b such that $-\ln b > 1 \Leftrightarrow \ln b < -1 \Leftrightarrow b < e^{-1} \Leftrightarrow b < 1/e$.

1. **(a)** We cannot say anything about Σa_n . If $a_n > b_n$ for all n and Σb_n is convergent, then Σa_n could be convergent or divergent. (See the note after Example 2.)
(b) If $a_n < b_n$ for all n , then Σa_n is convergent.

2. **(a)** If $a_n > b_n$ for all n , then Σa_n is divergent.
(b) We cannot say anything about Σa_n . If $a_n < b_n$ for all n and Σb_n is divergent, then Σa_n could be convergent or divergent.

3. $\frac{1}{n^2+n+1} < \frac{1}{n^2}$ for all $n \geq 1$, so $\Sigma_{n=1}^{\infty} \frac{1}{n^2+n+1}$ converges by comparison with $\Sigma_{n=1}^{\infty} \frac{1}{n^2}$, which converges because it is a p –series with $p=2>1$.

4. $\frac{2}{n^3+4} < \frac{2}{n^3}$ for all $n \geq 1$, so $\Sigma_{n=1}^{\infty} \frac{2}{n^3+4}$ converges by comparison with $\Sigma_{n=1}^{\infty} \frac{2}{n^3} = 2\Sigma_{n=1}^{\infty} \frac{1}{n^3}$, which converges because it is a constant multiple of a convergent p –series ($p=3>1$).

5. $\frac{5}{2+3^n} < \frac{5}{3^n}$ for all $n \geq 1$, so $\Sigma_{n=1}^{\infty} \frac{5}{2+3^n}$ converges by comparison with $\Sigma_{n=1}^{\infty} \frac{5}{3^n} = 5\Sigma_{n=1}^{\infty} \frac{1}{3^n}$, which converges because $\Sigma_{n=1}^{\infty} \frac{1}{3^n}$ is a convergent geometric series with $r=\frac{1}{3}$ ($|r|<1$).

6. $\frac{1}{n-\sqrt{n}} > \frac{1}{n}$ for all $n \geq 2$, so $\Sigma_{n=2}^{\infty} \frac{1}{n-\sqrt{n}}$ diverges by comparison with the divergent (partial) harmonic series $\Sigma_{n=2}^{\infty} \frac{1}{n}$.

7. $\frac{n+1}{n^2} > \frac{n}{n^2} = \frac{1}{n}$ for all $n \geq 1$, so $\Sigma_{n=1}^{\infty} \frac{n+1}{n^2}$ diverges by comparison with the harmonic series $\Sigma_{n=1}^{\infty} \frac{1}{n}$.

8. $\frac{4+3^n}{2^n} > \frac{3^n}{2^n} = \left(\frac{3}{2}\right)^n$ for all $n \geq 1$, so $\Sigma_{n=1}^{\infty} \frac{4+3^n}{2^n}$ diverges by comparison with the divergent geometric series $\Sigma_{n=1}^{\infty} \left(\frac{3}{2}\right)^n$.

9. $\frac{\cos^2 n}{n^2+1} \le \frac{1}{n^2+1} < \frac{1}{n^2}$, so the series $\sum_{n=1}^{\infty} \frac{\cos^2 n}{n^2+1}$ converges by comparison with the p –series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ $(p=2>1)$.

10. $\frac{n^2-1}{3n^4+1} < \frac{n^2}{3n^4+1} < \frac{n^2}{3n^4} = \frac{1}{3}\,\frac{1}{n^2}$. $\sum_{n=1}^{\infty} \frac{n^2-1}{3n^4+1}$ converges by comparison with $\sum_{n=1}^{\infty} \frac{1}{3n^2}$, which converges because it is a constant multiple of a convergent p –series $(p=2>1)$. The terms of the given series are positive for $n>1$, which is good enough.

11. If $a_n = \frac{n^2+1}{n^3-1}$ and $b_n = \frac{1}{n}$, then $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^3+n}{n^3-1} = \lim_{n\to\infty} \frac{1+1/n^2}{1-1/n^3} = 1$, so $\sum_{n=2}^{\infty} \frac{n^2+1}{n^3-1}$ diverges by the Limit Comparison Test with the divergent (partial) harmonic series $\sum_{n=2}^{\infty} \frac{1}{n}$. *Or:* Since $a_n = \frac{n^2+1}{n^3-1} > \frac{n^2+1}{n^3} > \frac{n^2}{n^3} = \frac{1}{n} = b_n$, we could use the Comparison Test.

12. $\frac{1+\sin n}{10^n} \le \frac{2}{10^n}$ and $\sum_{n=0}^{\infty} \frac{2}{10^n} = 2\sum_{n=0}^{\infty} \left(\frac{1}{10} \right)^n$, so the given series converges by comparison with a constant multiple of a convergent geometric series.

13. $\frac{n-1}{n4^n}$ is positive for $n>1$ and $\frac{n-1}{n4^n} < \frac{n}{n4^n} = \frac{1}{4^n}$, so $\sum_{n=1}^{\infty} \frac{n-1}{n4^n}$ converges by comparison with the convergent geometric series $\sum_{n=1}^{\infty} \left(\frac{1}{4} \right)^n$.

14. $\frac{\sqrt{n}}{n-1} > \frac{\sqrt{n}}{n} = \frac{1}{\sqrt{n}}$, so $\sum_{n=2}^{\infty} \frac{\sqrt{n}}{n-1}$ diverges by comparison with the divergent (partial) p –series $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}}$ $\left(p = \frac{1}{2} \le 1 \right)$.

15. $\frac{2+(-1)^n}{n\sqrt{n}} \le \frac{3}{n\sqrt{n}}$, and $\sum_{n=1}^{\infty} \frac{3}{n\sqrt{n}}$ converges because it is a constant multiple of the convergent p –series $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}}$ $\left(p = \frac{3}{2} > 1 \right)$, so the given series converges by the Comparison Test.

16. $\dfrac{1}{\sqrt{n^3+1}} < \dfrac{1}{\sqrt{n^3}} = \dfrac{1}{n^{3/2}}$, so $\sum_{n=1}^{\infty} \dfrac{1}{\sqrt{n^3+1}}$ converges by comparison with the convergent p – series $\sum_{n=1}^{\infty} \dfrac{1}{n^{3/2}} \left(p=\dfrac{3}{2}>1 \right)$.

17. Use the Limit Comparison Test with $a_n = \dfrac{1}{\sqrt{n^2+1}}$ and $b_n = \dfrac{1}{n}$:

$\lim_{n\to\infty} \dfrac{a_n}{b_n} = \lim_{n\to\infty} \dfrac{n}{\sqrt{n^2+1}} = \lim_{n\to\infty} \dfrac{1}{\sqrt{1+(1/n^2)}} = 1>0$. Since the harmonic series $\sum_{n=1}^{\infty} \dfrac{1}{n}$ diverges, so does $\sum_{n=1}^{\infty} \dfrac{1}{\sqrt{n^2+1}}$.

18. Use the Limit Comparison Test with $a_n = \dfrac{1}{2n+3}$ and $b_n = \dfrac{1}{n}$:

$\lim_{n\to\infty} \dfrac{a_n}{b_n} = \lim_{n\to\infty} \dfrac{n}{2n+3} = \lim_{n\to\infty} \dfrac{1}{2+(3/n)} = \dfrac{1}{2} >0$. Since the harmonic series $\sum_{n=1}^{\infty} \dfrac{1}{n}$ diverges, so does $\sum_{n=1}^{\infty} \dfrac{1}{2n+3}$.

19. $\dfrac{2^n}{1+3^n} < \dfrac{2^n}{3^n} = \left(\dfrac{2}{3} \right)^n$. $\sum_{n=1}^{\infty} \left(\dfrac{2}{3} \right)^n$ is a convergent geometric series ($|r| = \dfrac{2}{3} <1$), so $\sum_{n=1}^{\infty} \dfrac{2^n}{1+3^n}$ converges by the Comparison Test.

20. Use the Limit Comparison Test with $a_n = \dfrac{1+2^n}{1+3^n}$ and $b_n = \dfrac{2^n}{3^n}$: $\lim_{n\to\infty} \dfrac{a_n}{b_n} = \lim_{n\to\infty} \dfrac{(1/2)^n+1}{(1/3)^n+1} = 1>0$.
Since $\sum_{n=1}^{\infty} b_n$ converges (geometric series with $|r| = \dfrac{2}{3} <1$), $\sum_{n=1}^{\infty} \dfrac{1+2^n}{1+3^n}$ also converges.

21. Use the Limit Comparison Test with $a_n = \dfrac{1}{1+\sqrt{n}}$ and $b_n = \dfrac{1}{\sqrt{n}}$: $\lim_{n\to\infty} \dfrac{a_n}{b_n} = \lim_{n\to\infty} \dfrac{\sqrt{n}}{1+\sqrt{n}} = 1>0$.

Since $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is a divergent $p-$ series ($p=\frac{1}{2} \leq 1$), $\sum_{n=1}^{\infty} \frac{1}{1+\sqrt{n}}$ also diverges.

22. Use the Limit Comparison Test with $a_n = \frac{n+2}{(n+1)^3}$ and $b_n = \frac{1}{n^2}$:

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^2(n+2)}{(n+1)^3} = \lim_{n\to\infty} \frac{1+\frac{2}{n}}{\left(1+\frac{1}{n}\right)^3} = 1>0$. Since $\sum_{n=3}^{\infty} \frac{1}{n^2}$ is a convergent (partial) p –series $(p=2>1)$, the series $\sum_{n=3}^{\infty} \frac{n+2}{(n+1)^3}$ also converges.

23. Use the Limit Comparison Test with $a_n = \frac{5+2n}{(1+n^2)^2}$ and $b_n = \frac{1}{n^3}$:

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^3(5+2n)}{(1+n^2)^2} = \lim_{n\to\infty} \frac{5n^3+2n^4}{(1+n^2)^2} \cdot \frac{1/n^4}{1/(n^2)^2} = \lim_{n\to\infty} \frac{\frac{5}{n}+2}{\left(\frac{1}{n^2}+1\right)^2} = 2>0$. Since $\sum_{n=1}^{\infty} \frac{1}{n^3}$ is a convergent p –series $(p=3>1)$, the series $\sum_{n=1}^{\infty} \frac{5+2n}{(1+n^2)^2}$ also converges.

24. If $a_n = \frac{n^2-5n}{n^3+n+1}$ and $b_n = \frac{1}{n}$, then $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^3-5n^2}{n^3+n+1} = \lim_{n\to\infty} \frac{1-5/n}{1+1/n^2+1/n^3} = 1>0$, so $\sum_{n=1}^{\infty} \frac{n^2-5n}{n^3+n+1}$ diverges by the Limit Comparison Test with the divergent harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$.
(Note that $a_n > 0$ for $n \geq 6$.)

25. If $a_n = \frac{1+n+n^2}{\sqrt{1+n^2+n^6}}$ and $b_n = \frac{1}{n}$, then $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n+n^2+n^3}{\sqrt{1+n^2+n^6}} = \lim_{n\to\infty} \frac{1/n^2+1/n+1}{\sqrt{1/n^6+1/n^4+1}} = 1>0$, so $\sum_{n=1}^{\infty} \frac{1+n+n^2}{\sqrt{1+n^2+n^6}}$ diverges by the Limit Comparison Test with the divergent harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$.

26. If $a_n = \dfrac{n+5}{\sqrt[3]{n^7+n^2}}$ and $b_n = \dfrac{n}{\sqrt[3]{n^7}} = \dfrac{n}{n^{7/3}} = \dfrac{1}{n^{4/3}}$, then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^{7/3}+5n^{4/3}}{\left(n^7+n^2\right)^{1/3}} \cdot \frac{n^{-7/3}}{n^{-7/3}} = \lim_{n\to\infty} \frac{1+5/n}{\left[\left(n^7+n^2\right)/n^7\right]^{1/3}}$$

$$= \lim_{n\to\infty} \frac{1+5/n}{\left(1+1/n^5\right)^{1/3}} = \frac{1+0}{(1+0)^{1/3}} = 1>0,$$

so $\Sigma_{n=1}^{\infty} \dfrac{n+5}{\sqrt[3]{n^7+n^2}}$ converges by the Limit Comparison Test with the convergent p –series $\Sigma_{n=1}^{\infty} \dfrac{1}{n^{4/3}}$

.

27. Use the Limit Comparison Test with $a_n = \left(1+\dfrac{1}{n}\right)^2 e^{-n}$ and $b_n = e^{-n}$:

$\lim_{n\to\infty} \dfrac{a_n}{b_n} = \lim_{n\to\infty} \left(1+\dfrac{1}{n}\right)^2 = 1>0$. Since $\Sigma_{n=1}^{\infty} e^{-n} = \Sigma_{n=1}^{\infty} \dfrac{1}{e^n}$ is a convergent geometric series $\left(|r| = \dfrac{1}{e} <1\right)$, the series $\Sigma_{n=1}^{\infty} \left(1+\dfrac{1}{n}\right)^2 e^{-n}$ also converges.

28. Use the Limit Comparison Test with $a_n = \dfrac{2n^2+7n}{3^n\left(n^2+5n-1\right)}$ and $b_n = \dfrac{1}{3^n}$.

$\lim_{n\to\infty} \dfrac{a_n}{b_n} = \lim_{n\to\infty} \dfrac{2n^2+7n}{n^2+5n-1} = 2>0$, and since $\Sigma_{n=1}^{\infty} b_n$ is a convergent geometric series ($|r| = \dfrac{1}{3} <1$), $\Sigma_{n=1}^{\infty} \dfrac{2n^2+7n}{3^n(n^2+5n-1)}$ converges also.

29. Clearly $n! = n(n-1)(n-2)\cdot\cdot\cdot(3)(2) \geq 2\cdot 2\cdot 2\cdot \cdot\cdot\cdot\cdot 2\cdot 2 = 2^{n-1}$, so $\dfrac{1}{n!} \leq \dfrac{1}{2^{n-1}}$. $\Sigma_{n=1}^{\infty} \dfrac{1}{2^{n-1}}$ is a convergent geometric series ($|r| = \dfrac{1}{2} <1$), so $\Sigma_{n=1}^{\infty} \dfrac{1}{n!}$ converges by the Comparison Test.

30. $\dfrac{n!}{n^n} = \dfrac{1\cdot 2\cdot 3\cdot \cdot\cdot\cdot\cdot (n-1)n}{n\cdot n\cdot n\cdot \cdot\cdot\cdot\cdot n\cdot n} \leq \dfrac{1}{n}\cdot\dfrac{2}{n}\cdot 1\cdot 1\cdot \cdot\cdot\cdot\cdot 1$ for $n\geq 2$, so since $\Sigma_{n=1}^{\infty} \dfrac{2}{n^2}$ converges ($p=2>1$),

$\sum_{n=1}^{\infty} \frac{n!}{n^n}$ converges also by the Comparison Test.

31. Use the Limit Comparison Test with $a_n = \sin\left(\frac{1}{n}\right)$ and $b_n = \frac{1}{n}$. Then $\sum a_n$ and $\sum b_n$ are series with positive terms and $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sin(1/n)}{1/n} = \lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1 > 0$. Since $\sum_{n=1}^{\infty} b_n$ is the divergent harmonic series, $\sum_{n=1}^{\infty} \sin(1/n)$ also diverges. (Note that we could also use l'Hospital's Rule to evaluate the limit: $\lim_{x\to\infty} \frac{\sin(1/x)}{1/x} = \lim_{x\to\infty} \frac{\cos(1/x)\cdot\left(-1/x^2\right)}{-1/x^2} = \lim_{x\to\infty} \cos\frac{1}{x} = \cos 0 = 1$.)

32. Use the Limit Comparison Test with $a_n = \frac{1}{n^{1+1/n}}$ and $b_n = \frac{1}{n}$. $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n}{n^{1+1/n}} = \lim_{n\to\infty} \frac{1}{n^{1/n}} = 1$ (since $\lim_{x\to\infty} x^{1/x} = 1$ by l'Hospital's Rule), so $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (harmonic series) $\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^{1+1/n}}$ diverges.

33. $\sum_{n=1}^{10} \frac{1}{n^4+n^2} = \frac{1}{2} + \frac{1}{20} + \frac{1}{90} + \cdots + \frac{1}{10{,}100} \approx 0.567975$. Now $\frac{1}{n^4+n^2} < \frac{1}{n^4}$, so using the reasoning and notation of Example 5, the error is

$$R_{10} \le T_{10} = \sum_{n=11}^{\infty} \frac{1}{n^4} \le \int_{10}^{\infty} \frac{}{dx} x^4 = \lim_{t\to\infty} \left[-\frac{x^{-3}}{3} \right]_{10}^{t} = \frac{1}{3000} = 0.000\overline{3} .$$

34. $\sum_{n=1}^{10} \frac{1+\cos n}{n^5} = 1+\cos 1 + \frac{1+\cos 2}{32} + \frac{1+\cos 3}{243} + \cdots + \frac{1+\cos 10}{100{,}000} \approx 1.55972$. Now $\frac{1+\cos n}{n^5} \le \frac{2}{n^5}$,

so as in Example 5, $R_{10} \le T_{10} \le \int_{10}^{\infty} \frac{2}{x^5}\,dx = 2\lim_{t\to\infty} \left[-\frac{1}{4} x^{-4} \right]_{10}^{t} = 0.00005$.

35. $\sum_{n=1}^{10} \frac{1}{1+2^n} = \frac{1}{3} + \frac{1}{5} + \frac{1}{9} + \cdots + \frac{1}{1025} \approx 0.76352$. Now $\frac{1}{1+2^n} < \frac{1}{2^n}$, so the error is

$R_{10} \le T_{10} = \sum_{n=11}^{\infty} \frac{1}{2^n} = \frac{1/2^{11}}{1-1/2}$ (geometric series) ≈ 0.00098 .

36. $\sum_{n=1}^{10} \frac{n}{(n+1)3^n} = \frac{1}{6} + \frac{2}{27} + \frac{3}{108} + \cdot\cdot\cdot + \frac{10}{649{,}539} \approx 0.283597$. Now $\frac{n}{(n+1)3^n} < \frac{n}{n \cdot 3^n} = \frac{1}{3^n}$, so the error is $R_{10} \le T_{10} = \sum_{n=11}^{\infty} \frac{1}{3^n} = \frac{1/3^{11}}{1-1/3} \approx 0.0000085$.

37. Since $\frac{d_n}{10^n} \le \frac{9}{10^n}$ for each n , and since $\sum_{n=1}^{\infty} \frac{9}{10^n}$ is a convergent geometric series ($|r| = \frac{1}{10} < 1$), $0.d_1 d_2 d_3 \ldots = \sum_{n=1}^{\infty} \frac{d_n}{10^n}$ will always converge by the Comparison Test.

38. Clearly, if $p<0$ then the series diverges, since $\lim_{n \to \infty} \frac{1}{n^p \ln n} = \infty$. If $0 \le p \le 1$, then $n^p \ln n \le n \ln n \Rightarrow \frac{1}{n^p \ln n} \ge \frac{1}{n \ln n}$ and $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ diverges (Exercise .3.21), so $\sum_{n=2}^{\infty} \frac{1}{n^p \ln n}$ diverges. If $p>1$, use the Limit Comparison Test with $a_n = \frac{1}{n^p \ln n}$ and $b_n = \frac{1}{n^p}$. $\sum_{n=2}^{\infty} b_n$ converges, and $\lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{1}{\ln n} = 0$, so $\sum_{n=2}^{\infty} \frac{1}{n^p \ln n}$ also converges. (Or use the Comparison Test, since $n^p \ln n > n^p$ for $n > e$.) In summary, the series converges if and only if $p>1$.

39. Since $\sum a_n$ converges, $\lim_{n \to \infty} a_n = 0$, so there exists N such that $|a_n - 0| < 1$ for all $n>N \Rightarrow 0 \le a_n < 1$ for all $n>N \Rightarrow 0 \le a_n^2 \le a_n$. Since $\sum a_n$ converges, so does $\sum a_n^2$ by the Comparison Test.

40. **(a)** Since $\lim_{n \to \infty} \left(a_n / b_n\right) = 0$, there is a number $N>0$ such that $\left|a_n / b_n - 0\right| < 1$ for all $n>N$, and so $a_n < b_n$ since a_n and b_n are positive. Thus, since $\sum b_n$ converges, so does $\sum a_n$ by the Comparison Test.
(b)

(i) If $a_n = \frac{\ln n}{n^3}$ and $b_n = \frac{1}{n^2}$, then $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\ln n}{n} = \lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1/x}{1} = 0$, so $\Sigma_{n=1}^{\infty} \frac{\ln n}{n^3}$ converges by part (a).

(ii) If $a_n = \frac{\ln n}{\sqrt{n}\, e^n}$ and $b_n = \frac{1}{e^n}$, then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\ln n}{\sqrt{n}} = \lim_{x\to\infty} \frac{\ln x}{\sqrt{x}} = \lim_{x\to\infty} \frac{1/x}{1/(2\sqrt{x})} = \lim_{x\to\infty} \frac{2}{\sqrt{x}} = 0$$

Now Σb_n is a convergent geometric series with ratio $r = 1/e\,(|r| < 1)$, so Σa_n converges by part (a).

41. **(a)** Since $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$, there is an integer N such that $\frac{a_n}{b_n} > 1$ whenever $n > N$. (Take $M = 1$ in Definition .1.5.) Then $a_n > b_n$ whenever $n > N$ and since Σb_n is divergent, Σa_n is also divergent by the Comparison Test.

(b)

(c) If $a_n = \frac{1}{\ln n}$ and $b_n = \frac{1}{n}$ for $n \geq 2$, then $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n}{\ln n} = \lim_{x\to\infty} \frac{x}{\ln x} = \lim_{x\to\infty} \frac{1}{1/x} = \lim_{x\to\infty} x = \infty$, so by part (a), $\Sigma_{n=2}^{\infty} \frac{1}{\ln n}$ is divergent.

(d) If $a_n = \frac{\ln n}{n}$ and $b_n = \frac{1}{n}$, then $\Sigma_{n=1}^{\infty} b_n$ is the divergent harmonic series and $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \ln n = \lim_{x\to\infty} \ln x = \infty$, so $\Sigma_{n=1}^{\infty} a_n$ diverges by part (a).

42. Let $a_n = \frac{1}{n^2}$ and $b_n = \frac{1}{n}$. Then $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{n} = 0$, but Σb_n diverges while Σa_n converges.

43. $\lim_{n\to\infty} n a_n = \lim_{n\to\infty} \frac{a_n}{1/n}$, so we apply the Limit Comparison Test with $b_n = \frac{1}{n}$. Since $\lim_{n\to\infty} n a_n > 0$ we know that either both series converge or both series diverge, and we also know that

$\sum_{n=0}^{\infty} \frac{1}{n}$ diverges (p– series with $p=1$). Therefore, $\sum a_n$ must be divergent.

44. First we observe that, by l'Hospital's Rule, $\lim_{x\to 0} \frac{\ln(1+x)}{x} = \lim_{x\to 0} \frac{1}{1+x} = 1$. Also, if $\sum a_n$ converges, then $\lim_{n\to\infty} a_n = 0$ by Theorem .2.6. Therefore, $\lim_{n\to\infty} \frac{\ln\left(1+a_n\right)}{a_n} = \lim_{x\to 0} \frac{\ln(1+x)}{x} = 1 > 0$. We are given that $\sum a_n$ is convergent and $a_n > 0$. Thus, $\sum \ln\left(1+a_n\right)$ is convergent by the Limit Comparison Test.

45. Yes. Since $\sum a_n$ is a convergent series with positive terms, $\lim_{n\to\infty} a_n = 0$ by Theorem .2.6, and $\sum b_n = \sum \sin\left(a_n\right)$ is a series with positive terms (for large enough n). We have $\lim_{n\to\infty} \frac{b_n}{a_n} = \lim_{n\to\infty} \frac{\sin\left(a_n\right)}{a_n} = 1 > 0$ by Theorem 3. .2. Thus, $\sum b_n$ is also convergent by the Limit Comparison Test.

46. Yes. Since $\sum a_n$ converges, its terms approach 0 as $n\to\infty$, so for some integer N , $a_n \le 1$ for all $n \ge N$. But then $\sum_{n=1}^{\infty} a_n b_n = \sum_{n=1}^{N-1} a_n b_n + \sum_{n=N}^{\infty} a_n b_n \le \sum_{n=1}^{N-1} a_n b_n + \sum_{n=N}^{\infty} b_n$. The first term is a finite sum, and the second term converges since $\sum_{n=1}^{\infty} b_n$ converges. So $\sum a_n b_n$ converges by the Comparison Test.

1. **(a)** An alternating series is a series whose terms are alternately positive and negative.
(b) An alternating series $\sum_{n=1}^{\infty}(-1)^{n-1} b_n$ converges if $0<b_{n+1}\le b_n$ for all n and $\lim_{n\to\infty} b_n=0$. (This is the Alternating Series Test.)
(c) The error involved in using the partial sum s_n as an approximation to the total sum s is the remainder $R_n=s-s_n$ and the size of the error is smaller than b_{n+1} ; that is, $\left|R_n\right|\le b_{n+1}$. (This is the Alternating Series Estimation Theorem.)

2. $-\frac{1}{3}+\frac{2}{4}-\frac{3}{5}+\frac{4}{6}-\frac{5}{7}+\cdots=\sum_{n=1}^{\infty}(-1)^n \frac{n}{n+2}$. Here $a_n=(-1)^n \frac{n}{n+2}$. Since $\lim_{n\to\infty} a_n\ne 0$ (in fact the limit does not exist), the series diverges by the Test for Divergence.

3. $\frac{4}{7}-\frac{4}{8}+\frac{4}{9}-\frac{4}{10}+\frac{4}{11}-\cdots=\sum_{n=1}^{\infty}(-1)^{n-1} \frac{4}{n+6}$. Now $b_n=\frac{4}{n+6}>0$, $\left\{b_n\right\}$ is decreasing, and $\lim_{n\to\infty} b_n=0$, so the series converges by the Alternating Series Test.

4. $\sum_{n=2}^{\infty}(-1)^n \frac{1}{\ln n}$. $b_n=\frac{1}{\ln n}$ is positive and $\left\{b_n\right\}$ is decreasing; $\lim_{n\to\infty}\frac{1}{\ln n}=0$, so the series converges by the Alternating Series Test.

5. $b_n=\frac{1}{\sqrt{n}}>0$, $\left\{b_n\right\}$ is decreasing, and $\lim_{n\to\infty} b_n=0$, so the series $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{\sqrt{n}}$ converges by the Alternating Series Test.

6. $b_n=\frac{1}{3n-1}>0$, $\left\{b_n\right\}$ is decreasing, and $\lim_{n\to\infty} b_n=0$, so the series $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{3n-1}$ converges by the Alternating Series Test.

7. $\sum_{n=1}^{\infty} a_n=\sum_{n=1}^{\infty}(-1)^n \frac{3n-1}{2n+1}=\sum_{n=1}^{\infty}(-1)^n b_n$. Now $\lim_{n\to\infty} b_n=\lim_{n\to\infty}\frac{3-1/n}{2+1/n}=\frac{3}{2}\ne 0$. Since $\lim_{n\to\infty} a_n\ne 0$ (in fact the limit does not exist), the series diverges by the Test for Divergence.

8. $b_n=\frac{2n}{4n^2+1}>0$, $\left\{b_n\right\}$ is decreasing , and $\lim_{n\to\infty} b_n=\lim_{n\to\infty}\frac{2/n}{4+1/n^2}=0$, so the series $\sum_{n=1}^{\infty}(-1)^n \frac{2n}{4n^2+1}$ converges by the Alternating Series Test. Alternatively, to show that $\left\{b_n\right\}$ is decreasing, we could verify that $\frac{d}{dx}\left(\frac{2x}{4x^2+1}\right)<0$ for $x\ge 1$.

9. $b_n = \dfrac{1}{4n^2+1} > 0$, $\{b_n\}$ is decreasing, and $\lim_{n\to\infty} b_n = 0$, so the series $\sum_{n=1}^{\infty} \dfrac{(-1)^{n+1}}{4n^2+1}$ converges by the Alternating Series Test.

10. $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (-1)^n \dfrac{\sqrt{n}}{1+2\sqrt{n}} = \sum_{n=1}^{\infty} (-1)^n b_n$. Now $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \dfrac{1}{2+1/\sqrt{n}} = \dfrac{1}{2} \neq 0$. Since $\lim_{n\to\infty} a_n \neq 0$ (in fact the limit does not exist), the series diverges by the Test for Divergence.

11. $b_n = \dfrac{n^2}{n^3+4} > 0$ for $n \geq 1$. $\{b_n\}$ is decreasing for $n \geq 2$ since

$$\left(\frac{x^2}{x^3+4}\right)' = \frac{\left(x^3+4\right)(2x) - x^2\left(3x^2\right)}{\left(x^3+4\right)^2} = \frac{x\left(2x^3+8-3x^3\right)}{\left(x^3+4\right)^2} = \frac{x\left(8-x^3\right)}{\left(x^3+4\right)^2} < 0 \text{ for } x > 2 \text{ . Also,}$$

$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \dfrac{1/n}{1+4/n^3} = 0$. Thus, the series $\sum_{n=1}^{\infty} (-1)^{n+1} \dfrac{n^2}{n^3+4}$ converges by the Alternating Series Test.

12. $b_n = \dfrac{e^{1/n}}{n} > 0$ for $n \geq 1$. $\{b_n\}$ is decreasing since

$$\left(\frac{e^{1/x}}{x}\right)' = \frac{x \cdot e^{1/x}\left(-1/x^2\right) - e^{1/x} \cdot 1}{x^2} = \frac{-e^{1/x}(1+x)}{x^3} < 0 \text{ for } x > 0$$

. Also, $\lim_{n\to\infty} b_n = 0$ since $\lim_{n\to\infty} e^{1/n} = 1$. Thus, the series $\sum_{n=1}^{\infty} (-1)^{n-1} \dfrac{e^{1/n}}{n}$ converges by the Alternating Series Test.

13. $\sum_{n=2}^{\infty} (-1)^n \dfrac{n}{\ln n}$. $\lim_{n\to\infty} \dfrac{n}{\ln n} = \lim_{x\to\infty} \dfrac{x}{\ln x} = \lim_{x\to\infty} \dfrac{1}{1/x} = \infty$, so the series diverges by the Test for Divergence.

14. $\sum_{n=1}^{\infty} (-1)^{n-1} \left(\dfrac{\ln n}{n}\right) = 0 + \sum_{n=2}^{\infty} (-1)^{n-1} \left(\dfrac{\ln n}{n}\right)$. $b_n = \dfrac{\ln n}{n} > 0$ for $n \geq 2$, and if $f(x) = \dfrac{\ln x}{x}$, then $f'(x) = \dfrac{1-\ln x}{x^2} < 0$ for $x > e$, so $\{b_n\}$ is eventually decreasing. Also,

$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \dfrac{\ln n}{n} = \lim_{x\to\infty} \dfrac{\ln x}{x} = \lim_{x\to\infty} \dfrac{1/x}{1} = 0$, so the series converges by the Alternating Series Test.

15.

$\sum_{n=1}^{\infty} \frac{\cos n\pi}{n^{3/4}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^{3/4}}$. $b_n = \frac{1}{n^{3/4}}$ is decreasing and positive and $\lim_{n\to\infty} \frac{1}{n^{3/4}} = 0$, so the series converges by the Alternating Series Test.

16. $\sin\left(\frac{n\pi}{2}\right) = 0$ if n is even and $(-1)^k$ if $n=2k+1$, so the series is $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}$. $b_n = \frac{1}{(2n+1)!} > 0$, $\left\{ b_n \right\}$ is decreasing, and $\lim_{n\to\infty} \frac{1}{(2n+1)!} = 0$, so the series converges by the Alternating Series Test.

17. $\sum_{n=1}^{\infty} (-1)^n \sin \frac{\pi}{n}$. $b_n = \sin \frac{\pi}{n} > 0$ for $n \geq 2$ and $\sin \frac{\pi}{n} \geq \sin \frac{\pi}{n+1}$, and $\lim_{n\to\infty} \sin \frac{\pi}{n} = \sin 0 = 0$, so the series converges by the Alternating Series Test.

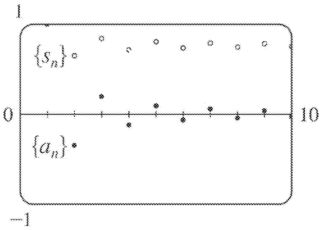
18. $\sum_{n=1}^{\infty} (-1)^n \cos\left(\frac{\pi}{n}\right)$. $\lim_{n\to\infty} \cos\left(\frac{\pi}{n}\right) = \cos(0) = 1$, so $\lim_{n\to\infty} (-1)^n \cos\left(\frac{\pi}{n}\right)$ does not exist and the series diverges by the Test for Divergence.

19. $\frac{n^n}{n!} = \frac{n \cdot n \cdot \cdots \cdot n}{1 \cdot 2 \cdot \cdots \cdot n} \geq n \Rightarrow \lim_{n\to\infty} \frac{n^n}{n!} = \infty \Rightarrow \lim_{n\to\infty} \frac{(-1)^n n^n}{n!}$ does not exist. So the series diverges by the Test for Divergence.

20. $\sum_{n=1}^{\infty} \left(-\frac{n}{5}\right)^n$ diverges by the Test for Divergence since $\lim_{n\to\infty} \left(\frac{n}{5}\right)^n = \infty \Rightarrow \lim_{n\to\infty} \left(-\frac{n}{5}\right)^n$ does not exist.

21.

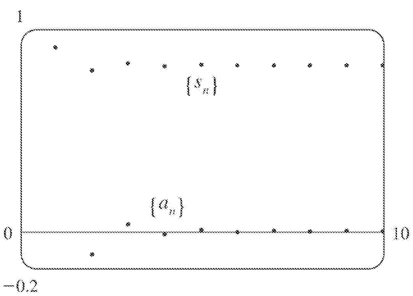
n	a_n	s_n
1	1	1
2	−0.35355	0.64645
3	0.19245	0.83890
4	−0.125	0.71390
5	0.08944	0.80334
6	−0.06804	0.73530
7	0.05399	0.78929
8	−0.04419	0.74510
9	0.03704	0.78214
10	−0.03162	0.75051



By the Alternating Series Estimation Theorem, the error in the approximation $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^{3/2}} \approx 0.75051$ is $\left|s-s_{10}\right| \le b_{11}=1/(11)^{3/2} \approx 0.0275$ (to four decimal places, rounded up).

22.

n	a_n	s_n
1	1	1
2	−0.125	0.875
3	0.03704	0.91204
4	−0.01563	0.89641
5	0.008	0.90441
6	−0.00463	0.89978
7	0.00292	0.90270
8	−0.00195	0.90074
9	0.00137	0.90212
10	−0.001	0.90112



By the Alternating Series Estimation Theorem, the error in the approximation $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^{3}} \approx 0.90112$ is $\left|s-s_{10}\right| \le b_{11}=1/11^{3} \approx 0.0007513$.

23. The series $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n^2}$ satisfies (i) of the Alternating Series Test because $\frac{1}{(n+1)^2} < \frac{1}{n^2}$ and (ii) $\lim_{n\to\infty} \frac{1}{n^2}=0$, so the series is convergent. Now $b_{10}=\frac{1}{10^2}=0.01$ and $b_{11}=\frac{1}{11^2}=\frac{1}{121}\approx 0.008<0.01$, so by the Alternating Series Estimation Theorem, $n=10$. (That is, since the 11 th term is less than the

desired error, we need to add the first 10 terms to get the sum to the desired accuracy.)

24. The series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^4}$ satisfies (i) of the Alternating Series Test because $\frac{1}{(n+1)^4}<\frac{1}{n^4}$ and (ii) $\lim_{n\to\infty}\frac{1}{n^4}=0$, so the series is convergent. Now $b_5=1/5^4=0.0016>0.001$ and $b_6=1/6^4\approx 0.00077<0.001$, so by the Alternating Series Estimation Theorem, $n=5$.

25. The series $\sum_{n=1}^{\infty}\frac{(-2)^n}{n!}=\sum_{n=1}^{\infty}(-1)^n\frac{2^n}{n!}$ satisfies (i) of the Alternating Series Test because $b_{n+1}=\frac{2^{n+1}}{(n+1)!}=\frac{2\cdot 2^n}{(n+1)n!}=\frac{2}{n+1}\cdot\frac{2^n}{n!}=\frac{2}{n+1}\cdot b_n\le b_n$ and (ii) $\lim_{n\to\infty}\frac{2^n}{n!}=\frac{2}{n}\cdot\frac{2}{n-1}\cdot\cdots\cdot\frac{2}{2}\cdot\frac{2}{1}=0$, so the series is convergent. Now $b_7=2^7/7!\approx 0.025>0.01$ and $b_8=2^8/8!\approx 0.006<0.01$, so by the Alternating Series Estimation Theorem, $n=7$. (That is, since the 8 th term is less than the desired error, we need to add the first 7 terms to get the sum to the desired accuracy.)

26. The series $\sum_{n=1}^{\infty}\frac{(-1)^n n}{4^n}=\sum_{n=1}^{\infty}(-1)^n\frac{n}{4^n}$ satisfies (i) of the Alternating Series Test because $b_{n+1}=\frac{n+1}{4^{n+1}}<\frac{n+3n}{4^n\cdot 4^1}=\frac{4n}{4\cdot 4^n}=\frac{n}{4^n}=b_n$ and (ii) $\lim_{n\to\infty}\frac{n}{4^n}=0$, so the series is convergent. Now $b_5=5/4^5\approx 0.0049>0.002$ and $b_6=6/4^6\approx 0.0015<0.002$, so by the Alternating Series Estimation Theorem, $n=5$.

27. $b_7=\frac{1}{7^5}=\frac{1}{16,807}\approx 0.0000595$, so $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^5}\approx s_6=\sum_{n=1}^{6}\frac{(-1)^{n+1}}{n^5}=1-\frac{1}{32}+\frac{1}{243}-\frac{1}{1024}+\frac{1}{3125}-\frac{1}{7776}\approx 0.972080$. Adding b_7 to s_6 does not change the fourth decimal place of s_6 , so the sum of the series, correct to four decimal places, is 0.9721 .

28. $b_6=\frac{6}{8^6}=\frac{6}{262,144}\approx 0.000023$, so

$\sum_{n=1}^{\infty} \frac{(-1)^n n}{8^n} \approx s_5 = \sum_{n=1}^{5} \frac{(-1)^n n}{8^n} = -\frac{1}{8} + \frac{2}{64} - \frac{3}{512} + \frac{4}{4096} - \frac{5}{32{,}768} \approx -0.098785$. Adding b_6 to s_5 does not change the fourth decimal place of s_5 , so the sum of the series, correct to four decimal places, is -0.0988 .

29. $b_7 = \frac{7^2}{10^7} = 0.0000049$, so

$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} n^2}{10^n} \approx s_6 = \sum_{n=1}^{6} \frac{(-1)^{n-1} n^2}{10^n} = \frac{1}{10} - \frac{4}{100} + \frac{9}{1000} - \frac{16}{10{,}000} + \frac{25}{100{,}000} - \frac{36}{1{,}000{,}000} = 0.067614$.

Adding b_7 to s_6 does not change the fourth decimal place of s_6 , so the sum of the series, correct to four decimal places, is 0.0676 .

30. $b_6 = \frac{1}{3^6 \cdot 6!} = \frac{1}{524{,}880} \approx 0.0000019$, so

$\sum_{n=1}^{\infty} \frac{(-1)^n}{3^n n!} \approx s_5 = \sum_{n=1}^{5} \frac{(-1)^n}{3^n n!} = -\frac{1}{3} + \frac{1}{18} - \frac{1}{162} + \frac{1}{1944} - \frac{1}{29{,}160} \approx -0.283471$. Adding b_6 to s_5 does not change the fourth decimal place of s_5 , so the sum of the series, correct to four decimal places, is -0.2835 .

31. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{49} - \frac{1}{50} + \frac{1}{51} - \frac{1}{52} + \cdots$. The 50 th partial sum of this series is an underestimate, since $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = s_{50} + \left(\frac{1}{51} - \frac{1}{52} \right) + \left(\frac{1}{53} - \frac{1}{54} \right) + \cdots$, and the terms in parentheses are all positive. The result can be seen geometrically in Figure 1.

32. If $p>0$, $\frac{1}{(n+1)^p} \le \frac{1}{n^p}$ ($\{1/n^p\}$ is decreasing) and $\lim_{n\to\infty} \frac{1}{n^p} = 0$, so the series converges by the Alternating Series Test. If $p \le 0$, $\lim_{n\to\infty} \frac{(-1)^{n-1}}{n^p}$ does not exist, so the series diverges by the Test for Divergence. Thus, $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^p}$ converges $\Leftrightarrow p>0$.

33. Clearly $b_n = \frac{1}{n+p}$ is decreasing and eventually positive and $\lim_{n\to\infty} b_n=0$ for any p . So the series converges (by the Alternating Series Test) for any p for which every b_n is defined, that is, $n+p\neq 0$ for $n\geq 1$, or p is not a negative integer.

34. Let $f(x)=\frac{(\ln x)^p}{x}$. Then $f'(x)=\frac{(\ln x)^{p-1}(p-\ln x)}{x^2}$ <0 if $x>e^p$ so f is eventually decreasing for every p . Clearly $\lim_{n\to\infty}\frac{(\ln n)^p}{n}=0$ if $p\leq 0$, and if $p>0$ we can apply l'Hospital's Rule $[p+1]$ times to get a limit of 0 as well. So the series converges for all p (by the Alternating Series Test).

35. $\Sigma b_{2n}=\Sigma 1/(2n)^2$ clearly converges (by comparison with the p– series for $p=2$). So suppose that $\Sigma(-1)^{n-1}b_n$ converges. Then by Theorem .2.8(ii), so does $\Sigma\left[(-1)^{n-1}b_n+b_n\right]=2\left(1+\frac{1}{3}+\frac{1}{5}+\cdots\right)=2\Sigma\frac{1}{2n-1}$. But this diverges by comparison with the harmonic series, a contradiction. Therefore, $\Sigma(-1)^{n-1}b_n$ must diverge. The Alternating Series Test does not apply since $\{b_n\}$ is not decreasing.

36. **(a)** We will prove this by induction. Let $P(n)$ be the proposition that $s_{2n}=h_{2n}-h_n$. $P(1)$ is the statement $s_2=h_2-h_1$, which is true since $1-\frac{1}{2}=\left(1+\frac{1}{2}\right)-1$. So suppose that $P(n)$ is true. We will show that $P(n+1)$ must be true as a consequence.

$$\begin{aligned} h_{2n+2}-h_{n+1} &= \left(h_{2n}+\frac{1}{2n+1}+\frac{1}{2n+2}\right)-\left(h_n+\frac{1}{n+1}\right)=\left(h_{2n}-h_n\right)+\frac{1}{2n+1}-\frac{1}{2n+2} \\ &= s_{2n}+\frac{1}{2n+1}-\frac{1}{2n+2}=s_{2n+2} \end{aligned}$$

which is $P(n+1)$, and proves that $s_{2n}=h_{2n}-h_n$ for all n .

(b) We know that $h_{2n}-\ln(2n)\to\gamma$ and $h_n-\ln n\to\gamma$ as $n\to\infty$. So $s_{2n}=h_{2n}-h_n=[h_{2n}-\ln(2n)]-\left(h_n-\ln n\right)+[\ln(2n)-\ln n]$, and $\lim_{n\to\infty}s_{2n}=\gamma-\gamma+\lim_{n\to\infty}[\ln(2n)-\ln n]=\lim_{n\to\infty}(\ln 2+\ln n-\ln n)=\ln 2$.

1. **(a)** Since $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=8>1$, part (b) of the Ratio Test tells us that the series $\Sigma\, a_n$ is divergent.

(b) Since $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=0.8<1$, part (a) of the Ratio Test tells us that the series $\Sigma\, a_n$ is absolutely convergent (and therefore convergent).

(c) Since $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=1$, the Ratio Test fails and the series $\Sigma\, a_n$ might converge or it might diverge.

2. The series $\Sigma_{n=1}^{\infty}\frac{n^2}{2^n}$ has positive terms and

$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\left[\frac{(n+1)^2}{2^{n+1}}\cdot\frac{2^n}{n^2}\right]=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^2\cdot\frac{1}{2}=\frac{1}{2}<1$, so the series is absolutely convergent by the Ratio Test.

3. $\Sigma_{n=0}^{\infty}\frac{(-10)^n}{\text{n!}}$. Using the Ratio Test,

$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{(-10)^{n+1}}{(n+1)!}\cdot\frac{\text{n!}}{(-10)^n}\right|=\lim_{n\to\infty}\left|\frac{-10}{n+1}\right|=0<1$, so the series is absolutely convergent.

4. $\Sigma_{n=1}^{\infty}(-1)^{n-1}\frac{2^n}{n^4}$ diverges by the Test for Divergence. $\lim_{n\to\infty}\frac{2^n}{n^4}=\infty$, so $\lim_{n\to\infty}(-1)^{n-1}\frac{2^n}{n^4}$ does not exist.

5. $\Sigma_{n=1}^{\infty}\frac{(-1)^{n+1}}{\sqrt[4]{n}}$ converges by the Alternating Series Test, but $\Sigma_{n=1}^{\infty}\frac{1}{\sqrt{[4]n}}$ is a divergent p –series $\left(p=\frac{1}{4}\leq 1\right)$, so the given series is conditionally convergent.

6. $\Sigma_{n=1}^{\infty}\frac{1}{n^4}$ is a convergent p –series $(p=4>1)$, so $\Sigma_{n=1}^{\infty}\frac{(-1)^n}{n^4}$ is absolutely convergent.

7. $\lim_{n\to\infty} \left| a_n \right| = \lim_{n\to\infty} \frac{n}{5+n} = \lim_{n\to\infty} \frac{1}{5/n+1} = 1$, so $\lim_{n\to\infty} a_n \neq 0$. Thus, the given series is divergent by the Test for Divergence.

8. $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ diverges by the Limit Comparison Test with the harmonic series:

$\lim_{n\to\infty} \frac{n/(n^2+1)}{1/n} = \lim_{n\to\infty} \frac{n^2}{n^2+1} = 1$. But $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{n^2+1}$ converges by the Alternating Series Test:

$\left\{ \frac{n}{n^2+1} \right\}$ has positive terms, is decreasing since $\left(\frac{x}{x^2+1} \right)' = \frac{1-x^2}{\left(x^2+1\right)^2} \leq 0$ for $x \geq 1$, and

$\lim_{n\to\infty} \frac{n}{n^2+1} = 0$. Thus, $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{n^2+1}$ is conditionally convergent.

9.

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{1/(2n+2)!}{1/(2n)!} = \lim_{n\to\infty} \frac{(2n)!}{(2n+2)!} = \lim_{n\to\infty} \frac{(2n)!}{(2n+2)(2n+1)(2n)!} = \lim_{n\to\infty} \frac{1}{(2n+2)(2n+1)} = 0 < 1$$

, so the series $\sum_{n=1}^{\infty} \frac{1}{(2n)!}$ is absolutely convergent by the Ratio Test. Of course, absolute convergence is the same as convergence for this series, since all of its terms are positive.

10. $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)!/e^{n+1}}{n!/e^n} \right| = \frac{1}{e} \lim_{n\to\infty} (n+1) = \infty$, so the series $\sum_{n=1}^{\infty} e^{-n} n!$ diverges by the Ratio Test.

11. Since $0 \leq \frac{e^{1/n}}{n^3} \leq \frac{e}{n^3} = e\left(\frac{1}{n^3} \right)$ and $\sum_{n=1}^{\infty} \frac{1}{n^3}$ is a convergent p -series $(p=3>1)$, $\sum_{n=1}^{\infty} \frac{e^{1/n}}{n^3}$ converges, and so $\sum_{n=1}^{\infty} \frac{(-1)^n e^{1/n}}{n^3}$ is absolutely convergent.

12. $\left| \frac{\sin 4n}{4^n} \right| \leq \frac{1}{4^n}$, so $\sum_{n=1}^{\infty} \left| \frac{\sin 4n}{4^n} \right|$ converges by comparison with the convergent geometric series

$\sum_{n=1}^{\infty} \frac{1}{4^n}$ $\left(|r| = \frac{1}{4} < 1 \right)$. Thus, $\sum_{n=1}^{\infty} \frac{\sin 4n}{4^n}$ is absolutely convergent.

13. $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left[\frac{(n+1)3^{n+1}}{4^n} \cdot \frac{4^{n-1}}{n \cdot 3^n} \right] = \lim_{n\to\infty} \left(\frac{3}{4} \cdot \frac{n+1}{n} \right) = \frac{3}{4} < 1$, so the series $\sum_{n=1}^{\infty} \frac{n(-3)^n}{4^{n-1}}$ is absolutely convergent by the Ratio Test.

14. $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left[\frac{(n+1)^2 2^{n+1}}{(n+1)!} \cdot \frac{n!}{n^2 2^n} \right] = \lim_{n\to\infty} \left[\left(1 + \frac{1}{n} \right)^2 \cdot \frac{2}{n+1} \right] = 0$, so the series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2 2^n}{n!}$ is absolutely convergent by the Ratio Test.

15. $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left[\frac{10^{n+1}}{(n+2)4^{2n+3}} \cdot \frac{(n+1)4^{2n+1}}{10^n} \right] = \lim_{n\to\infty} \left(\frac{10}{4^2} \cdot \frac{n+1}{n+2} \right) = \frac{5}{8} < 1$, so the series $\sum_{n=1}^{\infty} \frac{10^n}{(n+1)4^{2n+1}}$ is absolutely convergent by the Ratio Test. Since the terms of this series are positive, absolute convergence is the same as convergence.

16. $n^{2/3} - 2 > 0$ for $n \geq 3$, so $\frac{3 - \cos n}{n^{2/3} - 2} > \frac{1}{n^{2/3} - 2} > \frac{1}{n^{2/3}}$ for $n \geq 3$. Since $\sum_{n=1}^{\infty} \frac{1}{n^{2/3}}$ diverges $\left(p = \frac{2}{3} \leq 1 \right)$, so does $\sum_{n=1}^{\infty} \frac{3 - \cos n}{n^{2/3} - 2}$ by the Comparison Test.

17. $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$ converges by the Alternating Series Test since $\lim_{n\to\infty} \frac{1}{\ln n} = 0$ and $\left\{ \frac{1}{\ln n} \right\}$ is decreasing. Now $\ln n < n$, so $\frac{1}{\ln n} > \frac{1}{n}$, and since $\sum_{n=2}^{\infty} \frac{1}{n}$ is the divergent (partial) harmonic series, $\sum_{n=2}^{\infty} \frac{1}{\ln n}$ diverges by the Comparison Test. Thus, $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$ is conditionally convergent.

18.

$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{(n+1)!/(n+1)^{n+1}}{n!/n^n} = \lim_{n\to\infty}\frac{n^n}{(n+1)^n} = \lim_{n\to\infty}\frac{1}{(1+1/n)^n} = \frac{1}{e} < 1$, so the series $\sum_{n=1}^{\infty}\frac{n!}{n^n}$ converges absolutely by the Ratio Test.

19. $\frac{|\cos(n\pi/3)|}{n!} \le \frac{1}{n!}$ and $\sum_{n=1}^{\infty}\frac{1}{n!}$ converges (use the Ratio Test or the result of Exercise .4.29), so the series $\sum_{n=1}^{\infty}\frac{\cos(n\pi/3)}{n!}$ converges absolutely by the Comparison Test.

20. $\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\frac{1}{\ln n} = 0 < 1$, so the series $\sum_{n=2}^{\infty}\frac{(-1)^n}{(\ln n)^n}$ converges absolutely by the Root Test.

21. $\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\left(\frac{n^n}{3^{1+3n}}\right)^{1/n} = \lim_{n\to\infty}\frac{n}{\sqrt[n]{3}\cdot 3^3} = \infty$, so the series $\sum_{n=1}^{\infty}\frac{n^n}{3^{1+3n}}$ is divergent by the Root Test.

Or:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left[\frac{(n+1)^{n+1}}{3^{4+3n}}\cdot\frac{3^{1+3n}}{n^n}\right] = \lim_{n\to\infty}\left[\frac{1}{3^3}\cdot\left(\frac{n+1}{n}\right)^n(n+1)\right]$$

$$= \frac{1}{27}\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n\lim_{n\to\infty}(n+1) = \frac{1}{27}e\lim_{n\to\infty}(n+1) = \infty ,$$

so the series is divergent by the Ratio Test.

22. Since $\left\{\frac{1}{n\ln n}\right\}$ is decreasing and $\lim_{n\to\infty}\frac{1}{n\ln n} = 0$, the series $\sum_{n=2}^{\infty}\frac{(-1)^n}{n\ln n}$ converges by the Alternating Series Test. Since $\sum_{n=2}^{\infty}\frac{1}{n\ln n}$ diverges by the Integral Test (Exercise .3.21), the series $\sum_{n=2}^{\infty}\frac{(-1)^n}{n\ln n}$ is conditionally convergent.

23. $\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\frac{n^2+1}{2n^2+1} = \lim_{n\to\infty}\frac{1+1/n^2}{2+1/n^2} = \frac{1}{2} < 1$, so the series $\sum_{n=1}^{\infty}\left(\frac{n^2+1}{2n^2+1}\right)^n$ is absolutely

convergent by the Root Test.

24. $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \frac{1}{\arctan n} = \frac{1}{\pi/2} = \frac{2}{\pi} < 1$, so the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{(\arctan n)^n}$ is absolutely convergent by the Root Test.

25. Use the Ratio Test with the series

$1 - \frac{1\cdot 3}{3!} + \frac{1\cdot 3\cdot 5}{5!} - \frac{1\cdot 3\cdot 5\cdot 7}{7!} + \cdots + (-1)^{n-1} \frac{1\cdot 3\cdot 5\cdot \cdots \cdot (2n-1)}{(2n-1)!} + \cdots = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1\cdot 3\cdot 5\cdot \cdots \cdot (2n-1)}{(2n-1)!}$

.

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(-1)^n \cdot 1\cdot 3\cdot 5\cdot \cdots \cdot (2n-1)[2(n+1)-1]}{[2(n+1)-1]!} \cdot \frac{(2n-1)!}{(-1)^{n-1} \cdot 1\cdot 3\cdot 5\cdot \cdots \cdot (2n-1)} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(-1)(2n+1)(2n-1)!}{(2n+1)(2n)(2n-1)!} \right|$$

$$= \lim_{n\to\infty} \frac{1}{2n} = 0 < 1,$$

so the given series is absolutely convergent and therefore convergent.

26. Use the Ratio Test with the series

$\frac{2}{5} + \frac{2\cdot 6}{5\cdot 8} + \frac{2\cdot 6\cdot 10}{5\cdot 8\cdot 11} + \frac{2\cdot 6\cdot 10\cdot 14}{5\cdot 8\cdot 11\cdot 14} + \cdots = \sum_{n=1}^{\infty} \frac{2\cdot 6\cdot 10\cdot 14\cdot \cdots \cdot (4n-2)}{5\cdot 8\cdot 11\cdot 14\cdot \cdots \cdot (3n+2)}$.

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{2\cdot 6\cdot 10\cdot \cdots \cdot (4n-2)[4(n+1)-2]}{5\cdot 8\cdot 11\cdot \cdots \cdot (3n+2)[3(n+1)+2]} \cdot \frac{5\cdot 8\cdot 11\cdot \cdots \cdot (3n+2)}{2\cdot 6\cdot 10\cdot \cdots \cdot (4n-2)} \right|$$

$$= \lim_{n\to\infty} \frac{4n+2}{3n+5} = \frac{4}{3} > 1,$$

so the given series is divergent.

27. $\sum_{n=1}^{\infty} \frac{2\cdot 4\cdot 6\cdot \cdots \cdot (2n)}{n!} = \sum_{n=1}^{\infty} \frac{(2\cdot 1)\cdot (2\cdot 2)\cdot (2\cdot 3)\cdot \cdots \cdot (2\cdot n)}{n!} = \sum_{n=1}^{\infty} \frac{2^n n!}{n!} = \sum_{n=1}^{\infty} 2^n$, which diverges by the Test for Divergence since $\lim_{n\to\infty} 2^n = \infty$.

28.

$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\dfrac{2^{n+1}(n+1)!}{5\cdot 8\cdot 11\cdot\cdots\cdot(3n+5)}}{\dfrac{2^n n!}{5\cdot 8\cdot 11\cdot\cdots\cdot(3n+2)}}\right| = \lim_{n\to\infty}\frac{2(n+1)}{3n+5} = \frac{2}{3} < 1$, so the series converges absolutely by the Ratio Test.

29. By the recursive definition, $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{5n+1}{4n+3}\right| = \frac{5}{4} > 1$, so the series diverges by the Ratio Test.

30. By the recursive definition, $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{2+\cos n}{\sqrt{n}}\right| = 0 < 1$, so the series converges absolutely by the Ratio Test.

31. **(a)** $\lim_{n\to\infty}\left|\frac{1/(n+1)^3}{1/n^3}\right| = \lim_{n\to\infty}\frac{n^3}{(n+1)^3} = \lim_{n\to\infty}\frac{1}{(1+1/n)^3} = 1$. Inconclusive.

(b) $\lim_{n\to\infty}\left|\frac{(n+1)}{2^{n+1}}\cdot\frac{2^n}{n}\right| = \lim_{n\to\infty}\frac{n+1}{2n} = \lim_{n\to\infty}\left(\frac{1}{2}+\frac{1}{2n}\right) = \frac{1}{2}$. Conclusive (convergent).

(c) $\lim_{n\to\infty}\left|\frac{(-3)^n}{\sqrt{n+1}}\cdot\frac{\sqrt{n}}{(-3)^{n-1}}\right| = 3\lim_{n\to\infty}\sqrt{\frac{n}{n+1}} = 3\lim_{n\to\infty}\sqrt{\frac{1}{1+1/n}} = 3$. Conclusive (divergent).

(d) $\lim_{n\to\infty}\left|\frac{\sqrt{n+1}}{1+(n+1)^2}\cdot\frac{1+n^2}{\sqrt{n}}\right| = \lim_{n\to\infty}\left[\sqrt{1+\frac{1}{n}}\cdot\frac{1/n^2+1}{1/n^2+(1+1/n)^2}\right] = 1$. Inconclusive.

32. We use the Ratio Test:

$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{[(n+1)!]^2}{[k(n+1)]!}\cdot\frac{(kn)!}{(n!)^2}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^2}{[k(n+1)][k(n+1)-1]\cdots[kn+1]}\right|$ Now if $k=1$, then this is equal to $\lim_{n\to\infty}\left|\frac{(n+1)^2}{(n+1)}\right| = \infty$, so the series diverges; if $k=2$, the limit is $\lim_{n\to\infty}\left|\frac{(n+1)^2}{(2n+2)(2n+1)}\right| = \frac{1}{4} < 1$, so the series converges, and if $k>2$, then the highest power of n in the denominator is larger than 2 , and so the limit is 0 , indicating convergence. So the series converges for $k\geq 2$.

33. **(a)** $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)!}\cdot\frac{n!}{x^n}\right| = \lim_{n\to\infty}\left|\frac{x}{n+1}\right| = |x|\lim_{n\to\infty}\frac{1}{n+1} = |x|\cdot 0 = 0 < 1$, so by the Ratio Test the series $\sum_{n=0}^{\infty}\frac{x^n}{n!}$ converges for all x .

(b) Since the series of part (a) always converges, we must have $\lim_{n\to\infty}\frac{x^n}{n!} = 0$ by Theorem .2.6.

34. **(a)**

$$R_n = a_{n+1} + a_{n+2} + a_{n+3} + a_{n+4} + \cdots = a_{n+1}\left(1 + \frac{a_{n+2}}{a_{n+1}} + \frac{a_{n+3}}{a_{n+1}} + \frac{a_{n+4}}{a_{n+1}} + \cdots\right)$$

$$= a_{n+1}\left(1 + \frac{a_{n+2}}{a_{n+1}} + \frac{a_{n+3}}{a_{n+2}}\frac{a_{n+2}}{a_{n+1}} + \frac{a_{n+4}}{a_{n+3}}\frac{a_{n+3}}{a_{n+2}}\frac{a_{n+2}}{a_{n+1}} + \cdots\right)$$

$$= a_{n+1}\left(1 + r_{n+1} + r_{n+2}r_{n+1} + r_{n+3}r_{n+2}r_{n+1} + \cdots\right) \quad (*)$$

$$\le a_{n+1}\left(1 + r_{n+1} + r_{n+1}^2 + r_{n+1}^3 + \cdots\right) = \frac{a_{n+1}}{1 - r_{n+1}}$$

(b) Note that since $\{r_n\}$ is increasing and $r_n \to L$ as $n\to\infty$, we have $r_n < L$ for all n . So, starting with equation ($*$),

$$R_n = a_{n+1}\left(1 + r_{n+1} + r_{n+2}r_{n+1} + r_{n+3}r_{n+2}r_{n+1} + \cdots\right) \le a_{n+1}\left(1 + L + L^2 + L^3 + \cdots\right) = \frac{a_{n+1}}{1-L}$$

35. **(a)** $s_5 = \sum_{n=1}^{5}\frac{1}{n2^n} = \frac{1}{2} + \frac{1}{8} + \frac{1}{24} + \frac{1}{64} + \frac{1}{160} = \frac{661}{960} \approx 0.68854$. Now the ratios $r_n = \frac{a_{n+1}}{a_n} = \frac{n2^n}{(n+1)2^{n+1}} = \frac{n}{2(n+1)}$ form an increasing sequence, since $r_{n+1} - r_n = \frac{n+1}{2(n+2)} - \frac{n}{2(n+1)} = \frac{(n+1)^2 - n(n+2)}{2(n+1)(n+2)} = \frac{1}{2(n+1)(n+2)} > 0$. So by Exercise 34(b), the error in

using s_5 is $R_5 \le \dfrac{a_6}{1-\lim\limits_{n\to\infty} r_n} = \dfrac{1/\left(6\cdot 2^6\right)}{1-1/2} = \dfrac{1}{192} \approx 0.00521$.

(b) The error in using s_n as an approximation to the sum is $R_n = \dfrac{a_{n+1}}{1-\frac{1}{2}} = \dfrac{2}{(n+1)2^{n+1}}$. We want $R_n < 0.00005 \Leftrightarrow \dfrac{1}{(n+1)2^n} < 0.00005 \Leftrightarrow (n+1)2^n > 20,000$. To find such an n we can use trial and error or a graph. We calculate $(11+1)2^{11} = 24,576$, so $s_{11} = \sum_{n=1}^{11} \dfrac{1}{n2^n} \approx 0.693109$ is within 0.00005 of the actual sum.

36. $s_{10} = \sum_{n=1}^{10} \dfrac{n}{2^n} = \dfrac{1}{2} + \dfrac{2}{4} + \dfrac{3}{8} + \cdots + \dfrac{10}{1024} \approx 1.988$. The ratios $r_n = \dfrac{a_{n+1}}{a_n} = \dfrac{n+1}{2^{n+1}} \cdot \dfrac{2^n}{n} = \dfrac{n+1}{2n} = \dfrac{1}{2}\left(1+\dfrac{1}{n}\right)$ form a decreasing sequence, and $r_{11} = \dfrac{11+1}{2(11)} = \dfrac{12}{22} = \dfrac{6}{11} < 1$, so by Exercise 34(a), the error in using s_{10} to approximate the sum of the series $\sum_{n=1}^{\infty} \dfrac{n}{2^n}$ is

$$R_{10} \le \frac{a_{11}}{1-r_{11}} = \frac{\frac{11}{2048}}{1-\frac{6}{11}} = \frac{121}{10,240} \approx 0.0118 .$$

37. Summing the inequalities $-\left|a_i\right| \le a_i \le \left|a_i\right|$ for $i=1,2,\ldots,n$, we get $-\sum_{i=1}^{n}\left|a_i\right| \le \sum_{i=1}^{n} a_i \le \sum_{i=1}^{n}\left|a_i\right|$ $\Rightarrow -\lim\limits_{n\to\infty} \sum_{i=1}^{n}\left|a_i\right| \le \lim\limits_{n\to\infty} \sum_{i=1}^{n} a_i \le \lim\limits_{n\to\infty} \sum_{i=1}^{n}\left|a_i\right| \Rightarrow -\sum_{n=1}^{\infty}\left|a_n\right| \le \sum_{n=1}^{\infty} a_n \le \sum_{n=1}^{\infty}\left|a_n\right| \Rightarrow$ $\left|\sum_{n=1}^{\infty} a_n\right| \le \sum_{n=1}^{\infty}\left|a_n\right|$.

38. **(a)** Following the hint, we get that $\left|a_n\right| < r^n$ for $n \ge N$, and so since the geometric series $\sum_{n=1}^{\infty} r^n$ converges ($0<r<1$), the series $\sum_{n=N}^{\infty}\left|a_n\right|$ converges as well by the Comparison Test, and hence so does $\sum_{n=1}^{\infty}\left|a_n\right|$, so $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

(b) If $\lim\limits_{n\to\infty} \sqrt[n]{\left|a_n\right|} = L > 1$, then there is an integer N such that $\sqrt[n]{\left|a_n\right|} > 1$ for all $n \ge N$, so $\left|a_n\right| > 1$

for $n \geq N$. Thus, $\lim_{n\to\infty} a_n \neq 0$, so $\Sigma_{n=1}^{\infty} a_n$ diverges by the Test for Divergence.

39. **(a)** Since $\Sigma\, a_n$ is absolutely convergent, and since $\left|a_n^+\right| \leq \left|a_n\right|$ and $\left|a_n^-\right| \leq \left|a_n\right|$ (because a_n^+ and a_n^- each equal either a_n or 0), we conclude by the Comparison Test that both $\Sigma\, a_n^+$ and $\Sigma\, a_n^-$ must be absolutely convergent. (Or use Theorem .2.8.)

(b) We will show by contradiction that both $\Sigma\, a_n^+$ and $\Sigma\, a_n^-$ must diverge. For suppose that $\Sigma\, a_n^+$ converged. Then so would $\Sigma\left(a_n^+ - \frac{1}{2}a_n\right)$ by Theorem .2.8. But $\Sigma\left(a_n^+ - \frac{1}{2}a_n\right) = \Sigma\left[\frac{1}{2}\left(a_n + \left|a_n\right|\right) - \frac{1}{2}a_n\right] = \frac{1}{2}\Sigma\left|a_n\right|$, which diverges because $\Sigma\, a_n$ is only conditionally convergent. Hence, $\Sigma\, a_n^+$ can't converge. Similarly, neither can $\Sigma\, a_n^-$.

40. Let $\Sigma\, b_n$ be the rearranged series constructed in the hint. This series will have partial sums s_n that oscillate in value back and forth across r . Since $\lim_{n\to\infty} a_n = 0$ (by Theorem .2.6), and since the size of the oscillations $\left|s_n - r\right|$ is always less than $\left|a_n\right|$ because of the way $\Sigma\, b_n$ was constructed, we have that $\Sigma\, b_n = \lim_{n\to\infty} s_n = r$.

1. $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2-1}{n^2+1} = \lim_{n\to\infty} \frac{1-1/n^2}{1+1/n} = 1 \neq 0$, so the series $\Sigma_{n=1}^{\infty} \frac{n^2-1}{n^2+1}$ diverges by the Test for Divergence.

2. If $a_n = \frac{n-1}{n^2+n}$ and $b_n = \frac{1}{n}$, then $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^2-n}{n^2+n} = \lim_{n\to\infty} \frac{1-1/n}{1+1/n} = 1$, so the series $\Sigma_{n=1}^{\infty} \frac{n-1}{n^2+n}$ diverges by the Limit Comparison Test with the harmonic series.

3. $\frac{1}{n^2+n} < \frac{1}{n^2}$ for all $n \geq 1$, so $\Sigma_{n=1}^{\infty} \frac{1}{n^2+n}$ converges by the Comparison Test with $\Sigma_{n=1}^{\infty} \frac{1}{n^2}$, a p – series that converges because $p=2>1$.

4. Let $b_n = \frac{n-1}{n^2+n}$. Then $b_1=0$, and $b_2=b_3=\frac{1}{6}$, but $b_n > b_{n+1}$ for $n \geq 3$ since

$$\left(\frac{x-1}{x^2+x} \right)' = \frac{\left(x^2+x\right)-(x-1)(2x+1)}{\left(x^2+x\right)^2} = \frac{-x^2+2x+1}{\left(x^2+x\right)^2} = \frac{2-(x-1)^2}{(x^2+x)^2} < 0 \text{ for } x \geq 3 \text{ . Thus,}$$

$\left\{ b_n \mid n \geq 3 \right\}$ is decreasing and $\lim_{n\to\infty} b_n = 0$, so $\Sigma_{n=3}^{\infty} (-1)^{n-1} \frac{n-1}{n^2+n}$ converges by the Alternating Series Test. Hence, the full series $\Sigma_{n=1}^{\infty} (-1)^{n-1} \frac{n-1}{n^2+n}$ also converges.

5. $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(-3)^{n+2}}{2^{3(n+1)}} \cdot \frac{2^{3n}}{(-3)^{n+1}} \right| = \lim_{n\to\infty} \left| \frac{-3 \cdot 2^{3n}}{2^{3n} \cdot 2^3} \right| = \lim_{n\to\infty} \frac{3}{2^3} = \frac{3}{8} < 1$, so the series $\Sigma_{n=1}^{\infty} \frac{(-3)^{n+1}}{2^{3n}}$ is absolutely convergent by the Ratio Test.

6. $\lim_{n\to\infty} \sqrt[n]{\left| a_n \right|} = \lim_{n\to\infty} \frac{3n}{1+8n} = \lim_{n\to\infty} \frac{3}{1/n+8} = \frac{3}{8} < 1$, so $\Sigma_{n=1}^{\infty} \left(\frac{3n}{1+8n} \right)^n$ converges by the Root Test.

7. Let $f(x) = \frac{1}{x\sqrt{\ln x}}$. Then f is positive, continuous, and decreasing on [2,∞)] , so we can apply the Integral Test. Since $\int \frac{1}{x\sqrt{\ln x}}\, dx \left[\begin{array}{c} \text{u=ln x,} \\ \text{du=dx/x} \end{array} \right] = \int u^{-1/2} du = 2u^{1/2} + C = 2\sqrt{\ln x} + C$, we find

$\int_2^{\infty} \frac{dx}{x\sqrt{\ln x}} = \lim_{t\to\infty} \int_2^{t} \frac{dx}{x\sqrt{\ln x}} = \lim_{t\to\infty} \left[2\sqrt{\ln x}\,\right]_2^t = \lim_{t\to\infty} \left(2\sqrt{\ln t} - 2\sqrt{\ln 2}\,\right) = \infty$. Since the integral diverges, the given series $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$ diverges.

8. $\sum_{k=1}^{\infty} \frac{2^k k!}{(k+2)!} = \sum_{k=1}^{\infty} \frac{2^k}{(k+1)(k+2)}$. Using the Ratio Test, we get $\lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k\to\infty} \left| \frac{2^{k+1}}{(k+2)(k+3)} \cdot \frac{(k+1)(k+2)}{2^k} \right| = \lim_{k\to\infty} \left(2\cdot \frac{k+1}{k+3} \right) = 2 > 1$, so the series diverges.
Or: Use the Test for Divergence.

9. $\sum_{k=1}^{\infty} k^2 e^{-k} = \sum_{k=1}^{\infty} \frac{k^2}{e^k}$. Using the Ratio Test, we get $\lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k\to\infty} \left| \frac{(k+1)^2}{e^{k+1}} \cdot \frac{e^k}{k^2} \right| = \lim_{k\to\infty} \left[\left(\frac{k+1}{k} \right)^2 \cdot \frac{1}{e} \right] = 1^2 \cdot \frac{1}{e} = \frac{1}{e} < 1$, so the series converges.

10. Let $f(x) = x^2 e^{-x^3}$. Then f is continuous and positive on $[1,\infty)$, and $f'(x) = \frac{x(2-3x^3)}{e^{x^3}} < 0$ for $x \ge 1$, so f is decreasing on $[1,\infty)$ as well, and we can apply the Integral Test.
$\int_1^{\infty} x^2 e^{-x^3}\,dx = \lim_{t\to\infty} \left[-\frac{1}{3} e^{-x^3} \right]_1^t = \frac{1}{3e}$, so the integral converges, and hence, the series converges.

11. $b_n = \frac{1}{n\ln n} > 0$ for $n \ge 2$, $\left\{ b_n \right\}$ is decreasing, and $\lim_{n\to\infty} b_n = 0$, so the given series $\sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n\ln n}$ converges by the Alternating Series Test.

12. Let $b_n = \frac{n}{n^2+25}$. Then $b_n > 0$, $\lim_{n\to\infty} b_n = 0$, and $b_n - b_{n+1} = \frac{n}{n^2+25} - \frac{n+1}{n^2+2n+26} = \frac{n^2+n-25}{(n^2+25)(n^2+2n+26)}$, which is positive for $n \ge 5$, so the sequence $\left\{ b_n \right\}$ decreases from $n=5$ on. Hence, the given series $\sum_{n=1}^{\infty} (-1)^n \frac{n}{n^2+25}$ converges by the Alternating Series Test.

13. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{3^{n+1}(n+1)^2}{(n+1)!}\cdot\frac{n!}{3^n n^2}\right|=\lim_{n\to\infty}\left[\frac{3(n+1)^2}{(n+1)n^2}\right]=3\lim_{n\to\infty}\frac{n+1}{n^2}=0<1$, so the series $\Sigma_{n=1}^{\infty}\frac{3^n n^2}{n!}$ converges by the Ratio Test.

14. The series $\Sigma_{n=1}^{\infty}\sin n$ diverges by the Test for Divergence since $\lim_{n\to\infty}\sin n$ does not exist.

15.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{(n+1)!}{2\cdot 5\cdot 8\cdot\cdots\cdot(3n+2)[3(n+1)+2]}\cdot\frac{2\cdot 5\cdot 8\cdot\cdots\cdot(3n+2)}{n!}\right|$$

$$=\lim_{n\to\infty}\frac{n+1}{3n+5}=\frac{1}{3}<1$$

so the series $\Sigma_{n=0}^{\infty}\frac{n!}{2\cdot 5\cdot 8\cdot\cdots\cdot(3n+2)}$ converges by the Ratio Test.

16. Using the Limit Comparison Test with $a_n=\frac{n^2+1}{n^3+1}$ and $b_n=\frac{1}{n}$, we have

$\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\left(\frac{n^2+1}{n^3+1}\cdot\frac{n}{1}\right)=\lim_{n\to\infty}\frac{n^3+n}{n^3+1}=\lim_{n\to\infty}\frac{1+1/n^2}{1+1/n^3}=1>0$. Since $\Sigma_{n=1}^{\infty}b_n$ is the divergent harmonic series, $\Sigma_{n=1}^{\infty}a_n$ is also divergent.

17. $\lim_{n\to\infty}2^{1/n}=2^0=1$, so $\lim_{n\to\infty}(-1)^n 2^{1/n}$ does not exist and the series $\Sigma_{n=1}^{\infty}(-1)^n 2^{1/n}$ diverges by the Test for Divergence.

18. $b_n=\frac{1}{\sqrt{n}-1}$ for $n\geq 2$. $\{b_n\}$ is a decreasing sequence of positive numbers and $\lim_{n\to\infty}b_n=0$, so $\Sigma_{n=2}^{\infty}\frac{(-1)^{n-1}}{\sqrt{n}-1}$ converges by the Alternating Series Test.

19. Let $f(x)=\frac{\ln x}{\sqrt{x}}$. Then $f'(x)=\frac{2-\ln x}{2x^{3/2}}<0$ when $\ln x>2$ or $x>e^2$, so $\frac{\ln n}{\sqrt{n}}$ is decreasing for $n>e^2$.

By l'Hospital's Rule, $\lim_{n\to\infty}\frac{\ln n}{\sqrt{n}}=\lim_{n\to\infty}\frac{1/n}{1/\left(2\sqrt{n}\right)}=\lim_{n\to\infty}\frac{2}{\sqrt{n}}=0$, so the series $\sum_{n=1}^{\infty}(-1)^n\frac{\ln n}{\sqrt{n}}$ converges by the Alternating Series Test.

20. $\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right|=\lim_{k\to\infty}\left|\frac{k+6}{5^{k+1}}\cdot\frac{5^k}{k+5}\right|=\frac{1}{5}\lim_{k\to\infty}\frac{k+6}{k+5}=\frac{1}{5}<1$, so the series $\sum_{k=1}^{\infty}\frac{k+5}{5^k}$ converges by the Ratio Test.

21. $\sum_{n=1}^{\infty}\frac{(-2)^{2n}}{n^n}=\sum_{n=1}^{\infty}\left(\frac{4}{n}\right)^n$. $\lim_{n\to\infty}\sqrt[n]{\left|a_n\right|}=\lim_{n\to\infty}\frac{4}{n}=0<1$, so the given series is absolutely convergent by the Root Test.

22. $\frac{\sqrt{n^2-1}}{n^3+2n^2+5}<\frac{n}{n^3+2n^2+5}<\frac{n}{n^3}=\frac{1}{n^2}$ for $n\geq 1$, so $\sum_{n=1}^{\infty}\frac{\sqrt{n^2-1}}{n^3+2n^2+5}$ converges by the Comparison Test with the convergent p –series $\sum_{n=1}^{\infty}1/n^2\,(p=2>1)$.

23. Using the Limit Comparison Test with $a_n=\tan\left(\frac{1}{n}\right)$ and $b_n=\frac{1}{n}$, we have

$\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\frac{\tan(1/n)}{1/n}=\lim_{x\to\infty}\frac{\tan(1/x)}{1/x}=\lim_{x\to\infty}\frac{\sec^2(1/x)\cdot(-1/x^2)}{-1/x^2}=\lim_{x\to\infty}\sec^2(1/x)=1^2=1>0$. Since $\sum_{n=1}^{\infty}b_n$ is the divergent harmonic series, $\sum_{n=1}^{\infty}a_n$ is also divergent.

24. $\frac{|\cos(n/2)|}{n^2+4n}<\frac{1}{n^2+4n}<\frac{1}{n^2}$ and since $\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges ($p=2>1$), $\sum_{n=1}^{\infty}\frac{\cos(n/2)}{n^2+4n}$ converges absolutely by the Comparison Test.

25. Use the Ratio Test.

$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{(n+1)!}{e^{(n+1)^2}}\cdot\frac{e^{n^2}}{n!}\right|=\lim_{n\to\infty}\frac{(n+1)n!\cdot e^{n^2}}{e^{n^2+2n+1}n!}=\lim_{n\to\infty}\frac{n+1}{e^{2n+1}}=0<1$, so $\sum_{n=1}^{\infty}\frac{n!}{e^{n^2}}$ converges.

26.

$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\left(\frac{n^2+2n+2}{5^{n+1}}\cdot\frac{5^n}{n^2+1}\right)=\lim_{n\to\infty}\left(\frac{1+2/n+2/n^2}{1+1/n^2}\cdot\frac{1}{5}\right)=\frac{1}{5}<1$, so $\sum_{n=1}^{\infty}\frac{n^2+1}{5^n}$ converges by the Ratio Test.

27. $\int_2^{\infty}\frac{\ln x}{x^2}\,dx=\lim_{t\to\infty}\left[-\frac{\ln x}{x}-\frac{1}{x}\right]_1^t$ (using integration by parts) $=1$. So $\sum_{n=1}^{\infty}\frac{\ln n}{n^2}$ converges by the Integral Test, and since $\frac{k\ln k}{(k+1)^3}<\frac{k\ln k}{k^3}=\frac{\ln k}{k^2}$, the given series $\sum_{k=1}^{\infty}\frac{k\ln k}{(k+1)^3}$ converges by the Comparison Test.

28. Since $\left\{\frac{1}{n}\right\}$ is a decreasing sequence, $e^{1/n}\le e^{1/1}=e$ for all $n\ge 1$, and $\sum_{n=1}^{\infty}\frac{e}{n^2}$ converges ($p=2>1$), so $\sum_{n=1}^{\infty}\frac{e^{1/n}}{n^2}$ converges by the Comparison Test. (Or use the Integral Test.)

29. $0<\frac{\tan^{-1}n}{n^{3/2}}<\frac{\pi/2}{n^{3/2}}$. $\sum_{n=1}^{\infty}\frac{\pi/2}{n^{3/2}}=\frac{\pi}{2}\sum_{n=1}^{\infty}\frac{1}{n^{3/2}}$ which is a convergent p– series ($p=\frac{3}{2}>1$), so $\sum_{n=1}^{\infty}\frac{\tan^{-1}n}{n^{3/2}}$ converges by the Comparison Test.

30. Let $f(x)=\frac{\sqrt{x}}{x+5}$. Then $f(x)$ is continuous and positive on $[1,\infty)$, and since $f'(x)=\frac{5-x}{2\sqrt{x}\,(x+5)^2}<0$ for $x>5$, $f(x)$ is eventually decreasing, so we can use the Alternating Series Test.
$\lim_{n\to\infty}\frac{\sqrt{n}}{n+5}=\lim_{n\to\infty}\frac{1}{n^{1/2}+5n^{-1/2}}=0$, so the series $\sum_{j=1}^{\infty}(-1)^j\frac{\sqrt{j}}{j+5}$ converges.

31. $\lim_{k\to\infty}a_k=\lim_{k\to\infty}\frac{5^k}{3^k+4^k}=\left[\text{ divide by } 4^k\right]\lim_{k\to\infty}\frac{(5/4)^k}{(3/4)^k+1}=\infty$ since $\lim_{k\to\infty}\left(\frac{3}{4}\right)^k=0$ and $\lim_{k\to\infty}\left(\frac{5}{4}\right)^k=\infty$. Thus, $\sum_{k=1}^{\infty}\frac{5^k}{3^k+4^k}$ diverges by the Test for Divergence.

32. $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \frac{2n}{n^2} = \lim_{n\to\infty} \frac{2}{n} = 0$, so the series $\sum_{n=1}^{\infty} \frac{(2n)^n}{n^{2n}}$ converges by the Root Test.

33. Let $a_n = \frac{\sin(1/n)}{\sqrt{n}}$ and $b_n = \frac{1}{n\sqrt{n}}$. Then $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sin(1/n)}{1/n} = 1 > 0$, so $\sum_{n=1}^{\infty} \frac{\sin(1/n)}{\sqrt{n}}$ converges by limit comparison with the convergent p –series $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ $(p=3/2>1)$.

34. $0 \le n\cos^2 n \le n$, so $\frac{1}{n+n\cos^2 n} \ge \frac{1}{n+n} = \frac{1}{2n}$. Thus, $\sum_{n=1}^{\infty} \frac{1}{n+n\cos^2 n}$ diverges by comparison with $\sum_{n=1}^{\infty} \frac{1}{2n}$, which is a constant multiple of the (divergent) harmonic series.

35. $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^{n^2/n} = \lim_{n\to\infty} \frac{1}{[(n+1)/n]^n} = \frac{1}{\lim_{n\to\infty}(1+1/n)^n} = \frac{1}{e} < 1$, so the series $\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2}$ converges by the Root Test.

36. Note that $(\ln n)^{\ln n} = \left(e^{\ln \ln n}\right)^{\ln n} = \left(e^{\ln n}\right)^{\ln \ln n} = n^{\ln \ln n}$ and $\ln \ln n \to \infty$ as $n \to \infty$, so $\ln \ln n > 2$ for sufficiently large n . For these n we have $(\ln n)^{\ln n} > n^2$, so $\frac{1}{(\ln n)^{\ln n}} < \frac{1}{n^2}$. Since $\sum_{n=2}^{\infty} \frac{1}{n^2}$ converges ($p=2>1$), so does $\sum_{n=2}^{\infty} \frac{1}{(\ln n)^{\ln n}}$ by the Comparison Test.

37. $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \left(2^{1/n} - 1\right) = 1-1 = 0 < 1$, so the series $\sum_{n=1}^{\infty} \left(\sqrt[n]{2} - 1\right)^n$ converges by the Root Test.

38. Use the Limit Comparison Test with $a_n = \sqrt[n]{2} - 1$ and $b_n = 1/n$. Then

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2^{1/n}-1}{1/n} = \lim_{x\to\infty} \frac{2^{1/x}-1}{1/x} = \lim_{x\to\infty} \frac{2^{1/x} \cdot \ln 2 \cdot (-1/x^2)}{-1/x^2} = \lim_{x\to\infty} (2^{1/x} \cdot \ln 2) = 1 \cdot \ln 2 = \ln 2 > 0$. So since $\sum_{n=1}^{\infty} b_n$ diverges (harmonic series), so does $\sum_{n=1}^{\infty} \left(\sqrt[n]{2} - 1\right)$.

Alternate Solution:

$\sqrt[n]{2}-1=\dfrac{1}{2^{(n-1)/n}+2^{(n-2)/n}+2^{(n-3)/n}+\cdot\cdot\cdot+2^{1/n}+1}\geq\dfrac{1}{2n}$,

and since $\sum_{n=1}^{\infty}\dfrac{1}{2n}=\dfrac{1}{2}\sum_{n=1}^{\infty}\dfrac{1}{n}$ diverges (harmonic series), so does $\sum_{n=1}^{\infty}\left(\sqrt[n]{2}-1\right)$ by the Comparison Test.

1. A power series is a series of the form $\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$, where x is a variable and the c_n 's are constants called the coefficients of the series.

More generally, a series of the form $\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$ is called a power series in $(x-a)$ or a power series centered at a or a power series about a , where a is a constant.

2. **(a)** Given the power series $\sum_{n=0}^{\infty} c_n (x-a)^n$, the radius of convergence is:

(a) 0 if the series converges only when $x=a$

(b) ∞ if the series converges for all x , or

(c) a positive number R such that the series converges if $|x-a|<R$ and diverges if $|x-a|>R$.

In most cases, R can be found by using the Ratio Test.

(b) The interval of convergence of a power series is the interval that consists of all values of x for which the series converges. Corresponding to the cases in part (a), the interval of convergence is: (i) the single point $\{a\}$, (ii) all real numbers; that is, the real number line $(-\infty,\infty)$, or (iii) an interval with endpoints $a-R$ and $a+R$ which can contain neither, either, or both of the endpoints. In this case, we must test the series for convergence at each endpoint to determine the interval of convergence.

3. If $a_n = \frac{x^n}{\sqrt{n}}$, then

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{x^{n+1}}{\sqrt{n+1}} \cdot \frac{\sqrt{n}}{x} \right| = \lim_{n\to\infty} \left| \frac{x}{\sqrt{n+1}/\sqrt{n}} \right| = \lim_{n\to\infty} \frac{|x|}{\sqrt{1+1/n}} = |x| .$$

By the Ratio Test, the series $\sum_{n=1}^{\infty} \frac{x^n}{\sqrt{n}}$ converges when $|x|<1$, so the radius of convergence $R=1$.

Now we'll check the endpoints, that is, $x=\pm 1$. When $x=1$, the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges because it is a p –series with $p=\frac{1}{2} \le 1$. When $x=-1$, the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ converges by the Alternating Series Test. Thus, the interval of convergence is $I=[-1,1)$.

4. If $a_n = \frac{(-1)^n x^n}{n+1}$, then $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{x^{n+1}}{n+2} \cdot \frac{n+1}{x^n} \right| = \lim_{n\to\infty} \frac{|x|}{1+1/(n+1)} = |x|$. By the Ratio Test, the series $\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n+1}$ converges when $|x|<1$, so $R=1$. When $x=-1$, the series diverges because it is the harmonic series; when $x=1$, it is the alternating harmonic series, which converges by

the Alternating Series Test. Thus, $I=(-1,1]$.

5. If $a_n=\dfrac{(-1)^{n-1}x^n}{n^3}$, then

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{(-1)^n x^{n+1}}{(n+1)^3}\cdot\frac{n^3}{(-1)^{n-1}x^n}\right|=\lim_{n\to\infty}\left|\frac{(-1)xn^3}{(n+1)^3}\right|=\lim_{n\to\infty}\left[\left(\frac{n}{n+1}\right)^3|x|\right]=1^3\cdot|x|=|x|$$

. By the Ratio Test, the series $\sum_{n=1}^{\infty}\dfrac{(-1)^{n-1}x^n}{n^3}$ converges when $|x|<1$, so the radius of convergence $R=1$. Now we'll check the endpoints, that is, $x=\pm 1$. When $x=1$, the series $\sum_{n=1}^{\infty}\dfrac{(-1)^{n-1}}{n^3}$ converges by the Alternating Series Test. When $x=-1$, the series $\sum_{n=1}^{\infty}\dfrac{(-1)^{n-1}(-1)^n}{n^3}=-\sum_{n=1}^{\infty}\dfrac{1}{n^3}$ converges because it is a constant multiple of a convergent p –series $(p=3>1)$. Thus, the interval of convergence is $I=[-1,1]$.

6. $a_n=\sqrt{n}\,x^n$, so we need $\lim_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\dfrac{\sqrt{n+1}\,|x|^{n+1}}{\sqrt{n}\,|x|^n}=\lim_{n\to\infty}\sqrt{1+\dfrac{1}{n}}\,|x|=|x|<1$ for convergence (by the Ratio Test), so $R=1$. When $x=\pm 1$, $\lim_{n\to\infty}|a_n|=\lim_{n\to\infty}\sqrt{n}=\infty$, so the series diverges by the Test for Divergence. Thus, $I=(-1,1)$.

7. If $a_n=\dfrac{x^n}{n!}$, then

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)!}\cdot\frac{n!}{x^n}\right|=\lim_{n\to\infty}\left|\frac{x}{n+1}\right|=|x|\lim_{n\to\infty}\frac{1}{n+1}=|x|\cdot 0=0<1 \text{ for } all \text{ real } x .$$

So, by the Ratio Test, $R=\infty$, and $I=(-\infty,\infty)$.

8. Here the Root Test is easier. If $a_n=n^n x^n$ then $\lim_{n\to\infty}\sqrt[n]{|a_n|}=\lim_{n\to\infty} n|x|=\infty$ if $x\neq 0$, so $R=0$ and $I=\{0\}$.

9. $\lim_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\dfrac{(n+1)4^{n+1}|x|^{n+1}}{n4^n|x|^n}=\lim_{n\to\infty}\left(1+\dfrac{1}{n}\right)4|x|=4|x|$. Now $4|x|<1\Leftrightarrow|x|<\dfrac{1}{4}$, so

by the Ratio Test, $R=\frac{1}{4}$. When $x=\frac{1}{4}$, we get the divergent series $\sum_{n=1}^{\infty}(-1)^n n$, and when $x=-\frac{1}{4}$, we get the divergent series $\sum_{n=1}^{\infty} n$. Thus, $I=\left(-\frac{1}{4},\frac{1}{4}\right)$.

10. If $a_n=\frac{x^n}{n3^n}$, then

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)3^{n+1}}\cdot\frac{n3^n}{x^n}\right|=\lim_{n\to\infty}\left|\frac{xn}{(n+1)3}\right|=\frac{|x|}{3}\lim_{n\to\infty}\frac{n}{n+1}=\frac{|x|}{3}$$

. By the Ratio Test, the series converges when $\frac{|x|}{3}<1\Leftrightarrow|x|<3$, so $R=3$. When $x=-3$, the series is the alternating harmonic series, which converges by the Alternating Series Test. When $x=3$, it is the harmonic series, which diverges. Thus, $I=[-3,3)$.

11. $a_n=\frac{(-2)^n x^n}{\sqrt[4]{n}}$, so $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\frac{2^{n+1}|x|^{n+1}}{\sqrt[4]{n+1}}\cdot\frac{\sqrt[4]{n}}{2^n|x|^n}=\lim_{n\to\infty}2|x|\sqrt[4]{\frac{n}{n+1}}=2|x|$, so by the Ratio Test, the series converges when $2|x|<1\Leftrightarrow|x|<\frac{1}{2}$, so $R=\frac{1}{2}$. When $x=-\frac{1}{2}$, we get the divergent

p –series $\sum_{n=1}^{\infty}\frac{1}{\sqrt[4]{n}}\left(p=\frac{1}{4}\leq 1\right)$. When $x=\frac{1}{2}$, we get the series $\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt[4]{n}}$, which converges by the Alternating Series Test. Thus, $I=\left(-\frac{1}{2},\frac{1}{2}\right]$.

12. $a_n=\frac{x^n}{5^n n^5}$, so $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{x^{n+1}}{5^{n+1}(n+1)^5}\cdot\frac{5^n n^5}{x^n}\right|=\lim_{n\to\infty}\frac{|x|}{5}\left(\frac{n}{n+1}\right)^5=\frac{|x|}{5}$. By the Ratio Test, the series converges when $|x|/5<1\Leftrightarrow|x|<5$, so $R=5$. When $x=-5$, we get the series $\sum_{n=1}^{\infty}\frac{(-1)^n}{n^5}$, which converges by the Alternating Series Test. When $x=5$, we get the convergent p –series $\sum_{n=1}^{\infty}\frac{1}{n^5}\ (p=5>1)$. Thus, $I=[-5,5]$.

13. If $a_n=(-1)^n\frac{x^n}{4^n\ln n}$, then

$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{4^{n+1}\ln(n+1)}\cdot\frac{4^n\ln n}{x^n}\right| = \frac{|x|}{4}\lim_{n\to\infty}\frac{\ln n}{\ln(n+1)} = \frac{|x|}{4}\cdot 1$ (by l'Hospital's Rule) $= \frac{|x|}{4}$. By the Ratio Test, the series converges when $\frac{|x|}{4}<1 \Leftrightarrow |x|<4$, so $R=4$. When $x=-4$, $\sum_{n=2}^{\infty}(-1)^n\frac{x^n}{4^n\ln n} = \sum_{n=2}^{\infty}\frac{[(-1)(-4)]^n}{4^n\ln n} = \sum_{n=2}^{\infty}\frac{1}{\ln n}$. Since $\ln n<n$ for $n\geq 2$, $\frac{1}{\ln n}>\frac{1}{n}$ and $\sum_{n=2}^{\infty}\frac{1}{n}$ is the divergent harmonic series (without the $n=1$ term), $\sum_{n=2}^{\infty}\frac{1}{\ln n}$ is divergent by the Comparison Test. When $x=4$, $\sum_{n=2}^{\infty}(-1)^n\frac{x^n}{4^n\ln n} = \sum_{n=2}^{\infty}(-1)^n\frac{1}{\ln n}$, which converges by the Alternating Series Test. Thus, $I=(-4,4]$.

14. $a_n=(-1)^n\frac{x^{2n}}{(2n)!}$, so $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{|x|^{2n+2}}{(2n+2)!}\cdot\frac{(2n)!}{|x|^{2n}} = \lim_{n\to\infty}\frac{|x|^2}{(2n+1)(2n+2)} = 0$. Thus, by the Ratio Test, the series converges for *all* real x and we have $R=\infty$ and $I=(-\infty,\infty)$.

15. If $a_n=\sqrt{n}\,(x-1)^n$, then $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\sqrt{n+1}\,|x-1|^{n+1}}{\sqrt{n}\,|x-1|^n}\right| = \lim_{n\to\infty}\sqrt{1+\frac{1}{n}}\,|x-1| = |x-1|$. By the Ratio Test, the series converges when $|x-1|<1 \Leftrightarrow -1<x-1<1 \Leftrightarrow 0<x<2$. When $x=0$, the series becomes $\sum_{n=0}^{\infty}(-1)^n\sqrt{n}$, which diverges by the Test for Divergence. When $x=2$, the series becomes $\sum_{n=0}^{\infty}\sqrt{n}$, which also diverges by the Test for Divergence. Thus, $I=(0,2)$.

16. If $a_n=n^3(x-5)^n$, $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^3(x-5)^{n+1}}{n^3(x-5)^n}\right| = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^3|x-5| = |x-5|$. By the Ratio Test, the series converges when $|x-5|<1 \Leftrightarrow -1<x-5<1 \Leftrightarrow 4<x<6$. When $x=4$, the series becomes $\sum_{n=0}^{\infty}(-1)^n n^3$, which diverges by the Test for Divergence. When $x=6$, the series becomes $\sum_{n=0}^{\infty}n^3$, which also diverges by the Test for Divergence. Thus, $R=1$ and $I=(4,6)$.

17. If $a_n=(-1)^n\frac{(x+2)^n}{n2^n}$, then

$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left[\frac{|x+2|^{n+1}}{(n+1)2^{n+1}}\cdot\frac{n2^n}{|x+2|^n}\right] = \lim_{n\to\infty}\frac{n}{n+1}\cdot\frac{|x+2|}{2} = \frac{|x+2|}{2}$. By the Ratio Test, the series converges when $\frac{|x+2|}{2}<1 \Leftrightarrow |x+2|<2 \Leftrightarrow -2<x+2<2 \Leftrightarrow -4<x<0$. When $x=-4$, the series becomes $\sum_{n=1}^{\infty}(-1)^n\frac{(-2)^n}{n2^n} = \sum_{n=1}^{\infty}\frac{2^n}{n2^n} = \sum_{n=1}^{\infty}\frac{1}{n}$, which is the divergent harmonic series. When $x=0$, the series is $\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$, the alternating harmonic series, which converges by the Alternating Series Test. Thus, $I=(-4,0]$.

18. If $a_n = \frac{(-2)^n}{\sqrt{n}}(x+3)^n$, then

$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-2)^{n+1}(x+3)^{n+1}}{\sqrt{n+1}}\cdot\frac{\sqrt{n}}{(-2)^n(x+3)^n}\right| = \lim_{n\to\infty}\frac{2|x+3|}{\sqrt{1+1/n}} = 2|x+3|<1 \Leftrightarrow |x+3|<\frac{1}{2}$ $\left[\text{so} R=\frac{1}{2}\right] \Leftrightarrow -\frac{7}{2}<x<-\frac{5}{2}$. When $x=-\frac{7}{2}$, the series becomes $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$, which diverges because it is a p –series with $p=\frac{1}{2}\le 1$. When $x=-\frac{5}{2}$, the series becomes $\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}$, which converges by the Alternating Series Test. Thus, $I=\left(-\frac{7}{2},-\frac{5}{2}\right]$.

19. If $a_n = \frac{(x-2)^n}{n^n}$, then $\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\frac{|x-2|}{n} = 0$, so the series converges for all x (by the Root Test). $R=\infty$ and $I=(-\infty,\infty)$.

20.

$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(3x-2)^{n+1}}{(n+1)3^{n+1}}\cdot\frac{n3^n}{(3x-2)^n}\right| = \lim_{n\to\infty}\left(\frac{|3x-2|}{3}\cdot\frac{1}{1+1/n}\right) = \frac{|3x-2|}{3} = \left|x-\frac{2}{3}\right|$, so by the Ratio Test, the series converges when $\left|x-\frac{2}{3}\right|<1 \Leftrightarrow -\frac{1}{3}<x<\frac{5}{3}$. $R=1$. When $x=-\frac{1}{3}$, the series is $\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$, the convergent alternating harmonic series. When $x=\frac{5}{3}$, the series becomes the divergent harmonic series. Thus, $I=\left[-\frac{1}{3},\frac{5}{3}\right)$.

21. $a_n = \frac{n}{b^n}(x-a)^n$, where $b>0$.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{(n+1)|x-a|^{n+1}}{b^{n+1}}\cdot\frac{b^n}{n|x-a|^n} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)\frac{|x-a|}{b} = \frac{|x-a|}{b} .$$

By the Ratio Test, the series converges when $\frac{|x-a|}{b}<1 \Leftrightarrow |x-a|<b$ [so $R=b$] $\Leftrightarrow$ $-b<x-a<b \Leftrightarrow a-b<x<a+b$. When $|x-a|=b$, $\lim_{n\to\infty}\left|a_n\right| = \lim_{n\to\infty} n=\infty$, so the series diverges. Thus, $I=(a-\mathrm{b},\mathrm{a}+b)$.

22. $a_n = \frac{n(x-4)^n}{n^3+1}$, so

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{(n+1)|x-4|^{n+1}}{(n+1)^3+1}\cdot\frac{n^3+1}{n|x-4|^n} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)\frac{n^3+1}{n^3+3n^2+3n+2}|x-4| = |x-4| .$$

By the Ratio Test, the series converges when $|x-4|<1 \Leftrightarrow -1<x-4<1 \Leftrightarrow 3<x<5$.

When $|x-4|=1$, $\sum_{n=1}^{\infty}\left|a_n\right| = \sum_{n=1}^{\infty}\frac{n}{n^3+1}$, which converges by comparison with the convergent p – series $\sum_{n=1}^{\infty}\frac{1}{n^2}$ $(p=2>1)$. Thus, $I=[3,5]$.

23. If $a_n = \mathrm{n}!(2x-1)^n$, then $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!(2x-1)^{n+1}}{\mathrm{n}!(2x-1)^n}\right| = \lim_{n\to\infty}(n+1)|2x-1| \to \infty$ as $n\to\infty$ for all $x\neq\frac{1}{2}$. Since the series diverges for all $x\neq\frac{1}{2}$, $R=0$ and $I=\left\{\frac{1}{2}\right\}$.

24. $a_n = \frac{n^2x^n}{2\cdot 4\cdot 6\cdot \cdots \cdot(2n)} = \frac{n^2x^n}{2^n\mathrm{n}!} = \frac{nx^n}{2^n(n-1)!}$, so

$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{(n+1)|x|^{n+1}}{2^{n+1}\mathrm{n}!}\cdot\frac{2^n(n-1)!}{n|x|^n} = \lim_{n\to\infty}\frac{n+1}{n^2}\frac{|x|}{2} = 0$. Thus, by the Ratio Test, the series converges for *all* real x and we have $R=\infty$ and $I=(-\infty,\infty)$.

25. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left[\frac{|4x+1|^{n+1}}{(n+1)^2}\cdot\frac{n^2}{|4x+1|^n}\right] = \lim_{n\to\infty}\frac{|4x+1|}{(1+1/n)^2} = |4x+1|$, so by the Ratio

Test, the series converges when $|4x+1|<1 \Leftrightarrow -1<4x+1<1 \Leftrightarrow -2<4x<0 \Leftrightarrow -\frac{1}{2}<x<0$, so $R=\frac{1}{4}$. When $x=-\frac{1}{2}$, the series becomes $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$, which converges by the Alternating Series Test. When $x=0$, the series becomes $\sum_{n=1}^{\infty} \frac{1}{n^2}$, a convergent p –series ($p=2>1$). $I=\left[-\frac{1}{2},0\right]$.

26. If $a_n=\frac{(-1)^n(2x+3)^n}{n\ln n}$, then we need $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=|2x+3|\lim_{n\to\infty}\frac{n\ln n}{(n+1)\ln(n+1)}=|2x+3|<1$ for convergence, so $-2<x<-1$ and $R=\frac{1}{2}$. When $x=-2$, $\sum_{n=2}^{\infty} a_n=\sum_{n=2}^{\infty}\frac{1}{n\ln n}$, which diverges (Integral Test), and when $x=-1$, $\sum_{n=2}^{\infty} a_n=\sum_{n=2}^{\infty}\frac{(-1)^n}{n\ln n}$, which converges (Alternating Series Test), so $I=(-2,-1]$.

27. If $a_n=\frac{x^n}{(\ln n)^n}$, then $\lim_{n\to\infty}\sqrt[n]{|a_n|}=\lim_{n\to\infty}\frac{|x|}{\ln n}=0<1$ for all x , so $R=\infty$ and $I=(-\infty,\infty)$ by the Root Test.

28. If $a_n=\frac{2\cdot 4\cdot 6\cdot\cdots\cdot(2n)x^n}{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}$, then we need $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}|x|\left(\frac{2n+2}{2n+1}\right)=|x|<1$ for convergence, so $R=1$. If $x=\pm 1$, $\left|a_n\right|=\frac{2\cdot 4\cdot 6\cdot\cdots\cdot(2n)}{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}>1$ for all n since each integer in the numerator is larger than the corresponding one in the denominator, so $\sum a_n$ diverges in both cases by the Test for Divergence, and $I=(-1,1)$.

29. **(a)** We are given that the power series $\sum_{n=0}^{\infty} c_n x^n$ is convergent for $x=4$. So by Theorem 3, it must converge for at least $-4<x\leq 4$. In particular, it converges when $x=-2$; that is, $\sum_{n=0}^{\infty} c_n(-2)^n$ is convergent.
(b) It does not follow that $\sum_{n=0}^{\infty} c_n(-4)^n$ is necessarily convergent.

30. We are given that the power series $\sum_{n=0}^{\infty} c_n x^n$ is convergent for $x=-4$ and divergent when $x=6$. So

by Theorem 3 it converges for at least $-4\leq x<4$ and diverges for at least $x\geq 6$ and $x<-6$. Therefore:

(a) It converges when $x=1$; that is, Σc_n is convergent.

(b) It diverges when $x=8$; that is, $\Sigma c_n 8^n$ is divergent.

(c) It converges when $x=-3$; that is, $\Sigma c_n(-3)^n$ is convergent.

(d) It diverges when $x=-9$; that is, $\Sigma c_n(-9)^n=\Sigma(-1)^n c_n 9^n$ is divergent.

31. If $a_n=\dfrac{(n!)^k}{(kn)!}x^n$, then

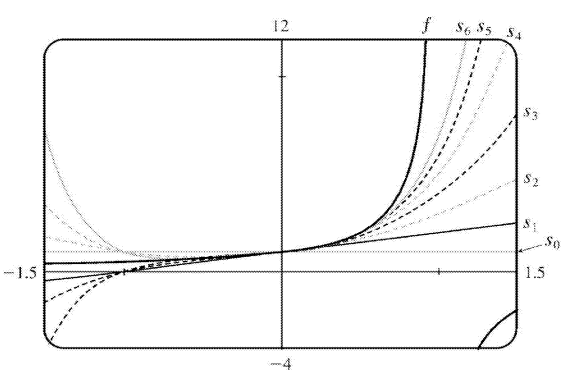
$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{[(n+1)!]^k(kn)!}{(n!)^k[k(n+1)]!}|x|=\lim_{n\to\infty}\frac{(n+1)^k}{(kn+k)(kn+k-1)\cdots(kn+2)(kn+1)}|x|$$

$$= \lim_{n\to\infty}\left[\frac{(n+1)}{(kn+1)}\frac{(n+1)}{(kn+2)}\cdots\frac{(n+1)}{(kn+k)}\right]|x|$$

$$= \lim_{n\to\infty}\left[\frac{n+1}{kn+1}\right]\lim_{n\to\infty}\left[\frac{n+1}{kn+2}\right]\cdots\lim_{n\to\infty}\left[\frac{n+1}{kn+k}\right]|x|=\left(\frac{1}{k}\right)^k|x|<1\Leftrightarrow$$

$|x|<k^k$ for convergence, and the radius of convergence is $R=k^k$.

32. The partial sums of the series $\Sigma_{n=0}^{\infty}x^n$ definitely do not converge to $f(x)=1/(1-x)$ for $x\geq 1$, since f is undefined at $x=1$ and negative on $(1,\infty)$, while all the partial sums are positive on this interval. The partial sums also fail to converge to f for $x\leq -1$, since $0<f(x)<1$ on this interval, while the partial sums are either larger than 1 or less than 0 . The partial sums seem to converge to f on $(-1,1)$. This graphical evidence is consistent with what we know about geometric series: convergence for $|x|<1$, divergence for $|x|\geq 1$ (see Example .2.5).

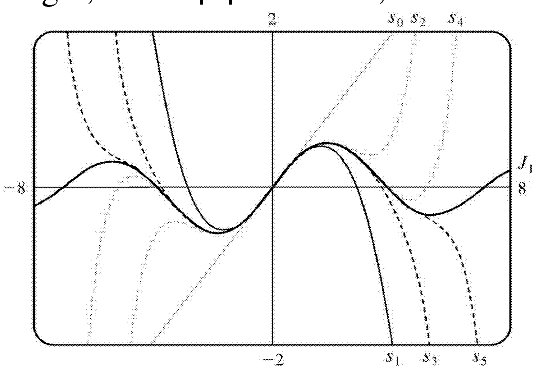


33. **(a)** If

$a_n = \dfrac{(-1)^n x^{2n+1}}{n!(n+1)!2^{2n+1}}$, then

$\lim_{n\to\infty} \left| \dfrac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \dfrac{x^{2n+3}}{(n+1)!(n+2)!2^{2n+3}} \cdot \dfrac{n!(n+1)!2^{2n+1}}{x^{2n+1}} \right| = \left(\dfrac{x}{2} \right)^2 \lim_{n\to\infty} \dfrac{1}{(n+1)(n+2)} = 0$ for all x . So $J_1(x)$ converges for all x and its domain is $(-\infty, \infty)$.

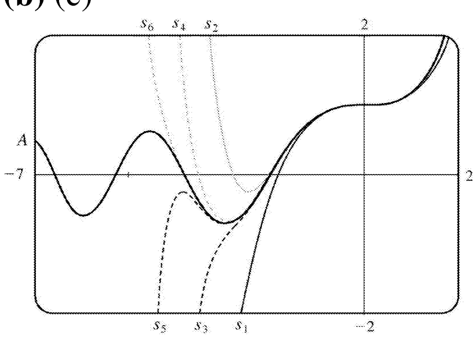
(b) p190pt (c) The initial terms of $J_1(x)$ up to n=5 are $a_0 = \dfrac{x}{2}$, $a_1 = -\dfrac{x^3}{16}$, $a_2 = \dfrac{x^5}{384}$, $a_3 = -\dfrac{x^7}{18{,}432}$, $a_4 = \dfrac{x^9}{1{,}474{,}560}$, and $a_5 = -\dfrac{x^{11}}{176{,}947{,}200}$. The partial sums seem to approximate $J_1(x)$ well near the origin, but as $|x|$ increases, we need to take a large number of terms to get a good approximation.



34. **(a)** $A(x) = 1 + \sum_{n=1}^{\infty} a_n$, where $a_n = \dfrac{x^{3n}}{2\cdot 3\cdot 5\cdot 6\cdot \cdots \cdot (3n-1)(3n)}$, so

$\lim_{n\to\infty} \left| \dfrac{a_{n+1}}{a_n} \right| = |x|^3 \lim_{n\to\infty} \dfrac{1}{(3n+2)(3n+3)} = 0$ for all x , so the domain is R .

(b) (c)



s_0=1 has been omitted from the graph. The partial sums seem to approximate $A(x)$ well near the origin, but as $|x|$ increases, we need to take a large number of terms to get a good approximation.

To plot A , we must first define $A(x)$ for the CAS. Note that for $n\geq 1$, the denominator of a_n is

$2\cdot 3\cdot 5\cdot 6\cdot \cdots \cdot (3n-1)\cdot 3n=\dfrac{(3n)!}{1\cdot 4\cdot 7\cdot \cdots \cdot (3n-2)}=\dfrac{(3n)!}{\prod_{k=1}^{n}(3k-2)}$, so $a_n=\dfrac{\prod_{k=1}^{n}(3k-2)}{(3n)!}x^{3n}$ and thus

$A(x)=1+\sum_{n=1}^{\infty}\dfrac{\prod_{k=1}^{n}(3k-2)}{(3n)!}x^{3n}$. Both Maple and Mathematica are able to plot A if we define it this way, and Derive is able to produce a similar graph using a suitable partial sum of $A(x)$. Derive, Maple and Mathematica all have two initially known Airy functions, called AI_SERIES(z,m) and BI_SERIES(z,m) from BESSEL.MTH in Derive and AiryAi and AiryBi in Maple and Mathematica (just Ai and Bi in older versions of Maple). However, it is very difficult to solve for A in terms of the CAS's Airy functions, although in fact

$A(x)=\dfrac{\sqrt{3}\ \mathrm{AiryAi}(x)+\mathrm{AiryBi}(x)}{\sqrt{3}\ \mathrm{AiryAi}(0)+\mathrm{AiryBi}(0)}$.

35.

$$\begin{aligned} s_{2n-1} &=1+2x+x^2+2x^3+x^4+2x^5+\cdots+x^{2n-2}+2x^{2n-1} \\ &=1(1+2x)+x^2(1+2x)+x^4(1+2x)+\cdots+x^{2n-2}(1+2x) \\ &=(1+2x)\left(1+x^2+x^4+\cdots+x^{2n-2}\right) \\ &=(1+2x)\frac{1-x^{2n}}{1-x^2}\ \text{ with } r=x^2\ ] \to \frac{1+2x}{1-x^2}\ \text{ as } n\to\infty\ , \end{aligned}$$

when $|x|<1$. Also $s_{2n}=s_{2n-1}+x^{2n}\to\dfrac{1+2x}{1-x^2}$ since $x^{2n}\to 0$ for $|x|<1$. Therefore, $s_n\to\dfrac{1+2x}{1-x^2}$ since s_{2n} and s_{2n-1} both approach $\dfrac{1+2x}{1-x^2}$ as $n\to\infty$. Thus, the interval of convergence is $(-1,1)$ and $f(x)=\dfrac{1+2x}{1-x^2}$.

36.

$$s_{4n-1}=c_0+c_1x+c_2x^2+c_3x^3+c_0x^4+c_1x^5+c_2x^6+c_3x^7+\cdots+c_3x^{4n-1}$$

$$=\left(c_0+c_1x+c_2x^2+c_3x^3\right)\left(1+x^4+x^8+\cdots+x^{4n-4}\right)\to\frac{c_0+c_1x+c_2x^2+c_3x^3}{1-x^4}\text{ as }n\to\infty$$

for $\left|x^4\right|<1\Leftrightarrow|x|<1$. Also s_{4n} , s_{4n+1} , s_{4n+2} have the same limits (for example, $s_{4n}=s_{4n-1}+c_0x^{4n}$ and $x^{4n}\to 0$ for $|x|<1$). So if at least one of c_0 , c_1 , c_2 , and c_3 is nonzero, then the interval of convergence is $(-1,1)$ and $f(x)=\dfrac{c_0+c_1x+c_2x^2+c_3x^3}{1-x^4}$.

37. We use the Root Test on the series Σc_nx^n . We need $\lim\limits_{n\to\infty}\sqrt[n]{\left|c_nx^n\right|}=|x|\lim\limits_{n\to\infty}\sqrt[n]{\left|c_n\right|}=c|x|<1$ for convergence, or $|x|<1/c$, so $R=1/c$.

38. Suppose $c_n\neq 0$. Applying the Ratio Test to the series $\Sigma c_n(x-a)^n$, we find that

$$L=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{c_{n+1}(x-a)^{n+1}}{c_n(x-a)^n}\right|=\lim_{n\to\infty}\frac{|x-a|}{\left|c_n/c_{n+1}\right|}\,(*)=\frac{|x-a|}{\lim\limits_{n\to\infty}\left|c_n/c_{n+1}\right|}$$

(if $\lim\limits_{n\to\infty}\left|c_n/c_{n+1}\right|\neq 0$), so the series converges when $\dfrac{|x-a|}{\lim\limits_{n\to\infty}\left|c_n/c_{n+1}\right|}<1\Leftrightarrow|x-a|<\lim\limits_{n\to\infty}\left|\dfrac{c_n}{c_{n+1}}\right|$.

Thus, $R=\lim\limits_{n\to\infty}\left|\dfrac{c_n}{c_{n+1}}\right|$. If $\lim\limits_{n\to\infty}\left|\dfrac{c_n}{c_{n+1}}\right|=0$ and $|x-a|\neq 0$, then $(*)$ shows that $L=\infty$ and so the series diverges, and hence, $R=0$. Thus, in all cases, $R=\lim\limits_{n\to\infty}\left|\dfrac{c_n}{c_{n+1}}\right|$.

39. For $2<x<3$, Σc_nx^n diverges and Σd_nx^n converges. By Exercise .2.61, $\Sigma\left(c_n+d_n\right)x^n$ diverges. Since both series converge for $|x|<2$, the radius of convergence of $\Sigma\left(c_n+d_n\right)x^n$ is 2 .

40. Since Σc_nx^n converges whenever $|x|<R$, $\Sigma c_nx^{2n}=\Sigma c_n\left(x^2\right)^n$ converges whenever $\left|x^2\right|<R\Leftrightarrow$ $|x|<\sqrt{R}$, so the second series has radius of convergence $\sqrt{R}$.

1. If $f(x)=\sum_{n=0}^{\infty} c_n x^n$ has radius of convergence 10 , then $f'(x)=\sum_{n=1}^{\infty} nc_n x^{n-1}$ also has radius of convergence 10 by Theorem 2.

2. If $f(x)=\sum_{n=0}^{\infty} b_n x^n$ converges on $(-2,2)$, then $\int f(x)dx=C+\sum_{n=0}^{\infty} \frac{b_n}{n+1} x^{n+1}$ has the same radius of convergence (by Theorem 2), but may not have the same interval of convergence — it may happen that the integrated series converges at an endpoint (or both endpoints).

3. Our goal is to write the function in the form $\frac{1}{1-r}$, and then use Equation (1) to represent the function as a sum of a power series. $f(x)=\frac{1}{1+x}=\frac{1}{1-(-x)}=\sum_{n=0}^{\infty}(-x)^n=\sum_{n=0}^{\infty}(-1)^n x^n$ with $|-x|<1\Leftrightarrow |x|<1$, so $R=1$ and $I=(-1,1)$.

4. $f(x)=\frac{3}{1-x^4}=3\left(\frac{1}{1-x^4}\right)=3(1+x^4+x^8+x^{12}+\cdots)=3\sum_{n=0}^{\infty}(x^4)^n=\sum_{n=0}^{\infty}3x^{4n}$ with $\left|x^4\right|<1\Leftrightarrow |x|<1$, so $R=1$ and $I=(-1,1)$.

5. Replacing x with x^3 in (1) gives $f(x)=\frac{1}{1-x^3}=\sum_{n=0}^{\infty}\left(x^3\right)^n=\sum_{n=0}^{\infty}x^{3n}$. The series converges when $\left|x^3\right|<1\Leftrightarrow |x|^3<1\Leftrightarrow |x|<\sqrt[3]{1}\Leftrightarrow |x|<1$. Thus, $R=1$ and $I=(-1,1)$.

6. $f(x)=\frac{1}{1+9x^2}=\frac{1}{1-\left(-9x^2\right)}=\sum_{n=0}^{\infty}\left(-9x^2\right)^n=\sum_{n=0}^{\infty}(-1)^n 3^{2n} x^{2n}$. The series converges when $\left|-9x^2\right|<1$; that is, when $|x|<\frac{1}{3}$, so $I=\left(-\frac{1}{3},\frac{1}{3}\right)$.

7. $f(x)=\frac{1}{x-5}=-\frac{1}{5}\left(\frac{1}{1-x/5}\right)=-\frac{1}{5}\sum_{n=0}^{\infty}\left(\frac{x}{5}\right)^n$ or equivalently, $-\sum_{n=0}^{\infty}\frac{1}{5^{n+1}}x^n$. The series converges when $\left|\frac{x}{5}\right|<1$; that is, when $|x|<5$, so $I=(-5,5)$.

8. $f(x)=\frac{x}{4x+1}=x\cdot\frac{1}{1-(-4x)}=x\sum_{n=0}^{\infty}(-4x)^n=\sum_{n=0}^{\infty}(-1)^n 2^{2n} x^{n+1}$. The series converges when $|-4x|<1$; that is, when $|x|<\frac{1}{4}$, so $I=\left(-\frac{1}{4},\frac{1}{4}\right)$.

9.

$$f(x)=\frac{x}{9+x^2}=\frac{x}{9}\left[\frac{1}{1+(x/3)^2}\right]=\frac{x}{9}\left[\frac{1}{1-\{-(x/3)^2\}}\right]=\frac{x}{9}\sum_{n=0}^{\infty}\left[-\left(\frac{x}{3}\right)^2\right]^n=\frac{x}{9}\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{9^n}=\sum_{n=0}^{\infty}(-1)^n\frac{x}{9}$$

. The geometric series $\sum_{n=0}^{\infty}\left[-\left(\frac{x}{3}\right)^2\right]^n$ converges when

$\left|-\left(\frac{x}{3}\right)^2\right|<1 \Leftrightarrow \frac{\left|x^2\right|}{9}<1 \Leftrightarrow |x|^2<9 \Leftrightarrow |x|<3$, so $R=3$ and $I=(-3,3)$.

10. $f(x)=\frac{x^2}{a^3-x^3}=\frac{x^2}{a^3}\cdot\frac{1}{1-x^3/a^3}=\frac{x^2}{a^3}\sum_{n=0}^{\infty}\left(\frac{x^3}{a^3}\right)^n=\sum_{n=0}^{\infty}\frac{x^{3n+2}}{a^{3n+3}}$. The series converges when $\left|x^3/a^3\right|<1 \Leftrightarrow \left|x^3\right|<\left|a^3\right| \Leftrightarrow |x|<|a|$, so $R=|a|$ and $I=(-|a|,|a|)$.

11. $f(x)=\frac{3}{x^2+x-2}=\frac{3}{(x+2)(x-1)}=\frac{A}{x+2}+\frac{B}{x-1} \Rightarrow 3=A(x-1)+B(x+2)$. Taking $x=-2$, we get $A=-1$. Taking $x=1$, we get $B=1$. Thus,

$$\frac{3}{x^2+x-2}=\frac{1}{x-1}-\frac{1}{x+2}=-\frac{1}{1-x}-\frac{1}{2}\,\frac{1}{1+x/2}=-\sum_{n=0}^{\infty}x^n-\frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{x}{2}\right)^n$$

$$=\sum_{n=0}^{\infty}\left[-1-\frac{1}{2}\left(-\frac{1}{2}\right)^n\right]x^n=\sum_{n=0}^{\infty}\left[-1+\left(-\frac{1}{2}\right)^{n+1}\right]x^n=\sum_{n=0}^{\infty}\left[\frac{(-1)^{n+1}}{2^{n+1}}-1\right]x^n$$

We represented the given function as the sum of two geometric series; the first converges for $x\in(-1,1)$ and the second converges for $x\in(-2,2)$. Thus, the sum converges for $x\in(-1,1)=I$.

12.

$$f(x)=\frac{7x-1}{3x^2+2x-1}=\frac{7x-1}{(3x-1)(x+1)}=\frac{A}{3x-1}+\frac{B}{x+1}=\frac{1}{3x-1}+\frac{2}{x+1}=2\cdot\frac{1}{1-(-x)}-\frac{1}{1-3x}$$

$$=2\sum_{n=0}^{\infty}(-x)^n-\sum_{n=0}^{\infty}(3x)^n=\sum_{n=0}^{\infty}\left[2(-1)^n-3^n\right]x^n$$

The series $\sum(-x)^n$ converges for $x\in(-1,1)$ and the series $\sum(3x)^n$ converges for $x\in\left(-\frac{1}{3},\frac{1}{3}\right)$, so their sum converges for $x\in\left(-\frac{1}{3},\frac{1}{3}\right)=I$.

13. **(a)**

$$f(x)=\frac{1}{(1+x)^2}=\frac{d}{dx}\left(\frac{-1}{1+x}\right)=-\frac{d}{dx}\left[\Sigma_{n=0}^{\infty}(-1)^n x^n\right]$$
$$=\Sigma_{n=1}^{\infty}(-1)^{n+1}nx^{n-1}=\Sigma_{n=0}^{\infty}(-1)^n(n+1)x^n \text{ with } R=1 .$$

In the last step, note that we *decreased* the initial value of the summation variable n by 1 , and then *increased* each occurrence of n in the term by 1 .

(b)

$$f(x)=\frac{1}{(1+x)^3}=-\frac{1}{2}\frac{d}{dx}\left[\frac{1}{(1+x)^2}\right]=-\frac{1}{2}\frac{d}{dx}\left[\Sigma_{n=0}^{\infty}(-1)^n(n+1)x^n\right]$$
$$=-\frac{1}{2}\Sigma_{n=1}^{\infty}(-1)^n(n+1)nx^{n-1}=\frac{1}{2}\Sigma_{n=0}^{\infty}(-1)^n(n+2)(n+1)x^n \text{ with } R=1 .$$

(c)

$$f(x)=\frac{x^2}{(1+x)^3}=x^2\cdot\frac{1}{(1+x)^3}=x^2\cdot\frac{1}{2}\Sigma_{n=0}^{\infty}(-1)^n(n+2)(n+1)x^n$$

$=\frac{1}{2}\Sigma_{n=0}^{\infty}(-1)^n(n+2)(n+1)x^{n+2}$. To write the power series with x^n rather than x^{n+2} ,

we will *decrease* each occurrence of n in the term by 2 and *increase* the initial value of the summation variable by 2 . This gives us $\frac{1}{2}\Sigma_{n=2}^{\infty}(-1)^n(n)(n-1)x^n$.

14. **(a)** $\frac{1}{1+x}=\frac{1}{1-(-x)}=\Sigma_{n=0}^{\infty}(-1)^n x^n$, so

$$f(x)=\ln(1+x)=\int\frac{dx}{1+x}=\int\left[\Sigma_{n=0}^{\infty}(-1)^n x^n\right]dx=C+\Sigma_{n=0}^{\infty}(-1)^n\frac{x^{n+1}}{n+1}$$
$$=\Sigma_{n=1}^{\infty}\frac{(-1)^{n-1}x^n}{n}\ C=0 \text{ since } f(0)=\ln 1=0],\text{ with } R=1$$

(b) $f(x)=x\ln(1+x)=x\left[\Sigma_{n=1}^{\infty}\frac{(-1)^{n-1}x^n}{n}\right]=\Sigma_{n=1}^{\infty}\frac{(-1)^{n-1}x^{n+1}}{n}=\Sigma_{n=2}^{\infty}\frac{(-1)^n x^n}{n-1}$ with $R=1$.

(c) $f(x)=\ln\left(x^2+1\right)=\Sigma_{n=1}^{\infty}\frac{(-1)^{n-1}\left(x^2\right)^n}{n}$ [by part (a)] $=\Sigma_{n=1}^{\infty}\frac{(-1)^{n-1}x^{2n}}{n}$ with $R=1$.

15.

$$f(x) = \ln(5-x) = -\int \frac{dx}{5-x} = -\frac{1}{5}\int \frac{dx}{1-x/5}$$

$$= -\frac{1}{5}\int \left[\Sigma_{n=0}^{\infty} \left(\frac{x}{5} \right)^n \right] dx = C - \frac{1}{5}\Sigma_{n=0}^{\infty} \frac{x^{n+1}}{5^n(n+1)} = C - \Sigma_{n=1}^{\infty} \frac{x^n}{n5^n}$$

Putting $x=0$, we get $C=\ln 5$. The series converges for $|x/5|<1 \Leftrightarrow |x|<5$, so $R=5$.

16. We know that $\frac{1}{1-2x} = \Sigma_{n=0}^{\infty}(2x)^n$. Differentiating, we get
$\frac{2}{(1-2x)^2} = \Sigma_{n=1}^{\infty} 2^n n x^{n-1} = \Sigma_{n=0}^{\infty} 2^{n+1}(n+1)x^n$, so
$f(x) = \frac{x^2}{(1-2x)^2} = \frac{x^2}{2} \cdot \frac{2}{(1-2x)^2} = \frac{x^2}{2}\Sigma_{n=0}^{\infty} 2^{n+1}(n+1)x^n = \Sigma_{n=0}^{\infty} 2^n(n+1)x^{n+2}$ or $\Sigma_{n=2}^{\infty} 2^{n-2}(n-1)x^n$, with
$R = \frac{1}{2}$.

17. $\frac{1}{2-x} = \frac{1}{2(1-x/2)} = \frac{1}{2}\Sigma_{n=0}^{\infty}\left(\frac{x}{2} \right)^n = \Sigma_{n=0}^{\infty} \frac{1}{2^{n+1}} x^n$ for $\left| \frac{x}{2} \right| < 1 \Leftrightarrow |x|<2$. Now
$\frac{1}{(x-2)^2} = \frac{d}{dx}\left(\frac{1}{2-x} \right) = \frac{d}{dx}\left(\Sigma_{n=0}^{\infty} \frac{1}{2^{n+1}} x^n \right) = \Sigma_{n=1}^{\infty} \frac{n}{2^{n+1}} x^{n-1} = \Sigma_{n=0}^{\infty} \frac{n+1}{2^{n+2}} x^n$. So
$f(x) = \frac{x^3}{(x-2)^2} = x^3\Sigma_{n=0}^{\infty} \frac{n+1}{2^{n+2}} x^n = \Sigma_{n=0}^{\infty} \frac{n+1}{2^{n+2}} x^{n+3}$ or $\Sigma_{n=3}^{\infty} \frac{n-2}{2^{n-1}} x^n$ for $|x|<2$. Thus, $R=2$ and $I=(-2,2)$.

18. From Example 7, $g(x) = \arctan x = \Sigma_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1}$. Thus,
$f(x) = \arctan(x/3) = \Sigma_{n=0}^{\infty}(-1)^n \frac{(x/3)^{2n+1}}{2n+1} = \Sigma_{n=0}^{\infty}(-1)^n \frac{1}{3^{2n+1}(2n+1)} x^{2n+1}$ for $\left| \frac{x}{3} \right| < 1 \Leftrightarrow |x|<3$, so $R=3$
.

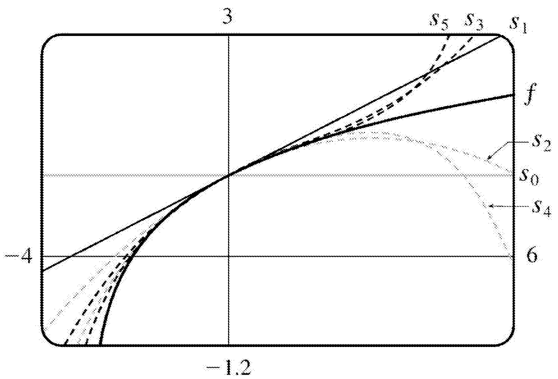
19.

$$f(x) = \ln(3+x) = \int \frac{dx}{3+x} = \frac{1}{3}\int \frac{dx}{1+x/3} = \frac{1}{3}\int \frac{dx}{1-(-x/3)} = \frac{1}{3}\int \Sigma_{n=0}^{\infty}\left(-\frac{x}{3} \right)^n dx$$

$$=C+\frac{1}{3}\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)3^n}x^{n+1}=\ln 3+\frac{1}{3}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n3^{n-1}}x^n$$

$=\ln 3+\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n3^n}x^n$. The series converges when $|-x/3|<1 \Leftrightarrow |x|<3$, so $R=3$.

The terms of the series are $a_0=\ln 3, a_1=\frac{x}{3}, a_2=-\frac{x^2}{18}, a_3=\frac{x^3}{81}, a_4=-\frac{x^4}{324}, a_5=\frac{x^5}{1215}, \ldots$.



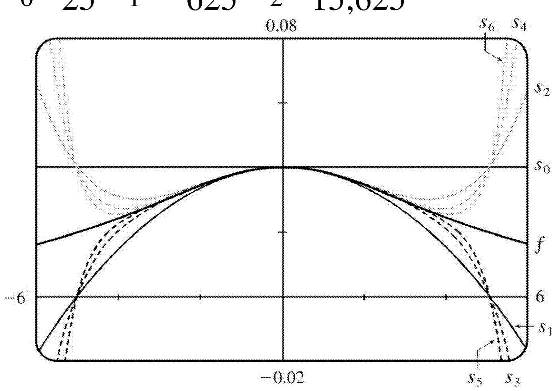
As n increases, $s_n(x)$ approximates f better on the interval of convergence, which is $(-3,3)$.

20.

$$f(x)=\frac{1}{x^2+25}=\frac{1}{25}\left(\frac{1}{1+x^2/25}\right)=\frac{1}{25}\left(\frac{1}{1-\left(-x^2/25\right)}\right)=\frac{1}{25}\sum_{n=0}^{\infty}\left(-\frac{x^2}{25}\right)^n=\frac{1}{25}\sum_{n=0}^{\infty}(-1)^n\left(\frac{x}{5}\right)^{2n}$$

. The series converges when $\left|-x^2/25\right|<1 \Leftrightarrow x^2<25 \Leftrightarrow |x|<5$, so $R=5$. The terms of the series are

$a_0=\frac{1}{25}, a_1=-\frac{x^2}{625}, a_2=\frac{x^4}{15{,}625}, \ldots$.

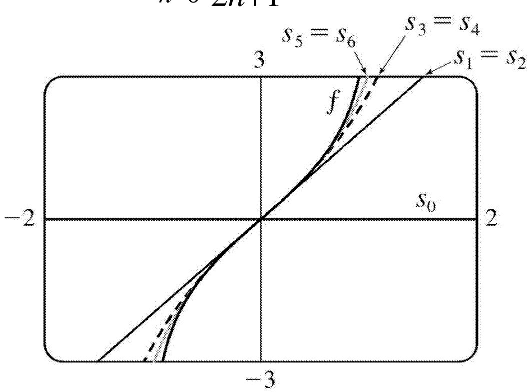


As n increases, $s_n(x)$ approximates f better on the interval of convergence, which is $(-5,5)$.

21.

$$f(x) = \ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x) = \int \frac{dx}{1+x} + \int \frac{dx}{1-x}$$
$$= \int \frac{dx}{1-(-x)} + \int \frac{dx}{1-x} = \int \left[\sum_{n=0}^{\infty} (-1)^n x^n + \sum_{n=0}^{\infty} x^n\right] dx$$
$$= \int \left[\left(1 - x + x^2 - x^3 + x^4 - \cdots\right) + \left(1 + x + x^2 + x^3 + x^4 + \cdots\right)\right] dx$$
$$= \int \left(2 + 2x^2 + 2x^4 + \cdots\right) dx = \int \sum_{n=0}^{\infty} 2x^{2n}\, dx = C + \sum_{n=0}^{\infty} \frac{2x^{2n+1}}{2n+1}$$

But $f(0) = \ln \frac{1}{1} = 0$, so $C=0$ and we have $f(x) = \sum_{n=0}^{\infty} \frac{2x^{2n+1}}{2n+1}$ with $R=1$. If $x = \pm 1$, then $f(x) = \pm 2\sum_{n=0}^{\infty} \frac{1}{2n+1}$, which both diverge by the Limit Comparison Test with $b_n = \frac{1}{n}$.

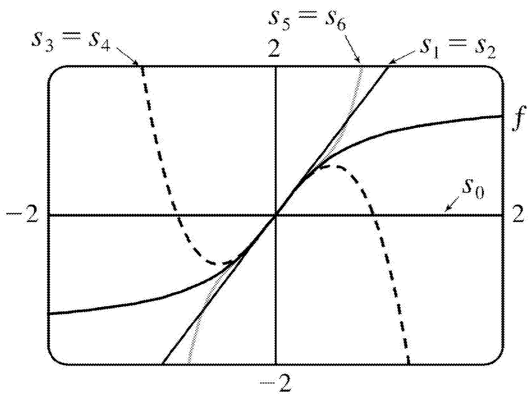


As n increases, $s_n(x)$ approximates f better on the interval of convergence, which is $(-1,1)$.

22.

$$f(x) = \tan^{-1}(2x) = 2\int \frac{dx}{1+4x^2} = 2\int \sum_{n=0}^{\infty} (-1)^n \left(4x^2\right)^n dx = 2\int \sum_{n=0}^{\infty} (-1)^n 4^n x^{2n}\, dx$$
$$= C + 2\sum_{n=0}^{\infty} \frac{(-1)^n 4^n x^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1} x^{2n+1}}{2n+1} \quad [\, f(0) = \tan^{-1} 0 = 0 \text{ , so } C=0 \,].$$

The series converges when $\left|4x^2\right| < 1 \Leftrightarrow |x| < \frac{1}{2}$, so $R = \frac{1}{2}$. If $x = \pm \frac{1}{2}$, then $f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1}$ and $f(x) = \sum_{n=0}^{\infty} (-1)^{n+1} \frac{1}{2n+1}$, respectively. Both series converge by the Alternating Series Test.



As n increases, $s_n(x)$ approximates f better on the interval of convergence, which is $\left[-\frac{1}{2}, \frac{1}{2} \right]$.

23. $\frac{t}{1-t^8} = t \cdot \frac{1}{1-t^8} = t\sum_{n=0}^{\infty} (t^8)^n = \sum_{n=0}^{\infty} t^{8n+1} \Rightarrow \int \frac{t}{1-t^8}\,dt = C + \sum_{n=0}^{\infty} \frac{t^{8n+2}}{8n+2}$. The series for $\frac{1}{1-t^8}$ converges when $|t^8| < 1 \Leftrightarrow |t| < 1$, so $R=1$ for that series and also the series for $t/(1-t^8)$. By Theorem 2, the series for $\int \frac{t}{1-t^8}\,dt$ also has $R=1$.

24. By Example 6, $\ln(1-t) = -\sum_{n=1}^{\infty} \frac{t^n}{n}$ for $|t| < 1$, so $\frac{\ln(1-t)}{t} = -\sum_{n=1}^{\infty} \frac{t^{n-1}}{n}$ and $\int \frac{\ln(1-t)}{t}\,dt = C - \sum_{n=1}^{\infty} \frac{t^n}{n^2}$. By Theorem 2, $R=1$.

25. By Example 7, $\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$ with $R=1$, so

$x - \tan^{-1} x = x - \left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \right) = \frac{x^3}{3} - \frac{x^5}{5} + \frac{x^7}{7} - \cdots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n+1}}{2n+1}$ and

$\frac{x - \tan^{-1} x}{x^3} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n-2}}{2n+1}$, so

$\int \frac{x - \tan^{-1} x}{x^3}\,dx = C + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n-1}}{(2n+1)(2n-1)} = C + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n-1}}{4n^2-1}$. By Theorem 2, $R=1$.

26. By Example 7, $\int \tan^{-1}\left(x^2\right) dx = \int \sum_{n=0}^{\infty} (-1)^n \frac{\left(x^2\right)^{2n+1}}{2n+1}\,dx = C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+3}}{(2n+1)(4n+3)}$ with $R=1$.

27. $\frac{1}{1+x^5}=\frac{1}{1-\left(-x^5\right)}=\sum_{n=0}^{\infty}\left(-x^5\right)^n=\sum_{n=0}^{\infty}(-1)^n x^{5n}\Rightarrow$

$\int \frac{1}{1+x^5}\,dx=\int\sum_{n=0}^{\infty}(-1)^n x^{5n}dx=C+\sum_{n=0}^{\infty}(-1)^n\frac{x^{5n+1}}{5n+1}$. Thus,

$I=\int_0^{0.2}\frac{1}{1+x^5}\,dx=\left[x-\frac{x^6}{6}+\frac{x^{11}}{11}-\cdots\right]_0^{0.2}=0.2-\frac{(0.2)^6}{6}+\frac{(0.2)^{11}}{11}-\cdots$. The series is alternating, so if we use the first two terms, the error is at most $(0.2)^{11}/11\approx 1.9\times 10^{-9}$. So $I\approx 0.2-(0.2)^6/6\approx 0.199989$ to six decimal places.

28. From Example 6 we know $\ln(1-x)=-\sum_{n=1}^{\infty}\frac{x^n}{n}$, so

$\ln\left(1+x^4\right)=\ln\left[1-\left(-x^4\right)\right]=-\sum_{n=1}^{\infty}\frac{\left(-x^4\right)^n}{n}=\sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^{4n}}{n}\Rightarrow$

$\int\ln\left(1+x^4\right)dx=\int\sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^{4n}}{n}\,dx=C+\sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^{4n+1}}{n(4n+1)}$. Thus,

$I=\int_0^{0.4}\ln\left(1+x^4\right)dx=\left[\frac{x^5}{5}-\frac{x^9}{18}+\frac{x^{13}}{39}-\frac{x^{17}}{68}+\cdots\right]_0^{0.4}=\frac{(0.4)^5}{5}-\frac{(0.4)^9}{18}+\frac{(0.4)^{13}}{39}-\frac{(0.4)^{17}}{68}+\cdots$. The series is alternating, so if we use the first three terms, the error is at most $(0.4)^{17}/68\approx 2.5\times 10^{-9}$. So $I\approx(0.4)^5/5-(0.4)^9/18+(0.9)^{13}/39\approx 0.002034$ to six decimal places.

29. We substitute x^4 for x in Example 7, and find that

$$\int x^2\tan^{-1}\left(x^4\right)dx=\int x^2\sum_{n=0}^{\infty}(-1)^n\frac{\left(x^4\right)^{2n+1}}{2n+1}\,dx$$

$$=\int\sum_{n=0}^{\infty}(-1)^n\frac{x^{8n+6}}{2n+1}\,dx=C+\sum_{n=0}^{\infty}(-1)^n\frac{x^{8n+7}}{(2n+1)(8n+7)}$$

So $\int_0^{1/3}x^2\tan^{-1}\left(x^4\right)dx=\left[\frac{x^7}{7}-\frac{x^{15}}{45}+\cdots\right]_0^{1/3}=\frac{1}{7\cdot 3^7}-\frac{1}{45\cdot 3^{15}}+\cdots$. The series is alternating, so if we use only one term, the error is at most $1/\left(45\cdot 3^{15}\right)\approx 1.5\times 10^{-9}$. So $\int_0^{1/3}x^2\tan^{-1}\left(x^4\right)dx\approx 1/\left(7\cdot 3^7\right)\approx 0.000065$ to six decimal places.

30. We substitute x^4 for x in Example 7, and find that

$$\int x^2\tan^{-1}\left(x^4\right)dx = \int x^2\sum_{n=0}^{\infty}(-1)^n\frac{\left(x^4\right)^{2n+1}}{2n+1}\,dx$$

$$= \int\sum_{n=0}^{\infty}(-1)^n\frac{x^{8n+6}}{2n+1}\,dx = C+\sum_{n=0}^{\infty}(-1)^n\frac{x^{8n+7}}{(2n+1)(8n+7)}$$

So $\int_0^{1/3} x^2\tan^{-1}\left(x^4\right)dx=\left[\frac{x^7}{7}-\frac{x^{15}}{45}+\cdots\right]_0^{1/3}=\frac{1}{7\cdot 3^7}-\frac{1}{45\cdot 3^{15}}+\cdots$. The series is alternating, so if we use only one term, the error is at most $1/\left(45\cdot 3^{15}\right)\approx 1.5\times 10^{-9}$. So $\int_0^{1/3} x^2\tan^{-1}\left(x^4\right)dx\approx 1/\left(7\cdot 3^7\right)\approx 0.000065$ to six decimal places.

31. Using the result of Example 6, $\ln(1-x)=-\sum_{n=1}^{\infty}\frac{x^n}{n}$, with $x=-0.1$, we have $\ln 1.1=\ln\left[1-(-0.1)\right]=0.1-\frac{0.01}{2}+\frac{0.001}{3}-\frac{0.0001}{4}+\frac{0.00001}{5}-\cdots$. The series is alternating, so if we use only the first four terms, the error is at most $\frac{0.00001}{5}=0.000002$. So $\ln 1.1\approx 0.1-\frac{0.01}{2}+\frac{0.001}{3}-\frac{0.0001}{4}\approx 0.09531$.

32. $f(x)=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}\Rightarrow f'(x)=\sum_{n=1}^{\infty}\frac{(-1)^n 2nx^{2n-1}}{(2n)!}$, so

$$f''(x) = \sum_{n=1}^{\infty}\frac{(-1)^n(2n)(2n-1)x^{2n-2}}{(2n)!}=\sum_{n=1}^{\infty}\frac{(-1)^n x^{2(n-1)}}{[2(n-1)]!}$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^{n+1}x^{2n}}{(2n)!}\qquad [\text{ subsistuting } n+1 \text{ for } n\ ]$$

$$= -\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}=-f(x)\Rightarrow f''(x)+f(x)=0.$$

33. **(a)** $J_0(x)=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$, $J_0'(x)=\sum_{n=1}^{\infty}\frac{(-1)^n 2nx^{2n-1}}{2^{2n}(n!)^2}$, and $J_0''(x)=\sum_{n=1}^{\infty}\frac{(-1)^n 2n(2n-1)x^{2n-2}}{2^{2n}(n!)^2}$, so

$$x^2J_0''(x)+xJ_0'(x)+x^2J_0(x)=\sum_{n=1}^{\infty}\frac{(-1)^n 2n(2n-1)x^{2n}}{2^{2n}(n!)^2}+\sum_{n=1}^{\infty}\frac{(-1)^n 2nx^{2n}}{2^{2n}(n!)^2}+\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+2}}{2^{2n}(n!)^2}$$

$$=\sum_{n=1}^{\infty}\frac{(-1)^n 2n(2n-1)x^{2n}}{2^{2n}(n!)^2}+\sum_{n=1}^{\infty}\frac{(-1)^n 2nx^{2n}}{2^{2n}(n!)^2}+\sum_{n=1}^{\infty}\frac{(-1)^{n-1} x^{2n}}{2^{2n-2}\left[(n-1)!\right]^2}$$

$$=\sum_{n=1}^{\infty}\frac{(-1)^n 2n(2n-1)x^{2n}}{2^{2n}(n!)^2}+\sum_{n=1}^{\infty}\frac{(-1)^n 2nx^{2n}}{2^{2n}(n!)^2}+\sum_{n=1}^{\infty}\frac{(-1)^n(-1)^{-1}2^2n^2x^{2n}}{2^{2n}(n!)^2}$$

$$=\sum_{n=1}^{\infty}(-1)^n\left[\frac{2n(2n-1)+2n-2^2n^2}{2^{2n}(n!)^2}\right]x^{2n}=\sum_{n=1}^{\infty}(-1)^n\left[\frac{4n^2-2n+2n-4n^2}{2^{2n}(n!)^2}\right]x^{2n}=0$$

(b)

$$\int_0^1 J_0(x)\,dx=\int_0^1\left[\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}\right]dx=\int_0^1\left(1-\frac{x^2}{4}+\frac{x^4}{64}-\frac{x^6}{2304}+\cdots\right)dx$$

$$=\left[x-\frac{x^3}{3\cdot 4}+\frac{x^5}{5\cdot 64}-\frac{x^7}{7\cdot 2304}+\cdots\right]_0^1=1-\frac{1}{12}+\frac{1}{320}-\frac{1}{16{,}128}+\cdots$$

Since $\frac{1}{16{,}128}\approx 0.000062$, it follows from The Alternating Series Estimation Theorem that, correct to three decimal places, $\int_0^1 J_0(x)\,dx\approx 1-\frac{1}{12}+\frac{1}{320}\approx 0.920$.

34. **(a)** $J_1(x)=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{n!(n+1)!2^{2n+1}}$, $J_1'(x)=\sum_{n=0}^{\infty}\frac{(-1)^n(2n+1)x^{2n}}{n!(n+1)!2^{2n+1}}$, and

$$J_1''(x)=\sum_{n=1}^{\infty}\frac{(-1)^n(2n+1)(2n)x^{2n-1}}{n!(n+1)!2^{2n+1}}.$$

$$x^2J_1''(x)+xJ_1'(x)+\left(x^2-1\right)J_1(x)$$

$$=\sum_{n=1}^{\infty}\frac{(-1)^n(2n+1)(2n)x^{2n+1}}{n!(n+1)!2^{2n+1}}+\sum_{n=0}^{\infty}\frac{(-1)^n(2n+1)x^{2n+1}}{n!(n+1)!2^{2n+1}}$$

$$+\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+3}}{n!(n+1)!2^{2n+1}}-\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{n!(n+1)!2^{2n+1}}$$

$$=\sum_{n=1}^{\infty}\frac{(-1)^n(2n+1)(2n)x^{2n+1}}{n!(n+1)!2^{2n+1}}+\sum_{n=0}^{\infty}\frac{(-1)^n(2n+1)x^{2n+1}}{n!(n+1)!2^{2n+1}}$$

$$-\sum_{n=1}^{\infty}\frac{(-1)^n x^{2n+1}}{(n-1)!n!2^{2n-1}}-\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{n!(n+1)!2^{2n+1}}\left[\begin{array}{c}\text{Replace n with n}-1\\ \text{in the third term}\end{array}\right]$$

$$=\frac{x}{2}-\frac{x}{2}+\sum_{n=1}^{\infty}(-1)^n\left[\frac{(2n+1)(2n)+(2n+1)-(n)(n+1)2^2-1}{n!(n+1)!2^{2n+1}}\right]x^{2n+1}=0$$

(b) $J_0(x)=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}\Rightarrow$

$$J_0'(x)=\sum_{n=1}^{\infty}\frac{(-1)^n(2n)x^{2n-1}}{2^{2n}(n!)^2}=\sum_{n=0}^{\infty}\frac{(-1)^{n+1}2(n+1)x^{2n+1}}{2^{2n+2}[(n+1)!]^2}\quad[\text{ Replace } n \text{ with } n+1\ ]$$

$$=-\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{2^{2n+1}(n+1)!n!}\quad[\text{ cancel 2 and } n+1\text{ ; take }-1\text{ outside sum}]=-J_1(x)$$

35. **(a)** $f(x)=\sum_{n=0}^{\infty}\frac{x^n}{n!}\Rightarrow f'(x)=\sum_{n=1}^{\infty}\frac{nx^{n-1}}{n!}=\sum_{n=1}^{\infty}\frac{x^{n-1}}{(n-1)!}=\sum_{n=0}^{\infty}\frac{x^n}{n!}=f(x)$

(b) By Theorem .4.2, the only solution to the differential equation $df(x)/dx=f(x)$ is $f(x)=Ke^x$, but $f(0)=1$, so $K=1$ and $f(x)=e^x$.
Or: We could solve the equation $df(x)/dx=f(x)$ as a separable differential equation.

36. $\frac{|\sin nx|}{n^2}\le\frac{1}{n^2}$, so $\sum_{n=1}^{\infty}\frac{\sin nx}{n^2}$ converges by the Comparison Test. $\frac{d}{dx}\left(\frac{\sin nx}{n^2}\right)=\frac{\cos nx}{n}$, so when $x=2k\pi$ (k an integer), $\sum_{n=1}^{\infty}f_n'(x)=\sum_{n=1}^{\infty}\frac{\cos(2kn\pi)}{n}=\sum_{n=1}^{\infty}\frac{1}{n}$, which diverges (harmonic series). $f_n''(x)=-\sin nx$, so $\sum_{n=1}^{\infty}f_n''(x)=-\sum_{n=1}^{\infty}\sin nx$, which converges only if $\sin nx=0$, or $x=k\pi$ (k an integer).

37. If $a_n=\frac{x^n}{n^2}$, then by the Ratio Test,

$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)^2}\cdot\frac{n^2}{x^n}\right| = |x|\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^2 = |x| < 1$ for convergence, so $R=1$.

When $x=\pm 1$, $\sum_{n=1}^{\infty}\left|\frac{x^n}{n^2}\right| = \sum_{n=1}^{\infty}\frac{1}{n^2}$ which is a convergent p– series ($p=2>1$), so the interval of convergence for f is $[-1,1]$. By Theorem 2, the radii of convergence of f' and f'' are both 1 , so we need only check the endpoints. $f(x)=\sum_{n=1}^{\infty}\frac{x^n}{n^2} \Rightarrow f'(x)=\sum_{n=1}^{\infty}\frac{nx^{n-1}}{n^2} = \sum_{n=0}^{\infty}\frac{x^n}{n+1}$, and this series diverges for $x=1$ (harmonic series) and converges for $x=-1$ (Alternating Series Test), so the interval of convergence is $[-1,1)$. $f''(x)=\sum_{n=1}^{\infty}\frac{nx^{n-1}}{n+1}$ diverges at both 1 and -1 (Test for Divergence) since $\lim_{n\to\infty}\frac{n}{n+1}=1\neq 0$, so its interval of convergence is $(-1,1)$.

38. **(a)** $\sum_{n=1}^{\infty}nx^{n-1} = \sum_{n=0}^{\infty}\frac{d}{dx}x^n = \frac{d}{dx}\left[\sum_{n=0}^{\infty}x^n\right] = \frac{d}{dx}\left[\frac{1}{1-x}\right] = -\frac{1}{(1-x)^2}(-1) = \frac{1}{(1-x)^2}$, $|x|<1$.

(b)

(i) $\sum_{n=1}^{\infty}nx^n = x\sum_{n=1}^{\infty}nx^{n-1} = x\left[\frac{1}{(1-x)^2}\right] = \frac{x}{(1-x)^2}$ for $|x|<1$.

(ii) Put $x=\frac{1}{2}$ in (i): $\sum_{n=1}^{\infty}\frac{n}{2^n} = \sum_{n=1}^{\infty}n\left(\frac{1}{2}\right)^n = \frac{1/2}{(1-1/2)^2} = 2$.

(c)

(i)

$$\sum_{n=2}^{\infty}n(n-1)x^n = x^2\sum_{n=2}^{\infty}n(n-1)x^{n-2} = x^2\frac{d}{dx}\left[\sum_{n=1}^{\infty}nx^{n-1}\right] = x^2\frac{d}{dx}\frac{1}{(1-x)^2}$$

$$= x^2\frac{2}{(1-x)^3} = \frac{2x^2}{(1-x)^3} \text{ for } |x|<1 .$$

(ii) Put $x=\frac{1}{2}$ in (i): $\sum_{n=2}^{\infty}\frac{n^2-n}{2^n} = \sum_{n=2}^{\infty}n(n-1)\left(\frac{1}{2}\right)^n = \frac{2(1/2)^2}{(1-1/2)^3} = 4$.

(iii)

From (b)(ii) and (c)(ii), we have $\sum_{n=1}^{\infty} \frac{n^2}{2^n} = \sum_{n=1}^{\infty} \frac{n^2-n}{2^n} + \sum_{n=1}^{\infty} \frac{n}{2^n} = 4+2=6$.

39. By Example 7, $\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$ for $|x|<1$. In particular, for $x = \frac{1}{\sqrt{3}}$, we have

$\frac{\pi}{6} = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \sum_{n=0}^{\infty} (-1)^n \frac{(1/\sqrt{3})^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} (-1)^n \left(\frac{1}{3}\right)^n \frac{1}{\sqrt{3}} \frac{1}{2n+1}$, so

$\pi = \frac{6}{\sqrt{3}} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)3^n} = 2\sqrt{3} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)3^n}$.

40. **(a)** $\int_0^{1/2} \frac{dx}{x^2-x+1} = \int_0^{1/2} \frac{dx}{(x-1/2)^2+3/4} \left[\begin{array}{l} x-1/2=(\sqrt{3}/2)\,u,\ \ u=(2/\sqrt{3})(x-1/2) \\ dx=(\sqrt{3}/2)\,du \end{array}\right]$

$= \int_{-1/\sqrt{3}}^{0} \frac{(\sqrt{3}/2)\,du}{(3/4)(u^2+1)} = \frac{2\sqrt{3}}{3} \left[\tan^{-1} u\right]_{-1/\sqrt{3}}^{0} = \frac{2}{\sqrt{3}} \left[0 - \left(-\frac{\pi}{6}\right)\right] = \frac{\pi}{3\sqrt{3}}$

(b) $\frac{1}{x^3+1} = \frac{1}{(x+1)(x^2-x+1)} \Rightarrow \frac{1}{x^2-x+1} = (x+1)\left(\frac{1}{1+x^3}\right) = (x+1)\frac{1}{1-(-x^3)}$

$= (x+1)\sum_{n=0}^{\infty} (-1)^n x^{3n} = \sum_{n=0}^{\infty} (-1)^n x^{3n+1} + \sum_{n=0}^{\infty} (-1)^n x^{3n}$ for $|x|<1 \Rightarrow \int \frac{dx}{x^2-x+1}$

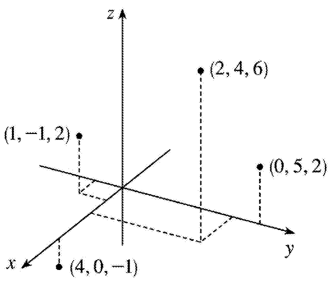
$= C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{3n+2}}{3n+2} + \sum_{n=0}^{\infty} (-1)^n \frac{x^{3n+1}}{3n+1}$ for $|x|<1 \Rightarrow \int_0^{1/2} \frac{dx}{x^2-x+1}$

$= \sum_{n=0}^{\infty} (-1)^n \left[\frac{1}{4\cdot 8^n(3n+2)} + \frac{1}{2\cdot 8^n(3n+1)}\right] = \frac{1}{4} \sum_{n=0}^{\infty} \frac{(-1)^n}{8^n} \left(\frac{2}{3n+1} + \frac{1}{3n+2}\right)$

By part (a), this equals $\frac{\pi}{3\sqrt{3}}$, so $\pi = \frac{3\sqrt{3}}{4} \sum_{n=0}^{\infty} \frac{(-1)^n}{8^n} \left(\frac{2}{3n+1} + \frac{1}{3n+2}\right)$.

1. We start at the origin, which has coordinates $(0,0,0)$. First we move 4 units along the positive x –axis, affecting only the x –coordinate, bringing us to the point $(4,0,0)$. We then move 3 units straight downward, in the negative z –direction. Thus only the z –coordinate is affected, and we arrive at $(4,0,-3)$.

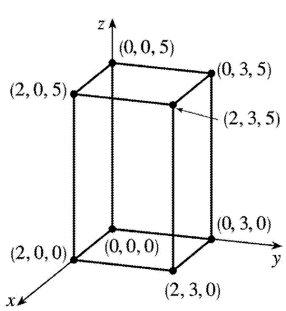
2.



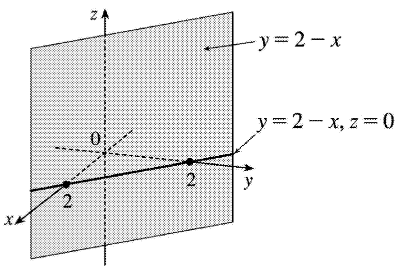
3. The distance from a point to the xz –plane is the absolute value of the y –coordinate of the point. $Q(-5,-1,4)$ has the y –coordinate with the smallest absolute value, so Q is the point closest to the xz –plane. $R(0,3,8)$ must lie in the yz –plane since the distance from R to the yz –plane, given by the x –coordinate of R, is 0.

4. The projection of $(2,3,5)$ on the xy –plane is $(2,3,0)$;
on the yz –plane, $(0,3,5)$; on the xz –plane, $(2,0,5)$.
The length of the diagonal of the box is the distance between the origin and $(2,3,5)$ given by

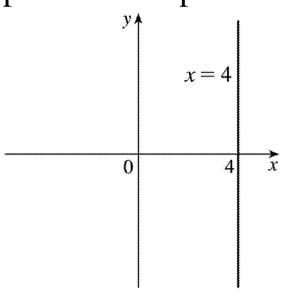
$$\sqrt{(2-0)^2+(3-0)^2+(5-0)^2}=\sqrt{38}\approx 6.16$$

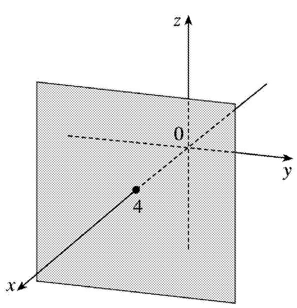


5. The equation $x+y=2$ represents the set of all points in R^3 whose x – and y –coordinates have a sum of 2 , or equivalently where $y=2-x$. This is the set $\{(x,2-x,z) \mid x\in R, z\in R\}$ which is a vertical plane that intersects the xy –plane in the line $y=2-x$, $z=0$.

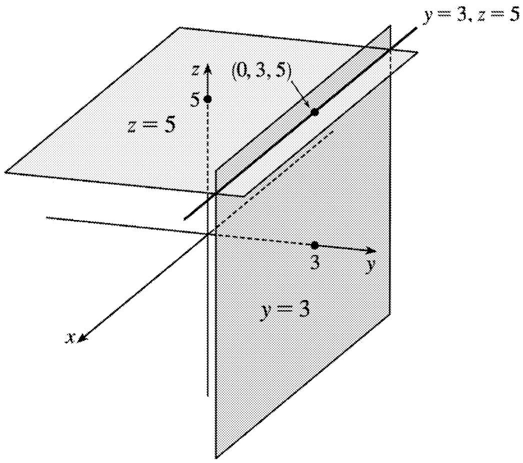


6. **(a)** In R^2 , the equation $x=4$ represents a line parallel to the y –axis. In R^3 , the equation $x=4$ represents the set $\{(x,y,z) \mid x=4\}$, the set of all points whose x –coordinate is 4 . This is the vertical plane that is parallel to the yz –plane and 4 units in front of it.





(b) In R^3 , the equation $y=3$ represents a vertical plane that is parallel to the xz –plane and 3 units to the right of it. The equation $z=5$ represents a horizontal plane parallel to the xy –plane and 5 units above it. The pair of equations $y=3$, $z=5$ represents the set of points that are simultaneously on both planes, or in other words, the line of intersection of the planes $y=3$, $z=5$. This line can also be described as the set $\{(x,3,5) \mid x\in R\}$, which is the set of all points in R^3 whose x –coordinate may vary but whose y – and z –coordinates are fixed at 3 and 5 , respectively. Thus the line is parallel to the x –axis and intersects the yz –plane in the point $(0,3,5)$.



7. We first find the lengths of the sides of the triangle by using the distance formula between pairs of vertices:

$$|PQ| = \sqrt{[1-(-2)]^2+(2-4)^2+(-1-0)^2} = \sqrt{9+4+1} = \sqrt{14}$$
$$|QR| = \sqrt{(-1-1)^2+(1-2)^2+[2-(-1)]^2} = \sqrt{4+1+9} = \sqrt{14}$$
$$|PR| = \sqrt{[-1-(-2)]^2+(1-4)^2+(2-0)^2} = \sqrt{1+9+4} = \sqrt{14}$$

Since all three sides have the same length, *PQR* is an equilateral triangle.

8. We can find the lengths of the sides of the triangle by using the distance formula between pairs of vertices:

$$|AB| = \sqrt{(3-1)^2+(4-2)^2+[-2-(-3)]^2} = \sqrt{4+4+1} = 3$$
$$|BC| = \sqrt{(3-3)^2+(-2-4)^2+[1-(-2)]^2} = \sqrt{0+36+9} = \sqrt{45} = 3\sqrt{5}$$
$$|AC| = \sqrt{(3-1)^2+(-2-2)^2+[1-(-3)]^2} = \sqrt{4+16+16} = 6$$

Since the Pythagorean Theorem is satisfied by $|AB|^2+|AC|^2=|BC|^2$, *ABC* is a right triangle. *ABC* is not ososceles, as no two sides have the same length.

9. **(a)** First we find the distances between points:

$$|AB| = \sqrt{(7-5)^2+(9-1)^2+(-1-3)^2} = \sqrt{84} = 2\sqrt{21}$$
$$|BC| = \sqrt{(1-7)^2+(-15-9)^2+[11-(-1)]^2} = \sqrt{756} = 6\sqrt{21}$$
$$|AC| = \sqrt{(1-5)^2+(-15-1)^2+(11-3)^2} = \sqrt{336} = 4\sqrt{21}$$

In order for the points to lie on a straight line, the sum of the two shortest distances must be equal to the longest distance. Since $|AB|+|AC|=|BC|$, the three points lie on a straight line.

(b) The distances between points are

$$|KL| = \sqrt{(1-0)^2+(2-3)^2+[-2-(-4)]^2}=\sqrt{6}$$
$$|LM| = \sqrt{(3-1)^2+(0-2)^2+[1-(-2)]^2}=\sqrt{17}$$
$$|KM| = \sqrt{(3-0)^2+(0-3)^2+[1-(-4)]^2}=\sqrt{43}$$

Since $\sqrt{6}+\sqrt{17}\neq\sqrt{43}$, the three points do not lie on a straight line.

10. **(a)** The distance from a point to the xy –plane is the absolute value of the z –coordinate of the point. Thus, the distance is $|-5|=5$.
(b) Similarly, the distance is the absolute value of the x –coordinate of the point: $|3|=3$.
(c) The distance is the absolute value of the y –coordinate of the point: $|7|=7$.
(d) The point on the x –axis closest to $(3,7,-5)$ is the point $(3,0,0)$. (Approach the x –axis perpendicularly.) The distance from $(3,7,-5)$ to the x –axis is the distance between these two points: $\sqrt{(3-3)^2+(7-0)^2+(-5-0)^2}=\sqrt{74}\approx 8.60$.
(e) The point on the y –axis closest to $(3,7,-5)$ is $(0,7,0)$. The distance between these points is $\sqrt{(3-0)^2+(7-7)^2+(-5-0)^2}=\sqrt{34}\approx 5.83$.
(f) The point on the z –axis closest to $(3,7,-5)$ is $(0,0,-5)$. The distance between these points is $\sqrt{(3-0)^2+(7-0)^2+[-5-(-5)]^2}=\sqrt{58}\approx 7.62$.

11. An equation of the sphere with center $(1,-4,3)$ and radius 5 is $(x-1)^2+[y-(-4)]^2+(z-3)^2=5^2$ or $(x-1)^2+(y+4)^2+(z-3)^2=25$. The intersection of this sphere with the xz –plane is the set of points on the sphere whose y –coordinate is 0 . Putting $y=0$ into the equation, we have $(x-1)^2+4^2+(z-3)^2=25,$ $y=0$ or $(x-1)^2+(z-3)^2=9, y=0$, which represents a circle in the xz –plane with center $(1,0,3)$ and radius 3 .

12. An equation of the sphere with center $(6,5,-2)$ and radius $\sqrt{7}$ is $(x-6)^2+(y-5)^2+[z-(-2)]^2=(\sqrt{7})^2$ or $(x-6)^2+(y-5)+(z+2)^2=7$. The intersection of this sphere with the xy –plane is the set of points on the sphere whose z –coordinate is 0 . Putting $z=0$ into the equation, we have $(x-6)^2+(y-5)^2=3, z=0$ which represents a circle in the xy –plane with center $(6,5,0)$ and radius $\sqrt{3}$. To find the intersection with the xz –plane, we set $y=0$: $(x-6)^2+(z+2)^2=-18$. Since no points satisfy this equation, the sphere does not intersect the xz –plane. (Also note that the distance from the center of the sphere to the xz –plane is greater than the radius of the sphere.) Similarly, the sphere does not intersect the yz –plane since substituting $x=0$ into the equation gives $(y-5)^2+(z+2)^2=-29$.

13. The radius of the sphere is the distance between $(4,3,-1)$ and $(3,8,1)$:
$r=\sqrt{(3-4)^2+(8-3)^2+[1-(-1)]^2}=\sqrt{30}$. Thus, an equation of the sphere is $(x-3)^2+(y-8)^2+(z-1)^2=30$.

14. If the sphere passes through the origin, the radius of the sphere must be the distance from the origin to the point $(1,2,3)$: $r=\sqrt{(1-0)^2+(2-0)^2+(3-0)^2}=\sqrt{14}$. Then an equation of the sphere is $(x-1)^2+(y-2)^2+(z-3)^2=14$.

15. Completing squares in the equation $x^2+y^2+z^2-6x+4y-2z=11$ gives
$(x^2-6x+9)+(y^2+4y+4)+(z^2-2z+1)=11+9+4+1 \Rightarrow$
$(x-3)^2+(y+2)^2+(z-1)^2=25$ which we recognize as an equation of a sphere with center $(3,-2,1)$ and radius 5 .

16. Completing squares in the equation gives $(x^2-4x+4)+(y^2+2y+1)+z^2=0+4+1 \Rightarrow (x-2)^2+(y+1)^2+z^2=5$ which we recognize as an equation of a sphere with center $(2,-1,0)$ and radius $\sqrt{5}$.

17. Completing squares in the equation gives
$\left(x^2-x+\frac{1}{4}\right)+\left(y^2-y+\frac{1}{4}\right)+\left(z^2-z+\frac{1}{4}\right)=\frac{1}{4}+\frac{1}{4}+\frac{1}{4} \Rightarrow$
$\left(x-\frac{1}{2}\right)^2+\left(y-\frac{1}{2}\right)^2+\left(z-\frac{1}{2}\right)^2=\frac{3}{4}$ which we recognize as an equation of a sphere with center $\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)$ and radius $\sqrt{\frac{3}{4}}=\frac{\sqrt{3}}{2}$.

18. Completing squares in the equation gives $4\left(x^2-2x+1\right)+4\left(y^2+4y+4\right)+4z^2=1+4+16 \Rightarrow$
$4(x-1)^2+4(y+2)^2+4z^2=21 \Rightarrow (x-1)^2+(y+2)^2+z^2=\frac{21}{4}$, which we recognize as an equation of a sphere with center $(1,-2,0)$ and radius $\sqrt{\frac{21}{4}}=\frac{\sqrt{21}}{2}$.

19. **(a)** If the midpoint of the line segment from $P_1\left(x_1,y_1,z_1\right)$ to $P_2\left(x_2,y_2,z_2\right)$ is $Q=\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2},\frac{z_1+z_2}{2}\right)$, then the distances $\left|P_1Q\right|$ and $\left|QP_2\right|$ are equal, and each is half of $\left|P_1P_2\right|$. We verify that this is the case:

$$\left|P_1P_2\right| = \sqrt{\left(x_2-x_1\right)^2+\left(y_2-y_1\right)^2+\left(z_2-z_1\right)^2}$$

$$\begin{aligned}
\left|P_1Q\right| &= \sqrt{\left[\frac{1}{2}\left(x_1+x_2\right)-x_1\right]^2+\left[\frac{1}{2}\left(y_1+y_2\right)-y_1\right]^2+\left[\frac{1}{2}\left(z_1+z_2\right)-z_1\right]^2} \\
&= \sqrt{\left(\frac{1}{2}x_2-\frac{1}{2}x_1\right)^2+\left(\frac{1}{2}y_2-\frac{1}{2}y_1\right)^2+\left(\frac{1}{2}z_2-\frac{1}{2}z_1\right)^2} \\
&= \sqrt{\left(\frac{1}{2}\right)^2\left[\left(x_2-x_1\right)^2+\left(y_2-y_1\right)^2+\left(z_2-z_1\right)^2\right]} \\
&= \frac{1}{2}\sqrt{\left(x_2-x_1\right)^2+\left(y_2-y_1\right)^2+\left(z_2-z_1\right)^2} \\
&= \frac{1}{2}\left|P_1P_2\right|
\end{aligned}$$

$$\begin{aligned}
\left|QP_2\right| &= \sqrt{\left[x_2-\frac{1}{2}\left(x_1+x_2\right)\right]^2+\left[y_2-\frac{1}{2}\left(y_1+y_2\right)\right]^2+\left[z_2-\frac{1}{2}\left(z_1+z_2\right)\right]^2} \\
&= \sqrt{\left(\frac{1}{2}x_2-\frac{1}{2}x_1\right)^2+\left(\frac{1}{2}y_2-\frac{1}{2}y_1\right)^2+\left(\frac{1}{2}z_2-\frac{1}{2}z_1\right)^2} \\
&= \sqrt{\left(\frac{1}{2}\right)^2\left[\left(x_2-x_1\right)^2+\left(y_2-y_1\right)^2+\left(z_2-z_1\right)^2\right]} \\
&= \frac{1}{2}\sqrt{\left(x_2-x_1\right)^2+\left(y_2-y_1\right)^2+\left(z_2-z_1\right)^2} \\
&= \frac{1}{2}\left|P_1P_2\right|
\end{aligned}$$

So Q is indeed the midpoint of P_1P_2.

(b) By part (a), the midpoints of sides AB, BC and CA are $P_1\left(-\frac{1}{2},1,4\right)$, $P_2\left(1,\frac{1}{2},5\right)$ and $P_3\left(\frac{5}{2},\frac{3}{2},4\right)$. (Recall that a median of a triangle is a line segment from a vertex to the midpoint of the opposite side.) Then the lengths of the medians are:

$$\left|AP_2\right| = \sqrt{0^2+\left(\frac{1}{2}-2\right)^2+(5-3)^2} = \sqrt{\frac{9}{4}+4} = \sqrt{\frac{25}{4}} = \frac{5}{2}$$

$$\left|BP_3\right| = \sqrt{\left(\frac{5}{2}+2\right)^2+\left(\frac{3}{2}\right)^2+(4-5)^2} = \sqrt{\frac{81}{4}+\frac{9}{4}+1} = \sqrt{\frac{94}{4}} = \frac{1}{2}\sqrt{94}$$

$$\left|CP_1\right| = \sqrt{\left(-\frac{1}{2}-4\right)^2+(1-1)^2+(4-5)^2} = \sqrt{\frac{81}{4}+1} = \frac{1}{2}\sqrt{85}$$

20. By Exercise 19(a), the midpoint of the diameter (and thus the center of the sphere) is $C\,(3,2,7)$. The radius is half the diameter, so $r=\frac{1}{2}\sqrt{(4-2)^2+(3-1)^2+(10-4)^2}=\frac{1}{2}\sqrt{44}=\sqrt{11}$. Therefore an equation of the sphere is $(x-3)^2+(y-2)^2+(z-7)^2=11$.

21. **(a)** Since the sphere touches the xy –plane, its radius is the distance from its center, $(2,-3,6)$, to the xy –plane, namely 6 . Therefore $r=6$ and an equation of the sphere is $(x-2)^2+(y+3)^2+(z-6)^2=6^2=36$.

(b) The radius of this sphere is the distance from its center $(2,-3,6)$ to the yz –plane, which is 2 . Therefore, an equation is $(x-2)^2+(y+3)^2+(z-6)^2=4$.

(c) Here the radius is the distance from the center $(2,-3,6)$ to the xz –plane, which is 3 . Therefore, an equation is $(x-2)^2+(y+3)^2+(z-6)^2=9$.

22. The largest sphere contained in the first octant must have a radius equal to the minimum distance from the center $(5,4,9)$ to any of the three coordinate planes. The shortest such distance is to the xz –plane, a distance of 4 . Thus an equation of the sphere is $(x-5)^2+(y-4)^2+(z-9)^2=16$.

23. The equation $y=-4$ represents a plane parallel to the xz –plane and 4 units to the left of it.

24. The equation $x=10$ represents a plane parallel to the yz –plane and 10 units in front of it.

25. The inequality $x>3$ represents a half–space consisting of all points in front of the plane $x=3$.

26. The inequality $y\geq 0$ represents a half–space consisting of all points on or to the right of the xz –plane.

27. The inequality $0\leq z\leq 6$ represents all points on or between the horizontal planes $z=0$ (the xy –plane) and $z=6$.

28. The equation $y=z$ represents a plane perpendicular to the yz – plane and intersecting the yz –plane in the line $y=z$, $x=0$.

29. The inequality $x^2+y^2+z^2>1$ is equivalent to $\sqrt{x^2+y^2+z^2}>1$, so the region consists of those points whose distance from the origin is greater than 1 . This is the set of all points outside the sphere with radius 1 and center $(0,0,0)$.

30. The inequality $1\leq x^2+y^2+z^2\leq 25$ is equivalent to $1\leq\sqrt{x^2+y^2+z^2}\leq 5$, so the region consists of those points whose distance from the origin is at least 1 and at most 5 . This is the set of all points on or between the concentric spheres with radii 1 and 5 and center $(0,0,0)$.

31. Completing the square in z gives $x^2+y^2+(z^2-2z+1)<3+1$ or $x^2+y^2+(z-1)^2<4$, which is equivalent to $\sqrt{x^2+y^2+(z-1)^2}<2$. Thus the region consists of those points whose distance from the point $(0,0,1)$ is less than 2 . This is the set of all points inside the sphere with radius 2 and center $(0,0,1)$.

32. The equation $x^2+y^2=1$ represents the set of all points in R^3 where $x^2+y^2=1$, a surface that intersects the xy –plane in the circle $x^2+y^2=1,z=0$. Since z can vary, the surface is a circular cylinder of radius 1 . Thus, the equation represents the region consisting of all points on a circular cylinder of radius 1 with axis the z –axis.

33. Here $x^2+z^2\leq 9$ or equivalently $\sqrt{x^2+z^2}\leq 3$ which describes the set of all points in R^3 whose distance from the y –axis is at most 3 . Thus, the inequality represents the region consisting of all points on or inside a circular cylinder of radius 3 with axis the y –axis.

34. The equation $xyz=0$ is satisfied when any of x , y , or z is 0 . Thus, the equation represents the region consisting of all points on the three coordinate planes $x=0$, $y=0$, and $z=0$.

35. This describes all points with negative y –coordinates, that is, $y<0$.

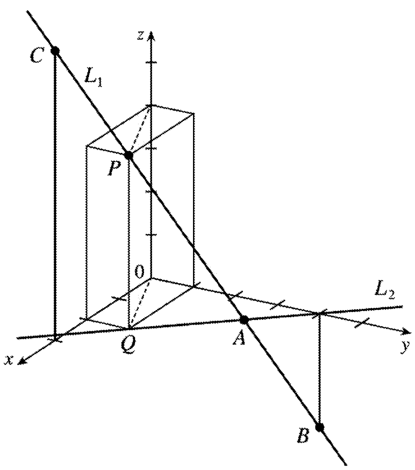
36. Because the box lies in the first quadrant, each point must comprise only nonnegative coordinates. So inequalities describing the region are $0\leq x\leq 1$, $0\leq y\leq 2$, $0\leq z\leq 3$.

37. This describes a region all of whose points have a distance to the origin which is greater than r , but smaller than R . So inequalities describing the region are $r<\sqrt{x^2+y^2+z^2}<R$, or $r^2<x^2+y^2+z^2<R^2$.

38. The solid sphere itself is represented by $\sqrt{x^2+y^2+z^2}\leq 2$. Since we want only the upper hemisphere, we restrict the z –coordinate to nonnegative values. Then inequalities describing the region are $\sqrt{x^2+y^2+z^2}\leq 2$, $z\geq 0$, or $x^2+y^2+z^2\leq 4$, $z\geq 0$.

39. **(a)** To find the x – and y –coordinates of the point P , we project it onto L_2 and project the resulting point Q onto the x – and y –axes. To find the z –coordinate, we project P onto either the xz –plane or the yz –plane (using our knowledge of its x – or
y –coordinate) and then project the resulting point onto the z –axis. (Or, we could draw a line parallel to QO from P to the z –axis.) The coordinates of P are $(2,1,4)$.

(b) A is the intersection of L_1 and L_2 , B is directly below the y –intercept of L_2 , and C is directly above the x –intercept of L_2 .

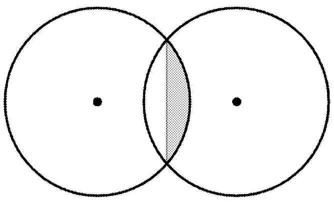


40. Let $P=(x,y,z)$. Then $2|PB|=|PA| \Leftrightarrow 4|PB|^2=|PA|^2 \Leftrightarrow$
$4\left((x-6)^2+(y-2)^2+(z+2)^2\right)=(x+1)^2+(y-5)^2+(z-3)^2 \Leftrightarrow$
$4\left(x^2-12x+36\right)-x^2-2x+4\left(y^2-4y+4\right)-y^2+10y+4\left(z^2+4z+4\right)-z^2+6z=35 \Leftrightarrow$
$3x^2-50x+3y^2-6y+3z^2+22z=35-144-16-16 \Leftrightarrow x^2-\frac{50}{3}x+y^2-2y+z^2+\frac{22}{3}z=-\frac{141}{3}$. By completing the square three times we get $\left(x-\frac{25}{3}\right)^2+(y-1)^2+\left(z+\frac{11}{3}\right)^2=\frac{332}{9}$, which is an equation of a sphere with center $\left(\frac{25}{3},1,-\frac{11}{3}\right)$ and radius $\frac{\sqrt{332}}{3}$.

41. We need to find a set of points $\{P(x,y,z) \mid |AP|=|BP|\}$.
$\sqrt{(x+1)^2+(y-5)^2+(z-3)^2}=\sqrt{(x-6)^2+(y-2)^2+(z+2)^2} \Rightarrow (x+1)^2+(y-5)+(z-3)^2=(x-6)^2+(y-2)^2+(z+2)^2 \Rightarrow$
$x^2+2x+1+y^2-10y+25+z^2-6z+9=x^2-12x+36+y^2-4y+4+z^2+4z+4 \Rightarrow 14x-6y-10z=9$. Thus the set of points is a plane perpendicular to the line segment joining A and B (since this plane must contain the perpendicular bisector of the line segment AB).

42. Completing the square three times in the first equation gives $(x+2)^2+(y-1)^2+(z+2)^2=2^2$, a sphere with center $(-2,1,2)$ and radius 2 . The second equation is that of a sphere with center $(0,0,0)$ and radius 2 . The distance between the centers of the spheres is $\sqrt{(-2-0)^2+(1-0)^2+(-2-0)^2}=\sqrt{4+1+4}=3$.

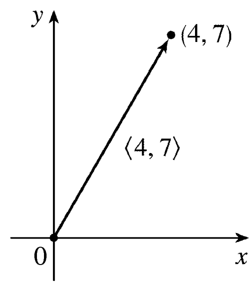
Since the spheres have the same radius, the volume inside both spheres is symmetrical about the plane containing the circle of intersection of the spheres. The distance from this plane to the center of the circles is $\frac{3}{2}$. So the region inside both spheres consists of two caps of spheres of height $h=2-\frac{3}{2}=\frac{1}{2}$.
From Exercise 6.2.49 [ET 6.2.49], the volume of a cap of a sphere is
$V=\frac{1}{3}\pi h^2(3r-h)=\frac{1}{3}\pi\left(\frac{1}{2}\right)^2\left(3\cdot 2-\frac{1}{2}\right)=\frac{11\pi}{24}$. So the total volume is $2\cdot\frac{11\pi}{24}=\frac{11\pi}{12}$.



1. **(a)** The cost of a theater ticket is a scalar, because it has only magnitude.
(b) The current in a river is a vector, because it has both magnitude (the speed of the current) and direction at any given location.
(c) If we assume that the initial path is linear, the initial flight path from Houston to Dallas is a vector, because it has both magnitude (distance) and direction.
(d) The population of the world is a scalar, because it has only magnitude.

2.

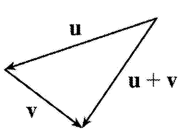
If the initial point of the vector $\langle 4,7\rangle$ is placed at the origin, then $\langle 4,7\rangle$ is the position vector of the point $(4,7)$.



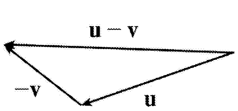
3. Vectors are equal when they share the same length and direction (but not necessarily location). Using the symmetry of the parallelogram as a guide, we see that $\overrightarrow{AB}=\overrightarrow{DC}$, $\overrightarrow{DA}=\overrightarrow{CB}$, $\overrightarrow{DE}=\overrightarrow{EB}$, and $\overrightarrow{EA}=\overrightarrow{CE}$.

4. **(a)** The initial point of $\overrightarrow{QR}$ is positioned at the terminal point of $\overrightarrow{PQ}$, so by the Triangle Law the sum $\overrightarrow{PQ}+\overrightarrow{QR}$ is the vector with initial point P and terminal point R , namely $\overrightarrow{PR}$.
(b) By the Triangle Law, $\overrightarrow{RP}+\overrightarrow{PS}$ is the vector with initial point R and terminal point S , namely $\overrightarrow{RS}$.
(c) First we consider $\overrightarrow{QS}-\overrightarrow{PS}$ as $\overrightarrow{QS}+\left(-\overrightarrow{PS}\right)$. Then since $-\overrightarrow{PS}$ has the same length as $\overrightarrow{PS}$ but points in the opposite direction, we have $-\overrightarrow{PS}=\overrightarrow{SP}$ and so $\overrightarrow{QS}-\overrightarrow{PS}=\overrightarrow{QS}+\overrightarrow{SP}=\overrightarrow{QP}$.
(d) We use the Triangle Law twice: $\overrightarrow{RS}+\overrightarrow{SP}+\overrightarrow{PQ}=\left(\overrightarrow{RS}+\overrightarrow{SP}\right)+\overrightarrow{PQ}=\overrightarrow{RP}+\overrightarrow{PQ}=\overrightarrow{RQ}$

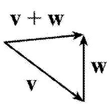
5. **(a)**



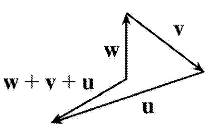
(b)



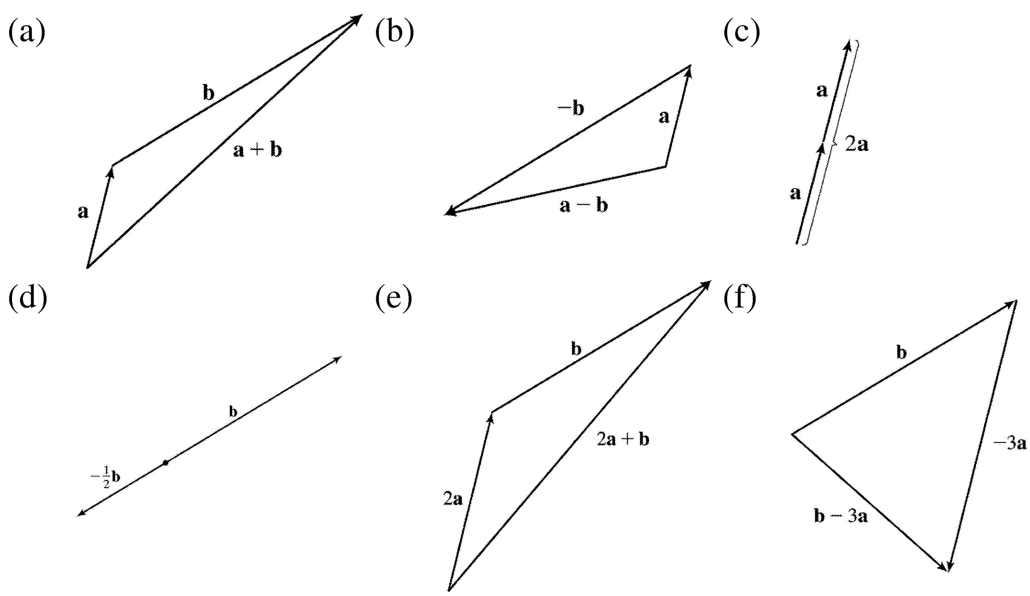
(c)



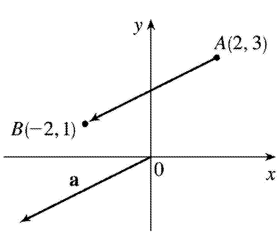
(d)



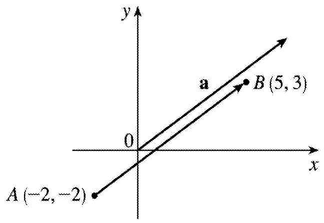
6.



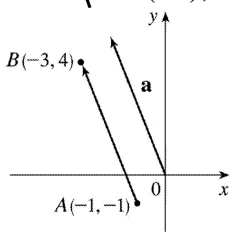
7. $\boldsymbol{a}=\langle -2-2, 1-3\rangle=\langle -4,-2\rangle$



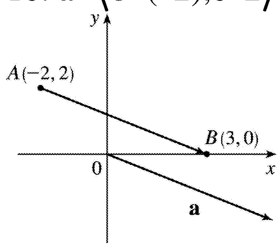
8. $\boldsymbol{a}=\langle 5-(-2), 3-(-2)\rangle=\langle 7,5\rangle$



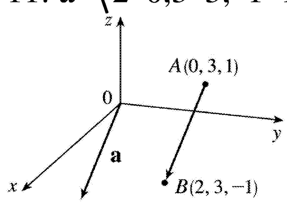
9. $\boldsymbol{a}=\left\langle -3-(-1),4-(-1)\right\rangle =\left\langle -2,5\right\rangle$



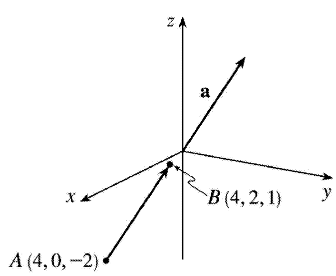
10. $\boldsymbol{a}=\left\langle 3-(-2),0-2\right\rangle =\left\langle 5,-2\right\rangle$



11. $\boldsymbol{a}=\left\langle 2-0,3-3,-1-1\right\rangle =\left\langle 2,0,-2\right\rangle$

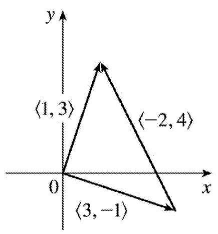


12. $\boldsymbol{a}=\left\langle 4-4,2-0,1-(-2)\right\rangle =\left\langle 0,2,3\right\rangle$



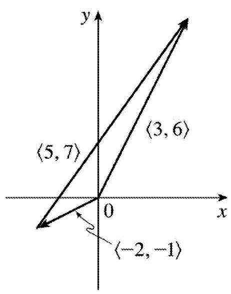
13.

$$\begin{aligned}\left\langle 3,-1\right\rangle +\left\langle -2,4\right\rangle &=\left\langle 3+(-2),-1+4\right\rangle \\ &=\left\langle 1,3\right\rangle\end{aligned}$$



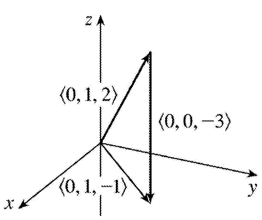
14.

$$\langle -2,-1\rangle+\langle 5,7\rangle =\langle -2+5,-1+7\rangle$$
$$=\langle 3,6\rangle$$



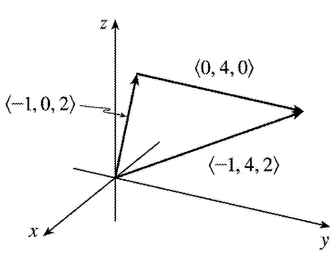
15.

$$\langle 0,1,2\rangle+\langle 0,0,-3\rangle =\langle 0+0,1+0,2+(-3)\rangle$$
$$=\langle 0,1,-1\rangle$$



16.

$$\langle -1,0,2\rangle+\langle 0,4,0\rangle =\langle -1+0,0+4,2+0\rangle$$
$$=\langle -1,4,2\rangle$$



17. $|\boldsymbol{a}|=\sqrt{(-4)^2+3^2}=\sqrt{25}=5$

$\boldsymbol{a}+\boldsymbol{b}=\langle -4+6,3+2\rangle=\langle 2,5\rangle$

$\boldsymbol{a}-\boldsymbol{b}=\langle -4-6,3-2\rangle=\langle -10,1\rangle$

$2\boldsymbol{a}=\langle 2(-4),2(3)\rangle=\langle -8,6\rangle$

$3\boldsymbol{a}+4\boldsymbol{b}=\langle -12,9\rangle+\langle 24,8\rangle=\langle 12,17\rangle$

18. $|\boldsymbol{a}|=\sqrt{2^2+(-3)^2}=\sqrt{13}$

$\boldsymbol{a}+\boldsymbol{b}=(2\boldsymbol{i}-3\boldsymbol{j})+(\boldsymbol{i}+5\boldsymbol{j})=3\boldsymbol{i}+2\boldsymbol{j}$

$\boldsymbol{a}-\boldsymbol{b}=(2\boldsymbol{i}-3\boldsymbol{j})-(\boldsymbol{i}+5\boldsymbol{j})=\boldsymbol{i}-8\boldsymbol{j}$

$2\boldsymbol{a}=2(2\boldsymbol{i}-3\boldsymbol{j})=4\boldsymbol{i}-6\boldsymbol{j}$

$$\begin{aligned}3\boldsymbol{a}+4\boldsymbol{b}&=3(2\boldsymbol{i}-3\boldsymbol{j})+4(\boldsymbol{i}+5\boldsymbol{j})\\&=6\boldsymbol{i}-9\boldsymbol{j}+4\boldsymbol{i}+20\boldsymbol{j}=10\boldsymbol{i}+11\boldsymbol{j}\end{aligned}$$

19. $|\boldsymbol{a}|=\sqrt{6^2+2^2+3^2}=\sqrt{49}=7$

$\boldsymbol{a}+\boldsymbol{b}=\langle 6+(-1),2+5,3+(-2)\rangle=\langle 5,7,1\rangle$

$$\begin{aligned}\boldsymbol{a}-\boldsymbol{b}&=\langle 6-(-1),2-5,3-(-2)\rangle\\&=\langle 7,-3,5\rangle\end{aligned}$$

$2\boldsymbol{a}=\langle 2(6),2(2),2(3)\rangle=\langle 12,4,6\rangle$

$$\begin{aligned}3\boldsymbol{a}+4\boldsymbol{b}&=\langle 18,6,9\rangle+\langle -4,20,-8\rangle\\&=\langle 14,26,1\rangle\end{aligned}$$

20. $|\boldsymbol{a}|=\sqrt{(-3)^2+(-4)^2+(-1)^2}=\sqrt{26}$

$$\begin{aligned}\boldsymbol{a}+\boldsymbol{b}&=\langle -3+6,-4+2,-1+(-3)\rangle\\&=\langle 3,-2,-4\rangle\end{aligned}$$

$$\begin{aligned}\boldsymbol{a}-\boldsymbol{b}&=\langle -3-6,-4-2,-1-(-3)\rangle\\&=\langle -9,-6,2\rangle\end{aligned}$$

$2\boldsymbol{a}=\langle 2(-3),2(-4),2(-1)\rangle=\langle -6,-8,-2\rangle$

$$\begin{aligned}3\boldsymbol{a}+4\boldsymbol{b}&=\langle -9,-12,-3\rangle+\langle 24,8,-12\rangle\\&=\langle 15,-4,-15\rangle\end{aligned}$$

21. $|\boldsymbol{a}|=\sqrt{1^2+(-2)^2+1^2}=\sqrt{6}$

$\boldsymbol{a}+\boldsymbol{b}=(\boldsymbol{i}-2\boldsymbol{j}+\boldsymbol{k})+(\boldsymbol{j}+2\boldsymbol{k})=\boldsymbol{i}-\boldsymbol{j}+3\boldsymbol{k}$
$\boldsymbol{a}-\boldsymbol{b}=(\boldsymbol{i}-2\boldsymbol{j}+\boldsymbol{k})-(\boldsymbol{j}+2\boldsymbol{k})=\boldsymbol{i}-3\boldsymbol{j}-\boldsymbol{k}$
$2\boldsymbol{a}=2(\boldsymbol{i}-2\boldsymbol{j}+\boldsymbol{k})=2\boldsymbol{i}-4\boldsymbol{j}+2\boldsymbol{k}$

$$\begin{aligned}3\boldsymbol{a}+4\boldsymbol{b}&=3(\boldsymbol{i}-2\boldsymbol{j}+\boldsymbol{k})+4(\boldsymbol{j}+2\boldsymbol{k})\\&=3\boldsymbol{i}-6\boldsymbol{j}+3\boldsymbol{k}+4\boldsymbol{j}+8\boldsymbol{k}\\&=3\boldsymbol{i}-2\boldsymbol{j}+11\boldsymbol{k}\end{aligned}$$

22. $|\boldsymbol{a}|=\sqrt{3^2+0^2+(-2)^2}=\sqrt{13}$
$\boldsymbol{a}+\boldsymbol{b}=(3\boldsymbol{i}-2\boldsymbol{k})+(\boldsymbol{i}-\boldsymbol{j}+\boldsymbol{k})=4\boldsymbol{i}-\boldsymbol{j}-\boldsymbol{k}$
$\boldsymbol{a}-\boldsymbol{b}=(3\boldsymbol{i}-2\boldsymbol{k})-(\boldsymbol{i}-\boldsymbol{j}+\boldsymbol{k})=2\boldsymbol{i}+\boldsymbol{j}-3\boldsymbol{k}$
$2\boldsymbol{a}=2(3\boldsymbol{i}-2\boldsymbol{k})=6\boldsymbol{i}-4\boldsymbol{k}$

$$\begin{aligned}3\boldsymbol{a}+4\boldsymbol{b}&=3(3\boldsymbol{i}-2\boldsymbol{k})+4(\boldsymbol{i}-\boldsymbol{j}+\boldsymbol{k})\\&=9\boldsymbol{i}-6\boldsymbol{k}+4\boldsymbol{i}-4\boldsymbol{j}+4\boldsymbol{k}\\&=13\boldsymbol{i}-4\boldsymbol{j}-2\boldsymbol{k}\end{aligned}$$

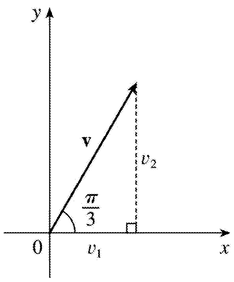
23. $|\langle 9,-5\rangle|=\sqrt{9^2+(-5)^2}=\sqrt{106}$, so $\boldsymbol{u}=\frac{1}{\sqrt{106}}\langle 9,-5\rangle=\left\langle \frac{9}{\sqrt{106}},\frac{-5}{\sqrt{106}}\right\rangle$.

24. $|12\boldsymbol{i}-5\boldsymbol{j}|=\sqrt{12^2+(-5)^2}=\sqrt{169}=13$, so $\boldsymbol{u}=\frac{1}{13}(12\boldsymbol{i}-5\boldsymbol{j})=\frac{12}{13}\boldsymbol{i}-\frac{5}{13}\boldsymbol{j}$.

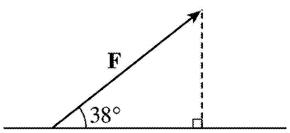
25. The vector $8\boldsymbol{i}-\boldsymbol{j}+4\boldsymbol{k}$ has length $|8\boldsymbol{i}-\boldsymbol{j}+4\boldsymbol{k}|=\sqrt{8^2+(-1)^2+4^2}=\sqrt{81}=9$, so by Equation 4 the unit vector with the same direction is $\frac{1}{9}(8\boldsymbol{i}-\boldsymbol{j}+4\boldsymbol{k})=\frac{8}{9}\boldsymbol{i}-\frac{1}{9}\boldsymbol{j}+\frac{4}{9}\boldsymbol{k}$.

26. $|\langle -2,4,2\rangle|=\sqrt{(-2)^2+4^2+2^2}=\sqrt{24}=2\sqrt{6}$, so a unit vector in the direction of $\langle -2,4,2\rangle$ is $\boldsymbol{u}=\frac{1}{2\sqrt{6}}\langle -2,4,2\rangle$. A vector in the same direction but with length 6 is
$6\boldsymbol{u}=6\cdot\frac{1}{2\sqrt{6}}\langle -2,4,2\rangle=\left\langle -\frac{6}{\sqrt{6}},\frac{12}{\sqrt{6}},\frac{6}{\sqrt{6}}\right\rangle$ or $\langle -\sqrt{6},2\sqrt{6},\sqrt{6}\rangle$.

27. From the figure, we see that the x –component of $\boldsymbol{v}$ is $v_1=|\boldsymbol{v}|\cos(\pi/3)=4\cdot\frac{1}{2}=2$ and the y –component is $v_2=|\boldsymbol{v}|\sin(\pi/3)=4\cdot\frac{\sqrt{3}}{2}=2\sqrt{3}$. Thus $\boldsymbol{v}=\langle v_1,v_2\rangle=\langle 2,2\sqrt{3}\rangle$.



28. From the figure, we see that the horizontal component of the force $\boldsymbol{F}$ is $|\boldsymbol{F}|\cos 38° = 50\cos 38° \approx 39.4$ N, and the vertical component is $|\boldsymbol{F}|\sin 38° = 50\sin 38° \approx 30.8$ N.



29. $\left|\boldsymbol{F}_1\right| = 10$ lb and $\left|\boldsymbol{F}_2\right| = 12$ lb.

$$\boldsymbol{F}_1 = -\left|\boldsymbol{F}_1\right|\cos 45°\boldsymbol{i} + \left|\boldsymbol{F}_1\right|\sin 45°\boldsymbol{j} = -10\cos 45°\boldsymbol{i} + 10\sin 45°\boldsymbol{j}$$
$$= -5\sqrt{2}\,\boldsymbol{i} + 5\sqrt{2}\,\boldsymbol{j}$$

$$\boldsymbol{F}_2 = \left|\boldsymbol{F}_2\right|\cos 30°\boldsymbol{i} + \left|\boldsymbol{F}_2\right|\sin 30°\boldsymbol{j} = 12\cos 30°\boldsymbol{i} + 12\sin 30°\boldsymbol{j} = 6\sqrt{3}\,\boldsymbol{i} + 6\boldsymbol{j}$$
$$\boldsymbol{F} = \boldsymbol{F}_1 + \boldsymbol{F}_2 = \left(6\sqrt{3} - 5\sqrt{2}\right)\boldsymbol{i} + \left(6 + 5\sqrt{2}\right)\boldsymbol{j} \approx 3.32\boldsymbol{i} + 13.07\boldsymbol{j}$$
$$|\boldsymbol{F}| \approx \sqrt{(3.32)^2 + (13.07)^2} \approx 13.5 \text{ lb. } \tan\theta = \frac{6+5\sqrt{2}}{6\sqrt{3}-5\sqrt{2}} \Rightarrow \theta = \tan^{-1}\frac{6+5\sqrt{2}}{6\sqrt{3}-5\sqrt{2}} \approx 76° .$$

30. Set up the coordinate axes so that north is the positive y –direction, and east is the positive x –direction. The wind is blowing at 50 km / h from the direction N45°W , so that its velocity vector is 50km/hS45°E , which can be written as $\boldsymbol{v}_{\text{wind}} = 50(\cos 45°\boldsymbol{i} - \sin 45°\boldsymbol{j})$. With respect to the still air, the velocity vector of the plane is 250km/hN60°E , or equivalently $\boldsymbol{v}_{\text{plane}} = 250(\cos 30°\boldsymbol{i} + \sin 30°\boldsymbol{j})$. The velocity of the plane relative to the ground is

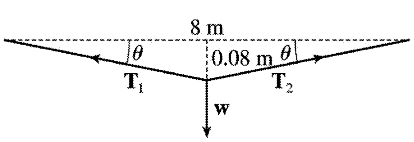
$$\boldsymbol{v} = \boldsymbol{v}_{\text{wind}} + \boldsymbol{v}_{\text{plane}} = (50\cos 45° + 250\cos 30°)\boldsymbol{i} + (-50\sin 45° + 250\sin 30°)\boldsymbol{j}$$
$$= \left(25\sqrt{2} + 125\sqrt{3}\right)\boldsymbol{i} + \left(125 - 25\sqrt{2}\right)\boldsymbol{j} \approx 251.9\boldsymbol{i} + 89.6\boldsymbol{j}$$

The ground speed is $|\boldsymbol{v}| \approx \sqrt{(251.9)^2 + (89.6)^2} \approx 267$km/h . The angle the velocity vector makes with the x –axis is $\theta \approx \tan^{-1}\left(\frac{89.6}{251.9}\right) \approx 20°$. Therefore, the true course of the plane is about N(90–20)°E=N70°E .

31. With respect to the water's surface, the woman's velocity is the vector sum of the velocity of the ship with respect to the water, and the woman's velocity with respect to the ship. If we let north be the positive y –direction, then $\boldsymbol{v}=\langle 0,22\rangle+\langle -3,0\rangle=\langle -3,22\rangle$. The woman's speed is $|\boldsymbol{v}|=\sqrt{9+484}\approx 22.2$ mi/h . The vector $\boldsymbol{v}$ makes an angle θ with the east, where $\theta=\tan^{-1}\left(\frac{22}{-3}\right)\approx 98°$. Therefore, the woman's direction is about N(98–90)°W=N8°W .

32. Call the two tensile forces $\boldsymbol{T}_3$ and $\boldsymbol{T}_5$, corresponding to the ropes of length 3 m and 5 m. In terms of vertical and horizontal components,
$\boldsymbol{T}_3=-\left|\boldsymbol{T}_3\right|\cos 52°\boldsymbol{i}+\left|\boldsymbol{T}_3\right|\sin 52°\boldsymbol{j}$ (1) and $\boldsymbol{T}_5=\left|\boldsymbol{T}_5\right|\cos 40°\boldsymbol{i}+\left|\boldsymbol{T}_5\right|\sin 40°\boldsymbol{j}$ (2)
The resultant of these forces, $\boldsymbol{T}_3+\boldsymbol{T}_5$, counterbalances the force of gravity acting on the decoration .
So $\boldsymbol{T}_3+\boldsymbol{T}_5=49\boldsymbol{j}$. Hence
$\boldsymbol{T}_3+\boldsymbol{T}_5=\left(-\left|\boldsymbol{T}_3\right|\cos 52°+\left|\boldsymbol{T}_5\right|\cos 40°\right)\boldsymbol{i}+\left(\left|\boldsymbol{T}_3\right|\sin 52°+\left|\boldsymbol{T}_5\right|\sin 40°\right)\boldsymbol{j}=49\boldsymbol{j}$. Thus
$-\left|\boldsymbol{T}_3\right|\cos 52°+\left|\boldsymbol{T}_5\right|\cos 40°=0$ and $\left|\boldsymbol{T}_3\right|\sin 52°+\left|\boldsymbol{T}_5\right|\sin 40°=49$.
From the first of these two equations $\left|\boldsymbol{T}_3\right|=\left|\boldsymbol{T}_5\right|\frac{\cos 40°}{\cos 52°}$. Substituting this into the second equation gives $\left|\boldsymbol{T}_5\right|=\frac{49}{\cos 40°\tan 52°+\sin 40°}\approx 30$ N. Therefore, $\left|\boldsymbol{T}_3\right|=\left|\boldsymbol{T}_5\right|\frac{\cos 40°}{\cos 52°}\approx 38$N . Finally, from (1) and (2), $\boldsymbol{T}_3\approx -23\boldsymbol{i}+30\boldsymbol{j}$, and $\boldsymbol{T}_5\approx 23\boldsymbol{i}+19\boldsymbol{j}$.

33. Let $\boldsymbol{T}_1$ and $\boldsymbol{T}_2$ represent the tension vectors in each side of the clothesline as shown in the figure.
$\boldsymbol{T}_1$ and $\boldsymbol{T}_2$ have equal vertical components and opposite horizontal components, so we can write

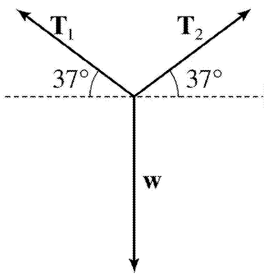


$\boldsymbol{T}_1=-a\boldsymbol{i}+b\boldsymbol{j}$ and $\boldsymbol{T}_2=a\boldsymbol{i}+b\boldsymbol{j}$ ($a,b>0$). By similar triangles, $\frac{b}{a}=\frac{0.08}{4}\Rightarrow a=50b$. The force due to gravity acting on the shirt has magnitude $0.8g\approx(0.8)(9.8)=7.84$ N, hence we have $\boldsymbol{w}=-7.84\boldsymbol{j}$. The resultant $\boldsymbol{T}_1+\boldsymbol{T}_2$ of the tensile forces counterbalances $\boldsymbol{w}$, so $\boldsymbol{T}_1+\boldsymbol{T}_2=-\boldsymbol{w}\Rightarrow(-a\boldsymbol{i}+b\boldsymbol{j})+(a\boldsymbol{i}+b\boldsymbol{j})=7.84\boldsymbol{j}$
$\Rightarrow(-50b\boldsymbol{i}+b\boldsymbol{j})+(50b\boldsymbol{i}+b\boldsymbol{j})=2b\boldsymbol{j}=7.84\boldsymbol{j}\Rightarrow b=\frac{7.84}{2}=3.92$ and $a=50b=196$. Thus the tensions are
$\boldsymbol{T}_1=-a\boldsymbol{i}+b\boldsymbol{j}=-196\boldsymbol{i}+3.92\boldsymbol{j}$ and $\boldsymbol{T}_2=a\boldsymbol{i}+b\boldsymbol{j}=196\boldsymbol{i}+3.92\boldsymbol{j}$.
Alternatively, we can find the value of θ and proceed as in Example 7.

34. We can consider the weight of the chain to be concentrated at its midpoint. The forces acting on the chain then are the tension vectors $\boldsymbol{T}_1$, $\boldsymbol{T}_2$ in each end of the chain and the weight $\boldsymbol{w}$, as shown in the figure. We know $\left|\boldsymbol{T}_1\right|=\left|\boldsymbol{T}_2\right|=25$ N so, in terms of vertical and horizontal components, we have

$\boldsymbol{T}_1 = -25\cos 37°\boldsymbol{i}+25\sin 37°\boldsymbol{j}$

$\boldsymbol{T}_2 = 25\cos 37°\boldsymbol{i}+25\sin 37°\boldsymbol{j}$



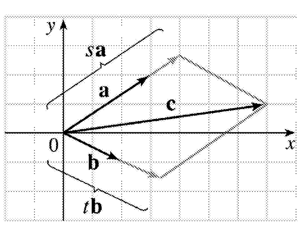
The resultant vector $\boldsymbol{T}_1+\boldsymbol{T}_2$ of the tensions counterbalances the weight $\boldsymbol{w}$, giving $\boldsymbol{T}_1+\boldsymbol{T}_2=-\boldsymbol{w}$. Since $\boldsymbol{w}=-|\boldsymbol{w}|\,\boldsymbol{j}$, we have $(-25\cos37°\boldsymbol{i}+25\sin37°\boldsymbol{j})+(25\cos37°\boldsymbol{i}+25\sin37°\boldsymbol{j})=|\boldsymbol{w}|\,\boldsymbol{j} \Rightarrow 50\sin37°\boldsymbol{j}=|\boldsymbol{w}|\,\boldsymbol{j} \Rightarrow$ $|\boldsymbol{w}|=50\sin37°\approx30.1$ So the weight is 30.1 N , and since $w=mg$, the mass is $\dfrac{30.1}{9.8}\approx3.07$kg .

35. By the Triangle Law, $\overrightarrow{AB}+\overrightarrow{BC}=\overrightarrow{AC}$. Then $\overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CA}=\overrightarrow{AC}+\overrightarrow{CA}$, but $\overrightarrow{AC}+\overrightarrow{CA}=\overrightarrow{AC}+\left(-\overrightarrow{AC}\right)=\boldsymbol{0}$. So $\overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CA}=\boldsymbol{0}$.

36. $\overrightarrow{AC}=\frac{1}{3}\overrightarrow{AB}$ and $\overrightarrow{BC}=\frac{2}{3}\overrightarrow{BA}$. $\boldsymbol{c}=\overrightarrow{OA}+\overrightarrow{AC}=\boldsymbol{a}+\frac{1}{3}\overrightarrow{AB}\Rightarrow\overrightarrow{AB}=3\boldsymbol{c}-3\boldsymbol{a}$.
$\boldsymbol{c}=\overrightarrow{OB}+\overrightarrow{BC}=\overrightarrow{OA}+\frac{2}{3}\overrightarrow{BA}\Rightarrow\overrightarrow{BA}=\frac{3}{2}\boldsymbol{c}-\frac{3}{2}\boldsymbol{b}$. $\overrightarrow{BA}=-\overrightarrow{AB}$, so $\frac{3}{2}\boldsymbol{c}-\frac{3}{2}\boldsymbol{b}=3\boldsymbol{a}-3\boldsymbol{c}\Leftrightarrow$
$\boldsymbol{c}+2\boldsymbol{c}=2\boldsymbol{a}+\boldsymbol{b}\Leftrightarrow\boldsymbol{c}=\frac{2}{3}\boldsymbol{a}+\frac{1}{3}\boldsymbol{b}$.

37.

(a), (b)

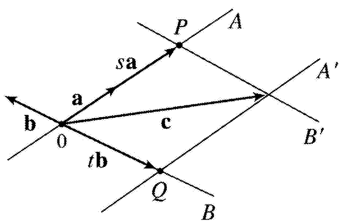


(c) From the sketch, we estimate that $s\approx1.3$ and $t\approx1.6$.

(d) $\boldsymbol{c}=s\,\boldsymbol{a}+t\,\boldsymbol{b}\Leftrightarrow7=3s+2t$ and $1=2s-t$.

Solving these equations gives $s=\frac{9}{7}$ and $t=\frac{11}{7}$.

38. Draw $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ emanating from the origin. Extend $\boldsymbol{a}$ and $\boldsymbol{b}$ to form lines A and B , and draw lines A' and B' parallel to these two lines through the terminal point of $\boldsymbol{c}$.



Since $\boldsymbol{a}$ and $\boldsymbol{b}$ are not parallel, A and B' must meet (at P), and A' and B must also meet (at Q). Now we see that $\overrightarrow{OP}+\overrightarrow{OQ}=\boldsymbol{c}$, so if $s=\frac{|\overrightarrow{OP}|}{|\boldsymbol{a}|}$ (or its negative, if $\boldsymbol{a}$ points in the direction opposite $\overrightarrow{OP}$) and $t=\frac{|\overrightarrow{OQ}|}{|\boldsymbol{b}|}$ (or its negative, as in the diagram), then $c=s\boldsymbol{a}+t\boldsymbol{b}$, as required.

Argument using components: Since $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ all lie in the same plane, we can consider them to be vectors in two dimensions. Let $\boldsymbol{a}=\langle a_1,a_2\rangle$, $\boldsymbol{b}=\langle b_1,b_2\rangle$, and $\boldsymbol{c}=\langle c_1,c_2\rangle$. We need $sa_1+tb_1=c_1$ and $sa_2+tb_2=c_2$. Multiplying the first equation by a_2 and the second by a_1 and subtracting, we get $t=\frac{c_2a_1-c_1a_2}{b_2a_1-b_1a_2}$. Similarly $s=\frac{b_2c_1-b_1c_2}{b_2a_1-b_1a_2}$. Since $\boldsymbol{a}\neq\boldsymbol{0}$ and $\boldsymbol{b}\neq\boldsymbol{0}$ and $\boldsymbol{a}$ is not a scalar multiple of $\boldsymbol{b}$, the denominator is not zero.

39. $|\boldsymbol{r}-\boldsymbol{r}_0|$ is the distance between the points (x,y,z) and (x_0,y_0,z_0) , so the set of points is a sphere with radius 1 and center (x_0,y_0,z_0) .

Alternate method: $|\boldsymbol{r}-\boldsymbol{r}_0|=1\Leftrightarrow\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}=1\Leftrightarrow(x-x_0)^2+(y-y_0)^2+(z-z_0)^2=1$, which is the equation of a sphere with radius 1 and center (x_0,y_0,z_0) .

40. Let P_1 and P_2 be the points with position vectors $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ respectively. Then $|\boldsymbol{r}-\boldsymbol{r}_1|+|\boldsymbol{r}-\boldsymbol{r}_2|$ is

the sum of the distances from (x,y) to P_1 and P_2. Since this sum is constant, the set of points (x,y) represents an ellipse with foci P_1 and P_2. The condition $k>\left|\boldsymbol{r}_1-\boldsymbol{r}_2\right|$ assures us that the ellipse is not degenerate.

41.

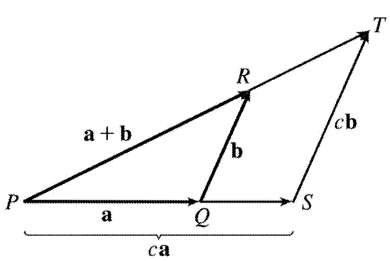
$$\begin{aligned}\boldsymbol{a}+(\boldsymbol{b}+\boldsymbol{c}) &=\left\langle a_1,a_2\right\rangle+\left(\left\langle b_1,b_2\right\rangle+\left\langle c_1,c_2\right\rangle\right)=\left\langle a_1,a_2\right\rangle+\left\langle b_1+c_1,b_2+c_2\right\rangle\\ &=\left\langle a_1+b_1+c_1,a_2+b_2+c_2\right\rangle=\left\langle \left(a_1+b_1\right)+c_1,\left(a_2+b_2\right)+c_2\right\rangle\\ &=\left\langle a_1+b_1,a_2+b_2\right\rangle+\left\langle c_1,c_2\right\rangle=\left(\left\langle a_1,a_2\right\rangle+\left\langle b_1,b_2\right\rangle\right)+\left\langle c_1,c_2\right\rangle\\ &=(\boldsymbol{a}+\boldsymbol{b})+\boldsymbol{c}\end{aligned}$$

42. *Algebraically:*

$$\begin{aligned}c(\boldsymbol{a}+\boldsymbol{b}) &=c\left(\left\langle a_1,a_2,a_3\right\rangle+\left\langle b_1,b_2,b_3\right\rangle\right)=c\left\langle a_1+b_1,a_2+b_2,a_3+b_3\right\rangle\\ &=\left\langle c\left(a_1+b_1\right),c\left(a_2+b_2\right),c\left(a_3+b_3\right)\right\rangle=\left\langle ca_1+cb_1,ca_2+cb_2,ca_3+cb_3\right\rangle\\ &=\left\langle ca_1,ca_2,ca_3\right\rangle+\left\langle cb_1,cb_2,cb_3\right\rangle=c\,\boldsymbol{a}+c\,\boldsymbol{b}\end{aligned}$$

Geometrically:

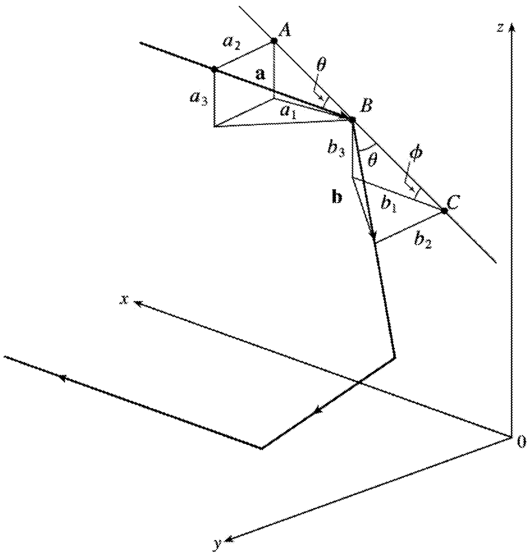
According to the Triangle Law, if $\boldsymbol{a}=\overrightarrow{PQ}$ and $\boldsymbol{b}=\overrightarrow{QR}$, then $\boldsymbol{a}+\boldsymbol{b}=\overrightarrow{PR}$. Construct triangle PST as shown so that $\overrightarrow{PS}=c\,\boldsymbol{a}$ and $\overrightarrow{ST}=c\,\boldsymbol{b}$. (We have drawn the case where $c>1$.) By the Triangle Law, $\overrightarrow{PT}=c\,\boldsymbol{a}+c\,\boldsymbol{b}$. But triangle PQR and triangle PST are similar triangles because $c\,\boldsymbol{b}$ is parallel to $\boldsymbol{b}$. Therefore, $\overrightarrow{PR}$ and $\overrightarrow{PT}$ are parallel and, in fact, $\overrightarrow{PT}=c\overrightarrow{PR}$. Thus, $c\,\boldsymbol{a}+c\,\boldsymbol{b}=c(\boldsymbol{a}+\boldsymbol{b})$.



43. Consider triangle ABC, where D and E are the midpoints of AB and BC. We know that $\overrightarrow{AB}+\overrightarrow{BC}=\overrightarrow{AC}$ (1) and $\overrightarrow{DB}+\overrightarrow{BE}=\overrightarrow{DE}$ (2). However, $\overrightarrow{DB}=\frac{1}{2}\overrightarrow{AB}$, and $\overrightarrow{BE}=\frac{1}{2}\overrightarrow{BC}$. Substituting these expressions for $\overrightarrow{DB}$ and $\overrightarrow{BE}$ into (2) gives

$\frac{1}{2}\overrightarrow{AB}+\frac{1}{2}\overrightarrow{BC}=\overrightarrow{DE}$. Comparing this with (1) gives $\overrightarrow{DE}=\frac{1}{2}\overrightarrow{AC}$. Therefore $\overrightarrow{AC}$ and $\overrightarrow{DE}$ are parallel and $\left|\overrightarrow{DE}\right|=\frac{1}{2}\left|\overrightarrow{AC}\right|$.

44.



The question states that the light ray strikes all three mirrors, so it is not parallel to any of them and $a_1\neq 0$, $a_2\neq 0$ and $a_3\neq 0$. Let $\boldsymbol{b}=\left\langle b_1,b_2,b_3\right\rangle$, as in the diagram. We can let $|\boldsymbol{b}|=|\boldsymbol{a}|$, since only its direction is important. Then $\frac{\left|b_2\right|}{|\boldsymbol{b}|}=\sin\theta=\frac{\left|a_2\right|}{|\boldsymbol{a}|}\Rightarrow\left|b_2\right|=\left|a_2\right|$. From the diagram $b_2\boldsymbol{j}$ and $a_2\boldsymbol{j}$ point in opposite directions, so $b_2=-a_2$. $|AB|=|BC|$, so $\left|b_3\right|=\sin\phi\ |BC|=\sin\phi\ |AB|=\left|a_3\right|$, and $\left|b_1\right|=\cos\phi\ |BC|=\cos\phi\ |AB|=\left|a_1\right|$.

$b_3\boldsymbol{k}$ and $a_3\boldsymbol{k}$ have the same direction, as do $b_1\boldsymbol{i}$ and $a_1\boldsymbol{i}$, so $\boldsymbol{b}=\left\langle a_1,-a_2,a_3\right\rangle$. When the ray hits the other mirrors, similar arguments show that these reflections will reverse the signs of the other two coordinates, so the final reflected ray will be $\left\langle -a_1,-a_2,-a_3\right\rangle=-\boldsymbol{a}$, which is parallel to $\boldsymbol{a}$.

1. **(a)** $\boldsymbol{a}\cdot\boldsymbol{b}$ is a scalar, and the dot product is defined only for vectors, so $(\boldsymbol{a}\cdot\boldsymbol{b})\cdot\boldsymbol{c}$ has no meaning.
(b) $(\boldsymbol{a}\cdot\boldsymbol{b})\boldsymbol{c}$ is a scalar multiple of a vector, so it does have meaning.
(c) Both $|\boldsymbol{a}|$ and $\boldsymbol{b}\cdot\boldsymbol{c}$ are scalars, so $|\boldsymbol{a}|\,(\boldsymbol{b}\cdot\boldsymbol{c})$ is an ordinary product of real numbers, and has meaning.
(d) Both $\boldsymbol{a}$ and $\boldsymbol{b}+\boldsymbol{c}$ are vectors, so the dot product $\boldsymbol{a}\cdot(\boldsymbol{b}+\boldsymbol{c})$ has meaning.
(e) $\boldsymbol{a}\cdot\boldsymbol{b}$ is a scalar, but $\boldsymbol{c}$ is a vector, and so the two quantities cannot be added and this expression has no meaning.
(f) $|\boldsymbol{a}|$ is a scalar, and the dot product is defined only for vectors, so $|\boldsymbol{a}|\cdot(\boldsymbol{b}+\boldsymbol{c})$ has no meaning.

2. Let the vectors be $\boldsymbol{a}$ and $\boldsymbol{b}$. Then by Theorem 3, $\boldsymbol{a}\cdot\boldsymbol{b}=|\boldsymbol{a}|\,|\boldsymbol{b}|\cos\theta=(6)\left(\frac{1}{3}\right)\cos\frac{\pi}{4}=\frac{6}{3\sqrt{2}}=\sqrt{2}$.

3. $\boldsymbol{a}\cdot\boldsymbol{b}=\langle 4,-1\rangle\cdot\langle 3,6\rangle=(4)\,(3)+(-1)\,(6)=6$

4. $\boldsymbol{a}\cdot\boldsymbol{b}=\left\langle \frac{1}{2},4\right\rangle\cdot\langle -8,-3\rangle=\left(\frac{1}{2}\right)(-8)+(4)(-3)=-16$

5. $\boldsymbol{a}\cdot\boldsymbol{b}=\langle 5,0,-2\rangle\cdot\langle 3,-1,10\rangle=(5)(3)+(0)(-1)+(-2)(10)=-5$

6. $\boldsymbol{a}\cdot\boldsymbol{b}=\langle s,2s,3s\rangle\cdot\langle t,-t,5t\rangle=(s)(t)+(2s)(-t)+(3s)(5t)=st-2st+15st=14st$

7. $\boldsymbol{a}\cdot\boldsymbol{b}=(\boldsymbol{i}-2\boldsymbol{j}+3\boldsymbol{k})\cdot(5\boldsymbol{i}+9\boldsymbol{k})=(1)(5)+(-2)(0)+(3)(9)=32$

8. $\boldsymbol{a}\cdot\boldsymbol{b}=(4\boldsymbol{j}-3\boldsymbol{k})\cdot(2\boldsymbol{i}+4\boldsymbol{j}+6\boldsymbol{k})=(0)(2)+(4)(4)+(-3)(6)=-2$

9. Use Theorem 3: $\boldsymbol{a}\cdot\boldsymbol{b}=|\boldsymbol{a}|\,|\boldsymbol{b}|\cos\theta=(12)(15)\cos\frac{\pi}{6}=180\cdot\frac{\sqrt{3}}{2}=90\sqrt{3}\approx 155.9$

10. Use Theorem 3: $\boldsymbol{a}\cdot\boldsymbol{b}=|\boldsymbol{a}|\,|\boldsymbol{b}|\cos\theta=(4)(10)\cos 120^{\circ}=40\left(-\frac{1}{2}\right)=-20$

11. $\boldsymbol{u}$,$\boldsymbol{v}$, and $\boldsymbol{w}$ are all unit vectors, so the triangle is an equilateral triangle. Thus the angle between $\boldsymbol{u}$ and $\boldsymbol{v}$ is 60° and $\boldsymbol{u}\cdot\boldsymbol{v}=|\boldsymbol{u}|\,|\boldsymbol{v}|\cos 60^{\circ}=(1)(1)\left(\frac{1}{2}\right)=\frac{1}{2}$. If $\boldsymbol{w}$ is moved so it has the same initial point as $\boldsymbol{u}$, we can see that the angle between them is 120° and we have
$\boldsymbol{u}\cdot\boldsymbol{w}=|\boldsymbol{u}|\,|\boldsymbol{w}|\cos 120^{\circ}=(1)(1)\left(-\frac{1}{2}\right)=-\frac{1}{2}$.

12. $\boldsymbol{u}$ is a unit vector, so $\boldsymbol{w}$ is also a unit vector, and $|\boldsymbol{v}|$ can be determined by examining the right triangle formed by $\boldsymbol{u}$ and $\boldsymbol{v}$. Since the angle between $\boldsymbol{u}$ and $\boldsymbol{v}$ is 45° , we have

$|\boldsymbol{v}|=|\boldsymbol{u}|\cos 45°=\frac{\sqrt{2}}{2}$. Then $\boldsymbol{u}\cdot\boldsymbol{v}=|\boldsymbol{u}|\,|\boldsymbol{v}|\cos 45°=(1)\left(\frac{\sqrt{2}}{2}\right)\frac{\sqrt{2}}{2}=\frac{1}{2}$. Since $\boldsymbol{u}$ and $\boldsymbol{w}$ are orthogonal, $\boldsymbol{u}\cdot\boldsymbol{w}=0$.

13. **(a)** $\boldsymbol{i}\cdot\boldsymbol{j}=\langle 1,0,0\rangle\cdot\langle 0,1,0\rangle=(1)(0)+(0)(1)+(0)(0)=0$. Similarly $\boldsymbol{j}\cdot\boldsymbol{k}=(0)(0)+(1)(0)+(0)(1)=0$ and $\boldsymbol{k}\cdot\boldsymbol{i}=(0)(1)+(0)(0)+(1)(0)=0$.
Another method: Because $\boldsymbol{i}$, $\boldsymbol{j}$, and $\boldsymbol{k}$ are mutually perpendicular, the cosine factor in each dot product (see Theorem 3) is $\cos\frac{\pi}{2}=0$.

(b) By Property 1 of the dot product, $\boldsymbol{i}\cdot\boldsymbol{i}=|\boldsymbol{i}|^2=1^2=1$ since $\boldsymbol{i}$ is a unit vector. Similarly, $\boldsymbol{j}\cdot\boldsymbol{j}=|\boldsymbol{j}|^2=1$ and $\boldsymbol{k}\cdot\boldsymbol{k}=|\boldsymbol{k}|^2=1$.

14. The dot product $\boldsymbol{A}\cdot\boldsymbol{P}$ is

$\langle a,b,c\rangle\cdot\langle 2,1.5,1\rangle=a(2)+b(1.5)+c(1)$
= (number of hamburgers sold) (price per hamburger)
+ (number of hot dogs sold) (price per hot dog)
+ (number of soft drinks sold) (price per soft drink)

so it is equal to the vendor's total revenue for that day.

15. $|\boldsymbol{a}|=\sqrt{3^2+4^2}=5$, $|\boldsymbol{b}|=\sqrt{5^2+12^2}=13$, and $\boldsymbol{a}\cdot\boldsymbol{b}=(3)(5)+(4)(12)=63$. Using Corollary 6, we have $\cos\theta=\frac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{a}|\,|\boldsymbol{b}|}=\frac{63}{5\cdot 13}=\frac{63}{65}$. So the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$ is $\theta=\cos^{-1}\left(\frac{63}{65}\right)\approx 14^\circ$.

16. $|\boldsymbol{a}|=\sqrt{(\sqrt{3})^2+1^2}=2$, $|\boldsymbol{b}|=\sqrt{0+25}=5$, and $\boldsymbol{a}\cdot\boldsymbol{b}=(\sqrt{3})(0)+(1)(5)=5$. Using Corollary 6, we have $\cos\theta=\frac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{a}|\,|\boldsymbol{b}|}=\frac{5}{2\cdot 5}=\frac{1}{2}$ and the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$ is $\cos^{-1}\left(\frac{1}{2}\right)=60°$.

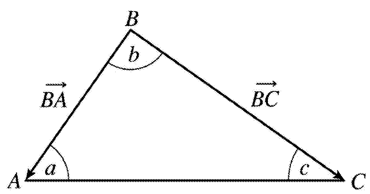
17. $|\boldsymbol{a}|=\sqrt{1^2+2^2+3^2}=\sqrt{14}$, $|\boldsymbol{b}|=\sqrt{4^2+0^2+(-1)^2}=\sqrt{17}$, and $\boldsymbol{a}\cdot\boldsymbol{b}=(1)(4)+(2)(0)+(3)(-1)=1$. Then $\cos\theta=\frac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{a}|\,|\boldsymbol{b}|}=\frac{1}{\sqrt{14}\cdot\sqrt{17}}=\frac{1}{\sqrt{238}}$ and the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$ is $\theta=\cos^{-1}\left(\frac{1}{\sqrt{238}}\right)\approx 86^\circ$.

18. $|\boldsymbol{a}|=\sqrt{6^2+(-3)^2+2^2}=7$, $|\boldsymbol{b}|=\sqrt{2^2+1^2+(-2)^2}=3$, and $\boldsymbol{a}\cdot\boldsymbol{b}=(6)(2)+(-3)(1)+(2)(-2)=5$. Then $\cos\theta=\frac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{a}|\,|\boldsymbol{b}|}=\frac{5}{7\cdot 3}=\frac{5}{21}$ and $\theta=\cos^{-1}\left(\frac{5}{21}\right)\approx 76^\circ$.

19. $|\boldsymbol{a}|=\sqrt{0^2+1^2+1^2}=\sqrt{2}$, $|\boldsymbol{b}|=\sqrt{1^2+2^2+(-3)^2}=\sqrt{14}$, and $\boldsymbol{a}\cdot\boldsymbol{b}=(0)(1)+(1)(2)+(1)(-3)=-1$. Then $\cos\theta=\frac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{a}|\,|\boldsymbol{b}|}=\frac{-1}{\sqrt{2}\cdot\sqrt{14}}=\frac{-1}{2\sqrt{7}}$ and $\theta=\cos^{-1}\left(-\frac{1}{2\sqrt{7}}\right)\approx 101^\circ$.

20. $|\boldsymbol{a}|=\sqrt{2^2+(-1)^2+1^2}=\sqrt{6}$, $|\boldsymbol{b}|=\sqrt{3^2+2^2+(-1)^2}=\sqrt{14}$, and $\boldsymbol{a}\cdot\boldsymbol{b}=(2)(3)+(-1)(2)+(1)(-1)=3$. Then $\cos\theta=\dfrac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{a}|\,|\boldsymbol{b}|}=\dfrac{3}{\sqrt{6}\cdot\sqrt{14}}=\dfrac{3}{2\sqrt{21}}$ and $\theta=\cos^{-1}\left(\dfrac{3}{2\sqrt{21}}\right)\approx 71^\circ$.

21. Let a , b , and c be the angles at vertices A , B , and C respectively. Then a is the angle between vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$, b is the angle between vectors $\overrightarrow{BA}$ and $\overrightarrow{BC}$, and c is the angle between vectors $\overrightarrow{CA}$ and $\overrightarrow{CB}$.



Thus $\cos a=\dfrac{\overrightarrow{AB}\cdot\overrightarrow{AC}}{|\overrightarrow{AB}|\,|\overrightarrow{AC}|}=\dfrac{\langle 2,6\rangle\cdot\langle -2,4\rangle}{\sqrt{2^2+6^2}\sqrt{(-2)^2+4^2}}=\dfrac{1}{\sqrt{40}\sqrt{20}}(-4+24)=\dfrac{20}{\sqrt{800}}=\dfrac{\sqrt{2}}{2}$ and

$a=\cos^{-1}\left(\dfrac{\sqrt{2}}{2}\right)=45^\circ$. Similarly,

$\cos b=\dfrac{\overrightarrow{BA}\cdot\overrightarrow{BC}}{|\overrightarrow{BA}|\,|\overrightarrow{BC}|}=\dfrac{\langle -2,-6\rangle\cdot\langle -4,-2\rangle}{\sqrt{4+36}\sqrt{16+4}}=\dfrac{1}{\sqrt{40}\sqrt{20}}(8+12)=\dfrac{20}{\sqrt{800}}=\dfrac{\sqrt{2}}{2}$ so $b=\cos^{-1}\left(\dfrac{\sqrt{2}}{2}\right)=45^\circ$ and $c=180^\circ-(45^\circ+45^\circ)=90^\circ$.

Alternate solution: Apply the Law of Cosines three times as follows: $\cos a=\dfrac{|\overrightarrow{BC}|^2-|\overrightarrow{AB}|^2-|\overrightarrow{AC}|^2}{2|\overrightarrow{AB}|\,|\overrightarrow{AC}|}$,

$\cos b=\dfrac{|\overrightarrow{AC}|^2-|\overrightarrow{AB}|^2-|\overrightarrow{BC}|^2}{2|\overrightarrow{AB}|\,|\overrightarrow{BC}|}$, and $\cos c=\dfrac{|\overrightarrow{AB}|^2-|\overrightarrow{AC}|^2-|\overrightarrow{BC}|^2}{2|\overrightarrow{AC}|\,|\overrightarrow{BC}|}$.

22. As in Exercise 21, let d , e , and f be the angles at vertices D , E , and F . Then d is the angle between vectors $\overrightarrow{DE}$ and $\overrightarrow{DF}$, e is the angle between vectors $\overrightarrow{ED}$ and $\overrightarrow{EF}$, and f is the angle between vectors $\overrightarrow{FD}$ and $\overrightarrow{FE}$. Thus

$\cos d=\dfrac{\overrightarrow{DE}\cdot\overrightarrow{DF}}{|\overrightarrow{DE}|\,|\overrightarrow{DF}|}=\dfrac{\langle -2,3,2\rangle\cdot\langle 1,1,-2\rangle}{\sqrt{(-2)^2+3^2+2^2}\sqrt{1^2+1^2+(-2)^2}}=\dfrac{1}{\sqrt{17}\sqrt{6}}(-2+3-4)=-\dfrac{3}{\sqrt{102}}$

and

$d=\cos^{-1}\left(-\frac{3}{\sqrt{102}}\right)\approx 107°$. Similarly,

$\cos e=\frac{\overrightarrow{ED}\cdot\overrightarrow{EF}}{|\overrightarrow{ED}|\,|\overrightarrow{EF}|}=\frac{\langle 2,-3,-2\rangle\cdot\langle 3,-2,-4\rangle}{\sqrt{4+9+4}\sqrt{9+4+16}}=\frac{1}{\sqrt{17}\sqrt{29}}(6+6+8)=\frac{20}{\sqrt{493}}$ so

$e=\cos^{-1}\left(\frac{20}{\sqrt{493}}\right)\approx 26°$ and $f\approx 180°-(107°+26°)=47°$.

Alternate solution: Apply the Law of Cosines three times as follows: $\cos d=\frac{|\overrightarrow{EF}|^2-|\overrightarrow{DE}|^2-|\overrightarrow{DF}|^2}{2|\overrightarrow{DE}|\,|\overrightarrow{DF}|}$,

$\cos e=\frac{|\overrightarrow{DF}|^2-|\overrightarrow{DE}|^2-|\overrightarrow{EF}|^2}{2|\overrightarrow{DE}|\,|\overrightarrow{EF}|}$, and $\cos f=\frac{|\overrightarrow{DE}|^2-|\overrightarrow{DF}|^2-|\overrightarrow{EF}|^2}{2|\overrightarrow{DF}|\,|\overrightarrow{EF}|}$.

23. **(a)** $\boldsymbol{a}\cdot\boldsymbol{b}=(-5)(6)+(3)(-8)+(7)(2)=-40\neq 0$, so $\boldsymbol{a}$ and $\boldsymbol{b}$ are not orthogonal. Also, since $\boldsymbol{a}$ is not a scalar multiple of $\boldsymbol{b}$, $\boldsymbol{a}$ and $\boldsymbol{b}$ are not parallel.
(b) $\boldsymbol{a}\cdot\boldsymbol{b}=(4)(-3)+(6)(2)=0$, so $\boldsymbol{a}$ and $\boldsymbol{b}$ are orthogonal (and not parallel).
(c) $\boldsymbol{a}\cdot\boldsymbol{b}=(-1)(3)+(2)(4)+(5)(-1)=0$, so $\boldsymbol{a}$ and $\boldsymbol{b}$ are orthogonal (and not parallel).
(d) Because $\boldsymbol{a}=-\frac{2}{3}\boldsymbol{b}$, $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel.

24. **(a)** Because $\boldsymbol{u}=-\frac{3}{4}\boldsymbol{v}$, $\boldsymbol{u}$ and $\boldsymbol{v}$ are parallel vectors (and thus not orthogonal).
(b) $\boldsymbol{u}\cdot\boldsymbol{v}=(1)(2)+(-1)(-1)+(2)(1)=5\neq 0$, so $\boldsymbol{u}$ and $\boldsymbol{v}$ are not orthogonal. Also, $\boldsymbol{u}$ is not a scalar multiple of $\boldsymbol{v}$, so $\boldsymbol{u}$ and $\boldsymbol{v}$ are not parallel.
(c) $\boldsymbol{u}\cdot\boldsymbol{v}=(a)(-b)+(b)(a)+(c)(0)=-ab+ab+0=0$, so $\boldsymbol{u}$ and $\boldsymbol{v}$ are orthogonal (and not parallel).

25. $\overrightarrow{QP}=\langle -1,-3,2\rangle$, $\overrightarrow{QR}=\langle 4,-2,-1\rangle$, and $\overrightarrow{QP}\cdot\overrightarrow{QR}=-4+6-2=0$. Thus $\overrightarrow{QP}$ and $\overrightarrow{QR}$ are orthogonal, so the angle of the triangle at vertex Q is a right angle.

26. $\langle -6,b,2\rangle$ and $\langle b,b^2,b\rangle$ are orthogonal when $\langle -6,b,2\rangle\cdot\langle b,b^2,b\rangle=0\Leftrightarrow(-6)(b)+(b)(b^2)+(2)(b)=0$ $\Leftrightarrow b^3-4b=0\Leftrightarrow b(b+2)(b-2)=0\Leftrightarrow b=0$ or $b=\pm 2$.

27. Let $\boldsymbol{a}=a_1\boldsymbol{i}+a_2\boldsymbol{j}+a_3\boldsymbol{k}$ be a vector orthogonal to both $\boldsymbol{i}+\boldsymbol{j}$ and $\boldsymbol{i}+\boldsymbol{k}$. Then $\boldsymbol{a}\cdot(\boldsymbol{i}+\boldsymbol{j})=0\Leftrightarrow a_1+a_2=0$ and $\boldsymbol{a}\cdot(\boldsymbol{i}+\boldsymbol{k})=0\Leftrightarrow a_1+a_3=0$, so $a_1=-a_2=-a_3$. Furthermore $\boldsymbol{a}$ is to be a unit vector, so $1=a_1^2+a_2^2+a_3^2=3a_1^2$ implies $a_1=\pm\frac{1}{\sqrt{3}}$. Thus $\boldsymbol{a}=\frac{1}{\sqrt{3}}\boldsymbol{i}-\frac{1}{\sqrt{3}}\boldsymbol{j}-\frac{1}{\sqrt{3}}\boldsymbol{k}$ and $\boldsymbol{a}=-\frac{1}{\sqrt{3}}\boldsymbol{i}+\frac{1}{\sqrt{3}}\boldsymbol{j}+\frac{1}{\sqrt{3}}\boldsymbol{k}$ are two such unit vectors.

28. Let

$\boldsymbol{u}=\langle a,b\rangle$ be a unit vector. By Theorem 3 we need $\boldsymbol{u}\cdot\boldsymbol{v}=|\boldsymbol{u}|\ |\boldsymbol{v}|\cos 60°\Leftrightarrow 3a+4b=(1)(5)\frac{1}{2}\Leftrightarrow b=\frac{5}{8}-\frac{3}{4}a$. Since $\boldsymbol{u}$ is a unit vector, $|\boldsymbol{u}|=\sqrt{a^2+b^2}=1\Leftrightarrow a^2+b^2=1\Leftrightarrow a^2+\left(\frac{5}{8}-\frac{3}{4}a\right)^2=1\Leftrightarrow\frac{25}{16}a^2-\frac{15}{16}a+\frac{25}{64}=1\Leftrightarrow$ $100a^2-60a-39=0$. By the quadratic formula, $a=\frac{-(-60)\pm\sqrt{(-60)^2-4(100)(-39)}}{2(100)}=\frac{60\pm\sqrt{19{,}200}}{200}=\frac{3\pm4\sqrt{3}}{10}$. If $a=\frac{3+4\sqrt{3}}{10}$ then $b=\frac{5}{8}-\frac{3}{4}\left(\frac{3+4\sqrt{3}}{10}\right)=\frac{4-3\sqrt{3}}{10}$, and if $a=\frac{3-4\sqrt{3}}{10}$ then $b=\frac{5}{8}-\frac{3}{4}\left(\frac{3-4\sqrt{3}}{10}\right)=\frac{4+3\sqrt{3}}{10}$. Thus the two unit vectors are $\left\langle\frac{3+4\sqrt{3}}{10},\frac{4-3\sqrt{3}}{10}\right\rangle\approx\langle 0.9928,-0.1196\rangle$ and $\left\langle\frac{3-4\sqrt{3}}{10},\frac{4+3\sqrt{3}}{10}\right\rangle\approx\langle -0.3928,0.9196\rangle$.

29. Since $|\langle 3,4,5\rangle|=\sqrt{9+16+25}=\sqrt{50}=5\sqrt{2}$, using Equations 8 and 9 we have $\cos\alpha=\frac{3}{5\sqrt{2}}$, $\cos\beta=\frac{4}{5\sqrt{2}}$, and $\cos\gamma=\frac{5}{5\sqrt{2}}=\frac{1}{\sqrt{2}}$. The direction angles are given by $\alpha=\cos^{-1}\left(\frac{3}{5\sqrt{2}}\right)\approx 65°$, $\beta=\cos^{-1}\left(\frac{4}{5\sqrt{2}}\right)\approx 56°$, and $\gamma=\cos^{-1}\left(\frac{1}{\sqrt{2}}\right)=45°$.

30. Since $|\langle 1,-2,-1\rangle|=\sqrt{1+4+1}=\sqrt{6}$, using Equations 8 and 9 we have $\cos\alpha=\frac{1}{\sqrt{6}}$, $\cos\beta=\frac{-2}{\sqrt{6}}$, and $\cos\gamma=\frac{-1}{\sqrt{6}}$. The direction angles are given by $\alpha=\cos^{-1}\left(\frac{1}{\sqrt{6}}\right)\approx 66°$, $\beta=\cos^{-1}\left(-\frac{2}{\sqrt{6}}\right)\approx 145°$, and $\gamma=\cos^{-1}\left(-\frac{1}{\sqrt{6}}\right)\approx 114°$.

31. Since $|2\boldsymbol{i}+3\boldsymbol{j}-6\boldsymbol{k}|=\sqrt{4+9+36}=\sqrt{49}=7$, Equations 8 and 9 give $\cos\alpha=\frac{2}{7}$, $\cos\beta=\frac{3}{7}$, and $\cos\gamma=\frac{-6}{7}$, while $\alpha=\cos^{-1}\left(\frac{2}{7}\right)\approx 73°$, $\beta=\cos^{-1}\left(\frac{3}{7}\right)\approx 65°$, and $\gamma=\cos^{-1}\left(-\frac{6}{7}\right)\approx 149°$.

32. Since $|2\boldsymbol{i}-\boldsymbol{j}+2\boldsymbol{k}|=\sqrt{4+1+4}=\sqrt{9}=3$, Equations 8 and 9 give $\cos\alpha=\frac{2}{3}$, $\cos\beta=\frac{-1}{3}$, and $\cos\gamma=\frac{2}{3}$, while $\alpha=\gamma=\cos^{-1}\left(\frac{2}{3}\right)\approx 48°$ and $\beta=\cos^{-1}\left(-\frac{1}{3}\right)\approx 109°$.

33. $|\langle c,c,c\rangle|=\sqrt{c^2+c^2+c^2}=\sqrt{3}\,c$ (since $c>0$), so $\cos\alpha=\cos\beta=\cos\gamma=\dfrac{c}{\sqrt{3}\,c}=\dfrac{1}{\sqrt{3}}$ and $\alpha=\beta=\gamma=\cos^{-1}\left(\dfrac{1}{\sqrt{3}}\right)\approx 55°$.

34. Since $\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1$, $\cos^2\gamma=1-\cos^2\alpha-\cos^2\beta=1-\cos^2\left(\frac{\pi}{4}\right)-\cos^2\left(\frac{\pi}{3}\right)=1-\left(\frac{1}{\sqrt{2}}\right)^2-\left(\frac{1}{2}\right)^2=\frac{1}{4}$. Thus $\cos\gamma=\pm\dfrac{1}{2}$ and $\gamma=\dfrac{\pi}{3}$ or $\gamma=\dfrac{2\pi}{3}$.

35. $|\boldsymbol{a}|=\sqrt{3^2+(-4)^2}=5$. The scalar projection of $\boldsymbol{b}$ onto $\boldsymbol{a}$ is $\operatorname{comp}_{\boldsymbol{a}}\boldsymbol{b}=\dfrac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{a}|}=\dfrac{3\cdot 5+(-4)\cdot 0}{5}=3$ and the vector projection of $\boldsymbol{b}$ onto $\boldsymbol{a}$ is $\operatorname{proj}_{\boldsymbol{a}}\boldsymbol{b}=\left(\dfrac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{a}|}\right)\dfrac{\boldsymbol{a}}{|\boldsymbol{a}|}=3\cdot\dfrac{1}{5}\langle 3,-4\rangle=\left\langle\dfrac{9}{5},-\dfrac{12}{5}\right\rangle$.

36. $|\boldsymbol{a}|=\sqrt{1^2+2^2}=\sqrt{5}$, so the scalar projection of $\boldsymbol{b}$ onto $\boldsymbol{a}$ is $\operatorname{comp}_{\boldsymbol{a}}\boldsymbol{b}=\dfrac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{a}|}=\dfrac{1(-4)+2\cdot 1}{\sqrt{5}}=-\dfrac{2}{\sqrt{5}}$ and the vector projection of $\boldsymbol{b}$ onto $\boldsymbol{a}$ is $\operatorname{proj}_{\boldsymbol{a}}\boldsymbol{b}=\left(\dfrac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{a}|}\right)\dfrac{\boldsymbol{a}}{|\boldsymbol{a}|}=-\dfrac{2}{\sqrt{5}}\cdot\dfrac{1}{\sqrt{5}}\langle 1,2\rangle=\left\langle-\dfrac{2}{5},-\dfrac{4}{5}\right\rangle$.

37. $|\boldsymbol{a}|=\sqrt{16+4+0}=2\sqrt{5}$ so the scalar projection of $\boldsymbol{b}$ onto $\boldsymbol{a}$ is $\operatorname{comp}_{\boldsymbol{a}}\boldsymbol{b}=\dfrac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{a}|}=\dfrac{1}{2\sqrt{5}}(4+2+0)=\dfrac{3}{\sqrt{5}}$. The vector projection of $\boldsymbol{b}$ onto $\boldsymbol{a}$ is $\operatorname{proj}_{\boldsymbol{a}}\boldsymbol{b}=\dfrac{3}{\sqrt{5}}\dfrac{\boldsymbol{a}}{|\boldsymbol{a}|}=\dfrac{3}{\sqrt{5}}\cdot\dfrac{1}{2\sqrt{5}}\langle 4,2,0\rangle=\dfrac{1}{5}\langle 6,3,0\rangle=\left\langle\dfrac{6}{5},\dfrac{3}{5},0\right\rangle$.

38. $|\boldsymbol{a}|=\sqrt{1+4+4}=3$ so the scalar projection of $\boldsymbol{b}$ onto $\boldsymbol{a}$ is $\operatorname{comp}_{\boldsymbol{a}}\boldsymbol{b}=\dfrac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{a}|}=\dfrac{-3+(-6)+8}{3}=-\dfrac{1}{3}$, while the vector projection is $\operatorname{proj}_{\boldsymbol{a}}\boldsymbol{b}=-\dfrac{1}{3}\dfrac{\boldsymbol{a}}{|\boldsymbol{a}|}=-\dfrac{1}{3}\cdot\dfrac{\langle -1,-2,2\rangle}{3}=\left\langle\dfrac{1}{9},\dfrac{2}{9},-\dfrac{2}{9}\right\rangle$.

39. $|\boldsymbol{a}|=\sqrt{1+0+1}=\sqrt{2}$ so the scalar projection of $\boldsymbol{b}$ onto $\boldsymbol{a}$ is $\operatorname{comp}_{\boldsymbol{a}}\boldsymbol{b}=\dfrac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{a}|}=\dfrac{1}{\sqrt{2}}(1+0+0)=\dfrac{1}{\sqrt{2}}$ while the vector projection of $\boldsymbol{b}$ onto $\boldsymbol{a}$ is $\operatorname{proj}_{\boldsymbol{a}}\boldsymbol{b}=\dfrac{1}{\sqrt{2}}\dfrac{\boldsymbol{a}}{|\boldsymbol{a}|}=\dfrac{1}{\sqrt{2}}\cdot\dfrac{1}{\sqrt{2}}(\boldsymbol{i}+\boldsymbol{k})=\dfrac{1}{2}(\boldsymbol{i}+\boldsymbol{k})$.

40. $|\boldsymbol{a}|=\sqrt{4+9+1}=\sqrt{14}$, so the scalar projection of $\boldsymbol{b}$ onto $\boldsymbol{a}$ is

$\text{comp}_{\boldsymbol{a}}\boldsymbol{b}=\dfrac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{a}|}=\dfrac{2-18-2}{\sqrt{14}}=-\dfrac{18}{\sqrt{14}}$ while the vector projection of $\boldsymbol{b}$ onto $\boldsymbol{a}$ is

$\text{proj}_{\boldsymbol{a}}\boldsymbol{b}=-\dfrac{18}{\sqrt{14}}\dfrac{\boldsymbol{a}}{|\boldsymbol{a}|}=-\dfrac{18}{\sqrt{14}}\cdot\dfrac{2\boldsymbol{i}-3\boldsymbol{j}+\boldsymbol{k}}{\sqrt{14}}=-\dfrac{9}{7}\left(2\boldsymbol{i}-3\boldsymbol{j}+\boldsymbol{k}\right)$.

41.

$$\left(\text{orth}_{\boldsymbol{a}}\boldsymbol{b}\right)\cdot\boldsymbol{a}=\left(\boldsymbol{b}-\text{proj}_{\boldsymbol{a}}\boldsymbol{b}\right)\cdot\boldsymbol{a}=\boldsymbol{b}\cdot\boldsymbol{a}-\left(\text{proj}_{\boldsymbol{a}}\boldsymbol{b}\right)\cdot\boldsymbol{a}=\boldsymbol{b}\cdot\boldsymbol{a}-\frac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{a}|^2}\,\boldsymbol{a}\cdot\boldsymbol{a}$$

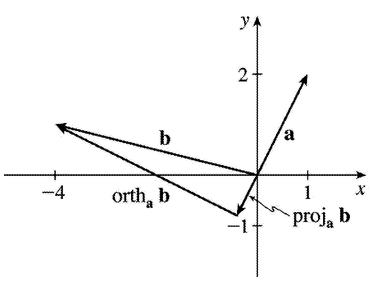
$$=\boldsymbol{b}\cdot\boldsymbol{a}-\frac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{a}|^2}\,|\boldsymbol{a}|^2=\boldsymbol{b}\cdot\boldsymbol{a}-\boldsymbol{a}\cdot\boldsymbol{b}=0$$

So they are orthogonal by (7).

42. Using the formula in Exercise 41 and the result of Exercise 36,
we have

$$\text{orth}_{\boldsymbol{a}}\boldsymbol{b}=\boldsymbol{b}-\text{proj}_{\boldsymbol{a}}\boldsymbol{b}=\langle -4,1\rangle-\left\langle -\frac{2}{5},-\frac{4}{5}\right\rangle$$

$$=\left\langle -\frac{18}{5},\frac{9}{5}\right\rangle$$



43. $\text{comp}_{\boldsymbol{a}}\boldsymbol{b}=\dfrac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{a}|}=2\Leftrightarrow\boldsymbol{a}\cdot\boldsymbol{b}=2|\boldsymbol{a}|=2\sqrt{10}$. If $\boldsymbol{b}=\left\langle b_1,b_2,b_3\right\rangle$, then we need $3b_1+0b_2-1b_3=2\sqrt{10}$. One possible solution is obtained by taking $b_1=0$, $b_2=0$, $b_3=-2\sqrt{10}$.
In general, $\boldsymbol{b}=\left\langle s,t,3s-2\sqrt{10}\right\rangle$, s , $t\in R$.

44. **(a)** $\text{comp}_{\boldsymbol{a}}\boldsymbol{b}=\text{comp}_{\boldsymbol{b}}\boldsymbol{a}\Leftrightarrow\dfrac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{a}|}=\dfrac{\boldsymbol{b}\cdot\boldsymbol{a}}{|\boldsymbol{b}|}\Leftrightarrow\dfrac{1}{|\boldsymbol{a}|}=\dfrac{1}{|\boldsymbol{b}|}$ or $\boldsymbol{a}\cdot\boldsymbol{b}=0\Leftrightarrow|\boldsymbol{b}|=|\boldsymbol{a}|$ or $\boldsymbol{a}\cdot\boldsymbol{b}=0$.

That is, if $\boldsymbol{a}$ and $\boldsymbol{b}$ are orthogonal or if they have the same length.

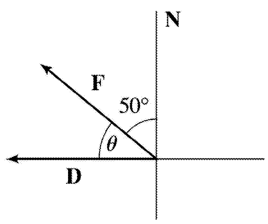
(b) $\text{proj}_{\boldsymbol{a}}\boldsymbol{b}=\text{proj}_{\boldsymbol{b}}\boldsymbol{a}\Leftrightarrow \dfrac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{a}|^2}\boldsymbol{a}=\dfrac{\boldsymbol{b}\cdot\boldsymbol{a}}{|\boldsymbol{b}|^2}\boldsymbol{b}\Leftrightarrow \boldsymbol{a}\cdot\boldsymbol{b}=0$ or $\dfrac{\boldsymbol{a}}{|\boldsymbol{a}|^2}=\dfrac{\boldsymbol{b}}{|\boldsymbol{b}|^2}$. But $\dfrac{\boldsymbol{a}}{|\boldsymbol{a}|^2}=\dfrac{\boldsymbol{b}}{|\boldsymbol{b}|^2}\Rightarrow\dfrac{|\boldsymbol{a}|}{|\boldsymbol{a}|^2}=\dfrac{|\boldsymbol{b}|}{|\boldsymbol{b}|^2}\Rightarrow$ $|\boldsymbol{a}|=|\boldsymbol{b}|$. Substituting this into the previous equation gives $\boldsymbol{a}=\boldsymbol{b}$.

So $\text{proj}_{\boldsymbol{a}}\boldsymbol{b}=\text{proj}_{\boldsymbol{b}}\boldsymbol{a}\Leftrightarrow$ $\boldsymbol{a}$ and $\boldsymbol{b}$ are orthogonal, or they are equal.

45. Here $\boldsymbol{D}=(4-2)\boldsymbol{i}+(9-3)\boldsymbol{j}+(15-0)\boldsymbol{k}=2\boldsymbol{i}+6\boldsymbol{j}+15\boldsymbol{k}$ so by Equation 12 we have $W=\boldsymbol{F}\cdot\boldsymbol{D}=20+108-90=38$ joules.

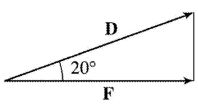
46.

$$W=|\boldsymbol{F}|\,|\boldsymbol{D}|\cos\theta=(20)(4)\cos 40^\circ \approx 61 \text{ ft-lb}$$



47.

$$W=|\boldsymbol{F}|\,|\boldsymbol{D}|\cos\theta=(25)(10)\cos 20^\circ \approx 235 \text{ ft-lb}$$

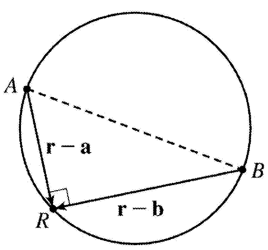


48. Here $|\boldsymbol{D}|=100$ m, $|\boldsymbol{F}|=50$ N, and $\theta=30^\circ$. Thus $W=|\boldsymbol{F}|\,|\boldsymbol{D}|\cos\theta=(50)(100)\left(\dfrac{\sqrt{3}}{2}\right)=2500\sqrt{3}$ joules.

49. First note that $\boldsymbol{n}=\langle a,b\rangle$ is perpendicular to the line, because if $Q_1=(a_1,b_1)$ and $Q_2=(a_2,b_2)$ lie on the line, then $\boldsymbol{n}\cdot\overrightarrow{Q_1Q_2}=aa_2-aa_1+bb_2-bb_1=0$, since $aa_2+bb_2=-c=aa_1+bb_1$ from the equation of the line. Let $P_2=(x_2,y_2)$ lie on the line. Then the distance from P_1 to the line is the absolute value of the scalar projection of

$\overrightarrow{P_1P_2}$ onto $\boldsymbol{n}$. $\text{comp}_{\boldsymbol{n}}\left(\overrightarrow{P_1P_2}\right)=\dfrac{\left|\boldsymbol{n}\cdot\left\langle x_2-x_1,y_2-y_1\right\rangle\right|}{|\boldsymbol{n}|}=\dfrac{\left|ax_2-ax_1+by_2-by_1\right|}{\sqrt{a^2+b^2}}=\dfrac{\left|ax_1+by_1+c\right|}{\sqrt{a^2+b^2}}$ since $ax_2+by_2=-c$. The required distance is $\dfrac{|3\cdot-2+-4\cdot 3+5|}{\sqrt{3^2+4^2}}=\dfrac{13}{5}$.

50. $(\boldsymbol{r}-\boldsymbol{a})\cdot(\boldsymbol{r}-\boldsymbol{b})=0$ implies that the vectors $\boldsymbol{r}-\boldsymbol{a}$ and $\boldsymbol{r}-\boldsymbol{b}$ are orthogonal. From the diagram (in which A , B and R are the terminal points of the vectors), we see that this implies that R lies on a sphere whose diameter is the line from A to B . The center of this circle is the midpoint of AB , that is,
$\frac{1}{2}(\boldsymbol{a}+\boldsymbol{b})=\left\langle \frac{1}{2}\left(a_1+b_1\right),\frac{1}{2}\left(a_2+b_2\right),\frac{1}{2}\left(a_3+b_3\right)\right\rangle$, and its radius is
$\frac{1}{2}|\boldsymbol{a}-\boldsymbol{b}|=\frac{1}{2}\sqrt{\left(a_1-b_1\right)^2+\left(a_2-b_2\right)^2+\left(a_3-b_3\right)^2}$.



Or: Expand the given equation, substitute $\boldsymbol{r}\cdot\boldsymbol{r}=x^2+y^2+z^2$ and complete the squares.

51. For convenience, consider the unit cube positioned so that its back left corner is at the origin, and its edges lie along the coordinate axes. The diagonal of the cube that begins at the origin and ends at $(1,1,1)$ has vector representation $\langle 1,1,1\rangle$. The angle θ between this vector and the vector of the edge which also begins at the origin and runs along the x –axis is given by
$\cos\theta=\dfrac{<1,1,1>\cdot<1,0,0>}{|<1,1,1>|\,|<1,0,0>|}=\dfrac{1}{\sqrt{3}}\Rightarrow\theta=\cos^{-1}\left(\dfrac{1}{\sqrt{3}}\right)\approx 55°$.

52. Consider a cube with sides of unit length, wholly within the first octant and with edges along each of the three coordinate axes. $\boldsymbol{i}+\boldsymbol{j}+\boldsymbol{k}$ and $\boldsymbol{i}+\boldsymbol{j}$ are vector representations of a diagonal of the cube and a diagonal of one of its faces. If θ is the angle between these diagonals, then
$\cos\theta=\dfrac{(\boldsymbol{i}+\boldsymbol{j}+\boldsymbol{k})\cdot(\boldsymbol{i}+\boldsymbol{j})}{|\boldsymbol{i}+\boldsymbol{j}+\boldsymbol{k}|\,|\boldsymbol{i}+\boldsymbol{j}|}=\dfrac{1+1}{\sqrt{3}\sqrt{2}}=\sqrt{\dfrac{2}{3}}\Rightarrow\theta=\cos^{-1}\sqrt{\dfrac{2}{3}}\approx 35°$.

53. Consider the H–C–H combination consisting of the sole carbon atom and the two hydrogen atoms that are at $(1,0,0)$ and $(0,1,0)$ (or any H–C–H combination, for that matter). Vector representations of

the line segments emanating from the carbon atom and extending to these two hydrogen atoms are $\left\langle 1-\frac{1}{2},0-\frac{1}{2},0-\frac{1}{2}\right\rangle=\left\langle \frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right\rangle$ and $\left\langle 0-\frac{1}{2},1-\frac{1}{2},0-\frac{1}{2}\right\rangle=\left\langle -\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right\rangle$. The bond angle, θ , is therefore given by

$$\cos\theta=\frac{\left\langle \frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right\rangle\cdot\left\langle -\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right\rangle}{\left|\left\langle \frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right\rangle\right|\left|\left\langle -\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right\rangle\right|}=\frac{-\frac{1}{4}-\frac{1}{4}+\frac{1}{4}}{\sqrt{\frac{3}{4}}\sqrt{\frac{3}{4}}}=-\frac{1}{3}\Rightarrow$$

$$\theta=\cos^{-1}\left(-\frac{1}{3}\right)\approx 109.5^\circ .$$

54. Let α be the angle between $\boldsymbol{a}$ and $\boldsymbol{c}$ and β be the angle between $\boldsymbol{c}$ and $\boldsymbol{b}$. We need to show that $\alpha=\beta$. Now $\cos\alpha=\frac{\boldsymbol{a}\cdot\boldsymbol{c}}{|\boldsymbol{a}|\,|\boldsymbol{c}|}=\frac{\boldsymbol{a}\cdot|\boldsymbol{a}|\,\boldsymbol{b}+\boldsymbol{a}\cdot|\boldsymbol{b}|\,\boldsymbol{a}}{|\boldsymbol{a}|\,|\boldsymbol{c}|}=\frac{|\boldsymbol{a}|\,\boldsymbol{a}\cdot\boldsymbol{b}+|\boldsymbol{a}|^2|\boldsymbol{b}|}{|\boldsymbol{a}|\,|\boldsymbol{c}|}=\frac{\boldsymbol{a}\cdot\boldsymbol{b}+|\boldsymbol{a}|\,|\boldsymbol{b}|}{|\boldsymbol{c}|}$. Similarly, $\cos\beta=\frac{\boldsymbol{b}\cdot\boldsymbol{c}}{|\boldsymbol{b}|\,|\boldsymbol{c}|}=\frac{|\boldsymbol{a}|\,|\boldsymbol{b}|+\boldsymbol{b}\cdot\boldsymbol{a}}{|\boldsymbol{c}|}$. Thus $\cos\alpha=\cos\beta$. However $0^\circ\le\alpha\le 180^\circ$ and $0^\circ\le\beta\le 180^\circ$, so $\alpha=\beta$ and $\boldsymbol{c}$ bisects the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$.

55. Let $\boldsymbol{a}=\left\langle a_1,a_2,a_3\right\rangle$ and $\boldsymbol{b}=\left\langle b_1,b_2,b_3\right\rangle$.

Property 2:

$$\begin{aligned}\boldsymbol{a}\cdot\boldsymbol{b}&=\left\langle a_1,a_2,a_3\right\rangle\cdot\left\langle b_1,b_2,b_3\right\rangle=a_1b_1+a_2b_2+a_3b_3\\&=b_1a_1+b_2a_2+b_3a_3=\left\langle b_1,b_2,b_3\right\rangle\cdot\left\langle a_1,a_2,a_3\right\rangle=\boldsymbol{b}\cdot\boldsymbol{a}\end{aligned}$$

Property 4:

$$\begin{aligned}(c\,\boldsymbol{a})\cdot\boldsymbol{b}&=\left\langle ca_1,ca_2,ca_3\right\rangle\cdot\left\langle b_1,b_2,b_3\right\rangle=\left(ca_1\right)b_1+\left(ca_2\right)b_2+\left(ca_3\right)b_3\\&=c\left(a_1b_1+a_2b_2+a_3b_3\right)=c\,(\boldsymbol{a}\cdot\boldsymbol{b})=a_1\left(cb_1\right)+a_2\left(cb_2\right)+a_3\left(cb_3\right)\\&=\left\langle a_1,a_2,a_3\right\rangle\cdot\left\langle cb_1,cb_2,cb_3\right\rangle=\boldsymbol{a}\cdot(c\,\boldsymbol{b})\end{aligned}$$

Property 5: $\boldsymbol{0}\cdot\boldsymbol{a}=\left\langle 0,0,0\right\rangle\cdot\left\langle a_1,a_2,a_3\right\rangle=(0)\left(a_1\right)+(0)\left(a_2\right)+(0)\left(a_3\right)=0$

56. Let the figure be called quadrilateral $ABCD$. The diagonals can be represented by $\overrightarrow{AC}$ and $\overrightarrow{BD}$. $\overrightarrow{AC}=\overrightarrow{AB}+\overrightarrow{BC}$ and $\overrightarrow{BD}=\overrightarrow{BC}+\overrightarrow{CD}=\overrightarrow{BC}-\overrightarrow{DC}=\overrightarrow{BC}-\overrightarrow{AB}$ (Since opposite sides of the object are of the same

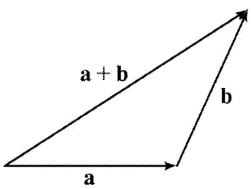
length and parallel, $\overrightarrow{AB}=\overrightarrow{DC}$.) Thus

$$\overrightarrow{AC}\cdot\overrightarrow{BD} = \left(\overrightarrow{AB}+\overrightarrow{BC}\right)\cdot\left(\overrightarrow{BC}-\overrightarrow{AB}\right)=\overrightarrow{AB}\cdot\left(\overrightarrow{BC}-\overrightarrow{AB}\right)+\overrightarrow{BC}\cdot\left(\overrightarrow{BC}-\overrightarrow{AB}\right)$$
$$= \overrightarrow{AB}\cdot\overrightarrow{BC}-\left|\overrightarrow{AB}\right|^2+\left|\overrightarrow{BC}\right|^2-\overrightarrow{AB}\cdot\overrightarrow{BC}=\left|\overrightarrow{BC}\right|^2-\left|\overrightarrow{AB}\right|^2$$

But $\left|\overrightarrow{AB}\right|^2=\left|\overrightarrow{BC}\right|^2$ because all sides of the quadrilateral are equal in length. Therefore $\overrightarrow{AC}\cdot\overrightarrow{BD}=0$, and since both of these vectors are nonzero this tells us that the diagonals of the quadrilateral are perpendicular.

57. $|\boldsymbol{a}\cdot\boldsymbol{b}|=\left|\,|\boldsymbol{a}|\,|\boldsymbol{b}|\cos\theta\,\right|=|\boldsymbol{a}|\,|\boldsymbol{b}|\,|\cos\theta|$. Since $|\cos\theta|\leq 1$, $|\boldsymbol{a}\cdot\boldsymbol{b}|=|\boldsymbol{a}|\,|\boldsymbol{b}|\,|\cos\theta|\leq|\boldsymbol{a}|\,|\boldsymbol{b}|$. *Note:* We have equality in the case of $\cos\theta=\pm 1$, so $\theta=0$ or $\theta=\pi$, thus equality when $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel.

58. **(a)**



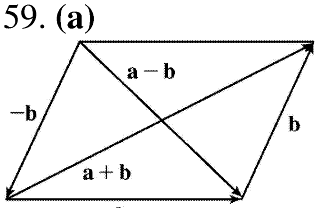
The Triangle Inequality states that the length of the longest side of a triangle is less than or equal to the sum of the lengths of the two shortest sides.

(b)

$$|\boldsymbol{a}+\boldsymbol{b}|^2=(\boldsymbol{a}+\boldsymbol{b})\cdot(\boldsymbol{a}+\boldsymbol{b})=(\boldsymbol{a}\cdot\boldsymbol{a})+2(\boldsymbol{a}\cdot\boldsymbol{b})+(\boldsymbol{b}\cdot\boldsymbol{b})=|\boldsymbol{a}|^2+2(\boldsymbol{a}\cdot\boldsymbol{b})+|\boldsymbol{b}|^2$$
$$\leq|\boldsymbol{a}|^2+2|\boldsymbol{a}|\,|\boldsymbol{b}|+|\boldsymbol{b}|^2$$
$$=(|\boldsymbol{a}|+|\boldsymbol{b}|)^2$$

Thus, taking the square root of both sides, $|\boldsymbol{a}+\boldsymbol{b}|\leq|\boldsymbol{a}|+|\boldsymbol{b}|$.

59. **(a)**



The Parallelogram Law states that the sum of the squares of the lengths of the diagonals of a parallelogram equals the sum of the squares of its (four) sides.

(b) $|\mathbf{a}+\mathbf{b}|^2=(\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}+\mathbf{b})=|\mathbf{a}|^2+2(\mathbf{a}\cdot\mathbf{b})+|\mathbf{b}|^2$ and $|\mathbf{a}-\mathbf{b}|^2=(\mathbf{a}-\mathbf{b})\cdot(\mathbf{a}-\mathbf{b})=|\mathbf{a}|^2-2(\mathbf{a}\cdot\mathbf{b})+|\mathbf{b}|^2$. Adding these two equations gives $|\mathbf{a}+\mathbf{b}|^2+|\mathbf{a}-\mathbf{b}|^2=2|\mathbf{a}|^2+2|\mathbf{b}|^2$.

1. $\mathbf{a}\times\mathbf{b}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & 2 & 0\\ 0 & 3 & 1\end{vmatrix}=\begin{vmatrix}2 & 0\\ 3 & 1\end{vmatrix}\mathbf{i}-\begin{vmatrix}1 & 0\\ 0 & 1\end{vmatrix}\mathbf{j}+\begin{vmatrix}1 & 2\\ 0 & 3\end{vmatrix}\mathbf{k}$

$=(2-0)\mathbf{i}-(1-0)\mathbf{j}+(3-0)\mathbf{k}=2\mathbf{i}-\mathbf{j}+3\mathbf{k}$

Now $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{a}=\langle 2,-1,3\rangle\cdot\langle 1,2,0\rangle=2-2+0=0$ and $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{b}=\langle 2,-1,3\rangle\cdot\langle 0,3,1\rangle=0-3+3=0$, so $\mathbf{a}\times\mathbf{b}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$.

2.

$$\mathbf{a}\times\mathbf{b}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 5 & 1 & 4\\ -1 & 0 & 2\end{vmatrix}=\begin{vmatrix}1 & 4\\ 0 & 2\end{vmatrix}\mathbf{i}-\begin{vmatrix}5 & 4\\ -1 & 2\end{vmatrix}\mathbf{j}+\begin{vmatrix}5 & 1\\ -1 & 0\end{vmatrix}\mathbf{k}$$

$=(2-0)\mathbf{i}-[10-(-4)]\mathbf{j}+[0-(-1)]\mathbf{k}=2\mathbf{i}-14\mathbf{j}+\mathbf{k}$

Now $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{a}=\langle 2,-14,1\rangle\cdot\langle 5,1,4\rangle=10-14+4=0$ and $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{b}=\langle 2,-14,1\rangle\cdot\langle -1,0,2\rangle=-2+0+2=0$, so $\mathbf{a}\times\mathbf{b}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$.

3.

$$\mathbf{a}\times\mathbf{b}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 2 & 1 & -1\\ 0 & 1 & 2\end{vmatrix}=\begin{vmatrix}1 & -1\\ 1 & 2\end{vmatrix}\mathbf{i}-\begin{vmatrix}2 & -1\\ 0 & 2\end{vmatrix}\mathbf{j}+\begin{vmatrix}2 & 1\\ 0 & 1\end{vmatrix}\mathbf{k}$$

$=[2-(-1)]\mathbf{i}-(4-0)\mathbf{j}+(2-0)\mathbf{k}=3\mathbf{i}-4\mathbf{j}+2\mathbf{k}$

Now $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{a}=(3\mathbf{i}-4\mathbf{j}+2\mathbf{k})\cdot(2\mathbf{i}+\mathbf{j}-\mathbf{k})=6-4-2=0$ and $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{b}=(3\mathbf{i}-4\mathbf{j}+2\mathbf{k})\cdot(\mathbf{j}+2\mathbf{k})=0-4+4=0$, so $\mathbf{a}\times\mathbf{b}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$.

4.

$$\mathbf{a}\times\mathbf{b}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & -1 & 1\\ 1 & 1 & 1\end{vmatrix}=\begin{vmatrix}-1 & 1\\ 1 & 1\end{vmatrix}\mathbf{i}-\begin{vmatrix}1 & 1\\ 1 & 1\end{vmatrix}\mathbf{j}+\begin{vmatrix}1 & -1\\ 1 & 1\end{vmatrix}\mathbf{k}$$

$=(-1-1)\mathbf{i}-(1-1)\mathbf{j}+[1-(-1)]\mathbf{k}=-2\mathbf{i}+2\mathbf{k}$

Now $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{a}=(-2\mathbf{i}+2\mathbf{k})\cdot(\mathbf{i}-\mathbf{j}+\mathbf{k})=-2+0+2=0$ and $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{b}=(-2\mathbf{i}+2\mathbf{k})\cdot(\mathbf{i}+\mathbf{j}+\mathbf{k})=-2+0+2=0$, so $\mathbf{a}\times\mathbf{b}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$.

5.

$$\mathbf{a}\times\mathbf{b}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 3 & 2 & 4\\ 1 & -2 & -3\end{vmatrix}=\begin{vmatrix}2 & 4\\ -2 & -3\end{vmatrix}\mathbf{i}-\begin{vmatrix}3 & 4\\ 1 & -3\end{vmatrix}\mathbf{j}+\begin{vmatrix}3 & 2\\ 1 & -2\end{vmatrix}\mathbf{k}$$

$=[-6-(-8)]\mathbf{i}-(-9-4)\mathbf{j}+(-6-2)\mathbf{k}=2\mathbf{i}+13\mathbf{j}-8\mathbf{k}$

Since $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{a}=(2\mathbf{i}+13\mathbf{j}-8\mathbf{k})\cdot(3\mathbf{i}+2\mathbf{j}+4\mathbf{k})=6+26-32=0$, $\mathbf{a}\times\mathbf{b}$ is orthogonal to $\mathbf{a}$.
Since $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{b}=(2\mathbf{i}+13\mathbf{j}-8\mathbf{k})\cdot(\mathbf{i}-2\mathbf{j}-3\mathbf{k})=2-26+24=0$, $\mathbf{a}\times\mathbf{b}$ is orthogonal to $\mathbf{b}$.

6.

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & e^{t} & e^{-t} \\ 2 & e^{t} & -e^{-t} \end{vmatrix} = \begin{vmatrix} e^{t} & e^{-t} \\ e^{t} & -e^{-t} \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & e^{-t} \\ 2 & -e^{-t} \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & e^{t} \\ 2 & e^{t} \end{vmatrix}\mathbf{k}$$

$$=(-1-1)\mathbf{i}-(-e^{-t}-2e^{-t})\mathbf{j}+(e^{t}-2e^{t})\mathbf{k}=-2\mathbf{i}+3e^{-t}\mathbf{j}-e^{t}\mathbf{k}$$

Since $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{a}=(-2\mathbf{i}+3e^{-t}\mathbf{j}-e^{t}\mathbf{k})\cdot(\mathbf{i}+e^{t}\mathbf{j}+e^{-t}\mathbf{k})=-2+3-1=0$, $\mathbf{a}\times\mathbf{b}$ is orthogonal to $\mathbf{a}$.
Since $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{b}=(-2\mathbf{i}+3e^{-t}\mathbf{j}-e^{t}\mathbf{k})\cdot(2\mathbf{i}+e^{t}\mathbf{j}-e^{-t}\mathbf{k})=-4+3+1=0$, $\mathbf{a}\times\mathbf{b}$ is orthogonal to $\mathbf{b}$.

7.

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ t & t^{2} & t^{3} \\ 1 & 2t & 3t^{2} \end{vmatrix} = \begin{vmatrix} t^{2} & t^{3} \\ 2t & 3t^{2} \end{vmatrix}\mathbf{i} - \begin{vmatrix} t & t^{3} \\ 1 & 3t^{2} \end{vmatrix}\mathbf{j} + \begin{vmatrix} t & t^{2} \\ 1 & 2t \end{vmatrix}\mathbf{k}$$

$$=(3t^{4}-2t^{4})\mathbf{i}-(3t^{3}-t^{3})\mathbf{j}+(2t^{2}-t^{2})\mathbf{k}=t^{4}\mathbf{i}-2t^{3}\mathbf{j}+t^{2}\mathbf{k}$$

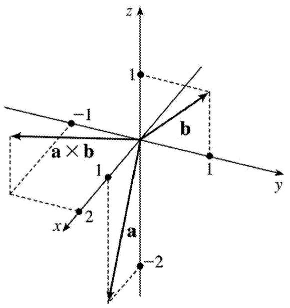
Since $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{a}=\langle t^{4},-2t^{3},t^{2}\rangle\cdot\langle t,t^{2},t^{3}\rangle=t^{5}-2t^{5}+t^{5}=0$, $\mathbf{a}\times\mathbf{b}$ is orthogonal to $\mathbf{a}$.
Since $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{b}=\langle t^{4},-2t^{3},t^{2}\rangle\cdot\langle 1,2t,3t^{2}\rangle=t^{4}-4t^{4}+3t^{4}=0$, $\mathbf{a}\times\mathbf{b}$ is orthogonal to $\mathbf{b}$.

8.

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -2 \\ 0 & 1 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & -2 \\ 1 & 1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & -2 \\ 0 & 1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}\mathbf{k}$$

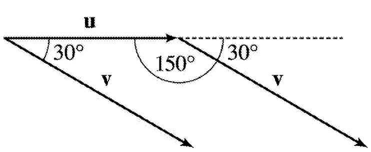
$$=2\mathbf{i}-\mathbf{j}+\mathbf{k}$$



9. **(a)** Since $\boldsymbol{b}\times\boldsymbol{c}$ is a vector, the dot product $\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c})$ is meaningful and is a scalar.
(b) $\boldsymbol{b}\cdot\boldsymbol{c}$ is a scalar, so $\boldsymbol{a}\times(\boldsymbol{b}\cdot\boldsymbol{c})$ is meaningless, as the cross product is defined only for two *vectors* .
(c) Since $\boldsymbol{b}\times\boldsymbol{c}$ is a vector, the cross product $\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c})$ is meaningful and results in another vector.
(d) $\boldsymbol{a}\cdot\boldsymbol{b}$ is a scalar, so the cross product $(\boldsymbol{a}\cdot\boldsymbol{b})\times\boldsymbol{c}$ is meaningless.
(e) Since $(\boldsymbol{a}\cdot\boldsymbol{b})$ and $(\boldsymbol{c}\cdot\boldsymbol{d})$ are both scalars, the cross product $(\boldsymbol{a}\cdot\boldsymbol{b})\times(\boldsymbol{c}\cdot\boldsymbol{d})$ is meaningless.
(f) $\boldsymbol{a}\times\boldsymbol{b}$ and $\boldsymbol{c}\times\boldsymbol{d}$ are both vectors, so the dot product $(\boldsymbol{a}\times\boldsymbol{b})\cdot(\boldsymbol{c}\times\boldsymbol{d})$ is meaningful and is a scalar.

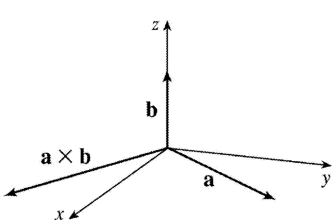
10. Using Theorem 6, we have $|\boldsymbol{u}\times\boldsymbol{v}|=|\boldsymbol{u}|\,|\boldsymbol{v}|\sin\theta=(5)(10)\sin 60°=25\sqrt{3}$. By the right–hand rule, $\boldsymbol{u}\times\boldsymbol{v}$ is directed into the page.

11. If we sketch $\boldsymbol{u}$ and $\boldsymbol{v}$ starting from the same initial point,
we see that the angle between them is $30°$. Using Theorem 6, we have
$|\boldsymbol{u}\times\boldsymbol{v}|=|\boldsymbol{u}|\,|\boldsymbol{v}|\sin 30°=(6)(8)\left(\frac{1}{2}\right)=24$
By the right–hand rule, $\boldsymbol{u}\times\boldsymbol{v}$ is directed into the page.



12. **(a)** $|\boldsymbol{a}\times\boldsymbol{b}|=|\boldsymbol{a}|\,|\boldsymbol{b}|\sin\theta=3\cdot 2\cdot\sin\frac{\pi}{2}=6$

(b) $\boldsymbol{a}\times\boldsymbol{b}$ is orthogonal to $\boldsymbol{k}$, so it lies in the xy –plane, and its z –coordinate is 0 . By the right–hand rule, its y –component is negative and its x –component is positive.



13. $\boldsymbol{a}\times\boldsymbol{b}=\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ 1 & 2 & 1\\ 0 & 1 & 3\end{vmatrix}=\begin{vmatrix}2 & 1\\ 1 & 3\end{vmatrix}\boldsymbol{i}-\begin{vmatrix}1 & 1\\ 0 & 3\end{vmatrix}\boldsymbol{j}+\begin{vmatrix}1 & 2\\ 0 & 1\end{vmatrix}\boldsymbol{k}$

$=(6-1)\boldsymbol{i}-(3-0)\boldsymbol{j}+(1-0)\boldsymbol{k}=5\boldsymbol{i}-3\boldsymbol{j}+\boldsymbol{k}$

$\boldsymbol{b}\times\boldsymbol{a}=\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ 0 & 1 & 3\\ 1 & 2 & 1\end{vmatrix}=\begin{vmatrix}1 & 3\\ 2 & 1\end{vmatrix}\boldsymbol{i}-\begin{vmatrix}0 & 3\\ 1 & 1\end{vmatrix}\boldsymbol{j}+\begin{vmatrix}0 & 1\\ 1 & 2\end{vmatrix}\boldsymbol{k}$

$=(1-6)\boldsymbol{i}-(0-3)\boldsymbol{j}+(0-1)\boldsymbol{k}=-5\boldsymbol{i}+3\boldsymbol{j}-\boldsymbol{k}$

Notice $\boldsymbol{a}\times\boldsymbol{b}=-\boldsymbol{b}\times\boldsymbol{a}$ here, as we know is always true by Theorem 8.

14. $\boldsymbol{b}\times\boldsymbol{c}=\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ -1 & 1 & 0\\ 0 & 0 & -4\end{vmatrix}=\begin{vmatrix}1 & 0\\ 0 & -4\end{vmatrix}\boldsymbol{i}-\begin{vmatrix}-1 & 0\\ 0 & -4\end{vmatrix}\boldsymbol{j}+\begin{vmatrix}-1 & 1\\ 0 & 0\end{vmatrix}\boldsymbol{k}=-4\boldsymbol{i}-4\boldsymbol{j}$ so

$\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c})=\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ 3 & 1 & 2\\ -4 & -4 & 0\end{vmatrix}=\begin{vmatrix}1 & 2\\ -4 & 0\end{vmatrix}\boldsymbol{i}-\begin{vmatrix}3 & 2\\ -4 & 0\end{vmatrix}\boldsymbol{j}+\begin{vmatrix}3 & 1\\ -4 & -4\end{vmatrix}\boldsymbol{k}=8\boldsymbol{i}-8\boldsymbol{j}-8\boldsymbol{k}$.

$\boldsymbol{a}\times\boldsymbol{b}=\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ 3 & 1 & 2\\ -1 & 1 & 0\end{vmatrix}=\begin{vmatrix}1 & 2\\ 1 & 0\end{vmatrix}\boldsymbol{i}-\begin{vmatrix}3 & 2\\ -1 & 0\end{vmatrix}\boldsymbol{j}+\begin{vmatrix}3 & 1\\ -1 & 1\end{vmatrix}\boldsymbol{k}=-2\boldsymbol{i}-2\boldsymbol{j}+4\boldsymbol{k}$ so

$(\boldsymbol{a}\times\boldsymbol{b})\times\boldsymbol{c}=\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ -2 & -2 & 4\\ 0 & 0 & -4\end{vmatrix}=\begin{vmatrix}-2 & 4\\ 0 & -4\end{vmatrix}\boldsymbol{i}-\begin{vmatrix}-2 & 4\\ 0 & -4\end{vmatrix}\boldsymbol{j}+\begin{vmatrix}-2 & -2\\ 0 & 0\end{vmatrix}\boldsymbol{k}=8\boldsymbol{i}-8\boldsymbol{j}$.

Thus $\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c})\neq(\boldsymbol{a}\times\boldsymbol{b})\times\boldsymbol{c}$.

15. We know that the cross product of two vectors is orthogonal to both. So we calculate

$\langle 1,-1,1\rangle\times\langle 0,4,4\rangle=\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ 1 & -1 & 1\\ 0 & 4 & 4\end{vmatrix}=\begin{vmatrix}-1 & 1\\ 4 & 4\end{vmatrix}\boldsymbol{i}-\begin{vmatrix}1 & 1\\ 0 & 4\end{vmatrix}\boldsymbol{j}+\begin{vmatrix}1 & -1\\ 0 & 4\end{vmatrix}\boldsymbol{k}=-8\boldsymbol{i}-4\boldsymbol{j}+4\boldsymbol{k}$.

So two unit vectors orthogonal to both are $\pm\dfrac{\langle -8,-4,4\rangle}{\sqrt{64+16+16}}=\pm\dfrac{\langle -8,-4,4\rangle}{4\sqrt{6}}$, that is,

$\left\langle -\dfrac{2}{\sqrt{6}},-\dfrac{1}{\sqrt{6}},\dfrac{1}{\sqrt{6}}\right\rangle$ and $\left\langle \dfrac{2}{\sqrt{6}},\dfrac{1}{\sqrt{6}},-\dfrac{1}{\sqrt{6}}\right\rangle$.

16. We know that the cross product of two vectors is orthogonal to both. So we calculate

$\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ 1 & 1 & 1\\ 2 & 0 & 1\end{vmatrix}=\begin{vmatrix}1 & 1\\ 0 & 1\end{vmatrix}\boldsymbol{i}-\begin{vmatrix}1 & 1\\ 2 & 1\end{vmatrix}\boldsymbol{j}+\begin{vmatrix}1 & 1\\ 2 & 0\end{vmatrix}\boldsymbol{k}=\boldsymbol{i}+\boldsymbol{j}-2\boldsymbol{k}$. Thus, two unit vectors

orthogonal to both are $\pm\dfrac{1}{\sqrt{6}}\langle 1,1,-2\rangle$, that is, $\left\langle \dfrac{1}{\sqrt{6}},\dfrac{1}{\sqrt{6}},-\dfrac{2}{\sqrt{6}}\right\rangle$ and $\left\langle -\dfrac{1}{\sqrt{6}},-\dfrac{1}{\sqrt{6}},\dfrac{2}{\sqrt{6}}\right\rangle$.

17. Let $\boldsymbol{a}=\left\langle a_1,a_2,a_3\right\rangle$. Then

$$\mathbf{0}\times \boldsymbol{a}=\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 0 & 0 & 0 \\ a_1 & a_2 & a_3 \end{vmatrix} = \begin{vmatrix} 0 & 0 \\ a_2 & a_3 \end{vmatrix} \boldsymbol{i}- \begin{vmatrix} 0 & 0 \\ a_1 & a_3 \end{vmatrix} \boldsymbol{j}+ \begin{vmatrix} 0 & 0 \\ a_1 & a_2 \end{vmatrix} \boldsymbol{k}=\mathbf{0}\ ,$$

$$\boldsymbol{a}\times \mathbf{0}=\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ a_1 & a_2 & a_3 \\ 0 & 0 & 0 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ 0 & 0 \end{vmatrix} \boldsymbol{i}- \begin{vmatrix} a_1 & a_3 \\ 0 & 0 \end{vmatrix} \boldsymbol{j}+ \begin{vmatrix} a_1 & a_2 \\ 0 & 0 \end{vmatrix} \boldsymbol{k}=\mathbf{0}\ .$$

18. Let $\boldsymbol{a}=\left\langle a_1,a_2,a_3\right\rangle$ and $\boldsymbol{b}=\left\langle b_1,b_2,b_3\right\rangle$.

$$(\boldsymbol{a}\times \boldsymbol{b})\cdot \boldsymbol{b} = \left\langle \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}, \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}, \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \right\rangle \cdot \left\langle b_1,b_2,b_3\right\rangle$$

$$= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} b_1 - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} b_2 + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} b_3$$

$$=\left(a_2b_3b_1-a_3b_2b_1\right)-\left(a_1b_3b_2-a_3b_1b_2\right)+\left(a_1b_2b_3-a_2b_1b_3\right)=0$$

19.

$$\boldsymbol{a}\times \boldsymbol{b} =\left\langle a_2b_3-a_3b_2,a_3b_1-a_1b_3,a_1b_2-a_2b_1\right\rangle$$

$$=\left\langle (-1)\left(b_2a_3-b_3a_2\right),(-1)\left(b_3a_1-b_1a_3\right),(-1)\left(b_1a_2-b_2a_1\right)\right\rangle$$

$$=-\left\langle b_2a_3-b_3a_2,b_3a_1-b_1a_3,b_1a_2-b_2a_1\right\rangle=-\boldsymbol{b}\times \boldsymbol{a}$$

20. $c\boldsymbol{a}=\left\langle ca_1,ca_2,ca_3\right\rangle$, so

$$(c\,\boldsymbol{a})\times \boldsymbol{b} =\left\langle ca_2b_3-ca_3b_2,ca_3b_1-ca_1b_3,ca_1b_2-ca_2b_1\right\rangle$$

$$=\left\langle a_2b_3-a_3b_2,a_3b_1-a_1b_3,a_1b_2-a_2b_1\right\rangle=c(\boldsymbol{a}\times \boldsymbol{b})$$

$$=\left\langle ca_2b_3-ca_3b_2,ca_3b_1-ca_1b_3,ca_1b_2-ca_2b_1\right\rangle$$

$$=\left\langle a_2\left(cb_3\right)-a_3\left(cb_2\right),a_3\left(cb_1\right)-a_1\left(cb_3\right),a_1\left(cb_2\right)-a_2\left(cb_1\right)\right\rangle$$
$$=\boldsymbol{a}\times c\,\boldsymbol{b}$$

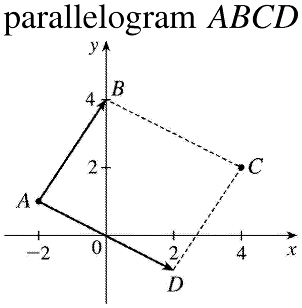
21.

$$\begin{aligned}\boldsymbol{a}\times(\boldsymbol{b}+\boldsymbol{c}) &=\boldsymbol{a}\times\left\langle b_1+c_1,b_2+c_2,b_3+c_3\right\rangle\\ &=\left\langle a_2\left(b_3+c_3\right)-a_3\left(b_2+c_2\right),a_3\left(b_1+c_1\right)-a_1\left(b_3+c_3\right),a_1\left(b_2+c_2\right)-a_2\left(b_1+c_1\right)\right\rangle\\ &=\left\langle a_2b_3+a_2c_3-a_3b_2-a_3c_2,a_3b_1+a_3c_1-a_1b_3-a_1c_3,a_1b_2+a_1c_2-a_2b_1-a_2c_1\right\rangle\\ &=\left\langle \left(a_2b_3-a_3b_2\right)+\left(a_2c_3-a_3c_2\right),\left(a_3b_1-a_1b_3\right)+\left(a_3c_1-a_1c_3\right),\right.\\ &\quad\left.\left(a_1b_2-a_2b_1\right)+\left(a_1c_2-a_2c_1\right)\right\rangle\\ &=\left\langle a_2b_3-a_3b_2,a_3b_1-a_1b_3,a_1b_2-a_2b_1\right\rangle+\left\langle a_2c_3-a_3c_2,a_3c_1-a_1c_3,a_1c_2-a_2c_1\right\rangle\\ &=(\boldsymbol{a}\times\boldsymbol{b})+(\boldsymbol{a}\times\boldsymbol{c})\end{aligned}$$

22.

$$\begin{aligned}(\boldsymbol{a}+\boldsymbol{b})\times\boldsymbol{c} &=-\boldsymbol{c}\times(\boldsymbol{a}+\boldsymbol{b}) && \text{by Property 1 of Theorem 8}\\ &=-(\boldsymbol{c}\times\boldsymbol{a}+\boldsymbol{c}\times\boldsymbol{b}) && \text{by Property 3 of Theorem 8}\\ &=-(-\boldsymbol{a}\times\boldsymbol{c}+(-\boldsymbol{b}\times\boldsymbol{c})) && \text{by Property 1 of Theorem 8}\\ &=\boldsymbol{a}\times\boldsymbol{c}+\boldsymbol{b}\times\boldsymbol{c} && \text{by Property 2 of Theorem 8}\end{aligned}$$

23. By plotting the vertices, we can see that the parallelogram is determined by the vectors $\overrightarrow{AB}=\langle 2,3\rangle$ and $\overrightarrow{AD}=\langle 4,-2\rangle$. We know that the area of the parallelogram determined by two vectors is equal to the length of the cross product of these vectors. In order to compute the cross product, we consider the vector $\overrightarrow{AB}$ as the three–dimensional vector $\langle 2,3,0\rangle$ (and similarly for $\overrightarrow{AD}$) , and then the area of parallelogram $ABCD$ is



$$\left|\overrightarrow{AB}\times\overrightarrow{AD}\right| = \left| \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 2 & 3 & 0 \\ 4 & -2 & 0 \end{vmatrix} \right| = |(0)\boldsymbol{i}-(0)\boldsymbol{j}+(-4-12)\boldsymbol{k}| = |-16\boldsymbol{k}| = 16$$

24. The parallelogram is determined by the vectors $\overrightarrow{KL}=\langle 0,1,3\rangle$ and $\overrightarrow{KN}=\langle 2,5,0\rangle$, so the area of parallelogram *KLMN* is

$$\left|\overrightarrow{KL}\times\overrightarrow{KN}\right| = \left| \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 0 & 1 & 3 \\ 2 & 5 & 0 \end{vmatrix} \right| = |(-15)\,\boldsymbol{i}-(-6)\,\boldsymbol{j}+(-2)\boldsymbol{k}| = |-15\boldsymbol{i}+6\boldsymbol{j}-2\boldsymbol{k}| = \sqrt{265}\approx 16.28$$

25. **(a)** Because the plane through P , Q , and R contains the vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$, a vector orthogonal to both of these vectors (such as their cross product) is also orthogonal to the plane. Here $\overrightarrow{PQ}=\langle -1,2,0\rangle$ and $\overrightarrow{PR}=\langle -1,0,3\rangle$, so

$$\overrightarrow{PQ}\times\overrightarrow{PR}=\langle (2)(3)-(0)(0),(0)(-1)-(-1)(3),(-1)(0)-(2)(-1)\rangle=\langle 6,3,2\rangle$$

Therefore, $\langle 6,3,2\rangle$ (or any scalar multiple thereof) is orthogonal to the plane through P , Q , and R .
(b) Note that the area of the triangle determined by P , Q , and R is equal to half of the area of the parallelogram determined by the three points. From part (a), the area of the parallelogram is $\left|\overrightarrow{PQ}\times\overrightarrow{PR}\right| = |\langle 6,3,2\rangle| = \sqrt{36+9+4} = 7$, so the area of the triangle is $\frac{1}{2}(7)=\frac{7}{2}$.

26. **(a)** $\overrightarrow{PQ}=\langle -3,2,-1\rangle$ and $\overrightarrow{PR}=\langle 1,-1,1\rangle$, so a vector orthogonal to the plane through P , Q , and R is $\overrightarrow{PQ}\times\overrightarrow{PR}=\langle (2)(1)-(-1)(-1),(-1)(1)-(-3)(1),(-3)(-1)-(2)(1)\rangle=\langle 1,2,1\rangle$
(or any scalar mutiple thereof).

(b) The area of the parallelogram determined by $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ is $\left|\overrightarrow{PQ}\times\overrightarrow{PR}\right| = |\langle 1,2,1\rangle| = \sqrt{1^2+2^2+1^2} = \sqrt{6}$, so the area of triangle *PQR* is $\frac{1}{2}\sqrt{6}$.

27. **(a)** $\overrightarrow{PQ}=\langle 4,3,-2\rangle$ and $\overrightarrow{PR}=\langle 5,5,1\rangle$, so a vector orthogonal to the plane through P , Q , and R is $\overrightarrow{PQ}\times\overrightarrow{PR}=\langle (3)(1)-(-2)(5),(-2)(5)-(4)(1),(4)(5)-(3)(5)\rangle=\langle 13,-14,5\rangle$
(or any scalar mutiple thereof).
(b) The area of the parallelogram determined by $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ is
$\left|\overrightarrow{PQ}\times\overrightarrow{PR}\right| = |\langle 13,-14,5\rangle| = \sqrt{13^2+(-14)^2+5^2} = \sqrt{390}$, so the area of triangle *PQR* is $\frac{1}{2}\sqrt{390}$.

28. **(a)** $\overrightarrow{PQ}=\langle 1,1,3\rangle$ and $\overrightarrow{PR}=\langle 3,2,5\rangle$, so a vector orthogonal to the plane through P , Q , and R is

$\overrightarrow{PQ}\times\overrightarrow{PR}=\langle(1)(5)-(3)(2),(3)(3)-(1)(5),(1)(2)-(1)(3)\rangle=\langle-1,4,-1\rangle$ (or any scalar multiple thereof).

(b) The area of the parallelogram determined by $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ is

$\left|\overrightarrow{PQ}\times\overrightarrow{PR}\right|=\left|\langle-1,4,-1\rangle\right|=\sqrt{1+16+1}=\sqrt{18}=3\sqrt{2}$, so the area of triangle PQR is $\frac{1}{2}\cdot 3\sqrt{2}=\frac{3}{2}\sqrt{2}$.

29. We know that the volume of the parallelepiped determined by $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ is the magnitude of their scalar triple product, which is

$$\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c}) = \begin{vmatrix} 6 & 3 & -1 \\ 0 & 1 & 2 \\ 4 & -2 & 5 \end{vmatrix} = 6\begin{vmatrix} 1 & 2 \\ -2 & 5 \end{vmatrix} - 3\begin{vmatrix} 0 & 2 \\ 4 & 5 \end{vmatrix} + (-1)\begin{vmatrix} 0 & 1 \\ 4 & -2 \end{vmatrix}$$

$$= 6(5+4)-3(0-8)-(0-4)=82$$

Thus the volume of the parallelepiped is 82 cubic units.

30.

$$\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c})=\begin{vmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{vmatrix} = 1\begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} + (-1)\begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} = -2-2+0=-4$$

.

So the volume of the parallelepiped determined by $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ is $|-4|=4$ cubic units.

31. $\boldsymbol{a}=\overrightarrow{PQ}=\langle 2,1,1\rangle$, $\boldsymbol{b}=\overrightarrow{PR}=\langle 1,-1,2\rangle$, and $\boldsymbol{c}=\overrightarrow{PS}=\langle 0,-2,3\rangle$.

$$\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c})=\begin{vmatrix} 2 & 1 & 1 \\ 1 & -1 & 2 \\ 0 & -2 & 3 \end{vmatrix} = 2\begin{vmatrix} -1 & 2 \\ -2 & 3 \end{vmatrix} - 1\begin{vmatrix} 1 & 2 \\ 0 & 3 \end{vmatrix} + 1\begin{vmatrix} 1 & -1 \\ 0 & -2 \end{vmatrix} = 2-3-2=-3 ,$$

so the volume of the parallelepiped is 3 cubic units.

32. $\boldsymbol{a}=\overrightarrow{PQ}=\langle 2,3,3\rangle$, $\boldsymbol{b}=\overrightarrow{PR}=\langle -1,-1,-1\rangle$ and $\boldsymbol{c}=\overrightarrow{PS}=\langle 6,-2,2\rangle$.

$$\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c})=\begin{vmatrix} 2 & 3 & 3 \\ -1 & -1 & -1 \\ 6 & -2 & 2 \end{vmatrix} = 2\begin{vmatrix} -1 & -1 \\ -2 & 2 \end{vmatrix} - 3\begin{vmatrix} -1 & -1 \\ 6 & 2 \end{vmatrix} + 3\begin{vmatrix} -1 & -1 \\ 6 & -2 \end{vmatrix} = -8-12+24=4$$

,

so the volume of the parallelepiped is 4 cubic units.

33. $$\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c})=\begin{vmatrix} 2 & 3 & 1 \\ 1 & -1 & 0 \\ 7 & 3 & 2 \end{vmatrix} = 2\begin{vmatrix} -1 & 0 \\ 3 & 2 \end{vmatrix} - 3\begin{vmatrix} 1 & 0 \\ 7 & 2 \end{vmatrix} + 1\begin{vmatrix} 1 & -1 \\ 7 & 3 \end{vmatrix} = -4-6+10=0 ,$$

which says that the volume of the parallelepiped determined by $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ is 0 , and thus these three vectors are coplanar.

34. $\boldsymbol{a}=\overrightarrow{PQ}=\langle 1,4,5\rangle$, $\boldsymbol{b}=\overrightarrow{PR}=\langle 2,-1,1\rangle$ and $\boldsymbol{c}=\overrightarrow{PS}=\langle 5,2,7\rangle$.

$$\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c})=\begin{vmatrix}1&4&5\\2&-1&1\\5&2&7\end{vmatrix}=1\begin{vmatrix}-1&1\\2&7\end{vmatrix}-4\begin{vmatrix}2&1\\5&7\end{vmatrix}+5\begin{vmatrix}2&-1\\5&2\end{vmatrix}=-9-36+45=0\ ,$$

so the volume of the parallelepiped determined by $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ is 0 , which says that these vectors lie in the same plane. Therefore, their initial and terminal points P , Q , R and S also lie in the same plane.

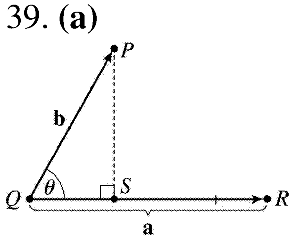
35. The magnitude of the torque is
$|\boldsymbol{\tau}|=|\boldsymbol{r}\times\boldsymbol{F}|=|\boldsymbol{r}|\,|\boldsymbol{F}|\sin\theta$ =(0.18m)(60N)sin (70+10)°=10.8sin 80°$\approx$10.6 J.

36. $|\boldsymbol{r}|=\sqrt{4^2+4^2}=4\sqrt{2}$ ft. A line drawn from the point P to the point of application of the force makes an angle of 180°–(45+30)°=105° with the force vector. Therefore,
$|\boldsymbol{\tau}|=|\boldsymbol{r}\times\boldsymbol{F}|=|\boldsymbol{r}|\,|\boldsymbol{F}|\sin\theta=(4\sqrt{2})(36)\sin 105°\approx 197$ ft–lb.

37. Using the notation of the text, $\boldsymbol{r}=\langle 0,0.3,0\rangle$ and $\boldsymbol{F}$ has direction $\langle 0,3,-4\rangle$. The angle θ between them can be determined by $\cos\theta=\dfrac{\langle 0,0.3,0\rangle\cdot\langle 0,3,-4\rangle}{|\langle 0,0.3,0\rangle|\,|\langle 0,3,-4\rangle|}\Rightarrow\cos\theta=\dfrac{0.9}{(0.3)(5)}\Rightarrow\cos\theta=0.6\Rightarrow$
$\theta\approx 53.1°$. Then $|\boldsymbol{\tau}|=|\boldsymbol{r}|\,|\boldsymbol{F}|\sin\theta\Rightarrow 100=0.3|\boldsymbol{F}|\sin 53.1°\Rightarrow|\boldsymbol{F}|\approx 417$ N.

38. Since $|\boldsymbol{u}\times\boldsymbol{v}|=|\boldsymbol{u}|\,|\boldsymbol{v}|\sin\theta$, $0\le\theta\le\pi$, $|\boldsymbol{u}\times\boldsymbol{v}|$ achieves its maximum value for $\sin\theta=1\Rightarrow\theta=\dfrac{\pi}{2}$, in which case $|\boldsymbol{u}\times\boldsymbol{v}|=|\boldsymbol{u}|\,|\boldsymbol{v}|=15$. The minimum value is zero, which occurs when $\sin\theta=0\Rightarrow\theta=0$ or π , so when $\boldsymbol{u}$, $\boldsymbol{v}$ are parallel. Thus, when $\boldsymbol{u}$ points in the same direction as $\boldsymbol{v}$, so $\boldsymbol{u}=3\boldsymbol{j}$, $|\boldsymbol{u}\times\boldsymbol{v}|=0$. As $\boldsymbol{u}$ rotates counterclockwise, $\boldsymbol{u}\times\boldsymbol{v}$ is directed in the negative z –direction (by the right–hand rule) and the length increases until $\theta=\dfrac{\pi}{2}$, in which case $\boldsymbol{u}=-3\boldsymbol{i}$ and $|\boldsymbol{u}\times\boldsymbol{v}|=15$. As $\boldsymbol{u}$ rotates to the negative y –axis, $\boldsymbol{u}\times\boldsymbol{v}$ remains pointed in the negative z –direction and the length of $\boldsymbol{u}\times\boldsymbol{v}$ decreases to 0, after which the direction of $\boldsymbol{u}\times\boldsymbol{v}$ reverses to point in the positive z –direction and $|\boldsymbol{u}\times\boldsymbol{v}|$ increases. When $\boldsymbol{u}=3\boldsymbol{i}$ (so $\theta=\dfrac{\pi}{2}$), $|\boldsymbol{u}\times\boldsymbol{v}|$ again reaches its maximum of 15 , after which $|\boldsymbol{u}\times\boldsymbol{v}|$ decreases to 0 as $\boldsymbol{u}$ rotates to the positive y –axis.

39. **(a)**

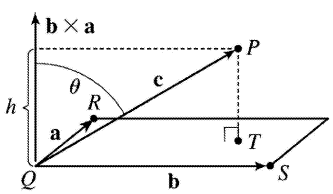


The distance between a point and a line is the length of the perpendicular from the point to the line, here $\left|\overrightarrow{PS}\right|=d$. But referring to triangle PQS , $d=\left|\overrightarrow{PS}\right|=\left|\overrightarrow{QP}\right|\sin\theta=|\boldsymbol{b}|\sin\theta$. But θ is the angle between $\overrightarrow{QP}=\boldsymbol{b}$ and $\overrightarrow{QR}=\boldsymbol{a}$. Thus by Theorem 6, $\sin\theta=\dfrac{|\boldsymbol{a}\times\boldsymbol{b}|}{|\boldsymbol{a}|\,|\boldsymbol{b}|}$ and so

$d=|\boldsymbol{b}|\sin\theta=\dfrac{|\boldsymbol{b}|\,|\boldsymbol{a}\times\boldsymbol{b}|}{|\boldsymbol{a}|\,|\boldsymbol{b}|}=\dfrac{|\boldsymbol{a}\times\boldsymbol{b}|}{|\boldsymbol{a}|}$.

(b) $\boldsymbol{a}=\overrightarrow{QR}=\langle -1,-2,-1\rangle$ and $\boldsymbol{b}=\overrightarrow{QP}=\langle 1,-5,-7\rangle$. Then
$\boldsymbol{a}\times\boldsymbol{b}=\langle(-2)(-7)-(-1)(-5),(-1)(1)-(-1)(-7),(-1)(-5)-(-2)(1)\rangle=\langle 9,-8,7\rangle$. Thus the distance is
$d=\dfrac{|\boldsymbol{a}\times\boldsymbol{b}|}{|\boldsymbol{a}|}=\dfrac{1}{\sqrt{6}}\sqrt{81+64+49}=\sqrt{\dfrac{194}{6}}=\sqrt{\dfrac{97}{3}}$.

40. **(a)** The distance between a point and a plane is the length of the perpendicular from the point to the plane, here $\left|\overrightarrow{TP}\right|=d$. But $\overrightarrow{TP}$ is parallel to $\boldsymbol{b}\times\boldsymbol{a}$ (because $\boldsymbol{b}\times\boldsymbol{a}$ is perpendicular to $\boldsymbol{b}$ and $\boldsymbol{a}$) and $d=\left|\overrightarrow{TP}\right|=$ the absolute value of the scalar projection of $\boldsymbol{c}$ along



$\boldsymbol{b}\times\boldsymbol{a}$, which is $|\boldsymbol{c}|\,|\cos\theta|$. (Notice that this is the same setup as the development of the volume of a parallelepiped with $h=|\boldsymbol{c}|\,|\cos\theta|$). Thus $d=|\boldsymbol{c}|\,|\cos\theta|=h=V/A$ where $A=|\boldsymbol{a}\times\boldsymbol{b}|$, the area of the base. So finally $d=\dfrac{V}{A}=\dfrac{|\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c})|}{|\boldsymbol{a}\times\boldsymbol{b}|}=\dfrac{|(\boldsymbol{a}\times\boldsymbol{b})\cdot\boldsymbol{c}|}{|\boldsymbol{a}\times\boldsymbol{b}|}$ by Theorem 8 # 5.

(b) $\boldsymbol{a}=\overrightarrow{QR}=\langle -1,2,0\rangle$, $\boldsymbol{b}=\overrightarrow{QS}=\langle -1,0,3\rangle$ and $\boldsymbol{c}=\overrightarrow{QP}=\langle 1,1,4\rangle$. Then

$$(\boldsymbol{a}\times\boldsymbol{b})\cdot\boldsymbol{c}=\begin{vmatrix}-1&2&0\\-1&0&3\\1&1&4\end{vmatrix}=(-1)\begin{vmatrix}0&3\\1&4\end{vmatrix}-2\begin{vmatrix}-1&3\\1&4\end{vmatrix}+0=17$$

and $\boldsymbol{a}\times\boldsymbol{b}=\begin{vmatrix}\boldsymbol{i}&\boldsymbol{j}&\boldsymbol{k}\\-1&2&0\\-1&0&3\end{vmatrix}=\begin{vmatrix}2&0\\0&3\end{vmatrix}\boldsymbol{i}-\begin{vmatrix}-1&0\\-1&3\end{vmatrix}\boldsymbol{j}+\begin{vmatrix}-1&2\\-1&0\end{vmatrix}\boldsymbol{k}=6\boldsymbol{i}+3\boldsymbol{j}+2\boldsymbol{k}$

Thus $d=\dfrac{|(\boldsymbol{a}\times\boldsymbol{b})\cdot\boldsymbol{c}|}{|\boldsymbol{a}\times\boldsymbol{b}|}=\dfrac{17}{\sqrt{36+9+4}}=\dfrac{17}{7}$.

41.

$$\begin{aligned}
(\mathbf{a}-\mathbf{b})\times(\mathbf{a}+\mathbf{b}) &= (\mathbf{a}-\mathbf{b})\times\mathbf{a}+(\mathbf{a}-\mathbf{b})\times\mathbf{b} && \text{by Theorem 8 \# 3}\\
&= \mathbf{a}\times\mathbf{a}+(-\mathbf{b})\times\mathbf{a}+\mathbf{a}\times\mathbf{b}+(-\mathbf{b})\times\mathbf{b} && \text{by Theorem 8 \# 4}\\
&= (\mathbf{a}\times\mathbf{a})-(\mathbf{b}\times\mathbf{a})+(\mathbf{a}\times\mathbf{b})-(\mathbf{b}\times\mathbf{b}) && \text{by Theorem 8 \# 2 (with } c=-1\text{)}\\
&= \mathbf{0}-(\mathbf{b}\times\mathbf{a})+(\mathbf{a}\times\mathbf{b})-\mathbf{0} && \text{by Example 2}\\
&= (\mathbf{a}\times\mathbf{b})+(\mathbf{a}\times\mathbf{b}) && \text{by Theorem 8 \# 1}\\
&= 2(\mathbf{a}\times\mathbf{b})
\end{aligned}$$

42. Let $\mathbf{a}=\left\langle a_1,a_2,a_3\right\rangle$, $\mathbf{b}=\left\langle b_1,b_2,b_3\right\rangle$ and $\mathbf{c}=\left\langle c_1,c_2,c_3\right\rangle$, so $\mathbf{b}\times\mathbf{c}=\left\langle b_2c_3-b_3c_2,b_3c_1-b_1c_3,b_1c_2-b_2c_1\right\rangle$ and

$$\begin{aligned}
\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) &= \langle\ a_2\left(b_1c_2-b_2c_1\right)-a_3\left(b_3c_1-b_1c_3\right),a_3\left(b_2c_3-b_3c_2\right)-a_1\left(b_1c_2-b_2c_1\right),\\
&\quad a_1\left(b_3c_1-b_1c_3\right)-a_2\left(b_2c_3-b_3c_2\right)\rangle\\
&= \langle\ a_2b_1c_2-a_2b_2c_1-a_3b_3c_1+a_3b_1c_3,a_3b_2c_3-a_3b_3c_2-a_1b_1c_2+a_1b_2c_1,\\
&\quad a_1b_3c_1-a_1b_1c_3-a_2b_2c_3+a_2b_3c_2\rangle\\
&= \langle\ \left(a_2c_2+a_3c_3\right)b_1-\left(a_2b_2+a_3b_3\right)c_1,\left(a_1c_1+a_3c_3\right)b_2-\left(a_1b_1+a_3b_3\right)c_2,\\
&\quad \left(a_1c_1+a_2c_2\right)b_3-\left(a_1b_1+a_2b_2\right)c_3\rangle
\end{aligned}$$

$$\begin{aligned}
(\ *\ )\quad &= \langle\ \left(a_2c_2+a_3c_3\right)b_1-\left(a_2b_2+a_3b_3\right)c_1+a_1b_1c_1-a_1b_1c_1,\\
&\quad \left(a_1c_1+a_3c_3\right)b_2-\left(a_1b_1+a_3b_3\right)c_2+a_2b_2c_2-a_2b_2c_2,\\
&\quad \left(a_1c_1+a_2c_2\right)b_3-\left(a_1b_1+a_2b_2\right)c_3+a_3b_3c_3-a_3b_3c_3\rangle\\
&= \langle\ \left(a_1c_1+a_2c_2+a_3c_3\right)b_1-\left(a_1b_1+a_2b_2+a_3b_3\right)c_1,\\
&\quad \left(a_1c_1+a_2c_2+a_3c_3\right)b_2-\left(a_1b_1+a_2b_2+a_3b_3\right)c_2,\\
&\quad \left(a_1c_1+a_2c_2+a_3c_3\right)b_3-\left(a_1b_1+a_2b_2+a_3b_3\right)c_3\rangle\\
&= \left(a_1c_1+a_2c_2+a_3c_3\right)\left\langle b_1,b_2,b_3\right\rangle-\left(a_1b_1+a_2b_2+a_3b_3\right)\left\langle c_1,c_2,c_3\right\rangle\\
&= (\mathbf{a}\cdot\mathbf{c})\mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\mathbf{c}
\end{aligned}$$

(*) Here we look ahead to see what terms are still needed to arrive at the desired equation. By adding and subtracting the same terms, we don’t change the value of the component.

43. $\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c})+\boldsymbol{b}\times(\boldsymbol{c}\times\boldsymbol{a})+\boldsymbol{c}\times(\boldsymbol{a}\times\boldsymbol{b})$
$=[(\boldsymbol{a}\cdot\boldsymbol{c})\boldsymbol{b}-(\boldsymbol{a}\cdot\boldsymbol{b})\boldsymbol{c}]+[(\boldsymbol{b}\cdot\boldsymbol{a})\boldsymbol{c}-(\boldsymbol{b}\cdot\boldsymbol{c})\boldsymbol{a}]+[(\boldsymbol{c}\cdot\boldsymbol{b})\boldsymbol{a}-(\boldsymbol{c}\cdot\boldsymbol{a})\boldsymbol{b}]$ by Exercise 42
$=(\boldsymbol{a}\cdot\boldsymbol{c})\boldsymbol{b}-(\boldsymbol{a}\cdot\boldsymbol{b})\boldsymbol{c}+(\boldsymbol{a}\cdot\boldsymbol{b})\boldsymbol{c}-(\boldsymbol{b}\cdot\boldsymbol{c})\boldsymbol{a}+(\boldsymbol{b}\cdot\boldsymbol{c})\boldsymbol{a}-(\boldsymbol{a}\cdot\boldsymbol{c})\boldsymbol{b}=\boldsymbol{0}$

44. Let $\boldsymbol{c}\times\boldsymbol{d}=\boldsymbol{v}$. Then

$(\boldsymbol{a}\times\boldsymbol{b})\cdot(\boldsymbol{c}\times\boldsymbol{d})=(\boldsymbol{a}\times\boldsymbol{b})\cdot\boldsymbol{v}=\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{v})$ by Theorem 8 # 5
$=\boldsymbol{a}\cdot[\boldsymbol{b}\times(\boldsymbol{c}\times\boldsymbol{d})]$
$=\boldsymbol{a}\cdot[(\boldsymbol{b}\cdot\boldsymbol{d})\boldsymbol{c}-(\boldsymbol{b}\cdot\boldsymbol{c})\boldsymbol{d}]$ by Exercise 42
$=(\boldsymbol{b}\cdot\boldsymbol{d})(\boldsymbol{a}\cdot\boldsymbol{c})-(\boldsymbol{b}\cdot\boldsymbol{c})(\boldsymbol{a}\cdot\boldsymbol{d})$ by Properties 3 and 4 of the dot product
$=\begin{vmatrix} \boldsymbol{a}\cdot\boldsymbol{c} & \boldsymbol{b}\cdot\boldsymbol{c} \\ \boldsymbol{a}\cdot\boldsymbol{d} & \boldsymbol{b}\cdot\boldsymbol{d} \end{vmatrix}$

45. **(a)** No. If $\boldsymbol{a}\cdot\boldsymbol{b}=\boldsymbol{a}\cdot\boldsymbol{c}$, then $\boldsymbol{a}\cdot(\boldsymbol{b}-\boldsymbol{c})=0$, so $\boldsymbol{a}$ is perpendicular to $\boldsymbol{b}-\boldsymbol{c}$, which can happen if $\boldsymbol{b}\neq\boldsymbol{c}$. For example, let $\boldsymbol{a}=\langle 1,1,1\rangle$, $\boldsymbol{b}=\langle 1,0,0\rangle$ and $\boldsymbol{c}=\langle 0,1,0\rangle$.
(b) No. If $\boldsymbol{a}\times\boldsymbol{b}=\boldsymbol{a}\times\boldsymbol{c}$ then $\boldsymbol{a}\times(\boldsymbol{b}-\boldsymbol{c})=\boldsymbol{0}$, which implies that $\boldsymbol{a}$ is parallel to $\boldsymbol{b}-\boldsymbol{c}$, which of course can happen if $\boldsymbol{b}\neq\boldsymbol{c}$.
(c) Yes. Since $\boldsymbol{a}\cdot\boldsymbol{c}=\boldsymbol{a}\cdot\boldsymbol{b}$, $\boldsymbol{a}$ is perpendicular to $\boldsymbol{b}-\boldsymbol{c}$, by part (a). From part (b), $\boldsymbol{a}$ is also parallel to $\boldsymbol{b}-\boldsymbol{c}$. Thus since $\boldsymbol{a}\neq\boldsymbol{0}$ but is both parallel and perpendicular to $\boldsymbol{b}-\boldsymbol{c}$, we have $\boldsymbol{b}-\boldsymbol{c}=\boldsymbol{0}$, so $\boldsymbol{b}=\boldsymbol{c}$.

46. **(a)** $\boldsymbol{k}_i$ is perpendicular to $\boldsymbol{v}_i$ if $i\neq j$ by the definition of $\boldsymbol{k}_i$ and Theorem 5.

(b) $\boldsymbol{k}_1\cdot\boldsymbol{v}_1=\dfrac{\boldsymbol{v}_2\times\boldsymbol{v}_3}{\boldsymbol{v}_1\cdot(\boldsymbol{v}_2\times\boldsymbol{v}_3)}\cdot\boldsymbol{v}_1=\dfrac{\boldsymbol{v}_1\cdot(\boldsymbol{v}_2\times\boldsymbol{v}_3)}{\boldsymbol{v}_1\cdot(\boldsymbol{v}_2\times\boldsymbol{v}_3)}=1$

$\boldsymbol{k}_2\cdot\boldsymbol{v}_2=\dfrac{\boldsymbol{v}_3\times\boldsymbol{v}_1}{\boldsymbol{v}_1\cdot(\boldsymbol{v}_2\times\boldsymbol{v}_3)}\cdot\boldsymbol{v}_2=\dfrac{\boldsymbol{v}_2\cdot(\boldsymbol{v}_3\times\boldsymbol{v}_1)}{\boldsymbol{v}_1\cdot(\boldsymbol{v}_2\times\boldsymbol{v}_3)}=\dfrac{(\boldsymbol{v}_2\times\boldsymbol{v}_3)\cdot\boldsymbol{v}_1}{\boldsymbol{v}_1\cdot(\boldsymbol{v}_2\times\boldsymbol{v}_3)}=1$ [by Theorem 8 # 5]

$\boldsymbol{k}_3\cdot\boldsymbol{v}_3=\dfrac{(\boldsymbol{v}_1\times\boldsymbol{v}_2)\cdot\boldsymbol{v}_3}{\boldsymbol{v}_1\cdot(\boldsymbol{v}_2\times\boldsymbol{v}_3)}=\dfrac{\boldsymbol{v}_1\cdot(\boldsymbol{v}_2\times\boldsymbol{v}_3)}{\boldsymbol{v}_1\cdot(\boldsymbol{v}_2\times\boldsymbol{v}_3)}=1$ [by Theorem 8 # 5]

(c)

$$\boldsymbol{k}_1\cdot(\boldsymbol{k}_2\times\boldsymbol{k}_3)=\boldsymbol{k}_1\cdot\left(\frac{\boldsymbol{v}_3\times\boldsymbol{v}_1}{\boldsymbol{v}_1\cdot(\boldsymbol{v}_2\times\boldsymbol{v}_3)}\times\frac{\boldsymbol{v}_1\times\boldsymbol{v}_2}{\boldsymbol{v}_1\cdot(\boldsymbol{v}_2\times\boldsymbol{v}_3)}\right)=\frac{\boldsymbol{k}_1}{[\boldsymbol{v}_1\cdot(\boldsymbol{v}_2\times\boldsymbol{v}_3)]^2}\cdot[(\boldsymbol{v}_3\times\boldsymbol{v}_1)\times(\boldsymbol{v}_1\times\boldsymbol{v}_2)]$$

$$=\frac{\boldsymbol{k}_1}{[\boldsymbol{v}_1\cdot(\boldsymbol{v}_2\times\boldsymbol{v}_3)]^2}\cdot\left([(\boldsymbol{v}_3\times\boldsymbol{v}_1)\cdot\boldsymbol{v}_2]\boldsymbol{v}_1-[(\boldsymbol{v}_3\times\boldsymbol{v}_1)\cdot\boldsymbol{v}_1]\boldsymbol{v}_2\right)\quad\text{[by Exercise 42]}$$

But $(\boldsymbol{v}_3\times\boldsymbol{v}_1)\cdot\boldsymbol{v}_1=0$ since $\boldsymbol{v}_3\times\boldsymbol{v}_1$ is orthogonal to $\boldsymbol{v}_1$, and

$\left(\boldsymbol{v}_3\times\boldsymbol{v}_1\right)\cdot\boldsymbol{v}_2=\boldsymbol{v}_2\cdot\left(\boldsymbol{v}_3\times\boldsymbol{v}_1\right)=\left(\boldsymbol{v}_2\times\boldsymbol{v}_3\right)\cdot\boldsymbol{v}_1=\boldsymbol{v}_1\cdot\left(\boldsymbol{v}_2\times\boldsymbol{v}_3\right)$. Thus

$$\boldsymbol{k}_1\cdot\left(\boldsymbol{k}_2\times\boldsymbol{k}_3\right) = \frac{\boldsymbol{k}_1}{\left[\boldsymbol{v}_1\cdot\left(\boldsymbol{v}_2\times\boldsymbol{v}_3\right)\right]^2}\cdot\left[\boldsymbol{v}_1\cdot\left(\boldsymbol{v}_2\times\boldsymbol{v}_3\right)\right]\boldsymbol{v}_1=\frac{\boldsymbol{k}_1\cdot\boldsymbol{v}_1}{\boldsymbol{v}_1\cdot\left(\boldsymbol{v}_2\times\boldsymbol{v}_3\right)}$$

$$= \frac{1}{\boldsymbol{v}_1\cdot\left(\boldsymbol{v}_2\times\boldsymbol{v}_3\right)} \qquad \text{[by part (b)]}$$

1. **(a)** True; each of the first two lines has a direction vector parallel to the direction vector of the third line, so these vectors are each scalar multiples of the third direction vector. Then the first two direction vectors are also scalar multiples of each other, so these vectors, and hence the two lines, are parallel.
(b) False; for example, the x – and y –axes are both perpendicular to the z –axis, yet the x – and y –axes are not parallel.
(c) True; each of the first two planes has a normal vector parallel to the normal vector of the third plane, so these two normal vectors are parallel to each other and the planes are parallel.
(d) False; for example, the xy – and yz –planes are not parallel, yet they are both perpendicular to the xz –plane.
(e) False; the x – and y –axes are not parallel, yet they are both parallel to the plane $z=1$.
(f) True; if each line is perpendicular to a plane, then the lines' direction vectors are both parallel to a normal vector for the plane. Thus, the direction vectors are parallel to each other and the lines are parallel.
(g) False; the planes $y=1$ and $z=1$ are not parallel, yet they are both parallel to the x –axis.
(h) True; if each plane is perpendicular to a line, then any normal vector for each plane is parallel to a direction vector for the line. Thus, the normal vectors are parallel to each other and the planes are parallel.
(i) True; see Figure 9 and the accompanying discussion.
(j) False; they can be skew, as in Example 3.
(k) True. Consider any normal vector for the plane and any direction vector for the line. If the normal vector is perpendicular to the direction vector, the line and plane are parallel. Otherwise, the vectors meet at an angle θ , $0° \le \theta < 90°$, and the line will intersect the plane at an angle $90° - \theta$.

2. For this line, we have $\mathbf{r}_0 = \mathbf{i} - 3\mathbf{k}$ and $\mathbf{v} = 2\mathbf{i} - 4\mathbf{j} + 5\mathbf{k}$, so a vector equation is $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v} = (\mathbf{i} - 3\mathbf{k}) + t(2\mathbf{i} - 4\mathbf{j} + 5\mathbf{k}) = (1+2t)\mathbf{i} - 4t\mathbf{j} + (-3+5t)\mathbf{k}$ and parametric equations are $x=1+2t$, $y=-4t$, $z=-3+5t$.

3. For this line, we have $\mathbf{r}_0 = -2\mathbf{i} + 4\mathbf{j} + 10\mathbf{k}$ and $\mathbf{v} = 3\mathbf{i} + \mathbf{j} - 8\mathbf{k}$, so a vector equation is $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v} = (-2\mathbf{i} + 4\mathbf{j} + 10\mathbf{k}) + t(3\mathbf{i} + \mathbf{j} - 8\mathbf{k}) = (-2+3t)\mathbf{i} + (4+t)\mathbf{j} + (10-8t)\mathbf{k}$ and parametric equations are $x=-2+3t$, $y=4+t$, $z=10-8t$.
4. This line has the same direction as the given line, $\mathbf{v} = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$. Here $\mathbf{r}_0 = 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k}$, so a vector equation is $\mathbf{r} = (0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k}) + t(2\mathbf{i} - \mathbf{j} + 3\mathbf{k}) = 2t\mathbf{i} - t\mathbf{j} + 3t\mathbf{k}$ and parametric equations are $x=2t$, $y=-t$, $z=3t$.

5. A line perpendicular to the given plane has the same direction as a normal vector to the plane, such as $\mathbf{n} = \langle 1,3,1 \rangle$. So $\mathbf{r}_0 = \mathbf{i} + 6\mathbf{k}$, and we can take $\mathbf{v} = \mathbf{i} + 3\mathbf{j} + \mathbf{k}$. Then a vector equation is $\mathbf{r} = (\mathbf{i} + 6\mathbf{k}) + t(\mathbf{i} + 3\mathbf{j} + \mathbf{k}) = (1+t)\mathbf{i} + 3t\mathbf{j} + (6+t)\mathbf{k}$, and parametric equations are $x=1+t$, $y=3t$, $z=6+t$.
6. The vector $\mathbf{v} = \langle 1-0, 2-0, 3-0 \rangle = \langle 1,2,3 \rangle$ is parallel to the line. Letting $P_0 = (0,0,0)$, parametric

equations are $x=0+1\cdot t=t$, $y=0+2\cdot t=2t$, $z=0+3\cdot t=3t$, while symmetric equations are $x=\frac{y}{2}=\frac{z}{3}$.

7. The vector $\boldsymbol{v}=\langle -4-1,3-3,0-2\rangle=\langle -5,0,-2\rangle$ is parallel to the line. Letting $P_0=(1,3,2)$, parametric equations are $x=1-5t$, $y=3+0t=3$, $z=2-2t$, while symmetric equations are $\frac{x-1}{-5}=\frac{z-2}{-2}$,$y=3$. Notice here that the direction number $b=0$, so rather than writing $\frac{y-3}{0}$ in the symmetric equation we must write the equation $y=3$ separately.

8. $\boldsymbol{v}=\langle 2-6,4-1,5-(-3)\rangle=\langle -4,3,8\rangle$, and letting $P_0=(6,1,-3)$, parametric equations are $x=6-4t$, $y=1+3t$, $z=-3+8t$, while symmetric equations are $\frac{x-6}{-4}=\frac{y-1}{3}=\frac{z+3}{8}$.

9. $\boldsymbol{v}=\left\langle 2-0,1-\frac{1}{2},-3-1\right\rangle=\left\langle 2,\frac{1}{2},-4\right\rangle$, and letting $P_0=(2,1,-3)$, parametric equations are $x=2+2t$, $y=1+\frac{1}{2}t$, $z=-3-4t$, while symmetric equations are $\frac{x-2}{2}=\frac{y-1}{1/2}=\frac{z+3}{-4}$ or $\frac{x-2}{2}=2y-2=\frac{z+3}{-4}$.

10. $\boldsymbol{v}=(\boldsymbol{i}+\boldsymbol{j})\times(\boldsymbol{j}+\boldsymbol{k})=\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix}=\boldsymbol{i}-\boldsymbol{j}+\boldsymbol{k}$ is the direction of the line perpendicular to both $\boldsymbol{i}+\boldsymbol{j}$ and $\boldsymbol{j}+\boldsymbol{k}$. With $P_0=(2,1,0)$, parametric equations are $x=2+t$, $y=1-t$, $z=t$ and symmetric equations are $x-2=\frac{y-1}{-1}=z$ or $x-2=1-y=z$.

11. The line has direction $\boldsymbol{v}=\langle 1,2,1\rangle$. Letting $P_0=(1,-1,1)$, parametric equations are $x=1+t$, $y=-1+2t$, $z=1+t$ and symmetric equations are $x-1=\frac{y+1}{2}=z-1$.

12. Setting $x=0$, we see that $(0,1,0)$ satisfies the equations of both planes, so they do in fact have a line of intersection. $\boldsymbol{v}=\boldsymbol{n}_1\times\boldsymbol{n}_2=\langle 1,1,1\rangle\times\langle 1,0,1\rangle=\langle 1,0,-1\rangle$ is the direction of this line. Taking the point $(0,1,0)$ as P_0 , parametric equations are $x=t$, $y=1$, $z=-t$, and symmetric equations are $x=-z$, $y=1$.

13. Direction vectors of the lines are $\boldsymbol{v}_1=\langle -2-(-4),0-(-6),-3-1\rangle=\langle 2,6,-4\rangle$ and $\boldsymbol{v}_2=\langle 5-10,3-18,14-4\rangle=\langle -5,-15,10\rangle$, and since $\boldsymbol{v}_2=-\frac{5}{2}\boldsymbol{v}_1$, the direction vectors and thus the lines are

parallel.

14. Direction vectors of the lines are $\boldsymbol{v}_1=\langle -2,4,4\rangle$ and $\boldsymbol{v}_2=\langle 8,-1,4\rangle$. Since $\boldsymbol{v}_1\cdot \boldsymbol{v}_2=-16-4+16\neq 0$, the vectors and thus the lines are not perpendicular.

15. **(a)** A direction vector of the line with parametric equations $x=1+2t$, $y=3t$, $z=5-7t$ is $\boldsymbol{v}=\langle 2,3,-7\rangle$ and the desired parallel line must also have $\boldsymbol{v}$ as a direction vector. Here $P_0=(0,2,-1)$, so symmetric equations for the line are $\frac{x}{2}=\frac{y-2}{3}=\frac{z+1}{-7}$.
(b) The line intersects the xy –plane when $z=0$, so we need $\frac{x}{2}=\frac{y-2}{3}=\frac{1}{-7}$ or $x=-\frac{2}{7}$, $y=\frac{11}{7}$. Thus the point of intersection with the xy –plane is $\left(-\frac{2}{7},\frac{11}{7},0\right)$. Similarly for the yz –plane, we need $x=0\Leftrightarrow 0=\frac{y-2}{3}=\frac{z+1}{-7}\Leftrightarrow y=2$, $z=-1$. Thus the line intersects the yz –plane at $(0,2,-1)$. For the xz –plane, we need $y=0\Leftrightarrow \frac{x}{2}=-\frac{2}{3}=\frac{z+1}{-7}\Leftrightarrow x=-\frac{4}{3}$, $z=\frac{11}{3}$. So the line intersects the xz –plane at $\left(-\frac{4}{3},0,\frac{11}{3}\right)$.

16. **(a)** A vector normal to the plane $2x-y+z=1$ is $\boldsymbol{n}=\langle 2,-1,1\rangle$, and since the line is to be perpendicular to the plane, $\boldsymbol{n}$ is also a direction vector for the line. Thus parametric equations of the line are $x=5+2t$, $y=1-t$, $z=t$.
(b) On the xy –plane, $z=0$. So $z=t=0$ in the parametric equations of the line, and therefore $x=5$ and $y=1$, giving the point of intersection $(5,1,0)$. For the yz –plane, $x=0$ which implies $t=-\frac{5}{2}$, so $y=\frac{7}{2}$ and $z=-\frac{5}{2}$ and the point is $\left(0,\frac{7}{2},-\frac{5}{2}\right)$. For the xz –plane, $y=0$ which implies $t=1$, so $x=7$ and $z=1$ and the point of intersection is $(7,0,1)$.

17. From Equation 4, the line segment from $\boldsymbol{r}_0=2\boldsymbol{i}-\boldsymbol{j}+4\boldsymbol{k}$ to $\boldsymbol{r}_1=4\boldsymbol{i}+6\boldsymbol{j}+\boldsymbol{k}$ is
$\boldsymbol{r}(t)=(1-t)\boldsymbol{r}_0+t\boldsymbol{r}_1=(1-t)(2\boldsymbol{i}-\boldsymbol{j}+4\boldsymbol{k})+t(4\boldsymbol{i}+6\boldsymbol{j}+\boldsymbol{k})=(2\boldsymbol{i}-\boldsymbol{j}+4\boldsymbol{k})+t(2\boldsymbol{i}+7\boldsymbol{j}-3\boldsymbol{k})$, $0\leq t\leq 1$.

18. From Equation 4, the line segment from $\boldsymbol{r}_0=10\boldsymbol{i}+3\boldsymbol{j}+\boldsymbol{k}$ to $\boldsymbol{r}_1=5\boldsymbol{i}+6\boldsymbol{j}-3\boldsymbol{k}$ is

$$\begin{aligned}\boldsymbol{r}(t)&=(1-t)\boldsymbol{r}_0+t\boldsymbol{r}_1=(1-t)(10\boldsymbol{i}+3\boldsymbol{j}+\boldsymbol{k})+t(5\boldsymbol{i}+6\boldsymbol{j}-3\boldsymbol{k})\\&=(10\boldsymbol{i}+3\boldsymbol{j}+\boldsymbol{k})+t(-5\boldsymbol{i}+3\boldsymbol{j}-4\boldsymbol{k})\ ,\ 0\leq t\leq 1\end{aligned}$$

The corresponding parametric equations are $x=10-5t$, $y=3+3t$, $z=1-4t$, $0\le t\le 1$.

19. Since the direction vectors are $\boldsymbol{v}_1=\langle -6,9,-3\rangle$ and $\boldsymbol{v}_2=\langle 2,-3,1\rangle$, we have $\boldsymbol{v}_1=-3\boldsymbol{v}_2$ so the lines are parallel.

20. The lines aren't parallel since the direction vectors $\langle 2,3,-1\rangle$ and $\langle 1,1,3\rangle$ aren't parallel. For the lines to intersect we must be able to find one value of t and one value of s that produce the same point from the respective parametric equations. Thus we need to satisfy the following three equations: $1+2t=-1+s$, $3t=4+s$, $2-t=1+3s$. Solving the first two equations we get $t=6$, $s=14$ and checking, we see that these values don't satisfy the third equation. Thus L_1 and L_2 aren't parallel and don't intersect, so they must be skew lines.

21. Since the direction vectors $\langle 1,2,3\rangle$ and $\langle -4,-3,2\rangle$ are not scalar multiples of each other, the lines are not parallel, so we check to see if the lines intersect. The parametric equations of the lines are L_1 : $x=t$, $y=1+2t$, $z=2+3t$ and L_2 : $x=3-4s$, $y=2-3s$, $z=1+2s$. For the lines to intersect, we must be able to find one value of t and one value of s that produce the same point from the respective parametric equations. Thus we need to satisfy the following three equations: $t=3-4s$, $1+2t=2-3s$, $2+3t=1+2s$. Solving the first two equations we get $t=-1$, $s=1$ and checking, we see that these values don't satisfy the third equation. Thus the lines aren't parallel and don't intersect, so they must be skew lines.

22. Since the direction vectors $\langle 2,2,-1\rangle$ and $\langle 1,-1,3\rangle$ aren't parallel, the lines aren't parallel. Here the parametric equations are L_1 : $x=1+2t$, $y=3+2t$, $z=2-t$ and L_2 : $x=2+s$, $y=6-s$, $z=-2+3s$. Thus, for the lines to intersect, the three equations $1+2t=2+s$, $3+2t=6-s$, and $2-t=-2+3s$ must be satisfied simultaneously. Solving the first two equations gives $t=1$, $s=1$ and, checking, we see that these values do satisfy the third equation, so the lines intersect when $t=1$ and $s=1$, that is, at the point $(3,5,1)$.

23. Since the plane is perpendicular to the vector $\langle -2,1,5\rangle$, we can take $\langle -2,1,5\rangle$ as a normal vector to the plane. $(6,3,2)$ is a point on the plane, so setting $a=-2$, $b=1$, $c=5$ and $x_0=6$, $y_0=3$, $z_0=2$ in Equation 7 gives $-2(x-6)+1(y-3)+5(z-2)=0$ or $-2x+y+5z=1$ to be an equation of the plane.

24. $\boldsymbol{j}+2\boldsymbol{k}=\langle 0,1,2\rangle$ is a normal vector to the plane and $(4,0,-3)$ is a point on the plane, so setting $a=0$, $b=1$, $c=2,x_0=4$, $y_0=0$, $z_0=-3$ in Equation 7 gives $0(x-4)+1(y-0)+2[z-(-3)]=0$ or $y+2z=-6$ to be an equation of the plane.

25. $\boldsymbol{i}+\boldsymbol{j}-\boldsymbol{k}=\langle 1,1,-1\rangle$ is a normal vector to the plane and $(1,-1,1)$ is a point on the plane, so setting $a=1$, $b=1$, $c=-1$, $x_0=1$, $y_0=-1$, $z_0=1$ in Equation 7 gives $1(x-1)+1[y-(-1)]-1(z-1)=0$ or $x+y-z=-1$ to be an equation of the plane.

26. Since the line is perpendicular to the plane, its direction vector $\langle 1,2,-3\rangle$ is a normal vector to the plane. An equation of the plane, then, is $1[x-(-2)]+2(y-8)-3(z-10)=0$ or $x+2y-3z=-16$.

27. Since the two planes are parallel, they will have the same normal vectors. So we can take $\boldsymbol{n}=\langle 2,-1,3\rangle$, and an equation of the plane is $2(x-0)-1(y-0)+3(z-0)=0$ or $2x-y+3z=0$.

28. Since the two planes are parallel, they will have the same normal vectors. So we can take $\boldsymbol{n}=\langle 1,1,1\rangle$, and an equation of the plane is $1[x-(-1)]+1(y-6)+1[z-(-5)]=0$ or $x+y+z=0$.

29. Since the two planes are parallel, they will have the same normal vectors. So we can take $\boldsymbol{n}=\langle 3,0,-7\rangle$, and an equation of the plane is $3(x-4)+0[y-(-2)]-7(z-3)=0$ or $3x-7z=-9$.

30. First, a normal vector for the plane $2x+4y+8z=17$ is $\boldsymbol{n}=\langle 2,4,8\rangle$. A direction vector for the line is $\boldsymbol{v}=\langle 2,1,-1\rangle$, and since $\boldsymbol{n}\cdot\boldsymbol{v}=0$ we know the line is perpendicular to $\boldsymbol{n}$ and hence parallel to the plane. Thus, there is a parallel plane which contains the line. By putting $t=0$, we know the point $(3,0,8)$ is on the line and hence the new plane. We can use the same normal vector $\boldsymbol{n}=\langle 2,4,8\rangle$, so an equation of the plane is $2(x-3)+4(y-0)+8(z-8)=0$ or $x+2y+4z=35$.

31. Here the vectors $\boldsymbol{a}=\langle 1-0,0-1,1-1\rangle=\langle 1,-1,0\rangle$ and $\boldsymbol{b}=\langle 1-0,1-1,0-1\rangle=\langle 1,0,-1\rangle$ lie in the plane, so $\boldsymbol{a}\times\boldsymbol{b}$ is a normal vector to the plane. Thus, we can take $\boldsymbol{n}=\boldsymbol{a}\times\boldsymbol{b}=\langle 1-0,0+1,0+1\rangle=\langle 1,1,1\rangle$. If P_0 is the point $(0,1,1)$, an equation of the plane is $1(x-0)+1(y-1)+1(z-1)=0$ or $x+y+z=2$.

32. Here the vectors $\boldsymbol{a}=\langle 2,-4,6\rangle$ and $\boldsymbol{b}=\langle 5,1,3\rangle$ lie in the plane, so $\boldsymbol{n}=\boldsymbol{a}\times\boldsymbol{b}=\langle -12-6,30-6,2+20\rangle=\langle -18,24,22\rangle$ is a normal vector to the plane and an equation of the plane is $-18(x-0)+24(y-0)+22(z-0)=0$ or $-18x+24y+22z=0$.

33. Here the vectors $\boldsymbol{a}=\langle 8-3,2-(-1),4-2\rangle=\langle 5,3,2\rangle$ and $\boldsymbol{b}=\langle -1-3,-2-(-1),-3-2\rangle=\langle -4,-1,-5\rangle$ lie in the plane, so a normal vector to the plane is $\boldsymbol{n}=\boldsymbol{a}\times\boldsymbol{b}=\langle -15+2,-8+25,-5+12\rangle=\langle -13,17,7\rangle$ and an equation of the plane is $-13(x-3)+17[y-(-1)]+7(z-2)=0$ or $-13x+17y+7z=-42$.

34. If we first find two nonparallel vectors in the plane, their cross product will be a normal vector to the plane. Since the given line lies in the plane, its direction vector $\boldsymbol{a}=\langle 3,1,-1\rangle$ is one vector in the plane. We can verify that the given point $(1,2,3)$ does not lie on this line, so to find another nonparallel vector $\boldsymbol{b}$ which lies in the plane, we can pick any point on the line and find a vector connecting the points. If we put $t=0$, we see that $(0,1,2)$ is on the line, so $\boldsymbol{b}=\langle 1-0,2-1,3-2\rangle=\langle 1,1,1\rangle$ and $\boldsymbol{n}=\boldsymbol{a}\times\boldsymbol{b}=\langle 1+1,-1-3,3-1\rangle=\langle 2,-4,2\rangle$. Thus, an equation of the plane is $2(x-1)-4(y-2)+2(z-3)=0$ or $2x-4y+2z=0$. (Equivalently, we can write $x-2y+z=0$.)

35. If we first find two nonparallel vectors in the plane, their cross product will be a normal vector to

the plane. Since the given line lies in the plane, its direction vector $\boldsymbol{a}=\langle -2,5,4\rangle$ is one vector in the plane. We can verify that the given point $(6,0,-2)$ does not lie on this line, so to find another nonparallel vector $\boldsymbol{b}$ which lies in the plane, we can pick any point on the line and find a vector connecting the points. If we put $t=0$, we see that $(4,3,7)$ is on the line, so $\boldsymbol{b}=\langle 6-4,0-3,-2-7\rangle=\langle 2,-3,-9\rangle$ and $\boldsymbol{n}=\boldsymbol{a}\times \boldsymbol{b}=\langle -45+12,8-18,6-10\rangle=\langle -33,-10,-4\rangle$. Thus, an equation of the plane is $-33(x-6)-10(y-0)-4[z-(-2)]=0$ or $33x+10y+4z=190$.

36. Since the line $x=2y=3z$, or $x=\frac{y}{1/2}=\frac{z}{1/3}$, lies in the plane, its direction vector $\boldsymbol{a}=\left\langle 1,\frac{1}{2},\frac{1}{3}\right\rangle$ is parallel to the plane. The point $(0,0,0)$ is on the line (put $t=0$), and we can verify that the given point $(1,-1,1)$ in the plane is not on the line. The vector connecting these two points, $\boldsymbol{b}=\langle 1,-1,1\rangle$, is therefore parallel to the plane, but not parallel to $\langle 1,2,3\rangle$. Then $\boldsymbol{a}\times \boldsymbol{b}=\left\langle \frac{1}{2}+\frac{1}{3},\frac{1}{3}-1,-1-\frac{1}{2}\right\rangle=\left\langle \frac{5}{6},-\frac{2}{3},-\frac{3}{2}\right\rangle$ is a normal vector to the plane, and an equation of the plane is $\frac{5}{6}(x-0)-\frac{2}{3}(y-0)-\frac{3}{2}(z-0)=0$ or $5x-4y-9z=0$.

37. A direction vector for the line of intersection is $\boldsymbol{a}=\boldsymbol{n}_1\times \boldsymbol{n}_2=\langle 1,1,-1\rangle\times\langle 2,-1,3\rangle=\langle 2,-5,-3\rangle$, and $\boldsymbol{a}$ is parallel to the desired plane. Another vector parallel to the plane is the vector connecting any point on the line of intersection to the given point $(-1,2,1)$ in the plane. Setting $x=0$, the equations of the planes reduce to $y-z=2$ and $-y+3z=1$ with simultaneous solution $y=\frac{7}{2}$ and $z=\frac{3}{2}$. So a point on the line is $\left(0,\frac{7}{2},\frac{3}{2}\right)$ and another vector parallel to the plane is $\left\langle -1,-\frac{3}{2},-\frac{1}{2}\right\rangle$. Then a normal vector to the plane is $\boldsymbol{n}=\langle 2,-5,-3\rangle\times\left\langle -1,-\frac{3}{2},-\frac{1}{2}\right\rangle=\langle -2,4,-8\rangle$ and an equation of the plane is $-2(x+1)+4(y-2)-8(z-1)=0$ or $x-2y+4z=-1$.

38. $\boldsymbol{n}_1=\langle 1,0,-1\rangle$ and $\boldsymbol{n}_2=\langle 0,1,2\rangle$. Setting $z=0$, it is easy to see that $(1,3,0)$ is a point on the line of intersection of $x-z=1$ and $y+2z=3$. The direction of this line is $\boldsymbol{v}_1=\boldsymbol{n}_1\times \boldsymbol{n}_2=\langle 1,-2,1\rangle$. A second vector parallel to the desired plane is $\boldsymbol{v}_2=\langle 1,1,-2\rangle$, since it is perpendicular to $x+y-2z=1$. Therefore, a normal of the plane in question is $\boldsymbol{n}=\boldsymbol{v}_1\times \boldsymbol{v}_2=\langle 4-1,1+2,1+2\rangle=\langle 3,3,3\rangle$, or we can use $\langle 1,1,1\rangle$. Taking $\left(x_0,y_0,z_0\right)=(1,3,0)$, the equation we are looking for is $(x-1)+(y-3)+z=0 \Leftrightarrow x+y+z=4$.

39. Substitute the parametric equations of the line into the equation of the plane: $(3-t)-(2+t)+2(5t)=9$

$\Rightarrow 8t=8 \Rightarrow t=1$. Therefore, the point of intersection of the line and the plane is given by $x=3-1=2$, $y=2+1=3$, and $z=5(1)=5$, that is, the point $(2,3,5)$.

40. Substitute the parametric equations of the line into the equation of the plane: $(1+2t)+2(4t)-(2-3t)+1=0 \Rightarrow 13t=0 \Rightarrow t=0$. Therefore, the point of intersection of the line and the plane is given by $x=1+2(0)=1$, $y=4(0)=0$, and $z=2-3(0)=2$, that is, the point $(1,0,2)$.

41. Parametric equations for the line are $x=t$, $y=1+t$, $z=\frac{1}{2}t$ and substituting into the equation of the plane gives $4(t)-(1+t)+3\left(\frac{1}{2}t\right)=8 \Rightarrow \frac{9}{2}t=9 \Rightarrow t=2$. Thus $x=2$, $y=1+2=3$, $z=\frac{1}{2}(2)=1$ and the point of intersection is $(2,3,1)$.

42. A direction vector for the line through $(1,0,1)$ and $(4,-2,2)$ is $\boldsymbol{v}=\langle 3,-2,1\rangle$ and, taking $P_0=(1,0,1)$, parametric equations for the line are $x=1+3t$, $y=-2t$, $z=1+t$. Substitution of the parametric equations into the equation of the plane gives $1+3t-2t+1+t=6 \Rightarrow t=2$. Then $x=1+3(2)=7$, $y=-2(2)=-4$, and $z=1+2=3$ so the point of intersection is $(7,-4,3)$.

43. Setting $x=0$, we see that $(0,1,0)$ satisfies the equations of both planes, so that they do in fact have a line of intersection. $\boldsymbol{v}=\boldsymbol{n}_1\times\boldsymbol{n}_2=\langle 1,1,1\rangle\times\langle 1,0,1\rangle=\langle 1,0,-1\rangle$ is the direction of this line. Therefore, direction numbers of the intersecting line are 1 , 0 , -1 .

44. The angle between the two planes is the same as the angle between their normal vectors. The normal vectors of the two planes are $\langle 1,1,1\rangle$ and $\langle 1,2,3\rangle$. The cosine of the angle θ between these two planes is $\cos\theta=\dfrac{\langle 1,1,1\rangle\cdot\langle 1,2,3\rangle}{|\langle 1,1,1\rangle|\,|\langle 1,2,3\rangle|}=\dfrac{1+2+3}{\sqrt{1+1+1}\sqrt{1+4+9}}=\dfrac{6}{\sqrt{42}}=\sqrt{\dfrac{6}{7}}$.

45. Normal vectors for the planes are $\boldsymbol{n}_1=\langle 1,4,-3\rangle$ and $\boldsymbol{n}_2=\langle -3,6,7\rangle$, so the normals (and thus the planes) aren't parallel. But $\boldsymbol{n}_1\cdot\boldsymbol{n}_2=-3+24-21=0$, so the normals (and thus the planes) are perpendicular.

46. Normal vectors for the planes are $\boldsymbol{n}_1=\langle -1,4,-2\rangle$ and $\boldsymbol{n}_2=\langle 3,-12,6\rangle$. Since $\boldsymbol{n}_2=-3\boldsymbol{n}_1$, the normals (and thus the planes) are parallel.

47. Normal vectors for the planes are $\boldsymbol{n}_1=\langle 1,1,1\rangle$ and $\boldsymbol{n}_2=\langle 1,-1,1\rangle$. The normals are not parallel, so neither are the planes. Furthermore, $\boldsymbol{n}_1\cdot\boldsymbol{n}_2=1-1+1=1\neq 0$, so the planes aren't perpendicular. The angle between them is given by

$\cos\theta = \dfrac{\boldsymbol{n}_1 \cdot \boldsymbol{n}_2}{|\boldsymbol{n}_1|\,|\boldsymbol{n}_2|} = \dfrac{1}{\sqrt{3}\sqrt{3}} = \dfrac{1}{3} \Rightarrow \theta = \cos^{-1}\left(\dfrac{1}{3}\right) \approx 70.5°$.

48. The normals are $\boldsymbol{n}_1 = \langle 2,-3,4\rangle$ and $\boldsymbol{n}_2 = \langle 1,6,4\rangle$ so the planes aren't parallel. Since $\boldsymbol{n}_1 \cdot \boldsymbol{n}_2 = 2-18+16=0$, the normals (and thus the planes) are perpendicular.

49. The normals are $\boldsymbol{n}_1 = \langle 1,-4,2\rangle$ and $\boldsymbol{n}_2 = \langle 2,-8,4\rangle$. Since $\boldsymbol{n}_2 = 2\boldsymbol{n}_1$, the normals (and thus the planes) are parallel.

50. The normal vectors are $\boldsymbol{n}_1 = \langle 1,2,2\rangle$ and $\boldsymbol{n}_2 = \langle 2,-1,2\rangle$. The normals are not parallel, so neither are the planes. Furthermore, $\boldsymbol{n}_1 \cdot \boldsymbol{n}_2 = 2-2+4=4 \neq 0$, so the planes aren't perpendicular. The angle between them is given by $\cos\theta = \dfrac{\boldsymbol{n}_1 \cdot \boldsymbol{n}_2}{|\boldsymbol{n}_1|\,|\boldsymbol{n}_2|} = \dfrac{4}{\sqrt{9}\sqrt{9}} = \dfrac{4}{9} \Rightarrow \theta = \cos^{-1}\left(\dfrac{4}{9}\right) \approx 63.6°$.

51. **(a)** To find a point on the line of intersection, set one of the variables equal to a constant, say $z=0$. (This will only work if the line of intersection crosses the xy –plane; otherwise, try setting x or y equal to 0 .) Then the equations of the planes reduce to $x+y=2$ and $3x-4y=6$. Solving these two equations gives $x=2$, $y=0$. So a point on the line of intersection is $(2,0,0)$. The direction of the line is $\boldsymbol{v} = \boldsymbol{n}_1 \times \boldsymbol{n}_2 = \langle 5-4,-3-5,-4-3\rangle = \langle 1,-8,-7\rangle$, and symmetric equations for the line are $x-2 = \dfrac{y}{-8} = \dfrac{z}{-7}$.

(b) The angle between the planes satisfies $\cos\theta = \dfrac{\boldsymbol{n}_1 \cdot \boldsymbol{n}_2}{|\boldsymbol{n}_1|\,|\boldsymbol{n}_2|} = \dfrac{3-4-5}{\sqrt{3}\sqrt{50}} = -\dfrac{\sqrt{6}}{5}$. Therefore $\theta = \cos^{-1}\left(-\dfrac{\sqrt{6}}{5}\right) \approx 119°$ (or $61°$).

52. **(a)** $x-2y+z=1 \Rightarrow \boldsymbol{n}_1 = \langle 1,-2,1\rangle$ and $2x+y+z=1 \Rightarrow \boldsymbol{n}_2 = \langle 2,1,1\rangle$. The vector that gives the direction of the line of intersection of these two planes is $\boldsymbol{v} = \boldsymbol{n}_1 \times \boldsymbol{n}_2 = \langle -2-1,2-1,1+4\rangle = \langle -3,1,5\rangle$. Setting $x=y=0$, we see that both planes contain $(0,0,1)$ so that this point must lie on their line of intersection. Then symmetric equations for this line are $\dfrac{x}{-3} = y = \dfrac{z-1}{5}$.

(b) $\cos\theta = \dfrac{\boldsymbol{n}_1 \cdot \boldsymbol{n}_2}{|\boldsymbol{n}_1|\,|\boldsymbol{n}_2|} = \dfrac{2-2+1}{\sqrt{1+4+1}\sqrt{4+1+1}} = \dfrac{1}{6} \Rightarrow \theta = \cos^{-1}\left(\dfrac{1}{6}\right) \approx 80°$.

53. Setting $x=0$, the equations of the two planes become $z=y$ and $5y+z=-1$, which intersect at $y=-\frac{1}{6}$ and $z=-\frac{1}{6}$. Thus we can choose $\left(x_0,y_0,z_0\right)=\left(0,-\frac{1}{6},-\frac{1}{6}\right)$. The vector giving the direction of this intersecting line, $\boldsymbol{v}$, is perpendicular to the normal vectors of both planes. So $\boldsymbol{v}=\boldsymbol{n}_1\times\boldsymbol{n}_2=\langle 2,-5,-1\rangle\times\langle 1,1,-1\rangle=\langle 6,1,7\rangle$. Therefore, by Equations 2, parametric equations for this line are $x=6t$, $y=-\frac{1}{6}+t$, $z=-\frac{1}{6}+7t$.

54. Setting $y=0$, the equations of the two planes become $2x+5z=-3$ and $x+z=-2$, which intersect at $x=-\frac{7}{3}$ and $z=\frac{1}{3}$. Thus we can choose $\left(x_0,y_0,z_0\right)=\left(-\frac{7}{3},0,\frac{1}{3}\right)$. The vector giving the direction of this intersecting line, $\boldsymbol{v}$, is perpendicular to the normal vectors of both planes. So $\boldsymbol{v}=\boldsymbol{n}_1\times\boldsymbol{n}_2=\langle 2,0,5\rangle\times\langle 1,-3,1\rangle=\langle 15,5-2,-6\rangle=3\langle 5,1,-2\rangle$. Therefore, by Equations 2, parametric equations of the line of intersection of the two planes are $x=-\frac{7}{3}+5t$, $y=t$, $z=\frac{1}{3}-2t$.

55. The plane contains all perpendicular bisectors of the line segment joining $(1,1,0)$ and $(0,1,1)$. All of these bisectors pass through the midpoint of this segment $\left(\frac{1}{2},\frac{1+1}{2},\frac{1}{2}\right)=\left(\frac{1}{2},1,\frac{1}{2}\right)$. The direction of this line segment $\langle 1-0,1-1,0-1\rangle=\langle 1,0,-1\rangle$ is perpendicular to the plane so that we can choose this to be $\boldsymbol{n}$. Therefore the equation of the plane is $1\left(x-\frac{1}{2}\right)+0(y-1)-1\left(z-\frac{1}{2}\right)=0\Leftrightarrow x=z$.

56. The plane will contain all perpendicular bisectors of the line segment joining the two points. Thus, a point in the plane is $P_0=(-1,-1,2)$, the midpoint of the line segment joining the two given points, and a normal to the plane is $\boldsymbol{n}=\langle 6,-6,2\rangle$, the vector connecting the two points. So an equation of the plane is $6(x+1)-6(y+1)+2(z-2)=0$ or $3x-3y+z=2$.

57. The plane contains the points $(a,0,0)$, $(0,b,0)$ and $(0,0,c)$. Thus the vectors $\boldsymbol{a}=\langle -a,b,0\rangle$ and $\boldsymbol{b}=\langle -a,0,c\rangle$ lie in the plane, and $\boldsymbol{n}=\boldsymbol{a}\times\boldsymbol{b}=\langle bc-0,0+ac,0+ab\rangle=\langle bc,ac,ab\rangle$ is a normal vector to the plane. The equation of the plane is therefore $bcx+acy+abz=abc+0+0$ or $bcx+acy+abz=abc$. Notice that if $a\neq 0$, $b\neq 0$ and $c\neq 0$ then we can rewrite the equation as $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$. This is a good equation to remember!

58. **(a)** For the lines to intersect, we must be able to find one value of t and one value of s satisfying the three equations $1+t=2-s$, $1-t=s$ and $2t=2$. From the third we get $t=1$, and putting this in the second gives $s=0$. These values of s and t do satisfy the first equation, so the lines intersect at the

point $P_0=(1+1,1-1,2(1))=(2,0,2)$.

(b) The direction vectors of the lines are $\langle 1,-1,2\rangle$ and $\langle -1,1,0\rangle$, so a normal vector for the plane is $\langle -1,1,0\rangle\times\langle 1,-1,2\rangle=\langle 2,2,0\rangle$ and it contains the point $(2,0,2)$. Then the equation of the plane is $2(x-2)+2(y-0)+0(z-2)=0\Leftrightarrow x+y=2$.

59. Two vectors which are perpendicular to the required line are the normal of the given plane, $\langle 1,1,1\rangle$, and a direction vector for the given line, $\langle 1,-1,2\rangle$. So a direction vector for the required line is $\langle 1,1,1\rangle\times\langle 1,-1,2\rangle=\langle 3,-1,-2\rangle$. Thus L is given by $\langle x,y,z\rangle=\langle 0,1,2\rangle+t\langle 3,-1,-2\rangle$, or in parametric form, $x=3t$, $y=1-t$, $z=2-2t$.

60. Let L be the given line. Then $(1,1,0)$ is the point on L corresponding to $t=0$. L is in the direction of $\boldsymbol{a}=\langle 1,-1,2\rangle$ and $\boldsymbol{b}=\langle -1,0,2\rangle$ is the vector joining $(1,1,0)$ and $(0,1,2)$. Then $\boldsymbol{b}-\text{proj}_{\boldsymbol{a}}\boldsymbol{b}=\langle -1,0,2\rangle-\dfrac{\langle 1,-1,2\rangle\cdot\langle -1,0,2\rangle}{1^2+(-1)^2+2^2}\langle 1,-1,2\rangle=\langle -1,0,2\rangle-\dfrac{1}{2}\langle 1,-1,2\rangle=\left\langle -\dfrac{3}{2},\dfrac{1}{2},1\right\rangle$ is a direction vector for the required line. Thus $2\left\langle -\dfrac{3}{2},\dfrac{1}{2},1\right\rangle=\langle -3,1,2\rangle$ is also a direction vector, and the line has parametric equations $x=-3t$, $y=1+t$, $z=2+2t$. (Notice that this is the same line as in Exercise 59.)

61. Let P_i have normal vector $\boldsymbol{n}_i$. Then $\boldsymbol{n}_1=\langle 4,-2,6\rangle$, $\boldsymbol{n}_2=\langle 4,-2,-2\rangle$, $\boldsymbol{n}_3=\langle -6,3,-9\rangle$, $\boldsymbol{n}_4=\langle 2,-1,-1\rangle$. Now $\boldsymbol{n}_1=-\dfrac{2}{3}\boldsymbol{n}_3$, so $\boldsymbol{n}_1$ and $\boldsymbol{n}_3$ are parallel, and hence P_1 and P_3 are parallel; similarly P_2 and P_4 are parallel because $\boldsymbol{n}_2=2\boldsymbol{n}_4$. However, $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$ are not parallel. $\left(0,0,\dfrac{1}{2}\right)$ lies on P_1 , but not on P_3 , so they are not the same plane, but both P_2 and P_4 contain the point $(0,0,-3)$, so these two planes are identical.

62. Let L_i have direction vector $\boldsymbol{v}_i$. Then $\boldsymbol{v}_1=\langle 1,1,-5\rangle$, $\boldsymbol{v}_2=\langle 1,1,-1\rangle$, $\boldsymbol{v}_3=\langle 1,1,-1\rangle$, $\boldsymbol{v}_4=\langle 2,2,-10\rangle$. $\boldsymbol{v}_2$ and $\boldsymbol{v}_3$ are equal so they're parallel. $\boldsymbol{v}_4=2\boldsymbol{v}_1$, so L_4 and L_1 are parallel. L_3 contains the point $(1,4,1)$, but this point does not lie on L_2 , so they're not equal. $(2,1,-3)$ lies on L_4 , and on L_1 , with $t=1$. So L_1 and L_4 are identical.

63. Let $Q=(2,2,0)$ and $R=(3,-1,5)$, points on the line corresponding to $t=0$ and $t=1$. Let $P=(1,2,3)$. Then $\boldsymbol{a}=\overrightarrow{QR}=\langle 1,-3,5\rangle$, $\boldsymbol{b}=\overrightarrow{QP}=\langle -1,0,3\rangle$. The distance is

$$d=\frac{|\boldsymbol{a}\times\boldsymbol{b}|}{|\boldsymbol{a}|}=\frac{|\langle 1,-3,5\rangle\times\langle -1,0,3\rangle|}{|\langle 1,-3,5\rangle|}=\frac{|\langle -9,-8,-3\rangle|}{|\langle 1,-3,5\rangle|}=\frac{\sqrt{9^2+8^2+3^2}}{\sqrt{1^2+3^2+5^2}}=\frac{\sqrt{154}}{\sqrt{35}}=\sqrt{\frac{22}{5}}\ .$$

64. Let $Q=(5,0,1)$ and $R=(4,3,3)$, points on the line corresponding to $t=0$ and $t=1$. Let $P=(1,0,-1)$. Then $\boldsymbol{a}=\overrightarrow{QR}=\langle -1,3,2\rangle$ and $\boldsymbol{b}=\overrightarrow{QP}=\langle -4,0,-2\rangle$. The distance is

$$d=\frac{|\boldsymbol{a}\times\boldsymbol{b}|}{|\boldsymbol{a}|}=\frac{|\langle -1,3,2\rangle\times\langle -4,0,-2\rangle|}{|\langle -1,3,2\rangle|}=\frac{|\langle -6,-10,12\rangle|}{|\langle -1,3,2\rangle|}=\frac{2\sqrt{3^2+5^2+6^2}}{\sqrt{1^2+3^2+2^2}}=\frac{2\sqrt{70}}{\sqrt{14}}=2\sqrt{5}\ .$$

65. By Equation 9, the distance is $D=\frac{1}{\sqrt{1+4+4}}\left[(1)(2)+(-2)(8)+(-2)(5)-1\right]=\frac{25}{3}$.

66. By Equation 9, the distance is $D=\frac{1}{\sqrt{16+36+1}}\left[4(3)+(-6)(-2)+1(7)-5\right]=\frac{26}{\sqrt{53}}$.

67. Put $y=z=0$ in the equation of the first plane to get the point $(-1,0,0)$ on the plane. Because the planes are parallel, the distance D between them is the distance from $(-1,0,0)$ to the second plane. By Equation 9, $D=\frac{|3(-1)+6(0)-3(0)-4|}{\sqrt{3^2+6^2+(-3)^2}}=\frac{7}{3\sqrt{6}}$ or $\frac{7\sqrt{6}}{18}$.

68. Put $y=z=0$ in the equation of the first plane to get the point $\left(\frac{4}{3},0,0\right)$ on the plane. Because the planes are parallel the distance D between them is the distance from $\left(\frac{4}{3},0,0\right)$ to the second plane. By Equation 9, $D=\frac{\left|1\left(\frac{4}{3}\right)+2(0)-3(0)-1\right|}{\sqrt{1^2+2^2+(-3)^2}}=\frac{1}{3\sqrt{14}}$.

69. The distance between two parallel planes is the same as the distance between a point on one of the planes and the other plane. Let $P_0=\left(x_0,y_0,z_0\right)$ be a point on the plane given by $ax+by+cz+d_1=0$. Then $ax_0+by_0+cz_0+d_1=0$ and the distance between P_0 and the plane given by $ax+by+cz+d_2=0$ is, from Equation 9, $D=\frac{\left|ax_0+by_0+cz_0+d_2\right|}{\sqrt{a^2+b^2+c^2}}=\frac{\left|-d_1+d_2\right|}{\sqrt{a^2+b^2+c^2}}=\frac{\left|d_1-d_2\right|}{\sqrt{a^2+b^2+c^2}}$.

70. The planes must have parallel normal vectors, so if $ax+by+cz+d=0$ is such a plane, then for some

$t\neq 0$, $\langle a,b,c\rangle = t\langle 1,2,-2\rangle = \langle t,2t,-2t\rangle$. So this plane is given by the equation $x+2y-2z+e=0$, where $e=d/t$. By Exercise 69, the distance between the planes is $2=\dfrac{|1-e|}{\sqrt{1^2+2^2+(-2)^2}} \Leftrightarrow 6=|1-e| \Leftrightarrow e=7$ or -5 . So the desired planes have equations $x+2y-2z=7$ and $x+2y-2z=-5$.

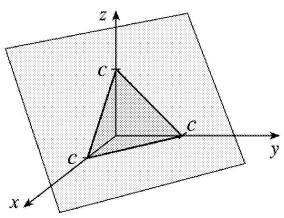
71. L_1 : $x=y=z \Rightarrow x=y$ (1). L_2 : $x+1=y/2=z/3 \Rightarrow x+1=y/2$ (2). The solution of (1) and (2) is $x=y=-2$. However, when $x=-2$, $x=z \Rightarrow z=-2$, but $x+1=z/3 \Rightarrow z=-3$, a contradiction. Hence the lines do not intersect. For L_1 , $\boldsymbol{v}_1=\langle 1,1,1\rangle$, and for L_2 , $\boldsymbol{v}_2=\langle 1,2,3\rangle$, so the lines are not parallel. Thus the lines are skew lines. If two lines are skew, they can be viewed as lying in two parallel planes and so the distance between the skew lines would be the same as the distance between these parallel planes. The common normal vector to the planes must be perpendicular to both $\langle 1,1,1\rangle$ and $\langle 1,2,3\rangle$, the direction vectors of the two lines. So set $\boldsymbol{n}=\langle 1,1,1\rangle \times \langle 1,2,3\rangle = \langle 3-2,-3+1,2-1\rangle = \langle 1,-2,1\rangle$. From above, we know that $(-2,-2,-2)$ and $(-2,-2,-3)$ are points of L_1 and L_2 respectively. So in the notation of Equation 8, $1(-2)-2(-2)+1(-2)+d_1=0 \Rightarrow d_1=0$ and $1(-2)-2(-2)+1(-3)+d_2=0 \Rightarrow d_2=1$.
By Exercise 69, the distance between these two skew lines is $D=\dfrac{|0-1|}{\sqrt{1+4+1}}=\dfrac{1}{\sqrt{6}}$.
Alternate solution (without reference to planes): A vector which is perpendicular to both of the lines is $\boldsymbol{n}=\langle 1,1,1\rangle \times \langle 1,2,3\rangle = \langle 1,-2,1\rangle$. Pick any point on each of the lines, say $(-2,-2,-2)$ and $(-2,-2,-3)$, and form the vector $\boldsymbol{b}=\langle 0,0,1\rangle$ connecting the two points. The distance between the two skew lines is the absolute value of the scalar projection of $\boldsymbol{b}$ along $\boldsymbol{n}$, that is, $D=\dfrac{|\boldsymbol{n}\cdot \boldsymbol{b}|}{|\boldsymbol{n}|}=\dfrac{|1\cdot 0-2\cdot 0+1\cdot 1|}{\sqrt{1+4+1}}=\dfrac{1}{\sqrt{6}}$.

72. First notice that if two lines are skew, they can be viewed as lying in two parallel planes and so the distance between the skew lines would be the same as the distance between these parallel planes. The common normal vector to the planes must be perpendicular to both $\boldsymbol{v}_1=\langle 1,6,2\rangle$ and $\boldsymbol{v}_2=\langle 2,15,6\rangle$, the direction vectors of the two lines respectively. Thus set $\boldsymbol{n}=\boldsymbol{v}_1\times \boldsymbol{v}_2=\langle 36-30,4-6,15-12\rangle = \langle 6,-2,3\rangle$. Setting $t=0$ and $s=0$ gives the points $(1,1,0)$ and $(1,5,-2)$. So in the notation of Equation 8, $6-2+0+d_1=0 \Rightarrow d_1=-4$ and $6-10-6+d_2=0 \Rightarrow d_2=10$. Then by Exercise 69 , the distance between the two skew lines is given by $D=\dfrac{|-4-10|}{\sqrt{36+4+9}}=\dfrac{14}{7}=2$.
Alternate solution (without reference to planes): We already know that the direction vectors of the two lines are $\boldsymbol{v}_1=\langle 1,6,2\rangle$ and $\boldsymbol{v}_2=\langle 2,15,6\rangle$. Then $\boldsymbol{n}=\boldsymbol{v}_1\times \boldsymbol{v}_2=\langle 6,-2,3\rangle$ is perpendicular to both lines. Pick any point on each of the lines, say $(1,1,0)$ and $(1,5,-2)$, and form the vector $\boldsymbol{b}=\langle 0,4,-2\rangle$ connecting the two points. Then the distance between the two skew lines is the absolute value of the scalar projection of $\boldsymbol{b}$ along $\boldsymbol{n}$, that is,

$$D=\frac{|\boldsymbol{n}\cdot \boldsymbol{b}|}{|\boldsymbol{n}|}=\frac{1}{\sqrt{36+4+9}}\ |0-8-6|=\frac{14}{7}=2\ .$$

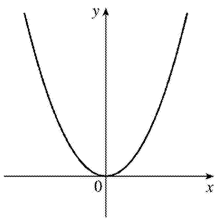
73. If $a\neq 0$, then $ax+by+cz+d=0\Rightarrow a\left(x+d/a\right)+b(y-0)+c(z-0)=0$ which by (7) is the scalar equation of the plane through the point $(-d/a,0,0)$ with normal vector $\langle a,b,c\rangle$. Similarly, if $b\neq 0$ (or if $c\neq 0$) the equation of the plane can be rewritten as $a(x-0)+b\left(y+d/b\right)+c(z-0)=0$ which by (7) is the scalar equation of a plane through the point $(0,-d/b,0)$ with normal vector $\langle a,b,c\rangle$.

74. **(a)** The planes $x+y+z=c$ have normal vector $\langle 1,1,1\rangle$, so they are all parallel. Their x –, y –, and z –intercepts are all c . When $c>0$ their intersection with the first octant is an equilateral triangle and when $c<0$ their intersection with the octant diagonally opposite the first is an equilateral triangle.

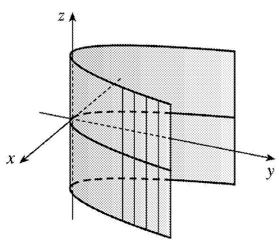


(b) The planes $x+y+cz=1$ have x –intercept 1 , y – intercept 1 , and z –intercept $1/c$. The plane with $c=0$ is parallel to the z –axis. As c gets larger, the planes get closer to the xy –plane.
(c) The planes $y\cos\theta+z\cos\theta=1$ have normal vectors $\langle 0,\cos\theta,\sin\theta\rangle$, which are perpendicular to the x –axis, and so the planes are parallel to the x –axis. We look at their intersection with the yz –plane. These are lines that are perpendicular to $\langle \cos\theta,\sin\theta\rangle$ and pass through $(\cos\theta,\sin\theta)$, since $\cos^2\theta+\sin^2\theta=1$. So these are the tangent lines to the unit circle. Thus the family consists of all planes tangent to the circular cylinder with radius 1 and axis the x –axis.

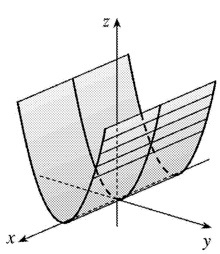
1. **(a)** In R^2, the equation $y=x^2$ represents a parabola.

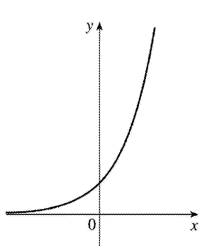


(b) In R^3, the equation $y=x^2$ doesn't involve z, so any horizontal plane with equation $z=k$ intersects the graph in a curve with equation $y=x^2$. Thus, the surface is a parabolic cylinder, made up of infinitely many shifted copies of the same parabola. The rulings are parallel to the z –axis.

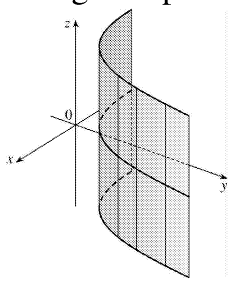


(c) In R^3, the equation $z=y^2$ also represents a parabolic cylinder. Since x doesn't appear, the graph is formed by moving the parabola $z=y^2$ in the direction of the x –axis. Thus, the rulings of the cylinder are parallel to the x –axis.

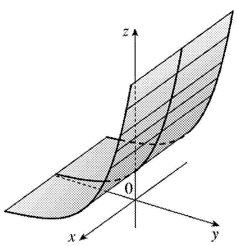


2. **(a)** 

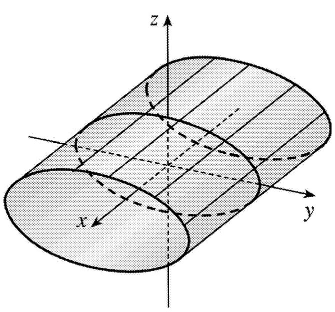
(b) Since the equation $y=e^x$ doesn't involve z , horizontal traces are copies of the curve $y=e^x$. The rulings are parallel to the z –axis.



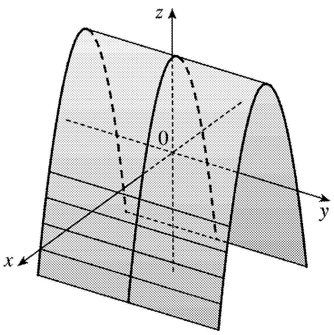
(c) The equation $z=e^y$ doesn't involve x , so vertical traces in $x=k$ (parallel to the yz –plane) are copies of the curve $z=e^y$. The rulings are parallel to the x –axis.



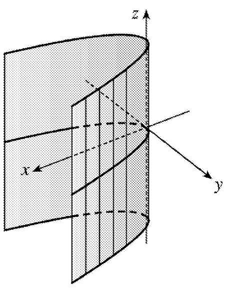
3. Since x is missing from the equation, the vertical traces $y^2+4z^2=4$, $x=k$, are copies of the same ellipse in the plane $x=k$. Thus, the surface $y^2+4z^2=4$ is an elliptic cylinder with rulings parallel to the x –axis.



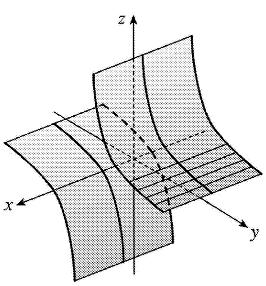
4. Since y is missing from the equation, each vertical trace $z=4-x^2$, $y=k$, is a copy of the same parabola in the plane $y=k$. Thus, the surface $z=4-x^2$ is a parabolic cylinder with rulings parallel to the y –axis.



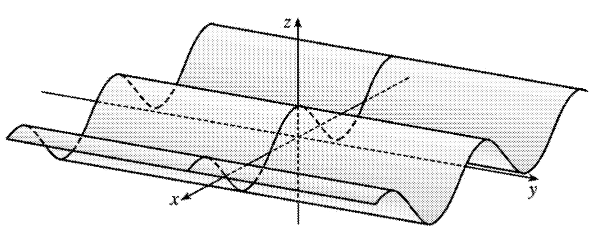
5. Since z is missing, each horizontal trace $x=y^2$, $z=k$, is a copy of the same parabola in the plane $z=k$. Thus, the surface $x-y^2=0$ is a parabolic cylinder with rulings parallel to the z –axis.



6. Since x is missing, each vertical trace $yz=4$, $x=k$ is a copy of the same hyperbola in the plane $x=k$. Thus, the surface $yz=4$ is a hyperbolic cylinder with rulings parallel to the x –axis.

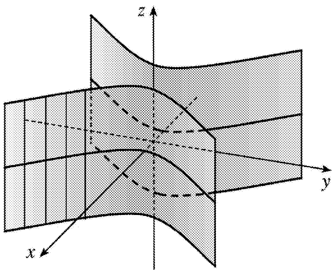


7. Since y is missing, each vertical trace $z=\cos x$, $y=k$ is a copy of a cosine curve in the plane $y=k$. Thus, the surface $z=\cos x$ is a cylindrical surface with rulings parallel to the y –axis.

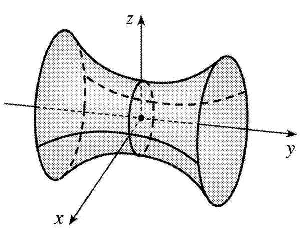


8. Since z is missing, each horizontal trace $x^2-y^2=1$, $z=k$ is a copy of the same hyperbola in the plane

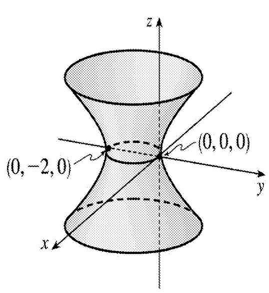
$z=k$. Thus, the surface $x^2-y^2=1$ is a hyperbolic cylinder with rulings parallel to the z –axis.



9. **(a)** The traces of $x^2+y^2-z^2=1$ in $x=k$ are $y^2-z^2=1-k^2$, a family of hyperbolas. (Note that the hyperbolas are oriented differently for $-1<k<1$ than for $k<-1$ or $k>1$.) The traces in $y=k$ are $x^2-z^2=1-k^2$, a similar family of hyperbolas. The traces in $z=k$ are $x^2+y^2=1+k^2$, a family of circles. For $k=0$, the trace in the xy –plane, the circle is of radius 1 . As $|k|$ increases, so does the radius of the circle. This behavior, combined with the hyperbolic vertical traces, gives the graph of the hyperboloid of one sheet in Table 1.
(b) The shape of the surface is unchanged, but the hyperboloid is rotated so that its axis is the y –axis. Traces in $y=k$ are circles, while traces in $x=k$ and $z=k$ are hyperbolas.

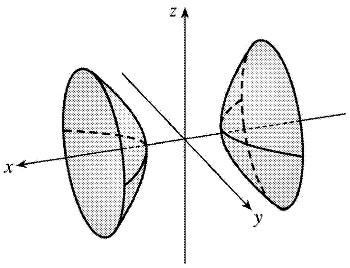


(c) Completing the square in y gives $x^2+(y+1)^2-z^2=1$. The surface is a hyperboloid identical to the one in part (a) but shifted one unit in the negative y –direction.

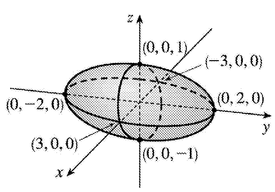


10. **(a)** The traces of $-x^2-y^2+z^2=1$ in $x=k$ are $-y^2+z^2=1+k^2$, a family of hyperbolas, as are the traces in $y=k$, $-x^2+z^2=1+k^2$. The traces in $z=k$ are $x^2+y^2=k^2-1$, a family of circles for $|k|>1$. As $|k|$ increases, the radii of the circles increase; the traces are empty for $|k|<1$. This behavior, combined with the vertical traces, gives the graph of the hyperboloid of two sheets in Table 1.

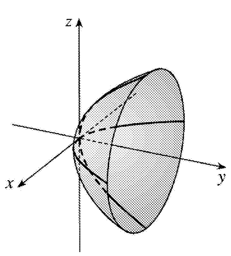
(b) The graph has the same shape as the hyperboloid in part (a) but is rotated so that its axis is the x –axis. Traces in $x=k$, $|k|>1$, are circles, while traces in $y=k$ and $z=k$ are hyperbolas.



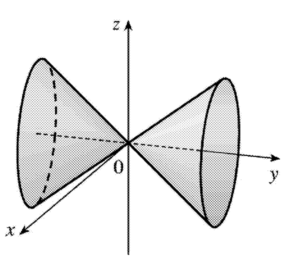
11. Traces: $x=k$, $9y^2+36z^2=36-4k^2$, an ellipse for $|k|<3$;
$y=k$, $4x^2+36z^2=36-9k^2$, an ellipse for $|k|<2$; $z=k$, $4x^2+9y^2=36(1-k^2)$, an ellipse for $|k|<1$. Thus the surface is an ellipsoid with center at the origin and axes along the x –, y – and z –axes.



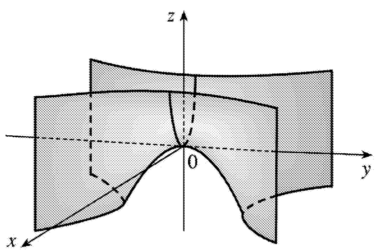
12. Traces: $x=k$, $4y=k^2+z^2$, a parabola; $y=k$, $4k=x^2+z^2$, a circle for $k>0$; $z=k$, $4y=x^2+k^2$ a parabola. Thus the surface is a circular paraboloid with axis the y –axis and vertex at $(0,0,0)$.



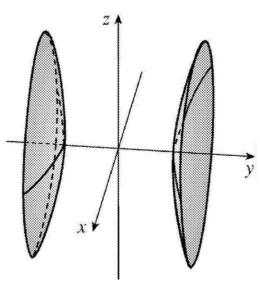
13. Traces: $x=k$, $y^2=k^2+z^2$ or $y^2-z^2=k^2$, a hyperbola for $k\neq 0$ and two intersecting lines for $k=0$; $y=k$, $x^2+z^2=k^2$, a circle for $k\neq 0$; $z=k$, $y^2=x^2+k^2$ or $y^2-x^2=k^2$, a hyperbola for $k\neq 0$ and two intersecting lines for $k=0$. Thus the surface is a cone (right circular) with axis the y –axis and vertex the origin.



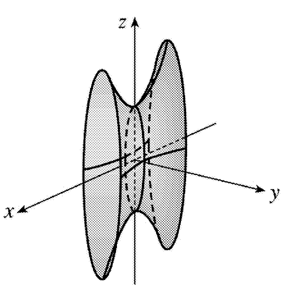
14. Traces: $x=k$, $z-k^2=-y^2$, a parabola; $y=k$, $z+k^2=x^2$, a parabola; $z=k$, $x^2-y^2=k$, a hyperbola. Thus the surface is a hyperbolic paraboloid with saddle point (0 , 0 , 0) (and since $c>0$, the saddle is upside down).



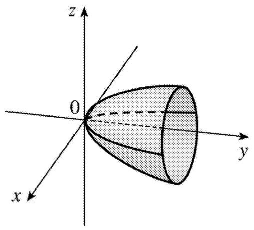
15. Traces: $x=k$, $4y^2-z^2=4+k^2$, a hyperbola; $y=k$, $x^2+z^2=4k^2-4$, a circle for $|k|>1$; $z=k$, $4y^2-x^2=4+k^2$, a hyperbola. Thus the surface is a hyperboloid of two sheets with axis the y –axis.



16. Traces: $x=k$, $25y^2+z^2=100+4k^2$, an ellipse; $y=k$, $25k^2+z^2=100+4x^2$ or $z^2-4x^2=100-25k^2$, a hyperbola for $|k|<2$; $z=k$, $25y^2+k^2=100+4x^2$ or $25y^2-4x^2=100-k^2$, a hyperbola for $|k|<10$. Thus the surface is a hyperboloid of one sheet with axis the x –axis.

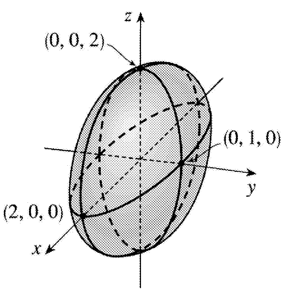


17. Traces: $x=k$, $k^2+4z^2-y=0$ or $y-k^2=4z^2$, a parabola;
$y=k$, $x^2+4z^2=k$, an ellipse for $k>0$; $z=k$, $x^2+4k^2-y=0$ or $y-4k^2=x^2$, a parabola. Thus the surface is an elliptic paraboloid with axis the y –axis and vertex the origin.

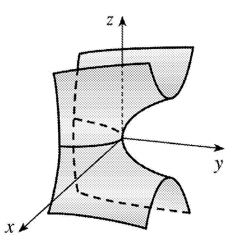


18. Traces: $x=k$, $|k|\leq 2\Rightarrow y^2+\frac{z^2}{4}=1-\frac{k^2}{4}$, ellipses;

$y=k$, $|k|\leq 1\Rightarrow x^2+z^2=4\left(1-k^2\right)$, circles; $z=k$, $|k|\leq 2\Rightarrow \frac{x^2}{4}+y^2=1-\frac{k^2}{4}$, ellipses. $x^2+4y^2+z^2=4\Leftrightarrow$ $\frac{x^2}{2^2}+\frac{y^2}{1^2}+\frac{z^2}{2^2}=1$, which is the equation of an ellipsoid.

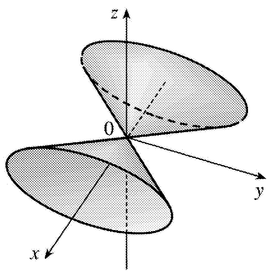


19. $y=z^2-x^2$. The traces in $x=k$ are the parabolas $y=z^2-k^2$;
the traces in $y=k$ are $k=z^2-x^2$, which are hyperbolas (note the hyperbolas are oriented differently for $k>0$ than for $k<0$); and the traces in $z=k$ are the parabolas $y=k^2-x^2$. Thus, $\frac{y}{1}=\frac{z^2}{1^2}-\frac{x^2}{1^2}$ is a hyperbolic paraboloid.



20. Traces: $x=k\Rightarrow y^2+4z^2=16k^2$, ellipses; $y=k\Rightarrow 16x^2-4z^2=k^2$, hyperbolas if $k\neq 0$ and two intersecting lines if $k=0$; $z=k\Rightarrow 16x^2-y^2=4k^2$, hyperbolas if $k\neq 0$ and two intersecting lines if $k=0$.
$16x^2=y^2+4z^2\Leftrightarrow$

$x^2 = \frac{y^2}{4^2} + \frac{z^2}{2^2}$ is an elliptic cone with axis the x –axis and vertex the origin.



21. This is the equation of an ellipsoid: $x^2+4y^2+9z^2=x^2+\frac{y^2}{(1/2)^2}+\frac{z^2}{(1/3)^2}=1$, with x –intercepts ± 1 , y –intercepts $\pm \frac{1}{2}$ and z –intercepts $\pm \frac{1}{3}$. So the major axis is the x –axis and the only possible graph is VII.

22. This is the equation of an ellipsoid: $9x^2+4y^2+z^2=\frac{x^2}{(1/3)^2}+\frac{y^2}{(1/2)^2}+z^2=1$, with x –intercepts $\pm \frac{1}{3}$, y –intercepts $\pm \frac{1}{2}$ and z –intercepts ± 1 . So the major axis is the z –axis and the only possible graph is IV.

23. This is the equation of a hyperboloid of one sheet, with $a=b=c=1$. Since the coefficient of y^2 is negative, the axis of the hyperboloid is the y – axis, hence the correct graph is II.

24. This is a hyperboloid of two sheets, with $a=b=c=1$. This surface does not intersect the xz –plane at all, so the axis of the hyperboloid is the y –axis and the graph is III.

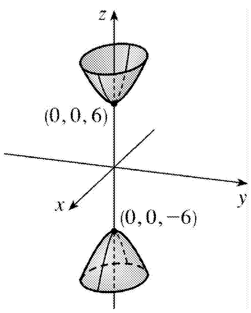
25. There are no real values of x and z that satisfy this equation for $y<0$, so this surface does not extend to the left of the xz –plane. The surface intersects the plane $y=k>0$ in an ellipse. Notice that y occurs to the first power whereas x and z occur to the second power. So the surface is an elliptic paraboloid with axis the y –axis. Its graph is VI.

26. This is the equation of a cone with axis the y –axis, so the graph is I.

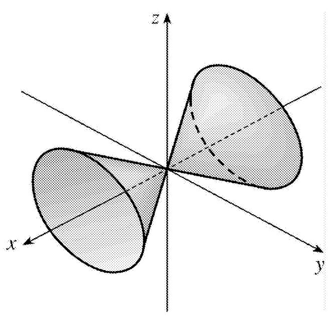
27. This surface is a cylinder because the variable y is missing from the equation. The intersection of the surface and the xz –plane is an ellipse. So the graph is VIII.

28. This is the equation of a hyperbolic paraboloid. The trace in the $xy-$ plane is the parabola $y=x^2$. So the correct graph is V.

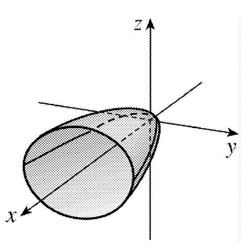
29. $z^2=4x^2+9y^2+36$ or $-4x^2-9y^2+z^2=36$ or $-\frac{x^2}{9}-\frac{y^2}{4}+\frac{z^2}{36}=1$ represents a hyperboloid of two sheets with axis the z –axis.



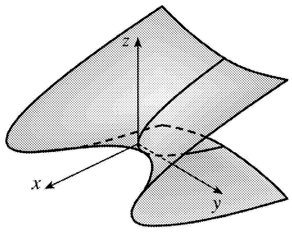
30. $x^2=2y^2+3z^2$ or $x^2=\frac{y^2}{1/2}+\frac{z^2}{1/3}$ or $\frac{x^2}{6}=\frac{y^2}{3}+\frac{z^2}{2}$ represents an elliptic cone with vertex $(0,0,0)$ and axis the x –axis.



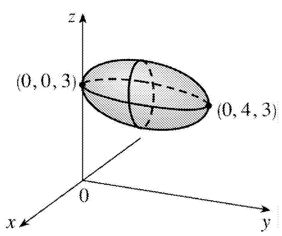
31. $x=2y^2+3z^2$ or $x=\frac{y^2}{1/2}+\frac{z^2}{1/3}$ or $\frac{x}{6}=\frac{y^2}{3}+\frac{z^2}{2}$ represents an elliptic paraboloid with vertex $(0,0,0)$ and axis the x –axis.



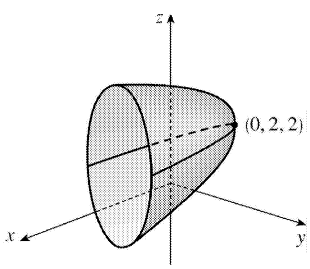
32. $4x-y^2+4z^2=0$ or $4x=y^2-4z^2$ or $x=\frac{y^2}{4}-z^2$ represents a hyperbolic paraboloid with center $(0,0,0)$.



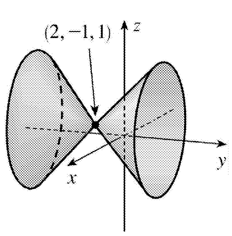
33. Completing squares in y and z gives $4x^2+(y-2)^2+4(z-3)^2=4$ or $x^2+\frac{(y-2)^2}{4}+(z-3)^2=1$, an ellipsoid with center $(0,2,3)$.



34. Completing squares in y and z gives $4(y-2)^2+(z-2)^2-x=0$ or $\frac{x}{4}=(y-2)^2+\frac{(z-2)^2}{4}$, an elliptic paraboloid with vertex $(0,2,2)$ and axis the horizontal line $y=2$, $z=2$.

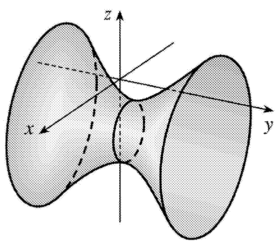


35. Completing squares in all three variables gives $(x-2)^2-(y+1)^2+(z-1)^2=0$ or $(y+1)^2=(x-2)^2+(z-1)^2$, a circular cone with center $(2,-1,1)$ and axis the horizontal line $x=2$, $z=1$.

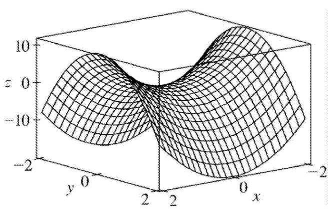


36. Completing squares in all three variables gives $(x-1)^2-(y-1)^2+(z+2)^2=2$ or

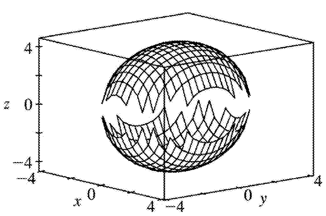
$\frac{(x-1)^2}{2} - \frac{(y-1)^2}{2} + \frac{(z+2)^2}{2} = 1$, a hyperboloid of one sheet with center $(1,1,-2)$ and axis the horizontal line $x=1$, $z=-2$.



37.

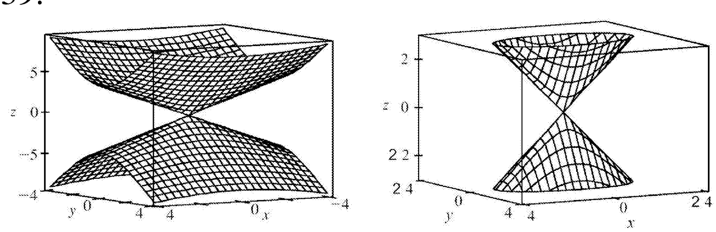


38.



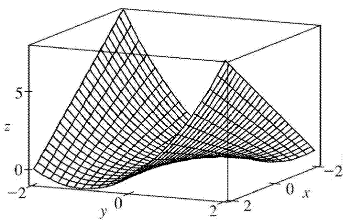
In Section 17.6 [ET 16.6], we will be able to graph ellipsoids without gaps; see Exercise 17.6.53 [ET 16.6.53].

39.

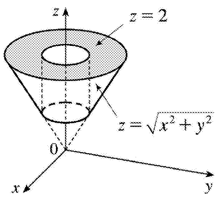


To restrict the z –range as in the second graph, we can use the option *view* = −2 . . 2 in Maple's *plot3d* command, or *PlotRange* –> { −2, 2} in Mathematica's *Plot3D* command.

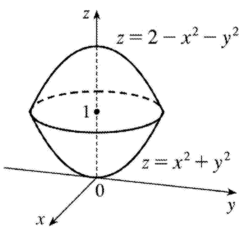
40.



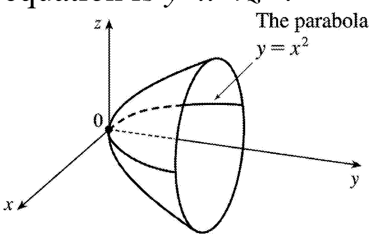
41.



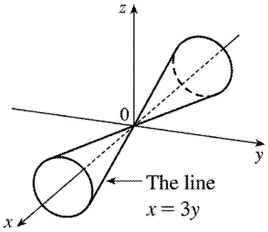
42.



43. The surface is a paraboloid of revolution (circular paraboloid) with vertex at the origin, axis the y –axis and opens to the right. Thus the trace in the yz –plane is also a parabola: $y=z^2$, $x=0$. The equation is $y=x^2+z^2$.



44. The surface is a right circular cone with vertex at $(0,0,0)$ and axis the x –axis. For $x=k\neq 0$, the trace is a circle with center $(k,0,0)$ and radius $r=y=\frac{x}{3}=\frac{k}{3}$. Thus the equation is $\frac{1}{3}x^2=y^2+z^2$.



45. Let $P=(x,y,z)$ be an arbitrary point equidistant from $(-1,0,0)$ and the plane $x=1$. Then the

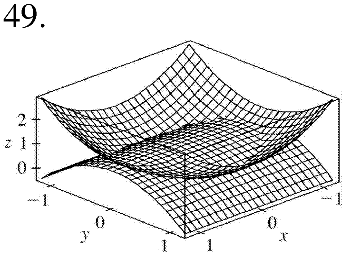
distance from P to $(-1,0,0)$ is $\sqrt{(x+1)^2+y^2+z^2}$ and the distance from P to the plane $x=1$ is $|x-1|/\sqrt{1^2}=|x-1|$ (by Equation 13.5.9 [ET 12.5.9]). So $|x-1|=\sqrt{(x+1)^2+y^2+z^2} \Leftrightarrow$ $(x-1)^2=(x+1)^2+y^2+z^2 \Leftrightarrow x^2-2x+1=x^2+2x+1+y^2+z^2 \Leftrightarrow -4x=y^2+z^2$. Thus the collection of all such points P is a circular paraboloid with vertex at the origin, axis the x –axis, which opens in the negative direction.

46. Let $P=(x,y,z)$ be an arbitrary point whose distance from the x –axis is twice its distance from the yz –plane. The distance from P to the x –axis is $\sqrt{(x-x)^2+y^2+z^2}=\sqrt{y^2+z^2}$ and the distance from P to the yz –plane ($x=0$) is $|x|/1=|x|$. Thus $\sqrt{y^2+z^2}=2|x| \Leftrightarrow y^2+z^2=4x^2 \Leftrightarrow x^2=\left(y^2/2^2\right)+\left(z^2/2^2\right)$. So the surface is a right circular cone with vertex the origin and axis the x –axis.

47. If (a,b,c) satisfies $z=y^2-x^2$, then $c=b^2-a^2$. L_1 : $x=a+t$, $y=b+t$, $z=c+2(b-a)t$, L_2 : $x=a+t$, $y=b-t$, $z=c-2(b+a)t$. Substitute the parametric equations of L_1 into the equation of the hyperbolic paraboloid in order to find the points of intersection: $z=y^2-x^2 \Rightarrow$ $c+2(b-a)t=(b+t)^2-(a+t)^2=b^2-a^2+2(b-a)t \Rightarrow c=b^2-a^2$. As this is true for all values of t , L_1 lies on $z=y^2-x^2$. Performing similar operations with L_2 gives: $z=y^2-x^2 \Rightarrow$ $c-2(b+a)t=(b-t)^2-(a+t)^2=b^2-a^2-2(b+a)t \Rightarrow c=b^2-a^2$. This tells us that all of L_2 also lies on $z=y^2-x^2$.

48. Any point on the curve of intersection must satisfy both $2x^2+4y^2-2z^2+6x=2$ and $2x^2+4y^2-2z^2-5y=0$. Subtracting, we get $6x+5y=2$, which is linear and therefore the equation of a plane. Thus the curve of intersection lies in this plane.

49.



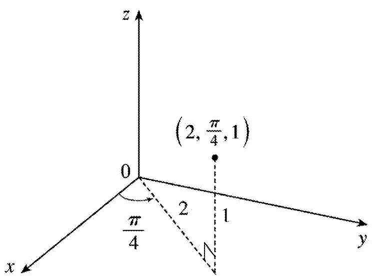
The curve of intersection looks like a bent ellipse. The projection of this curve onto the xy –plane is the set of points $(x,y,0)$ which satisfy $x^2+y^2=1-y^2 \Leftrightarrow x^2+2y^2=1 \Leftrightarrow$

$x^2 + \frac{y^2}{(1/\sqrt{2})^2} = 1$. This is an equation of an ellipse.

1. See Figure 1 and the accompanying discussion; see the paragraph accompanying Figure 3.

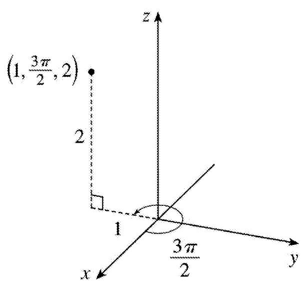
2. See Figure 5 and the accompanying discussion.

3.



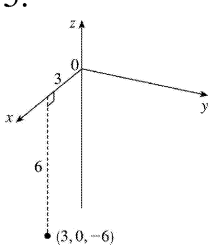
$x=2\cos\frac{\pi}{4}=\sqrt{2}$, $y=2\sin\frac{\pi}{4}=\sqrt{2}$, $z=1$, so the point is $(\sqrt{2},\sqrt{2},1)$ in rectangular coordinates.

4.



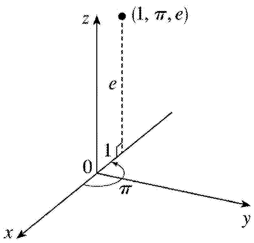
$x=1\cos\frac{3\pi}{2}=0$, $y=1\sin\frac{3\pi}{2}=-1$, $z=2$, so the point is $(0,-1,2)$ in rectangular coordinates.

5.



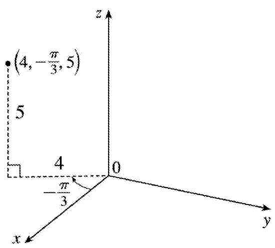
$x=3\cos 0=3$, $y=3\sin 0=0$, and $z=-6$, so the point is $(3,0,-6)$ in rectangular coordinates.

6.



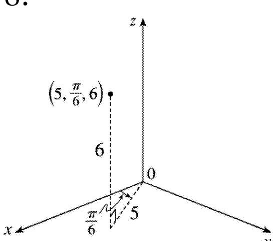
$x = 1\cos\pi = -1$, $y = 1\sin\pi = 0$, and $z = e$, so the point is $(-1,0,e)$ in rectangular coordinates.

7.



$x = 4\cos\left(-\frac{\pi}{3}\right) = 2$, $y = 4\sin\left(-\frac{\pi}{3}\right) = -2\sqrt{3}$, and $z = 5$, so the point is $(2,-2\sqrt{3},5)$ in rectangular coordinates.

8.



$x = 5\cos\left(\frac{\pi}{6}\right) = \frac{5\sqrt{3}}{2}$, $y = 5\sin\left(\frac{\pi}{6}\right) = \frac{5}{2}$, and $z = 6$, so the point is $\left(\frac{5\sqrt{3}}{2}, \frac{5}{2}, 6\right)$ in rectangular coordinates.

9. $r^2 = x^2 + y^2 = 1^2 + (-1)^2 = 2$ so $r = \sqrt{2}$; $\tan\theta = \frac{y}{x} = \frac{-1}{1} = -1$ and the point $(1,-1)$ is in the fourth quadrant of the xy –plane, so $\theta = \frac{7\pi}{4} + 2n\pi$; $z = 4$. Thus, one set of cylindrical coordinates is $\left(\sqrt{2}, \frac{7\pi}{4}, 4\right)$.

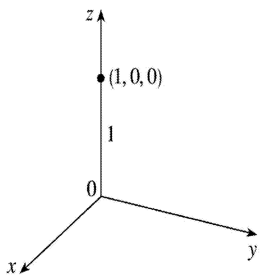
10. $r^2 = x^2 + y^2 = 3^2 + 3^2 = 18$ so $r = \sqrt{18} = 3\sqrt{2}$; $\tan\theta = \frac{y}{x} = \frac{3}{3} = 1$ and the point $(3,3)$ is in the first quadrant of the xy –plane, so $\theta = \frac{\pi}{4} + 2n\pi$; $z = -2$. Thus, one set of cylindrical coordinates is $\left(3\sqrt{2}, \frac{\pi}{4}, -2\right)$.

11. $r^2 = (-1)^2 + (-\sqrt{3})^2 = 4$ so $r = 2$; $\tan\theta = \frac{-\sqrt{3}}{-1} = \sqrt{3}$ and the point $(-1,-\sqrt{3})$ is in the third quadrant of

the xy –plane, so $\theta=\frac{4\pi}{3}+2n\pi$; $z=2$. Thus, one set of cylindrical coordinates is $\left(2,\frac{4\pi}{3},2\right)$.

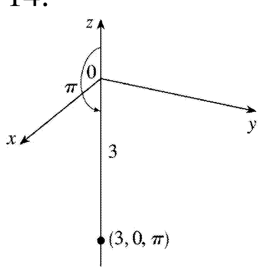
12. $r^2=3^2+4^2=25$ so $r=5$; $\tan\theta=\frac{4}{3}$ and the point $(3,4)$ is in the first quadrant of the xy –plane, so $\theta=\tan^{-1}\left(\frac{4}{3}\right)+2n\pi\approx0.93+2n\pi$; $z=5$. Thus, one set of cylindrical coordinates is $\left(5,\tan^{-1}\left(\frac{4}{3}\right),5\right)\approx(5,0.93,5)$.

13.



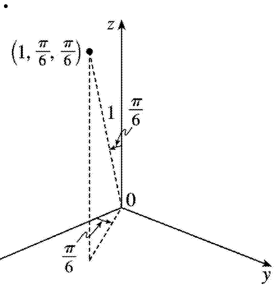
$x=\rho\sin\phi\cos\theta=(1)\sin 0\cos 0=0$, $y=\rho\sin\phi\sin\theta=(1)\sin 0\sin 0=0$, and $z=\rho\cos\phi=(1)\cos 0=1$ so the point is $(0,0,1)$ in rectangular coordinates.

14.



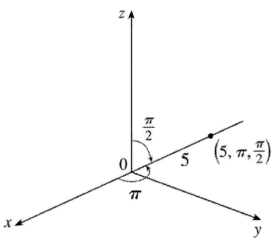
$x=3\sin\pi\cos 0=0$, $y=3\sin\pi\sin 0=0$, $z=3\cos\pi=-3$ and in rectangular coordinates the point is $(0,0,-3)$.

15.



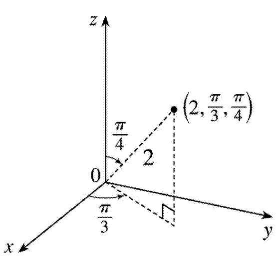
$x = \sin\frac{\pi}{6}\cos\frac{\pi}{6} = \frac{\sqrt{3}}{4}$, $y = \sin\frac{\pi}{6}\sin\frac{\pi}{6} = \frac{1}{4}$, and $z = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$, so the point is $\left(\frac{\sqrt{3}}{4}, \frac{1}{4}, \frac{\sqrt{3}}{2}\right)$ in rectangular coordinates.

16.

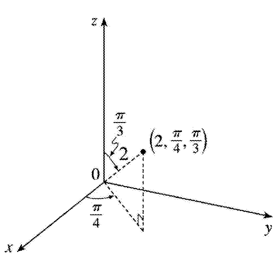


$x = 5\sin\frac{\pi}{2}\cos\pi = -5$, $y = 5\sin\frac{\pi}{2}\sin\pi = 0$, $z = 5\cos\frac{\pi}{2} = 0$ so the point is $(-5,0,0)$ in rectangular coordinates.

17. $x = 2\sin\frac{\pi}{4}\cos\frac{\pi}{3} = \frac{\sqrt{2}}{2}$, $y = 2\sin\frac{\pi}{4}\sin\frac{\pi}{3} = \frac{\sqrt{6}}{2}$, $z = 2\cos\frac{\pi}{4} = \sqrt{2}$ so the point is $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{6}}{2}, \sqrt{2}\right)$ in rectangular coordinates.



18. $x = 2\sin\frac{\pi}{3}\cos\frac{\pi}{4} = \frac{\sqrt{6}}{2}$, $y = 2\sin\frac{\pi}{3}\sin\frac{\pi}{4} = \frac{\sqrt{6}}{2}$, $z = 2\cos\frac{\pi}{3} = 1$ so the point is $\left(\frac{\sqrt{6}}{2}, \frac{\sqrt{6}}{2}, 1\right)$ in rectangular coordinates.



19. $\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{1+3+12} = 4$, $\cos\phi = \frac{z}{\rho} = \frac{2\sqrt{3}}{4} = \frac{\sqrt{3}}{2} \Rightarrow \phi = \frac{\pi}{6}$, and

$\cos\theta = \frac{x}{\rho\sin\phi} = \frac{1}{4\sin(\pi/6)} = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}$ (since $y>0$). Thus spherical coordinates are $\left(4, \frac{\pi}{3}, \frac{\pi}{6}\right)$.

20. $\rho=\sqrt{x^2+y^2+z^2}=\sqrt{0+3+1}=2$, $\cos\phi=\frac{z}{\rho}=\frac{1}{2}\Rightarrow\phi=\frac{\pi}{3}$, and $\cos\theta=\frac{x}{\rho\sin\phi}=\frac{0}{2\sin(\pi/3)}=0\Rightarrow$ $\theta=\frac{\pi}{2}$ (since $y>0$). Thus spherical coordinates are $\left(2,\frac{\pi}{2},\frac{\pi}{3}\right)$.

21. $\rho=\sqrt{0+1+1}=\sqrt{2}$, $\cos\phi=\frac{-1}{\sqrt{2}}\Rightarrow\phi=\frac{3\pi}{4}$, and $\cos\theta=\frac{0}{\sqrt{2}\sin(3\pi/4)}=0\Rightarrow\theta=\frac{3\pi}{2}$ (since $y<0$). Thus spherical coordinates are $\left(\sqrt{2},\frac{3\pi}{2},\frac{3\pi}{4}\right)$.

22. $\rho=\sqrt{1+1+6}=2\sqrt{2}$, $\cos\phi=\frac{\sqrt{6}}{2\sqrt{2}}=\frac{\sqrt{3}}{2}\Rightarrow\phi=\frac{\pi}{6}$, and $\cos\theta=\frac{-1}{2\sqrt{2}\sin(\pi/6)}=-\frac{1}{\sqrt{2}}\Rightarrow\theta=\frac{3\pi}{4}$ (since $y>0$). Thus spherical coordinates are $\left(2\sqrt{2},\frac{3\pi}{4},\frac{\pi}{6}\right)$.

23. $\rho=\sqrt{x^2+y^2+z^2}=\sqrt{r^2+z^2}=\sqrt{1+3}=2$; $\theta=\frac{\pi}{6}$; $\cos\phi=\frac{z}{\rho}=\frac{\sqrt{3}}{2}\Rightarrow\phi=\frac{\pi}{6}$, thus in spherical coordinates the point is $\left(2,\frac{\pi}{6},\frac{\pi}{6}\right)$.

24. $\rho=\sqrt{r^2+z^2}=\sqrt{6+2}=2\sqrt{2}$; $\theta=\frac{\pi}{4}$; $\cos\phi=\frac{z}{\rho}=\frac{\sqrt{2}}{2\sqrt{2}}=\frac{1}{2}\Rightarrow\phi=\frac{\pi}{3}$, thus in spherical coordinates the point is $\left(2\sqrt{2},\frac{\pi}{4},\frac{\pi}{3}\right)$.

25. $\rho=\sqrt{r^2+z^2}=\sqrt{3+1}=2$; $\theta=\frac{\pi}{2}$; $\cos\phi=\frac{z}{\rho}=\frac{-1}{2}\Rightarrow\phi=\frac{2\pi}{3}$, so in spherical coordinates the point is $\left(2,\frac{\pi}{2},\frac{2\pi}{3}\right)$.

26. $\rho=\sqrt{16+9}=5$; $\theta=\frac{\pi}{8}$; $\cos\phi=\frac{3}{5}\Rightarrow\phi=\cos^{-1}\left(\frac{3}{5}\right)$, so in spherical coordinates the point is $\left(5,\frac{\pi}{8},\cos^{-1}\left(\frac{3}{5}\right)\right)\approx\left(5,\frac{\pi}{8},0.927\right)$.

27. $z=\rho\cos\phi=2\cos 0=2$, $\rho^2=x^2+y^2+z^2=r^2+z^2\Rightarrow r=\sqrt{\rho^2-z^2}=\sqrt{2^2-2^2}=0$, (or $r=2\sin 0=0$), $\theta=0$ and the point is $(0,0,2)$.

28. $z=2\sqrt{2}\cos\frac{\pi}{2}=0$, $r=2\sqrt{2}\sin\frac{\pi}{2}=2\sqrt{2}$, $\theta=\frac{3\pi}{2}$ and the point is $\left(2\sqrt{2},\frac{3\pi}{2},0\right)$.

29.

$z=8\cos\frac{\pi}{2}=0$, $r=8\sin\frac{\pi}{2}=8$, $\theta=\frac{\pi}{6}$ and the point is $\left(8,\frac{\pi}{6},0\right)$.

30. $z=4\cos\frac{\pi}{3}=2$, $r=4\sin\frac{\pi}{3}=2\sqrt{3}$, $\theta=\frac{\pi}{4}$ and the point is $\left(2\sqrt{3},\frac{\pi}{4},2\right)$.

31. Since $r=3$, $x^2+y^2=9$ and the surface is a circular cylinder with radius 3 and axis the z –axis.

32. Since $\rho=3$, $x^2+y^2+z^2=9$ and the surface is a sphere with center the origin and radius 3 .

33. Since $\phi=0$, $x=0$ and $y=0$ while $z=\rho\geq 0$. Thus the ''surface'' is the positive z –axis including the origin.

34. Since $\phi=\frac{\pi}{2}$, $z=0$ but there are no restrictions on x and y ($x=\rho\cos\theta$, $y=\rho\sin\theta$). Thus the surface is the xy –plane.

35. Since $\phi=\frac{\pi}{3}$, the surface is the top half of the right circular cone with vertex at the origin and axis the positive z –axis.

36. Whether spherical or cylindrical coordinates, since $\theta=\frac{\pi}{3}$ the surface is a half–plane including the z –axis and intersecting the xy –plane in the half–line $y=\sqrt{3}\,x$, $x>0$.

37. $z=r^2=x^2+y^2$, so the surface is a circular paraboloid with vertex at the origin and axis the positive z –axis.

38. Since $r=4\sin\theta$ and $y=r\sin\theta$, $y=4\sin^2\theta$. Also $r^2=x^2+y^2$ so $x^2+y^2=16\sin^2\theta$. Thus $x^2+y^2-4y=16\sin^2\theta-16\sin^2\theta=0$ or $x^2+(y-2)^2=4$, a circular cylinder of radius 2 and with axis parallel to the z –axis.

39. $2=\rho\cos\phi=z$ is a plane through the point $(0,0,2)$ and parallel to the xy –plane.

40. Since $\rho\sin\phi=2$ and $x=\rho\sin\phi\cos\theta$, $x=2\cos\theta$. Also $y=\rho\sin\phi\sin\theta$ so $y=2\sin\theta$. Then $x^2+y^2=4\cos^2\theta+4\sin^2\theta=4$, a circular cylinder of radius 2 about the z –axis.

41. $r=2\cos\theta\Rightarrow r^2=x^2+y^2=2r\cos\theta=2x\Leftrightarrow(x-1)^2+y^2=1$, which is the equation of a circular cylinder with radius 1 , whose axis is the vertical line $x=1$, $y=0$, $z=z$.

42. $\rho=2\cos\phi \Rightarrow \rho^2=2\rho\cos\phi=2z \Leftrightarrow x^2+y^2+z^2=2z \Leftrightarrow x^2+y^2+(z-1)^2=1$. Therefore, the surface is a sphere of radius 1 centered at $(0,0,1)$.

43. Since $r^2+z^2=25$ and $r^2=x^2+y^2$, we have $x^2+y^2+z^2=25$, a sphere with radius 5 and center at the origin.

44. Since $r^2-2z^2=4$ and $r^2=x^2+y^2$, we have $x^2+y^2-2z^2=4$ or $\frac{1}{4}x^2+\frac{1}{4}y^2-\frac{1}{2}z^2=1$, a hyperboloid of one sheet with axis the z –axis.

45. Since $x^2=\rho^2\sin^2\phi\cos^2\theta$ and $z^2=\rho^2\cos^2\phi$, the equation of the surface in rec tan gular coordinates is $x^2+z^2=4$. Thus the surface is a circular cylinder of radius 2 about the y –axis.

46. Since $\rho^2\left(\sin^2\phi-4\cos^2\phi\right)=1$, $\rho^2\left(\sin^2\phi-4\cos^2\phi\right)+\rho^2\cos^2\phi-\rho^2\cos^2\phi=1$ or $\rho^2\left(\sin^2\phi+\cos^2\phi-5\cos^2\phi\right)=1$ or $\rho^2\left(1-5\cos^2\phi\right)=1$. But $\rho^2=x^2+y^2+z^2$ and $z^2=\rho^2\cos^2\phi$, so we can rewrite the equation of the surface as $x^2+y^2+z^2-5z^2=1$ or $x^2+y^2-4z^2=1$. Thus the surface is a hyperboloid of one sheet with axis the z –axis.

47. Since $r^2-r=0$, $r=0$ or $r=1$. But $x^2+y^2=r^2$. Thus the surface consists of the right circular cylinder of radius 1 and axis the z –axis along with the surface given by $x^2+y^2=0$, that is, the z –axis.

48. Since $\rho^2-6\rho+8=0$, either $\rho=2$ or $\rho=4$. Thus the surface consists of two concentric spheres (centered at the origin), one with radius 2 and the other with radius 4 .

49. **(a)** $x^2+y^2=r^2$, so the equation becomes $z=r^2$.
(b) $x=\rho\sin\phi\cos\theta$, $y=\rho\sin\phi\sin\theta$, and $z=\rho\cos\phi$, so the equation becomes $\rho\cos\phi=(\rho\sin\phi\cos\theta)^2+(\rho\sin\phi\sin\theta)^2$ or $\rho\cos\phi=\rho^2\sin^2\phi$ or $\rho\sin^2\phi=\cos\phi$.

50. **(a)** $x^2+y^2=r^2$, so the equation becomes $r^2+z^2=2$.
(b) $x^2+y^2+z^2=\rho^2$, so the equation becomes $\rho^2=2$ or $\rho=\sqrt{2}$.

51. **(a)** $x=r\cos\theta$, so the equation becomes $r\cos\theta=3$ or $r=3\sec\theta$ (since $\cos\theta\neq 0$ here).
(b) $x=\rho\sin\phi\cos\theta$, so the equation becomes $\rho\sin\phi\cos\theta=3$.

52. **(a)** $x^2+y^2=r^2$, so the equation becomes $r^2+z^2+2z=0$ or $r^2+(z+1)^2=1$.

(b) $x^2+y^2+z^2=\rho^2$ and $z=\rho\cos\phi$, so the equation becomes $\rho^2+2\rho\cos\phi=0$ or $\rho=-2\cos\phi$.

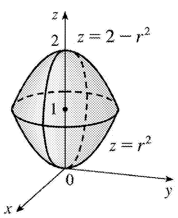
53. **(a)** $r^2(\cos^2\theta-\sin^2\theta)-2z^2=4$ or $2z^2=r^2\cos 2\theta-4$.
(b) $\rho^2\left(\sin^2\phi\cos^2\theta-\sin^2\phi\sin^2\theta-2\cos^2\phi\right)=4$ or $\rho^2\left(\sin^2\phi\cos 2\theta-2\cos^2\phi\right)=4$.

54. **(a)** $r^2\sin^2\theta+z^2=1$
(b) $\rho^2\sin^2\phi\sin^2\theta+\rho^2\cos^2\phi=1$ or $\rho^2\left(\sin^2\phi\sin^2\theta+\cos^2\phi\right)=1$.

55. **(a)** $r^2=2r\sin\theta$ or $r=2\sin\theta$.
(b) $\rho^2\sin^2\phi\left(\cos^2\theta+\sin^2\theta\right)=2\rho\sin\phi\sin\theta$ or $\rho\sin^2\phi=2\sin\phi\sin\theta$ or $\rho\sin\phi=2\sin\theta$.

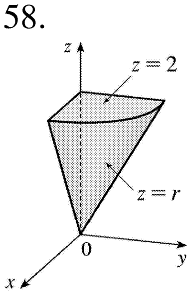
56. **(a)** $z=r^2(\cos^2\theta-\sin^2\theta)$ or $z=r^2\cos 2\theta$.
(b) $\rho\cos\phi=\rho^2\sin^2\phi(\cos^2\theta-\sin^2\theta)$ or $\cos\phi=\rho\sin^2\phi\cos 2\theta$.

57.



$z=r^2=x^2+y^2$ is a circular paraboloid with vertex (0,0,0) , opening upward. $z=2-r^2 \Rightarrow z-2=-\left(x^2+y^2\right)$ is a circular paraboloid with vertex (0,0,2) opening downward. Thus $r^2\leq z\leq 2-r^2$ is the solid region enclosed by these two surfaces.

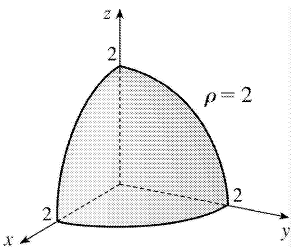
58.



$z=r=\sqrt{x^2+y^2}$ is a cone that opens upward. Thus $r\leq z\leq 2$ is the region above this cone and beneath the horizontal plane $z=2$.

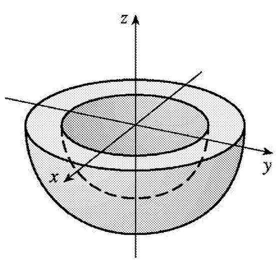
$0 \leq \theta \leq \frac{\pi}{2}$ restricts the solid to that part of this region in the first octant.

59.



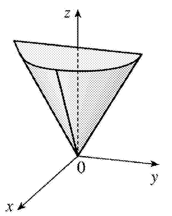
$\rho=2$ represents a sphere of radius 2, centered at the origin, so $\rho \leq 2$ is this sphere and its interior. $0 \leq \phi \leq \frac{\pi}{2}$ restricts the solid to that portion of the region that lies on or above the xy –plane, and $0 \leq \theta \leq \frac{\pi}{2}$ further restricts the solid to the first octant. Thus the solid is the portion in the first octant of the solid ball centered at the origin with radius 2.

60.

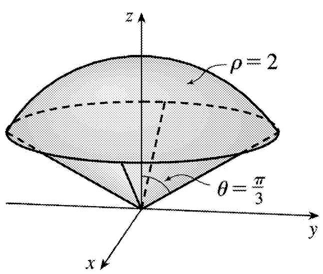


$2 \leq \rho \leq 3$ represents the solid region between and including the spheres of radii 2 and 3, centered at the origin. $\frac{\pi}{2} \leq \phi \leq \pi$ restricts the solid to that portion on or below the xy –plane.

61. $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ restricts the solid to the 4 octants in which x is positive. $\rho = \sec \phi \Rightarrow \rho \cos \phi = z = 1$, which is the equation of a horizontal plane. $0 \leq \phi \leq \frac{\pi}{6}$ describes a cone, opening upward. So the solid lies above the cone $\phi = \frac{\pi}{6}$ and below the plane $z=1$.



62. $\rho=2 \Leftrightarrow x^2+y^2+z^2=4$, which is a sphere of radius 2 , centered at the origin. Hence $\rho \le 2$ is this sphere and its interior. $0 \le \phi \le \frac{\pi}{3}$ restricts the solid to that section of this ball that lies above the cone $\phi = \frac{\pi}{3}$.

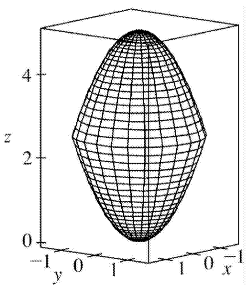


63. We can position the cylindrical shell vertically so that its axis coincides with the z –axis and its base lies in the xy –plane. If we use centimeters as the unit of measurement, then cylindrical coordinates conveniently describe the shell as $6 \le r \le 7$, $0 \le \theta \le 2\pi$, $0 \le z \le 20$.

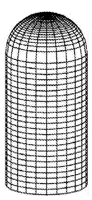
64. **(a)** The hollow ball is a spherical shell with outer radius 15 cm and inner radius 14.5 cm. If we center the ball at the origin of the coordinate system and use centimeters as the unit of measurement, then spherical coordinates conveniently describe the hollow ball as $14.5 \le \rho \le 15$, $0 \le \theta \le 2\pi$, $0 \le \phi \le \pi$.
(b) If we position the ball as in part (a), one possibility is to take the half of the ball that is above the xy –plane which is described by $14.5 \le \rho \le 15$, $0 \le \theta \le 2\pi$, $0 \le \phi \le \pi/2$.

65. $z \ge \sqrt{x^2+y^2}$ because the solid lies above the cone. Squaring both sides of this inequality gives $z^2 \ge x^2+y^2 \Rightarrow 2z^2 \ge x^2+y^2+z^2=\rho^2 \Rightarrow z^2=\rho^2\cos^2\phi \ge \frac{1}{2}\rho^2 \Rightarrow \cos^2\phi \ge \frac{1}{2}$. The cone opens upward so that the inequality is $\cos\phi \ge \frac{1}{\sqrt{2}}$, or equivalently $0 \le \phi \le \frac{\pi}{4}$. In spherical coordinates the sphere $z=x^2+y^2+z^2$ is $\rho\cos\phi=\rho^2 \Rightarrow \rho=\cos\phi$. $0 \le \rho \le \cos\phi$ because the solid lies below the sphere. The solid can therefore be described as the region in spherical coordinates satisfying $0 \le \rho \le \cos\phi$, $0 \le \phi \le \frac{\pi}{4}$.

66. In cylindrical coordinates, the equations are $z=r^2$ and $z=5-r^2$. The curve of intersection is $r^2=5-r^2$ or $r=\sqrt{5/2}$. So we graph the surfaces in cylindrical coordinates, with $0\le r\le\sqrt{5/2}$. In Maple, we can use either the *coords=cylindrical* option in a regular *plot* command, or the *plots[cylinderplot]* command. In Mathematica, we can use *ParametricPlot3d* .



67. In cylindrical coordinates, the equation of the cylinder is $r=3$, $0\le z\le 10$. The hemisphere is the upper part of the sphere radius 3 , center $(0,0,10)$, equation $r^2+(z-10)^2=3^2$, $z\ge 10$. In Maple, we can use either the *coords=cylindrical* option in a regular *plot* command, or the *plots[cylinderplot]* command. In Mathematica, we can use *ParametricPlot3d* .



68. We begin by finding the positions of Los Angeles and Montreal in spherical coordinates, using the method described in the exercise:

Montreal	Los Angeles
$\rho=3960$ mi $\theta=360°-73.60°=286.40°$ $\phi=90°-45.50°=44.50°$	$\rho=3960$ mi $\theta=360°-118.25°=241.75°$ $\phi=90°-34.06°=55.94°$

Now we change the above to Cartesian coordinates using $x=\rho\cos\theta\sin\phi$, $y=\rho\sin\theta\sin\phi$ and $z=\rho\cos\phi$ to get two position vectors of length 3960 mi (since both cities must lie on the surface of the Earth). In particular:

Montreal: $\langle 783.67,-2662.67,2824.47\rangle$ Los Angeles: $\langle -1552.80,-2889.91,2217.84\rangle$

To find the angle α between these two vectors we use the dot product:

$\langle 783.67, -2662.67, 2824.47 \rangle \cdot \langle -1552.80, -2889.91, 2217.84 \rangle = (3960)^2 \cos\alpha \Rightarrow \cos\alpha \approx 0.8126 \Rightarrow$ $\alpha \approx 0.6223$ rad. The great circle distance between the cities is $s = \rho\theta \approx 3960(0.6223) \approx 2464$ mi.

1. The component functions t^2, $\sqrt{t-1}$, and $\sqrt{5-t}$ are all defined when $t-1\geq 0\Rightarrow t\geq 1$ and $5-t\geq 0\Rightarrow t\leq 5$, so the domain of $\boldsymbol{r}(t)$ is $[1,5]$.

2. The component functions $\frac{t-2}{t+2}$, $\sin t$, and $\ln(9-t^2)$ are all defined when $t\neq -2$ and $9-t^2>0\Rightarrow -3<t<3$, so the domain of $\boldsymbol{r}(t)$ is $(-3,-2)\cup(-2,3)$.

3. $\lim_{t\to 0^+}\cos t=\cos 0=1$, $\lim_{t\to 0^+}\sin t=\sin 0=0$, $\lim_{t\to 0^+} t\ln t=\lim_{t\to 0^+}\frac{\ln t}{1/t}=\lim_{t\to 0^+}\frac{1/t}{-1/t^2}=\lim_{t\to 0^+} -t=0$. Thus $\lim_{t\to 0^+}\langle\cos t,\sin t,t\ln t\rangle=\left\langle \lim_{t\to 0^+}\cos t,\lim_{t\to 0^+}\sin t,\lim_{t\to 0^+} t\ln t\right\rangle=\langle 1,0,0\rangle$.

4. $\lim_{t\to 0}\frac{e^t-1}{t}=\lim_{t\to 0}\frac{e^t}{1}=1$ [using l'Hospital's Rule],

$\lim_{t\to 0}\frac{\sqrt{1+t}-1}{t}=\lim_{t\to 0}\frac{\sqrt{1+t}-1}{t}\cdot\frac{\sqrt{1+t}+1}{\sqrt{1+t}+1}=\lim_{t\to 0}\frac{1}{\sqrt{1+t}+1}=\frac{1}{2}$, $\lim_{t\to 0}\frac{3}{1+t}=3$.

Thus the given limit equals $\left\langle 1,\frac{1}{2},3\right\rangle$.

5. $\lim_{t\to 1}\sqrt{t+3}=2$, $\lim_{t\to 1}\frac{t-1}{t^2-1}=\lim_{t\to 1}\frac{1}{t+1}=\frac{1}{2}$, $\lim_{t\to 1}\left(\frac{\tan t}{t}\right)=\tan 1$.

Thus the given limit equals $\left\langle 2,\frac{1}{2},\tan 1\right\rangle$.

6. $\lim_{t\to\infty}\arctan t=\frac{\pi}{2}$, $\lim_{t\to\infty}e^{-2t}=0$, $\lim_{t\to\infty}\frac{\ln t}{t}=\lim_{t\to\infty}\frac{1/t}{1}=0$ [by l'Hospital's Rule].

Thus $\lim_{t\to\infty}\left\langle \arctan t,e^{-2t},\frac{\ln t}{t}\right\rangle=\left\langle\frac{\pi}{2},0,0\right\rangle$.

7. The corresponding parametric equations for this curve are $x=t^4+1$, $y=t$. We can make a table of values, or we can eliminate the parameter: $t=y\Rightarrow x=y^4+1$, with $y\in R$. By comparing different values of t, we find the direction in which t increases as indicated in the graph.

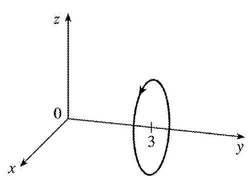
8. The corresponding parametric equations for this curve are $x=t^3, y=t^2$. We can make a table of values, or we can eliminate the parameter: $x=t^3\Rightarrow t=\sqrt[3]{x}\Rightarrow y=t^2=\left(\sqrt[3]{x}\right)^2=x^{2/3}$, with $t\in R\Rightarrow x\in R$. By

comparing different values of t , we find the direction in which t increases as indicated in the graph.

9. The corresponding parametric equations are $x=t$, $y=\cos 2t$, $z=\sin 2t$. Note that $y^2+z^2=\cos^2 2t+\sin^2 2t=1$, so the curve lies on the circular cylinder $y^2+z^2=1$. Since $x=t$, the curve is a helix.

10. The corresponding parametric equations are $x=1+t$, $y=3t$, $z=-t$, which are parametric equations of a line through the point $(1,0,0)$ and with direction vector $\langle 1,3,-1\rangle$.

11. The parametric equations give $x^2+z^2=\sin^2 t+\cos^2 t=1$, $y=3$, which is a circle of radius 1 , center $(0,3,0)$ in the plane $y=3$.



12. The parametric equations are $x=t$, $y=t$, $z=\cos t$. Thus $x=y$, so the curve must lie in the plane $x=y$. Combine this with $z=\cos t$ to determine that the curve traces out the cosine curve in the vertical plane $x=y$.

13. The parametric equations are $x=t^2$, $y=t^4$, $z=t^6$. These are positive for $t\neq 0$ and 0 when $t=0$. So the curve lies entirely in the first quadrant. The projection of the graph onto the xy –plane is $y=x^2$, $y>0$, a half parabola. On the xz –plane $z=x^3$, $z>0$, a half cubic, and the yz –plane, $y^3=z^2$.

14. The parametric equations give $x^2+y^2+z^2=2\sin^2 t+2\cos^2 t=2$, so the curve lies on the sphere with radius $\sqrt{2}$ and center $(0,0,0)$. Furthermore $x=y=\sin t$, so the curve is the intersection of this sphere with the plane $x=y$, that is, the curve is the circle of radius $\sqrt{2}$, center $(0,0,0)$ in the plane $x=y$.

15. Taking $\boldsymbol{r}_0=\langle 0,0,0\rangle$ and $\boldsymbol{r}_1=\langle 1,2,3\rangle$, we have from Equation 13.5.4 $\boldsymbol{r}(t)=(1-t)\boldsymbol{r}_0+t\boldsymbol{r}_1=(1-t)\langle 0,0,0\rangle+t\langle 1,2,3\rangle$, $0\leq t\leq 1$ or $\boldsymbol{r}(t)=\langle \text{t,2t,3}t\rangle$, $0\leq t\leq 1$.
Parametric equations are $x=t$, $y=2t$, $z=3t$, $0\leq t\leq 1$.

16. Taking $\boldsymbol{r}_0=\langle 1,0,1\rangle$ and $\boldsymbol{r}_1=\langle 2,3,1\rangle$, we have from Equation 13.5.4 $\boldsymbol{r}(t)=(1-t)\boldsymbol{r}_0+t\boldsymbol{r}_1=(1-t)\langle 1,0,1\rangle+t\langle 2,3,1\rangle$, $0\le t\le 1$ or $\boldsymbol{r}(t)=\langle 1+t,3t,1\rangle$, $0\le t\le 1$. Parametric equations are $x=1+t$, $y=3t$, $z=1$, $0\le t\le 1$.

17. Taking $\boldsymbol{r}_0=\langle 1,-1,2\rangle$ and $\boldsymbol{r}_1=\langle 4,1,7\rangle$, we have $\boldsymbol{r}(t)=(1-t)\boldsymbol{r}_0+t\boldsymbol{r}_1=(1-t)\langle 1,-1,2\rangle+t\langle 4,1,7\rangle$, $0\le t\le 1$ or $\boldsymbol{r}(t)=\langle 1+3t,-1+2t,2+5t\rangle$, $0\le t\le 1$. Parametric equations are $x=1+3t$, $y=-1+2t$, $z=2+5t$, $0\le t\le 1$.

18. Taking $\boldsymbol{r}_0=\langle -2,4,0\rangle$ and $\boldsymbol{r}_1=\langle 6,-1,2\rangle$, we have $\boldsymbol{r}(t)=(1-t)\boldsymbol{r}_0+t\boldsymbol{r}_1=(1-t)\langle -2,4,0\rangle+t\langle 6,-1,2\rangle$, $0\le t\le 1$ or $\boldsymbol{r}(t)=\langle -2+8t,4-5t,2t\rangle$, $0\le t\le 1$. Parametric equations are $x=-2+8t$, $y=4-5t$, $z=2t$, $0\le t\le 1$.

19. $x=\cos 4t$, $y=t$, $z=\sin 4t$. At any point (x,y,z) on the curve, $x^2+z^2=\cos^2 4t+\sin^2 4t=1$. So the curve lies on a circular cylinder with axis the y –axis. Since $y=t$, this is a helix. So the graph is VI.

20. $x=t$, $y=t^2$, $z=e^{-t}$. At any point on the curve, $y=x^2$. So the curve lies on the parabolic cylinder $y=x^2$. Note that y and z are positive for all t , and the point $(0,0,1)$ is on the curve (when $t=0$). As $t\to\infty$, $(x,y,z)\to(\infty,\infty,0)$, while as $t\to-\infty$, $(x,y,z)\to(-\infty,\infty,\infty)$, so the graph must be II.

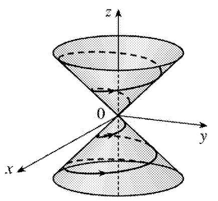
21. $x=t$, $y=1/(1+t^2)$, $z=t^2$. Note that y and z are positive for all t . The curve passes through $(0,1,0)$ when $t=0$. As $t\to\infty$, $(x,y,z)\to(\infty,0,\infty)$, and as $t\to-\infty$, $(x,y,z)\to(-\infty,0,\infty)$. So the graph is IV.

22. $x=e^{-t}\cos 10t$, $y=e^{-t}\sin 10t$, $z=e^{-t}$. $x^2+y^2=e^{-2t}\cos^2 10t+e^{-2t}\sin^2 10t=e^{-2t}(\cos^2 10t+\sin^2 10t)=e^{-2t}=z^2$, so the curve lies on the cone $x^2+y^2=z^2$. Also, z is always positive; the graph must be I.

23. $x=\cos t$, $y=\sin t$, $z=\sin 5t$. $x^2+y^2=\cos^2 t+\sin^2 t=1$, so the curve lies on a circular cylinder with axis the z –axis. Each of x , y and z is periodic, and at $t=0$ and $t=2\pi$ the curve passes through the same point, so the curve repeats itself and the graph is V.

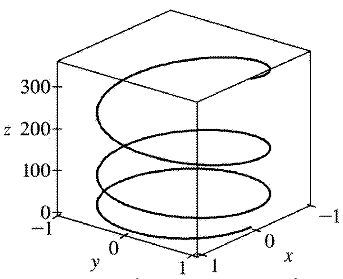
24. $x=\cos t$, $y=\sin t$, $z=\ln t$. $x^2+y^2=\cos^2 t+\sin^2 t=1$, so the curve lies on a circular cylinder with axis the z –axis. As $t\to 0$, $z\to-\infty$, so the graph is III.

25. If $x=t\cos t$, $y=t\sin t$, and $z=t$, then $x^2+y^2=t^2\cos^2 t+t^2\sin^2 t=t^2=z^2$, so the curve lies on the cone $z^2=x^2+y^2$. Since $z=t$, the curve is a spiral on this cone.

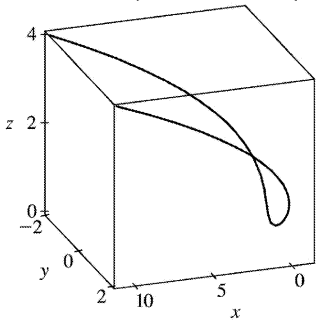


26. Here $x^2=\sin^2 t=z$ and $x^2+y^2=\sin^2 t+\cos^2 t=1$, so the curve is the intersection of the parabolic cylinder $z=x^2$ with the circular cylinder $x^2+y^2=1$.

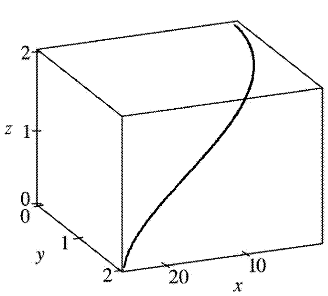
27. $\boldsymbol{r}(t)=\left\langle \sin t,\cos t,t^2\right\rangle$



28. $\boldsymbol{r}(t)=\left\langle t^4-t^2+1,t,t^2\right\rangle$

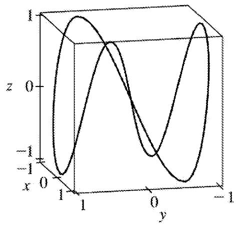


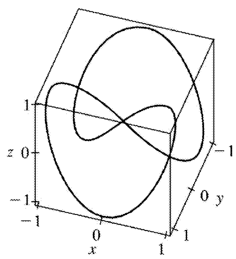
29. $\boldsymbol{r}(t)=\left\langle t^2,\sqrt{t-1},\sqrt{5-t}\right\rangle$



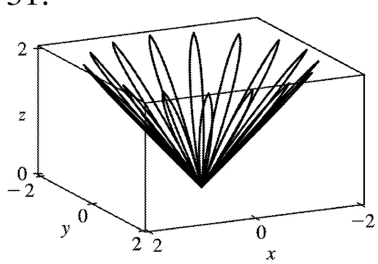
30. We have the computer plot the parametric equations $x=\sin t$, $y=\sin 2t$, $z=\sin 3t$, $0\leq t\leq 2\pi$. The

shape of the curve is not clear from just one viewpoint, so we include a second plot drawn from a different angle.





31.



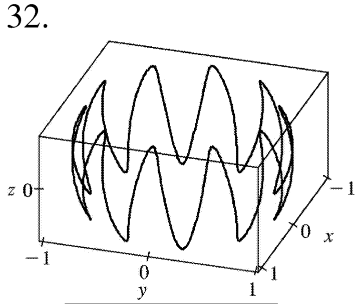
$x=(1+\cos 16t)\cos t$, $y=(1+\cos 16t)\sin t$, $z=1+\cos 16t$. At any point on the graph,

$$x^2+y^2=(1+\cos 16t)^2\cos^2 t+(1+\cos 16t)^2\sin^2 t$$

$$=(1+\cos 16t)^2=z^2 \text{ , so the graph lies on the cone}$$

$x^2+y^2=z^2$. From the graph at left, we see that this curve looks like the projection of a leaved two–dimensional curve onto a cone.

32.



$x=\sqrt{1-0.25\cos^2 10t}\,\cos t$, $y=\sqrt{1-0.25\cos^2 10t}\,\sin t$, $z=0.5\cos 10t$. At any point on the graph,

$$\begin{aligned} x^2+y^2+z^2 &= (1-0.25\cos^2 10t)\cos^2 t \\ &\quad +(1-0.25\cos^2 10t)\sin^2 t+0.25\cos^2 t \\ &= 1-0.25\cos^2 10t+0.25\cos^2 10t=1 , \end{aligned}$$

so the graph lies on the sphere $x^2+y^2+z^2=1$, and since $z=0.5\cos 10t$ the graph resembles a trigonometric curve with ten peaks projected onto the sphere. The graph is generated by $t\in[0,2\pi]$.

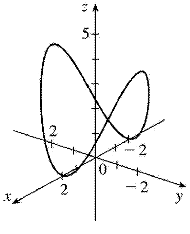
33. If $t=-1$, then $x=1,y=4,z=0$, so the curve passes through the point $(1,4,0)$. If $t=3$, then $x=9,y=-8,z=28$, so the curve passes through the point $(9,-8,28)$. For the point $(4,7,-6)$ to be on the curve, we require $y=1-3t=7\Rightarrow t=-2$. But then $z=1+(-2)^3=-7\neq-6$, so $(4,7,-6)$ is not on the curve.

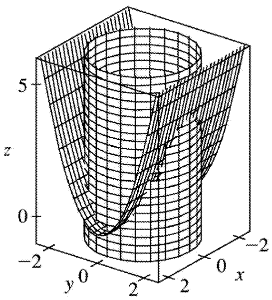
34. The projection of the curve C of intersection onto the xy –plane is the circle $x^2+y^2=4,z=0$. Then we can write $x=2\cos t,y=2\sin t,0\leq t\leq 2\pi$. Since C also lies on the surface $z=xy$, we have $z=xy=(2\cos t)(2\sin t)=4\cos t\sin t$, or $2\sin(2t)$. Then parametric equations for C are $x=2\cos t,y=2\sin t,z=2\sin(2t),0\leq t\leq 2\pi$, and the corresponding vector function is $\boldsymbol{r}(t)=2\cos t\,\boldsymbol{i}+2\sin t\,\boldsymbol{j}+2\sin(2t)\boldsymbol{k}$, $0\leq t\leq 2\pi$.

35. Both equations are solved for z , so we can substitute to eliminate z : $\sqrt{x^2+y^2}=1+y\Rightarrow x^2+y^2=1+2y+y^2\Rightarrow x^2=1+2y\Rightarrow y=\frac{1}{2}(x^2-1)$. We can form parametric equations for the curve C of intersection by choosing a parameter $x=t$, then $y=\frac{1}{2}(t^2-1)$ and $z=1+y=1+\frac{1}{2}(t^2-1)=\frac{1}{2}(t^2+1)$. Thus a vector function representing C is $\boldsymbol{r}(t)=t\,\boldsymbol{i}+\frac{1}{2}(t^2-1)\boldsymbol{j}+\frac{1}{2}(t^2+1)\boldsymbol{k}$.

36. The projection of the curve C of intersection onto the xy –plane is the parabola $y=x^2,z=0$. Then we can choose the parameter $x=t\Rightarrow y=t^2$. Since C also lies on the surface $z=4x^2+y^2$, we have $z=4x^2+y^2=4t^2+(t^2)^2$. Then parametric equations for C are $x=t$, $y=t^2$, $z=4t^2+t^4$, and the corresponding vector function is $\boldsymbol{r}(t)=t\,\boldsymbol{i}+t^2\boldsymbol{j}+(4t^2+t^4)\boldsymbol{k}$.

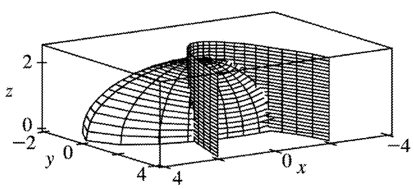
37.

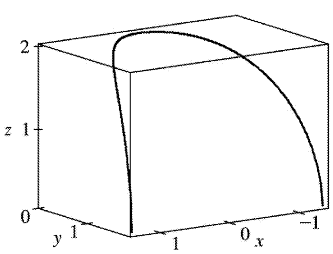




The projection of the curve C of intersection onto the xy –plane is the circle $x^2+y^2=4,z=0$. Then we can write $x=2\cos t$, $y=2\sin t$, $0\leq t\leq 2\pi$. Since C also lies on the surface $z=x^2$, we have $z=x^2=(2\cos t)^2=4\cos^2 t$. Then parametric equations for C are $x=2\cos t$, $y=2\sin t$, $z=4\cos^2 t, 0\leq t\leq 2\pi$.

38.





$x=t\Rightarrow y=t^2\Rightarrow 4z^2=16-x^2-4y^2=16-t^2-4t^4\Rightarrow z=\sqrt{4-\left(\frac{1}{2}t\right)^2-t^4}$. Note that z is positive because the intersection is with the top half of the ellipsoid. Hence the curve is given by $x=t$, $y=t^2$, $z=\sqrt{4-\frac{1}{4}t^2-t^4}$.

39. For the particles to collide, we require $\boldsymbol{r}_1(t)=\boldsymbol{r}_2(t)\Leftrightarrow\left\langle t^2,7t-12,t^2\right\rangle=\left\langle 4t-3,t^2,5t-6\right\rangle$. Equating components gives $t^2=4t-3$, $7t-12=t^2$, and $t^2=5t-6$. From the first equation, $t^2-4t+3=0\Leftrightarrow(t-3)(t-1)=0$ so $t=1$ or $t=3$. $t=1$ does not satisfy the other two equations, but $t=3$ does. The particles collide when $t=3$, at the point $(9,9,9)$.

40. The particles collide provided $\boldsymbol{r}_1(t)=\boldsymbol{r}_2(t)\Leftrightarrow\left\langle t,t^2,t^3\right\rangle=\left\langle 1+2t,1+6t,1+14t\right\rangle$. Equating components

gives $t=1+2t$, $t^2=1+6t$, and $t^3=1+14t$. The first equation gives $t=-1$, but this does not satisfy the other equations, so the particles do not collide. For the paths to intersect, we need to find a value for t and a value for s where $\boldsymbol{r}_1(t)=\boldsymbol{r}_2(s)\Leftrightarrow\left\langle t,t^2,t^3\right\rangle=\left\langle 1+2s,1+6s,1+14s\right\rangle$. Equating components, $t=1+2s$, $t^2=1+6s$, and $t^3=1+14s$. Substituting the first equation into the second gives $(1+2s)^2=1+6s\Rightarrow$ $4s^2-2s=0\Rightarrow 2s(2s-1)=0\Rightarrow s=0$ or $s=\frac{1}{2}$. From the first equation, $s=0\Rightarrow t=1$ and $s=\frac{1}{2}\Rightarrow t=2$. Checking, we see that both pairs of values satisfy the third equation. Thus the paths intersect twice, at the point $(1,1,1)$ when $s=0$ and $t=1$, and at $(2,4,8)$ when $s=\frac{1}{2}$ and $t=2$.

41. Let $\boldsymbol{u}(t)=\left\langle u_1(t),u_2(t),u_3(t)\right\rangle$ and $\boldsymbol{v}(t)=\left\langle v_1(t),v_2(t),v_3(t)\right\rangle$. In each part of this problem the basic procedure is to use Equation 1 and then analyze the individual component functions using the limit properties we have already developed for real–valued functions.

(a) $\lim_{t\to a}\boldsymbol{u}(t)+\lim_{t\to a}\boldsymbol{v}(t)=\left\langle \lim_{t\to a}u_1(t),\lim_{t\to a}u_2(t),\lim_{t\to a}u_3(t)\right\rangle+\left\langle \lim_{t\to a}v_1(t),\lim_{t\to a}v_2(t),\lim_{t\to a}v_3(t)\right\rangle$ and the limits of these component functions must each exist since the vector functions both possess limits as $t\to a$. Then
adding the two vectors and using the addition property of limits for real–valued functions, we have that

$$\begin{aligned}\lim_{t\to a}\boldsymbol{u}(t)+\lim_{t\to a}\boldsymbol{v}(t) &= \left\langle \lim_{t\to a}u_1(t)+\lim_{t\to a}v_1(t),\lim_{t\to a}u_2(t)+\lim_{t\to a}v_2(t),\lim_{t\to a}u_3(t)+\lim_{t\to a}v_3(t)\right\rangle\\ &= \left\langle \lim_{t\to a}\left[u_1(t)+v_1(t)\right],\lim_{t\to a}\left[u_2(t)+v_2(t)\right],\lim_{t\to a}\left[u_3(t)+v_3(t)\right]\right\rangle\\ &= \lim_{t\to a}\left\langle u_1(t)+v_1(t),u_2(t)+v_2(t),u_3(t)+v_3(t)\right\rangle \text{ [using(1)backward}\\ &= \lim_{t\to a}\left[\boldsymbol{u}(t)+\boldsymbol{v}(t)\right]\end{aligned}$$

(b)

$$\begin{aligned}\lim_{t\to a}c\boldsymbol{u}(t) &=\lim_{t\to a}\left\langle cu_1(t),cu_2(t),cu_3(t)\right\rangle=\left\langle \lim_{t\to a}cu_1(t),\lim_{t\to a}cu_2(t),\lim_{t\to a}cu_3(t)\right\rangle\\ &=\left\langle c\lim_{t\to a}u_1(t),c\lim_{t\to a}u_2(t),c\lim_{t\to a}u_3(t)\right\rangle=c\left\langle \lim_{t\to a}u_1(t),\lim_{t\to a}u_2(t),\lim_{t\to a}u_3(t)\right\rangle\\ &=c\lim_{t\to a}\left\langle u_1(t),u_2(t),u_3(t)\right\rangle=c\lim_{t\to a}\boldsymbol{u}(t)\end{aligned}$$

(c)

$$\lim_{t\to a} \boldsymbol{u}(t)\cdot\lim_{t\to a} \boldsymbol{v}(t) = \left\langle \lim_{t\to a} u_1(t),\lim_{t\to a} u_2(t),\lim_{t\to a} u_3(t)\right\rangle \cdot \left\langle \lim_{t\to a} v_1(t),\lim_{t\to a} v_2(t),\lim_{t\to a} v_3(t)\right\rangle$$

$$=\left[\lim_{t\to a} u_1(t)\right]\left[\lim_{t\to a} v_1(t)\right]+\left[\lim_{t\to a} u_2(t)\right]\left[\lim_{t\to a} v_2(t)\right]+\left[\lim_{t\to a} u_3(t)\right]\left[\lim_{t\to a} v_3(t)\right]$$

$$=\lim_{t\to a} u_1(t)v_1(t)+\lim_{t\to a} u_2(t)v_2(t)+\lim_{t\to a} u_3(t)v_3(t)$$

$$=\lim_{t\to a} \left[u_1(t)v_1(t)+u_2(t)v_2(t)+u_3(t)v_3(t)\right]=\lim_{t\to a} \left[\boldsymbol{u}(t)\cdot \boldsymbol{v}(t)\right]$$

(d)

$$\lim_{t\to a} \boldsymbol{u}(t)\times\lim_{t\to a} \boldsymbol{v}(t) = \left\langle \lim_{t\to a} u_1(t),\lim_{t\to a} u_2(t),\lim_{t\to a} u_3(t)\right\rangle \times \left\langle \lim_{t\to a} v_1(t),\lim_{t\to a} v_2(t),\lim_{t\to a} v_3(t)\right\rangle$$

$$=\left\langle \left[\lim_{t\to a} u_2(t)\right]\left[\lim_{t\to a} v_3(t)\right]-\left[\lim_{t\to a} u_3(t)\right]\left[\lim_{t\to a} v_2(t)\right],\right.$$

$$\left[\lim_{t\to a} u_3(t)\right]\left[\lim_{t\to a} v_1(t)\right]-\left[\lim_{t\to a} u_1(t)\right]\left[\lim_{t\to a} v_3(t)\right],$$

$$\left.\left[\lim_{t\to a} u_1(t)\right]\left[\lim_{t\to a} v_2(t)\right]-\left[\lim_{t\to a} u_2(t)\right]\left[\lim_{t\to a} v_1(t)\right]\right\rangle$$

$$=\left\langle \lim_{t\to a} \left[u_2(t)v_3(t)-u_3(t)v_2(t)\right],\lim_{t\to a} \left[u_3(t)v_1(t)-u_1(t)v_3(t)\right],\right.$$

$$\left.\lim_{t\to a} \left[u_1(t)v_2(t)-u_2(t)v_1(t)\right]\right\rangle$$

$$=\lim_{t\to a} \left\langle u_2(t)v_3(t)-u_3(t)v_2(t),\mathrm{u}_3(t)v_1(t)-u_1(t)v_3(t),\right.$$

$$\left.u_1(t)v_2(t)-u_2(t)v_1(t)\right\rangle$$

$$=\lim_{t\to a} \left[\boldsymbol{u}(t)\times \boldsymbol{v}(t)\right]$$

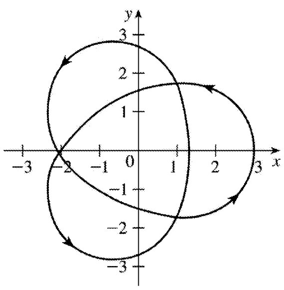
42. The projection of the curve onto the xy –plane is given by the parametric equations $x=(2+\cos 1.5t)\cos t$, $y=(2+\cos 1.5t)\sin t$. If we convert to polar coordinates, we have

$$r^2 = x^2+y^2=[(2+\cos 1.5t)\cos t]^2+[(2+\cos 1.5t)\sin t]^2$$

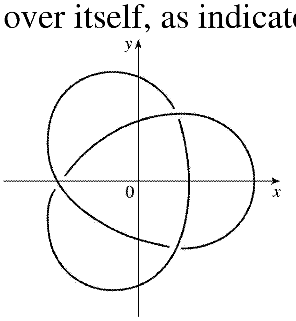
$$= (2+\cos 1.5t)^2(\cos^2 t+\sin^2 t)$$

$$= (2+\cos 1.5t)^2$$

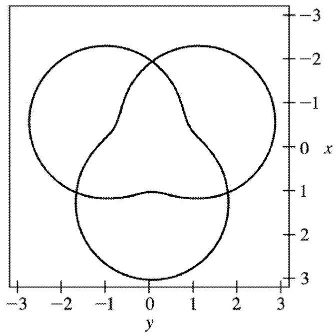
$\Rightarrow r=2+\cos 1.5t$. Also, $\tan\theta=\dfrac{y}{x}=\dfrac{(2+\cos 1.5t)\sin t}{(2+\cos 1.5t)\cos t}=\tan t\Rightarrow\theta=t$.
Thus the polar equation of the curve is $r=2+\cos 1.5\theta$. At $\theta=0$, we have $r=3$, and r decreases to 1 as θ increases to $\dfrac{2\pi}{3}$. For $\dfrac{2\pi}{3}\le\theta\le\dfrac{4\pi}{3}$, r increases to 3 ; r decreases to 1 again at $\theta=2\pi$, increases to 3 at $\theta=\dfrac{8\pi}{3}$, decreases to 1 at $\theta=\dfrac{10\pi}{3}$, and completes the closed curve by increasing to 3 at $\theta=4\pi$. We sketch an approximate graph as shown in the figure.



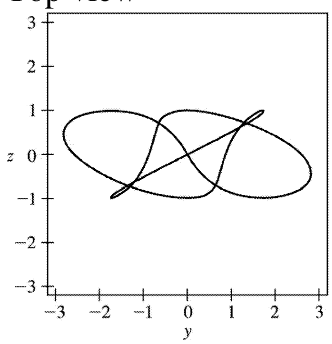
We can determine how the curve passes over itself by investigating the maximum and minimum values of z for $t=\theta\in[0,4\pi]$. Since $z=\sin 1.5t$, z is maximized where $\sin 1.5t=1\Rightarrow 1.5t=\dfrac{\pi}{2}$, $\dfrac{5\pi}{2}$, or $\dfrac{9\pi}{2}\Rightarrow$
$t=\dfrac{\pi}{3}$, $\dfrac{5\pi}{3}$, or 3π . z is minimized where $\sin 1.5t=-1\Rightarrow 1.5t=\dfrac{3\pi}{2}$, $\dfrac{7\pi}{2}$, or $\dfrac{11\pi}{2}\Rightarrow t=\pi$, $\dfrac{7\pi}{3}$, or $\dfrac{11\pi}{3}$. Note that these are precisely the values for which $\cos 1.5t=0\Rightarrow r=2$, and on the graph of the projection, these six points appear to be at the three self–intersections we see. Comparing the maximum and minimum values of z at these intersections, we can determine where the curve passes over itself, as indicated in the figure.



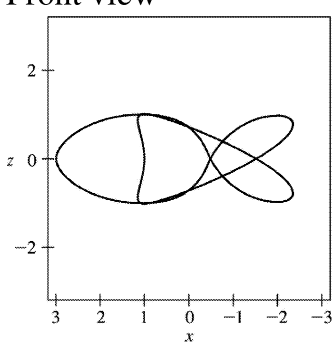
We show a computer–drawn graph of the curve from above, as well as views from the front and from the right side.



Top view

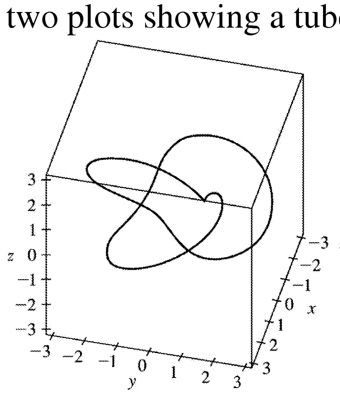


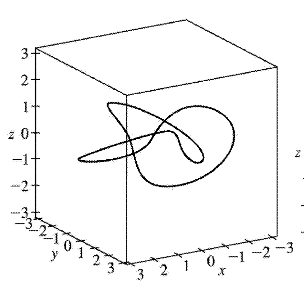
Front view

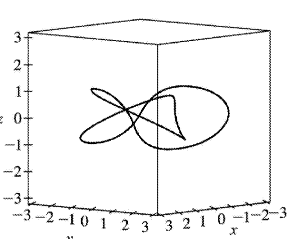


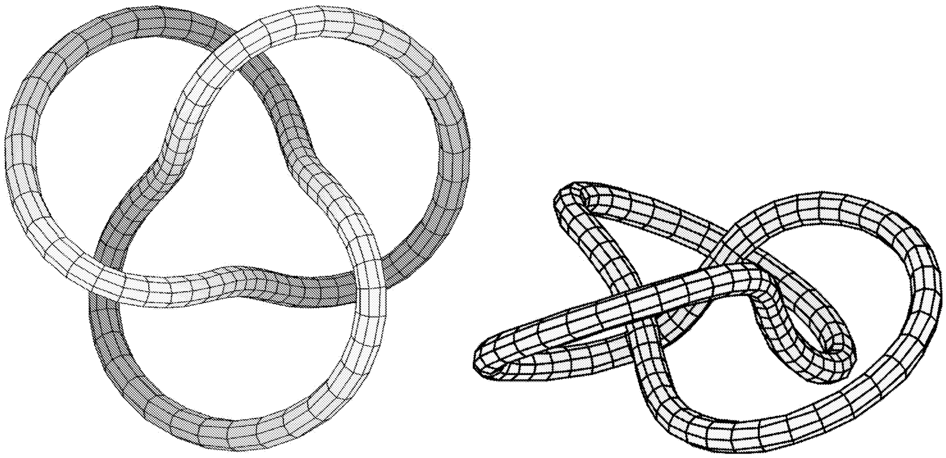
Side view

The top view graph shows a more accurate representation of the projection of the trefoil knot on the xy –plane (the axes are rotated 90°). Notice the indentations the graph exhibits at the points corresponding to $r=1$. Finally, we graph several additional viewpoints of the trefoil knot, along with two plots showing a tube of radius 0.2 around the curve.









43. Let $\boldsymbol{r}(t)=\langle f(t),g(t),h(t)\rangle$ and $\boldsymbol{b}=\langle b_1,b_2,b_3\rangle$. If $\lim_{t\to a}\boldsymbol{r}(t)=\boldsymbol{b}$, then $\lim_{t\to a}\boldsymbol{r}(t)$ exists, so by (1), $\boldsymbol{b}=\lim_{t\to a}\boldsymbol{r}(t)=\left\langle \lim_{t\to a}f(t),\lim_{t\to a}g(t),\lim_{t\to a}h(t)\right\rangle$. By the definition of equal vectors we have $\lim_{t\to a}f(t)=b_1$, $\lim_{t\to a}g(t)=b_2$ and $\lim_{t\to a}h(t)=b_3$. But these are limits of real–valued functions, so by the definition of limits, for every $\varepsilon>0$ there exists $\delta_1>0$, $\delta_2>0$, $\delta_3>0$ so $\left|f(t)-b_1\right|<\varepsilon/3$ whenever $0<|t-a|<\delta_1$, $\left|g(t)-b_2\right|<\varepsilon/3$ whenever $0<|t-a|<\delta_2$, and $\left|h(t)-b_3\right|<\varepsilon/3$ whenever $0<|t-a|<\delta_3$. Letting $\delta=$ minimum of $\{\delta_1,\delta_2,\delta_3\}$, we have $\left|f(t)-b_1\right|+\left|g(t)-b_2\right|+\left|h(t)-b_3\right|<\varepsilon/3+\varepsilon/3+\varepsilon/3=\varepsilon$ whenever $0<|t-a|<\delta$. But $|\boldsymbol{r}(t)-\boldsymbol{b}|=\left|\left\langle f(t)-b_1,g(t)-b_2,h(t)-b_3\right\rangle\right|$

$=\sqrt{(f(t)-b_1)^2+(g(t)-b_2)^2+(h(t)-b_3)^2}\le\sqrt{[f(t)-b_1]^2}+\sqrt{[g(t)-b_2]^2}+\sqrt{[h(t)-b_3]^2}$

$=\left|f(t)-b_1\right|+\left|g(t)-b_2\right|+\left|h(t)-b_3\right|$. Thus for every $\varepsilon>0$ there exists $\delta>0$ such that $|\boldsymbol{r}(t)-b|\le\left|f(t)-b_1\right|+\left|g(t)-b_2\right|+\left|h(t)-b_3\right|<\varepsilon$ whenever $0<|t-a|<\delta$. Conversely, if for every $\varepsilon>0$, there exists $\delta>0$ such that $|\boldsymbol{r}(t)-\boldsymbol{b}|<\varepsilon$ whenever $0<|t-a|<\delta$, then

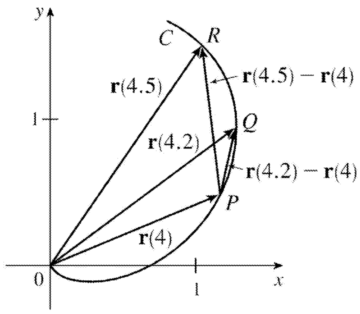
$\left|\left\langle f(t)-b_1,g(t)-b_2,h(t)-b_3\right\rangle\right|<\varepsilon\Leftrightarrow\sqrt{[f(t)-b_1]^2+[g(t)-b_2]^2+[h(t)-b_3]^2}<\varepsilon\Leftrightarrow$

$[f(t)-b_1]^2+[g(t)-b_2]^2+[h(t)-b_3]^2<\varepsilon^2$ whenever $0<|t-a|<\delta$. But each term on the left side of this inequality is positive so $[f(t)-b_1]^2<\varepsilon^2$, $[g(t)-b_2]^2<\varepsilon^2$ and $[h(t)-b_3]^2<\varepsilon^2$ whenever $0<|t-a|<\delta$, or taking the square root of both sides in each of the above we have $\left|f(t)-b_1\right|<\varepsilon$, $\left|g(t)-b_2\right|<\varepsilon$ and $\left|h(t)-b_3\right|<\varepsilon$ whenever $0<|t-a|<\delta$. And by definition of limits of real–valued functions we have

$\lim_{t\to a} f(t)=b_1$, $\lim_{t\to a} g(t)=b_2$ and $\lim_{t\to a} h(t)=b_3$. But by (1), $\lim_{t\to a} \boldsymbol{r}(t)=\left\langle \lim_{t\to a} f(t),\lim_{t\to a} g(t),\lim_{t\to a} h(t)\right\rangle$, so $\lim_{t\to a} \boldsymbol{r}(t)=\left\langle b_1,\mathrm{b}_2,\mathrm{b}_3\right\rangle=\boldsymbol{b}$.

1.

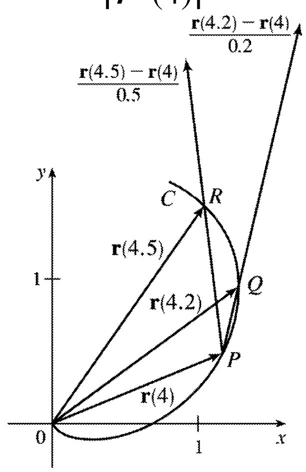
(a)



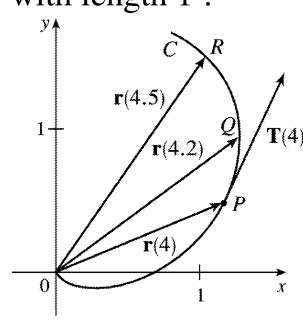
(b) $\dfrac{\boldsymbol{r}(4.5)-\boldsymbol{r}(4)}{0.5}=2[\boldsymbol{r}(4.5)-\boldsymbol{r}(4)]$, so we draw a vector in the same direction but with twice the length of the vector $\boldsymbol{r}(4.5)-\boldsymbol{r}(4)$. $\dfrac{\boldsymbol{r}(4.2)-\boldsymbol{r}(4)}{0.2}=5[\boldsymbol{r}(4.2)-\boldsymbol{r}(4)]$, so we draw a vector in the same direction but with 5 times the length of the vector $\boldsymbol{r}(4.2)-\boldsymbol{r}(4)$

(c) By Definition 1, $\boldsymbol{r}'(4)=\lim_{h\to 0}\dfrac{\boldsymbol{r}(4+h)-\boldsymbol{r}(4)}{h}$.

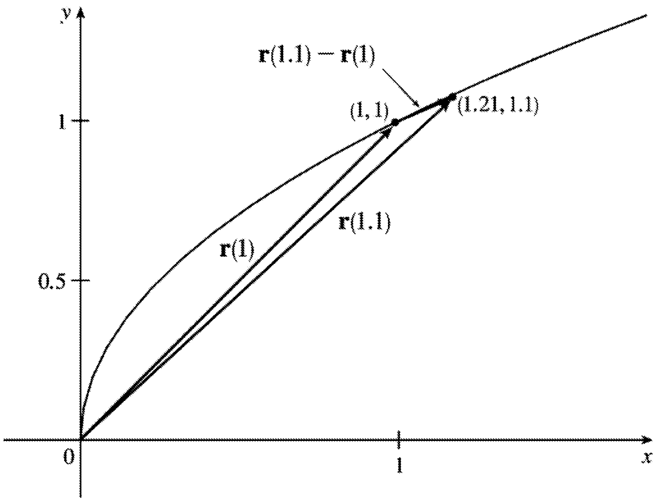
$\boldsymbol{T}(4)=\dfrac{\boldsymbol{r}'(4)}{\left|\boldsymbol{r}'(4)\right|}$.



(d) $\boldsymbol{T}(4)$ is a unit vector in the same direction as $\boldsymbol{r}'(4)$, that is, parallel to the tangent line to the curve at $\boldsymbol{r}(4)$ with length 1 .

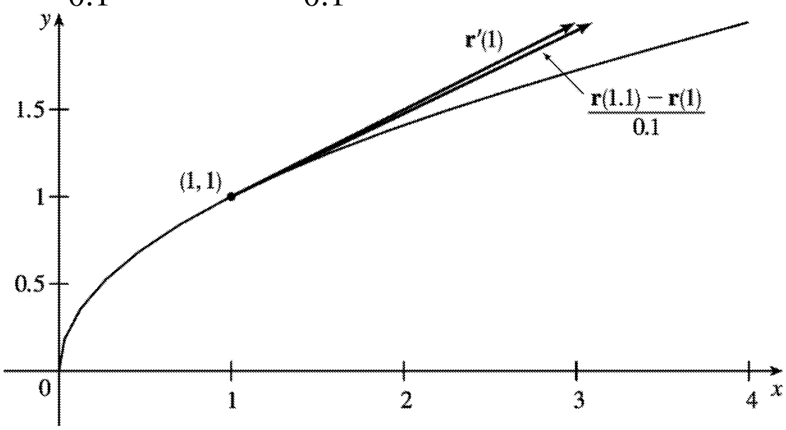


2. **(a)** The curve can be represented by the parametric equations $x=t^2$, $y=t$, $0\leq t\leq 2$. Eliminating the parameter, we have $x=y^2$, $0\leq y\leq 2$, a portion of which we graph here, along with the vectors $\boldsymbol{r}(1)$, $\boldsymbol{r}(1.1)$, and $\boldsymbol{r}(1.1)-\boldsymbol{r}(1)$.



(b) Since $\boldsymbol{r}(t)=\left\langle t^2,\mathrm{t}\right\rangle$, we differentiate components, giving $\boldsymbol{r}'(t)=\left\langle 2\mathrm{t},1\right\rangle$, so $\boldsymbol{r}'(1)=\left\langle 2,1\right\rangle$.

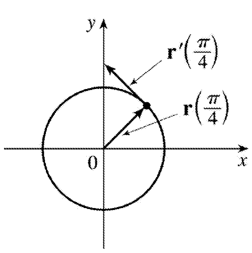
$$\frac{\boldsymbol{r}(1.1)-\boldsymbol{r}(1)}{0.1}=\frac{\left\langle 1.21,1.1\right\rangle-\left\langle 1,1\right\rangle}{0.1}=10\left\langle 0.21,0.1\right\rangle=\left\langle 2.1,1\right\rangle .$$



As we can see from the graph, these vectors are very close in length and direction. $\boldsymbol{r}'(1)$ is defined to be $\lim_{h\to 0}\frac{\boldsymbol{r}(1+h)-\boldsymbol{r}(1)}{h}$, and we recognize $\frac{\boldsymbol{r}(1.1)-\boldsymbol{r}(1)}{0.1}$ as the expression after the limit sign with $h=0.1$. Since h is close to 0 , we would expect $\frac{\boldsymbol{r}(1.1)-\boldsymbol{r}(1)}{0.1}$ to be a vector close to $\boldsymbol{r}'(1)$.

3.

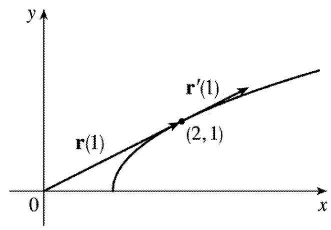
(a), (c)



(b) $\boldsymbol{r}'(t)=\langle -\sin t, \cos t\rangle$

4.

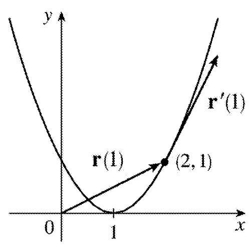
(a), (c)



(b) $\boldsymbol{r}'(t)=\left\langle 1, \frac{1}{2\sqrt{t}} \right\rangle$

5. Since $(x-1)^2=t^2=y$, the curve is a parabola.

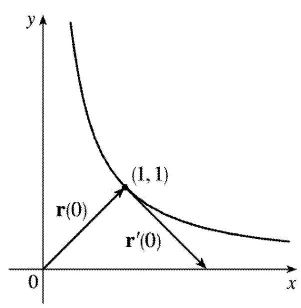
(a), (c)



(b) $\boldsymbol{r}'(t)=\boldsymbol{i}+2t\,\boldsymbol{j}$

6.

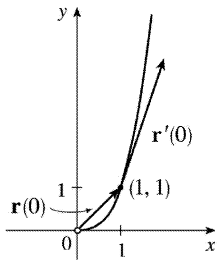
(a), (c)



(b) $\boldsymbol{r}'(t)=e^{t}\,\boldsymbol{i}-e^{-t}\,\boldsymbol{j}$

7.

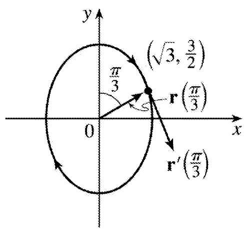
(a), (c)



(b) $\boldsymbol{r}'(t)=e^{t}\boldsymbol{i}+3e^{3t}\boldsymbol{j}$

8. $x=2\sin t$, $y=3\cos t$, so $(x/2)^2+(y/3)^2=\sin^2 t+\cos^2 t=1$ and the curve is an ellipse .

(a), (c)



(b) $\boldsymbol{r}'(t)=2\cos t\,\boldsymbol{i}-3\sin t\,\boldsymbol{j}$

9. $\boldsymbol{r}'(t)=\left\langle \frac{d}{dt}\left[t^2\right], \frac{d}{dt}\left[1-t\right], \frac{d}{dt}\left[\sqrt{t}\,\right] \right\rangle = \left\langle 2\text{t}, -1, \frac{1}{2\sqrt{t}} \right\rangle$

10. $\boldsymbol{r}(t)=\langle\cos 3\text{t},\text{t},\sin 3t\rangle \Rightarrow \boldsymbol{r}'(t)=\langle -3\sin 3\text{t},1,3\cos 3t\rangle$

11. $\boldsymbol{r}(t)=\boldsymbol{i}-\boldsymbol{j}+e^{4t}\boldsymbol{k}\Rightarrow \boldsymbol{r}'(t)=0\boldsymbol{i}+0\boldsymbol{j}+4e^{4t}\boldsymbol{k}=4e^{4t}\boldsymbol{k}$

12. $\boldsymbol{r}(t)=\sin^{-1} t\,\boldsymbol{i}+\sqrt{1-t^2}\,\boldsymbol{j}+\boldsymbol{k}\Rightarrow \boldsymbol{r}'(t)=\frac{1}{\sqrt{1-t^2}}\,\boldsymbol{i}-\frac{t}{\sqrt{1-t^2}}\,\boldsymbol{j}$

13. $\boldsymbol{r}(t)=e^{t^2}\boldsymbol{i}-\boldsymbol{j}+\ln(1+3t)\,\boldsymbol{k} \Rightarrow \boldsymbol{r}'(t)=2te^{t^2}\boldsymbol{i}+\frac{3}{1+3t}\,\boldsymbol{k}$

14.

$$\begin{aligned}\boldsymbol{r}'(t) &=[at(-3\sin 3t)+a\cos 3t]\boldsymbol{i}+b\cdot 3\sin^2 t\cos t\,\boldsymbol{j}+c\cdot 3\cos^2 t(-\sin t)\boldsymbol{k}\\ &=(a\cos 3t-3at\sin 3t)\boldsymbol{i}+3b\sin^2 t\cos t\,\boldsymbol{j}-3c\cos^2 t\sin t\,\boldsymbol{k}\end{aligned}$$

15. $\boldsymbol{r}'(t)=\boldsymbol{0}+\boldsymbol{b}+2t\,\boldsymbol{c}=\boldsymbol{b}+2t\,\boldsymbol{c}$ by Formulas 1 and 3 of Theorem 3.

16. To find $\boldsymbol{r}'(t)$, we first expand $\boldsymbol{r}(t)=t\,\boldsymbol{a}\times(\boldsymbol{b}+t\,\boldsymbol{c})=t(\boldsymbol{a}\times\boldsymbol{b})+t^2(\boldsymbol{a}\times\boldsymbol{c})$, so $\boldsymbol{r}'(t)=\boldsymbol{a}\times\boldsymbol{b}+2t(\boldsymbol{a}\times\boldsymbol{c})$.

17. $\boldsymbol{r}'(t)=\left\langle 30t^4,12t^2,2\right\rangle\Rightarrow\boldsymbol{r}'(1)=\langle 30,12,2\rangle$. So $\left|\boldsymbol{r}'(1)\right|=\sqrt{30^2+12^2+2^2}=\sqrt{1048}=2\sqrt{262}$ and $\boldsymbol{T}(1)=\dfrac{\boldsymbol{r}'(1)}{\left|\boldsymbol{r}'(1)\right|}=\dfrac{1}{2\sqrt{262}}\langle 30,12,2\rangle=\left\langle \dfrac{15}{\sqrt{262}},\dfrac{6}{\sqrt{262}},\dfrac{1}{\sqrt{262}}\right\rangle$.

18. $\boldsymbol{r}'(t)=\dfrac{2}{\sqrt{t}}\,\boldsymbol{i}+2t\,\boldsymbol{j}+\boldsymbol{k}\Rightarrow\boldsymbol{r}'(1)=2\boldsymbol{i}+2\boldsymbol{j}+\boldsymbol{k}$. Thus $\boldsymbol{T}(1)=\dfrac{\boldsymbol{r}'(1)}{\left|\boldsymbol{r}'(1)\right|}=\dfrac{1}{\sqrt{2^2+2^2+1^2}}(2\boldsymbol{i}+2\boldsymbol{j}+\boldsymbol{k})=\dfrac{1}{3}(2\boldsymbol{i}+2\boldsymbol{j}+\boldsymbol{k})=\dfrac{2}{3}\,\boldsymbol{i}+\dfrac{2}{3}\,\boldsymbol{j}+\dfrac{1}{3}\,\boldsymbol{k}$.

19. $\boldsymbol{r}'(t)=-\sin t\,\boldsymbol{i}+3\,\boldsymbol{j}+4\cos 2t\,\boldsymbol{k}\Rightarrow\boldsymbol{r}'(0)=3\,\boldsymbol{j}+4\,\boldsymbol{k}$. Thus $\boldsymbol{T}(0)=\dfrac{\boldsymbol{r}'(0)}{\left|\boldsymbol{r}'(0)\right|}=\dfrac{1}{\sqrt{0^2+3^2+4^2}}(3\,\boldsymbol{j}+4\boldsymbol{k})=\dfrac{1}{5}(3\,\boldsymbol{j}+4\boldsymbol{k})=\dfrac{3}{5}\,\boldsymbol{j}+\dfrac{4}{5}\,\boldsymbol{k}$.

20. $\boldsymbol{r}'(t)=2\cos t\,\boldsymbol{i}-2\sin t\,\boldsymbol{j}+\sec^2 t\,\boldsymbol{k}\Rightarrow\boldsymbol{r}'\left(\dfrac{\pi}{4}\right)=\sqrt{2}\,\boldsymbol{i}-\sqrt{2}\,\boldsymbol{j}+2\boldsymbol{k}$ and $\left|\boldsymbol{r}'\left(\dfrac{\pi}{4}\right)\right|=\sqrt{2+2+4}=2\sqrt{2}$. Thus $\boldsymbol{T}\left(\dfrac{\pi}{4}\right)=\dfrac{\boldsymbol{r}'\left(\dfrac{\pi}{4}\right)}{\left|\boldsymbol{r}'\left(\dfrac{\pi}{4}\right)\right|}=\dfrac{1}{2\sqrt{2}}\left(\sqrt{2}\,\boldsymbol{i}-\sqrt{2}\,\boldsymbol{j}+2\boldsymbol{k}\right)=\dfrac{1}{2}\,\boldsymbol{i}-\dfrac{1}{2}\,\boldsymbol{j}+\dfrac{1}{\sqrt{2}}\,\boldsymbol{k}$.

21. $\boldsymbol{r}(t)=\left\langle t,t^2,t^3\right\rangle\Rightarrow\boldsymbol{r}'(t)=\left\langle 1,2t,3t^2\right\rangle$. Then $\boldsymbol{r}'(1)=\langle 1,2,3\rangle$ and $\left|\boldsymbol{r}'(1)\right|=\sqrt{1^2+2^2+3^2}=\sqrt{14}$, so $\boldsymbol{T}(1)=\dfrac{\boldsymbol{r}'(1)}{\left|\boldsymbol{r}'(1)\right|}=\dfrac{1}{\sqrt{14}}\langle 1,2,3\rangle=\left\langle \dfrac{1}{\sqrt{14}},\dfrac{2}{\sqrt{14}},\dfrac{3}{\sqrt{14}}\right\rangle$. $\boldsymbol{r}''(t)=\langle 0,2,6t\rangle$, so

$$\boldsymbol{r}'(t)\times\boldsymbol{r}''(t)=\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ 1 & 2t & 3t^2\\ 0 & 2 & 6t\end{vmatrix}=\begin{vmatrix}2t & 3t^2\\ 2 & 6t\end{vmatrix}\boldsymbol{i}-\begin{vmatrix}1 & 3t^2\\ 0 & 6t\end{vmatrix}\boldsymbol{j}+\begin{vmatrix}1 & 2t\\ 0 & 2\end{vmatrix}\boldsymbol{k}$$

$$=(12t^2-6t^2)\boldsymbol{i}-(6t-0)\,\boldsymbol{j}+(2-0)\boldsymbol{k}=\left\langle 6t^2,-6t,2\right\rangle .$$

22. $\boldsymbol{r}(t)=\left\langle e^{2t},e^{-2t},te^{2t}\right\rangle\Rightarrow\boldsymbol{r}'(t)=\left\langle 2e^{2t},-2e^{-2t},(2t+1)e^{2t}\right\rangle\Rightarrow\boldsymbol{r}'(0)=\left\langle 2e^0,-2e^0,(0+1)e^0\right\rangle=\langle 2,-2,1\rangle$ and

$\left|\boldsymbol{r}'(0)\right|=\sqrt{2^2+(-2)^2+1^2}=3$. Then $\boldsymbol{T}(0)=\dfrac{\boldsymbol{r}'(0)}{\left|\boldsymbol{r}'(0)\right|}=\dfrac{1}{3}\langle 2,-2,1\rangle=\left\langle \dfrac{2}{3},-\dfrac{2}{3},\dfrac{1}{3}\right\rangle$.

$\boldsymbol{r}''(t)=\left\langle 4e^{2t},4e^{-2t},(4t+4)e^{2t}\right\rangle \Rightarrow \boldsymbol{r}''(0)=\left\langle 4e^{0},4e^{0},(0+4)e^{0}\right\rangle=\langle 4,4,4\rangle$.

$$\begin{aligned}\boldsymbol{r}'(t)\cdot\boldsymbol{r}''(t) &=\left\langle 2e^{2t},-2e^{-2t},(2t+1)e^{2t}\right\rangle\cdot\left\langle 4e^{2t},4e^{-2t},(4t+4)e^{2t}\right\rangle\\ &=(2e^{2t})(4e^{2t})+(-2e^{-2t})(4e^{-2t})+((2t+1)e^{2t})((4t+4)e^{2t})\\ &=8e^{4t}-8e^{-4t}+(8t^2+12t+4)e^{4t}=(8t^2+12t+12)e^{4t}-8e^{-4t}\end{aligned}$$

23. The vector equation for the curve is $\boldsymbol{r}(t)=\left\langle t^5,t^4,t^3\right\rangle$, so $\boldsymbol{r}'(t)=\left\langle 5t^4,4t^3,3t^2\right\rangle$. The point $(1,1,1)$ corresponds to $t=1$, so the tangent vector there is $\boldsymbol{r}'(1)=\langle 5,4,3\rangle$. Thus, the tangent line goes through the point $(1,1,1)$ and is parallel to the vector $\langle 5,4,3\rangle$. Parametric equations are $x=1+5t$, $y=1+4t$, $z=1+3t$.

24. The vector equation for the curve is $\boldsymbol{r}(t)=\left\langle t^2-1,t^2+1,t+1\right\rangle$, so $\boldsymbol{r}'(t)=\langle 2t,2t,1\rangle$. The point $(-1,1,1)$ corresponds to $t=0$, so the tangent vector there is $\boldsymbol{r}'(0)=\langle 0,0,1\rangle$. Thus, the tangent line is parallel to the vector $\langle 0,0,1\rangle$ and parametric equations are $x=-1+0\cdot t=-1$, $y=1+0\cdot t=1$, $z=1+1\cdot t=1+t$.

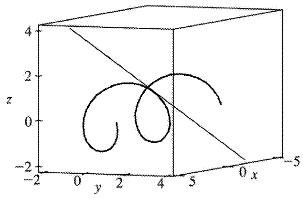
25. The vector equation for the curve is $\boldsymbol{r}(t)=\left\langle e^{-t}\cos t,e^{-t}\sin t,e^{-t}\right\rangle$, so

$$\begin{aligned}\boldsymbol{r}'(t) &=\left\langle e^{-t}(-\sin t)+(\cos t)(-e^{-t}),e^{-t}\cos t+(\sin t)(-e^{-t}),(-e^{-t})\right\rangle\\ &=\left\langle -e^{-t}(\cos t+\sin t),e^{-t}(\cos t-\sin t),-e^{-t}\right\rangle .\end{aligned}$$

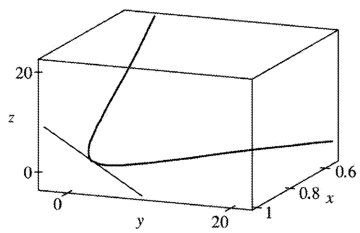
The point $(1,0,1)$ corresponds to $t=0$, so the tangent vector there is $\boldsymbol{r}'(0)=\left\langle -e^{0}(\cos 0+\sin 0),e^{0}(\cos 0-\sin 0),-e^{0}\right\rangle=\langle -1,1,-1\rangle$. Thus, the tangent line is parallel to the vector $\langle -1,1,-1\rangle$ and parametric equations are $x=1+(-1)t=1-t$, $y=0+1\cdot t=t$, $z=1+(-1)t=1-t$.

26. $\boldsymbol{r}(t)=\left\langle \ln t,2\sqrt{t},t^2\right\rangle$, $\boldsymbol{r}'(t)=\left\langle 1/t,1/\sqrt{t},2t\right\rangle$. At $(0,2,1)$, $t=1$ and $\boldsymbol{r}'(1)=\langle 1,1,2\rangle$. Thus, parametric equations of the tangent line are $x=t$, $y=2+t$, $z=1+2t$.

27. $\boldsymbol{r}(t)=\left\langle t,\sqrt{2}\cos t,\sqrt{2}\sin t\right\rangle \Rightarrow \boldsymbol{r}'(t)=\left\langle 1,-\sqrt{2}\sin t,\sqrt{2}\cos t\right\rangle$. At $\left(\dfrac{\pi}{4},1,1\right)$, $t=\dfrac{\pi}{4}$ and $\boldsymbol{r}'\left(\dfrac{\pi}{4}\right)=\langle 1,-1,1\rangle$. Thus, parametric equations of the tangent line are $x=\dfrac{\pi}{4}+t$, $y=1-t$, $z=1+t$.



28. $\boldsymbol{r}(t)=\langle \cos t, 3e^{2t}, 3e^{-2t}\rangle$, $\boldsymbol{r}'(t)=\langle -\sin t, 6e^{2t}, -6e^{-2t}\rangle$. At $(1,3,3)$, $t=0$ and $\boldsymbol{r}'(0)=\langle 0,6,-6\rangle$. Thus, parametric equations of the tangent line are $x=1$, $y=3+6t$, $z=3-6t$.

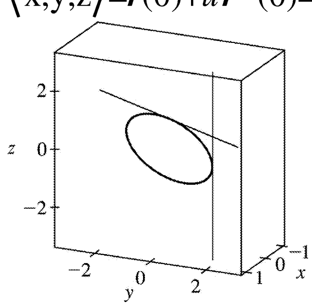


29. **(a)** $\boldsymbol{r}(t)=\langle t^3, t^4, t^5\rangle \Rightarrow \boldsymbol{r}'(t)=\langle 3t^2, 4t^3, 5t^4\rangle$, and since $\boldsymbol{r}'(0)=\langle 0,0,0\rangle=\mathbf{0}$, the curve is not smooth.

(b) $\boldsymbol{r}(t)=\langle t^3+t, t^4, t^5\rangle \Rightarrow \boldsymbol{r}'(t)=\langle 3t^2+1, 4t^3, 5t^4\rangle$. $\boldsymbol{r}'(t)$ is continuous since its component functions are continuous. Also, $\boldsymbol{r}'(t)\neq\mathbf{0}$, as the y – and z –components are 0 only for $t=0$, but $\boldsymbol{r}'(0)=\langle 1,0,0\rangle\neq\mathbf{0}$. Thus, the curve is smooth.

(c) $\boldsymbol{r}(t)=\langle \cos^3 t, \sin^3 t\rangle \Rightarrow \boldsymbol{r}'(t)=\langle -3\cos^2 t\sin t, 3\sin^2 t\cos t\rangle$. Since $\boldsymbol{r}'(0)=\langle -3\cos^2 0\sin 0, 3\sin^2 0\cos 0\rangle=\langle 0,0\rangle=\mathbf{0}$, the curve is not smooth.

30. **(a)** p250pt The tangent line at $t=0$ is the line through the point with position vector $\boldsymbol{r}(0)=\langle \sin 0, 2\sin 0, \cos 0\rangle=\langle 0,0,1\rangle$,
and in the direction of the tangent vector, $\boldsymbol{r}'(0)=\langle \pi\cos 0, 2\pi\cos 0, -\pi\sin 0\rangle=\langle \pi, 2\pi, 0\rangle$.
So an equation of the line is
$\langle x,y,z\rangle=\boldsymbol{r}(0)+u\boldsymbol{r}'(0)=\langle 0+\pi u, 0+2\pi u, 1\rangle=\langle \pi u, 2\pi u, 1\rangle$. (b)



$\boldsymbol{r}\left(\frac{1}{2}\right)=\left\langle \sin\frac{\pi}{2},2\sin\frac{\pi}{2},\cos\frac{\pi}{2}\right\rangle=\langle 1,2,0\rangle$,

$\boldsymbol{r}'\left(\frac{1}{2}\right)=\left\langle \pi\cos\frac{\pi}{2},2\pi\cos\frac{\pi}{2},-\pi\sin\frac{\pi}{2}\right\rangle=\langle 0,0,-\pi\rangle$.

So the equation of the second line is $\langle x,y,z\rangle=\langle 1,2,0\rangle+v\langle 0,0,-\pi\rangle=\langle 1,2,-\pi v\rangle$. The lines intersect where $\langle \pi u,2\pi u,1\rangle=\langle 1,2,-\pi v\rangle$, so the point of intersection is $(1,2,1)$.

31. The angle of intersection of the two curves is the angle between the two tangent vectors to the curves at the point of intersection. Since $\boldsymbol{r}_1'(t)=\langle 1,2t,3t^2\rangle$ and $t=0$ at $(0,0,0)$, $\boldsymbol{r}_1'(0)=\langle 1,0,0\rangle$ is a tangent vector to $\boldsymbol{r}_1$ at $(0,0,0)$. Similarly, $\boldsymbol{r}_2'(t)=\langle \cos t,2\cos 2t,1\rangle$ and since $\boldsymbol{r}_2(0)=\langle 0,0,0\rangle$, $\boldsymbol{r}_2'(0)=\langle 1,2,1\rangle$ is a tangent vector to $\boldsymbol{r}_2$ at $(0,0,0)$. If θ is the angle between these two tangent vectors, then $\cos\theta=\frac{1}{\sqrt{1}\sqrt{6}}\langle 1,0,0\rangle\cdot\langle 1,2,1\rangle=\frac{1}{\sqrt{6}}$ and $\theta=\cos^{-1}\left(\frac{1}{\sqrt{6}}\right)\approx 66°$.

32. To find the point of intersection, we must find the values of t and s which satisfy the following three equations simultaneously: $t=3-s$, $1-t=s-2$, $3+t^2=s^2$. Solving the last two equations gives $t=1$, $s=2$ (check these in the first equation). Thus the point of intersection is $(1,0,4)$. To find the angle θ of intersection, we proceed as in Exercise 31 . The tangent vectors to the respective curves at $(1,0,4)$ are $\boldsymbol{r}_1'(1)=\langle 1,-1,2\rangle$ and $\boldsymbol{r}_2'(2)=\langle -1,1,4\rangle$. So $\cos\theta=\frac{1}{\sqrt{6}\sqrt{18}}(-1-1+8)=\frac{6}{6\sqrt{3}}=\frac{1}{\sqrt{3}}$ and $\theta=\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)\approx 55°$.

Note: In Exercise 31 , the curves intersect when the value of both parameters is zero. However, as seen in this exercise, it is not necessary for the parameters to be of equal value at the point of intersection.

33.

$$\int_0^1 (16t^3\boldsymbol{i}-9t^2\boldsymbol{j}+25t^4\boldsymbol{k})\,dt=\left(\int_0^1 16t^3\,dt\right)\boldsymbol{i}-\left(\int_0^1 9t^2\,dt\right)\boldsymbol{j}+\left(\int_0^1 25t^4\,dt\right)\boldsymbol{k}$$
$$=\left[4t^4\right]_0^1\boldsymbol{i}-\left[3t^3\right]_0^1\boldsymbol{j}+\left[5t^5\right]_0^1\boldsymbol{k}=4\boldsymbol{i}-3\boldsymbol{j}+5\boldsymbol{k}$$

34. $\int_0^1\left(\frac{4}{1+t^2}\boldsymbol{j}+\frac{2t}{1+t^2}\boldsymbol{k}\right)dt=\left[4\tan^{-1}t\,\boldsymbol{j}+\ln(1+t^2)\boldsymbol{k}\right]_0^1$

$=\left[4\tan^{-1}1\,\boldsymbol{j}+\ln 2\,\boldsymbol{k}\right]-\left[4\tan^{-1}0\,\boldsymbol{j}+\ln 1\,\boldsymbol{k}\right]=4\left(\frac{\pi}{4}\right)\boldsymbol{j}+\ln 2\,\boldsymbol{k}-0\,\boldsymbol{j}-0\,\boldsymbol{k}=\pi\,\boldsymbol{j}+\ln 2\,\boldsymbol{k}$

35. $\int_0^{\pi/2}(3\sin^2 t\cos t\,\boldsymbol{i}+3\sin t\cos^2 t\,\boldsymbol{j}+2\sin t\cos t\,\boldsymbol{k})\,dt$

$=\left(\int_0^{\pi/2}3\sin^2 t\cos t\,dt\right)\boldsymbol{i}+\left(\int_0^{\pi/2}3\sin t\cos^2 t\,dt\right)\boldsymbol{j}+\left(\int_0^{\pi/2}2\sin t\cos t\,dt\right)\boldsymbol{k}$

$=\left[\sin^3 t\right]_0^{\pi/2}\boldsymbol{i}+\left[-\cos^3 t\right]_0^{\pi/2}\boldsymbol{j}+\left[\sin^2 t\right]_0^{\pi/2}\boldsymbol{k}$

$=(1-0)\,\boldsymbol{i}+(0+1)\,\boldsymbol{j}+(1-0)\,\boldsymbol{k}=\boldsymbol{i}+\boldsymbol{j}+\boldsymbol{k}$

36. $\int_1^4\left(\sqrt{t}\,\boldsymbol{i}+te^{-t}\,\boldsymbol{j}+t^{-2}\boldsymbol{k}\right)dt=\left[\frac{2}{3}t^{3/2}\boldsymbol{i}-t^{-1}\boldsymbol{k}\right]_1^4+\left(\left[-te^{-t}\right]_1^4+\int_1^4 e^{-t}\,dt\right)\boldsymbol{j}$

$=\left(\frac{16}{3}-\frac{2}{3}\right)\boldsymbol{i}-\left(\frac{1}{4}-1\right)\boldsymbol{k}+(-4e^{-4}+e^{-1}-e^{-4}+e^{-1})\,\boldsymbol{j}=\frac{14}{3}\,\boldsymbol{i}+e^{-1}(2-5e^{-3})\,\boldsymbol{j}+\frac{3}{4}\,\boldsymbol{k}$

37. $\int\left(e^t\boldsymbol{i}+2t\,\boldsymbol{j}+\ln t\,\boldsymbol{k}\right)dt=\left(\int e^t\,dt\right)\boldsymbol{i}+\left(\int 2t\,dt\right)\boldsymbol{j}+\left(\int \ln t\,dt\right)\boldsymbol{k}$

$=e^t\boldsymbol{i}+t^2\,\boldsymbol{j}+(t\ln t-t)\,\boldsymbol{k}+\boldsymbol{C}$, where $\boldsymbol{C}$ is a vector constant of integration.

38.

$$\int(\cos\pi t\,\boldsymbol{i}+\sin\pi t\,\boldsymbol{j}+t\,\boldsymbol{k})\,dt=\left(\int\cos\pi t\,dt\right)\boldsymbol{i}+\left(\int\sin\pi t\,dt\right)\boldsymbol{j}+\left(\int t\,dt\right)\boldsymbol{k}$$
$$=\frac{1}{\pi}\sin\pi t\,\boldsymbol{i}-\frac{1}{\pi}\cos\pi t\,\boldsymbol{j}+\frac{1}{2}t^2\boldsymbol{k}+\boldsymbol{C}$$

39. $\boldsymbol{r}'(t)=t^2\boldsymbol{i}+4t^3\boldsymbol{j}-t^2\boldsymbol{k}\Rightarrow\boldsymbol{r}(t)=\frac{1}{3}t^3\boldsymbol{i}+t^4\boldsymbol{j}-\frac{1}{3}t^3\boldsymbol{k}+\boldsymbol{C}$, where $\boldsymbol{C}$ is a constant vector.

But $\boldsymbol{j}=\boldsymbol{r}(0)=(0)\,\boldsymbol{i}+(0)\,\boldsymbol{j}-(0)\,\boldsymbol{k}+\boldsymbol{C}$. Thus $\boldsymbol{C}=\boldsymbol{j}$ and $\boldsymbol{r}(t)=\frac{1}{3}t^3\boldsymbol{i}+t^4\boldsymbol{j}-\frac{1}{3}t^3\boldsymbol{k}+\boldsymbol{j}=\frac{1}{3}t^3\boldsymbol{i}+(t^4+1)\,\boldsymbol{j}-\frac{1}{3}t^3\boldsymbol{k}$.

40. $\boldsymbol{r}'(t)=\sin t\,\boldsymbol{i}-\cos t\,\boldsymbol{j}+2t\,\boldsymbol{k}\Rightarrow\boldsymbol{r}(t)=(-\cos t)\,\boldsymbol{i}-(\sin t)\,\boldsymbol{j}+t^2\boldsymbol{k}+\boldsymbol{C}$. But $\boldsymbol{i}+\boldsymbol{j}+2\boldsymbol{k}=\boldsymbol{r}(0)=-\boldsymbol{i}+(0)\,\boldsymbol{j}+(0)\,\boldsymbol{k}+\boldsymbol{C}$.

Thus $\boldsymbol{C}=2\boldsymbol{i}+\boldsymbol{j}+2\boldsymbol{k}$ and $\boldsymbol{r}(t)=(2-\cos t)\,\boldsymbol{i}+(1-\sin t)\,\boldsymbol{j}+(2+t^2)\,\boldsymbol{k}$.

41.

$$\frac{d}{dt}\left[\boldsymbol{u}(t)+\boldsymbol{v}(t)\right]=\frac{d}{dt}\left\langle u_1(t)+v_1(t),\mathrm{u}_2(t)+v_2(t),\mathrm{u}_3(t)+v_3(t)\right\rangle$$
$$=\left\langle\frac{d}{dt}\left[u_1(t)+v_1(t)\right],\frac{d}{dt}\left[u_2(t)+v_2(t)\right],\frac{d}{dt}\left[u_3(t)+v_3(t)\right]\right\rangle$$

$$=\left\langle u_1'(t)+v_1'(t), u_2'(t)+v_2'(t), u_3'(t)+v_3'(t)\right\rangle$$

$$=\left\langle u_1'(t), u_2'(t), u_3'(t)\right\rangle+\left\langle v_1'(t), v_2'(t), v_3'(t)\right\rangle=\boldsymbol{u}'(t)+\boldsymbol{v}'(t)\ .$$

42.

$$\frac{d}{dt}\left[f(t)\,\boldsymbol{u}(t)\right]=\frac{d}{dt}\left\langle f(t)u_1(t), f(t)u_2(t), f(t)u_3(t)\right\rangle$$

$$=\left\langle \frac{d}{dt}\left[f(t)u_1(t)\right], \frac{d}{dt}\left[f(t)u_2(t)\right], \frac{d}{dt}\left[f(t)u_3(t)\right]\right\rangle$$

$$=\left\langle f'(t)u_1(t)+f(t)u_1'(t), f'(t)u_2(t)+f(t)u_2'(t), f'(t)u_3(t)+f(t)u_3'(t)\right\rangle$$

$$=f'(t)\left\langle u_1(t), u_2(t), u_3(t)\right\rangle+f(t)\left\langle u_1'(t), u_2'(t), u_3'(t)\right\rangle$$

$$=f'(t)\,\boldsymbol{u}(t)+f(t)\,\boldsymbol{u}'(t)$$

43. $\frac{d}{dt}\left[\boldsymbol{u}(t)\times\boldsymbol{v}(t)\right]$

$$=\frac{d}{dt}\left\langle u_2(t)v_3(t)-u_3(t)v_2(t), u_3(t)v_1(t)-u_1(t)v_3(t), u_1(t)v_2(t)-u_2(t)v_1(t)\right\rangle$$

$$=\left\langle u_2'v_3(t)+u_2(t)v_3'(t)-u_3'(t)v_2(t)-u_3(t)v_2'(t),\right.$$

$$u_3'(t)v_1(t)+u_3(t)v_1'(t)-u_1'(t)v_3(t)-u_1(t)v_3'(t),$$

$$\left. u_1'(t)v_2(t)+u_1(t)v_2'(t)-u_2'(t)v_1(t)-u_2(t)v_1'(t)\right\rangle$$

$$=\left\langle u_2'(t)v_3(t)-u_3'(t)v_2(t), u_3'(t)v_1(t)-u_1'(t)v_3(t), u_1'(t)v_2(t)-u_2'(t)v_1(t)\right\rangle$$

$$+\left\langle u_2(t)v_3'(t)-u_3(t)v_2'(t), u_3(t)v_1'(t)-u_1(t)v_3'(t), u_1(t)v_2'(t)-u_2(t)v_1'(t)\right\rangle$$

$$=\boldsymbol{u}'(t)\times\boldsymbol{v}(t)+\boldsymbol{u}(t)\times\boldsymbol{v}'(t)$$

Alternate solution: Let $\boldsymbol{r}(t)=\boldsymbol{u}(t)\times\boldsymbol{v}(t)$. Then

$$\boldsymbol{r}(t+h)-\boldsymbol{r}(t)=\left[\boldsymbol{u}(t+h)\times\boldsymbol{v}(t+h)\right]-\left[\boldsymbol{u}(t)\times\boldsymbol{v}(t)\right]$$

$$=\left[\boldsymbol{u}(t+h)\times\boldsymbol{v}(t+h)\right]-\left[\boldsymbol{u}(t)\times\boldsymbol{v}(t)\right]+\left[\boldsymbol{u}(t+h)\times\boldsymbol{v}(t)\right]-\left[\boldsymbol{u}(t+h)\times\boldsymbol{v}(t)\right]$$

$$=\boldsymbol{u}(t+h)\times[\boldsymbol{v}(t+h)-\boldsymbol{v}(t)]+[\boldsymbol{u}(t+h)-\boldsymbol{u}(t)]\times\boldsymbol{v}(t)$$

(Be careful of the order of the cross product.)
Dividing through by h and taking the limit as $h\to 0$ we have

$$\boldsymbol{r}'(t)=\lim_{h\to 0}\frac{\boldsymbol{u}(t+h)\times[\boldsymbol{v}(t+h)-\boldsymbol{v}(t)]}{h}+\lim_{h\to 0}\frac{[\boldsymbol{u}(t+h)-\boldsymbol{u}(t)]\times\boldsymbol{v}(t)}{h}$$

$$=\boldsymbol{u}(t)\times\boldsymbol{v}'(t)+\boldsymbol{u}'(t)\times\boldsymbol{v}(t)$$

by Exercise 14.1.41(a) and Definition 1.

44.

$$\frac{d}{dt}[\boldsymbol{u}(f(t))]=\frac{d}{dt}\left\langle u_1(f(t)),\mathrm{u}_2(f(t)),\mathrm{u}_3(f(t))\right\rangle=\left\langle\frac{d}{dt}\left[u_1(f(t))\right],\frac{d}{dt}\left[u_2(f(t))\right],\frac{d}{dt}\left[u_3(f(t))\right]\right\rangle$$

$$=\left\langle f'(t)u_1'(f(t)),\mathrm{f}'(t)u_2'(f(t)),\mathrm{f}'(t)u_3'(f(t))\right\rangle=f'(t)\,\boldsymbol{u}'(t)$$

45.

$$\frac{d}{dt}[\boldsymbol{u}(t)\cdot\boldsymbol{v}(t)]=\boldsymbol{u}'(t)\cdot\boldsymbol{v}(t)+\boldsymbol{u}(t)\cdot\boldsymbol{v}'(t)$$

$$=(-4t\,\boldsymbol{j}+9t^2\boldsymbol{k})\cdot(t\,\boldsymbol{i}+\cos t\,\boldsymbol{j}+\sin t\,\boldsymbol{k})+(\boldsymbol{i}-2t^2\boldsymbol{j}+3t^3\boldsymbol{k})\cdot(\boldsymbol{i}-\sin t\,\boldsymbol{j}+\cos t\,\boldsymbol{k})$$

$$=-4t\cos t+9t^2\sin t+1+2t^2\sin t+3t^3\cos t$$

$$=1-4t\cos t+11t^2\sin t+3t^3\cos t$$

46.

$$\frac{d}{dt}[\boldsymbol{u}(t)\times\boldsymbol{v}(t)]=\boldsymbol{u}'(t)\times\boldsymbol{v}(t)+\boldsymbol{u}(t)\times\boldsymbol{v}'(t)$$

$$=(-4t\,\boldsymbol{j}+9t^2\boldsymbol{k})\times(t\,\boldsymbol{i}+\cos t\,\boldsymbol{j}+\sin t\,\boldsymbol{k})+(\boldsymbol{i}-2t^2\boldsymbol{j}+3t^3\boldsymbol{k})\times(\boldsymbol{i}-\sin t\,\boldsymbol{j}+\cos t\,\boldsymbol{k})$$

$$=(-4t\sin t-9t^2\cos t)\,\boldsymbol{i}+(9t^3-0)\,\boldsymbol{j}+(0+4t^2)\,\boldsymbol{k}$$

$$+(-2t^2\cos t+3t^3\sin t)\,\boldsymbol{i}+(3t^3-\cos t)\,\boldsymbol{j}+(-\sin t+2t^2)\,\boldsymbol{k}$$

$$=[(\sin t)(3t^3-4t)-11t^2\cos t]\,\boldsymbol{i}+(12t^3-\cos t)\,\boldsymbol{j}+(6t^2-\sin t)\,\boldsymbol{k}$$

47. $\frac{d}{dt}\left[\boldsymbol{r}(t)\times\boldsymbol{r}'(t)\right]=\boldsymbol{r}'(t)\times\boldsymbol{r}'(t)+\boldsymbol{r}(t)\times\boldsymbol{r}''(t)$ by Formula 5 of Theorem 3. But $\boldsymbol{r}'(t)\times\boldsymbol{r}'(t)=\mathbf{0}$ (see Example 13.4.2). Thus,

$\frac{d}{dt}\left[\boldsymbol{r}(t)\times\boldsymbol{r}'(t)\right]=\boldsymbol{r}(t)\times\boldsymbol{r}''(t)$.

48.

$$\begin{aligned}\frac{d}{dt}\left(\boldsymbol{u}(t)\cdot[\boldsymbol{v}(t)\times\boldsymbol{w}(t)]\right) &=\boldsymbol{u}'(t)\cdot[\boldsymbol{v}(t)\times\boldsymbol{w}(t)]+\boldsymbol{u}(t)\cdot\frac{d}{dt}[\boldsymbol{v}(t)\times\boldsymbol{w}(t)]\\ &=\boldsymbol{u}'(t)\cdot[\boldsymbol{v}(t)\times\boldsymbol{w}(t)]+\boldsymbol{u}(t)\cdot\left[\boldsymbol{v}'(t)\times\boldsymbol{w}(t)+\boldsymbol{v}(t)\times\boldsymbol{w}'(t)\right]\\ &=\boldsymbol{u}'(t)\cdot[\boldsymbol{v}(t)\times\boldsymbol{w}(t)]+\boldsymbol{u}(t)\cdot\left[\boldsymbol{v}'(t)\times\boldsymbol{w}(t)\right]+\boldsymbol{u}(t)\cdot\left[\boldsymbol{v}(t)\times\boldsymbol{w}'(t)\right]\\ &=\boldsymbol{u}'(t)\cdot[\boldsymbol{v}(t)\times\boldsymbol{w}(t)]-\boldsymbol{v}'(t)\cdot[\boldsymbol{u}(t)\times\boldsymbol{w}(t)]+\boldsymbol{w}'(t)\cdot[\boldsymbol{u}(t)\times\boldsymbol{v}(t)]\end{aligned}$$

49. $\frac{d}{dt}|\boldsymbol{r}(t)|=\frac{d}{dt}[\boldsymbol{r}(t)\cdot\boldsymbol{r}(t)]^{1/2}=\frac{1}{2}[\boldsymbol{r}(t)\cdot\boldsymbol{r}(t)]^{-1/2}\left[2\boldsymbol{r}(t)\cdot\boldsymbol{r}'(t)\right]=\frac{1}{|\boldsymbol{r}(t)|}\boldsymbol{r}(t)\cdot\boldsymbol{r}'(t)$

50. Since $\boldsymbol{r}(t)\cdot\boldsymbol{r}'(t)=0$, we have $0=2\boldsymbol{r}(t)\cdot\boldsymbol{r}'(t)=\frac{d}{dt}[\boldsymbol{r}(t)\cdot\boldsymbol{r}(t)]=\frac{d}{dt}|\boldsymbol{r}(t)|^2$. Thus $|\boldsymbol{r}(t)|^2$, and so $|\boldsymbol{r}(t)|$, is a constant, and hence the curve lies on a sphere with center the origin.

51. Since $\boldsymbol{u}(t)=\boldsymbol{r}(t)\cdot\left[\boldsymbol{r}'(t)\times\boldsymbol{r}''(t)\right]$, $\boldsymbol{u}'(\mathrm{t})=\boldsymbol{r}'(\mathrm{t})\cdot\left[\boldsymbol{r}'(t)\times\boldsymbol{r}''(t)\right]+\boldsymbol{r}(\mathrm{t})\cdot\frac{d}{dt}\left[\boldsymbol{r}'(t)\times\boldsymbol{r}''(t)\right]$ $=0+\boldsymbol{r}(\mathrm{t})\cdot\left[\boldsymbol{r}''(t)\times\boldsymbol{r}''(t)+\boldsymbol{r}'(t)\times\boldsymbol{r}'''(t)\right]=\boldsymbol{r}(\mathrm{t})\cdot\left[\boldsymbol{r}'(t)\times\boldsymbol{r}'''(t)\right]$

1. $\mathbf{r}'(t)=\langle 2\cos t, 5, -2\sin t\rangle \Rightarrow |\mathbf{r}'(t)|=\sqrt{(2\cos t)^2+5^2+(-2\sin t)^2}=\sqrt{29}$. Then using Formula 3, we have

$L=\int_{-10}^{10} |\mathbf{r}'(t)|\,dt=\int_{-10}^{10} \sqrt{29}\,dt=\left[\sqrt{29}\,t\right]_{-10}^{10}=20\sqrt{29}$.

2. $\mathbf{r}'(t)=\langle 2t, \cos t+t\sin t-\cos t, -\sin t+t\cos t+\sin t\rangle=\langle 2t, t\sin t, t\cos t\rangle \Rightarrow$
$|\mathbf{r}'(t)|=\sqrt{(2t)^2+(t\sin t)^2+(t\cos t)^2}=\sqrt{4t^2+t^2(\sin^2 t+\cos^2 t)}=\sqrt{5}\,|t|=\sqrt{5}\,t$ for $0\le t\le \pi$. Then using Formula 3, we have $L=\int_0^{\pi} |\mathbf{r}'(t)|\,dt=\int_0^{\pi} \sqrt{5}\,t\,dt=\left[\sqrt{5}\,\frac{t^2}{2}\right]_0^{\pi}=\frac{\sqrt{5}}{2}\pi^2$.

3. $\mathbf{r}'(t)=\sqrt{2}\,\mathbf{i}+e^t\mathbf{j}-e^{-t}\mathbf{k} \Rightarrow$
$|\mathbf{r}'(t)|=\sqrt{(\sqrt{2})^2+(e^t)^2+(-e^{-t})^2}=\sqrt{2+e^{2t}+e^{-2t}}=\sqrt{(e^t+e^{-t})^2}=e^t+e^{-t}$ (since $e^t+e^{-t}>0$).
Then $L=\int_0^1 |\mathbf{r}'(t)|\,dt=\int_0^1 (e^t+e^{-t})\,dt=\left[e^t-e^{-t}\right]_0^1=e-e^{-1}$.

4. $\mathbf{r}'(t)=\langle 2t, 2, 1/t\rangle$, $|\mathbf{r}'(t)|=\sqrt{4t^2+4+(1/t)^2}=\frac{1+2t^2}{|t|}=\frac{1+2t^2}{t}$ for $1\le t\le e$.
$L=\int_1^e \frac{1+2t^2}{t}\,dt=\int_1^e \left(\frac{1}{t}+2t\right)dt=\left[\ln t+t^2\right]_1^e=e^2$

5. $\mathbf{r}'(t)=2t\,\mathbf{j}+3t^2\mathbf{k} \Rightarrow |\mathbf{r}'(t)|=\sqrt{4t^2+9t^4}=t\sqrt{4+9t^2}$ (since $t\ge 0$).
Then $L=\int_0^1 |\mathbf{r}'(t)|\,dt=\int_0^1 t\sqrt{4+9t^2}\,dt=\left[\frac{1}{18}\cdot\frac{2}{3}(4+9t^2)^{3/2}\right]_0^1=\frac{1}{27}(13^{3/2}-4^{3/2})=\frac{1}{27}(13^{3/2}-8)$.

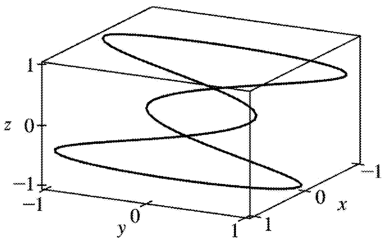
6. $\mathbf{r}'(t)=12\mathbf{i}+12\sqrt{t}\,\mathbf{j}+6t\,\mathbf{k} \Rightarrow |\mathbf{r}'(t)|=\sqrt{144+144t+36t^2}=\sqrt{36(t+2)^2}=6|t+2|=6(t+2)$ for $0\le t\le 1$.
Then $L=\int_0^1 |\mathbf{r}'(t)|\,dt=\int_0^1 6(t+2)\,dt=\left[3t^2+12t\right]_0^1=15$.

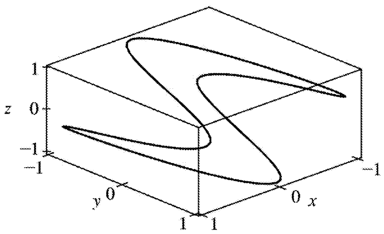
7. The point $(2,4,8)$ corresponds to $t=2$, so by Equation 2, $L=\int_0^2 \sqrt{(1)^2+(2t)^2+(3t^2)^2}\,dt$. If

$f(t)=\sqrt{1+4t^2+9t^4}$, then Simpson's Rule gives

$L \approx \frac{2-0}{10\cdot 3}\left[f(0)+4f(0.2)+2f(0.4)+\cdots+4f(1.8)+f(2)\right]\approx 9.5706$.

8. Here are two views of the curve with parametric equations $x=\cos t$, $y=\sin 3t$, $z=\sin t$:





The complete curve is given by the parameter interval $[0,2\pi]$, so

$L=\int_0^{2\pi}\sqrt{(-\sin t)^2+(3\cos 3t)^2+(\cos t)^2}\,dt=\int_0^{2\pi}\sqrt{1+9\cos^2 3t}\,dt\approx 13.9744$.

9. $\boldsymbol{r}'(t)=2\boldsymbol{i}-3\boldsymbol{j}+4\boldsymbol{k}$ and $\frac{ds}{dt}=\left|\boldsymbol{r}'(t)\right|=\sqrt{4+9+16}=\sqrt{29}$. Then $s=s(t)=\int_0^t\left|\boldsymbol{r}'(u)\right|du=\int_0^t\sqrt{29}\,du=\sqrt{29}\,t$.

Therefore, $t=\frac{1}{\sqrt{29}}s$, and substituting for t in the original equation, we have

$\boldsymbol{r}(t(s))=\frac{2}{\sqrt{29}}s\boldsymbol{i}+\left(1-\frac{3}{\sqrt{29}}s\right)\boldsymbol{j}+\left(5+\frac{4}{\sqrt{29}}s\right)\boldsymbol{k}$.

10. $\boldsymbol{r}'(t)=2e^{2t}(\cos 2t-\sin 2t)\boldsymbol{i}+2e^{2t}(\cos 2t+\sin 2t)\boldsymbol{k}$,

$\frac{ds}{dt}=\left|\boldsymbol{r}'(t)\right|=2e^{2t}\sqrt{(\cos 2t-\sin 2t)^2+(\cos 2t+\sin 2t)^2}=2e^{2t}\sqrt{2\cos^2 2t+2\sin^2 2t}=2\sqrt{2}\,e^{2t}$.

$s=s(t)=\int_0^t\left|\boldsymbol{r}'(u)\right|du=\int_0^t 2\sqrt{2}\,e^{2u}du=\left[\sqrt{2}\,e^{2u}\right]_0^t=\sqrt{2}\,(e^{2t}-1)\Rightarrow$

$\frac{s}{\sqrt{2}}+1=e^{2t}\Rightarrow t=\frac{1}{2}\ln\left(\frac{s}{\sqrt{2}}+1\right)$. Substituting, we have

$\boldsymbol{r}(t(s))=e^{2\left(\frac{1}{2}\ln\left(\frac{s}{\sqrt{2}}+1\right)\right)}\cos 2\left(\frac{1}{2}\ln\left(\frac{s}{\sqrt{2}}+1\right)\right)\boldsymbol{i}+2\boldsymbol{j}+e^{2\left(\frac{1}{2}\ln\left(\frac{s}{\sqrt{2}}+1\right)\right)}\sin 2\left(\frac{1}{2}\ln\left(\frac{s}{\sqrt{2}}+1\right)\right)\boldsymbol{k}$

$$=\left(\frac{s}{\sqrt{2}}+1\right)\cos\left(\ln\left(\frac{s}{\sqrt{2}}+1\right)\right)\boldsymbol{i}+2\boldsymbol{j}+\left(\frac{s}{\sqrt{2}}+1\right)\sin\left(\ln\left(\frac{s}{\sqrt{2}}+1\right)\right)\boldsymbol{k}\ .$$

11. $\left|\boldsymbol{r}'(t)\right|=\sqrt{(3\cos t)^2+16+(-3\sin t)^2}=\sqrt{9+16}=5$ and $s(t)=\int_0^t\left|\boldsymbol{r}'(u)\right|du=\int_0^t 5\,du=5t\Rightarrow t(s)=\frac{1}{5}s$.
Therefore,
$\boldsymbol{r}(t(s))=3\sin\left(\frac{1}{5}s\right)\boldsymbol{i}+\frac{4}{5}s\,\boldsymbol{j}+3\cos\left(\frac{1}{5}s\right)\boldsymbol{k}$.

12. $\boldsymbol{r}'(t)=\frac{-4t}{(t^2+1)^2}\boldsymbol{i}+\frac{-2t^2+2}{(t^2+1)^2}\boldsymbol{j}$,

$$\frac{ds}{dt}=\left|\boldsymbol{r}'(t)\right|=\sqrt{\left[\frac{-4t}{(t^2+1)^2}\right]^2+\left[\frac{-2t^2+2}{(t^2+1)^2}\right]^2}=\sqrt{\frac{4t^4+8t^2+4}{(t^2+1)^4}}=\sqrt{\frac{4(t^2+1)^2}{(t^2+1)^4}}$$
$$=\sqrt{\frac{4}{(t^2+1)^2}}=\frac{2}{t^2+1}$$

Since the initial point $(1,0)$ corresponds to $t=0$, the arc length function
$s(t)=\int_0^t\left|\boldsymbol{r}'(u)\right|du=\int_0^t\frac{2}{u^2+1}du=2\arctan t$. Then $\arctan t=\frac{1}{2}s\Rightarrow t=\tan\frac{1}{2}s$. Substituting, we have

$$\boldsymbol{r}(t(s))=\left[\frac{2}{\tan^2\left(\frac{1}{2}s\right)+1}-1\right]\boldsymbol{i}+\frac{2\tan\left(\frac{1}{2}s\right)}{\tan^2\left(\frac{1}{2}s\right)+1}\boldsymbol{j}=\frac{1-\tan^2\left(\frac{1}{2}s\right)}{1+\tan^2\left(\frac{1}{2}s\right)}\boldsymbol{i}+\frac{2\tan\left(\frac{1}{2}s\right)}{\sec^2\left(\frac{1}{2}s\right)}\boldsymbol{j}$$
$$=\frac{1-\tan^2\left(\frac{1}{2}s\right)}{\sec^2\left(\frac{1}{2}s\right)}\boldsymbol{i}+2\tan\left(\frac{1}{2}s\right)\cos^2\left(\frac{1}{2}s\right)\boldsymbol{j}$$
$$=\left[\cos^2\left(\frac{1}{2}s\right)-\sin^2\left(\frac{1}{2}s\right)\right]\boldsymbol{i}+2\sin\left(\frac{1}{2}s\right)\cos\left(\frac{1}{2}s\right)\boldsymbol{j}=\cos s\,\boldsymbol{i}+\sin s\,\boldsymbol{j}$$

With this parametrization, we recognize the function as representing the unit circle. Note here that the curve approaches, but does not include, the point $(-1,0)$, since $\cos s=-1$ for $s=\pi+2k\pi$ (k an integer) but then $t=\tan\left(\frac{1}{2}s\right)$ is undefined.

13. **(a)** $\boldsymbol{r}'(t)=\langle 2\cos t, 5, -2\sin t\rangle \Rightarrow |\boldsymbol{r}'(t)|=\sqrt{4\cos^2 t+25+4\sin^2 t}=\sqrt{29}$. Then

$\boldsymbol{T}(t)=\dfrac{\boldsymbol{r}'(t)}{|\boldsymbol{r}'(t)|}=\dfrac{1}{\sqrt{29}}\langle 2\cos t, 5, -2\sin t\rangle$ or $\left\langle \dfrac{2}{\sqrt{29}}\cos t, \dfrac{5}{\sqrt{29}}, -\dfrac{2}{\sqrt{29}}\sin t\right\rangle$.

$\boldsymbol{T}'(t)=\dfrac{1}{\sqrt{29}}\langle -2\sin t, 0, -2\cos t\rangle \Rightarrow |\boldsymbol{T}'(t)|=\dfrac{1}{\sqrt{29}}\sqrt{4\sin^2 t+0+4\cos^2 t}=\dfrac{2}{\sqrt{29}}$. Thus

$\boldsymbol{N}(t)=\dfrac{\boldsymbol{T}'(t)}{|\boldsymbol{T}'(t)|}=\dfrac{1/\sqrt{29}}{2/\sqrt{29}}\langle -2\sin t, 0, -2\cos t\rangle=\langle -\sin t, 0, -\cos t\rangle$.

(b) $\kappa(t)=\dfrac{|\boldsymbol{T}'(t)|}{|\boldsymbol{r}'(t)|}=\dfrac{2/\sqrt{29}}{\sqrt{29}}=\dfrac{2}{29}$.

14. **(a)** $\boldsymbol{r}'(t)=\langle 2t, t\sin t, t\cos t\rangle \Rightarrow |\boldsymbol{r}'(t)|=\sqrt{4t^2+t^2\sin^2 t+t^2\cos^2 t}=\sqrt{5t^2}=\sqrt{5}\,t$ (since $t>0$).

Then $\boldsymbol{T}(t)=\dfrac{\boldsymbol{r}'(t)}{|\boldsymbol{r}'(t)|}=\dfrac{1}{\sqrt{5}\,t}\langle 2t, t\sin t, t\cos t\rangle=\dfrac{1}{\sqrt{5}}\langle 2, \sin t, \cos t\rangle$. $\boldsymbol{T}'(t)=\dfrac{1}{\sqrt{5}}\langle 0, \cos t, -\sin t\rangle \Rightarrow$

$|\boldsymbol{T}'(t)|=\dfrac{1}{\sqrt{5}}\sqrt{0+\cos^2 t+\sin^2 t}=\dfrac{1}{\sqrt{5}}$.

Thus $\boldsymbol{N}(t)=\dfrac{\boldsymbol{T}'(t)}{|\boldsymbol{T}'(t)|}=\dfrac{1/\sqrt{5}}{1/\sqrt{5}}\langle 0, \cos t, -\sin t\rangle=\langle 0, \cos t, -\sin t\rangle$.

(b) $\kappa(t)=\dfrac{|\boldsymbol{T}'(t)|}{|\boldsymbol{r}'(t)|}=\dfrac{1/\sqrt{5}}{\sqrt{5}\,t}=\dfrac{1}{5t}$.

15. **(a)** $\boldsymbol{r}'(t)=\left\langle \sqrt{2}, e^t, -e^{-t}\right\rangle \Rightarrow |\boldsymbol{r}'(t)|=\sqrt{2+e^{2t}+e^{-2t}}=\sqrt{(e^t+e^{-t})^2}=e^t+e^{-t}$. Then

$\boldsymbol{T}(t)=\dfrac{\boldsymbol{r}'(t)}{|\boldsymbol{r}'(t)|}=\dfrac{1}{e^t+e^{-t}}\left\langle \sqrt{2}, e^t, -e^{-t}\right\rangle=\dfrac{1}{e^{2t}+1}\left\langle \sqrt{2}\,e^t, e^{2t}, -1\right\rangle$ $\left(\textit{after multiplying by } \dfrac{e^t}{e^t}\right)$ and

$$\begin{aligned}
\boldsymbol{T}'(t) &= \frac{1}{e^{2t}+1}\left\langle \sqrt{2}\,e^t, 2e^{2t}, 0\right\rangle-\frac{2e^{2t}}{\left(e^{2t}+1\right)^2}\left\langle \sqrt{2}\,e^t, e^{2t}, -1\right\rangle \\
&= \frac{1}{(e^{2t}+1)^2}\left[(e^{2t}+1)\left\langle \sqrt{2}\,e^t, 2e^{2t}, 0\right\rangle-2e^{2t}\left\langle \sqrt{2}\,e^t, e^{2t}, -1\right\rangle\right] \\
&= \frac{1}{(e^{2t}+1)^2}\left\langle \sqrt{2}\,e^t\left(1-e^{2t}\right), 2e^{2t}, 2e^{2t}\right\rangle
\end{aligned}$$

Then $\left|\boldsymbol{T}'(t)\right| = \dfrac{1}{(e^{2t}+1)^2}\sqrt{2e^{2t}(1-2e^{2t}+e^{4t})+4e^{4t}+4e^{4t}} = \dfrac{1}{\left(e^{2t}+1\right)^2}\sqrt{2e^{2t}(1+2e^{2t}+e^{4t})}$

$= \dfrac{1}{(e^{2t}+1)^2}\sqrt{2e^{2t}\left(1+e^{2t}\right)^2} = \dfrac{\sqrt{2}\,e^{t}(1+e^{2t})}{(e^{2t}+1)^2} = \dfrac{\sqrt{2}\,e^{t}}{e^{2t}+1}$

Therefore $\boldsymbol{N}(t) = \dfrac{\boldsymbol{T}'(t)}{\left|\boldsymbol{T}'(t)\right|} = \dfrac{e^{2t}+1}{\sqrt{2}\,e^{t}}\,\dfrac{1}{(e^{2t}+1)^2}\left\langle\sqrt{2}\,e^{t}(1-e^{2t}),2e^{2t},2e^{2t}\right\rangle$

$= \dfrac{1}{\sqrt{2}\,e^{t}(e^{2t}+1)}\left\langle\sqrt{2}\,e^{t}(1-e^{2t}),2e^{2t},2e^{2t}\right\rangle$

$= \dfrac{1}{e^{2t}+1}\left\langle 1-e^{2t},\sqrt{2}\,e^{t},\sqrt{2}\,e^{t}\right\rangle$

(b) $\kappa(t) = \dfrac{\left|\boldsymbol{T}'(t)\right|}{\left|\boldsymbol{r}'(t)\right|} = \dfrac{\sqrt{2}\,e^{t}}{e^{2t}+1}\cdot\dfrac{1}{e^{t}+e^{-t}} = \dfrac{\sqrt{2}\,e^{t}}{e^{3t}+2e^{t}+e^{-t}} = \dfrac{\sqrt{2}\,e^{2t}}{e^{4t}+2e^{2t}+1} = \dfrac{\sqrt{2}\,e^{2t}}{(e^{2t}+1)^2}$.

16. **(a)** $\boldsymbol{T}(t) = \dfrac{\boldsymbol{r}'(t)}{\left|\boldsymbol{r}'(t)\right|} = \dfrac{1}{\sqrt{4t^2+4+(1/t)^2}}\left\langle 2t,2,1/t\right\rangle = \dfrac{|t|}{2t^2+1}\left\langle 2t,2,1/t\right\rangle$. But since the $\boldsymbol{k}$ –component is $\ln t$, t is positive, $|t|=t$ and

$\boldsymbol{T}(t) = \dfrac{1}{2t^2+1}\left\langle 2t^2,2t,1\right\rangle$. Then

$\boldsymbol{T}'(t) = \dfrac{1}{2t^2+1}\left\langle 4t,2,0\right\rangle - \left(2t^2+1\right)^{-2}(4t)\left\langle 2t^2,2t,1\right\rangle = \dfrac{1}{(2t^2+1)^2}\left\langle 4t,2-4t^2,-4t\right\rangle$, so

$\boldsymbol{N}(t) = \dfrac{\boldsymbol{T}'(t)}{\left|\boldsymbol{T}'(t)\right|} = \dfrac{\left\langle 4t,2-4t^2,-4t\right\rangle}{\sqrt{(4t)^2+(2-4t^2)^2+(-4t)^2}} = \dfrac{1}{2t^2+1}\left\langle 2t,1-2t^2,-2t\right\rangle$.

(b) $\kappa(t) = \dfrac{\left|\boldsymbol{T}'(t)\right|}{\left|\boldsymbol{r}'(t)\right|} = \dfrac{2}{2t^2+1}\left(\dfrac{t}{2t^2+1}\right) = \dfrac{2t}{(2t^2+1)^2}$

17. $\boldsymbol{r}'(t)=2t\,\boldsymbol{i}+\boldsymbol{k}$, $\boldsymbol{r}''(t)=2\boldsymbol{i}$, $\left|\boldsymbol{r}'(t)\right| = \sqrt{(2t)^2+0^2+1^2} = \sqrt{4t^2+1}$, $\boldsymbol{r}'(t)\times\boldsymbol{r}''(t)=2\boldsymbol{j}$, $\left|\boldsymbol{r}'(t)\times\boldsymbol{r}''(t)\right|=2$
.
Then

$\kappa(t)=\dfrac{\left|\boldsymbol{r}'(t)\times\boldsymbol{r}''(t)\right|}{\left|\boldsymbol{r}'(t)\right|^3}=\dfrac{2}{\left(\sqrt{4t^2+1}\right)^3}=\dfrac{2}{(4t^2+1)^{3/2}}$.

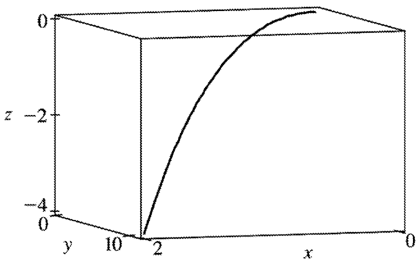
18. $\boldsymbol{r}'(t)=\boldsymbol{i}+\boldsymbol{j}+2t\boldsymbol{k}$, $\boldsymbol{r}''(t)=2\boldsymbol{k}$, $\left|\boldsymbol{r}'(t)\right|=\sqrt{1^2+1^2+(2t)^2}=\sqrt{4t^2+2}$, $\boldsymbol{r}'(t)\times\boldsymbol{r}''(t)=2\boldsymbol{i}-2\boldsymbol{j}$, $\left|\boldsymbol{r}'(t)\times\boldsymbol{r}''(t)\right|=\sqrt{2^2+2^2+0^2}=\sqrt{8}=2\sqrt{2}$. Then
$\kappa(t)=\dfrac{\left|\boldsymbol{r}'(t)\times\boldsymbol{r}''(t)\right|}{\left|\boldsymbol{r}'(t)\right|^3}=\dfrac{2\sqrt{2}}{\left(\sqrt{4t^2+2}\right)^3}=\dfrac{2\sqrt{2}}{\left(\sqrt{2}\sqrt{2t^2+1}\right)^3}=\dfrac{1}{(2t^2+1)^{3/2}}$.

19. $\boldsymbol{r}'(t)=3\boldsymbol{i}+4\cos t\,\boldsymbol{j}-4\sin t\,\boldsymbol{k}$, $\boldsymbol{r}''(t)=-4\sin t\,\boldsymbol{j}-4\cos t\,\boldsymbol{k}$, $\left|\boldsymbol{r}'(t)\right|=\sqrt{9+16\cos^2 t+16\sin^2 t}=\sqrt{9+16}=5$, $\boldsymbol{r}'(t)\times\boldsymbol{r}''(t)=-16\boldsymbol{i}+12\cos t\,\boldsymbol{j}-12\sin t\,\boldsymbol{k}$, $\left|\boldsymbol{r}'(t)\times\boldsymbol{r}''(t)\right|=\sqrt{256+144\cos^2 t+144\sin^2 t}=\sqrt{400}=20$.
Then $\kappa(t)=\dfrac{\left|\boldsymbol{r}'(t)\times\boldsymbol{r}''(t)\right|}{\left|\boldsymbol{r}'(t)\right|^3}=\dfrac{20}{5^3}=\dfrac{4}{25}$.

20. $\boldsymbol{r}'(t)=\left\langle e^t\cos t-e^t\sin t, e^t\cos t+e^t\sin t, 1\right\rangle$. The point $(1,0,0)$ corresponds to $t=0$, and
$\boldsymbol{r}'(0)=\langle 1,1,1\rangle\Rightarrow\left|\boldsymbol{r}'(0)\right|=\sqrt{1^2+1^2+1^2}=\sqrt{3}$.
$\boldsymbol{r}''(t)=\left\langle e^t\cos t-e^t\sin t-e^t\cos t-e^t\sin t, e^t\cos t-e^t\sin t+e^t\cos t+e^t\sin t, 0\right\rangle$
$=\left\langle -2e^t\sin t, 2e^t\cos t, 0\right\rangle\Rightarrow\boldsymbol{r}''(0)=\langle 0,2,0\rangle$.
$\boldsymbol{r}'(0)\times\boldsymbol{r}''(0)=\langle -2,0,2\rangle$. $\left|\boldsymbol{r}'(0)\times\boldsymbol{r}''(0)\right|=\sqrt{(-2)^2+0^2+2^2}=\sqrt{8}=2\sqrt{2}$.
Then $\kappa(0)=\dfrac{\left|\boldsymbol{r}'(0)\times\boldsymbol{r}''(0)\right|}{\left|\boldsymbol{r}'(0)\right|^3}=\dfrac{2\sqrt{2}}{\left(\sqrt{3}\right)^3}=\dfrac{2\sqrt{2}}{3\sqrt{3}}$ or $\dfrac{2\sqrt{6}}{9}$.

21. $\boldsymbol{r}'(t)=\left\langle 1, 2t, 3t^2\right\rangle$. The point $(1,1,1)$ corresponds to $t=1$, and
$\boldsymbol{r}'(1)=\langle 1,2,3\rangle\Rightarrow\left|\boldsymbol{r}'(1)\right|=\sqrt{1+4+9}=\sqrt{14}$. $\boldsymbol{r}''(t)=\langle 0,2,6t\rangle\Rightarrow\boldsymbol{r}''(1)=\langle 0,2,6\rangle$.
$\boldsymbol{r}'(1)\times\boldsymbol{r}''(1)=\langle 6,-6,2\rangle$, so $\left|\boldsymbol{r}'(1)\times\boldsymbol{r}''(1)\right|=\sqrt{36+36+4}=\sqrt{76}$. Then
$\kappa(1)=\dfrac{\left|\boldsymbol{r}'(1)\times\boldsymbol{r}''(1)\right|}{\left|\boldsymbol{r}'(1)\right|^3}=\dfrac{\sqrt{76}}{\sqrt{14}^3}=\dfrac{1}{7}\sqrt{\dfrac{19}{14}}$.

22.



$\mathbf{r}(t)=\left\langle t,4t^{3/2},-t^{2}\right\rangle \Rightarrow \mathbf{r}'(t)=\left\langle 1,6t^{1/2},-2t\right\rangle$,
$\mathbf{r}''(t)=\left\langle 0,3t^{-1/2},-2\right\rangle$, $\left|\mathbf{r}'(t)\right|^{3}=\left(1+36t+4t^{2}\right)^{3/2}$,
$\mathbf{r}'(t)\times\mathbf{r}''(t)=\left\langle -12t^{1/2}+6t^{1/2},2,3t^{-1/2}\right\rangle \Rightarrow$
$\left|\mathbf{r}'(t)\times\mathbf{r}''(t)\right|=\sqrt{36t+4+9t^{-1}}=\left[\dfrac{36t^{2}+4t+9}{t}\right]^{1/2}$

$$\kappa(t)=\frac{\left|\mathbf{r}'(t)\times\mathbf{r}''(t)\right|}{\left|\mathbf{r}'(t)\right|^{3}}=\left(\frac{36t^{2}+4t+9}{t}\right)^{1/2}\frac{1}{(1+36t+4t^{2})^{3/2}}=\frac{\sqrt{36t^{2}+4t+9}}{t^{1/2}(1+36t+4t^{2})^{3/2}} .$$

The point $(1,4,-1)$ corresponds to $t=1$, so the curvature at this point is $\kappa(1)=\dfrac{\sqrt{36+4+9}}{(1+36+4)^{3/2}}=\dfrac{7}{41\sqrt{41}}$.

23. $f(x)=x^{3}$, $f'(x)=3x^{2}$, $f''(x)=6x$, $\kappa(x)=\dfrac{\left|f''(x)\right|}{\left[1+\left(f'(x)\right)^{2}\right]^{3/2}}=\dfrac{6|x|}{\left(1+9x^{4}\right)^{3/2}}$

24. $f(x)=\cos x$, $f'(x)=-\sin x$, $f''(x)=-\cos x$,
$$\kappa(x)=\frac{\left|f''(x)\right|}{\left[1+(f'(x))^{2}\right]^{3/2}}=\frac{|-\cos x|}{\left[1+(-\sin x)^{2}\right]^{3/2}}=\frac{|\cos x|}{(1+\sin^{2}x)^{3/2}}$$

25. $f(x)=4x^{5/2}$, $f'(x)=10x^{3/2}$, $f''(x)=15x^{1/2}$,
$$\kappa(x)=\frac{\left|f''(x)\right|}{\left[1+(f'(x))^{2}\right]^{3/2}}=\frac{\left|15x^{1/2}\right|}{\left[1+(10x^{3/2})^{2}\right]^{3/2}}=\frac{15\sqrt{x}}{(1+100x^{3})^{3/2}}$$

26. $y'=\dfrac{1}{x}$, $y''=-\dfrac{1}{x^{2}}$,

$$\kappa(x)=\frac{\left|y''(x)\right|}{\left[1+\left(y'(x)\right)^2\right]^{3/2}}=\left|\frac{-1}{x^2}\right|\frac{1}{(1+1/x^2)^{3/2}}=\frac{1}{x^2}\frac{(x^2)^{3/2}}{(x^2+1)^{3/2}}=\frac{|x|}{(x^2+1)^{3/2}}=\frac{x}{(x^2+1)^{3/2}}$$

(since $x>0$). To find the maximum curvature, we first find the critical numbers of $\kappa(x)$:

$$\kappa'(x)=\frac{(x^2+1)^{3/2}-x\left(\frac{3}{2}\right)(x^2+1)^{1/2}(2x)}{\left[(x^2+1)^{3/2}\right]^2}=\frac{(x^2+1)^{1/2}[(x^2+1)-3x^2]}{(x^2+1)^3}=\frac{1-2x^2}{(x^2+1)^{5/2}}\ ;$$

$\kappa'(x)=0\Rightarrow 1-2x^2=0$, so the only critical number in the domain is $x=\frac{1}{\sqrt{2}}$. Since $\kappa'(x)>0$ for $0<x<\frac{1}{\sqrt{2}}$ and $\kappa'(x)<0$ for $x>\frac{1}{\sqrt{2}}$, $\kappa(x)$ attains its maximum at $x=\frac{1}{\sqrt{2}}$. Thus, the maximum curvature occurs at $\left(\frac{1}{\sqrt{2}},\ln\frac{1}{\sqrt{2}}\right)$. Since $\lim_{x\to\infty}\frac{x}{(x^2+1)^{3/2}}=0$, $\kappa(x)$ approaches 0 as $x\to\infty$.

27. Since $y'=y''=e^x$, the curvature is $\kappa(x)=\frac{\left|y''(x)\right|}{\left[1+(y'(x))^2\right]^{3/2}}=\frac{e^x}{(1+e^{2x})^{3/2}}=e^x(1+e^{2x})^{-3/2}$.

To find the maximum curvature, we first find the critical numbers of $\kappa(x)$:

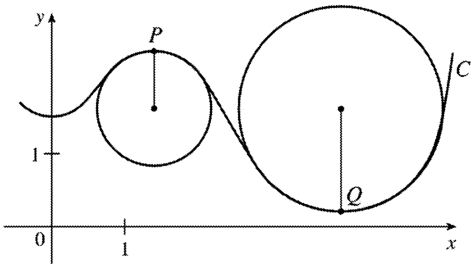
$$\kappa'(x)=e^x(1+e^{2x})^{-3/2}+e^x\left(-\frac{3}{2}\right)(1+e^{2x})^{-5/2}(2e^{2x})=e^x\frac{1+e^{2x}-3e^{2x}}{(1+e^{2x})^{5/2}}=e^x\frac{1-2e^{2x}}{(1+e^{2x})^{5/2}}\ .$$

$\kappa'(x)=0$ when $1-2e^{2x}=0$, so $e^{2x}=\frac{1}{2}$ or $x=-\frac{1}{2}\ln 2$. And since $1-2e^{2x}>0$ for $x<-\frac{1}{2}\ln 2$ and $1-2e^{2x}<0$ for $x>-\frac{1}{2}\ln 2$, the maximum curvature is attained at the point $\left(-\frac{1}{2}\ln 2,e^{(-\ln 2)/2}\right)=\left(-\frac{1}{2}\ln 2,\frac{1}{\sqrt{2}}\right)$. Since $\lim_{x\to\infty}e^x(1+e^{2x})^{-3/2}=0,\kappa(x)$ approaches 0 as $x\to\infty$.

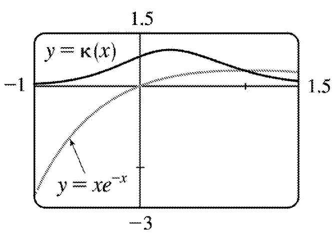
28. We can take the parabola as having its vertex at the origin and opening upward, so the equation is $f(x)=ax^2,a>0$. Then by Equation 11, $\kappa(x)=\frac{\left|f''(x)\right|}{[1+(f'(x))^2]^{3/2}}=\frac{|2a|}{[1+(2ax)^2]^{3/2}}=\frac{2a}{(1+4a^2x^2)^{3/2}}$, thus $\kappa(0)=2a$. We want $\kappa(0)=4$, so $a=2$ and the equation is $y=2x^2$.

29. **(a)** C appears to be changing direction more quickly at P than Q , so we would expect the curvature to be greater
at P .
(b) First we sketch approximate osculating circles at P and Q . Using the axes scale as a guide, we measure the radius of the osculating circle at P to be approximately 0.8 units, thus

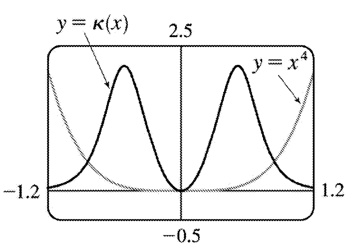
$\rho=\frac{1}{\kappa}\Rightarrow\kappa=\frac{1}{\rho}\approx\frac{1}{0.8}\approx1.3$. Similarly, we estimate the radius of the osculating circle at Q to be 1.4 units, so $\kappa=\frac{1}{\rho}\approx\frac{1}{1.4}\approx0.7$.



30. $y=xe^{-x}\Rightarrow y'=e^{-x}(1-x)$, $y''=e^{-x}(x-2)$, and $\kappa(x)=\frac{|y''|}{[1+(y')^2]^{3/2}}=\frac{e^{-x}|x-2|}{[1+e^{-2x}(1-x)^2]^{3/2}}$. The graph of the curvature here is what we would expect. The graph of xe^{-x} is bending most sharply slightly to the right of the origin. As $x\to\infty$, the graph of xe^{-x} is asymptotic to the x –axis, and so the curvature approaches zero.



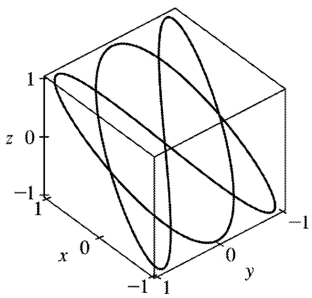
31. $y=x^4\Rightarrow y'=4x^3$, $y''=12x^2$, and $\kappa(x)=\frac{|y''|}{[1+(y')^2]^{3/2}}=\frac{12x^2}{(1+16x^6)^{3/2}}$. The appearance of the two humps in this graph is perhaps a little surprising, but it is explained by the fact that $y=x^4$ is very flat around the origin, and so here the curvature is zero.



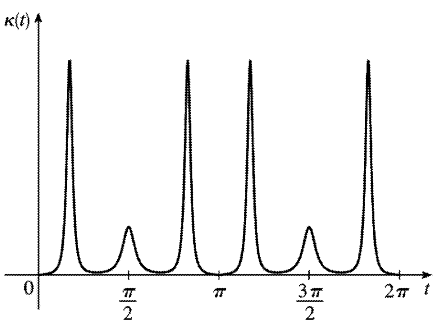
32. Notice that the curve a is highest for the same x –values at which curve b is turning more sharply, and a is 0 or near 0 where b is nearly straight. So, a must be the graph of $y=\kappa(x)$, and b is the graph of $y=f(x)$.

33. Notice that the curve b has two inflection points at which the graph appears almost straight. We would expect the curvature to be 0 or nearly 0 at these values, but the curve a isn't near 0 there. Thus, a must be the graph of $y=f(x)$ rather than the graph of curvature, and b is the graph of $y=\kappa(x)$.

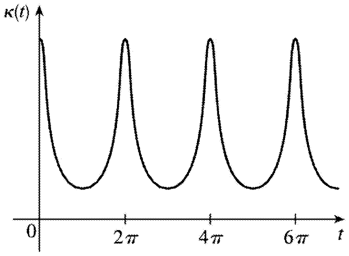
34. **(a)** The complete curve is given by $0\leq t\leq 2\pi$. Curvature appears to have a local (or absolute) maximum at 6 points. (Look at points where the curve appears to turn more sharply.)



(b) Using a CAS, we find (after simplifying) $\kappa(t)=\dfrac{3\sqrt{2}\sqrt{(5\sin t+\sin 5t)^2}}{(9\cos 6t+2\cos 4t+11)^{3/2}}$. (To compute cross products in Maple, use the Linalg package and the crossprod(a,b) command; in Mathematica, use Cross.) The graph shows 6 local (or absolute) maximum points for $0\leq t\leq 2\pi$, as observed in part (a).



35. Using a CAS, we find (after simplifying) $\kappa(t)=\dfrac{6\sqrt{4\cos^2 t-12\cos t+13}}{(17-12\cos t)^{3/2}}$. (To compute cross products in Maple, use the Linalg package and the crossprod (a,b) command; in Mathematica, use Cross.) Curvature is largest at integer multiples of 2π .



36. Here $\boldsymbol{r}(t)=\langle f(t),g(t)\rangle$, $\boldsymbol{r}'(t)=\langle f'(t),g'(t)\rangle$,

$\boldsymbol{r}''(t)=\langle f''(t),g''(t)\rangle$, $|\boldsymbol{r}'(t)|^3=\left[\sqrt{(f'(t))^2+(g'(t))^2}\,\right]^3=[(f'(t))^2+(g'(t))^2]^{3/2}=(\dot{x}^2+\dot{y}^2)^{3/2}$, and

$|\boldsymbol{r}'(t)\times\boldsymbol{r}''(t)|=\left|\left\langle 0,0,f'(t)g''(t)-f''(t)g'(t)\right\rangle\right|=\left[(\dot{x}\ddot{y}-\ddot{x}\dot{y})^2\right]^{1/2}=|\dot{x}\ddot{y}-\dot{y}\ddot{x}|$. Thus

$$\kappa(t)=\frac{|\dot{x}\ddot{y}-\dot{y}\ddot{x}|}{(\dot{x}^2+\dot{y}^2)^{3/2}} .$$

37. $x=e^t\cos t\Rightarrow\dot{x}=e^t(\cos t-\sin t)\Rightarrow\ddot{x}=e^t(-\sin t-\cos t)+e^t(\cos t-\sin t)=-2e^t\sin t$,

$y=e^t\sin t\Rightarrow\dot{y}=e^t(\cos t+\sin t)\Rightarrow\ddot{y}=e^t(-\sin t+\cos t)+e^t(\cos t+\sin t)=2e^t\cos t$. Then

$$\kappa(t)=\frac{|\dot{x}\ddot{y}-\dot{y}\ddot{x}|}{\left(\dot{x}^2+\dot{y}^2\right)^{3/2}}=\frac{|e^t(\cos t-\sin t)(2e^t\cos t)-e^t(\cos t+\sin t)(-2e^t\sin t)|}{\left([e^t(\cos t-\sin t)]^2+[e^t(\cos t+\sin t)]^2\right)^{3/2}}$$

$$=\frac{|2e^{2t}(\cos^2 t-\sin t\cos t+\sin t\cos t+\sin^2 t)|}{\left[e^{2t}(\cos^2 t-2\cos t\sin t+\sin^2 t+\cos^2 t+2\cos t\sin t+\sin^2 t)\right]^{3/2}}$$

$$=\frac{|2e^{2t}(1)|}{\left[e^{2t}(1+1)\right]^{3/2}}=\frac{2e^{2t}}{e^{3t}(2)^{3/2}}=\frac{1}{\sqrt{2}\,e^t}$$

38. $x=1+t^3\Rightarrow\dot{x}=3t^2\Rightarrow\ddot{x}=6t$, $y=t+t^2\Rightarrow\dot{y}=1+2t\Rightarrow\ddot{y}=2$. Then

$$\kappa(t)=\frac{|\dot{x}\ddot{y}-\dot{y}\ddot{x}|}{\left(\dot{x}^2+\dot{y}^2\right)^{3/2}}=\frac{|(3t^2)(2)-(1+2t)(6t)|}{\left[(3t^2)^2+(1+2t)^2\right]^{3/2}}=\frac{|-6t^2-6t|}{(9t^4+4t^2+4t+1)^{3/2}}$$

$$= \frac{6\left|t^2+t\right|}{(9t^4+4t^2+4t+1)^{3/2}}$$

39. $\left(1,\frac{2}{3},1\right)$ corresponds to $t=1$. $\boldsymbol{T}(t)=\dfrac{\boldsymbol{r}'(t)}{\left|\boldsymbol{r}'(t)\right|}=\dfrac{\langle 2\text{t},2t^2,1\rangle}{\sqrt{4t^2+4t^4+1}}=\dfrac{\langle 2\text{t},2t^2,1\rangle}{2t^2+1}$, so

$\boldsymbol{T}(1)=\left\langle \frac{2}{3},\frac{2}{3},\frac{1}{3}\right\rangle$.

$\boldsymbol{T}'(t)=-4t(2t^2+1)^{-2}\langle 2\text{t},2t^2,1\rangle+(2t^2+1)^{-1}\langle 2,4\text{t},0\rangle$ [by Theorem 14.2.3 [ET 13.2.3] #3]

$=(2t^2+1)^{-2}\left\langle -8t^2+4t^2+2,-8t^3+8t^3+4\text{t},-4t\right\rangle=2(2t^2+1)^{-2}\left\langle 1-2t^2,2\text{t},-2t\right\rangle$

$$\boldsymbol{N}(t)=\frac{\boldsymbol{T}'(t)}{\left|\boldsymbol{T}'(t)\right|}=\frac{2(2t^2+1)^{-2}\left\langle 1-2t^2,2\text{t},-2t\right\rangle}{2(2t^2+1)^{-2}\sqrt{(1-2t^2)^2+(2t)^2+(-2t)^2}}=\frac{\left\langle 1-2t^2,2\text{t},-2t\right\rangle}{\sqrt{1-4t^2+4t^4+8t^2}}$$

$$=\frac{\left\langle 1-2t^2,2\text{t},-2t\right\rangle}{1+2t^2}$$

$\boldsymbol{N}(1)=\left\langle -\frac{1}{3},\frac{2}{3},-\frac{2}{3}\right\rangle$ and $\boldsymbol{B}(1)=\boldsymbol{T}(1)\times\boldsymbol{N}(1)=\left\langle -\frac{4}{9}-\frac{2}{9},-\left(-\frac{4}{9}+\frac{1}{9}\right),\frac{4}{9}+\frac{2}{9}\right\rangle=\left\langle -\frac{2}{3},\frac{1}{3},\frac{2}{3}\right\rangle$.

40. $(1,0,1)$ corresponds to $t=0$. $\boldsymbol{r}(t)=e^t\langle 1,\sin \text{t},\cos t\rangle$, so

$\boldsymbol{r}'(t)=e^t\langle 1,\sin \text{t},\cos t\rangle+e^t\langle 0,\cos \text{t},-\sin t\rangle=e^t\langle 1,\sin t+\cos \text{t},\cos t-\sin t\rangle$ and

$$\boldsymbol{T}(t)=\frac{\boldsymbol{r}'(t)}{\left|\boldsymbol{r}'(t)\right|}=\frac{e^t\langle 1,\sin t+\cos \text{t},\cos t-\sin t\rangle}{e^t\sqrt{1+\sin^2 t+2\sin t\cos t+\cos^2 t+\cos^2 t-2\sin t\cos t+\sin^2 t}}$$

$$=\frac{\langle 1,\sin t+\cos \text{t},\cos t-\sin t\rangle}{\sqrt{3}}\ ,$$

$\boldsymbol{T}(0)=\left\langle \frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right\rangle$. $\boldsymbol{T}'(t)=\frac{1}{\sqrt{3}}\langle 0,\cos t-\sin \text{t},-\sin t-\cos t\rangle$, so

$$\boldsymbol{N}(t)=\frac{\boldsymbol{T}'(t)}{\left|\boldsymbol{T}'(t)\right|}=\frac{\frac{1}{\sqrt{3}}\left\langle 0,\cos t-\sin t,-\sin t-\cos t\right\rangle}{\frac{1}{\sqrt{3}}\sqrt{0^2+\cos^2 t-2\cos t\sin t+\sin^2 t+\sin^2 t+2\sin t\cos t+\cos^2 t}}$$

$$=\frac{1}{\sqrt{2}}\left\langle 0,\cos t-\sin t,-\sin t-\cos t\right\rangle .$$

$\boldsymbol{N}(0)=\left\langle 0,\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\right\rangle$ and $\boldsymbol{B}(0)=\boldsymbol{T}(0)\times\boldsymbol{N}(0)=\left\langle -\frac{2}{\sqrt{6}},\frac{1}{\sqrt{6}},\frac{1}{\sqrt{6}}\right\rangle$.

41. $(0,\pi,-2)$ corresponds to $t=\pi$. $\boldsymbol{r}(t)=\left\langle 2\sin 3t,t,2\cos 3t\right\rangle \Rightarrow$

$$\boldsymbol{T}(t)=\frac{\boldsymbol{r}'(t)}{\left|\boldsymbol{r}'(t)\right|}=\frac{\left\langle 6\cos 3t,1,-6\sin 3t\right\rangle}{\sqrt{36\cos^2 3t+1+36\sin^2 3t}}=\frac{1}{\sqrt{37}}\left\langle 6\cos 3t,1,-6\sin 3t\right\rangle .$$

$\boldsymbol{T}(\pi)=\frac{1}{\sqrt{37}}\left\langle -6,1,0\right\rangle$ is a normal vector for the normal plane, and so $\left\langle -6,1,0\right\rangle$ is also normal. Thus an equation for the plane is $-6(x-0)+1(y-\pi)+0(z+2)=0$ or $y-6x=\pi$.

$$\boldsymbol{T}'(t)=\frac{1}{\sqrt{37}}\left\langle -18\sin 3t,0,-18\cos 3t\right\rangle \Rightarrow \left|\boldsymbol{T}'(t)\right|=\frac{\sqrt{18^2\sin^2 3t+18^2\cos^2 3t}}{\sqrt{37}}=\frac{18}{\sqrt{37}}\Rightarrow$$

$\boldsymbol{N}(t)=\frac{\boldsymbol{T}'(t)}{\left|\boldsymbol{T}'(t)\right|}=\left\langle -\sin 3t,0,-\cos 3t\right\rangle$. So $\boldsymbol{N}(\pi)=\left\langle 0,0,1\right\rangle$ and

$\boldsymbol{B}(\pi)=\frac{1}{\sqrt{37}}\left\langle -6,1,0\right\rangle\times\left\langle 0,0,1\right\rangle=\frac{1}{\sqrt{37}}\left\langle 1,6,0\right\rangle$. Since $\boldsymbol{B}(\pi)$ is a normal to the osculating plane, so is $\left\langle 1,6,0\right\rangle$ and an equation for the plane is $1(x-0)+6(y-\pi)+0(z+2)=0$ or $x+6y=6\pi$.

42. $t=1$ at $(1,1,1)$. $\boldsymbol{r}'(t)=\left\langle 1,2t,3t^2\right\rangle$. $\boldsymbol{r}'(1)=\left\langle 1,2,3\right\rangle$ is normal to the normal plane, so an equation for this plane is $1(x-1)+2(y-1)+3(z-1)=0$, or $x+2y+3z=6$.

$\boldsymbol{T}(t)=\frac{\boldsymbol{r}'(t)}{\left|\boldsymbol{r}'(t)\right|}=\frac{1}{\sqrt{1+4t^2+9t^4}}\left\langle 1,2t,3t^2\right\rangle$. Using the product rule on each term of $\boldsymbol{T}(t)$ gives

$$\boldsymbol{T}'(t)=\frac{1}{\left(1+4t^2+9t^4\right)^{3/2}}\left\langle -\frac{1}{2}(8t+36t^3),2(1+4t^2+9t^4)-\frac{1}{2}(8t+36t^3)2t,\right.$$

$$\left. 6t(1+4t^2+9t^4)-\frac{1}{2}(8t+36t^3)3t^2\right\rangle$$

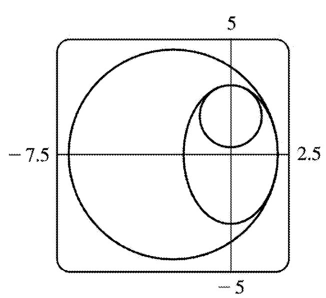
$$=\frac{1}{(1+4t^2+9t^4)^{3/2}}\left\langle -4t-18t^3, 2-18t^4, 6t+12t^3\right\rangle = \frac{-2}{(14)^{3/2}}\left\langle 11,8,-9\right\rangle \text{ when } t=1 \ .$$

$\boldsymbol{N}(1)\parallel \boldsymbol{T}'(1)\parallel \langle 11,8,-9\rangle$ and $\boldsymbol{T}(1)\parallel \boldsymbol{r}'(1)=\langle 1,2,3\rangle \Rightarrow$ a normal vector to the osculating plane is $\langle 11,8,-9\rangle\times\langle 1,2,3\rangle=\langle 42,-42,14\rangle$ or equivalently $\langle 3,-3,1\rangle$. An equation for the plane is $3(x-1)-3(y-1)+(z-1)=0$ or $3x-3y+z=1$.

43. The ellipse is given by the parametric equations $x=2\cos t$, $y=3\sin t$, so using the result from Exercise 36,

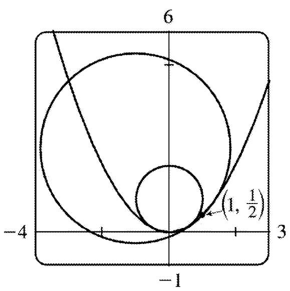
$$\kappa(t)=\frac{\left|\dot{x}\ddot{y}-\ddot{x}\dot{y}\right|}{\left(\dot{x}^2+\dot{y}^2\right)^{3/2}}=\frac{\left|(-2\sin t)(-3\sin t)-(3\cos t)(-2\cos t)\right|}{(4\sin^2 t+9\cos^2 t)^{3/2}}=\frac{6}{(4\sin^2 t+9\cos^2 t)^{3/2}}$$

At $(2,0)$, $t=0$. Now $\kappa(0)=\frac{6}{27}=\frac{2}{9}$, so the radius of the osculating circle is $1/\kappa(0)=\frac{9}{2}$ and its center is $\left(-\frac{5}{2},0\right)$. Its equation is therefore $\left(x+\frac{5}{2}\right)^2+y^2=\frac{81}{4}$. At $(0,3)$, $t=\frac{\pi}{2}$, and $\kappa\left(\frac{\pi}{2}\right)=\frac{6}{8}=\frac{3}{4}$. So the radius of the osculating circle is $\frac{4}{3}$ and its center is $\left(0,\frac{5}{3}\right)$. Hence its equation is $x^2+\left(y-\frac{5}{3}\right)^2=\frac{16}{9}$.



44. $y=\frac{1}{2}x^2\Rightarrow y'=x$ and $y''=1$, so Formula 11 gives $\kappa(x)=\frac{1}{(1+x^2)^{3/2}}$. So the curvature at $(0,0)$ is $\kappa(0)=1$ and the osculating circle has radius 1 and center $(0,1)$, and hence equation $x^2+(y-1)^2=1$. The curvature at $\left(1,\frac{1}{2}\right)$ is $\kappa(1)=\frac{1}{(1+1^2)^{3/2}}=\frac{1}{2\sqrt{2}}$. The tangent line to the parabola at $\left(1,\frac{1}{2}\right)$ has slope 1 ,
so the normal line has slope -1 . Thus the center of the osculating circle lies in the direction of the

unit vector $\left\langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$. The circle has radius $2\sqrt{2}$, so its center has position vector $\left\langle 1, \frac{1}{2} \right\rangle + 2\sqrt{2}\left\langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle = \left\langle -1, \frac{5}{2} \right\rangle$. So the equation of the circle is $(x+1)^2 + \left(y - \frac{5}{2}\right)^2 = 8$.



45. The tangent vector is normal to the normal plane, and the vector $\langle 6,6,-8\rangle$ is normal to the given plane. But $\boldsymbol{T}(t) \parallel \boldsymbol{r}'(t)$ and $\langle 6,6,-8\rangle \parallel \langle 3,3,-4\rangle$, so we need to find t such that $\boldsymbol{r}'(t) \parallel \langle 3,3,-4\rangle$. $\boldsymbol{r}(t) = \langle t^3, 3t, t^4\rangle \Rightarrow \boldsymbol{r}'(t) = \langle 3t^2, 3, 4t^3\rangle \parallel \langle 3,3,-4\rangle$ when $t=-1$. So the planes are parallel at the point $\boldsymbol{r}(-1) = (-1,-3,1)$.

46. To find the osculating plane, we first calculate the tangent and normal vectors.
In Maple, we set x:=t^3; y:=3*t; and z:=t^4; and then calculate the components of the tangent vector $\boldsymbol{T}(t)$ using the diff command. We find that $\boldsymbol{T}(t) = \frac{\langle 3t^2, 3, 4t^3\rangle}{\sqrt{16t^6+9t^4+9}}$. Differentiating the components of $\boldsymbol{T}(t)$, we find that $\boldsymbol{N}(t) = \frac{\boldsymbol{T}'(t)}{|\boldsymbol{T}'(t)|} = \frac{\langle -6t(8t^6-9), 3(48t^5+18t^3), 36t^2(t^4+3)\rangle}{\sqrt{144t(8t^6-9)^2 + 9(96t^5+36t^3)^2 + 5{,}184t^{12} + 31{,}104t^8 + 46{,}656t^4}}$.
In Maple, we can calculate $\boldsymbol{B}(t) = \boldsymbol{T}(t) \times \boldsymbol{N}(t)$ using the linalg package. First we define $\boldsymbol{T}$ and $\boldsymbol{N}$ using T:=array(); and N:=array(); where f, g, h, F, G, and H are the components of $\boldsymbol{T}$ and $\boldsymbol{N}$. Then we use the command B:=crossprod(T,N);. After normalization and simplification, we find that $\boldsymbol{B}(t) = b\langle 6t, -2t^3, -3\rangle$, where

$$b = \frac{t\sqrt{16t^6+9t^4+9}}{\sqrt{16t^2(8t^6-9)^2 + (96t^5+36t^3)^2 + 576t^{12} + 3456t^8 + 5184t^4}}$$

In Mathematica, we use the command Dt to differentiate the components of $\boldsymbol{r}(t)$ and subsequently $\boldsymbol{T}(t)$, and then load the vector analysis package with the command << Calculus‘VectorAnalysis‘. After setting T={f,g,h} and N={F,G,H} , we use CrossProduct [T, N] to find $\boldsymbol{B}$ (before normalization).

Now $\boldsymbol{B}(t)$ is parallel to $\left\langle 6t,-2t^3,-3\right\rangle$, so if $\boldsymbol{B}(t)$ is parallel to $\langle 1,1,1\rangle$ for some t , then $6t=1\Rightarrow t=\frac{1}{6}$, but $-2\left(\frac{1}{6}\right)^3\neq 1$. So there is no such osculating plane.

47. $\kappa=\left|\frac{d\boldsymbol{T}}{ds}\right|=\left|\frac{d\boldsymbol{T}/dt}{ds/dt}\right|=\frac{|d\boldsymbol{T}/dt|}{ds/dt}$ and $\boldsymbol{N}=\frac{d\boldsymbol{T}/dt}{|d\boldsymbol{T}/dt|}$, so $\kappa\boldsymbol{N}=\frac{\left|\frac{d\boldsymbol{T}}{dt}\right|\frac{d\boldsymbol{T}}{dt}}{\left|\frac{d\boldsymbol{T}}{dt}\right|\frac{ds}{dt}}=\frac{d\boldsymbol{T}/dt}{ds/dt}=\frac{d\boldsymbol{T}}{ds}$ by the Chain Rule.

48. For a plane curve, $\boldsymbol{T}=|\boldsymbol{T}|\cos\phi\,\boldsymbol{i}+|\boldsymbol{T}|\sin\phi\,\boldsymbol{j}=\cos\phi\,\boldsymbol{i}+\sin\phi\,\boldsymbol{j}$. Then
$\frac{d\boldsymbol{T}}{ds}=\left(\frac{d\boldsymbol{T}}{d\phi}\right)\left(\frac{d\phi}{ds}\right)=(-\sin\phi\,\boldsymbol{i}+\cos\phi\,\boldsymbol{j})\left(\frac{d\phi}{ds}\right)$ and
$\left|\frac{d\boldsymbol{T}}{ds}\right|=|-\sin\phi\,\boldsymbol{i}+\cos\phi\,\boldsymbol{j}|\left|\frac{d\phi}{ds}\right|=\left|\frac{d\phi}{ds}\right|$. Hence for a plane curve, the curvature is $\kappa=|d\phi/ds|$.

49. **(a)** $|\boldsymbol{B}|=1\Rightarrow\boldsymbol{B}\cdot\boldsymbol{B}=1\Rightarrow\frac{d}{ds}(\boldsymbol{B}\cdot\boldsymbol{B})=0\Rightarrow 2\frac{d\boldsymbol{B}}{ds}\cdot\boldsymbol{B}=0\Rightarrow\frac{d\boldsymbol{B}}{ds}\perp\boldsymbol{B}$

(b) $\boldsymbol{B}=\boldsymbol{T}\times\boldsymbol{N}\Rightarrow$

$$\frac{d\boldsymbol{B}}{ds}=\frac{d}{ds}(\boldsymbol{T}\times\boldsymbol{N})=\frac{d}{dt}(\boldsymbol{T}\times\boldsymbol{N})\frac{1}{ds/dt}=\frac{d}{dt}(\boldsymbol{T}\times\boldsymbol{N})\frac{1}{|\boldsymbol{r}'(t)|}$$

$$=\left[\left(\boldsymbol{T}'\times\boldsymbol{N}\right)+\left(\boldsymbol{T}\times\boldsymbol{N}'\right)\right]\frac{1}{|\boldsymbol{r}'(t)|}=\left[\left(\boldsymbol{T}'\times\frac{\boldsymbol{T}'}{|\boldsymbol{T}'|}\right)+\left(\boldsymbol{T}\times\boldsymbol{N}'\right)\right]\frac{1}{|\boldsymbol{r}'(t)|}=\frac{\boldsymbol{T}\times\boldsymbol{N}'}{|\boldsymbol{r}'(t)|}$$

$\Rightarrow\frac{d\boldsymbol{B}}{ds}\perp\boldsymbol{T}$

(c) $\boldsymbol{B}=\boldsymbol{T}\times\boldsymbol{N}\Rightarrow\boldsymbol{T}\perp\boldsymbol{N}$, $\boldsymbol{B}\perp\boldsymbol{T}$ and $\boldsymbol{B}\perp\boldsymbol{N}$. So $\boldsymbol{B}$, $\boldsymbol{T}$ and $\boldsymbol{N}$ form an orthogonal set of vectors in the three–dimensional space R^3 . From parts (a) and (b), $d\boldsymbol{B}/ds$ is perpendicular to both $\boldsymbol{B}$ and $\boldsymbol{T}$, so $d\boldsymbol{B}/ds$ is parallel to $\boldsymbol{N}$. Therefore, $d\boldsymbol{B}/ds=-\tau(s)\boldsymbol{N}$, where $\tau(s)$ is a scalar.

(d) Since $\boldsymbol{B}=\boldsymbol{T}\times\boldsymbol{N}$, $\boldsymbol{T}\perp\boldsymbol{N}$ and both $\boldsymbol{T}$ and $\boldsymbol{N}$ are unit vectors, $\boldsymbol{B}$ is a unit vector mutually perpendicular to both $\boldsymbol{T}$ and $\boldsymbol{N}$. For a plane curve, $\boldsymbol{T}$ and $\boldsymbol{N}$ always lie in the plane of the curve, so that $\boldsymbol{B}$ is a constant unit vector always perpendicular to the plane. Thus $d\boldsymbol{B}/ds=\boldsymbol{0}$, but $d\boldsymbol{B}/ds=-\tau(s)\boldsymbol{N}$ and $\boldsymbol{N}\neq\boldsymbol{0}$, so $\tau(s)=0$.

50. $\boldsymbol{N}=\boldsymbol{B}\times\boldsymbol{T}\Rightarrow$

$$\begin{aligned}\frac{d\boldsymbol{N}}{ds} &= \frac{d}{ds}(\boldsymbol{B}\times\boldsymbol{T})=\frac{d\boldsymbol{B}}{ds}\times\boldsymbol{T}+\boldsymbol{B}\times\frac{d\boldsymbol{T}}{ds} && \text{[by Theorem 14.2.3 [ET 13.2.3]\#5]}\\ &=-\tau\boldsymbol{N}\times\boldsymbol{T}+\boldsymbol{B}\times\kappa\boldsymbol{N} && \text{[by Formulas 3 and 1]}\\ &=-\tau(\boldsymbol{N}\times\boldsymbol{T})+\kappa(\boldsymbol{B}\times\boldsymbol{N}) && \text{[by Theorem 13.4.8 [ET 12.4.8]\#2]}\end{aligned}$$

But $\boldsymbol{B}\times\boldsymbol{N}=\boldsymbol{B}\times(\boldsymbol{B}\times\boldsymbol{T})=(\boldsymbol{B}\cdot\boldsymbol{T})\boldsymbol{B}-(\boldsymbol{B}\cdot\boldsymbol{B})\boldsymbol{T}$ [by Theorem 13.4.8 [ET 12.4.8]#6] $=-\boldsymbol{T}\Rightarrow$
$d\boldsymbol{N}/ds=\tau(\boldsymbol{T}\times\boldsymbol{N})-\kappa\boldsymbol{T}=-\kappa\boldsymbol{T}+\tau\boldsymbol{B}$.

51. **(a)** $\boldsymbol{r}'=s'\boldsymbol{T}\Rightarrow\boldsymbol{r}''=s''\boldsymbol{T}+s'\boldsymbol{T}'=s''\boldsymbol{T}+s'\frac{d\boldsymbol{T}}{ds}s'=s''\boldsymbol{T}+\kappa(s')^2\boldsymbol{N}$ by the first Serret–Frenet formula.

(b) Using part (a), we have

$$\begin{aligned}\boldsymbol{r}'\times\boldsymbol{r}'' &= (s'\boldsymbol{T})\times[s''\boldsymbol{T}+\kappa(s')^2\boldsymbol{N}]\\ &= \left[(s'\boldsymbol{T})\times(s''\boldsymbol{T})\right]+\left[(s'\boldsymbol{T})\times(\kappa(s')^2\boldsymbol{N})\right]3]\\ &= (s's'')(\boldsymbol{T}\times\boldsymbol{T})+\kappa(s')^3(\boldsymbol{T}\times\boldsymbol{N})=\boldsymbol{0}+\kappa(s')^3\boldsymbol{B}=\kappa(s')^3\boldsymbol{B}\end{aligned}$$

(c) Using part (a), we have

$$\begin{aligned}\boldsymbol{r}''' &= [s''\boldsymbol{T}+\kappa(s')^2\boldsymbol{N}]'=s'''\boldsymbol{T}+s''\boldsymbol{T}'+\kappa'(s')^2\boldsymbol{N}+2\kappa s's''\boldsymbol{N}+\kappa(s')^2\boldsymbol{N}'\\ &= s'''\boldsymbol{T}+s''\frac{d\boldsymbol{T}}{ds}s'+\kappa'(s')^2\boldsymbol{N}+2\kappa s's''\boldsymbol{N}+\kappa(s')^2\frac{d\boldsymbol{N}}{ds}s'\\ &= s'''\boldsymbol{T}+s''s'\kappa\boldsymbol{N}+\kappa'(s')^2\boldsymbol{N}+2\kappa s's''\boldsymbol{N}+\kappa(s')^3(-\kappa\boldsymbol{T}+\tau\boldsymbol{B}) \text{ [by the second formula]}\\ &= [s'''-\kappa^2(s')^3]\boldsymbol{T}+[3\kappa s's''+\kappa'(s')^2]\boldsymbol{N}+\kappa\tau(s')^3\boldsymbol{B}\end{aligned}$$

(d) Using parts (b) and (c) and the facts that $\boldsymbol{B}\cdot\boldsymbol{T}=0$, $\boldsymbol{B}\cdot\boldsymbol{N}=0$, and $\boldsymbol{B}\cdot\boldsymbol{B}=1$, we get

$$\begin{aligned}\frac{(\boldsymbol{r}'\times\boldsymbol{r}'')\cdot\boldsymbol{r}'''}{\left|\boldsymbol{r}'\times\boldsymbol{r}''\right|^2} &= \frac{\kappa(s')^3\boldsymbol{B}\cdot\left\{[s'''-\kappa^2(s')^3]\boldsymbol{T}+[3\kappa s's''+\kappa'(s')^2]\boldsymbol{N}+\kappa\tau(s')^3\boldsymbol{B}\right\}}{\left|\kappa(s')^3\boldsymbol{B}\right|^2}\\ &= \frac{\kappa(s')^3\kappa\tau(s')^3}{\left[\kappa(s')^3\right]^2}=\tau\end{aligned}$$

52. First we find the quantities required to compute κ :

$$\boldsymbol{r}'(t)=\left\langle -a\sin t,a\cos t,b\right\rangle\Rightarrow\boldsymbol{r}''(t)=\left\langle -a\cos t,-a\sin t,0\right\rangle\Rightarrow\boldsymbol{r}'''(t)=\left\langle a\sin t,-a\cos t,0\right\rangle$$

$\left|\boldsymbol{r}'(t)\right| = \sqrt{(-a\sin t)^2+(a\cos t)^2+b^2}=\sqrt{a^2+b^2}$

$\boldsymbol{r}'(t)\times\boldsymbol{r}''(t) =$

$\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ -a\sin t & a\cos t & b \\ -a\cos t & -a\sin t & 0 \end{vmatrix}$

$=ab\sin t\,\boldsymbol{i}-ab\cos t\,\boldsymbol{j}+a^2\boldsymbol{k}$

$\left|\boldsymbol{r}'(t)\times\boldsymbol{r}''(t)\right| = \sqrt{(ab\sin t)^2+(-ab\cos t)^2+(a^2)^2}=\sqrt{a^2b^2+a^4}$

$\left(\boldsymbol{r}'(t)\times\boldsymbol{r}''(t)\right)\cdot\boldsymbol{r}'''(t) = (ab\sin t)(a\sin t)+(-ab\cos t)(-a\cos t)+(a^2)(0)=a^2b$

Then by Theorem 10,

$$\kappa(t)=\frac{\left|\boldsymbol{r}'(t)\times\boldsymbol{r}''(t)\right|}{\left|\boldsymbol{r}'(t)\right|^3}=\frac{\sqrt{a^2b^2+a^4}}{\left(\sqrt{a^2+b^2}\right)^3}=\frac{a\sqrt{a^2+b^2}}{\left(\sqrt{a^2+b^2}\right)^3}=\frac{a}{a^2+b^2}$$

which is a constant.

From Exercise 51(d), the torsion τ is given by

$$\tau=\frac{(\boldsymbol{r}'\times\boldsymbol{r}'')\cdot\boldsymbol{r}'''}{\left|\boldsymbol{r}'\times\boldsymbol{r}''\right|^2}=\frac{a^2b}{\left(\sqrt{a^2b^2+a^4}\right)^2}=\frac{b}{a^2+b^2}$$

which is also a constant.

53. $\boldsymbol{r}=\left\langle t,\frac{1}{2}t^2,\frac{1}{3}t^3\right\rangle \Rightarrow \boldsymbol{r}'=\left\langle 1,t,t^2\right\rangle$, $\boldsymbol{r}''=\langle 0,1,2t\rangle$, $\boldsymbol{r}'''=\langle 0,0,2\rangle \Rightarrow \boldsymbol{r}'\times\boldsymbol{r}''=\left\langle t^2,-2t,1\right\rangle \Rightarrow$

$\tau=\dfrac{\left(\boldsymbol{r}'\times\boldsymbol{r}''\right)\cdot\boldsymbol{r}'''}{\left|\boldsymbol{r}'\times\boldsymbol{r}''\right|^2}=\dfrac{\left\langle t^2,-2t,1\right\rangle\cdot\langle 0,0,2\rangle}{t^4+4t^2+1}=\dfrac{2}{t^4+4t^2+1}$

54. $\boldsymbol{r}=\langle \sinh t,\cosh t,t\rangle \Rightarrow \boldsymbol{r}'=\langle \cosh t,\sinh t,1\rangle$, $\boldsymbol{r}''=\langle \sinh t,\cosh t,0\rangle$, $\boldsymbol{r}'''=\langle \cosh t,\sinh t,0\rangle \Rightarrow$

$\boldsymbol{r}'\times\boldsymbol{r}''=\left\langle -\cosh t,\sinh t,\cosh^2 t-\sinh^2 t\right\rangle=\langle -\cosh t,\sinh t,1\rangle \Rightarrow$

$\kappa=\dfrac{\left|\boldsymbol{r}'\times\boldsymbol{r}''\right|}{\left|\boldsymbol{r}'\right|^3}=\dfrac{\left|\langle -\cosh t,\sinh t,1\rangle\right|}{\left|\langle \cosh t,\sinh t,1\rangle\right|^3}=\dfrac{\sqrt{\cosh^2 t+\sinh^2 t+1}}{\left(\cosh^2 t+\sinh^2 t+1\right)^{3/2}}=\dfrac{1}{\cosh^2 t+\sinh^2 t+1}=\dfrac{1}{2\cosh^2 t}$,

$$\tau=\frac{\left(\mathbf{r}'\times\mathbf{r}''\right)\cdot\mathbf{r}'''}{\left|\mathbf{r}'\times\mathbf{r}''\right|^{2}}=\frac{\langle -\cosh t,\sinh t,1\rangle\cdot\langle\cosh t,\sinh t,0\rangle}{\cosh^{2}t+\sinh^{2}t+1}=\frac{-\cosh^{2}t+\sinh^{2}t}{2\cosh^{2}t}=\frac{-1}{2\cosh^{2}t}$$

So at the point $(0,1,0)$, $t=0$, and $\kappa=\frac{1}{2}$ and $\tau=-\frac{1}{2}$.

55. For one helix, the vector equation is $\mathbf{r}(t)=\langle 10\cos t,10\sin t,34t/(2\pi)\rangle$ (measuring in angstroms), because the radius of each helix is 10 angstroms, and z increases by 34 angstroms for each increase of 2π in t . Using the arc length formula, letting t go from 0 to $2.9\times 10^{8}\times 2\pi$, we find the approximate length of each helix to be

$$L=\int_0^{2.9\times 10^{8}\times 2\pi}\left|\mathbf{r}'(t)\right|dt=\int_0^{2.9\times 10^{8}\times 2\pi}\sqrt{(-10\sin t)^{2}+(10\cos t)^{2}+\left(\frac{34}{2\pi}\right)^{2}}\,dt$$

$$=\sqrt{100+\left(\frac{34}{2\pi}\right)^{2}}\,t\,\Bigg]_0^{2.9\times 10^{8}\times 2\pi}=2.9\times 10^{8}\times 2\pi\sqrt{100+\left(\frac{34}{2\pi}\right)^{2}}$$

$\approx 2.07\times 10^{10}$—more than two meters!

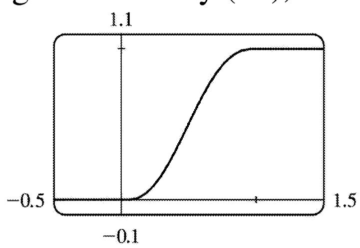
56. **(a)** For the function $F(x)=\begin{cases}0 & \text{if } x<0\\ P(x) & \text{if } 0<x<1\\ 1 & \text{if } x\geq 1\end{cases}$ to be continuous, we must have $P(0)=0$ and $P(1)=1$.

For F' to be continuous, we must have $P'(0)=P'(1)=0$. The curvature of the curve $y=F(x)$ at the point $(x,F(x))$ is $\kappa(x)=\frac{\left|F''(x)\right|}{\left(1+\left[F'(x)\right]^{2}\right)^{3/2}}$. For $\kappa(x)$ to be continuous, we must have $P''(0)=P''(1)=0$.

Write $P(x)=ax^{5}+bx^{4}+cx^{3}+dx^{2}+ex+f$. Then $P'(x)=5ax^{4}+4bx^{3}+3cx^{2}+2dx+e$ and $P''(x)=20ax^{3}+12bx^{2}+6cx+2d$. Our six conditions are:

$P(0)=0\Rightarrow f=0$ (1)
$P(1)=1\Rightarrow a+b+c+d+e+f=1$ (2)
$P'(0)=0\Rightarrow e=0$ (3)
$P'(1)=0\Rightarrow 5a+4b+3c+2d+e=0$ (4)
$P''(0)=0\Rightarrow d=0$ (5)
$P''(1)=0\Rightarrow 20a+12b+6c+2d=0$ (6)

From (1), (3), and (5), we have $d=e=f=0$. Thus (2), (4) and (6) become (7) $a+b+c=1$, (8) $5a+4b+3c=0$, and (9) $10a+6b+3c=0$. Subtracting (8) from (9) gives (10) $5a+2b=0$. Multiplying (7) by 3 and subtracting from (8) gives (11) $2a+b=-3$. Multiplying (11) by 2 and subtracting from (10) gives $a=6$. By (10), $b=-15$. By (7), $c=10$. Thus, $P(x)=6x^5-15x^4+10x^3$.

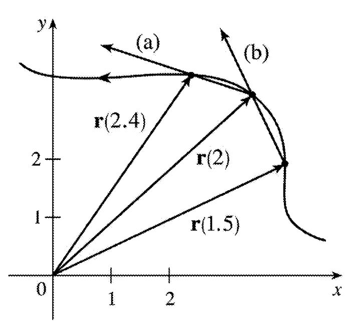
(b) 

1. **(a)** If $\boldsymbol{r}(t)=x(t)\boldsymbol{i}+y(t)\,\boldsymbol{j}+z(t)\boldsymbol{k}$ is the position vector of the particle at time t , then the average velocity over the time interval $[0,1]$ is
$\boldsymbol{v}_{\text{ave}}=\dfrac{\boldsymbol{r}(1)-\boldsymbol{r}(0)}{1-0}=\dfrac{(4.5\boldsymbol{i}+6.0\boldsymbol{j}+3.0\boldsymbol{k})-(2.7\boldsymbol{i}+9.8\boldsymbol{j}+3.7\boldsymbol{k})}{1}=1.8\boldsymbol{i}-3.8\boldsymbol{j}-0.7\boldsymbol{k}$. Similarly, over the other intervals we have

$$
\begin{aligned}
[0.5,1]:\boldsymbol{v}_{\text{ave}} &= \frac{\boldsymbol{r}(1)-\boldsymbol{r}(0.5)}{1-0.5}=\frac{(4.5\boldsymbol{i}+6.0\boldsymbol{j}+3.0\boldsymbol{k})-(3.5\boldsymbol{i}+7.2\boldsymbol{j}+3.3\boldsymbol{k})}{0.5}\\
&= 2.0\boldsymbol{i}-2.4\boldsymbol{j}-0.6\boldsymbol{k}\\
[1,2]:\boldsymbol{v}_{\text{ave}} &= \frac{\boldsymbol{r}(2)-\boldsymbol{r}(1)}{2-1}=\frac{(7.3\boldsymbol{i}+7.8\boldsymbol{j}+2.7\boldsymbol{k})-(4.5\boldsymbol{i}+6.0\boldsymbol{j}+3.0\boldsymbol{k})}{1}\\
&= 2.8\boldsymbol{i}+1.8\boldsymbol{j}-0.3\boldsymbol{k}\\
[1,1.5]:\boldsymbol{v}_{\text{ave}} &= \frac{\boldsymbol{r}(1.5)-\boldsymbol{r}(1)}{1.5-1}=\frac{(5.9\boldsymbol{i}+6.4\boldsymbol{j}+2.8\boldsymbol{k})-(4.5\boldsymbol{i}+6.0\boldsymbol{j}+3.0\boldsymbol{k})}{0.5}\\
&= 2.8\boldsymbol{i}+0.8\boldsymbol{j}-0.4\boldsymbol{k}
\end{aligned}
$$

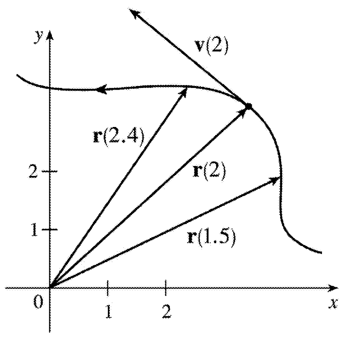
(b) We can estimate the velocity at $t=1$ by averaging the average velocities over the time intervals $[0.5,1]$ and $[1,1.5]$: $\boldsymbol{v}(1)\approx\frac{1}{2}\left[(2\boldsymbol{i}-2.4\boldsymbol{j}-0.6\boldsymbol{k})+(2.8\boldsymbol{i}+0.8\boldsymbol{j}-0.4\boldsymbol{k})\right]=2.4\boldsymbol{i}-0.8\boldsymbol{j}-0.5\boldsymbol{k}$. Then the speed is $|\boldsymbol{v}(1)|\approx\sqrt{(2.4)^2+(-0.8)^2+(-0.5)^2}\approx2.58$.

2. **(a)** The average velocity over $2\leq t\leq 2.4$ is
$\dfrac{\boldsymbol{r}(2.4)-\boldsymbol{r}(2)}{2.4-2}=2.5[\boldsymbol{r}(2.4)-\boldsymbol{r}(2)]$, so we sketch a vector in the same direction but 2.5 times the length of $[\boldsymbol{r}(2.4)-\boldsymbol{r}(2)]$.



(b) The average velocity over $1.5\leq t\leq 2$ is $\dfrac{\boldsymbol{r}(2)-\boldsymbol{r}(1.5)}{2-1.5}=2[\boldsymbol{r}(2)-\boldsymbol{r}(1.5)]$, so we sketch a vector in the same direction but twice the length of $[\boldsymbol{r}(2)-\boldsymbol{r}(1.5)]$.
(c) Using Equation 2 we have $\boldsymbol{v}(2)=\lim\limits_{h\to0}\dfrac{\boldsymbol{r}(2+h)-\boldsymbol{r}(2)}{h}$.

(d) $\boldsymbol{v}(2)$ is tangent to the curve at $\boldsymbol{r}(2)$ and points in the direction of increasing t . Its length is the speed of the particle at $t=2$. We can estimate the speed by averaging the lengths of the vectors found in parts (a) and (b) which represent the average speed over $2\leq t\leq 2.4$ and $1.5\leq t\leq 2$ respectively. Using the axes scale as a guide, we estimate the vectors to have lengths 2.8 and 2.7 . Thus, we estimate the speed at $t=2$ to be $|\boldsymbol{v}(2)|\approx \frac{1}{2}(2.8+2.7)=2.75$ and we draw the velocity vector $\boldsymbol{v}(2)$ with this length.

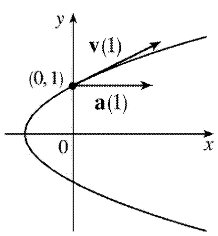


3. $\boldsymbol{r}(t)=\left\langle t^2-1,t\right\rangle \Rightarrow$ At $t=1$:

$\boldsymbol{v}(t)=\boldsymbol{r}'(t)=\langle 2t,1\rangle$, $\boldsymbol{v}(1)=\langle 2,1\rangle$

$\boldsymbol{a}(t)=\boldsymbol{r}''(t)=\langle 2,0\rangle$, $\boldsymbol{a}(1)=\langle 2,0\rangle$

$|\boldsymbol{v}(t)|=\sqrt{4t^2+1}$

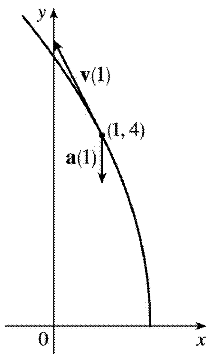


4. $\boldsymbol{r}(t)=\left\langle 2-t,4\sqrt{t}\right\rangle \Rightarrow$ At $t=1$:

$\boldsymbol{v}(t)=\boldsymbol{r}'(t)=\left\langle -1,2/\sqrt{t}\right\rangle$, $\boldsymbol{v}(1)=\langle -1,2\rangle$

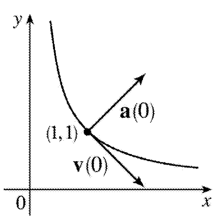
$\boldsymbol{a}(t)=\boldsymbol{r}''(t)=\left\langle 0,-1/t^{3/2}\right\rangle$, $\boldsymbol{a}(1)=\langle 0,-1\rangle$

$|\boldsymbol{v}(t)|=\sqrt{1+4/t}$



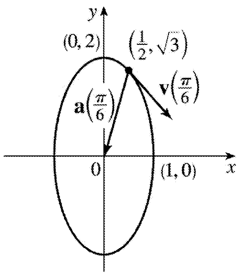
5. $\boldsymbol{r}(t)=e^{t}\boldsymbol{i}+e^{-t}\boldsymbol{j}\Rightarrow$
$\boldsymbol{v}(t)=e^{t}\boldsymbol{i}-e^{-t}\boldsymbol{j}$,
$\boldsymbol{a}(t)=e^{t}\boldsymbol{i}+e^{-t}\boldsymbol{j}$
At $t=0$:
$\boldsymbol{v}(0)=\boldsymbol{i}-\boldsymbol{j}$,
$\boldsymbol{a}(0)=\boldsymbol{i}+\boldsymbol{j}$

$|\boldsymbol{v}(t)|=\sqrt{e^{2t}+e^{-2t}}=e^{-t}\sqrt{e^{4t}+1}$

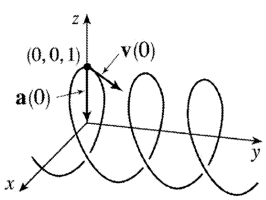


Since $x=e^{t}$, $t=\ln x$ and $y=e^{-t}=e^{-\ln x}=1/x$, and $x>0$, $y>0$.

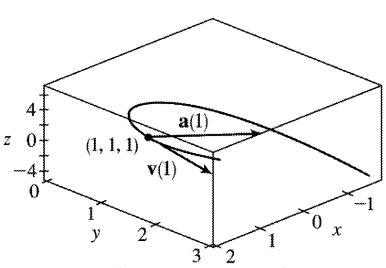
6. $\boldsymbol{r}(t)=\sin t\,\boldsymbol{i}+2\cos t\,\boldsymbol{j}\Rightarrow$
$\boldsymbol{v}(t)=\cos t\,\boldsymbol{i}-2\sin t\,\boldsymbol{j}$, $\boldsymbol{v}\left(\frac{\pi}{6}\right)=\frac{\sqrt{3}}{2}\,\boldsymbol{i}-\boldsymbol{j}$
$\boldsymbol{a}(t)=-\sin t\,\boldsymbol{i}-2\cos t\,\boldsymbol{j}$, $\boldsymbol{a}\left(\frac{\pi}{6}\right)=-\frac{1}{2}\,\boldsymbol{i}-\sqrt{3}\,\boldsymbol{j}$
$|\boldsymbol{v}(t)|=\sqrt{\cos^{2}t+4\sin^{2}t}=\sqrt{1+3\sin^{2}t}$
And $x^{2}+y^{2}/4=\sin^{2}t+\cos^{2}t=1$, an ellipse.



7. $\boldsymbol{r}(t)=\sin t\,\boldsymbol{i}+t\,\boldsymbol{j}+\cos t\,\boldsymbol{k}\Rightarrow$
$\boldsymbol{v}(t)=\cos t\,\boldsymbol{i}+\boldsymbol{j}-\sin t\,\boldsymbol{k}$, $\boldsymbol{v}(0)=\boldsymbol{i}+\boldsymbol{j}$
$\boldsymbol{a}(t)=-\sin t\,\boldsymbol{i}-\cos t\,\boldsymbol{k}$, $\boldsymbol{a}(0)=-\boldsymbol{k}$
$|\boldsymbol{v}(t)|=\sqrt{\cos^2 t+1+\sin^2 t}=\sqrt{2}$
Since $x^2+z^2=1$, $y=t$, the path of the particle is a helix about the y –axis.



8. $\boldsymbol{r}(t)=t\,\boldsymbol{i}+t^2\,\boldsymbol{j}+t^3\,\boldsymbol{k}\Rightarrow$
$\boldsymbol{v}(t)=\boldsymbol{i}+2t\,\boldsymbol{j}+3t^2\,\boldsymbol{k}$, $\boldsymbol{v}(1)=\boldsymbol{i}+2\,\boldsymbol{j}+3\,\boldsymbol{k}\,\boldsymbol{a}(t)=2\,\boldsymbol{j}+6t\,\boldsymbol{k}$, $\boldsymbol{a}(1)=2\,\boldsymbol{j}+6\,\boldsymbol{k}$
$|\boldsymbol{v}(t)|=\sqrt{1+4t^2+9t^4}$
The path is a ‘‘twisted cubic’’
(see Example 14.1.7).



9. $\boldsymbol{r}(t)=\left\langle t^2+1,t^3,t^2-1\right\rangle \Rightarrow \boldsymbol{v}(t)=\boldsymbol{r}'(t)=\left\langle 2t,3t^2,2t\right\rangle$, $\boldsymbol{a}(t)=\boldsymbol{v}'(t)=\left\langle 2,6t,2\right\rangle$,
$|\boldsymbol{v}(t)|=\sqrt{(2t)^2+(3t^2)^2+(2t)^2}=\sqrt{9t^4+8t^2}=|t|\sqrt{9t^2+8}$.
10. $\boldsymbol{r}(t)=\left\langle 2\cos t,3t,2\sin t\right\rangle \Rightarrow \boldsymbol{v}(t)=\boldsymbol{r}'(t)=\left\langle -2\sin t,3,2\cos t\right\rangle$, $\boldsymbol{a}(t)=\boldsymbol{v}'(t)=\left\langle -2\cos t,0,-2\sin t\right\rangle$,
$|\boldsymbol{v}(t)|=\sqrt{4\sin^2 t+9+4\cos^2 t}=\sqrt{13}$.

11. $\boldsymbol{r}(t)=\sqrt{2}\,t\,\boldsymbol{i}+e^{t}\boldsymbol{j}+e^{-t}\boldsymbol{k}\Rightarrow\boldsymbol{v}(t)=\boldsymbol{r}'(t)=\sqrt{2}\,\boldsymbol{i}+e^{t}\boldsymbol{j}-e^{-t}\boldsymbol{k}$, $\boldsymbol{a}(t)=\boldsymbol{v}'(t)=e^{t}\boldsymbol{j}+e^{-t}\boldsymbol{k}$,
$|\boldsymbol{v}(t)|=\sqrt{2+e^{2t}+e^{-2t}}=\sqrt{(e^{t}+e^{-t})^{2}}=e^{t}+e^{-t}$.

12. $\boldsymbol{r}(t)=t^{2}\boldsymbol{i}+\ln t\,\boldsymbol{j}+t\,\boldsymbol{k}\Rightarrow\boldsymbol{v}(t)=\boldsymbol{r}'(t)=2t\,\boldsymbol{i}+t^{-1}\boldsymbol{j}+\boldsymbol{k}$, $\boldsymbol{a}(t)=\boldsymbol{v}'(t)=2\boldsymbol{i}-t^{-2}\boldsymbol{j}$, $|\boldsymbol{v}(t)|=\sqrt{4t^{2}+t^{-2}+1}$.

13. $\boldsymbol{r}(t)=e^{t}\langle\cos t,\sin t,t\rangle\Rightarrow\boldsymbol{v}(t)=\boldsymbol{r}'(t)=e^{t}\langle\cos t,\sin t,t\rangle+e^{t}\langle-\sin t,\cos t,1\rangle=e^{t}\langle\cos t-\sin t,\sin t+\cos t,t+1\rangle$

$$\begin{aligned}\boldsymbol{a}(t)&=\boldsymbol{v}'(t)=e^{t}\langle\cos t-\sin t-\sin t-\cos t,\sin t+\cos t+\cos t-\sin t,t+1+1\rangle\\&=e^{t}\langle-2\sin t,2\cos t,t+2\rangle\end{aligned}$$

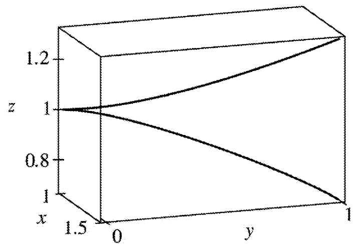
$$\begin{aligned}|\boldsymbol{v}(t)|&=e^{t}\sqrt{\cos^{2}t+\sin^{2}t-2\cos t\sin t+\sin^{2}t+\cos^{2}t+2\sin t\cos t+t^{2}+2t+1}\\&=e^{t}\sqrt{t^{2}+2t+3}\end{aligned}$$

14. $\boldsymbol{r}(t)=t\sin t\,\boldsymbol{i}+t\cos t\,\boldsymbol{j}+t^{2}\boldsymbol{k}\Rightarrow\boldsymbol{v}(t)=\boldsymbol{r}'(t)=(\sin t+t\cos t)\,\boldsymbol{i}+(\cos t-t\sin t)\,\boldsymbol{j}+2t\,\boldsymbol{k}$,
$\boldsymbol{a}(t)=\boldsymbol{v}'(t)=(2\cos t-t\sin t)\,\boldsymbol{i}+(-2\sin t-t\cos t)\,\boldsymbol{j}+2\boldsymbol{k}$,
$|\boldsymbol{v}(t)|=\sqrt{(\sin^{2}t+2t\sin t\cos t+t^{2}\cos^{2}t)+(\cos^{2}t-2t\sin t\cos t+t^{2}\sin^{2}t)+4t^{2}}=\sqrt{5t^{2}+1}$.

15. $\boldsymbol{a}(t)=\boldsymbol{k}\Rightarrow\boldsymbol{v}(t)=\int\boldsymbol{a}(t)\,dt=\int\boldsymbol{k}\,dt=t\,\boldsymbol{k}+\boldsymbol{c}_{1}$ and $\boldsymbol{i}-\boldsymbol{j}=\boldsymbol{v}(0)=0\,\boldsymbol{k}+\boldsymbol{c}_{1}$, so $\boldsymbol{c}_{1}=\boldsymbol{i}-\boldsymbol{j}$ and $\boldsymbol{v}(t)=\boldsymbol{i}-\boldsymbol{j}+t\,\boldsymbol{k}$.
$\boldsymbol{r}(t)=\int\boldsymbol{v}(t)\,dt=\int(\boldsymbol{i}-\boldsymbol{j}+t\,\boldsymbol{k})\,dt=t\,\boldsymbol{i}-t\,\boldsymbol{j}+\frac{1}{2}t^{2}\boldsymbol{k}+\boldsymbol{c}_{2}$. But $\boldsymbol{0}=\boldsymbol{r}(0)=\boldsymbol{0}+\boldsymbol{c}_{2}$, so $\boldsymbol{c}_{2}=\boldsymbol{0}$ and $\boldsymbol{r}(t)=t\,\boldsymbol{i}-t\,\boldsymbol{j}+\frac{1}{2}t^{2}\boldsymbol{k}$.

16. $\boldsymbol{a}(t)=-10\boldsymbol{k}\Rightarrow\boldsymbol{v}(t)=\int(-10\boldsymbol{k})\,dt=-10t\,\boldsymbol{k}+\boldsymbol{c}_{1}$, and $\boldsymbol{i}+\boldsymbol{j}-\boldsymbol{k}=\boldsymbol{v}(0)=\boldsymbol{0}+\boldsymbol{c}_{1}$, so $\boldsymbol{c}_{1}=\boldsymbol{i}+\boldsymbol{j}-\boldsymbol{k}$ and
$\boldsymbol{v}(t)=\boldsymbol{i}+\boldsymbol{j}-(10t+1)\,\boldsymbol{k}$.
$\boldsymbol{r}(t)=\int[\boldsymbol{i}+\boldsymbol{j}-(10t+1)\boldsymbol{k}]\,dt=t\,\boldsymbol{i}+t\,\boldsymbol{j}-(5t^{2}+t)\,\boldsymbol{k}+\boldsymbol{c}_{2}$. But $2\boldsymbol{i}+3\boldsymbol{j}=\boldsymbol{r}(0)=\boldsymbol{0}+\boldsymbol{c}_{2}$, so $\boldsymbol{c}_{2}=2\boldsymbol{i}+3\boldsymbol{j}$ and
$\boldsymbol{r}(t)=(t+2)\,\boldsymbol{i}+(t+3)\,\boldsymbol{j}-(5t^{2}+t)\,\boldsymbol{k}$.

17. **(a)** $\boldsymbol{a}(t)=\boldsymbol{i}+2\boldsymbol{j}+2t\,\boldsymbol{k}\Rightarrow\boldsymbol{v}(t)=\int(\boldsymbol{i}+2\boldsymbol{j}+2t\,\boldsymbol{k})\,dt=t\,\boldsymbol{i}+2t\,\boldsymbol{j}+t^{2}\boldsymbol{k}+\boldsymbol{c}_{1}$, and $\boldsymbol{0}=\boldsymbol{v}(0)=\boldsymbol{0}+\boldsymbol{c}_{1}$, so $\boldsymbol{c}_{1}=\boldsymbol{0}$ and
$\boldsymbol{v}(t)=\boldsymbol{i}+2t\,\boldsymbol{j}+t^{2}\boldsymbol{k}$. $\boldsymbol{r}(t)=\int(t\,\boldsymbol{i}+2t\,\boldsymbol{j}+t^{2}\boldsymbol{k})\,dt=\frac{1}{2}t^{2}\boldsymbol{i}+t^{2}\boldsymbol{j}+\frac{1}{3}t^{3}\boldsymbol{k}+\boldsymbol{c}_{2}$. But $\boldsymbol{i}+\boldsymbol{k}=\boldsymbol{r}(0)=\boldsymbol{0}+\boldsymbol{c}_{2}$, so $\boldsymbol{c}_{2}=\boldsymbol{i}+\boldsymbol{k}$ and
$\boldsymbol{r}(t)=\left(1+\frac{1}{2}t^{2}\right)\boldsymbol{i}+t^{2}\boldsymbol{j}+\left(1+\frac{1}{3}t^{3}\right)\boldsymbol{k}$.

(b) 

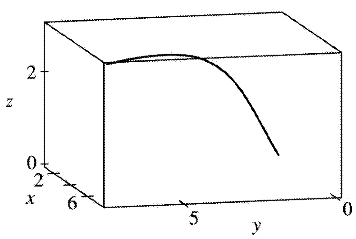
18. **(a)** $\boldsymbol{a}(t)=t\,\boldsymbol{i}+t^2\,\boldsymbol{j}+\cos 2t\,\boldsymbol{k}\Rightarrow$

$\boldsymbol{v}(t)=\int (t\,\boldsymbol{i}+t^2\,\boldsymbol{j}+\cos 2t\,\boldsymbol{k})\,dt$

$=\frac{t^2}{2}\,\boldsymbol{i}+\frac{t^3}{3}\,\boldsymbol{j}+\frac{\sin 2t}{2}\,\boldsymbol{k}+\boldsymbol{c}_1$

and $\boldsymbol{i}+\boldsymbol{k}=\boldsymbol{v}(0)=\boldsymbol{0}+\boldsymbol{c}_1$, so $\boldsymbol{c}_1=\boldsymbol{i}+\boldsymbol{k}$ and

$\boldsymbol{v}(t)=\left(\frac{1}{2}t^2+1\right)\boldsymbol{i}+\frac{1}{3}t^3\boldsymbol{j}+\left(1+\frac{1}{2}\sin 2t\right)\boldsymbol{k}$.



(b)

$\boldsymbol{r}(t)=\int\left[\left(\frac{1}{2}t^2+1\right)\boldsymbol{i}+\frac{1}{3}t^3\boldsymbol{j}+\left(1+\frac{1}{2}\sin 2t\right)\boldsymbol{k}\right]dt=\left(\frac{1}{6}t^3+t\right)\boldsymbol{i}+\frac{1}{12}t^4\boldsymbol{j}+\left(t-\frac{1}{4}\cos 2t\right)\boldsymbol{k}+\boldsymbol{c}_2$

But $\boldsymbol{j}=\boldsymbol{r}(0)=-\frac{1}{4}\boldsymbol{k}+\boldsymbol{c}_2$, so $\boldsymbol{c}_2=\boldsymbol{j}+\frac{1}{4}\boldsymbol{k}$ and $\boldsymbol{r}(t)=\left(\frac{1}{6}t^3+t\right)\boldsymbol{i}+\left(1+\frac{1}{12}t^4\right)\boldsymbol{j}+\left(\frac{1}{4}+t-\frac{1}{4}\cos 2t\right)\boldsymbol{k}$.

19. $\boldsymbol{r}(t)=\left\langle t^2,5t,t^2-16t\right\rangle\Rightarrow\boldsymbol{v}(t)=\langle 2t,5,2t-16\rangle$, $|\boldsymbol{v}(t)|=\sqrt{4t^2+25+4t^2-64t+256}=\sqrt{8t^2-64t+281}$ and $\frac{d}{dt}|\boldsymbol{v}(t)|=\frac{1}{2}(8t^2-64t+281)^{-1/2}(16t-64)$. This is zero if and only if the numerator is zero, that is, $16t-64=0$ or $t=4$. Since $\frac{d}{dt}|\boldsymbol{v}(t)|<0$ for $t<4$ and $\frac{d}{dt}|\boldsymbol{v}(t)|>0$ for $t>4$, the minimum speed of $\sqrt{153}$ is attained at $t=4$ units of time.

20. Since $\boldsymbol{r}(t)=t^3\boldsymbol{i}+t^2\boldsymbol{j}+t^3\boldsymbol{k}$, $\boldsymbol{a}(t)=\boldsymbol{r}''(t)=6t\,\boldsymbol{i}+2\,\boldsymbol{j}+6t\,\boldsymbol{k}$. By Newton's Second Law, $\boldsymbol{F}(t)=m\,\boldsymbol{a}(t)=6mt\,\boldsymbol{i}+2m\,\boldsymbol{j}+6mt\,\boldsymbol{k}$ is the required force.

21. $|\boldsymbol{F}(t)|=20$ N in the direction of the positive z –axis, so $\boldsymbol{F}(t)=20\boldsymbol{k}$. Also $m=4$ kg, $\boldsymbol{r}(0)=\boldsymbol{0}$ and

$\boldsymbol{v}(0)=\boldsymbol{i}-\boldsymbol{j}$. Since $20\boldsymbol{k}=\boldsymbol{F}(t)=4\,\boldsymbol{a}(t)$, $\boldsymbol{a}(t)=5\,\boldsymbol{k}$. Then $\boldsymbol{v}(t)=5t\,\boldsymbol{k}+\boldsymbol{c}_1$ where $\boldsymbol{c}_1=\boldsymbol{i}-\boldsymbol{j}$ so $\boldsymbol{v}(t)=\boldsymbol{i}-\boldsymbol{j}+5t\,\boldsymbol{k}$ and the speed is $|\boldsymbol{v}(t)|=\sqrt{1+1+25t^2}=\sqrt{25t^2+2}$. Also $\boldsymbol{r}(t)=t\,\boldsymbol{i}-t\,\boldsymbol{j}+\frac{5}{2}\,t^2\,\boldsymbol{k}+\boldsymbol{c}_2$ and $\boldsymbol{0}=\boldsymbol{r}(0)$, so $\boldsymbol{c}_2=\boldsymbol{0}$ and $\boldsymbol{r}(t)=t\,\boldsymbol{i}-t\,\boldsymbol{j}+\frac{5}{2}\,t^2\,\boldsymbol{k}$.

22. The argument here is the same as that in Example 14.2.5 with $\boldsymbol{r}(t)$ replaced by $\boldsymbol{v}(t)$ and $\boldsymbol{r}'(t)$ replaced
by $\boldsymbol{a}(t)$.

23. $|\boldsymbol{v}(0)|=500$ m / s and since the angle of elevation is 30° , the direction of the velocity is $\frac{1}{2}\left(\sqrt{3}\,\boldsymbol{i}+\boldsymbol{j}\right)$. Thus $\boldsymbol{v}(0)=250\left(\sqrt{3}\,\boldsymbol{i}+\boldsymbol{j}\right)$ and if we set up the axes so the projectile starts at the origin, then $\boldsymbol{r}(0)=\boldsymbol{0}$. Ignoring air resistance, the only force is that due to gravity, so $\boldsymbol{F}(t)=-mg\,\boldsymbol{j}$ where $g\approx 9.8$ m / s^2 . Thus $\boldsymbol{a}(t)=-g\,\boldsymbol{j}$ and $\boldsymbol{v}(t)=-gt\,\boldsymbol{j}+\boldsymbol{c}_1$. But $250\left(\sqrt{3}\,\boldsymbol{i}+\boldsymbol{j}\right)=\boldsymbol{v}(0)=\boldsymbol{c}_1$, so $\boldsymbol{v}(t)=250\sqrt{3}\,\boldsymbol{i}+(250-gt)\,\boldsymbol{j}$ and $\boldsymbol{r}(t)=250\sqrt{3}\,t\,\boldsymbol{i}+\left(250t-\frac{1}{2}\,gt^2\right)\boldsymbol{j}+\boldsymbol{c}_2$ where $\boldsymbol{0}=\boldsymbol{r}(0)=\boldsymbol{c}_2$. Thus $\boldsymbol{r}(t)=250\sqrt{3}\,t\,\boldsymbol{i}+\left(250t-\frac{1}{2}\,gt^2\right)\boldsymbol{j}$.

(a) Setting $250t-\frac{1}{2}\,gt^2=0$ gives $t=0$ or $t=\frac{500}{g}\approx 51.0$ s. So the range is $250\sqrt{3}\cdot\frac{500}{g}\approx 22$ km.

(b) $0=\frac{d}{dt}\left(250t-\frac{1}{2}\,gt^2\right)=250-gt$ implies that the maximum height is attained when $t=250/g\approx 25.5$ s. Thus, the maximum height is $(250)(250/g)-g(250/g)^2\,\frac{1}{2}=(250)^2/(2g)\approx 3.2$ km.

(c) From part (a), impact occurs at $t=500/g\approx 51.0$. Thus, the velocity at impact is $\boldsymbol{v}(500/g)=250\sqrt{3}\,\boldsymbol{i}+[250-g(500/g)]\,\boldsymbol{j}=250\sqrt{3}\,\boldsymbol{i}-250\,\boldsymbol{j}$ and the speed is $|\boldsymbol{v}(500/g)|=250\sqrt{3+1}=500$ m / s.

24. As in Exercise 23, $\boldsymbol{v}(t)=250\sqrt{3}\,\boldsymbol{i}+(250-gt)\,\boldsymbol{j}$ and $\boldsymbol{r}(t)=250\sqrt{3}\,t\,\boldsymbol{i}+\left(250t-\frac{1}{2}\,gt^2\right)\boldsymbol{j}+\boldsymbol{c}_2$. But $\boldsymbol{r}(0)=200\,\boldsymbol{j}$, so $\boldsymbol{c}_2=200\,\boldsymbol{j}$ and $\boldsymbol{r}(t)=250\sqrt{3}\,t\,\boldsymbol{i}+\left(200+250t-\frac{1}{2}\,gt^2\right)\boldsymbol{j}$.

(a) $200+250t-\frac{1}{2}\,gt^2=0$ implies that $gt^2-500t-400=0$ or $t=\frac{500\pm\sqrt{500^2+1600g}}{2g}$. Taking the positive t -value gives $t=\frac{500+\sqrt{250{,}000+1600g}}{2g}\approx 51.8$ s. Thus the range is

$\left(250\sqrt{3}\right)\frac{500+\sqrt{250{,}000+1600g}}{2g}\approx 22.4$ km.

(b)

$0=\frac{d}{dt}\left(200+250t-\frac{1}{2}gt^2\right)=250-gt$ implies that the maximum height is attained when $t=250/g\approx 25.5$ s and thus the maximum height is $\left[200+(250)\left(\frac{250}{g}\right)-\frac{g}{2}\left(\frac{250}{g}\right)^2\right]=200+\frac{(250)^2}{2g}\approx 3.4$ km.
Alternate solution: Because the projectile is fired in the same direction and with the same velocity as in Exercise 23, but from a point 200 m higher, the maximum height reached is 200 m higher than that found in Exercise 23, that is, 3.2 km +200 m =3.4 km.
(c) From part (a), impact occurs at $t=\frac{500+\sqrt{250{,}000+1600g}}{2g}$. Thus the velocity at impact is $250\sqrt{3}\,\boldsymbol{i}+\left[250-g\,\frac{500+\sqrt{250{,}000+1600g}}{2g}\right]\boldsymbol{j}$, so $|\boldsymbol{v}|\approx\sqrt{(250)^2(3)+(250-51.8g)^2}\approx 504$ m / s.

25. As in Example 5, $\boldsymbol{r}(t)=(v_0\cos 45°)t\,\boldsymbol{i}+\left[(v_0\sin 45°)t-\frac{1}{2}gt^2\right]\boldsymbol{j}=\frac{1}{2}\left[v_0\sqrt{2}\,t\,\boldsymbol{i}+\left(v_0\sqrt{2}\,t-gt^2\right)\boldsymbol{j}\right]$.
Then the ball lands at $t=\frac{v_0\sqrt{2}}{g}$ s. Now since it lands 90 m away, $90=\frac{1}{2}v_0\sqrt{2}\,\frac{v_0\sqrt{2}}{g}$ or $v_0^2=90g$ and the initial velocity is $v_0=\sqrt{90g}\approx 30$ m / s.

26. As in Example 5, $\boldsymbol{r}(t)=(v_0\cos 30°)t\,\boldsymbol{i}+\left[(v_0\sin 30°)t-\frac{1}{2}gt^2\right]\boldsymbol{j}=\frac{1}{2}\left[v_0\sqrt{3}\,t\,\boldsymbol{i}+(v_0t-gt^2)\boldsymbol{j}\right]$ and then $\boldsymbol{v}(t)=\boldsymbol{r}'(t)=\frac{1}{2}\left[v_0\sqrt{3}\,\boldsymbol{i}+(v_0-2gt)\boldsymbol{j}\right]$. The shell reaches its maximum height when the vertical component of velocity is zero, so $\frac{1}{2}(v_0-2gt)=0\Rightarrow t=\frac{v_0}{2g}$. The vertical height of the shell at that time is 500 m, so $\frac{1}{2}\left[v_0\left(\frac{v_0}{2g}\right)-g\left(\frac{v_0}{2g}\right)^2\right]=500\Rightarrow\frac{v_0^2}{8g}=500\Rightarrow v_0=\sqrt{4000g}=\sqrt{4000(9.8)}\approx 198$ m / s.

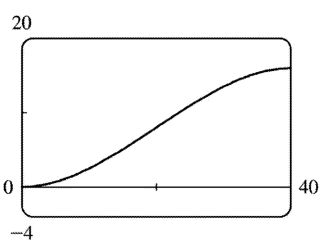
27. Let α be the angle of elevation. Then $v_0=150$ m / s and from Example 5, the horizontal distance traveled by the projectile is $d=\frac{v_0^2\sin 2\alpha}{g}$. Thus $\frac{150^2\sin 2\alpha}{g}=800\Rightarrow\sin 2\alpha=\frac{800g}{150^2}\approx 0.3484\Rightarrow$ $2\alpha\approx 20.4°$ or $180-20.4=159.6°$. Two angles of elevation then are $\alpha\approx 10.2°$ and $\alpha\approx 79.8°$.

28. Here $v_0=115$ ft / s, the angle of elevation is $\alpha=50°$, and if we place the origin at home plate, then $\boldsymbol{r}(0)=3\boldsymbol{j}$. As in Example 5, we have

$\boldsymbol{r}(t)=-\frac{1}{2}gt^2\boldsymbol{j}+t\boldsymbol{v}_0+\boldsymbol{D}$ where $\boldsymbol{D}=\boldsymbol{r}(0)=3\boldsymbol{j}$ and $\boldsymbol{v}_0=v_0\cos\alpha\,\boldsymbol{i}+v_0\sin\alpha\,\boldsymbol{j}$, so $\boldsymbol{r}(t)=(v_0\cos\alpha)t\,\boldsymbol{i}+\left[(v_0\sin\alpha)t-\frac{1}{2}gt^2+3\right]\boldsymbol{j}$. Thus, parametric equations for the trajectory of the ball are $x=(v_0\cos\alpha)t$, $y=(v_0\sin\alpha)t-\frac{1}{2}gt^2+3$. The ball reaches the fence when $x=400\Rightarrow(v_0\cos\alpha)t=400\Rightarrow$ $t=\frac{400}{v_0\cos\alpha}=\frac{400}{115\cos 50^\circ}\approx 5.41$ s. At this time, the height of the ball is

$y=(v_0\sin\alpha)t-\frac{1}{2}gt^2+3\approx(115\sin 50^\circ)(5.41)-\frac{1}{2}(32)(5.41)^2+3\approx 11.2$ ft. Since the fence is 10 ft high, the ball clears the fence.

29. **(a)** After t seconds, the boat will be $5t$ meters west of point A. The velocity of the water at that location is $\frac{3}{400}(5t)(40-5t)\boldsymbol{j}$. The velocity of the boat in still water is $5\boldsymbol{i}$, so the resultant velocity of the boat is $\boldsymbol{v}(t)=5\boldsymbol{i}+\frac{3}{400}(5t)(40-5t)\boldsymbol{j}=5\boldsymbol{i}+\left(\frac{3}{2}t-\frac{3}{16}t^2\right)\boldsymbol{j}$. Integrating, we obtain $\boldsymbol{r}(t)=5t\,\boldsymbol{i}+\left(\frac{3}{4}t^2-\frac{1}{16}t^3\right)\boldsymbol{j}+\boldsymbol{C}$.

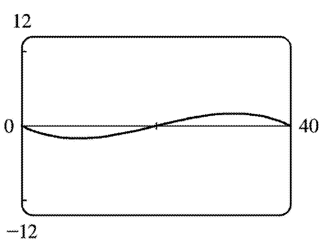


If we place the origin at A (and consider $\boldsymbol{j}$ to coincide with the northern direction) then $\boldsymbol{r}(0)=\boldsymbol{0}\Rightarrow\boldsymbol{C}=\boldsymbol{0}$ and we have $\boldsymbol{r}(t)=5t\,\boldsymbol{i}+\left(\frac{3}{4}t^2-\frac{1}{16}t^3\right)\boldsymbol{j}$. The boat reaches the east bank after 8 s, and it is located at $\boldsymbol{r}(8)=5(8)\boldsymbol{i}+\left(\frac{3}{4}(8)^2-\frac{1}{16}(8)^3\right)\boldsymbol{j}=40\boldsymbol{i}+16\boldsymbol{j}$. Thus the boat is 16 m downstream.

(b) Let α be the angle north of east that the boat heads. Then the velocity of the boat in still water is given by $5(\cos\alpha)\boldsymbol{i}+5(\sin\alpha)\boldsymbol{j}$. At t seconds, the boat is $5(\cos\alpha)t$ meters from the west bank, at which point the velocity of the water is $\frac{3}{400}[5(\cos\alpha)t][40-5(\cos\alpha)t]\,\boldsymbol{j}$. The resultant velocity of the boat is given by

$$\boldsymbol{v}(t)=5(\cos\alpha)\boldsymbol{i}+\left[5\sin\alpha+\frac{3}{400}(5t\cos\alpha)(40-5t\cos\alpha)\right]\boldsymbol{j}$$
$$=(5\cos\alpha)\boldsymbol{i}+\left(5\sin\alpha+\frac{3}{2}t\cos\alpha-\frac{3}{16}t^2\cos^2\alpha\right)\boldsymbol{j}$$

Integrating, $\boldsymbol{r}(t)=(5t\cos\alpha)\boldsymbol{i}+\left(5t\sin\alpha+\frac{3}{4}t^2\cos\alpha-\frac{1}{16}t^3\cos^2\alpha\right)\boldsymbol{j}$ (where we have again placed the origin at A). The boat will reach the east bank when $5t\cos\alpha=40\Rightarrow t=\frac{40}{5\cos\alpha}=\frac{8}{\cos\alpha}$.
In order to land at point $B(40,0)$ we need $5t\sin\alpha+\frac{3}{4}t^2\cos\alpha-\frac{1}{16}t^3\cos^2\alpha=0\Rightarrow$
$5\left(\frac{8}{\cos\alpha}\right)\sin\alpha+\frac{3}{4}\left(\frac{8}{\cos\alpha}\right)^2\cos\alpha-\frac{1}{16}\left(\frac{8}{\cos\alpha}\right)^3\cos^2\alpha=0\Rightarrow\frac{1}{\cos\alpha}(40\sin\alpha+48-32)=0\Rightarrow$
$40\sin\alpha+16=0\Rightarrow\sin\alpha=-\frac{2}{5}$. Thus $\alpha=\sin^{-1}\left(-\frac{2}{5}\right)\approx-23.6^\circ$, so the boat should head 23.6° south of east (upstream).
The path does seem realistic. The boat initially heads upstream to counteract the effect of the current. Near the center of the river, the current is stronger and the boat is pushed downstream. When the boat nears the eastern bank, the current is slower and the boat is able to progress upstream to arrive at point B.



30. As in Exercise 29(b), let α be the angle north of east that the boat heads, so the velocity of the boat in still water is given by $5(\cos\alpha)\boldsymbol{i}+5(\sin\alpha)\boldsymbol{j}$. At t seconds, the boat is $5(\cos\alpha)t$ meters from the west bank, at which point the velocity of the water is
$3\sin(\pi x/40)\boldsymbol{j}=3\sin[\pi\cdot5(\cos\alpha)t/40]\boldsymbol{j}=3\sin\left(\frac{\pi}{8}t\cos\alpha\right)\boldsymbol{j}$. The resultant velocity of the boat then is given by $\boldsymbol{v}(t)=5(\cos\alpha)\boldsymbol{i}+\left[5\sin\alpha+3\sin\left(\frac{\pi}{8}t\cos\alpha\right)\right]\boldsymbol{j}$. Integrating,
$\boldsymbol{r}(t)=(5t\cos\alpha)\boldsymbol{i}+\left[5t\sin\alpha-\frac{24}{\pi\cos\alpha}\cos\left(\frac{\pi}{8}t\cos\alpha\right)\right]\boldsymbol{j}+\boldsymbol{C}$.
If we place the origin at A then $\boldsymbol{r}(0)=\boldsymbol{0}\Rightarrow-\frac{24}{\pi\cos\alpha}\boldsymbol{j}+\boldsymbol{C}=\boldsymbol{0}\Rightarrow\boldsymbol{C}=\frac{24}{\pi\cos\alpha}\boldsymbol{j}$ and
$\boldsymbol{r}(t)=(5t\cos\alpha)\boldsymbol{i}+\left[5t\sin\alpha-\frac{24}{\pi\cos\alpha}\cos\left(\frac{\pi}{8}t\cos\alpha\right)+\frac{24}{\pi\cos\alpha}\right]\boldsymbol{j}$.

The boat will reach the east bank when $5t\cos\alpha=40\Rightarrow t=\dfrac{8}{\cos\alpha}$. In order to land at point $B(40,0)$ we need $5t\sin\alpha-\dfrac{24}{\pi\cos\alpha}\cos\left(\dfrac{\pi}{8}t\cos\alpha\right)+\dfrac{24}{\pi\cos\alpha}=0\Rightarrow$

$5\left(\dfrac{8}{\cos\alpha}\right)\sin\alpha-\dfrac{24}{\pi\cos\alpha}\cos\left[\dfrac{\pi}{8}\left(\dfrac{8}{\cos\alpha}\right)\cos\alpha\right]+\dfrac{24}{\pi\cos\alpha}=0\Rightarrow$

$\dfrac{1}{\cos\alpha}\left(40\sin\alpha-\dfrac{24}{\pi}\cos\pi+\dfrac{24}{\pi}\right)=0\Rightarrow 40\sin\alpha+\dfrac{48}{\pi}=0\Rightarrow\sin\alpha=-\dfrac{6}{5\pi}$. Thus

$\alpha=\sin^{-1}\left(-\dfrac{6}{5\pi}\right)\approx-22.5^\circ$, so the boat should head 22.5° south of east.

31. $\boldsymbol{r}(t)=(3t-t^3)\boldsymbol{i}+3t^2\boldsymbol{j}\Rightarrow\boldsymbol{r}'(t)=(3-3t^2)\boldsymbol{i}+6t\,\boldsymbol{j}$,

$|\boldsymbol{r}'(t)|=\sqrt{(3-3t^2)^2+(6t)^2}=\sqrt{9+18t^2+9t^4}=\sqrt{(3-3t^2)^2}=3+3t^2$, $\boldsymbol{r}''(t)=-6t\,\boldsymbol{i}+6\,\boldsymbol{j}$,

$\boldsymbol{r}'(t)\times\boldsymbol{r}''(t)=(18+18t^2)\boldsymbol{k}$. Then Equation 9 gives

$a_T=\dfrac{\boldsymbol{r}'(t)\cdot\boldsymbol{r}''(t)}{|\boldsymbol{r}'(t)|}=\dfrac{(3-3t^2)(-6t)+(6t)(6)}{3+3t^2}=\dfrac{18t+18t^3}{3+3t^2}=\dfrac{18t(1+t^2)}{3(1+t^2)}=6t$ [or by Equation 8,

$a_T=v'=\dfrac{d}{dt}\left[3+3t^2\right]=6t$] and Equation 10 gives $a_N=\dfrac{|\boldsymbol{r}'(t)\times\boldsymbol{r}''(t)|}{|\boldsymbol{r}'(t)|}=\dfrac{18+18t^2}{3+3t^2}=\dfrac{18(1+t^2)}{3(1+t^2)}=6$.

32. $\boldsymbol{r}(t)=(1+t)\boldsymbol{i}+(t^2-2t)\boldsymbol{j}\Rightarrow\boldsymbol{r}'(t)=\boldsymbol{i}+(2t-2)\boldsymbol{j}$, $|\boldsymbol{r}'(t)|=\sqrt{1^2+(2t-2)^2}=\sqrt{4t^2-8t+5}$, $\boldsymbol{r}''(t)=2\boldsymbol{j}$,

$\boldsymbol{r}'(t)\times\boldsymbol{r}''(t)=2\boldsymbol{k}$. Then Equation 9 gives $a_T=\dfrac{\boldsymbol{r}'(t)\cdot\boldsymbol{r}''(t)}{|\boldsymbol{r}'(t)|}=\dfrac{2(2t-2)}{\sqrt{4t^2-8t+5}}$ and Equation 10 gives

$a_N=\dfrac{|\boldsymbol{r}'(t)\times\boldsymbol{r}''(t)|}{|\boldsymbol{r}'(t)|}=\dfrac{2}{\sqrt{4t^2-8t+5}}$.

33. $\boldsymbol{r}(t)=\cos t\,\boldsymbol{i}+\sin t\,\boldsymbol{j}+t\,\boldsymbol{k}\Rightarrow\boldsymbol{r}'(t)=-\sin t\,\boldsymbol{i}+\cos t\,\boldsymbol{j}+\boldsymbol{k}$, $|\boldsymbol{r}'(t)|=\sqrt{\sin^2t+\cos^2t+1}=\sqrt{2}$,

$\boldsymbol{r}''(t)=-\cos t\,\boldsymbol{i}-\sin t\,\boldsymbol{j}$, $\boldsymbol{r}'(t)\times\boldsymbol{r}''(t)=\sin t\,\boldsymbol{i}-\cos t\,\boldsymbol{j}+\boldsymbol{k}$. Then

$a_T=\dfrac{\boldsymbol{r}'(t)\cdot\boldsymbol{r}''(t)}{|\boldsymbol{r}'(t)|}=\dfrac{\sin t\cos t-\sin t\cos t}{\sqrt{2}}=0$ and $a_N=\dfrac{|\boldsymbol{r}'(t)\times\boldsymbol{r}''(t)|}{|\boldsymbol{r}'(t)|}=\dfrac{\sqrt{\sin^2t+\cos^2t+1}}{\sqrt{2}}=\dfrac{\sqrt{2}}{\sqrt{2}}=1$.

34. $\boldsymbol{r}(t)=t\,\boldsymbol{i}+t^2\boldsymbol{j}+3t\,\boldsymbol{k}\Rightarrow\boldsymbol{r}'(t)=\boldsymbol{i}+2t\,\boldsymbol{j}+3\boldsymbol{k}$, $|\boldsymbol{r}'(t)|=\sqrt{1^2+(2t)^2+3^2}=\sqrt{4t^2+10}$, $\boldsymbol{r}''(t)=2\boldsymbol{j}$,

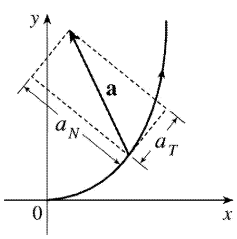
$\boldsymbol{r}'(t)\times\boldsymbol{r}''(t)=-6\boldsymbol{i}+2\boldsymbol{k}$. Then $a_T=\dfrac{\boldsymbol{r}'(t)\cdot\boldsymbol{r}''(t)}{\left|\boldsymbol{r}'(t)\right|}=\dfrac{4t}{\sqrt{4t^2+10}}$ and $a_N=\dfrac{\left|\boldsymbol{r}'(t)\times\boldsymbol{r}''(t)\right|}{\left|\boldsymbol{r}'(t)\right|}=\dfrac{2\sqrt{10}}{\sqrt{4t^2+10}}$.

35. $\boldsymbol{r}(t)=e^t\boldsymbol{i}+\sqrt{2}\,t\,\boldsymbol{j}+e^{-t}\boldsymbol{k}\Rightarrow\boldsymbol{r}'(t)=e^t\boldsymbol{i}+\sqrt{2}\,\boldsymbol{j}-e^{-t}\boldsymbol{k}$, $|\boldsymbol{r}(t)|=\sqrt{e^{2t}+2+e^{-2t}}=\sqrt{(e^t+e^{-t})^2}=e^t+e^{-t}$,
$\boldsymbol{r}''(t)=e^t\boldsymbol{i}+e^{-t}\boldsymbol{k}$. Then $a_T=\dfrac{e^{2t}-e^{-2t}}{e^t+e^{-t}}=\dfrac{(e^t+e^{-t})(e^t-e^{-t})}{e^t+e^{-t}}=e^t-e^{-t}=2\sinh t$ and

$$a_N=\frac{\left|\sqrt{2}\,e^{-t}\boldsymbol{i}-2\boldsymbol{j}-\sqrt{2}\,e^t\boldsymbol{k}\right|}{e^t+e^{-t}}=\frac{\sqrt{2(e^{-2t}+2+e^{2t})}}{e^t+e^{-t}}=\sqrt{2}\;\frac{e^t+e^{-t}}{e^t+e^{-t}}=\sqrt{2}\ .$$

36. $\boldsymbol{r}(t)=t\,\boldsymbol{i}+\cos^2t\,\boldsymbol{j}+\sin^2t\,\boldsymbol{k}\Rightarrow\boldsymbol{r}'(t)=\boldsymbol{i}-2\cos t\sin t\,\boldsymbol{j}+2\sin t\cos t\,\boldsymbol{k}=\boldsymbol{i}-\sin 2t\,\boldsymbol{j}+\sin 2t\,\boldsymbol{k}$,
$\left|\boldsymbol{r}'(t)\right|=\sqrt{1+2\sin^2 2t}$, $\boldsymbol{r}''(t)=2(\sin^2t-\cos^2t)\,\boldsymbol{j}+2(\cos^2t-\sin^2t)\,\boldsymbol{k}=-2\cos 2t\,\boldsymbol{j}+2\cos 2t\,\boldsymbol{k}$. So
$a_T=\dfrac{2\sin 2t\cos 2t+2\sin 2t\cos 2t}{\sqrt{1+2\sin^2 2t}}=\dfrac{4\sin 2t\cos 2t}{\sqrt{1+2\sin^2 2t}}$ and $a_N=\dfrac{|-2\cos 2t\,\boldsymbol{j}-2\cos 2t\,\boldsymbol{k}|}{\sqrt{1+2\sin^2 2t}}=\dfrac{2\sqrt{2}\,|\cos 2t|}{\sqrt{1+2\sin^2 t}}$.

37. The tangential component of $\boldsymbol{a}$ is the length of the projection of $\boldsymbol{a}$ onto $\boldsymbol{T}$, so we sketch the scalar projection of $\boldsymbol{a}$ in the tangential direction to the curve and estimate its length to be 4.5 (using the fact that $\boldsymbol{a}$ has length 10 as a guide). Similarly, the normal component of $\boldsymbol{a}$ is the length of the projection of $\boldsymbol{a}$ onto $\boldsymbol{N}$, so we sketch the scalar projection of $\boldsymbol{a}$ in the normal direction to the curve and estimate its length to be 9.0 . Thus $a_T\approx 4.5\text{ cm / s}^2$ and $a_N\approx 9.0\text{ cm / s}^2$.



38. $\boldsymbol{L}(t)=m\boldsymbol{r}(t)\times\boldsymbol{v}(t)\Rightarrow$

$$\boldsymbol{L}'(t)=m[\boldsymbol{r}'(t)\times\boldsymbol{v}(t)+\boldsymbol{r}(t)\times\boldsymbol{v}'(t)]\text{[by Theorem 14.2.3 [ET 13.2.3]\#5]}$$
$$=m[\boldsymbol{v}(t)\times\boldsymbol{v}(t)+\boldsymbol{r}(t)\times\boldsymbol{v}'(t)]=m[\boldsymbol{0}+\boldsymbol{r}(t)\times\boldsymbol{a}(t)]=\boldsymbol{\tau}(t)$$

So if the torque is always $\boldsymbol{0}$, then $\boldsymbol{L}'(t)=\boldsymbol{0}$ for all t , and so $\boldsymbol{L}(t)$ is constant.

39. If the engines are turned off at time t , then the spacecraft will continue to travel in the direction of

$\boldsymbol{v}(t)$, so we need a t such that for some scalar $s>0$, $\boldsymbol{r}(t)+s\,\boldsymbol{v}(t)=\langle 6,4,9\rangle$. $\boldsymbol{v}(t)=\boldsymbol{r}'(t)=\boldsymbol{i}+\frac{1}{t}\,\boldsymbol{j}+\frac{8t}{(t^2+1)^2}\,\boldsymbol{k}$

$\Rightarrow \boldsymbol{r}(t)+s\,\boldsymbol{v}(t)=\left\langle 3+t+s, 2+\ln t+\frac{s}{t}, 7-\frac{4}{t^2+1}+\frac{8st}{(t^2+1)^2}\right\rangle \Rightarrow 3+t+s=6 \Rightarrow s=3-t$, so $7-\frac{4}{t^2+1}+\frac{8(3-t)t}{(t^2+1)^2}=9 \Leftrightarrow$

$\frac{24t-12t^2-4}{(t^2+1)^2}=2 \Leftrightarrow t^4+8t^2-12t+3=0$. It is easily seen that $t=1$ is a root of this polynomial. Also

$2+\ln 1+\frac{3-1}{1}=4$, so $t=1$ is the desired solution.

40. **(a)** $m\,\frac{d\boldsymbol{v}}{dt}=\frac{dm}{dt}\,\boldsymbol{v}_e \Leftrightarrow \frac{d\boldsymbol{v}}{dt}=\frac{1}{m}\,\frac{dm}{dt}\,\boldsymbol{v}_e$. Integrating both sides of this equation with respect to t gives

$\int_0^t \frac{d\boldsymbol{v}}{du}\,du=\boldsymbol{v}_e\int_0^t \frac{1}{m}\,\frac{dm}{du}\,du \Rightarrow \int_{\boldsymbol{v}(0)}^{\boldsymbol{v}(t)} d\boldsymbol{v}=\boldsymbol{v}_e\int_{m(0)}^{m(t)} \frac{dm}{m} \Rightarrow \boldsymbol{v}(t)-\boldsymbol{v}(0)=\ln\left(\frac{m(t)}{m(0)}\right)\boldsymbol{v}_e \Rightarrow$

$\boldsymbol{v}(t)=\boldsymbol{v}(0)-\ln\left(\frac{m(0)}{m(t)}\right)\boldsymbol{v}_e$.

(b) $|\boldsymbol{v}(t)|=2\left|\boldsymbol{v}_e\right|$, and $|\boldsymbol{v}(0)|=0$. Therefore, by part (a), $2\left|\boldsymbol{v}_e\right|=\left|-\ln\left(\frac{m(0)}{m(t)}\right)\boldsymbol{v}_e\right| \Rightarrow$

$2\left|\boldsymbol{v}_e\right|=\ln\left(\frac{m(0)}{m(t)}\right)\left|\boldsymbol{v}_e\right|$. $\left[\text{Note:}\,m(0)>m(t)\text{ so that }\ln\left(\frac{m(0)}{m(t)}\right)>0\right] \Rightarrow m(t)=e^{-2}m(0)$.

Thus $\frac{m(0)-e^{-2}m(0)}{m(0)}=1-e^{-2}$ is the fraction of the initial mass that is burned as fuel.

1. **(a)** From Table 1, $f(-15,40)=-27$, which means that if the temperature is $-15°$ C and the wind speed is 40 km / h, then the air would feel equivalent to approximately $-27°$ C without wind.
(b) The question is asking: when the temperature is $-20°$ C , what wind speed gives a wind–chill index of $-30°$ C ? From Table 1, the speed is 20 km / h.
(c) The question is asking: when the wind speed is 20 km / h, what temperature gives a wind–chill index of $-49°$ C ? From Table 1, the temperature is $-35°$ C .
(d) The function $W=f(-5,v)$ means that we fix T at -5 and allow v to vary, resulting in a function of one variable. In other words, the function gives wind–chill index values for different wind speeds when the temperature is $-5°$ C . From Table 1 (look at the row corresponding to $T=-5$), the function decreases and appears to approach a constant value as v increases.
(e) The function $W=f(T,50)$ means that we fix v at 50 and allow T to vary, again giving a function of one variable. In other words, the function gives wind–chill index values for different temperatures when the wind speed is 50 km / h . From Table 1 (look at the column corresponding to $v=50$), the function increases almost linearly as T increases.

2. **(a)** From the table, $f(95,70)=124$, which means that when the actual temperature is 95 and the relative humidity is 70% , the perceived air temperature is approximately 124° F .
(b) Looking at the row corresponding to $T=90$, we see that $f(90,h)=100$ when $h=60$.
(c) Looking at the column corresponding to $h=50$, we see that $f(T,50)=88$ when $T=85$.
(d) $I=f(80,h)$ means that T is fixed at 80 and h is allowed to vary, resulting in a function of h that gives the humidex values for different relative humidities when the actual temperature is 80° F . Similarly, $I=f(100,h)$ is a function of one variable that gives the humidex values for different relative humidities when the actual temperature is 100° F . Looking at the rows of the table corresponding to $T=80$ and $T=100$, we see that $f(80,h)$ increases at a relatively constant rate of approximately 1° F per 10% relative humidity, while $f(100,h)$ increases more quickly (at first with an average rate of change of 5° F per 10% relative humidity) and at an increasing rate (approximately 12° F per 10% relative humidity for larger values of h).

3. If the amounts of labor and capital are both doubled, we replace L,K in the function with 2L,2K , giving

$$P(2L,2K) = 1.01(2L)^{0.75}(2K)^{0.25} = 1.01(2^{0.75})(2^{0.25})L^{0.75}K^{0.25} = (2^{1})1.01L^{0.75}K^{0.25}$$
$$= 2P(L,K)$$

Thus, the production is doubled. It is also true for the general case $P(L,K)=bL^{\alpha}K^{1-\alpha}$:
$P(2L,2K)=b(2L)^{\alpha}(2K)^{1-\alpha}=b(2^{\alpha})(2^{1-\alpha})L^{\alpha}K^{1-\alpha}=(2^{\alpha+1-\alpha})bL^{\alpha}K^{1-\alpha}=2P(L,K)$.

4. We compare the values for the wind–chill index given by Table 1 with those given by the model function:

Modeled Wind–Chill Index Values $W(T,v)$

Wind Speed (km/h)

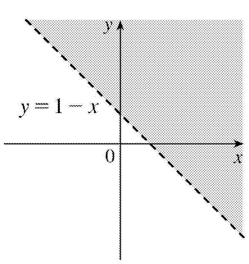
Actual temperature (°C)

T \ V	5	10	15	20	25	30	40	50	60	70	80
5	4.08	2.66	1.74	1.07	0.52	0.05	−0.71	−1.33	−1.85	−2.30	−2.70
0	−1.59	−3.31	−4.42	−5.24	−5.91	−6.47	−7.40	−8.14	−8.77	−9.32	−9.80
−5	−7.26	−9.29	−10.58	−11.55	−12.34	−13.00	−14.08	−14.96	−15.70	−16.34	−16.91
−10	−12.93	−15.26	−16.75	−17.86	−18.76	−19.52	−20.77	−21.77	−22.62	−23.36	−24.01
−15	−18.61	−21.23	−22.91	−24.17	−25.19	−26.04	−27.45	−28.59	−29.54	−30.38	−31.11
−20	−24.28	−27.21	−29.08	−30.48	−31.61	−32.57	−34.13	−35.40	−36.47	−37.40	−38.22
−25	−29.95	−33.18	−35.24	−36.79	−38.04	−39.09	−40.82	−42.22	−43.39	−44.42	−45.32
−30	−35.62	−39.15	−41.41	−43.10	−44.46	−45.62	−47.50	−49.03	−50.32	−51.44	−52.43
−35	−41.30	−45.13	−47.57	−49.41	−50.89	−52.14	−54.19	−55.84	−57.24	−58.46	−59.53
−40	−46.97	−51.10	−53.74	−55.72	−57.31	−58.66	−60.87	−62.66	−64.17	−65.48	−66.64

The values given by the function appear to be fairly close (within 0.5) to the values in Table 1.

5. **(a)** According to the table, $f(40,15)=25$, which means that if a 40 –knot wind has been blowing in the open sea for 15 hours, it will create waves with estimated heights of 25 feet.
(b) $h=f(30,t)$ means we fix v at 30 and allow t to vary, resulting in a function of one variable. Thus here, $h=f(30,t)$ gives the wave heights produced by 30 –knot winds blowing for t hours. From the table (look at the row corresponding to $v=30$), the function increases but at a declining rate as t increases. In fact, the function values appear to be approaching a limiting value of approximately 19 , which suggests that 30 –knot winds cannot produce waves higher than about 19 feet.
(c) $h=f(v,30)$ means we fix t at 30 , again giving a function of one variable. So, $h=f(v,30)$ gives the wave heights produced by winds of speed v blowing for 30 hours. From the table (look at the column corresponding to $t=30$), the function appears to increase at an increasing rate, with no apparent limiting value. This suggests that faster winds (lasting 30 hours) always create higher waves.

6. **(a)** $f(1,1)=\ln(1+1-1)=\ln 1=0$
(b) $f(e,1)=\ln(e+1-1)=\ln e=1$
(c) $\ln(x+y-1)$ is defined only when $x+y-1>0$, that is, $y>1-x$. So the domain of f is $\{(x,y)\mid y>1-x\}$.



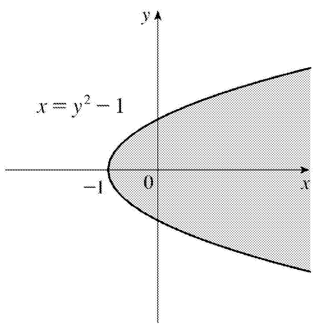
(d) Since $\ln(x+y-1)$ can be any real number, the range is R .

7. **(a)** $f(2,0)=2^2 e^{3(2)(0)}=4(1)=4$

(b) Since both x^2 and the exponential function are defined everywhere, $x^2 e^{3xy}$ is defined for all choices of values for x and y . Thus, the domain of f is R^2 .

(c) Because the range of $g(x,y)=3xy$ is R , and the range of e^x is $(0,\infty)$, the range of $e^{g(x,y)}=e^{3xy}$ is $(0,\infty)$. The range of x^2 is $[0,\infty)$, so the range of the product $x^2 e^{3xy}$ is $[0,\infty)$.

8. $\sqrt{1+x-y^2}$ is defined only when $1+x-y^2 \geq 0 \Rightarrow x \geq y^2-1$, so the domain of f is $\{(x,y) \mid x \geq y^2-1\}$, all those points on or to the right of the parabola $x=y^2-1$.
The range of f is $[0,\infty)$.



9. **(a)** $f(2,-1,6)=e^{\sqrt{6-2^2-(-1)^2}}=e^{\sqrt{1}}=e$.

(b) $e^{\sqrt{z-x^2-y^2}}$ is defined when $z-x^2-y^2 \geq 0 \Rightarrow z \geq x^2+y^2$. Thus the domain of f is $\{(x,y,z) \mid z \geq x^2+y^2\}$.

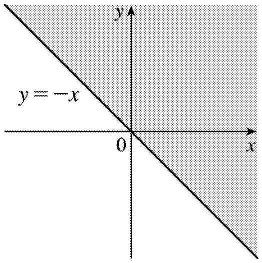
(c) Since $\sqrt{z-x^2-y^2} \geq 0$, we have $e^{\sqrt{z-x^2-y^2}} \geq 1$. Thus the range of f is $[1,\infty)$.

10. **(a)** $g(2,-2,4)=\ln\left(25-2^2-(-2)^2-4^2\right)=\ln 1=0.$

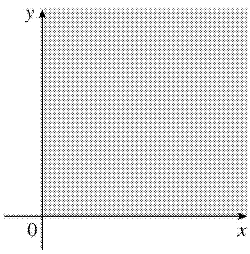
(b) For the logarithmic function to be defined, we need $25-x^2-y^2-z^2>0$. Thus the domain of g is $\{(x,y,z) \mid x^2+y^2+z^2<25\}$, the interior of the sphere $x^2+y^2+z^2=25$.

(c) Since $0<25-x^2-y^2-z^2 \leq 25$ for (x,y,z) in the domain of g , $\ln\left(25-x^2-y^2-z^2\right) \leq \ln 25$. Thus the range of g is $(-\infty,\ln 25]$.

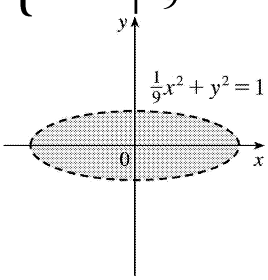
11. $\sqrt{x+y}$ is defined only when $x+y \geq 0$, or $y \geq -x$. So the domain of f is $\{(x,y) \mid y \geq -x\}$.



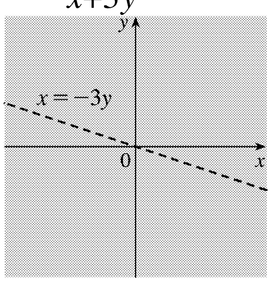
12. We need $x \geq 0$ and $y \geq 0$, so $D = \{ (x,y) \mid x \geq 0 \text{ and } y \geq 0 \}$, the first quadrant.



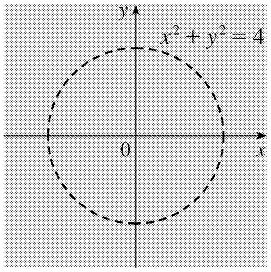
13. $\ln \left(9 - x^2 - 9y^2 \right)$ is defined only when $9 - x^2 - 9y^2 > 0$, or $\frac{1}{9} x^2 + y^2 < 1$. So the domain of f is $\left\{ (x,y) \,\middle|\, \frac{1}{9} x^2 + y^2 < 1 \right\}$, the interior of an ellipse.



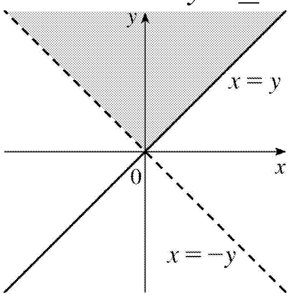
14. $\frac{x-3y}{x+3y}$ is defined only when $x+3y \neq 0$, or $x \neq -3y$. So the domain of f is $\{ (x,y) \mid x \neq -3y \}$.



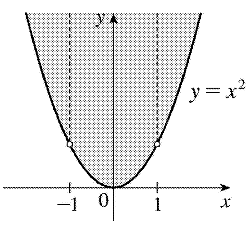
15. $\frac{3x+5y}{x^2+y^2-4}$ is defined only when $x^2+y^2-4 \neq 0$, or $x^2+y^2 \neq 4$. So the domain of f is $\left\{ (x,y) \mid x^2+y^2 \neq 4 \right\}$.



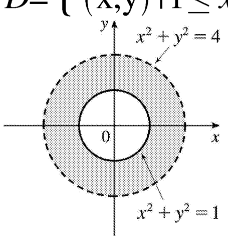
16. We need $y-x\geq 0$ or $y\geq x$ and $y+x>0$ or $x>-y$. Thus $D=\{(x,y)\mid -y<x\leq y, y>0\}$.



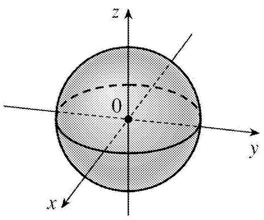
17. $\sqrt{y-x^2}$ is defined only when $y-x^2\geq 0$, or $y\geq x^2$. In addition, f is not defined if $1-x^2=0\Rightarrow x=\pm 1$. Thus the domain of f is $\left\{(x,y)\mid y\geq x^2, x\neq \pm 1\right\}$.



18. f is defined only when $x^2+y^2-1\geq 0\Rightarrow x^2+y^2\geq 1$ and $4-x^2-y^2>0\Rightarrow x^2+y^2<4$. Thus $D=\left\{(x,y)\mid 1\leq x^2+y^2<4\right\}$.

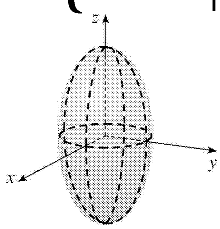


19. We need $1-x^2-y^2-z^2\geq 0$ or $x^2+y^2+z^2\leq 1$, so $D=\left\{(x,y,z)\mid x^2+y^2+z^2\leq 1\right\}$ (the points inside or on the sphere of radius 1 , center the origin).

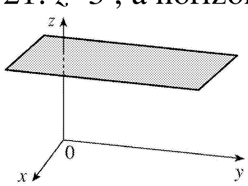


20. f is defined only when $16-4x^2-4y^2-z^2>0 \Rightarrow \frac{x^2}{4}+\frac{y^2}{4}+\frac{z^2}{16}<1$. Thus,

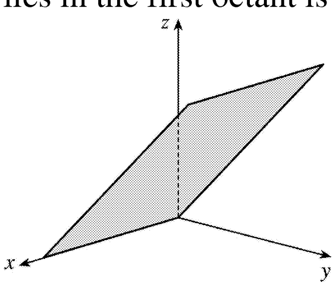
$D=\left\{ (x,y,z) \;\middle|\; \frac{x^2}{4}+\frac{y^2}{4}+\frac{z^2}{16}<1 \right\}$, that is, the points inside the ellipsoid $\frac{x^2}{4}+\frac{y^2}{4}+\frac{z^2}{16}=1$.



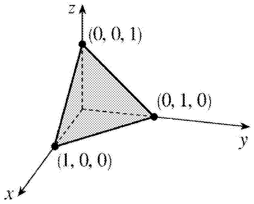
21. $z=3$, a horizontal plane through the point $(0,0,3)$.



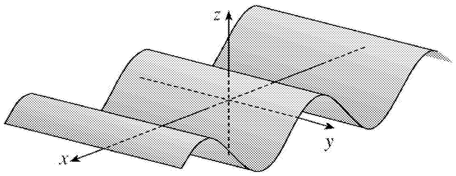
22. $z=y$, a plane which intersects the yz –plane in the line $z=y$, $x=0$. The portion of this plane that lies in the first octant is shown.



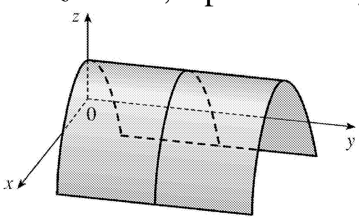
23. $z=1-x-y$ or $x+y+z=1$, a plane with intercepts 1 , 1 , and 1 .



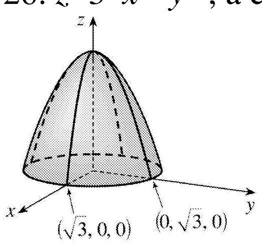
24. $z=\cos x$, a "wave."



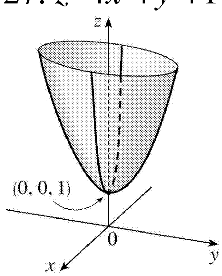
25. $z=1-x^2$, a parabolic cylinder.



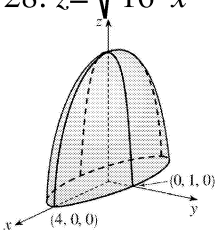
26. $z=3-x^2-y^2$, a circular paraboloid with vertex at $(0,0,3)$.



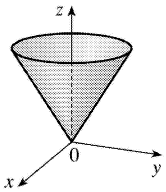
27. $z=4x^2+y^2+1$, an elliptic paraboloid with vertex at $(0,0,1)$.



28. $z=\sqrt{16-x^2-16y^2}$ so $z\geq 0$ and $z^2+x^2+16y^2=16$, the top half of an ellipsoid.



29. $z=\sqrt{x^2+y^2}$ so $x^2+y^2=z^2$ and $z\geq 0$, the top half of a right circular cone.



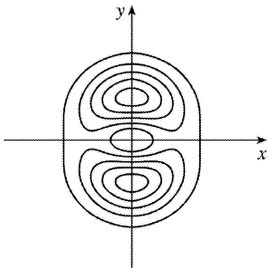
30. All six graphs have different traces in the planes $x=0$ and $y=0$, so we investigate these for each function.
(a) $f(x,y)=|x|+|y|$. The trace in $x=0$ is $z=|y|$, and in $y=0$ is $z=|x|$, so it must be graph VI.
(b) $f(x,y)=|xy|$. The trace in $x=0$ is $z=0$, and in $y=0$ is $z=0$, so it must be graph V.
(c) $f(x,y)=\dfrac{1}{1+x^2+y^2}$. The trace in $x=0$ is $z=\dfrac{1}{1+y^2}$, and in $y=0$ is $z=\dfrac{1}{1+x^2}$. In addition, we can see that f is close to 0 for large values of x and y , so this is graph I.
(d) $f(x,y)=(x^2-y^2)^2$. The trace in $x=0$ is $z=y^4$, and in $y=0$ is $z=x^4$. Both graph II and graph IV seem plausible; notice the trace in $z=0$ is $0=(x^2-y^2)^2 \Rightarrow y=\pm x$, so it must be graph IV.
(e) $f(x,y)=(x-y)^2$. The trace in $x=0$ is $z=y^2$, and in $y=0$ is $z=x^2$. Both graph II and graph IV seem plausible; notice the trace in $z=0$ is $0=(x-y)^2 \Rightarrow y=x$, so it must be graph II.
(f) $f(x,y)=\sin(|x|+|y|)$. The trace in $x=0$ is $z=\sin|y|$, and in $y=0$ is $z=\sin|x|$. In addition, notice that the oscillating nature of the graph is characteristic of trigonometric functions. So this is graph III.

31. The point $(-3,3)$ lies between the level curves with z –values 50 and 60 . Since the point is a little closer to the level curve with $z=60$, we estimate that $f(-3,3)\approx 56$. The point $(3,-2)$ appears to be just about halfway between the level curves with z –values 30 and 40 , so we estimate $f(3,-2)\approx 35$. The graph rises as we approach the origin, gradually from above, steeply from below.

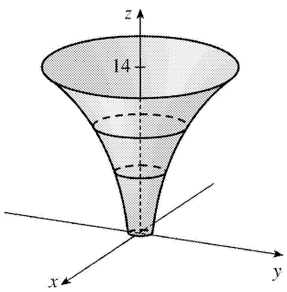
32. If we start at the origin and move along the x –axis, for example, the z –values of a cone centered at the origin increase at a constant rate, so we would expect its level curves to be equally spaced. A paraboloid with vertex the origin, on the other hand, has z –values which change slowly near the origin and more quickly as we move farther away. Thus, we would expect its level curves near the origin to be spaced more widely apart than those farther from the origin. Therefore contour map I must correspond to the paraboloid, and contour map II the cone.

33. Near A , the level curves are very close together, indicating that the terrain is quite steep. At B , the level curves are much farther apart, so we would expect the terrain to be much less steep than near A , perhaps almost flat.

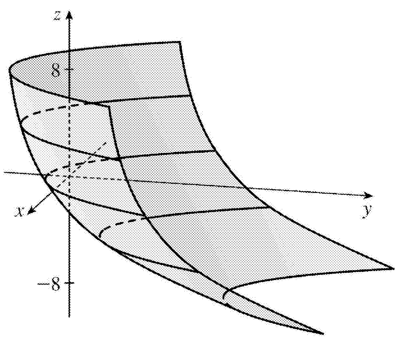
34.



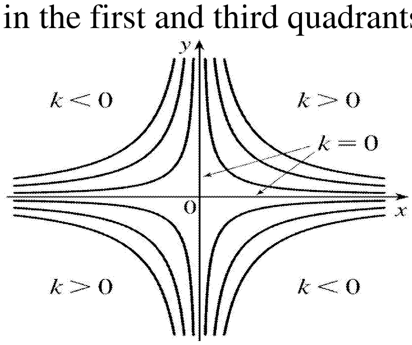
35.



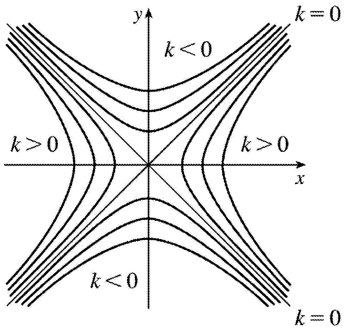
36.



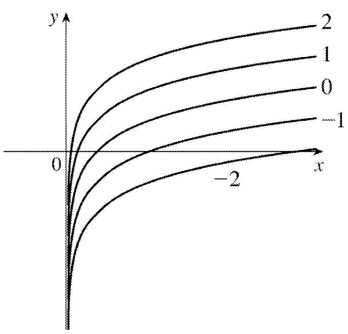
37. The level curves are $xy=k$. For $k=0$ the curves are the coordinate axis; if $k>0$, they are hyperbolas in the first and third quadrants; if $k<0$, they are hyperbolas in the second and fourth quadrants.



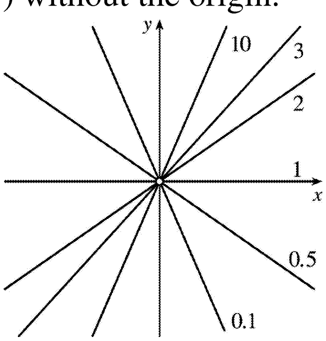
38. The level curves are $k=x^2-y^2$. When $k=0$, these are the lines $y=\pm x$. When $k>0$, the curves are hyperbolas with axis the x –axis and when $k<0$, they are hyperbolas with axis the y –axis.



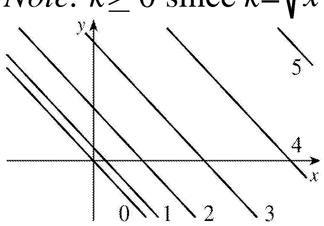
39. The level curves are $y - \ln x = k$ or $y = \ln x + k$.



40. The level curves are $e^{y/x} = k$ or equivalently $y = x \ln k$ ($x \neq 0$), a family of lines with slope $\ln k$ ($k > 0$) without the origin.

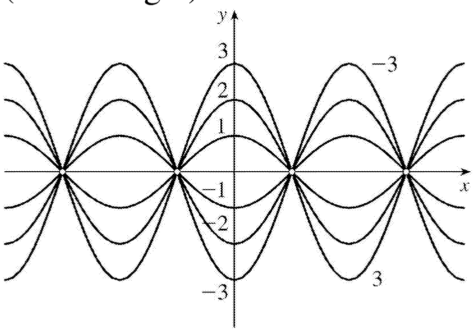


41. $k = \sqrt{x+y}$ or for $x + y \geq 0$, $k^2 = x + y$, or $y = -x + k^2$.
Note: $k \geq 0$ since $k = \sqrt{x+y}$.

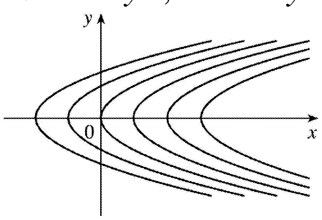


42. $k = y \sec x$ or $y = k \cos x$, $x \neq \dfrac{\pi}{2} + n\pi$

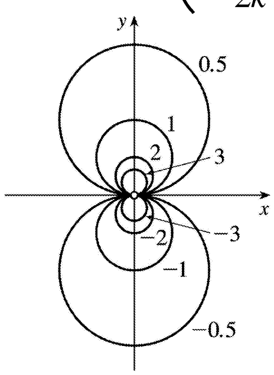
(n an integer).



43. $k=x-y^2$, or $x-k=y^2$, a family of parabolas with vertex $(k,0)$.

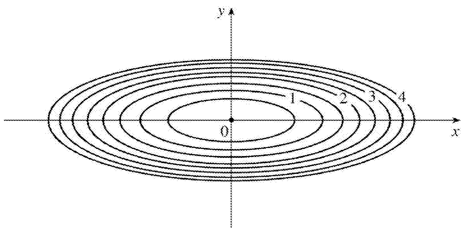


44. For $k\neq 0$ and $(x,y)\neq(0,0)$, $k=\dfrac{y}{x^2+y^2} \Leftrightarrow x^2+y^2-\dfrac{y}{k}=0 \Leftrightarrow x^2+\left(y-\dfrac{1}{2k}\right)^2=\dfrac{1}{4k^2}$, a family of circles with center $\left(0,\dfrac{1}{2k}\right)$ and radius $\dfrac{1}{2k}$ (without the origin). If $k=0$, the level curve is the x –axis.

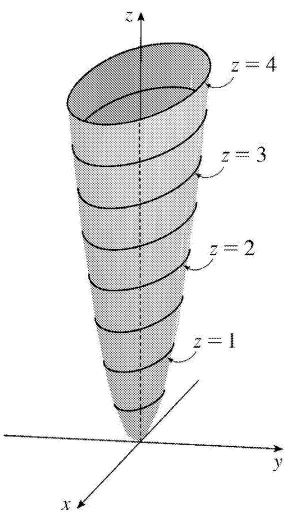


45. The contour map consists of the level curves $k=x^2+9y^2$, a family of ellipses with major axis the x –axis. (Or, if $k=0$, the origin.)

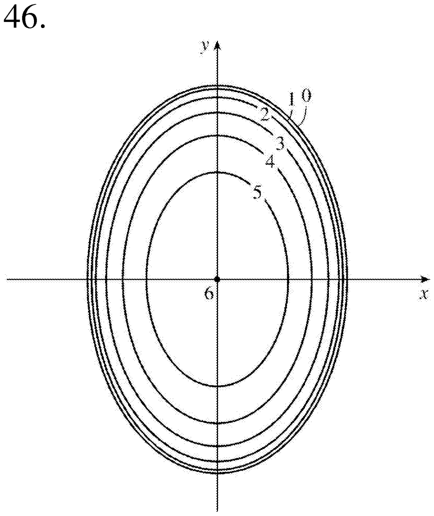
The graph of $f(x,y)$ is the surface $z=x^2+9y^2$, an elliptic paraboloid.



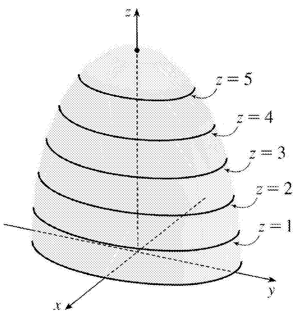
If we visualize lifting each ellipse $k=x^2+9y^2$ of the contour map to the plane $z=k$, we have horizontal traces that indicate the shape of the graph of f .



46.

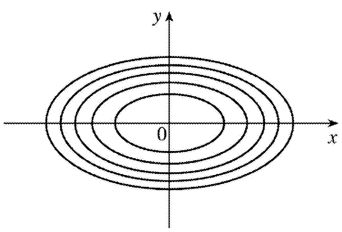


The contour map consists of the level curves $k=\sqrt{36-9x^2-4y^2} \Rightarrow 9x^2+4y^2=36-k^2$, $k\geq 0$, a family of ellipses with major axis the y –axis. (Or, if $k=6$, the origin.)



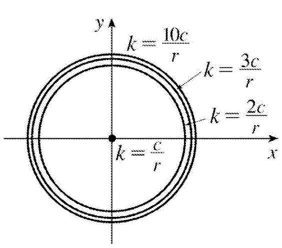
The graph of $f(x,y)$ is the surface $z=\sqrt{36-9x^2-4y^2}$, or equivalently the upper half of the ellipsoid $9x^2+4y^2+z^2=36$. If we visualize lifting each ellipse $k=\sqrt{36-9x^2-4y^2}$ of the contour map to the plane $z=k$, we have horizontal traces that indicate the shape of the graph of f .

47. The isothermals are given by $k=100/\left(1+x^2+2y^2\right)$ or $x^2+2y^2=(100-k)/k$ ($0<k\leq 100$), a family of ellipses.

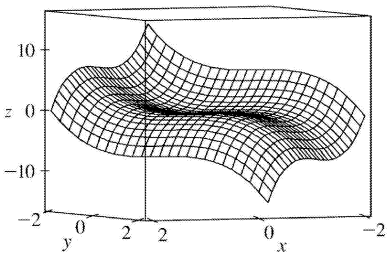


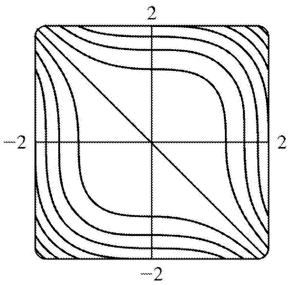
48. The equipotential curves are $k=\dfrac{c}{\sqrt{r^2-x^2-y^2}}$ or $x^2+y^2=r^2-\left(\dfrac{c}{k}\right)^2$, a family of circles ($k\geq c/r$).

Note: As $k\to\infty$, the radius of the circle approaches r .



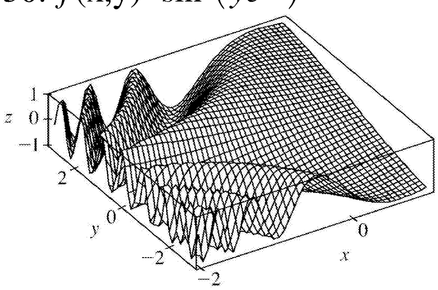
49. $f(x,y)=x^3+y^3$

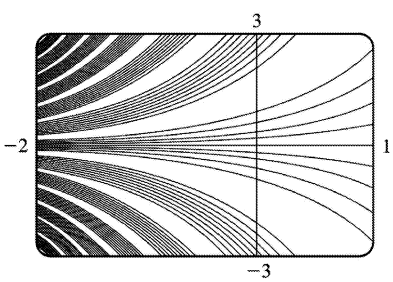




Note that the function is 0 along the line $y=-x$.

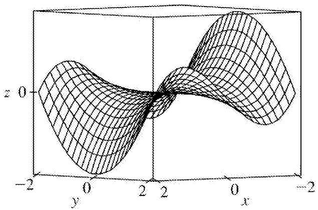
50. $f(x,y)=\sin\left(ye^{-x}\right)$

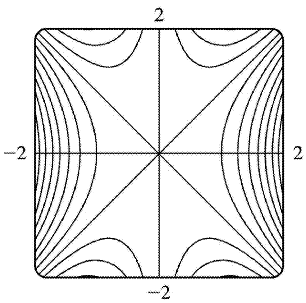




Cross–sections parallel to the yz –plane (such as the left–front trace in the first graph above) are sine–like curves. The periods of these curves decrease as x decreases.

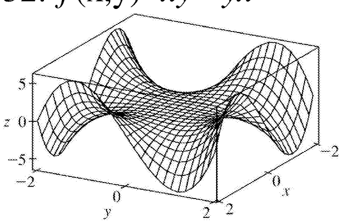
51. $f(x,y)=xy^2-x^3$

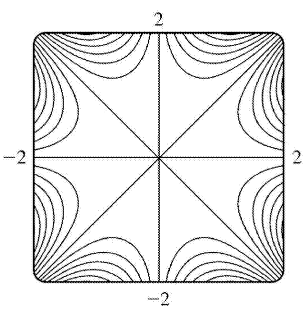




The traces parallel to the *yz* –plane (such as the left–front trace in the graph above) are parabolas; those parallel to
the *xz* –plane (such as the right–front trace) are cubic curves. The surface is called a monkey saddle because a monkey sitting on the surface near the origin has places for both legs and tail to rest.

52. $f(x,y)=xy^3-yx^3$





The traces parallel to either the *yz* –plane or the *xz* –plane are cubic curves.

53. **(a)** B
(b) III
Reasons: This function is constant on any circle centered at the origin, a description which matches only B and III.

54. **(a)** C
(b) II
Reasons: This function is the same if x is interchanged with y , so its graph is symmetric about the plane $x=y$. Also, $z(0,0)=0$ and the values of z approach 0 as we use points farther from the origin. These conditions are satisfied only by C and II.

55. **(a)** F
(b) V
Reasons: z increases without bound as we use points closer to the origin, a condition satisfied only by

F and V.

56. **(a)** A
(b) VI
Reasons: Along the lines $y=\pm\frac{1}{\sqrt{3}}x$ and $x=0$, this function is 0 .

57. **(a)** D
(b) IV
Reasons: This function is periodic in both x and y, with period 2π in each variable.

58. **(a)** E
(b) I
Reasons: This function is periodic along the x –axis, and increases as $|y|$ increases.

59. $k=x+3y+5z$ is a family of parallel planes with normal vector $\langle 1,3,5\rangle$.

60. $k=x^2+3y^2+5z^2$ is a family of ellipsoids for $k>0$ and the origin for $k=0$.

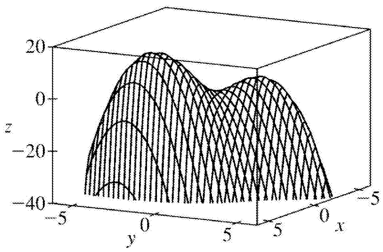
61. $k=x^2-y^2+z^2$ are the equations of the level surfaces. For $k=0$, the surface is a right circular cone with vertex the origin and axis the y– axis. For $k>0$, we have a family of hyperboloids of one sheet with axis the y –axis. For $k<0$, we have a family of hyperboloids of two sheets with axis the y –axis.

62. $k=x^2-y^2$ is a family of hyperbolic cylinders. The cross section of this family in the xy –plane has the same graph as the level curves in Exercise 38.

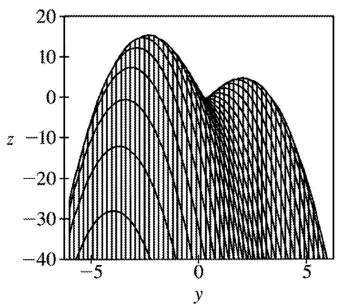
63. **(a)** The graph of g is the graph of f shifted upward 2 units.
(b) The graph of g is the graph of f stretched vertically by a factor of 2.
(c) The graph of g is the graph of f reflected about the xy –plane.
(d) The graph of $g(x,y)=-f(x,y)+2$ is the graph of f reflected about the xy –plane and then shifted upward 2 units.

64. **(a)** The graph of g is the graph of f shifted 2 units in the positive x –direction.
(b) The graph of g is the graph of f shifted 2 units in the negative y –direction.
(c) The graph of g is the graph of f shifted 3 units in the negative x –direction and 4 units in the positive y –direction.

65. $f(x,y)=3x-x^4-4y^2-10xy$



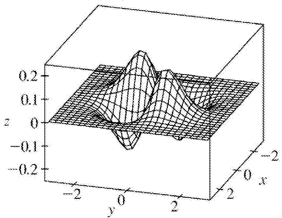
Three–dimensional view



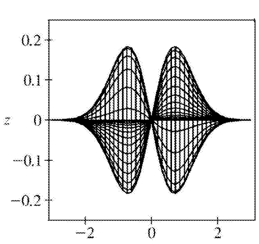
Front view

It does appear that the function has a maximum value, at the higher of the two "hilltops." From the front view graph, the maximum value appears to be approximately 15 . Both hilltops could be considered local maximum points, as
the values of f there are larger than at the neighboring points. There does not appear to be any local minimum point; although the valley shape between the two peaks looks like a minimum of some kind, some neighboring points have lower function values.

66. $f(x,y)=xye^{-x^2-y^2}$



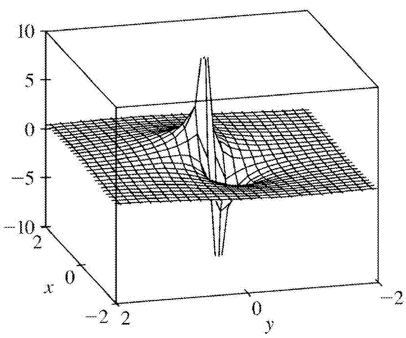
Three–dimensional view



Front view

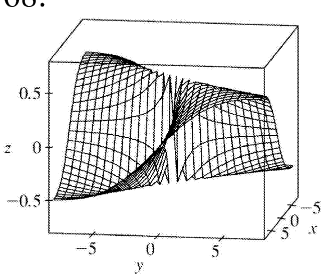
The function does have a maximum value, which it appears to achieve at two different points (the two "hilltops"). From the front view graph, we can estimate the maximum value to be approximately 0.18 . These same two points can also be considered local maximum points. The two "valley bottoms" visible in the graph can be considered local minimum points, as all the neighboring points give greater values of f .

67.



$f(x,y)=\dfrac{x+y}{x^2+y^2}$. As both x and y become large, the function values appear to approach 0 , regardless of which direction is considered. As (x,y) approaches the origin, the graph exhibits asymptotic behavior. From some directions, $f(x,y)\to\infty$, while in others $f(x,y)\to-\infty$. (These are the vertical spikes visible in the graph.) If the graph is examined carefully, however, one can see that $f(x,y)$ approaches 0 along the line $y=-x$.

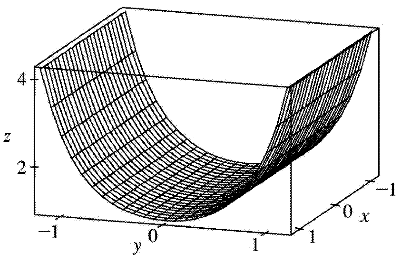
68.



$f(x,y)=\dfrac{xy}{x^2+y^2}$. The graph exhibits different limiting values as x and y become large or as (x,y) approaches the origin, depending on the direction being examined. For example, although f is undefined at the origin, the function values appear to be $\frac{1}{2}$ along the line $y=x$, regardless of the distance from the origin. Along the line $y=-x$, the value is always

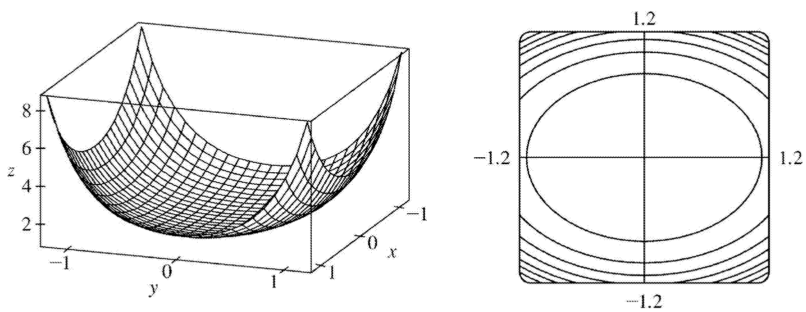
$-\frac{1}{2}$. Along the axes, $f(x,y)=0$ for all values of (x,y) except the origin. Other directions, heading toward the origin or away from the origin, give various limiting values between $-\frac{1}{2}$ and $\frac{1}{2}$.

69. $f(x,y)=e^{cx^2+y^2}$. First, if $c=0$, the graph is the cylindrical surface $z=e^{y^2}$ (whose level curves are parallel lines). When $c>0$, the vertical trace above the y –axis remains fixed while the sides of the surface in the x –direction ''curl'' upward, giving the graph a shape resembling an elliptic paraboloid. The level curves of the surface are ellipses centered at the origin.



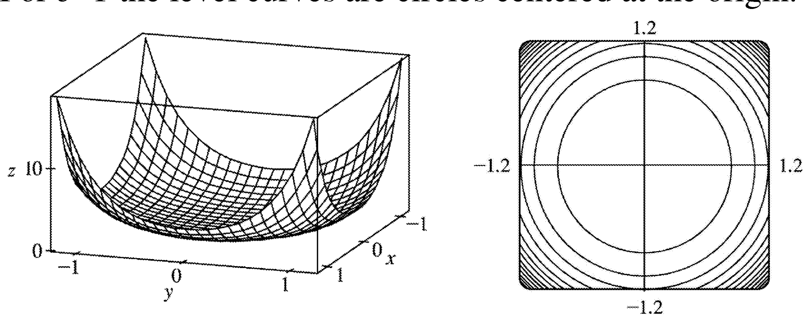
$c=0$

For $0<c<1$, the ellipses have major axis the x –axis and the eccentricity increases as $c\to 0$.



$c=0.5$ (level curves in increments of 1)

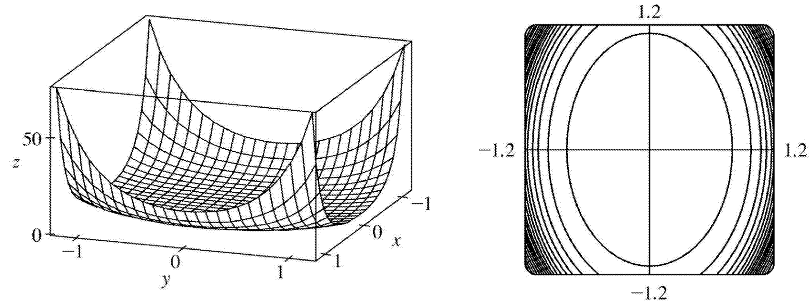
For $c=1$ the level curves are circles centered at the origin.



$c=1$ (level curves in increments of 1)

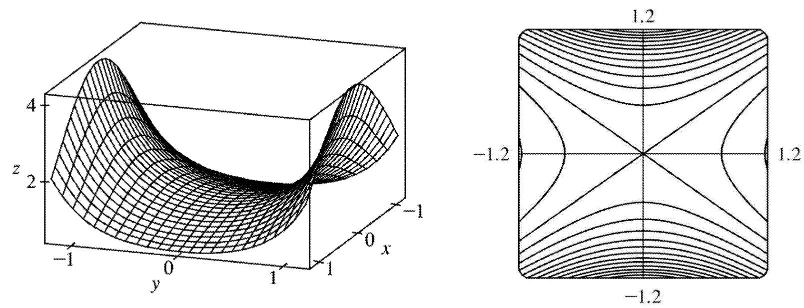
When $c>1$, the level curves are ellipses with major axis the y –axis, and the eccentricity increases as c

increases.

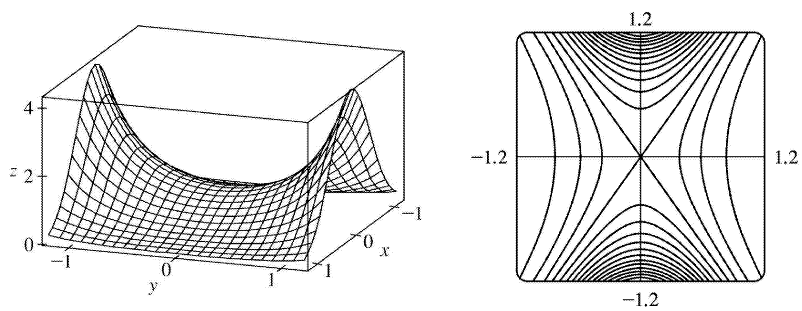


$c=2$ (level curves in increments of 4)

For values of $c<0$, the sides of the surface in the x –direction curl downward and approach the xy –plane (while the vertical trace $x=0$ remains fixed), giving a saddle–shaped appearance to the graph near the point $(0,0,1)$. The level curves consist of a family of hyperbolas. As c decreases, the surface becomes flatter in the x –direction and the surface's approach to the curve in the trace $x=0$ becomes steeper, as the graphs demonstrate.

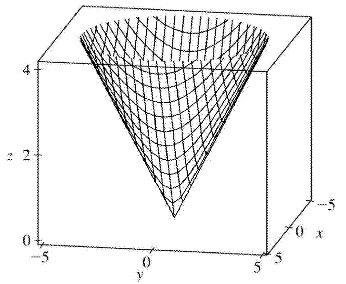


$c=-0.5$ (level curves in increments of 0.25)

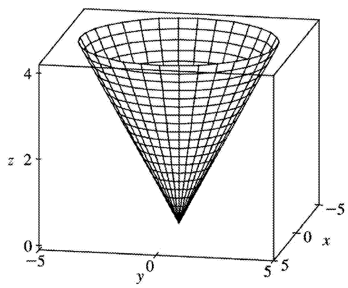


$c=-2$ (level curves in increments of 0.25)

70. First, we graph $f(x,y)=\sqrt{x^2+y^2}$. As an alternative, the x^2+y^2 expression suggests that cylindrical coordinates may be appropriate, giving the equivalent equation $z=\sqrt{r^2}=r$, $r\geq 0$ which we graph as well. Notice that the graph in cylindrical coordinates better demonstrates the symmetry of the surface.

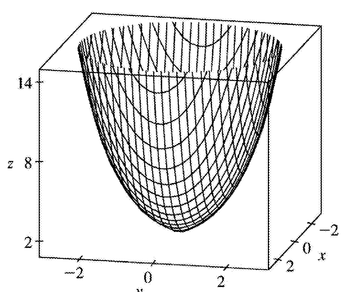


$f(x,y)=\sqrt{x^2+y^2}$

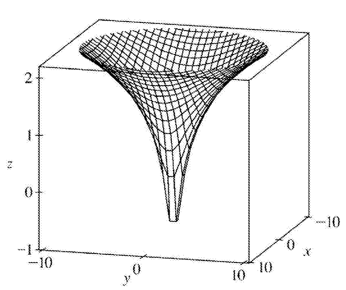


$z=r, r\geq 0$

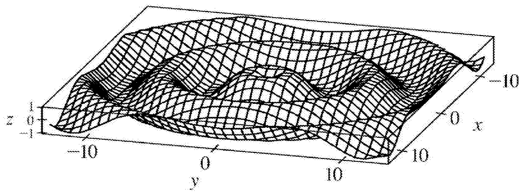
Graphs of the other four functions follow.



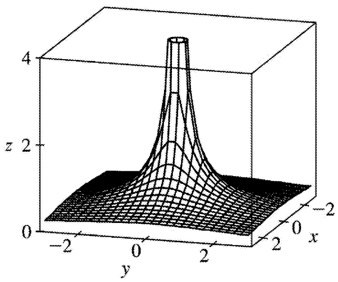
$f(x,y)=e^{\sqrt{x^2+y^2}}$



$f(x,y)=\ln\sqrt{x^2+y^2}$



$f(x,y)=\sin\left(\sqrt{x^2+y^2}\right)$



$f(x,y)=\dfrac{1}{\sqrt{x^2+y^2}}$

Notice that each graph $f(x,y)=g\left(\sqrt{x^2+y^2}\right)$ exhibits radial symmetry about the z –axis and the trace in the xz –plane for $x\geq 0$ is the graph of $z=g(x)$, $x\geq 0$. This suggests that the graph of $f(x,y)=g\left(\sqrt{x^2+y^2}\right)$ is obtained from the graph of g by graphing $z=g(x)$ in the xz –plane and rotating the curve about the z –axis.

71. **(a)** $P=bL^{\alpha}K^{1-\alpha} \Rightarrow \dfrac{P}{K}=bL^{\alpha}K^{-\alpha} \Rightarrow \dfrac{P}{K}=b\left(\dfrac{L}{K}\right)^{\alpha} \Rightarrow \ln\dfrac{P}{K}=\ln\left(b\left(\dfrac{L}{K}\right)^{\alpha}\right) \Rightarrow$
$\ln\dfrac{P}{K}=\ln b+\alpha\ln\left(\dfrac{L}{K}\right)$

(b) We list the values for ln (L/K) and ln (P/K) for the years 1899—1922. (Historically, these values were rounded to 2 decimal places.)

Year	$x=\ln(L/K)$	$y=\ln(P/K)$
1899	0	0
1900	–0.02	–0.06
1901	–0.04	–0.02
1902	–0.04	0
1903	–0.07	–0.05
1904	–0.13	–0.12
1905	–0.18	–0.04

1906	–0.20	–0.07
1907	–0.23	–0.15
1908	–0.41	–0.38
1909	–0.33	–0.24
1910	–0.35	–0.27
1911	–0.38	–0.34
1912	–0.38	–0.24
1913	–0.41	–0.25
1914	–0.47	–0.37
1915	–0.53	–0.34
1915	–0.53	–0.34
1916	–0.49	–0.28
1917	–0.53	–0.39
1918	–0.60	–0.50
1919	–0.68	–0.57
1920	–0.74	–0.57
1921	–1.05	–0.85
1922	–0.98	–0.59

After entering the (x,y) pairs into a calculator or CAS, the resulting least squares regression line through the points is approximately $y=0.75136x+0.01053$, which we round to $y=0.75x+0.01$.
(c) Comparing the regression line from part (b) to the equation $y=\ln b+\alpha x$ with $x=\ln (L/K)$ and $y=\ln (P/K)$, we have $\alpha =0.75$ and $\ln b=0.01\Rightarrow b=e^{0.01}\approx 1.01$. Thus, the Cobb–Douglas production function is $P=bL^{\alpha}K^{1-\alpha}=1.01L^{0.75}K^{0.25}$.

1. In general, we can't say anything about $f(3,1)$! $\lim_{(x,y)\to(3,1)} f(x,y)=6$ means that the values of $f(x,y)$ approach 6 as (x,y) approaches, but is not equal to, (3,1) . If f is continuous, we know that $\lim_{(x,y)\to(a,b)} f(x,y)=f(a,b)$, so $\lim_{(x,y)\to(3,1)} f(x,y)=f(3,1)=6$.

2. **(a)** The outdoor temperature as a function of longitude, latitude, and time is continuous. Small changes in longitude, latitude, or time can produce only small changes in temperature, as the temperature doesn't jump abruptly from one value to another.
(b) Elevation is not necessarily continuous. If we think of a cliff with a sudden drop–off, a very small change in longitude or latitude can produce a comparatively large change in elevation, without all the intermediate values being attained. Elevation *can* jump from one value to another.
(c) The cost of a taxi ride is usually discontinuous. The cost normally increases in jumps, so small changes in distance traveled or time can produce a jump in cost. A graph of the function would show breaks in the surface.

3. We make a table of values of $f(x,y)=\dfrac{x^2y^3+x^3y^2-5}{2-xy}$ for a set of (x,y) points near the origin.

x \ y	−0.2	−0.1	−0.05	0	0.05	0.1	0.2
−0.2	−2.551	−2.525	−2.513	−2.500	−2.488	−2.475	−2.451
−0.1	−2.525	−2.513	−2.506	−2.500	−2.494	−2.488	−2.475
−0.05	−2.513	−2.506	−2.503	−2.500	−2.497	−2.494	−2.488
0	−2.500	−2.500	−2.500		−2.500	−2.500	−2.500
0.05	−2.488	−2.494	−2.497	−2.500	−2.503	−2.506	−2.513
0.1	−2.475	−2.488	−2.494	−2.500	−2.506	−2.513	−2.525
0.2	−2.451	−2.475	−2.488	−2.500	−2.513	−2.525	−2.551

As the table shows, the values of $f(x,y)$ seem to approach −2.5 as (x,y) approaches the origin from a variety of different directions. This suggests that $\lim_{(x,y)\to(0,0)} f(x,y)=-2.5$.
Since f is a rational function, it is continuous on its domain. f is defined at (0,0) , so we can use direct substitution to establish that $\lim_{(x,y)\to(0,0)} f(x,y)=\dfrac{0^2 0^3+0^3 0^2-5}{2-0\cdot 0}=-\dfrac{5}{2}$, verifying our guess.

4. We make a table of values of $f(x,y)=\dfrac{2xy}{x^2+2y^2}$ for a set of (x,y) points near the origin.

$x \backslash y$	−0.3	−0.2	−0.1	0	0.1	0.2	0.3
−0.3	0.667	0.706	0.545	0.000	−0.545	−0.706	−0.667
−0.2	0.545	0.667	0.667	0.000	−0.667	−0.667	−0.545
−0.1	0.316	0.444	0.667	0.000	−0.667	−0.444	−0.316
0	0.000	0.000	0.000		0.000	0.000	0.000
0.1	−0.316	−0.444	−0.667	0.000	0.667	0.444	0.316
0.2	−0.545	−0.667	−0.667	0.000	0.667	0.667	0.545
0.3	−0.667	−0.706	−0.545	0.000	0.545	0.706	0.667

It appears from the table that the values of $f(x,y)$ are not approaching a single value as (x,y) approaches the origin. For verification, if we first approach (0,0) along the x –axis, we have $f(x,0)=0$, so $f(x,y)\to 0$. But if we approach (0,0) along the line $y=x$, $f(x,x)=\dfrac{2x^2}{x^2+2x^2}=\dfrac{2}{3}$ ($x\neq 0$), so $f(x,y)\to \dfrac{2}{3}$. Since f approaches different values along different paths to the origin, this limit does not exist.

5. $f(x,y)=x^5+4x^3y-5xy^2$ is a polynomial, and hence continuous, so
$\lim_{(x,y)\to(5,-2)} f(x,y)=f(5,-2)=5^5+4(5)^3(-2)-5(5)(-2)^2=2025$.

6. $x-2y$ is a polynomial and therefore continuous. Since $\cos t$ is a continuous function, the composition $\cos(x-2y)$ is also continuous. xy is also a polynomial, and hence continuous, so the product $f(x,y)=xy\cos(x-2y)$ is a continuous function. Then
$\lim_{(x,y)\to(6,3)} f(x,y)=f(6,3)=(6)(3)\cos(6-2\cdot 3)=18$.

7. $f(x,y)=x^2/\left(x^2+y^2\right)$. First approach (0,0) along the x –axis. Then $f(x,0)=x^2/x^2=1$ for $x\neq 0$, so $f(x,y)\to 1$. Now approach (0,0) along the y –axis. Then for $y\neq 0$, $f(0,y)=0$, so $f(x,y)\to 0$. Since f has two different limits along two different lines, the limit does not exist.

8. $f(x,y)=\left(x^2+\sin^2 y\right)/\left(2x^2+y^2\right)$. First approach $(0,0)$ along the x –axis. Then $f(x,0)=x^2/2x^2=\dfrac{1}{2}$ for $x\neq 0$, so $f(x,y)\to \dfrac{1}{2}$. Next approach $(0,0)$ along the y –axis. For $y\neq 0$, $f(0,y)=\dfrac{\sin^2 y}{y^2}=\left(\dfrac{\sin y}{y}\right)^2$ and $\lim_{y\to 0}\dfrac{\sin y}{y}=1$, so $f(x,y)\to 1$. Since f has two different limits along two different lines, the limit does not exist.

9. $f(x,y)=(xy\cos y)/(3x^2+y^2)$. On the x –axis, $f(x,0)=0$ for $x\neq 0$, so $f(x,y)\to 0$ as $(x,y)\to(0,0)$ along the x –axis. Approaching $(0,0)$ along the line $y=x$, $f(x,x)=(x^2\cos x)/4x^2=\frac{1}{4}\cos x$ for $x\neq 0$, so $f(x,y)\to\frac{1}{4}$ along this line. Thus the limit does not exist.

10. $f(x,y)=6x^3y/\left(2x^4+y^4\right)$. On the x –axis, $f(x,0)=0$ for $x\neq 0$, so $f(x,y)\to 0$ as $(x,y)\to(0,0)$ along the x –axis. Approaching $(0,0)$ along the line $y=x$ gives $f(x,x)=6x^4/\left(3x^4\right)=2$ for $x\neq 0$, so along this line $f(x,y)\to 2$ as $(x,y)\to(0,0)$. Thus the limit does not exist.

11. $f(x,y)=\dfrac{xy}{\sqrt{x^2+y^2}}$. We can see that the limit along any line through $(0,0)$ is 0 , as well as along other paths through $(0,0)$ such as $x=y^2$ and $y=x^2$. So we suspect that the limit exists and equals 0 ; we use the Squeeze Theorem to prove our assertion. $0\leq\left|\dfrac{xy}{\sqrt{x^2+y^2}}\right|\leq|x|$ since $|y|\leq\sqrt{x^2+y^2}$, and $|x|\to 0$ as $(x,y)\to(0,0)$. So $\lim_{(x,y)\to(0,0)} f(x,y)=0$.

12. $f(x,y)=\left(x^4-y^4\right)/\left(x^2+y^2\right)=\left(x^2+y^2\right)\left(x^2-y^2\right)/(x^2+y^2\big)=x^2-y^2$ for $(x,y)\neq(0,0)$. Thus the limit as $(x,y)\to(0,0)$ is 0 .

13. Let $f(x,y)=\dfrac{2x^2y}{x^4+y^2}$. Then $f(x,0)=0$ for $x\neq 0$, so $f(x,y)\to 0$ as $(x,y)\to(0,0)$ along the x –axis. But $f\left(x,x^2\right)=\dfrac{2x^4}{2x^4}=1$ for $x\neq 0$, so $f(x,y)\to 1$ as $(x,y)\to(0,0)$ along the parabola $y=x^2$. Thus the limit doesn't exist.

14. We can use the Squeeze Theorem to show that $\lim_{(x,y)\to(0,0)}\dfrac{x^2\sin^2 y}{x^2+2y^2}=0$:

$0\leq\dfrac{x^2\sin^2 y}{x^2+2y^2}\leq\sin^2 y$ since $\dfrac{x^2}{x^2+2y^2}\leq 1$, and $\sin^2 y\to 0$ as $(x,y)\to(0,0)$, so $\lim_{(x,y)\to(0,0)}\dfrac{x^2\sin^2 y}{x^2+2y^2}=0$.

15.

$$\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{\sqrt{x^2+y^2+1}-1} = \lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{\sqrt{x^2+y^2+1}-1} \cdot \frac{\sqrt{x^2+y^2+1}+1}{\sqrt{x^2+y^2+1}+1}$$
$$= \lim_{(x,y)\to(0,0)} \frac{\left(x^2+y^2\right)\left(\sqrt{x^2+y^2+1}+1\right)}{x^2+y^2}$$
$$= \lim_{(x,y)\to(0,0)} \left(\sqrt{x^2+y^2+1}+1\right)=2$$

16. $f(x,y)=xy^4/\left(x^2+y^8\right)$. On the x –axis, $f(x,0)=0$ for $x\neq 0$, so $f(x,y)\to 0$ as $(x,y)\to(0,0)$ along the x –axis. Approaching $(0,0)$ along the curve $x=y^4$ gives $f(y^4,y)=y^8/2y^8=\frac{1}{2}$ for $y\neq 0$, so along this path $f(x,y)\to \frac{1}{2}$ as $(x,y)\to(0,0)$. Thus the limit does not exist.

17. e^{-xy} and $\sin(\pi z/2)$ are each compositions of continuous functions, and hence continuous, so their product $f(x,y,z)=e^{-xy}\sin(\pi z/2)$ is a continuous function. Then $\lim_{(x,y,z)\to(3,0,1)} f(x,y,z)=f(3,0,1)=e^{-(3)(0)}\sin(\pi\cdot 1/2)=1$.

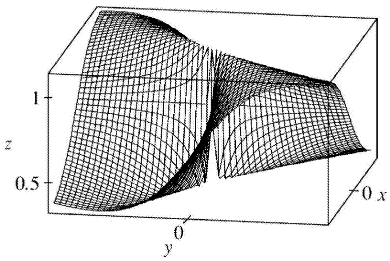
18. $f(x,y,z)=\dfrac{x^2+2y^2+3z^2}{x^2+y^2+z^2}$. Then $f(x,0,0)=\dfrac{x^2+0+0}{x^2+0+0}=1$ for $x\neq 0$, so $f(x,y,z)\to 1$ as $(x,y,z)\to(0,0,0)$ along the x –axis. But $f(0,y,0)=\dfrac{0+2y^2+0}{0+y^2+0}=2$ for $y\neq 0$, so $f(x,y,z)\to 2$ as $(x,y,z)\to(0,0,0)$ along the y –axis. Thus, the limit doesn't exist.

19. $f(x,y,z)=\dfrac{xy+yz^2+xz^2}{x^2+y^2+z^4}$. Then $f(x,0,0)=0/x^2=0$ for $x\neq 0$, so as $(x,y,z)\to(0,0,0)$ along the x –axis, $f(x,y,z)\to 0$. But $f(x,x,0)=x^2/\left(2x^2\right)=\frac{1}{2}$ for $x\neq 0$, so as $(x,y,z)\to(0,0,0)$ along the line $y=x$, $z=0$, $f(x,y,z)\to \frac{1}{2}$. Thus the limit doesn't exist.

20. $f(x,y,z)=\dfrac{xy+yz+zx}{x^2+y^2+z^2}$. Then $f(x,0,0)=0$ for $x\neq 0$, so as $(x,y,z)\to(0,0,0)$ along the x –axis,

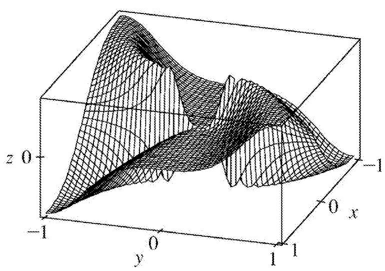
$f(x,y,z)\to 0$. But $f(x,x,0)=x^2/\left(2x^2\right)=\frac{1}{2}$ for $x\neq 0$, so as $(x,y,z)\to(0,0,0)$ along the line $y=x$, $z=0$, $f(x,y,z)\to\frac{1}{2}$. Thus the limit doesn't exist.

21.



From the ridges on the graph, we see that as $(x,y)\to(0,0)$ along the lines under the two ridges, $f(x,y)$ approaches different values. So the limit does not exist.

22.



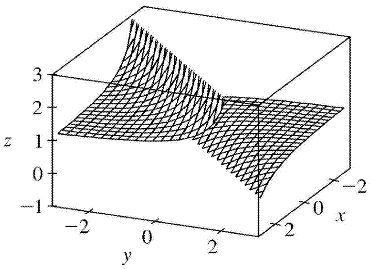
From the graph, it appears that as we approach the origin along the lines $x=0$ or $y=0$, the function is everywhere 0 , whereas if we approach the origin along a certain curve it has a constant value of about $\frac{1}{2}$. Since the function approaches different values depending on the path of approach, the limit does not exist.

23. $h(x,y)=g\,(f(x,y))=(2x+3y-6)^2+\sqrt{2x+3y-6}$. Since f is a polynomial, it is continuous on R^2 and g is continuous on its domain $\{t\,|\,t\geq 0\}$. Thus h is continuous on its domain $D=\{(x,y)\,|\,2x+3y-6\geq 0\}=\left\{(x,y)\,|\,y\geq -\frac{2}{3}x+2\right\}$, which consists of all points on or above the line $y=-\frac{2}{3}x+2$.

24. $h(x,y)=g(f(x,y))=\left(\sqrt{x^2-y}-1\right)\Big/\left(\sqrt{x^2-y}+1\right)$. Since f is a polynomial, it is continuous on R^2 and g is continuous on its domain $\{t\,|\,t\geq 0\}$. Thus h is continuous on its domain

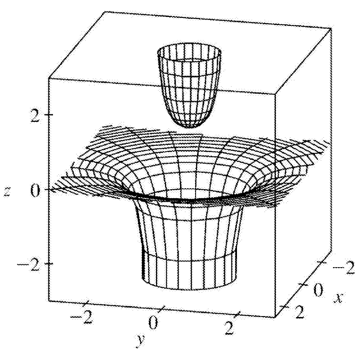
$D=\{(x,y) \mid x^2-y\geq 0\}=\{(x,y) \mid y\leq x^2\}$ which consists of all points below or on the parabola $y=x^2$.

25.



From the graph, it appears that f is discontinuous along the line $y=x$. If we consider $f(x,y)=e^{1/(x-y)}$ as a composition of functions, $g(x,y)=1/(x-y)$ is a rational function and therefore continuous except where $x-y=0\Rightarrow y=x$. Since the function $h(t)=e^t$ is continuous everywhere, the composition $h(g(x,y))=e^{1/(x-y)}=f(x,y)$ is continuous except along the line $y=x$, as we suspected.

26.



We can see a circular break in the graph, corresponding approximately to the unit circle, where f is discontinuous.
[*Note:* For a more accurate graph, try converting to cylindrical coordinates first.] Since $f(x,y)=\dfrac{1}{1-x^2-y^2}$ is a rational function, it is continuous except where $1-x^2-y^2=0\Rightarrow x^2+y^2=1$, confirming our observation that f is discontinuous on the circle $x^2+y^2=1$.

27. The functions $\sin(xy)$ and e^x-y^2 are continuous everywhere, so $F(x,y)=\dfrac{\sin(xy)}{e^x-y^2}$ is continuous except where $e^x-y^2=0\Rightarrow y^2=e^x\Rightarrow y=\pm\sqrt{e^x}=\pm e^{\frac{1}{2}x}$. Thus F is continuous on its domain $\{(x,y) \mid y\neq\pm e^{x/2}\}$.

28. $F(x,y)=\dfrac{x-y}{1+x^2+y^2}$ is a rational function and thus is continuous on its domain R^2 (since the denominator is never zero).

29. $F(x,y)=\arctan\left(x+\sqrt{y}\right)=g(f(x,y))$ where $f(x,y)=x+\sqrt{y}$, continuous on its domain $\{(x,y)\mid y\geq 0\}$, and $g(t)=\arctan t$ is continuous everywhere. Thus F is continuous on its domain $\{(x,y)\mid y\geq 0\}$.

30. e^{x^2y} is continuous on R^2 and $\sqrt{x+y^2}$ is continuous on its domain $\left\{(x,y)\mid x+y^2\geq 0\right\}=\left\{(x,y)\mid x\geq -y^2\right\}$, so $F(x,y)=e^{x^2y}+\sqrt{x+y^2}$ is continuous on the set $\left\{(x,y)\mid x\geq -y^2\right\}$.

31. $G(x,y)=\ln\left(x^2+y^2-4\right)=g(f(x,y))$ where $f(x,y)=x^2+y^2-4$, continuous on R^2 , and $g(t)=\ln t$, continuous on its domain $\{t\mid t>0\}$. Thus G is continuous on its domain $\left\{(x,y)\mid x^2+y^2-4>0\right\}=\left\{(x,y)\mid x^2+y^2>4\right\}$, the exterior of the circle $x^2+y^2=4$.

32. $G(x,y)=g\left(f(x,y)\right)$ where $f(x,y)=x^2+y^2$, continuous on R^2 , and $g(t)=\sin^{-1}t$, continuous on its domain $\{t\mid -1\leq t\leq 1\}$. Thus G is continuous on its domain $D=\left\{(x,y)\mid -1\leq x^2+y^2\leq 1\right\}=\left\{(x,y)\mid x^2+y^2\leq 1\right\}$, inside and on the circle $x^2+y^2=1$.

33. $\sqrt{y}$ is continuous on its domain $\{y\mid y\geq 0\}$ and $x^2-y^2+z^2$ is continuous everywhere, so $f(x,y,z)=\dfrac{\sqrt{y}}{x^2-y^2+z^2}$ is continuous for $y\geq 0$ and $x^2-y^2+z^2\neq 0\Rightarrow y^2\neq x^2+z^2$, that is, $\left\{(x,y,z)\mid y\geq 0, y\neq\sqrt{x^2+z^2}\right\}$.

34. $f(x,y,z)=\sqrt{x+y+z}=h(g(x,y,z))$ where $g(x,y,z)=x+y+z$, continuous everywhere, and $h(t)=\sqrt{t}$ is continuous on its domain $\{t\mid t\geq 0\}$. Thus f is continuous on its domain $\{(x,y,z)\mid x+y+z\geq 0\}$, so f is continuous on and above the plane $z=-x-y$.

35. $f(x,y)=\begin{cases}\dfrac{x^2y^3}{2x^2+y^2} & \text{if } (x,y)\neq(0,0)\\ 1 & \text{if } (x,y)=(0,0)\end{cases}$ The first piece of f is a rational function defined everywhere except at the origin, so f is continuous on R^2 except possibly at the origin. Since $x^2\leq 2x^2+y^2$, we have $\left|x^2y^3/(2x^2+y^2)\right|\leq\left|y^3\right|$. We know that $\left|y^3\right|\rightarrow 0$ as $(x,y)\rightarrow(0,0)$. So, by the

Squeeze Theorem, $\lim_{(x,y)\to(0,0)} f(x,y)=\lim_{(x,y)\to(0,0)} \frac{x^2y^3}{2x^2+y^2}=0$. But $f(0,0)=1$, so f is discontinuous at $(0,0)$. Therefore, f is continuous on the set $\{(x,y)\mid(x,y)\neq(0,0)\}$.

36. $f(x,y)=\begin{cases} \frac{xy}{x^2+xy+y^2} & \text{if}(x,y)\neq(0,0) \\ 0 & \text{if}(x,y)=(0,0) \end{cases}$ The first piece of f is a rational function defined everywhere except at the origin, so f is continuous on R^2 except possibly at the origin. $f(x,0)=0/x^2=0$ for $x\neq0$, so $f(x,y)\to0$ as $(x,y)\to(0,0)$ along the x –axis. But $f(x,x)=x^2/\left(3x^2\right)=\frac{1}{3}$ for $x\neq0$, so $f(x,y)\to\frac{1}{3}$ as $(x,y)\to(0,0)$ along the line $y=x$. Thus $\lim_{(x,y)\to(0,0)} f(x,y)$ doesn't exist, so f is not continuous at $(0,0)$ and the largest set on which f is continuous is $\{(x,y)\mid(x,y)\neq(0,0)\}$.

37. $\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y^2}=\lim_{r\to0^+} \frac{(r\cos\theta)^3+(r\sin\theta)^3}{r^2}=\lim_{r\to0^+}\left(r\cos^3\theta+r\sin^3\theta\right)=0$

38.

$$\lim_{(x,y)\to(0,0)}\left(x^2+y^2\right)\ln\left(x^2+y^2\right)=\lim_{r\to0^+} r^2\ln r^2=\lim_{r\to0^+}\frac{\ln r^2}{1/r^2}$$

$$=\lim_{r\to0^+}\frac{\left(1/r^2\right)(2r)}{-2/r^3}\text{ [using l'Hospital's Rule] }=\lim_{r\to0^+}\left(-r^2\right)=0$$

39.

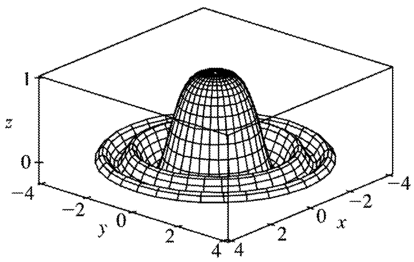
$$\lim_{(x,y,z)\to(0,0,0)}\frac{xyz}{x^2+y^2+z^2}=\lim_{\rho\to0^+}\frac{(\rho\sin\phi\cos\theta)(\rho\sin\phi\sin\theta)(\rho\cos\phi)}{\rho^2}$$

$$=\lim_{\rho\to0^+}\left(\rho\sin^2\phi\cos\phi\sin\theta\cos\theta\right)=0$$

40. $\lim_{(x,y)\to(0,0)}\frac{\sin\left(x^2+y^2\right)}{x^2+y^2}=\lim_{r\to0^+}\frac{\sin\left(r^2\right)}{r^2}$, which is an indeterminate form of type 0/0 . Using l'Hospital's Rule, we get

$$\lim_{r\to 0^+} \frac{\sin\left(r^2\right)}{r^2} = \lim_{r\to 0^+} \frac{2r\cos\left(r^2\right)}{2r} = \lim_{r\to 0^+} \cos\left(r^2\right) = 1 .$$

Or: Use the fact that $\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$.



41. Since $|\mathbf{x}-\mathbf{a}|^2 = |\mathbf{x}|^2 + |\mathbf{a}|^2 - 2|\mathbf{x}|\,|\mathbf{a}|\cos\theta \geq |\mathbf{x}|^2 + |\mathbf{a}|^2 - 2|\mathbf{x}|\,|\mathbf{a}| = \left(|\mathbf{x}| - |\mathbf{a}|\right)^2$, we have $\left||\mathbf{x}| - |\mathbf{a}|\right| \leq |\mathbf{x}-\mathbf{a}|$. Let $\varepsilon > 0$ be given and set $\delta = \varepsilon$. Then whenever $0 < |\mathbf{x}-\mathbf{a}| < \delta$, $\left||\mathbf{x}| - |\mathbf{a}|\right| \leq |\mathbf{x}-\mathbf{a}| < \delta = \varepsilon$. Hence $|\mathbf{x}| = |\mathbf{a}|$ and $f(\mathbf{x}) = |\mathbf{x}|$ is continuous on R^n .

42. Let $\varepsilon > 0$ be given. We need to find $\delta > 0$ such that $|f(\mathbf{x}) - f(\mathbf{a})| < \varepsilon$ whenever $|\mathbf{x}-\mathbf{a}| < \delta$ or $|\mathbf{c}\cdot\mathbf{x} - \mathbf{c}\cdot\mathbf{a}| < \varepsilon$ whenever $|\mathbf{x}-\mathbf{a}| < \delta$. But $|\mathbf{c}\cdot\mathbf{x} - \mathbf{c}\cdot\mathbf{a}| = |\mathbf{c}\cdot(\mathbf{x}-\mathbf{a})|$ and $|\mathbf{c}\cdot(\mathbf{x}-\mathbf{a})| \leq |\mathbf{c}|\,|\mathbf{x}-\mathbf{a}|$ by Exercise 13.3.57 [ET 12.3.57] (the Cauchy–Schwartz Inequality). Let $\varepsilon > 0$ be given and set $\delta = \varepsilon / |\mathbf{c}|$. Then whenever $0 < |\mathbf{ax}-\mathbf{a}| < \delta$, $|f(\mathbf{x}) - f(\mathbf{a})| = |\mathbf{c}\cdot\mathbf{x} - \mathbf{c}\cdot\mathbf{a}| \leq |\mathbf{c}|\,|\mathbf{x}-\mathbf{a}| < |\mathbf{c}|\,\delta = |\mathbf{c}|\left(\varepsilon / |\mathbf{c}|\right) = \varepsilon$. So f is continuous on R^n .

1. **(a)** $\partial T/\partial x$ represents the rate of change of T when we fix y and t and consider T as a function of the single variable x, which describes how quickly the temperature changes when longitude changes but latitude and time are constant. $\partial T/\partial y$ represents the rate of change of T when we fix x and t and consider T as a function of y, which describes how quickly the temperature changes when latitude changes but longitude and time are constant. $\partial T/\partial t$ represents the rate of change of T when we fix x and y and consider T as a function of t, which describes how quickly the temperature changes over time for a constant longitude and latitude.
(b) $f_x(158,21,9)$ represents the rate of change of temperature at longitude $158°$ W, latitude $21°$ N at 9:00 A.M. when only longitude varies. Since the air is warmer to the west than to the east, increasing longitude results in an increased air temperature, so we would expect $f_x(158,21,9)$ to be positive. $f_y(158,21,9)$ represents the rate of change of temperature at the same time and location when only latitude varies. Since the air is warmer to the south and cooler to the north, increasing latitude results in a decreased air temperature, so we would expect $f_y(158,21,9)$ to be negative. $f_t(158,21,9)$ represents the rate of change of temperature at the same time and location when only time varies. Since typically air temperature increases from the morning to the afternoon as the sun warms it, we would expect $f_t(158,21,9)$ to be positive.

2. By Definition 4, $f_T(92,60)=\lim_{h\to 0}\dfrac{f(92+h,60)-f(92,60)}{h}$, which we can approximate by considering $h=2$ and $h=-2$ and using the values given in Table 1:
$f_T(92,60)\approx\dfrac{f(94,60)-f(92,60)}{2}=\dfrac{111-105}{2}=3$, $f_T(92,60)\approx\dfrac{f(90,60)-f(92,60)}{-2}=\dfrac{100-105}{-2}=2.5$.
Averaging these values, we estimate $f_T(92,60)$ to be approximately 2.75. Thus, when the actual temperature is $92°$ F and the relative humidity is 60%, the apparent temperature rises by about $2.75°$ F for every degree that the actual temperature rises.
Similarly, $f_H(92,60)=\lim_{h\to 0}\dfrac{f(92,60+h)-f(92,60)}{h}$ which we can approximate by considering $h=5$ and $h=-5$: $f_H(92,60)\approx\dfrac{f(92,65)-f(92,60)}{5}=\dfrac{108-105}{5}=0.6$,
$f_H(92,60)\approx\dfrac{f(92,55)-f(92,60)}{-5}=\dfrac{103-105}{-5}=0.4$. Averaging these values, we estimate $f_H(92,60)$ to be approximately 0.5. Thus, when the actual temperature is $92°$ F and the relative humidity is 60%, the apparent temperature rises by about $0.5°$ F for every percent that the relative humidity increases.

3. **(a)** By Definition 4, $f_T(-15,30)=\lim_{h\to 0}\dfrac{f(-15+h,30)-f(-15,30)}{h}$, which we can approximate by considering $h=5$ and $h=-5$ and using the values given in the table:

$f_T(-15,30) \approx \dfrac{f(-10,30)-f(-15,30)}{5} = \dfrac{-20-(-26)}{5} = \dfrac{6}{5} = 1.2$,
$f_T(-15,30) \approx \dfrac{f(-20,30)-f(-15,30)}{-5} = \dfrac{-33-(-26)}{-5} = \dfrac{-7}{-5} = 1.4$. Averaging these values, we estimate $f_T(-15,30)$ to be approximately 1.3 . Thus, when the actual temperature is $-15°$ F and the wind speed is 30 km / h, the apparent temperature rises by about 1.3° F for every degree that the actual temperature rises.

Similarly, $f_v(-15,30) = \lim_{h \to 0} \dfrac{f(-15,30+h)-f(-15,30)}{h}$ which we can approximate by considering $h=10$ and $h=-10$: $f_v(-15,30) \approx \dfrac{f(-15,40)-f(-15,30)}{10} = \dfrac{-27-(-26)}{10} = \dfrac{-1}{10} = -0.1$,
$f_v(-15,30) \approx \dfrac{f(-15,20)-f(-15,30)}{-10} = \dfrac{-24-(-26)}{-10} = \dfrac{2}{-10} = -0.2$. Averaging these values, we estimate $f_v(-15,30)$ to be approximately -0.15 . Thus, when the actual temperature is $-15°$ F and the wind speed is 30 km / h, the apparent temperature decreases by about 0.15° F for every km / h that the wind speed increases.

(b) For a fixed wind speed v , the values of the wind–chill index W increase as temperature T increases (look at a column of the table), so $\dfrac{\partial W}{\partial T}$ is positive. For a fixed temperature T , the values of W decrease (or remain constant) as v increases (look at a row of the table), so $\dfrac{\partial W}{\partial v}$ is negative (or perhaps 0).

(c) For fixed values of T , the function values $f(T,v)$ appear to become constant (or nearly constant) as v increases, so the corresponding rate of change is 0 or near 0 as v increases. This suggests that $\lim_{v \to \infty} (\partial W / \partial v) = 0$.

4. **(a)** $\partial h / \partial v$ represents the rate of change of h when we fix t and consider h as a function of v , which describes how quickly the wave heights change when the wind speed changes for a fixed time duration. $\partial h / \partial t$ represents the rate of change of h when we fix v and consider h as a function of t , which describes how quickly the wave heights change when the duration of time changes, but the wind speed is constant.

(b) By Definition 4, $f_v(40,15) = \lim_{h \to 0} \dfrac{f(40+h,15)-f(40,15)}{h}$ which we can approximate by considering $h=10$ and $h=-10$ and using the values given in the table: $f_v(40,15) \approx \dfrac{f(50,15)-f(40,15)}{10} = \dfrac{36-25}{10} = 1.1$, $f_v(40,15) \approx \dfrac{f(30,15)-f(40,15)}{-10} = \dfrac{16-25}{-10} = 0.9$. Averaging these values, we have $f_v(40,15) \approx 1.0$. Thus, when a 40 –knot wind has been blowing for 15 hours, the wave heights should increase by about 1 foot for every knot that the wind speed increases (with the same time duration). Similarly,

$f_t(40,15)=\lim_{h\to 0}\dfrac{f(40,15+h)-f(40,15)}{h}$ which we can approximate by considering $h=5$ and $h=-5$:
$f_t(40,15)\approx\dfrac{f(40,20)-f(40,15)}{5}=\dfrac{28-25}{5}=0.6$, $f_t(40,15)\approx\dfrac{f(40,10)-f(40,15)}{-5}=\dfrac{21-25}{-5}=0.8$.
Averaging these values, we have $f_t(40,15)\approx 0.7$. Thus, when a 40 –knot wind has been blowing for 15 hours, the wave heights increase by about 0.7 feet for every additional hour that the wind blows.
(c) For fixed values of v , the function values $f(v,t)$ appear to increase in smaller and smaller increments, becoming nearly constant as t increases. Thus, the corresponding rate of change is nearly 0 as t increases, suggesting that $\lim_{t\to\infty}(\partial h/\partial t)=0$.

5. **(a)** If we start at $(1,2)$ and move in the positive x –direction, the graph of f increases. Thus $f_x(1,2)$ is positive.
(b) If we start at $(1,2)$ and move in the positive y –direction, the graph of f decreases. Thus $f_y(1,2)$ is negative.

6. **(a)** The graph of f decreases if we start at $(-1,2)$ and move in the positive x –direction, so $f_x(-1,2)$ is negative.
(b) The graph of f decreases if we start at $(-1,2)$ and move in the positive y –direction, so $f_y(-1,2)$ is negative.
(c) $f_{xx}=\dfrac{\partial}{\partial x}\left(f_x\right)$, so f_{xx} is the rate of change of f_x in the x –direction. f_x is negative at $(-1,2)$ and if we move in the positive x –direction, the surface becomes less steep. Thus the values of f_x are increasing and $f_{xx}(-1,2)$ is positive.
(d) f_{yy} is the rate of change of f_y in the y –direction. f_y is negative at $(-1,2)$ and if we move in the positive y –direction, the surface becomes steeper. Thus the values of f_y are decreasing, and $f_{yy}(-1,2)$ is negative.

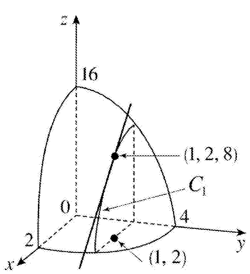
7. First of all, if we start at the point $(3,-3)$ and move in the positive y –direction, we see that both b and c decrease, while a increases. Both b and c have a low point at about $(3,-1.5)$, while a is 0 at this point. So a is definitely the graph of f_y , and one of b and c is the graph of f . To see which is which, we start at the point $(-3,-1.5)$ and move in the positive x –direction. b traces out a line with negative slope, while c traces out a parabola opening downward. This tells us that b is the x –derivative of c . So c is the graph of f , b is the graph of f_x , and a is the graph of f_y .

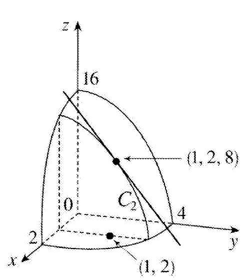
8.

$f_x(2,1)$ is the rate of change of f at $(2,1)$ in the x –direction. If we start at $(2,1)$, where $f(2,1)=10$, and move in the positive x –direction, we reach the next contour line (where $f(x,y)=12$) after approximately 0.6 units. This represents an average rate of change of about $\frac{2}{0.6}$. If we approach the point $(2,1)$ from the left (moving in the positive x –direction) the output values increase from 8 to 10 with an increase in x of approximately 0.9 units, corresponding to an average rate of change of $\frac{2}{0.9}$. A good estimate for $f_x(2,1)$ would be the average of these two, so $f_x(2,1)\approx 2.8$. Similarly, $f_y(2,1)$ is the rate of change of f at $(2,1)$ in the y –direction. If we approach $(2,1)$ from below, the output values decrease from 12 to 10 with a change in y of approximately 1 unit, corresponding to an average rate of change of -2 . If we start at $(2,1)$ and move in the positive y –direction, the output values decrease from 10 to 8 after approximately 0.9 units, a rate of change of $\frac{-2}{0.9}$. Averaging these two results, we estimate $f_y(2,1)\approx -2.1$.

9. $f(x,y)=16-4x^2-y^2 \Rightarrow f_x(x,y)=-8x$ and $f_y(x,y)=-2y \Rightarrow f_x(1,2)=-8$ and $f_y(1,2)=-4$. The graph of f is the paraboloid $z=16-4x^2-y^2$ and the vertical plane $y=2$ intersects it in the parabola $z=12-4x^2$, $y=2$ (the curve C_1 in the first figure).

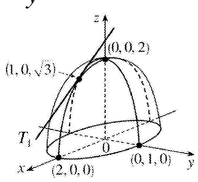
The slope of the tangent line to this parabola at $(1,2,8)$ is $f_x(1,2)=-8$. Similarly the plane $x=1$ intersects the paraboloid in the parabola $z=12-y^2$, $x=1$ (the curve C_2 in the second figure) and the slope of the tangent line at $(1,2,8)$ is $f_y(1,2)=-4$.

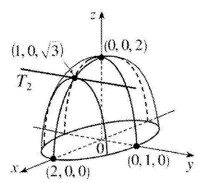




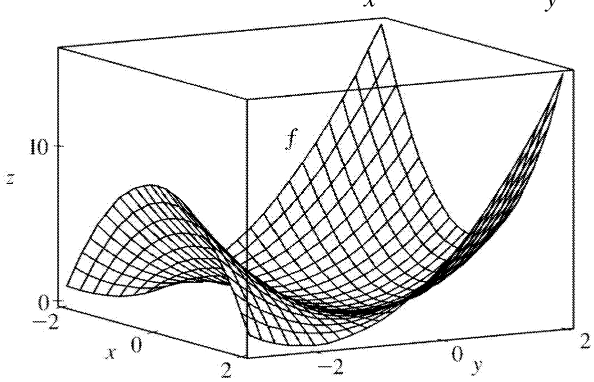
10. $f(x,y)=(4-x^2-4y^2)^{1/2} \Rightarrow f_x(x,y)=-x(4-x^2-4y^2)^{-1/2}$ and $f_y(x,y)=-4y(4-x^2-4y^2)^{-1/2} \Rightarrow$

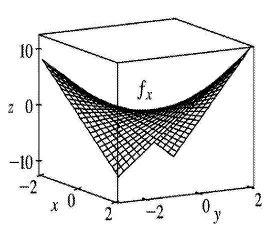
$f_x(1,0)=-\dfrac{1}{\sqrt{3}}$, $f_y(1,0)=0$. The graph of f is the upper half of the ellipsoid $z^2+x^2+4y^2=4$ and the plane $y=0$ intersects the graph in the semicircle $x^2+z^2=4$, $z\geq 0$ and the slope of the tangent line T_1 to this semicircle at $\left(1,0,\sqrt{3}\right)$ is $f_x(1,0)=-\dfrac{1}{\sqrt{3}}$. Similarly the plane $x=1$ intersects the graph in the semi–ellipse $z^2+4y^2=3$, $z\geq 0$ and the slope of the tangent line T_2 to this semi–ellipse at $\left(1,0,\sqrt{3}\right)$ is $f_y(1,0)=0$.

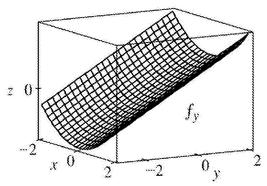




11. $f(x,y)=x^2+y^2+x^2y\Rightarrow f_x=2x+2xy$, $f_y=2y+x^2$

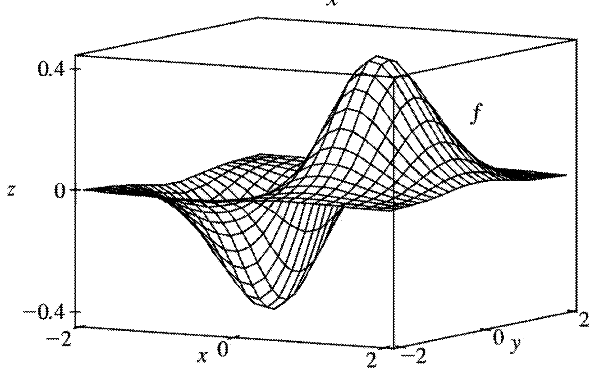


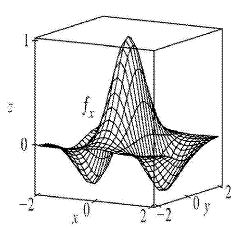


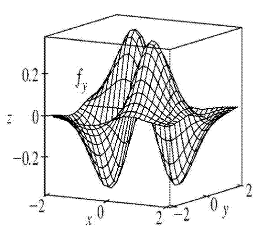


Note that the traces of f in planes parallel to the xz –plane are parabolas which open downward for $y<-1$ and
upward for $y>-1$, and the traces of f_x in these planes are straight lines, which have negative slopes for $y<-1$ and positive slopes for $y>-1$. The traces of f in planes parallel to the yz –plane are parabolas which always open upward, and the traces of f_y in these planes are straight lines with positive slopes.

12. $f(\mathrm{x,y})=xe^{-x^2-y^2} \Rightarrow f_x=x\left(-2xe^{-x^2-y^2}\right)+e^{-x^2-y^2}=e^{-x^2-y^2}(1-2x^2)$, $f_y=-2xye^{-x^2-y^2}$







Note that traces of f in planes parallel to the xz –plane have two extreme values, while traces of f_x in these planes have two zeros. Traces of f in planes parallel to the yz –plane have only one extreme value (a minimum if $x<0$, a maximum if $x>0$), and traces of f_y in these planes have only one zero (going from negative to positive if $x<0$ and from positive to negative if $x>0$).

13. $f(x,y)=3x-2y^4 \Rightarrow f_x(x,y)=3-0=3$, $f_y(x,y)=0-8y^3=-8y^3$

14. $f(x,y)=x^5+3x^3y^2+3xy^4 \Rightarrow f_x(x,y)=5x^4+3\cdot 3x^2\cdot y^2+3\cdot 1\cdot y^4=5x^4+9x^2y^2+3y^4$,
$f_y(x,y)=0+3x^3\cdot 2y+3x\cdot 4y^3=6x^3y+12xy^3$.

15. $z=xe^{3y} \Rightarrow \dfrac{\partial z}{\partial x}=e^{3y}$, $\dfrac{\partial z}{\partial y}=3xe^{3y}$

16. $z=y\ln x \Rightarrow \dfrac{\partial z}{\partial x}=\dfrac{y}{x}$, $\dfrac{\partial z}{\partial y}=\ln x$

17. $f(x,y)=\dfrac{x-y}{x+y} \Rightarrow f_x(x,y)=\dfrac{(1)(x+y)-(x-y)(1)}{(x+y)^2}=\dfrac{2y}{(x+y)^2}$, $f_y(x,y)=\dfrac{(-1)(x+y)-(x-y)(1)}{(x+y)^2}=-\dfrac{2x}{(x+y)^2}$

18. $f(x,y)=x^y \Rightarrow f_x(x,y)=yx^{y-1}$, $f_y(x,y)=x^y\ln x$

19. $w=\sin\alpha\cos\beta \Rightarrow \dfrac{\partial w}{\partial\alpha}=\cos\alpha\cos\beta$, $\dfrac{\partial w}{\partial\beta}=-\sin\alpha\sin\beta$

20. $f(s,t)=\dfrac{st^2}{s^2+t^2} \Rightarrow$
$f_s(s,t)=\dfrac{t^2\left(s^2+t^2\right)-st^2(2s)}{\left(s^2+t^2\right)^2}=\dfrac{t^4-s^2t^2}{\left(s^2+t^2\right)^2}$, $f_t(s,t)=\dfrac{2st\left(s^2+t^2\right)-st^2(2t)}{\left(s^2+t^2\right)^2}=\dfrac{2s^3t}{\left(s^2+t^2\right)^2}$

21. $f(r,s)=r\ln\left(r^2+s^2\right) \Rightarrow f_r(r,s)=r\cdot\dfrac{2r}{r^2+s^2}+\ln\left(r^2+s^2\right)\cdot 1=\dfrac{2r^2}{r^2+s^2}+\ln\left(r^2+s^2\right)$,
$f_s(r,s)=r\cdot\dfrac{2s}{r^2+s^2}+0=\dfrac{2rs}{r^2+s^2}$

22. $f(\text{x,t})=\arctan\left(x\sqrt{t}\right)\Rightarrow f_x(\text{x,t})=\dfrac{1}{1+\left(x\sqrt{t}\right)^2}\cdot\sqrt{t}=\dfrac{\sqrt{t}}{1+x^2t}$,
$f_t(\text{x,t})=\dfrac{1}{1+\left(x\sqrt{t}\right)^2}\cdot x\left(\dfrac{1}{2}t^{-1/2}\right)=\dfrac{x}{2\sqrt{t}\left(1+x^2t\right)}$

23. $u=te^{w/t}\Rightarrow\dfrac{\partial u}{\partial t}=t\cdot e^{w/t}(-wt^{-2})+e^{w/t}\cdot 1=e^{w/t}-\dfrac{w}{t}e^{w/t}=e^{w/t}\left(1-\dfrac{w}{t}\right)$, $\dfrac{\partial u}{\partial w}=te^{w/t}\cdot\dfrac{1}{t}=e^{w/t}$

24. $f(\text{x,y})=\int_y^x\cos(t^2)\,dt\Rightarrow f_x(\text{x,y})=\dfrac{\partial}{\partial x}\int_y^x\cos\left(t^2\right)dt=\cos\left(x^2\right)$ by the Fundamental Theorem of Calculus, Part 1; $f_y(\text{x,y})=\dfrac{\partial}{\partial y}\int_y^x\cos\left(t^2\right)dt=-\dfrac{\partial}{\partial y}\cos\left(t^2\right)dt=-\cos\left(y^2\right)$.

25. $f(\text{x,y,z})=xy^2z^3+3yz\Rightarrow f_x(\text{x,y,z})=y^2z^3$, $f_y(\text{x,y,z})=2xyz^3+3z$, $f_z(\text{x,y,z})=3xy^2z^2+3y$

26. $f(\text{x,y,z})=x^2e^{yz}\Rightarrow f_x(\text{x,y,z})=2xe^{yz}$, $f_y(\text{x,y,z})=x^2e^{yz}(z)=x^2ze^{yz}$, $f_z(\text{x,y,z})=x^2e^{yz}(y)=x^2ye^{yz}$

27. $w=\ln\left(x+2y+3z\right)\Rightarrow\dfrac{\partial w}{\partial x}=\dfrac{1}{x+2y+3z}$, $\dfrac{\partial w}{\partial y}=\dfrac{2}{x+2y+3z}$, $\dfrac{\partial w}{\partial z}=\dfrac{3}{x+2y+3z}$

28. $w=\sqrt{r^2+s^2+t^2}\Rightarrow\dfrac{\partial w}{\partial r}=\dfrac{1}{2}(r^2+s^2+t^2)^{-1/2}(2r)=\dfrac{r}{\sqrt{r^2+s^2+t^2}}$, $\dfrac{\partial w}{\partial s}=\dfrac{s}{\sqrt{r^2+s^2+t^2}}$, $\dfrac{\partial w}{\partial t}=\dfrac{t}{\sqrt{r^2+s^2+t^2}}$

29. $u=xe^{-t}\sin\theta\Rightarrow\dfrac{\partial u}{\partial x}=e^{-t}\sin\theta$, $\dfrac{\partial u}{\partial t}=-xe^{-t}\sin\theta$, $\dfrac{\partial u}{\partial\theta}=xe^{-t}\cos\theta$

30. $u=x^{y/z}\Rightarrow u_x=\dfrac{y}{z}x^{(y/z)-1}$, $u_y=x^{y/z}\ln x\cdot\dfrac{1}{z}=\dfrac{x^{y/z}}{z}\ln x$, $u_z=x^{y/z}\ln x\cdot\dfrac{-y}{z^2}=-\dfrac{yx^{y/z}}{z^2}\ln x$

31. $f(\text{x,y,z,t})=xyz^2\tan(yt)\Rightarrow f_x(\text{x,y,z,t})=yz^2\tan(yt)$,
$f_y(\text{x,y,z,t})=xyz^2\cdot\sec^2(yt)\cdot t+xz^2\tan(yt)=xyz^2t\sec^2(yt)+xz^2\tan(yt)$,
$f_z(\text{x,y,z,t})=2xyz\tan(yt)$, $f_t(\text{x,y,z,t})=xyz^2\sec^2(yt)\cdot y=xy^2z^2\sec^2(yt)$.

32.

$f(x,y,z,t)=\dfrac{xy^2}{t+2z} \Rightarrow$

$f_x(x,y,z,t)=\dfrac{y^2}{t+2z}$, $f_y(x,y,z,t)=\dfrac{2xy}{t+2z}$,

$f_z(x,y,z,t)=xy^2(-1)(t+2z)^{-2}(2)=-\dfrac{2xy^2}{(t+2z)^2}$, $f_t(x,y,z,t)=xy^2(-1)(t+2z)^{-2}(1)=-\dfrac{xy^2}{(t+2z)^2}$.

33. $u=\sqrt{x_1^2+x_2^2+\cdot\cdot\cdot+x_n^2}$. For each $i=1\ ,\ \ldots\ ,\ n$,

$u_{x_i}=\dfrac{1}{2}\left(x_1^2+x_2^2+\cdot\cdot\cdot+x_n^2\right)^{-1/2}\left(2x_i\right)=\dfrac{x_i}{\sqrt{x_1^2+x_2^2+\cdot\cdot\cdot+x_n^2}}$.

34. $u=\sin\left(x_1+2x_2+\cdot\cdot\cdot+nx_n\right)$. For each $i=1\ ,\ \ldots\ ,\ n$, $u_{x_i}=i\cos\left(x_1+2x_2+\cdot\cdot\cdot+nx_n\right)$.

35. $f(x,y)=\sqrt{x^2+y^2} \Rightarrow f_x(x,y)=\dfrac{1}{2}\left(x^2+y^2\right)^{-1/2}(2x)=\dfrac{x}{\sqrt{x^2+y^2}}$, so $f_x(3,4)=\dfrac{3}{\sqrt{3^2+4^2}}=\dfrac{3}{5}$.

36. $f(x,y)=\sin\ (2x+3y) \Rightarrow f_y(x,y)=\cos\ (2x+3y)\cdot 3=3\cos\ (2x+3y)$, so

$f_y(-6,4)=3\cos\ \left[2(-6)+3(4)\right]=3\cos 0=3$.

37. $f(x,y,z)=\dfrac{x}{y+z}=x(y+z)^{-1} \Rightarrow f_z(x,y,z)=x(-1)(y+z)^{-2}=-\dfrac{x}{(y+z)^2}$, so $f_z(3,2,1)=-\dfrac{3}{(2+1)^2}=-\dfrac{1}{3}$.

38. $f(u,v,w)=w\tan(uv) \Rightarrow f_v(u,v,w)=w\sec^2(uv)\cdot u=uw\sec^2(uv)$, so $f_v(2,0,3)=(2)(3)\sec^2(2\cdot 0)=6.$

39. $f(x,y)=x^2-xy+2y^2 \Rightarrow$

$$\begin{aligned} f_x(x,y) &= \lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h}=\lim_{h\to 0}\frac{(x+h)^2-(x+h)y+2y^2-\left(x^2-xy+2y^2\right)}{h} \\ &= \lim_{h\to 0}\frac{h(2x-y+h)}{h}=\lim_{h\to 0}(2x-y+h)=2x-y \end{aligned}$$

$$f_y(x,y) = \lim_{h\to 0} \frac{f(x,y+h)-f(x,y)}{h} = \lim_{h\to 0} \frac{x^2-x(y+h)+2(y+h)^2-\left(x^2-xy+2y^2\right)}{h}$$

$$= \lim_{h\to 0} \frac{h(4y-x+2h)}{h} = \lim_{h\to 0} (4y-x+2h) = 4y-x$$

40. $f(x,y)=\sqrt{3x-y} \Rightarrow$

$$f_x(x,y) = \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{3(x+h)-y}-\sqrt{3x-y}}{h} \cdot \frac{\sqrt{3(x+h)-y}+\sqrt{3x-y}}{\sqrt{3(x+h)-y}+\sqrt{3x-y}}$$

$$= \lim_{h\to 0} \frac{3}{\sqrt{3(x+h)-y}+\sqrt{3x-y}} = \frac{3}{2\sqrt{3x-y}}$$

$$f_y(x,y) = \lim_{h\to 0} \frac{f(x,y+h)-f(x,y)}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{3x-(y+h)}-\sqrt{3x-y}}{h} \cdot \frac{\sqrt{3x-(y+h)}+\sqrt{3x-y}}{\sqrt{3x-(y+h)}+\sqrt{3x-y}}$$

$$= \lim_{h\to 0} \frac{-1}{\sqrt{3x-(y+h)}+\sqrt{3x-y}} = \frac{-1}{2\sqrt{3x-y}}$$

41. $x^2+y^2+z^2=3xyz \Rightarrow \frac{\partial}{\partial x}\left(x^2+y^2+z^2\right)=\frac{\partial}{\partial x}(3xyz) \Rightarrow 2x+0+2z\frac{\partial z}{\partial x}=3y\left(x\frac{\partial z}{\partial x}+z\cdot 1\right) \Leftrightarrow$
$2z\frac{\partial z}{\partial x}-3xy\frac{\partial z}{\partial x}=3yz-2x \Leftrightarrow (2z-3xy)\frac{\partial z}{\partial x}=3yz-2x$, so $\frac{\partial z}{\partial x}=\frac{3yz-2x}{2z-3xy}$.
$\frac{\partial}{\partial y}\left(x^2+y^2+z^2\right)=\frac{\partial}{\partial y}(3xyz) \Rightarrow 0+2y+2z\frac{\partial z}{\partial y}=3x\left(y\frac{\partial z}{\partial y}+z\cdot 1\right) \Leftrightarrow 2z\frac{\partial z}{\partial y}-3xy\frac{\partial z}{\partial y}=3xz-2y \Leftrightarrow$
$(2z-3xy)\frac{\partial z}{\partial y}=3xz-2y$, so $\frac{\partial z}{\partial y}=\frac{3xz-2y}{2z-3xy}$.

42. $yz=\ln(x+z) \Rightarrow \frac{\partial}{\partial x}(yz)=\frac{\partial}{\partial x}(\ln(x+z)) \Rightarrow y\frac{\partial z}{\partial x}=\frac{1}{x+z}\left(1+\frac{\partial z}{\partial x}\right) \Leftrightarrow \left(y-\frac{1}{x+z}\right)\frac{\partial z}{\partial x}=\frac{1}{x+z}$, so
$\frac{\partial z}{\partial x}=\frac{1/(x+z)}{y-1/(x+z)}=\frac{1}{y(x+z)-1}$.
$\frac{\partial}{\partial y}(yz)=\frac{\partial}{\partial y}(\ln(x+z)) \Rightarrow y\frac{\partial z}{\partial y}+z\cdot 1=\frac{1}{x+z}\left(0+\frac{\partial z}{\partial y}\right) \Leftrightarrow \left(y-\frac{1}{x+z}\right)\frac{\partial z}{\partial y}=-z$, so
$\frac{\partial z}{\partial y}=\frac{-z}{y-1/(x+z)}=\frac{z(x+z)}{1-y(x+z)}$.

43. $x-z=\arctan(yz) \Rightarrow \frac{\partial}{\partial x}(x-z)=\frac{\partial}{\partial x}(\arctan(yz)) \Rightarrow 1-\frac{\partial z}{\partial x}=\frac{1}{1+(yz)^2}\cdot y\frac{\partial z}{\partial x} \Leftrightarrow 1=\left(\frac{y}{1+y^2z^2}+1\right)\frac{\partial z}{\partial x} \Leftrightarrow$

$1=\left(\frac{y+1+y^2z^2}{1+y^2z^2}\right)\frac{\partial z}{\partial x}$, so $\frac{\partial z}{\partial x}=\frac{1+y^2z^2}{1+y+y^2z^2}$.

$\frac{\partial}{\partial y}(x-z)=\frac{\partial}{\partial y}(\arctan(yz)) \Rightarrow 0-\frac{\partial z}{\partial y}=\frac{1}{1+(yz)^2}\cdot\left(y\frac{\partial z}{\partial y}+z\cdot 1\right) \Leftrightarrow -\frac{z}{1+y^2z^2}=\left(\frac{y}{1+y^2z^2}+1\right)\frac{\partial z}{\partial y} \Leftrightarrow$

$-\frac{z}{1+y^2z^2}=\left(\frac{y+1+y^2z^2}{1+y^2z^2}\right)\frac{\partial z}{\partial y} \Leftrightarrow \frac{\partial z}{\partial y}=-\frac{z}{1+y+y^2z^2}$.

44. $\sin(xyz)=x+2y+3z \Rightarrow \frac{\partial}{\partial x}(\sin(xyz))=\frac{\partial}{\partial x}(x+2y+3z) \Rightarrow \cos(xyz)\cdot y\left(x\frac{\partial z}{\partial x}+z\right)=1+3\frac{\partial z}{\partial x} \Leftrightarrow ($

$xy\cos(xyz)-3)\frac{\partial z}{\partial x}=1-yz\cos(xyz)$, so $\frac{\partial z}{\partial x}=\frac{1-yz\cos(xyz)}{xy\cos(xyz)-3}$.

$\frac{\partial}{\partial y}(\sin(xyz))=\frac{\partial}{\partial y}(x+2y+3z) \Rightarrow \cos(xyz)\cdot x\left(y\frac{\partial z}{\partial y}+z\right)=2+3\frac{\partial z}{\partial y} \Leftrightarrow (xy\cos(xyz)-3)\frac{\partial z}{\partial y}=2-xz\cos(xyz)$

, so $\frac{\partial z}{\partial y}=\frac{2-xz\cos(xyz)}{xy\cos(xyz)-3}$.

45. **(a)** $z=f(x)+g(y) \Rightarrow \frac{\partial z}{\partial x}=f'(x)$, $\frac{\partial z}{\partial y}=g'(y)$

(b) $z=f(x+y)$. Let $u=x+y$. Then $\frac{\partial z}{\partial x}=\frac{df}{du}\frac{\partial u}{\partial x}=\frac{df}{du}(1)=f'(u)=f'(x+y)$,

$\frac{\partial z}{\partial y}=\frac{df}{du}\frac{\partial u}{\partial y}=\frac{df}{du}(1)=f'(u)=f'(x+y)$.

46. **(a)** $z=f(x)g(y) \Rightarrow \frac{\partial z}{\partial x}=f'(x)g(y)$, $\frac{\partial z}{\partial y}=f(x)g'(y)$

(b) $z=f(xy)$. Let $u=xy$. Then $\frac{\partial u}{\partial x}=y$ and $\frac{\partial u}{\partial y}=x$. Hence $\frac{\partial z}{\partial x}=\frac{df}{du}\frac{\partial u}{\partial x}=\frac{df}{du}\cdot y=yf'(u)=yf'(xy)$ and

$\frac{\partial z}{\partial y}=\frac{df}{du}\frac{\partial u}{\partial y}=\frac{df}{du}\cdot x=xf'(u)=xf'(xy)$.

(c) $z=f\left(\frac{x}{y}\right)$. Let $u=\frac{x}{y}$. Then $\frac{\partial u}{\partial x}=\frac{1}{y}$ and $\frac{\partial u}{\partial y}=-\frac{x}{y^2}$. Hence $\frac{\partial z}{\partial x}=\frac{df}{du}\frac{\partial u}{\partial x}=f'(u)\frac{1}{y}=\frac{f'(x/y)}{y}$

and $\frac{\partial z}{\partial y}=\frac{df}{du}\frac{\partial u}{\partial y}=f'(u)\left(-\frac{x}{y^2}\right)=-\frac{xf'(x/y)}{y^2}$.

47. $f(\mathrm{x,y})=x^4-3x^2y^3 \Rightarrow f_x(\mathrm{x,y})=4x^3-6xy^3$, $f_y(\mathrm{x,y})=-9x^2y^2$. Then $f_{xx}(\mathrm{x,y})=12x^2-6y^3$,

$f_{xy}(x,y)=-18xy^2$, $f_{yx}(x,y)=-18xy^2$, and $f_{yy}(x,y)=-18x^2y$.

48. $f(x,y)=\ln(3x+5y) \Rightarrow f_x(x,y)=\dfrac{3}{3x+5y}$, $f_y(x,y)=\dfrac{5}{3x+5y}$. Then
$f_{xx}(x,y)=3(-1)(3x+5y)^{-2}(3)=-\dfrac{9}{(3x+5y)^2}$, $f_{xy}(x,y)=-\dfrac{15}{(3x+5y)^2}$, $f_{yx}(x,y)=-\dfrac{15}{(3x+5y)^2}$, and
$f_{yy}(x,y)=-\dfrac{25}{(3x+5y)^2}$.

49. $z=\dfrac{x}{x+y}=x(x+y)^{-1} \Rightarrow z_x=\dfrac{1(x+y)-1(x)}{(x+y)^2}=\dfrac{y}{(x+y)^2}$, $z_y=x(-1)(x+y)^{-2}=-\dfrac{x}{(x+y)^2}$. Then
$z_{xx}=y(-2)(x+y)^{-3}=-\dfrac{2y}{(x+y)^3}$, $z_{xy}=\dfrac{1(x+y)^2-y(2)(x+y)}{\left[(x+y)^2\right]^2}=\dfrac{x+y-2y}{(x+y)^3}=\dfrac{x-y}{(x+y)^3}$,
$z_{yx}=-\dfrac{1(x+y)^2-x(2)(x+y)}{\left[(x+y)^2\right]^2}=-\dfrac{-x^2+xy+y^2}{(x+y)^2}=\dfrac{(x+y)(x-y)}{(x+y)^2}=\dfrac{x-y}{(x+y)^3}$, and $z_{yy}=-x(-2)(x+y)^{-3}=\dfrac{2x}{(x+y)^3}$.

50. $z=y\tan 2x \Rightarrow z_x=y\sec^2(2x)\cdot 2=2y\sec^2(2x)$, $z_y=\tan 2x$. Then
$z_{xx}=2y(2)\sec(2x)\cdot\sec(2x)\tan(2x)\cdot 2=8y\sec^2(2x)\tan(2x)$, $z_{xy}=2\sec^2(2x)$,
$z_{yx}=\sec^2(2x)\cdot 2=2\sec^2(2x)$, and $z_{yy}=0$.

51. $u=e^{-s}\sin t \Rightarrow u_s=-e^{-s}\sin t$, $u_t=e^{-s}\cos t$. Then $u_{ss}=e^{-s}\sin t$, $u_{st}=-e^{-s}\cos t$, $u_{ts}=-e^{-s}\cos t$, and
$u_{tt}=-e^{-s}\sin t$.

52. $v=\sqrt{x+y^2} \Rightarrow v_x=\dfrac{1}{2}\left(x+y^2\right)^{-1/2}=\dfrac{1}{2\sqrt{x+y^2}}$, $v_y=\dfrac{1}{2}\left(x+y^2\right)^{-1/2}(2y)=\dfrac{y}{\sqrt{x+y^2}}$. Then
$v_{xx}=\dfrac{1}{2}\left(-\dfrac{1}{2}\right)\left(x+y^2\right)^{-3/2}=-\dfrac{1}{4\left(x+y^2\right)^{3/2}}$, $v_{xy}=\dfrac{1}{2}\left(-\dfrac{1}{2}\right)\left(x+y^2\right)^{-3/2}(2y)=-\dfrac{y}{2\left(x+y^2\right)^{3/2}}$,
$v_{yx}=y\left(-\dfrac{1}{2}\right)\left(x+y^2\right)^{-3/2}=-\dfrac{y}{2\left(x+y^2\right)^{3/2}}$, and

$$v_{yy}=\frac{1\sqrt{x+y^2}-y\left(\frac{1}{2}\right)\left(x+y^2\right)^{-1/2}(2y)}{\left(\sqrt{x+y^2}\right)^2}=\frac{\left(x+y^2\right)-y^2}{\left(x+y^2\right)^{3/2}}=\frac{x}{\left(x+y^2\right)^{3/2}}\ .$$

53. $u=x\sin(x+2y)\Rightarrow u_x=x\cdot\cos(x+2y)(1)+\sin(x+2y)\cdot 1=x\cos(x+2y)+\sin(x+2y)$, $u_{xy}=x(-\sin(x+2y)(2))+\cos(x+2y)(2)=2\cos(x+2y)-2x\sin(x+2y)$ and $u_y=x\cos\left(x+2y\right)(2)=2x\cos(x+2y)$, $u_{yx}=2x\cdot(-\sin(x+2y)(1))+\cos\left(x+2y\right)\cdot 2=2\cos(x+2y)-2x\sin(x+2y)$. Thus $u_{xy}=u_{yx}$.

54. $u=x^4y^2-2xy^5\Rightarrow u_x=4x^3y^2-2y^5$, $u_{xy}=8x^3y-10y^4$ and $u_y=2x^4y-10xy^4$, $u_{yx}=8x^3y-10y^4$. Thus $u_{xy}=u_{yx}$.

55. $u=\ln\sqrt{x^2+y^2}=\ln\left(x^2+y^2\right)^{1/2}=\frac{1}{2}\ln\left(x^2+y^2\right)\Rightarrow u_x=\frac{1}{2}\ \frac{1}{x^2+y^2}\cdot 2x=\frac{x}{x^2+y^2}$,
$u_{xy}=x(-1)\left(x^2+y^2\right)^{-2}(2y)=-\frac{2xy}{\left(x^2+y^2\right)^2}$ and $u_y=\frac{1}{2}\ \frac{1}{x^2+y^2}\cdot 2y=\frac{y}{x^2+y^2}$,
$u_{yx}=y(-1)\left(x^2+y^2\right)^{-2}(2x)=-\frac{2xy}{\left(x^2+y^2\right)^2}$. Thus $u_{xy}=u_{yx}$.

56. $u=xye^y\Rightarrow u_x=ye^y$, $u_{xy}=ye^y+e^y=(y+1)e^y$ and $u_y=x\left(ye^y+e^y\right)=x(y+1)e^y$, $u_{yx}=(y+1)e^y$. Thus $u_{xy}=u_{yx}$.

57. $f(\mathrm{x,y})=3xy^4+x^3y^2\Rightarrow f_x=3y^4+3x^2y^2$, $f_{xx}=6xy^2$, $f_{xxy}=12xy$ and $f_y=12xy^3+2x^3y$, $f_{yy}=36xy^2+2x^3$, $f_{yyy}=72xy$.

58. $f(\mathrm{x,t})=x^2e^{-ct}\Rightarrow f_t=x^2(-ce^{-ct})$, $f_{tt}=x^2(c^2e^{-ct})$, $f_{ttt}=x^2(-c^3e^{-ct})=-c^3x^2e^{-ct}$ and $f_{tx}=2x(-ce^{-ct})$, $f_{txx}=2(-ce^{-ct})=-2ce^{-ct}$.

59. $f(\mathrm{x,y,z})=\cos(4x+3y+2z)\Rightarrow$
$f_x=-\sin(4x+3y+2z)(4)=-4\sin(4x+3y+2z)$,
$f_{xy}=-4\cos(4x+3y+2z)(3)=-12\cos(4x+3y+2z)$,
$f_{xyz}=-12(-\sin(4x+3y+2z))(2)=24\sin(4x+3y+2z)$ and $f_y=-\sin(4x+3y+2z)(3)=-3\sin(4x+3y+2z)$,
$f_{yz}=-3\cos(4x+3y+2z)(2)=-6\cos(4x+3y+2z)$,

$f_{yzz}=-6(-\sin(4x+3y+2z))(2)=12\sin(4x+3y+2z)$.

60. $f(r,s,t)=r\ln(rs^2t^3)\Rightarrow$

$f_r=r\cdot\dfrac{1}{rs^2t^3}(s^2t^3)+\ln(rs^2t^3)\cdot 1=\dfrac{rs^2t^3}{rs^2t^3}+\ln(rs^2t^3)=1+\ln(rs^2t^3)$,

$f_{rs}=\dfrac{1}{rs^2t^3}(2rst^3)=\dfrac{2}{s}=2s^{-1}$, $f_{rss}=-2s^{-2}=-\dfrac{2}{s^2}$ and $f_{rst}=0$.

61. $u=e^{r\theta}\sin\theta\Rightarrow\dfrac{\partial u}{\partial\theta}=e^{r\theta}\cos\theta+\sin\theta\cdot e^{r\theta}(r)=e^{r\theta}(\cos\theta+r\sin\theta)$,

$\dfrac{\partial^2u}{\partial r\partial\theta}=e^{r\theta}(\sin\theta)+(\cos\theta+r\sin\theta)e^{r\theta}(\theta)=e^{r\theta}(\sin\theta+\theta\cos\theta+r\theta\sin\theta)$,

$\dfrac{\partial^3u}{\partial r^2\partial\theta}=e^{r\theta}(\theta\sin\theta)+(\sin\theta+\theta\cos\theta+r\theta\sin\theta)\cdot e^{r\theta}(\theta)=\theta e^{r\theta}(2\sin\theta+\theta\cos\theta+r\theta\sin\theta)$.

62. $z=u\sqrt{v-w}=u(v-w)^{1/2}\Rightarrow\dfrac{\partial z}{\partial w}=u\left[\dfrac{1}{2}(v-w)^{-1/2}(-1)\right]=-\dfrac{1}{2}u(v-w)^{-1/2}$,

$\dfrac{\partial^2z}{\partial v\partial w}=-\dfrac{1}{2}u\left(-\dfrac{1}{2}(v-w)^{-3/2}(1)\right)=\dfrac{1}{4}u(v-w)^{-3/2}$, $\dfrac{\partial^3z}{\partial u\partial v\partial w}=\dfrac{1}{4}(v-w)^{-3/2}$.

63. $w=\dfrac{x}{y+2z}=x(y+2z)^{-1}\Rightarrow\dfrac{\partial w}{\partial x}=(y+2z)^{-1}$, $\dfrac{\partial^2w}{\partial y\partial x}=-(y+2z)^{-2}(1)=-(y+2z)^{-2}$,

$\dfrac{\partial^3w}{\partial z\partial y\partial x}=-(-2)(y+2z)^{-3}(2)=4(y+2z)^{-3}=\dfrac{4}{(y+2z)^3}$ and $\dfrac{\partial w}{\partial y}=x(-1)(y+2z)^{-2}(1)=-x(y+2z)^{-2}$,

$\dfrac{\partial^2w}{\partial x\partial y}=-(y+2z)^{-2}$, $\dfrac{\partial^3w}{\partial x^2\partial y}=0$.

64. $u=x^ay^bz^c$. If $a=0$, or if $b=0$ or 1 , or if $c=0$, 1 , or 2 , then $\dfrac{\partial^6u}{\partial x\partial y^2\partial z^3}=0$. Otherwise

$\dfrac{\partial u}{\partial z}=cx^ay^bz^{c-1}$, $\dfrac{\partial^2u}{\partial z^2}=c(c-1)x^ay^bz^{c-2}$, $\dfrac{\partial^3u}{\partial z^3}=c(c-1)(c-2)x^ay^bz^{c-3}$, $\dfrac{\partial^4u}{\partial y\partial z^3}=bc(c-1)(c-2)x^ay^{b-1}z^{c-3}$,

$\dfrac{\partial^5 u}{\partial y^2 \partial z^3} = b(b-1)c(c-1)(c-2)x^a y^{b-2} z^{c-3}$, and $\dfrac{\partial^6 u}{\partial x \partial y^2 \partial z^3} = ab(b-1)c(c-1)(c-2)x^{a-1} y^{b-2} z^{c-3}$.

65. By Definition 4, $f_x(3,2) = \lim_{h \to 0} \dfrac{f(3+h,2) - f(3,2)}{h}$ which we can approximate by considering $h=0.5$ and $h=-0.5$: $f_x(3,2) \approx \dfrac{f(3.5,2)-f(3,2)}{0.5} = \dfrac{22.4-17.5}{0.5} = 9.8$, $f_x(3,2) \approx \dfrac{f(2.5,2)-f(3,2)}{-0.5} = \dfrac{10.2-17.5}{-0.5} = 14.6$. Averaging these values, we estimate $f_x(3,2)$ to be approximately 12.2 . Similarly, $f_x(3,2.2) = \lim_{h \to 0} \dfrac{f(3+h,2.2)-f(3,2.2)}{h}$ which we can approximate by considering $h=0.5$ and $h=-0.5$: $f_x(3,2.2) \approx \dfrac{f(3.5,2.2)-f(3,2.2)}{0.5} = \dfrac{26.1-15.9}{0.5} = 20.4$, $f_x(3,2.2) \approx \dfrac{f(2.5,2.2)-f(3,2.2)}{-0.5} = \dfrac{9.3-15.9}{-0.5} = 13.2$. Averaging these values, we have $f_x(3,2.2) \approx 16.8$.

To estimate $f_{xy}(3,2)$, we first need an estimate for $f_x(3,1.8)$:

$f_x(3,1.8) \approx \dfrac{f(3.5,1.8)-f(3,1.8)}{0.5} = \dfrac{20.0-18.1}{0.5} = 3.8$,

$f_x(3,1.8) \approx \dfrac{f(2.5,1.8)-f(3,1.8)}{-0.5} = \dfrac{12.5-18.1}{-0.5} = 11.2$. Averaging these values, we get $f_x(3,1.8) \approx 7.5$.

Now $f_{xy}(x,y) = \dfrac{\partial}{\partial y}\left[f_x(x,y)\right]$ and $f_x(x,y)$ is itself a function of 2 variables, so Definition 4 says that $f_{xy}(x,y) = \dfrac{\partial}{\partial y}\left[f_x(x,y)\right] = \lim_{h \to 0} \dfrac{f_x(x,y+h)-f_x(x,y)}{h} \Rightarrow f_{xy}(3,2) = \lim_{h \to 0} \dfrac{f_x(3,2+h)-f_x(3,2)}{h}$.

We can estimate this value using our previous work with $h=0.2$ and $h=-0.2$:

$f_{xy}(3,2) \approx \dfrac{f_x(3,2.2)-f_x(3,2)}{0.2} = \dfrac{16.8-12.2}{0.2} = 23$, $f_{xy}(3,2) \approx \dfrac{f_x(3,1.8)-f_x(3,2)}{-0.2} = \dfrac{7.5-12.2}{-0.2} = 23.5$.

Averaging these values, we estimate $f_{xy}(3,2)$ to be approximately 23.25 .

66. **(a)** If we fix y and allow x to vary, the level curves indicate that the value of f decreases as we move through P in the positive x –direction, so f_x is negative at P .

(b) If we fix x and allow y to vary, the level curves indicate that the value of f increases as we move through P in the positive y –direction, so f_y is positive at P .

(c) $f_{xx}=\frac{\partial}{\partial x}\left(f_x\right)$, so if we fix y and allow x to vary, f_{xx} is the rate of change of f_x as x increases. Note that at points to the right of P the level curves are spaced farther apart (in the x –direction) than at points to the left of P , demonstrating that f decreases less quickly with respect to x to the right of P. So as we move through P in the positive x –direction the (negative) value of f_x increases, hence $\frac{\partial}{\partial x}\left(f_x\right)=f_{xx}$ is positive at P .

(d) $f_{xy}=\frac{\partial}{\partial y}\left(f_x\right)$, so if we fix x and allow y to vary, f_{xy} is the rate of change of f_x as y increases. The level curves are closer together (in the x –direction) at points above P than at those below P , demonstrating that f decreases more quickly with respect to x for y –values above P . So as we move through P in the positive y –direction, the (negative) value of f_x decreases, hence f_{xy} is negative.

(e) $f_{yy}=\frac{\partial}{\partial y}\left(f_y\right)$, so if we fix x and allow y to vary, f_{yy} is the rate of change of f_y as y increases. The level curves are closer together (in the y –direction) at points above P than at those below P , demonstrating that f increases more quickly with respect to y above P . So as we move through P in the positive y –direction the (positive) value of f_y increases, hence $\frac{\partial}{\partial y}\left(f_y\right)=f_{yy}$ is positive at P .

67. $u=e^{-\alpha^2k^2t}\sin kx \Rightarrow u_x=ke^{-\alpha^2k^2t}\cos kx$, $u_{xx}=-k^2e^{-\alpha^2k^2t}\sin kx$, and $u_t=-\alpha^2k^2e^{-\alpha^2k^2t}\sin kx$. Thus $\alpha^2u_{xx}=u_t$.

68. **(a)** $u=x^2+y^2 \Rightarrow u_x=2x$, $u_{xx}=2$; $u_y=2y$, $u_{yy}=2$. Thus $u_{xx}+u_{yy}\neq 0$ and $u=x^2+y^2$ does not satisfy Laplace’s Equation.

(b) $u=x^2-y^2$ is a solution: $u_{xx}=2$, $u_{yy}=-2$ so $u_{xx}+u_{yy}=0$.

(c) $u=x^3+3xy^2$ is not a solution: $u_x=3x^2+3y^2$, $u_{xx}=6x$; $u_y=6xy$, $u_{yy}=6x$.

(d) $u=\ln\sqrt{x^2+y^2}$ is a solution: $u_x=\frac{1}{\sqrt{x^2+y^2}}\left(\frac{1}{2}\right)\left(x^2+y^2\right)^{-1/2}(2x)=\frac{x}{x^2+y^2}$,

$u_{xx}=\frac{\left(x^2+y^2\right)-x(2x)}{\left(x^2+y^2\right)^2}=\frac{y^2-x^2}{\left(x^2+y^2\right)^2}$. By symmetry, $u_{yy}=\frac{x^2-y^2}{\left(x^2+y^2\right)^2}$, so $u_{xx}+u_{yy}=0$.

(e) $u=\sin x\cosh y+\cos x\sinh y$ is a solution: $u_x=\cos x\cosh y-\sin x\sinh y$, $u_{xx}=-\sin x\cosh y-\cos x\sinh y$, and $u_y=\sin x\sinh y+\cos x\cosh y$, $u_{yy}=\sin x\cosh y+\cos x\sinh y$.

(f) $u=e^{-x}\cos y-e^{-y}\cos x$ is a solution:

$u_x=-e^{-x}\cos y+e^{-y}\sin x$, $u_{xx}=e^{-x}\cos y+e^{-y}\cos x$, and $u_y=-e^{-x}\sin y+e^{-y}\cos x$, $u_{yy}=-e^{-x}\cos y-e^{-y}\cos x$.

69. $u=\dfrac{1}{\sqrt{x^2+y^2+z^2}}\Rightarrow u_x=\left(-\dfrac{1}{2}\right)\left(x^2+y^2+z^2\right)^{-3/2}(2x)=-x\left(x^2+y^2+z^2\right)^{-3/2}$ and

$u_{xx}=-\left(x^2+y^2+z^2\right)^{-3/2}-x\left(-\dfrac{3}{2}\right)\left(x^2+y^2+z^2\right)^{-5/2}(2x)=\dfrac{2x^2-y^2-z^2}{\left(x^2+y^2+z^2\right)^{5/2}}$.

By symmetry, $u_{yy}=\dfrac{2y^2-x^2-z^2}{\left(x^2+y^2+z^2\right)^{5/2}}$ and $u_{zz}=\dfrac{2z^2-x^2-y^2}{\left(x^2+y^2+z^2\right)^{5/2}}$. Thus

$u_{xx}+u_{yy}+u_{zz}=\dfrac{2x^2-y^2-z^2+2y^2-x^2-z^2+2z^2-x^2-y^2}{\left(x^2+y^2+z^2\right)^{5/2}}=0$.

70. **(a)** $u=\sin(kx)\sin(akt)\Rightarrow u_t=ak\sin(kx)\cos(akt)$, $u_{tt}=-a^2k^2\sin(kx)\sin(akt)$, $u_x=k\cos(kx)\sin(akt)$,

$u_{xx}=-k^2\sin(kx)\sin(akt)$. Thus $u_{tt}=a^2u_{xx}$.

(b) $u=\dfrac{t}{a^2t^2-x^2}\Rightarrow u_t=\dfrac{\left(a^2t^2-x^2\right)-t\left(2a^2t\right)}{\left(a^2t^2-x^2\right)^2}=-\dfrac{a^2t^2+x^2}{\left(a^2t^2-x^2\right)^2}$,

$u_{tt}=\dfrac{-2a^2t\left(a^2t^2-x^2\right)^2+\left(a^2t^2-x^2\right)(2)\left(a^2t^2-x^2\right)\left(2a^2t\right)}{\left(a^2t^2-x^2\right)^4}=\dfrac{2a^4t^3+6a^2tx^2}{a^2t^2-x^2}$,

$u_x=t(-1)\left(a^2t^2-x^2\right)^{-2}(2x)=\dfrac{2tx}{\left(a^2t^2-x^2\right)^2}$,

$u_{xx}=\dfrac{2t\left(a^2t^2-x^2\right)^2-2tx(2)\left(a^2t^2-x^2\right)(-2x)}{\left(a^2t^2-x^2\right)^4}=\dfrac{2a^2t^3-2tx^2+8tx^2}{\left(a^2t^2-x^2\right)^3}=\dfrac{2a^2t^3+6tx^2}{\left(a^2t^2-x^2\right)^4}$.

Thus $u_{tt}=a^2u_{xx}$.

(c) $u=(x-at)^6+(x+at)^6\Rightarrow u_t=-6a(x-at)^5+6a(x+at)^5$, $u_{tt}=30a^2(x-at)^4+30a^2(x+at)^4$,

$u_x=6(x-at)^5+6(x+at)^5$, $u_{xx}=30(x-at)^4+30(x+at)^4$. Thus $u_{tt}=a^2u_{xx}$.

(d) $u=\sin(x-at)+\ln(x+at)\Rightarrow u_t=-a\cos(x-at)+\dfrac{a}{x+at}$, $u_{tt}=-a^2\sin(x-at)-\dfrac{a^2}{(x+at)^2}$,

$u_x=\cos(x-at)+\dfrac{1}{x+at}$,

$u_{xx}=-\sin(x-at)-\dfrac{1}{(x+at)^2}$. Thus $u_{tt}=a^2u_{xx}$.

71. Let $v=x+at$, $w=x-at$. Then $u_t=\dfrac{\partial[f(v)+g(w)]}{\partial t}=\dfrac{df(v)}{dv}\dfrac{\partial v}{\partial t}+\dfrac{dg(w)}{dw}\dfrac{\partial w}{\partial t}=af'(v)-ag'(w)$ and $u_{tt}=\dfrac{\partial[af'(v)-ag'(w)]}{\partial t}=a[af''(v)+ag''(w)]=a^2[f''(v)+g''(w)]$. Similarly, by using the Chain Rule we have $u_x=f'(v)+g'(w)$ and $u_{xx}=f''(v)+g''(w)$. Thus $u_{tt}=a^2u_{xx}$.

72. For each i , $i=1,\ldots,n$, $\partial u/\partial x_i=a_ie^{a_1x_1+a_2x_2+\cdots+a_nx_n}$ and $\partial^2u/\partial x_i^2=a_i^2e^{a_1x_1+a_2x_2+\cdots+a_nx_n}$. Then $\dfrac{\partial^2u}{\partial x_1^2}+\dfrac{\partial^2u}{\partial x_2^2}+\cdots+\dfrac{\partial^2u}{\partial x_n^2}=\left(a_1^2+a_2^2+\cdots+a_n^2\right)e^{a_1x_1+a_2x_2+\cdots+a_nx_n}=e^{a_1x_1+a_2x_2+\cdots+a_nx_n}=u$ since $a_1^2+a_2^2+\cdots+a_n^2=1$.

73. $z_x=e^y+ye^x$, $z_{xx}=ye^x$, $\partial^3z/\partial x^3=ye^x$. By symmetry $z_y=xe^y+e^x$, $z_{yy}=xe^y$ and $\partial^3z/\partial y^3=xe^y$. Then $\partial^3z/\partial x\partial y^2=e^y$ and $\partial^3z/\partial x^2\partial y=e^x$. Thus $z=xe^y+ye^x$ satisfies the given partial differential equation.

74. $P=bL^\alpha K^\beta$, so $\dfrac{\partial P}{\partial L}=\alpha bL^{\alpha-1}K^\beta$ and $\dfrac{\partial P}{\partial K}=\beta bL^\alpha K^{\beta-1}$. Then

$$\begin{aligned}L\frac{\partial P}{\partial L}+K\frac{\partial P}{\partial K}&=L\left(\alpha bL^{\alpha-1}K^\beta\right)+K\left(\beta bL^\alpha K^{\beta-1}\right)=\alpha bL^{1+\alpha-1}K^\beta+\beta bL^\alpha K^{1+\beta-1}\\&=(\alpha+\beta)bL^\alpha K^\beta=(\alpha+\beta)P\end{aligned}$$

75. If we fix $K=K_0, P\left(L,K_0\right)$ is a function of a single variable L , and $\dfrac{dP}{dL}=\alpha\dfrac{P}{L}$ is a separable differential equation. Then $\dfrac{dP}{P}=\alpha\dfrac{dL}{L}\Rightarrow\int\dfrac{dP}{P}=\int\alpha\dfrac{dL}{L}\Rightarrow\ln|P|=\alpha\ln|L|+C\left(K_0\right)$, where $C\left(K_0\right)$ can depend on K_0 . Then $|P|=e^{\alpha\ln|L|+C\left(K_0\right)}$, and since $P>0$ and $L>0$, we have $P=e^{\alpha\ln L}e^{C\left(K_0\right)}=e^{C\left(K_0\right)}e^{\ln L^\alpha}=C_1\left(K_0\right)L^\alpha$ where $C_1\left(K_0\right)=e^{C\left(K_0\right)}$.

76. **(a)** $\partial T/\partial x=-60(2x)/\left(1+x^2+y^2\right)^2$, so at $(2,1)$, $T_x=-240/(1+4+1)^2=-\dfrac{20}{3}$.

(b) $\partial T/\partial y = -60(2y)/\left(1+x^2+y^2\right)^2$, so at $(2,1)$, $T_y = -120/36 = -\frac{10}{3}$. Thus from the point $(2,1)$ the temperature is decreasing at a rate of $\frac{20}{3}$ °F/m in the x-direction and is decreasing at a rate of $\frac{10}{3}$ °F/m in the y-direction.

77. By the Chain Rule, taking the partial derivative of both sides with respect to R_1 gives

$$\frac{\partial R^{-1}}{\partial R}\frac{\partial R}{\partial R_1} = \frac{\partial\left[\left(1/R_1\right)+\left(1/R_2\right)+\left(1/R_3\right)\right]}{\partial R_1} \text{ or } -R^{-2}\frac{\partial R}{\partial R_1} = -R_1^{-2}. \text{ Thus } \frac{\partial R}{\partial R_1} = \frac{R^2}{R_1^2}.$$

78. $P = \frac{mRT}{V}$ so $\frac{\partial P}{\partial V} = \frac{-mRT}{V^2}$; $V = \frac{mRT}{P}$, so $\frac{\partial V}{\partial T} = \frac{mR}{P}$; $T = \frac{PV}{mR}$, so $\frac{\partial T}{\partial P} = \frac{V}{mR}$.

Thus $\frac{\partial P}{\partial V}\frac{\partial V}{\partial T}\frac{\partial T}{\partial P} = \frac{-mRT}{V^2}\frac{mR}{P}\frac{V}{mR} = \frac{-mRT}{PV} = -1$, since $PV = mRT$.

79. By Exercise 78, $PV = mRT \Rightarrow P = \frac{mRT}{V}$, so $\frac{\partial P}{\partial T} = \frac{mR}{V}$. Also, $PV = mRT \Rightarrow V = \frac{mRT}{P}$ and $\frac{\partial V}{\partial T} = \frac{mR}{P}$. Since $T = \frac{PV}{mR}$, we have $T\frac{\partial P}{\partial T}\frac{\partial V}{\partial T} = \frac{PV}{mR}\cdot\frac{mR}{V}\cdot\frac{mR}{P} = mR$.

80. $\frac{\partial W}{\partial T} = 0.6215 + 0.3965v^{0.16}$. When $T = -15$° F and $v = 30$ km / h,
$\frac{\partial W}{\partial T} = 0.6215 + 0.3965(30)^{0.16} \approx 1.3048$, so we would expect the apparent temperature to drop by approximately 1.3° F if the actual temperature decreases by 1° F.
$\frac{\partial W}{\partial v} = -11.37(0.16)v^{-0.84} + 0.3965T(0.16)v^{-0.84}$ and when $T = -15$° F and $v = 30$ km / h,
$\frac{\partial W}{\partial v} = -11.37(0.16)(30)^{-0.84} + 0.3965(-15)(0.16)(30)^{-0.84} \approx -0.1592$, so we would expect the apparent temperature to drop by approximately 0.16° F if the wind speed increases by 1 km / h.

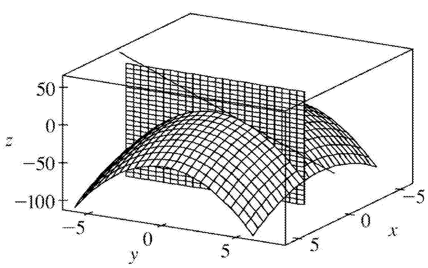
81. $\frac{\partial K}{\partial m} = \frac{1}{2}V^2$, $\frac{\partial K}{\partial V} = mV$, $\frac{\partial^2 K}{\partial V^2} = m$. Thus $\frac{\partial K}{\partial m}\cdot\frac{\partial^2 K}{\partial V^2} = \frac{1}{2}V^2 m = K$.

82. The Law of Cosines says that $a^2 = b^2 + c^2 - 2bc\cos A$. Thus

$\frac{\partial\left(a^2\right)}{\partial a}=\frac{\partial\left(b^2+c^2-2ab\cos A\right)}{\partial a}$ or $2a=-2bc(-\sin A)\frac{\partial A}{\partial a}$, implying that $\frac{\partial A}{\partial a}=\frac{a}{bc\sin A}$. Taking the partial derivative of both sides with respect to b gives $0=2b-2c(\cos A)-2bc(-\sin A)\frac{\partial A}{\partial b}$. Thus $\frac{\partial A}{\partial b}=\frac{c\cos A-b}{bc\sin A}$. By symmetry $\frac{\partial A}{\partial c}=\frac{b\cos A-c}{bc\sin A}$.

83. $f_x(x,y)=x+4y \Rightarrow f_{xy}(x,y)=4$ and $f_y(x,y)=3x-y \Rightarrow f_{yx}(x,y)=3$. Since f_{xy} and f_{yx} are continuous everywhere but $f_{xy}(x,y)\neq f_{yx}(x,y)$, Clairaut's Theorem implies that such a function $f(x,y)$ does not exist.

84. Setting $x=1$, the equation of the parabola of intersection is $z=6-1-1-2y^2=4-2y^2$. The slope of the tangent is $\partial z/\partial y=-4y$, so at $(1,2,-4)$ the slope is -8 . Parametric equations for the line are therefore $x=1$, $y=2+t$, $z=-4-8t$.



85. By the geometry of partial derivatives, the slope of the tangent line is $f_x(1,2)$. By implicit differentiation of $4x^2+2y^2+z^2=16$, we get $8x+2z(\partial z/\partial x)=0 \Rightarrow \partial z/\partial x=-4x/z$, so when $x=1$ and $z=2$ we have $\partial z/\partial x=-2$. So the slope is $f_x(1,2)=-2$. Thus the tangent line is given by $z-2=-2(x-1)$, $y=2$. Taking the parameter to be $t=x-1$, we can write parametric equations for this line: $x=1+t$, $y=2$, $z=2-2t$.

86. $T(x,t)=T_0+T_1e^{-\lambda x}\sin(\omega t-\lambda x)$

(a)

$$\partial T/\partial x = T_1e^{-\lambda x}[\cos(\omega t-\lambda x)(-\lambda)]+T_1(-\lambda e^{-\lambda x})\sin(\omega t-\lambda x)$$
$$= -\lambda T_1e^{-\lambda x}[\sin(\omega t-\lambda x)+\cos(\omega t-\lambda x)]$$

This quantity represents the rate of change of temperature with respect to depth below the surface, at a given time t .

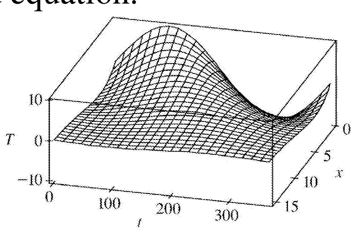
(b) $\partial T/\partial t = T_1 e^{-\lambda x}[\cos(\omega t-\lambda x)(\omega)] = \omega T_1 e^{-\lambda x}\cos(\omega t-\lambda x)$. This quantity represents the rate of change of temperature with respect to time at a fixed depth x .

(c)

$$
\begin{aligned}
T_{xx} &= \frac{\partial}{\partial x}\left(\frac{\partial T}{\partial x}\right) \\
&= -\lambda T_1 \Big(e^{-\lambda x}[\cos(\omega t-\lambda x)(-\lambda) - \sin(\omega t-\lambda x)(-\lambda)] \\
&\quad \Big) + e^{-\lambda x}(-\lambda)[\sin(\omega t-\lambda x)+\cos(\omega t-\lambda x)] \\
&= 2\lambda^2 T_1 e^{-\lambda x}\cos(\omega t-\lambda x)
\end{aligned}
$$

But from part (b), $T_t = \omega T_1 e^{-\lambda x}\cos(\omega t-\lambda x) = \dfrac{\omega}{2\lambda^2} T_{xx}$. So with $k = \dfrac{\omega}{2\lambda^2}$, the function T satisfies the heat equation.

(d)



Note that near the surface (that is, for small x) the temperature varies greatly as t changes, but deeper (for large x) the temperature is more stable.

(e) The term $-\lambda x$ is a phase shift: it represents the fact that since heat diffuses slowly through soil, it takes time for changes in the surface temperature to affect the temperature at deeper points. As x increases, the phase shift also increases. For example, at the surface the highest temperature is reached at $t \approx 100$, whereas at a depth of 5 feet the peak temperature is attained at $t \approx 150$, and at a depth of 10 feet, at $t \approx 220$.

87. By Clairaut's Theorem, $f_{xyy} = (f_{xy})_y = (f_{yx})_y = f_{yxy} = (f_y)_{xy} = (f_y)_{yx} = f_{yyx}$.

88. **(a)** Since we are differentiating n times, with two choices of variable at each differentiation, there are 2^n n th order partial derivatives.

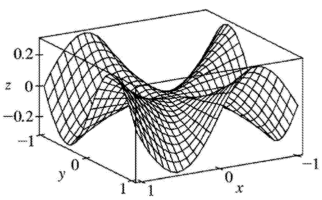
(b) If these partial derivatives are all continuous, then the order in which the partials are taken doesn't affect the value of the result, that is, all n th order partial derivatives with p partials with respect to x and $n-p$ partials with respect to y are equal. Since the number of partials taken with respect to x for an n th order partial derivative can range from 0 to n , a function of two variables has $n+1$ distinct partial derivatives of order n if these partial derivatives are all continuous.

(c) Since n differentiations are to be performed with three choices of variable at each differentiation, there are 3^n n th order partial derivatives of a function of three variables.

89. Let $g(x)=f(x,0)=x(x^2)^{-3/2}e^0=x|x|^{-3}$. But we are using the point $(1,0)$, so near $(1,0)$, $g(x)=x^{-2}$. Then $g'(x)=-2x^{-3}$ and $g'(1)=-2$, so using (1) we have $f_x(1,0)=g'(1)=-2$.

90. $f_x(0,0)=\lim_{h\to 0}\dfrac{f(0+h,0)-f(0,0)}{h}=\lim_{h\to 0}\dfrac{\left(h^3+0\right)^{1/3}-0}{h}=\lim_{h\to 0}\dfrac{h}{h}=1$.

Or: Let $g(x)=f(x,0)=\sqrt[3]{x^3+0}=x$. Then $g'(x)=1$ and $g'(0)=1$ so, by (1), $f_x(0,0)=g'(0)=1$.

91. **(a)** 

(b) For $(x,y)\neq(0,0)$, $f_x(x,y)=\dfrac{\left(3x^2y-y^3\right)\left(x^2+y^2\right)-\left(x^3y-xy^3\right)(2x)}{\left(x^2+y^2\right)^2}=\dfrac{x^4y+4x^2y^3-y^5}{\left(x^2+y^2\right)^2}$, and by symmetry

$f_y(x,y)=\dfrac{x^5-4x^3y^2-xy^4}{\left(x^2+y^2\right)^2}$.

(c) $f_x(0,0)=\lim_{h\to 0}\dfrac{f(h,0)-f(0,0)}{h}=\lim_{h\to 0}\dfrac{\left(0/h^2\right)-0}{h}=0$ and $f_y(0,0)=\lim_{h\to 0}\dfrac{f(0,h)-f(0,0)}{h}=0$.

(d) By (3), $f_{xy}(0,0)=\dfrac{\partial f_x}{\partial y}=\lim_{h\to 0}\dfrac{f_x(0,h)-f_x(0,0)}{h}=\lim_{h\to 0}\dfrac{\left(-h^5-0\right)/h^4}{h}=-1$ while by (2),

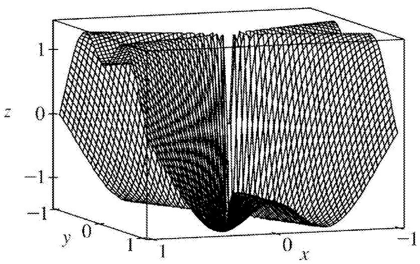
$f_{yx}(0,0)=\dfrac{\partial f_y}{\partial x}=\lim_{h\to 0}\dfrac{f_y(h,0)-f_y(0,0)}{h}=\lim_{h\to 0}\dfrac{h^5/h^4}{h}=1$.

(e) For $(x,y)\neq(0,0)$, we use a CAS to compute

$f_{xy}(x,y)=\dfrac{x^6+9x^4y^2-4x^2y^4+4y^6}{\left(x^2+y^2\right)^3}$.

Now as $(x,y)\to(0,0)$ along the x –axis, $f_{xy}(x,y)\to 1$ while as $(x,y)\to(0,0)$ along the y –axis,

$f_{xy}(x,y) \to 4$. Thus f_{xy} isn't continuous at (0,0) and Clairaut's Theorem doesn't apply, so there is no contradiction. The graphs of f_{xy} and f_{yx} are identical except at the origin, where we observe the discontinuity.



1. $z=f(x,y)=4x^2-y^2+2y \Rightarrow f_x(x,y)=8x$, $f_y(x,y)=-2y+2$, so $f_x(-1,2)=-8$, $f_y(-1,2)=-2$. By Equation 2, an equation of the tangent plane is $z-4=f_x(-1,2)[x-(-1)]+f_y(-1,2)(y-2) \Rightarrow z-4=-8(x+1)-2(y-2)$ or $z=-8x-2y$.

2. $z=f(x,y)=9x^2+y^2+6x-3y+5 \Rightarrow f_x(x,y)=18x+6$, $f_y(x,y)=2y-3$, so $f_x(1,2)=24$ and $f_y(1,2)=1$. By Equation 2, an equation of the tangent plane is $z-18=f_x(1,2)(x-1)+f_y(1,2)(y-2) \Rightarrow$ $z-18=24(x-1)+1(y-2)$ or $z=24x+y-8$.

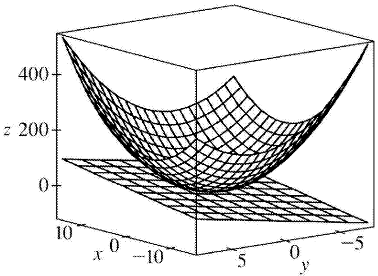
3. $z=f(x,y)=\sqrt{4-x^2-2y^2} \Rightarrow f_x(x,y)=\frac{1}{2}(4-x^2-2y^2)^{-1/2}(-2x)=-\frac{x}{\sqrt{4-x^2-2y^2}}$,
$f_y(x,y)=\frac{1}{2}(4-x^2-2y^2)^{-1/2}(-4y)=-\frac{2y}{\sqrt{4-x^2-2y^2}}$, so $f_x(1,-1)=-1$ and $f_y(1,-1)=2$. Thus, an equation of the tangent plane is $z-1=f_x(1,-1)(x-1)+f_y(1,-1)[y-(-1)] \Rightarrow z-1=-1(x-1)+2(y+1)$ or $x-2y+z=4$.

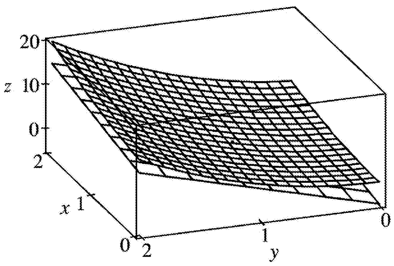
4. $z=f(x,y)=y\ln x \Rightarrow f_x(x,y)=y/x$, $f_y(x,y)=\ln x$, so $f_x(1,4)=4$, $f_y(1,4)=0$, and an equation of the tangent plane is $z-0=f_x(1,4)(x-1)+f_y(1,4)(y-4) \Rightarrow z=4(x-1)+0(y-4)$ or $z=4x-4$.

5. $z=f(x,y)=y\cos(x-y) \Rightarrow f_x=y(-\sin(x-y)(1))=-y\sin(x-y)$,
$f_y=y(-\sin(x-y)(-1))+\cos(x-y)=y\sin(x-y)+\cos(x-y)$, so $f_x(2,2)=-2\sin(0)=0$,
$f_y(2,2)=2\sin(0)+\cos(0)=1$ and an equation of the tangent plane is $z-2=0(x-2)+1(y-2)$ or $z=y$.

6. $z=f(x,y)=e^{x^2-y^2} \Rightarrow f_x(x,y)=2xe^{x^2-y^2}$, $f_y(x,y)=-2ye^{x^2-y^2}$, so $f_x(1,-1)=2$, $f_y(1,-1)=2$. By Equation 2, an equation of the tangent plane is $z-1=f_x(1,-1)(x-1)+f_y(1,-1)[y-(-1)] \Rightarrow z-1=2(x-1)+2(y+1)$ or $z=2x+2y+1$.

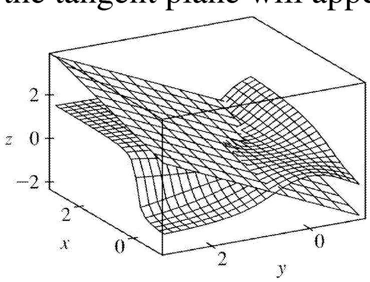
7. $z=f(x,y)=x^2+xy+3y^2$, so $f_x(x,y)=2x+y \Rightarrow f_x(1,1)=3$, $f_y(x,y)=x+6y \Rightarrow f_y(1,1)=7$ and an equation of the tangent plane is $z-5=3(x-1)+7(y-1)$ or $z=3x+7y-5$. After zooming in, the surface and the tangent plane become almost indistinguishable. (Here, the tangent plane is below the surface.) If we zoom in farther, the surface and the tangent plane will appear to coincide.

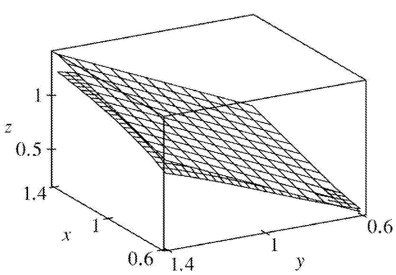




8. $z=f(x,y)=\arctan\left(xy^2\right)\Rightarrow f_x=\frac{1}{1+\left(xy^2\right)^2}(y^2)=\frac{y^2}{1+x^2y^4}$, $f_y=\frac{1}{1+\left(xy^2\right)^2}(2xy)=\frac{2xy}{1+x^2y^4}$,

$f_x(1,1)=\frac{1}{1+1}=\frac{1}{2}$, $f_y(1,1)=\frac{2}{1+1}=1$, so an equation of the tangent plane is $z-\frac{\pi}{4}=\frac{1}{2}(x-1)+1(y-1)$ or $z=\frac{1}{2}x+y-\frac{3}{2}+\frac{\pi}{4}$. After zooming in, the surface and the tangent plane become almost indistinguishable. (Here the tangent plane is above the surface.) If we zoom in farther, the surface and the tangent plane will appear to coincide.

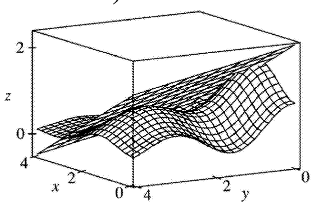


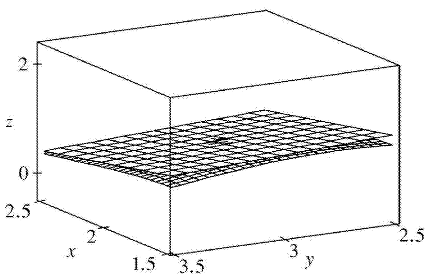


9. $f(x,y)=e^{-\left(x^2+y^2\right)/15}\left(\sin^2 x+\cos^2 y\right)$. A CAS gives

$f_x = -\frac{2}{15} e^{-\left(x^2+y^2\right)/15}\left(x\sin^2 x + x\cos^2 y - 15\sin x\cos x\right)$ and

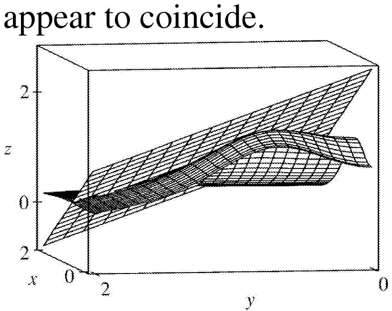
$f_y = -\frac{2}{15} e^{-\left(x^2+y^2\right)/15}\left(y\sin^2 x + y\cos^2 y + 15\sin y\cos y\right)$. We use the CAS to evaluate these at $(2,3)$, and then substitute the results into Equation 2 in order to plot the tangent plane. After zooming in, the surface and the tangent plane become almost indistinguishable. (Here, the tangent plane is above the surface.) If we zoom in farther, the surface and the tangent plane will appear to coincide.

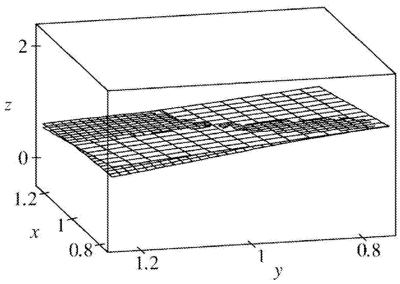




10. $f(x,y) = \frac{\sqrt{1+4x^2+4y^2}}{1+x^4+y^4}$. A CAS gives $f_x = \frac{4x\left(1-3x^4+y^4-x^2-4x^2y^2\right)}{\sqrt{1+4x^2+4y^2}\left(1+x^4+y^4\right)^2}$ and

$f_y = \frac{4y\left(1-3y^4+x^4-y^2-4x^2y^2\right)}{\sqrt{1+4x^2+4y^2}\left(1+x^4+y^4\right)^2}$. We use the CAS to evaluate these at $(1,1)$, and then substitute the results into Equation 2 to get an equation of the tangent plane: $z = \frac{25-8x-8y}{9}$. After zooming in, the surface and the tangent plane become almost indistinguishable. (Here, the tangent plane is shown with fewer traces than the surface.) If we zoom in farther, the surface and the tangent plane will appear to coincide.





11. $f(x,y)=x\sqrt{y}$. The partial derivatives are $f_x(x,y)=\sqrt{y}$ and $f_y(x,y)=\dfrac{x}{2\sqrt{y}}$, so $f_x(1,4)=2$ and $f_y(1,4)=\dfrac{1}{4}$. Both f_x and f_y are continuous functions for $y>0$, so by Theorem 8, f is differentiable at $(1,4)$. By Equation 3, the linearization of f at $(1,4)$ is given by
$L(x,y)=f(1,4)+f_x(1,4)(x-1)+f_y(1,4)(y-4)=2+2(x-1)+\frac{1}{4}(y-4)=2x+\frac{1}{4}y-1$.

12. $f(x,y)=\dfrac{x}{y}$. The partial derivatives are $f_x(x,y)=\dfrac{1}{y}$ and $f_y(x,y)=-\dfrac{x}{y^2}$, so $f_x(6,3)=\dfrac{1}{3}$ and $f_y(6,3)=-\dfrac{2}{3}$. Both f_x and f_y are continuous functions for $y\neq 0$, so f is differentiable at $(6,3)$ by Theorem 8. The linearization of f at $(6,3)$ is given by
$L(x,y)=f(6,3)+f_x(6,3)(x-6)+f_y(6,3)(y-3)=2+\frac{1}{3}(x-6)-\frac{2}{3}(y-3)=\frac{1}{3}x-\frac{2}{3}y+2$.

13. $f(x,y)=e^x\cos xy$. The partial derivatives are $f_x(x,y)=e^x(\cos xy-y\sin xy)$ and $f_y(x,y)=-xe^x\sin xy$, so $f_x(0,0)=1$ and $f_y(0,0)=0$. Both f_x and f_y are continuous functions, so f is differentiable at $(0,0)$ by Theorem 8. The linearization of f at $(0,0)$ is given by
$L(x,y)=f(0,0)+f_x(0,0)(x-0)+f_y(0,0)(y-0)=1+1(x-0)+0(y-0)=x+1$.

14. $f(x,y)=\sqrt{x+e^{4y}}=\left(x+e^{4y}\right)^{1/2}$. The partial derivatives are $f_x(x,y)=\frac{1}{2}\left(x+e^{4y}\right)^{-1/2}$ and $f_y(x,y)=\frac{1}{2}\left(x+e^{4y}\right)^{-1/2}(4e^{4y})=2e^{4y}\left(x+e^{4y}\right)^{-1/2}$, so $f_x(3,0)=\frac{1}{2}\left(3+e^0\right)^{-1/2}=\frac{1}{4}$ and $f_y(3,0)=2e^0\left(3+e^0\right)^{-1/2}=1$. Both f_x and f_y are continuous functions near $(3,0)$, so f is differentiable at $(3,0)$ by Theorem 8. The linearization of f at $(3,0)$ is
$L(x,y)=f(3,0)+f_x(3,0)(x-3)+f_y(3,0)(y-0)=2+\frac{1}{4}(x-3)+1(y-0)=\frac{1}{4}x+y+\frac{5}{4}$.

15. $f(x,y)=\tan^{-1}(x+2y)$. The partial derivatives are $f_x(x,y)=\dfrac{1}{1+(x+2y)^2}$ and $f_y(x,y)=\dfrac{2}{1+(x+2y)^2}$, so $f_x(1,0)=\dfrac{1}{2}$ and $f_y(1,0)=1$. Both f_x and f_y are continuous functions, so f is differentiable at $(1,0)$, and the linearization of f at $(1,0)$ is
$L(x,y)=f(1,0)+f_x(1,0)(x-1)+f_y(1,0)(y-0)=\dfrac{\pi}{4}+\dfrac{1}{2}(x-1)+1(y)=\dfrac{1}{2}x+y+\dfrac{\pi}{4}-\dfrac{1}{2}$.

16. $f(x,y)=\sin(2x+3y)$. The partial derivatives are $f_x(x,y)=2\cos(2x+3y)$ and $f_y(x,y)=3\cos(2x+3y)$, so $f_x(-3,2)=2$ and $f_y(-3,2)=3$. Both f_x and f_y are continuous functions, so f is differentiable at $(-3,2)$, and the linearization of f at $(-3,2)$ is
$L(x,y)=f(-3,2)+f_x(-3,2)(x+3)+f_y(-3,2)(y-2)=0+2(x+3)+3(y-2)=2x+3y$.

17. $f(x,y)=\sqrt{20-x^2-7y^2}\Rightarrow f_x(x,y)=-\dfrac{x}{\sqrt{20-x^2-7y^2}}$ and $f_y(x,y)=-\dfrac{7y}{\sqrt{20-x^2-7y^2}}$, so $f_x(2,1)=-\dfrac{2}{3}$ and $f_y(2,1)=-\dfrac{7}{3}$. Then the linear approximation of f at $(2,1)$ is given by

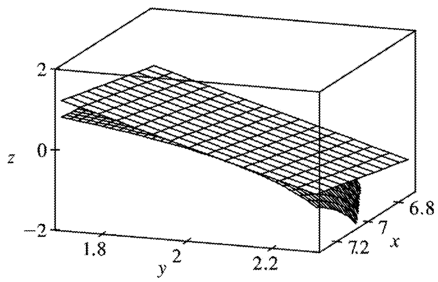
$$\begin{aligned} f(x,y) &\approx f(2,1)+f_x(2,1)(x-2)+f_y(2,1)(y-1)=3-\frac{2}{3}(x-2)-\frac{7}{3}(y-1) \\ &= -\frac{2}{3}x-\frac{7}{3}y+\frac{20}{3} \end{aligned}$$

Thus $f(1.95,1.08)\approx-\dfrac{2}{3}(1.95)-\dfrac{7}{3}(1.08)+\dfrac{20}{3}=2.84\overline{6}$.

18. $f(x,y)=\ln(x-3y)\Rightarrow f_x(x,y)=\dfrac{1}{x-3y}$ and $f_y(x,y)=-\dfrac{3}{x-3y}$, so $f_x(7,2)=1$ and $f_y(7,2)=-3$. Then the linear approximation of f at $(7,2)$ is given by

$$\begin{aligned} f(x,y) &\approx f(7,2)+f_x(7,2)(x-7)+f_y(7,2)(y-2) \\ &= 0+1(x-7)-3(y-2)=x-3y-1 \end{aligned}$$

Thus $f(6.9,2.06)\approx 6.9-3(2.06)-1=-0.28$. The graph shows that our approximated value is slightly greater than the actual value.



19. $f(x,y,z)=\sqrt{x^2+y^2+z^2} \Rightarrow f_x(x,y,z)=\dfrac{x}{\sqrt{x^2+y^2+z^2}}$, $f_y(x,y,z)=\dfrac{y}{\sqrt{x^2+y^2+z^2}}$, and $f_z(x,y,z)=\dfrac{z}{\sqrt{x^2+y^2+z^2}}$, so $f_x(3,2,6)=\dfrac{3}{7}$, $f_y(3,2,6)=\dfrac{2}{7}$, and $f_z(3,2,6)=\dfrac{6}{7}$. Then the linear approximation of f at (3,2,6) is given by

$$\begin{aligned} f(x,y,z) &\approx f(3,2,6)+f_x(3,2,6)(x-3)+f_y(3,2,6)(y-2)+f_z(3,2,6)(z-6) \\ &= 7+\frac{3}{7}(x-3)+\frac{2}{7}(y-2)+\frac{6}{7}(z-6)=\frac{3}{7}x+\frac{2}{7}y+\frac{6}{7}z \end{aligned}$$

Thus $\sqrt{(3.02)^2+(1.97)^2+(5.99)^2}=f(3.02,1.97,5.99)\approx\frac{3}{7}(3.02)+\frac{2}{7}(1.97)+\frac{6}{7}(5.99)\approx 6.9914$.

20. From the table, $f(40,20)=28$. To estimate $f_v(40,20)$ and $f_t(40,20)$ we follow the procedure used in Exercise 15.3.4 . Since $f_v(40,20)=\lim\limits_{h\to 0}\dfrac{f(40+h,20)-f(40,20)}{h}$, we approximate this quantity with $h=\pm 10$ and use the values given in the table: $f_v(40,20)\approx\dfrac{f(50,20)-f(40,20)}{10}=\dfrac{40-28}{10}=1.2$, $f_v(40,20)\approx\dfrac{f(30,20)-f(40,20)}{-10}=\dfrac{17-28}{-10}=1.1$. Averaging these values gives $f_v(40,20)\approx 1.15$. Similarly, $f_t(40,20)=\lim\limits_{h\to 0}\dfrac{f(40,20+h)-f(40,20)}{h}$, so we use $h=10$ and $h=-5$: $f_t(40,20)\approx\dfrac{f(40,30)-f(40,20)}{10}=\dfrac{31-28}{10}=0.3$, $f_t(40,20)\approx\dfrac{f(40,15)-f(40,20)}{-5}=\dfrac{25-28}{-5}=0.6$. Averaging these values gives $f_t(40,15)\approx 0.45$. The linear approximation, then, is

$$\begin{aligned} f(v,t) &\approx f(40,20)+f_v(40,20)(v-40)+f_t(40,20)(t-20) \\ &\approx 28+1.15(v-40)+0.45(t-20) \end{aligned}$$

When $v=43$ and $t=24$, we estimate $f(43,24)\approx 28+1.15(43-40)+0.45(24-20)=33.25$, so we would expect the wave heights to be approximately 33.25 ft.

21. From the table, $f(94,80)=127$. To estimate $f_T(94,80)$ and $f_H(94,80)$ we follow the procedure used in Section 15.3 [ET 14.3]. Since $f_T(94,80)=\lim_{h\to 0} \frac{f(94+h,80)-f(94,80)}{h}$, we approximate this quantity
with $h=\pm 2$ and use the values given in the table: $f_T(94,80)\approx \frac{f(96,80)-f(94,80)}{2} = \frac{135-127}{2} =4$,
$f_T(94,80)\approx \frac{f(92,80)-f(94,80)}{-2} = \frac{119-127}{-2} =4$.
Averaging these values gives $f_T(94,80)\approx 4$. Similarly, $f_H(94,80)=\lim_{h\to 0} \frac{f(94,80+h)-f(94,80)}{h}$,
so we use $h=\pm 5$: $f_H(94,80)\approx \frac{f(94,85)-f(94,80)}{5} = \frac{132-127}{5} =1$,
$f_H(94,80)\approx \frac{f(94,75)-f(94,80)}{-5} = \frac{122-127}{-5} =1$. Averaging these values gives $f_H(94,80)\approx 1$. The linear approximation, then, is

$$\begin{aligned} f(\mathrm{T,H}) &\approx f(94,80)+f_T(94,80)(T-94)+f_H(94,80)(H-80) \\ &\approx 127+4(T-94)+1(H-80) \end{aligned}$$

Thus when $T=95$ and $H=78$, $f(95,78)\approx 127+4(95-94)+1(78-80)=129$, so we estimate the heat index to be approximately $129°$ F .

22. From the table, $f(-15,50)=-29$. To estimate $f_T(-15,50)$ and $f_v(-15,50)$ we follow the procedure used in Section 15.3 . Since $f_T(-15,50)=\lim_{h\to 0} \frac{f(-15+h,50)-f(-15,50)}{h}$, we approximate this quantity with $h=\pm 5$ and use the values given in the table:
$f_T(-15,50)\approx \frac{f(-10,50)-f(-15,50)}{5} = \frac{-22-(-29)}{5} =1.4$,
$f_T(-15,50)\approx \frac{f(-20,50)-f(-15,50)}{-5} = \frac{-35-(-29)}{-5} =1.2$.
Averaging these values gives $f_T(-15,50)\approx 1.3$. Similarly $f_v(-15,50)=\lim_{h\to 0} \frac{f(-15,50+h)-f(-15,50)}{h}$
so we use $h=\pm 10$: $f_v(-15,50)\approx \frac{f(-15,60)-f(-15,50)}{10} = \frac{-30-(-29)}{10} =-0.1$,
$f_v(-15,50)\approx \frac{f(-15,40)-f(-15,50)}{-10} = \frac{-27-(-29)}{-10} =-0.2.$
Averaging these values gives $f_v(-15,50)\approx -0.15$. The linear approximation to the wind–chill index function, then, is

$$f(\mathrm{T,v}) \approx f(-15,50)+f_T(-15,50)(T-(-15))+f_v(-15,50)(v-50)$$

$\approx -29+(1.3)(T+15)-(0.15)(v-50)$

Thus when $T=-17°$ C and $v=55$ km / h, $f(-17,55)\approx -29+(1.3)(-17+15)-(0.15)(55-50)=-32.35$, so we estimate the wind–chill index to be approximately $-32.35°$ C .

23. $z=x^3\ln(y^2)\Rightarrow$

$$dz=\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy=3x^2\ln\left(y^2\right)dx+x^3\cdot\frac{1}{y^2}(2y)dy=3x^2\ln\left(y^2\right)dx+\frac{2x^3}{y}dy\ .$$

24. $v=y\cos xy\Rightarrow$

$$dv=\frac{\partial v}{\partial x}dx+\frac{\partial v}{\partial y}dy=y(-\sin xy)y\,dx+[y(-\sin xy)x+\cos xy]\,dy=-y^2\sin xy\,dx+(\cos xy-xy\sin xy)\,dy$$

25. $u=e^t\sin\theta\Rightarrow du=\frac{\partial u}{\partial t}dt+\frac{\partial u}{\partial\theta}d\theta=e^t\sin\theta\,dt+e^t\cos\theta\,d\theta$

26. $u=\frac{r}{s+2t}\Rightarrow$

$$\begin{aligned}du &= \frac{\partial u}{\partial r}dr+\frac{\partial u}{\partial s}ds+\frac{\partial u}{\partial t}dt=\frac{1}{s+2t}dr+r(-1)(s+2t)^{-2}ds+r(-1)(s+2t)^{-2}(2)\,dt\\ &= \frac{1}{s+2t}dr-\frac{r}{(s+2t)^2}ds-\frac{2r}{(s+2t)^2}dt\end{aligned}$$

27. $w=\ln\sqrt{x^2+y^2+z^2}\Rightarrow$

$$\begin{aligned}dw &= \frac{\partial w}{\partial x}dx+\frac{\partial w}{\partial y}dy+\frac{\partial w}{\partial z}dz\\ &= \left(\frac{1}{2}\right)\frac{2x\left(x^2+y^2+z^2\right)^{-1/2}dx+2y\left(x^2+y^2+z^2\right)^{-1/2}dy+2z\left(x^2+y^2+z^2\right)^{-1/2}dz}{\left(x^2+y^2+z^2\right)^{1/2}}\\ &= \frac{x\,dx+y\,dy+z\,dz}{x^2+y^2+z^2}\end{aligned}$$

28. $w=xye^{xz}\Rightarrow$

$$\begin{aligned}dw &= \frac{\partial w}{\partial x}dx+\frac{\partial w}{\partial y}dy+\frac{\partial w}{\partial z}dz=(xyze^{xz}+ye^{xz})dx+xe^{xz}dy+x^2ye^{xz}dz\\ &=(xz+1)ye^{xz}dx+xe^{xz}dy+x^2ye^{xz}dz\ .\end{aligned}$$

29. $dx=\Delta x=0.05$, $dy=\Delta y=0.1$, $z=5x^2+y^2$, $z_x=10x$, $z_y=2y$. Thus when $x=1$ and $y=2$, $dz=z_x(1,2)dx+z_y(1,2)dy=(10)(0.05)+(4)(0.1)=0.9$ while $\Delta z=f(1.05,2.1)-f(1,2)=5(1.05)^2+(2.1)^2-5-4=0.9225$.

30. $dx=\Delta x=-0.04$, $dy=\Delta y=0.05$, $z=x^2-xy+3y^2$, $z_x=2x-y$, $z_y=6y-x$. Thus when $x=3$ and $y=-1$, $dz=(7)(-0.04)+(-9)(0.05)=-0.73$ while $\Delta z=(2.96)^2-(2.96)(-0.95)+3(-0.95)^2-(9+3+3)=-0.7189$.

31. $dA=\frac{\partial A}{\partial x}dx+\frac{\partial A}{\partial y}dy=y\,dx+x\,dy$ and $|\Delta x|\leq 0.1$, $|\Delta y|\leq 0.1$. We use $dx=0.1$, $dy=0.1$ with $x=30$, $y=24$; then the maximum error in the area is about $dA=24(0.1)+30(0.1)=5.4\text{ cm}^2$.

32. Let S be surface area. Then $S=2(xy+xz+yz)$ and $dS=2(y+z)dx+2(x+z)dy+2(x+y)dz$. The maximum error occurs with $\Delta x=\Delta y=\Delta z=0.2$. Using $dx=\Delta x$, $dy=\Delta y$, $dz=\Delta z$ we find the maximum error in calculated surface area to be about $dS=(220)(0.2)+(260)(0.2)+(280)(0.2)=152\text{ cm}^2$.

33. The volume of a can is $V=\pi r^2h$ and $\Delta V\approx dV$ is an estimate of the amount of tin. Here $dV=2\pi rh\,dr+\pi r^2dh$, so put $dr=0.04$, $dh=0.08$ (0.04 on top, 0.04 on bottom) and then $\Delta V\approx dV=2\pi(48)(0.04)+\pi(16)(0.08)\approx 16.08\text{ cm}^3$. Thus the amount of tin is about 16 cm^3 .

34. Let V be the volume. Then $V=\pi r^2h$ and $\Delta V\approx dV=2\pi rh\,dr+\pi r^2dh$ is an estimate of the amount of metal. With $dr=0.05$ and $dh=0.2$ we get $dV=2\pi(2)(10)(0.05)+\pi(2)^2(0.2)=2.80\pi\approx 8.8\text{ cm}^3$.

35. The area of the rectangle is $A=xy$, and $\Delta A\approx dA$ is an estimate of the area of paint in the stripe. Here $dA=y\,dx+x\,dy$, so with $dx=dy=\frac{3+3}{12}=\frac{1}{2}$, $\Delta A\approx dA=(100)\left(\frac{1}{2}\right)+(200)\left(\frac{1}{2}\right)=150\text{ ft}^2$. Thus there are approximately 150 ft^2 of paint in the stripe.

36. Here $dV=\Delta V=0.3$, $dT=\Delta T=-5$, $P=8.31\frac{T}{V}$, so

$$dP=\left(\frac{8.31}{V}\right)dT-\frac{8.31\cdot T}{V^2}dV=8.31\left[-\frac{5}{12}-\frac{310}{144}\cdot\frac{3}{10}\right]\approx -8.83 .$$

Thus the pressure will drop by about 8.83 kPa .

37. First we find $\frac{\partial R}{\partial R_1}$ implicitly by taking partial derivatives of both sides with respect to R_1 :

$\frac{\partial}{\partial R_1}\left[\frac{1}{R}\right]=\frac{\partial\left[\left(1/R_1\right)+\left(1/R_2\right)+\left(1/R_3\right)\right]}{\partial R_1}\Rightarrow -R^{-2}\frac{\partial R}{\partial R_1}=-R_1^{-2}\Rightarrow\frac{\partial R}{\partial R_1}=\frac{R^2}{R_1^2}$. Then by symmetry,

$\frac{\partial R}{\partial R_2}=\frac{R^2}{R_2^2}$, $\frac{\partial R}{\partial R_3}=\frac{R^2}{R_3^2}$. When $R_1=25$, $R_2=40$ and $R_3=50$, $\frac{1}{R}=\frac{17}{200}\Leftrightarrow R=\frac{200}{17}$ ohms. Since the possible error for each R_i is 0.5% , the maximum error of R is attained by setting $\Delta R_i=0.005R_i$. So

$$\Delta R\approx dR=\frac{\partial R}{\partial R_1}\Delta R_1+\frac{\partial R}{\partial R_2}\Delta R_2+\frac{\partial R}{\partial R_3}\Delta R_3=(0.005)R^2\left[\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3}\right]$$

$$=(0.005)R=\frac{1}{17}\approx 0.059\text{ohms}$$

38. Let x,y,z and w be the four numbers with $p(x,y,z,w)=xyzw$. Since the largest error due to rounding for each number is 0.05 , the maximum error in the calculated product is approximated by $dp=(yzw)(0.05)+(xzw)(0.05)+(xyw)(0.05)+(xyz)(0.05)$. Furthermore, each of the numbers is positive but less than 50 , so the product of any three is between 0 and $(50)^3$. Thus $dp\le 4(50)^3(0.05)=25,000$.

39.

$$\Delta z=f(a+\Delta x,b+\Delta y)-f(a,b)=(a+\Delta x)^2+(b+\Delta y)^2-\left(a^2+b^2\right)$$
$$=a^2+2a\Delta x+(\Delta x)^2+b^2+2b\Delta y+(\Delta y)^2-a^2-b^2=2a\Delta x+(\Delta x)^2+2b\Delta y+(\Delta y)^2$$

But $f_x(a,b)=2a$ and $f_y(a,b)=2b$ and so $\Delta z=f_x(a,b)\Delta x+f_y(a,b)\Delta y+\Delta x\Delta x+\Delta y\Delta y$, which is Definition 7 with $\varepsilon_1=\Delta x$ and $\varepsilon_2=\Delta y$. Hence f is differentiable.

40.

$$\Delta z=f(a+\Delta x,b+\Delta y)-f(a,b)=(a+\Delta x)(b+\Delta y)-5(b+\Delta y)^2-\left(ab-5b^2\right)$$
$$=ab+a\Delta y+b\Delta x+\Delta x\Delta y-5b^2-10b\Delta y-5(\Delta y)^2-ab+5b^2$$
$$=(a-10b)\Delta y+b\Delta x+\Delta x\Delta y-5\Delta y\Delta y,$$

but $f_x(a,b)=b$ and $f_y(a,b)=a-10b$ and so $\Delta z=f_x(a,b)\Delta x+f_y(a,b)\Delta y+\Delta x\Delta y-5\Delta y\Delta y$, which is Definition 7 with $\varepsilon_1=\Delta y$ and $\varepsilon_2=-5\Delta y$. Hence f is differentiable.

41. To show that f is continuous at (a,b) we need to show that

$\lim_{(x,y)\to(a,b)} f(x,y)=f(a,b)$ or equivalently $\lim_{(\Delta x,\Delta y)\to(0,0)} f(a+\Delta x,b+\Delta y)=f(a,b)$. Since f is differentiable at (a,b) , $f(a+\Delta x,b+\Delta y)-f(a,b)=\Delta z=f_x(a,b)\Delta x+f_y(a,b)\Delta y+\varepsilon_1\Delta x+\varepsilon_2\Delta y$, where ε_1 and $\varepsilon_2\to 0$ as $(\Delta x,\Delta y)\to(0,0)$. Thus $f(a+\Delta x,b+\Delta y)=f(a,b)+f_x(a,b)\Delta x+f_y(a,b)\Delta y+\varepsilon_1\Delta x+\varepsilon_2\Delta y$. Taking the limit of both sides as $(\Delta x,\Delta y)\to(0,0)$ gives $\lim_{(\Delta x,\Delta y)\to(0,0)} f(a+\Delta x,b+\Delta y)=f(a,b)$. Thus f is continuous at (a,b) .

42. **(a)** $\lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h}=\lim_{h\to 0}\frac{0-0}{h}=0$ and $\lim_{h\to 0}\frac{f(0,h)-f(0,0)}{h}=\lim_{h\to 0}\frac{0-0}{h}=0$. Thus $f_x(0,0)=f_y(0,0)=0$. To show that f isn't differentiable at (0,0) we need only show that f is not continuous at (0,0) and apply Exercise 41 . As $(x,y)\to(0,0)$ along the x –axis $f(x,y)=0/x^2=0$ for $x\neq 0$ so $f(x,y)\to 0$ as $(x,y)\to(0,0)$ along the x –axis. But as $(x,y)\to(0,0)$ along the line $y=x$, $f(x,x)=x^2/\left(2x^2\right)=\frac{1}{2}$ for $x\neq 0$ so $f(x,y)\to\frac{1}{2}$ as $(x,y)\to(0,0)$ along this line. Thus $\lim_{(x,y)\to(0,0)} f(x,y)$ doesn't exist, so f is discontinuous at (0,0) and thus not differentiable there.

(b) For $(x,y)\neq(0,0)$, $f_x(x,y)=\frac{\left(x^2+y^2\right)y-xy(2x)}{\left(x^2+y^2\right)^2}=\frac{y\left(y^2+x^2\right)}{\left(x^2+y^2\right)^2}$. If we approach (0,0) along the y –axis, then $f_x(x,y)=f_x(0,y)=\frac{y^3}{y^4}=\frac{1}{y}$, so $f_x(x,y)\to\pm\infty$ as $(x,y)\to(0,0)$. Thus $\lim_{(x,y)\to(0,0)} f_x(x,y)$ does not exist and $f_x(x,y)$ is not continuous at (0,0). Similarly, $f_y(x,y)=\frac{\left(x^2+y^2\right)x-xy(2y)}{\left(x^2+y^2\right)^2}=\frac{x(x^2-y^2)}{\left(x^2+y^2\right)^2}$ for $(x,y)\neq(0,0)$, and if we approach (0,0) along the x –axis, then $f_y(x,y)=f_x(x,0)=\frac{x^3}{x^4}=\frac{1}{x}$. Thus $\lim_{(x,y)\to(0,0)} f_y(x,y)$ does not exist and $f_y(x,y)$ is not continuous at (0,0).

1. $z=x^2y+xy^2$, $x=2+t^4$, $y=1-t^3 \Rightarrow$

$$\frac{dz}{dt}=\frac{\partial z}{\partial x}\frac{dx}{dt}+\frac{\partial z}{\partial y}\frac{dy}{dt}=\left(2xy+y^2\right)\left(4t^3\right)+\left(x^2+2xy\right)\left(-3t^2\right)=4\left(2xy+y^2\right)^3-3\left(x^2+2xy\right)t^2$$

2. $z=\sqrt{x^2+y^2}$, $x=e^{2t}$, $y=e^{-2t} \Rightarrow$

$$\frac{dz}{dt}=\frac{\partial z}{\partial x}\frac{dx}{dt}+\frac{\partial z}{\partial y}\frac{dy}{dt}=\frac{1}{2}\left(x^2+y^2\right)^{-1/2}(2x)\cdot e^{2t}(2)+\frac{1}{2}\left(x^2+y^2\right)^{-1/2}(2y)\cdot e^{-2t}(-2)=\frac{2xe^{2t}-2ye^{2t}}{\sqrt{x^2+y^2}}$$

3. $z=\sin x\cos y$, $x=\pi t$, $y=\sqrt{t} \Rightarrow$

$$\frac{dz}{dt}=\frac{\partial z}{\partial x}\frac{dx}{dt}+\frac{\partial z}{\partial y}\frac{dy}{dt}=\cos x\cos y\cdot\pi+\sin x(-\sin y)\cdot\frac{1}{2}t^{-1/2}=\pi\cos x\cos y-\frac{1}{2\sqrt{t}}\sin x\sin y$$

4. $z=x\ln(x+2y)$, $x=\sin t$, $y=\cos t \Rightarrow$

$$\frac{dz}{dt}=\frac{\partial z}{\partial x}\frac{dx}{dt}+\frac{\partial z}{\partial y}\frac{dy}{dt}=\left[x\cdot\frac{1}{x+2y}+1\cdot\ln(x+2y)\right]\cos t+x\cdot\frac{1}{x+2y}(2)\cdot(-\sin t)$$
$$=\left[\frac{x}{x+2y}+\ln(x+2y)\right]\cos t-\frac{2x}{x+2y}(\sin t)$$

5. $w=xe^{y/z}$, $x=t^2$, $y=1-t$, $z=1+2t \Rightarrow$

$$\frac{dw}{dt}=\frac{\partial w}{\partial x}\frac{dx}{dt}+\frac{\partial w}{\partial y}\frac{dy}{dt}+\frac{\partial w}{\partial z}\frac{dz}{dt}=e^{y/z}\cdot 2t+xe^{y/z}\left(\frac{1}{z}\right)\cdot(-1)+xe^{y/z}\left(-\frac{y}{z^2}\right)\cdot 2=e^{y/z}\left(2t-\frac{x}{z}-\frac{2xy}{z^2}\right)$$

6. $w=xy+yz^2$, $x=e^t$, $y=e^t\sin t$, $z=e^t\cos t \Rightarrow$

$$\frac{dw}{dt}=\frac{\partial w}{\partial x}\frac{dx}{dt}+\frac{\partial w}{\partial y}\frac{dy}{dt}+\frac{\partial w}{\partial z}\frac{dz}{dt}=y\cdot e^t+\left(x+z^2\right)\cdot\left(e^t\cos t+e^t\sin t\right)+2yz\cdot(-e^t\sin t+e^t\cos t)$$
$$=e^t\left[y+\left(x+z^2\right)(\cos t+\sin t)+2yz(\cos t-\sin t)\right]$$

7. $z=x^2+xy+y^2$, $x=s+t$, $y=st \Rightarrow$

$$\frac{\partial z}{\partial s}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial s}=(2x+y)(1)+(x+2y)(t)=2x+y+xt+2yt$$
$$\frac{\partial z}{\partial t}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial t}=(2x+y)(1)+(x+2y)(s)=2x+y+xs+2ys$$

8. $z=x/y$, $x=se^t$, $y=1+se^{-t} \Rightarrow$

$$\frac{\partial z}{\partial s}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial s}=\frac{1}{y}\left(e^t\right)+\left(-\frac{x}{y^2}\right)\left(e^{-t}\right)=\frac{1}{y}e^t-\frac{x}{y^2}e^{-t}$$

$$\frac{\partial z}{\partial t}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial t}=\frac{1}{y}\left(se^{t}\right)+\left(-\frac{x}{y^2}\right)\left(-se^{-t}\right)=\frac{s}{y}e^{t}+\frac{xs}{y^2}e^{-t}$$

9. $z=\arctan(2x+y)$, $x=s^2t$, $y=s\ln t\Rightarrow$

$$\frac{\partial z}{\partial s}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial s}=\frac{2}{1+(2x+y)^2}\cdot 2st+\frac{1}{1+(2x+y)^2}\cdot\ln t=\frac{4st+\ln t}{1+(2x+y)^2}$$

$$\frac{\partial z}{\partial t}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial t}=\frac{2}{1+(2x+y)^2}\cdot s^2+\frac{1}{1+(2x+y)^2}\cdot\frac{s}{t}=\frac{2s^2+s/t}{1+(2x+y)^2}$$

10. $z=e^{xy}\tan y$, $x=s+2t$, $y=s/t\Rightarrow$

$$\frac{\partial z}{\partial s}=ye^{xy}\tan y\cdot 1+\left(e^{xy}\sec^2 y+xe^{xy}\tan y\right)\cdot\frac{1}{t}=ye^{xy}\tan y+\frac{e^{xy}}{t}\left(\sec^2 y+x\tan y\right)$$

$$\frac{\partial z}{\partial t}=ye^{xy}\tan y\cdot 2+\left(e^{xy}\sec^2 y+xe^{xy}\tan y\right)\left(\frac{-s}{t^2}\right)=2ye^{xy}\tan y-\frac{se^{xy}}{t^2}\left(\sec^2 y+x\tan y\right)$$

11. $z=e^{r}\cos\theta$, $r=st$, $\theta=\sqrt{s^2+t^2}\Rightarrow$

$$\frac{\partial z}{\partial s}=\frac{\partial z}{\partial r}\frac{\partial r}{\partial s}+\frac{\partial z}{\partial\theta}\frac{\partial\theta}{\partial s}=e^{r}\cos\theta\cdot t+e^{r}(-\sin\theta)\cdot\frac{1}{2}\left(s^2+t^2\right)^{-1/2}(2s)$$

$$=te^{r}\cos\theta-e^{r}\sin\theta\cdot\frac{s}{\sqrt{s^2+t^2}}=e^{r}\left(t\cos\theta-\frac{s}{\sqrt{s^2+t^2}}\sin\theta\right)$$

$$\frac{\partial z}{\partial t}=\frac{\partial z}{\partial r}\frac{\partial r}{\partial t}+\frac{\partial z}{\partial\theta}\frac{\partial\theta}{\partial t}=e^{r}\cos\theta\cdot s+e^{r}(-\sin\theta)\cdot\frac{1}{2}\left(s^2+t^2\right)^{-1/2}(2t)$$

$$=se^{r}\cos\theta-e^{r}\sin\theta\cdot\frac{t}{\sqrt{s^2+t^2}}=e^{r}\left(s\cos\theta-\frac{t}{\sqrt{s^2+t^2}}\sin\theta\right)$$

12. $z=\sin\alpha\tan\beta$, $\alpha=3s+t$, $\beta=s-t\Rightarrow$

$$\frac{\partial z}{\partial s}=\frac{\partial z}{\partial\alpha}\frac{\partial\alpha}{\partial s}+\frac{\partial z}{\partial\beta}\frac{\partial\beta}{\partial s}=\cos\alpha\tan\beta\cdot 3+\sin\alpha\sec^2\beta\cdot 1=3\cos\alpha\tan\beta+\sin\alpha\sec^2\beta$$

$$\frac{\partial z}{\partial t}=\frac{\partial z}{\partial\alpha}\frac{\partial\alpha}{\partial t}+\frac{\partial z}{\partial\beta}\frac{\partial\beta}{\partial t}=\cos\alpha\tan\beta\cdot 1+\sin\alpha\sec^2\beta\cdot(-1)=\cos\alpha\tan\beta-\sin\alpha\sec^2\beta$$

13. When $t=3$, $x=g(3)=2$ and $y=h(3)=7$. By the Chain Rule (2),

$$\frac{dz}{dt}=\frac{\partial f}{\partial x}\frac{dx}{dt}+\frac{\partial f}{\partial y}\frac{dy}{dt}=f_x(2,7)g'(3)+f_y(2,7)h'(3)=(6)(5)+(-8)(-4)=62\ .$$

14. By the Chain Rule (3), $\frac{\partial W}{\partial s}=\frac{\partial W}{\partial u}\frac{\partial u}{\partial s}+\frac{\partial W}{\partial v}\frac{\partial v}{\partial s}$. Then

$W_s(1,0) = F_u(u(1,0),\text{v}(1,0))\,u_s(1,0)+F_v(u(1,0),\text{v}(1,0))\,v_s(1,0)$

$= F_u(2,3)u_s(1,0)+F_v(2,3)v_s(1,0)=(-1)(-2)+(10)(5)=52$

Similarly, $\frac{\partial W}{\partial t}=\frac{\partial W}{\partial u}\frac{\partial u}{\partial t}+\frac{\partial W}{\partial v}\frac{\partial v}{\partial t}\Rightarrow$

$W_t(1,0) = F_u(u(1,0),\text{v}(1,0))\,u_t(1,0)+F_v(u(1,0),\text{v}(1,0))\,v_t(1,0)$

$= F_u(2,3)u_t(1,0)+F_v(2,3)v_t(1,0)=(-1)(6)+(10)(4)=34$

15. $g(\text{u,v})=f(x(\text{u,v}),\text{y(u,v)})$ where $x=e^u+\sin v$, $y=e^u+\cos v\Rightarrow \frac{\partial x}{\partial u}=e^u$, $\frac{\partial x}{\partial v}=\cos v$, $\frac{\partial y}{\partial u}=e^u$, $\frac{\partial y}{\partial v}=-\sin v$. By the Chain Rule (3), $\frac{\partial g}{\partial u}=\frac{\partial f}{\partial x}\frac{\partial x}{\partial u}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial u}$. Then

$g_u(0,0) = f_x(x(0,0),\text{y}(0,0))\,x_u(0,0)+f_y(x(0,0),\text{y}(0,0))\,y_u(0,0)$

$= f_x(1,2)\left(e^0\right)+f_y(1,2)\left(e^0\right)=2(1)+5(1)=7$

Similarly $\frac{\partial g}{\partial v}=\frac{\partial f}{\partial x}\frac{\partial x}{\partial v}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial v}$. Then

$g_v(0,0) = f_x(x(0,0),\text{y}(0,0))\,x_v(0,0)+f_y(x(0,0),\text{y}(0,0))\,y_v(0,0)$

$= f_x(1,2)(\cos 0)+f_y(1,2)(-\sin 0)=2(1)+5(0)=2$

16. $g(\text{r,s})=f(x(\text{r,s}),\text{y(r,s)})$ where $x=2r-s$, $y=s^2-4r\Rightarrow \frac{\partial x}{\partial r}=2$, $\frac{\partial x}{\partial s}=-1$, $\frac{\partial y}{\partial r}=-4$, $\frac{\partial y}{\partial s}=2s$. By the Chain Rule (3) $\frac{\partial g}{\partial r}=\frac{\partial f}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial r}$. Then

$g_r(1,2) = f_x(x(1,2),\text{y}(1,2))\,x_r(1,2)+f_y(x(1,2),\text{y}(1,2))\,y_r(1,2)$

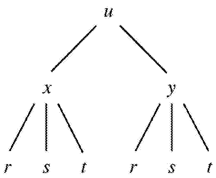
$= f_x(0,0)(2)+f_y(0,0)(-4)=4(2)+8(-4)=-24$

Similarly $\frac{\partial g}{\partial s}=\frac{\partial f}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial s}$. Then

$g_s(1,2) = f_x(x(1,2),\text{y}(1,2))\,x_s(1,2)+f_y(x(1,2),\text{y}(1,2))\,y_s(1,2)$

$= f_x(0,0)(-1)+f_y(0,0)(4)=4(-1)+8(4)=28$

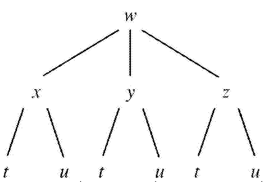
17.



$u=f(x,y)$, $x=x(r,s,t)$, $y=y(r,s,t) \Rightarrow$

$$\frac{\partial u}{\partial r}=\frac{\partial u}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial r}\ ,\ \frac{\partial u}{\partial s}=\frac{\partial u}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial s}\ ,\ \frac{\partial u}{\partial t}=\frac{\partial u}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial t}$$

18.

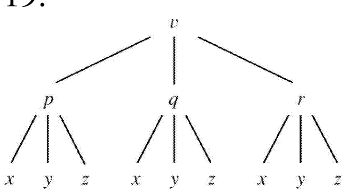


$w=f(x,y,z)$, $x=x(t,u)$, $y=y(t,u)$, $z=z(t,u) \Rightarrow$

$$\frac{\partial w}{\partial t}=\frac{\partial w}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial w}{\partial y}\frac{\partial y}{\partial t}+\frac{\partial w}{\partial z}\frac{\partial z}{\partial t}\ ,$$

$$\frac{\partial w}{\partial u}=\frac{\partial w}{\partial x}\frac{\partial x}{\partial u}+\frac{\partial w}{\partial y}\frac{\partial y}{\partial u}+\frac{\partial w}{\partial z}\frac{\partial z}{\partial u}$$

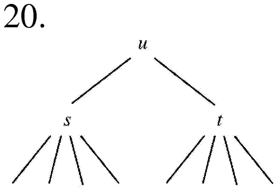
19.



$v=f(p,q,r)$, $p=p(x,y,z)$, $q=q(x,y,z)$, $r=r(x,y,z) \Rightarrow$

$$\frac{\partial v}{\partial x}=\frac{\partial v}{\partial p}\frac{\partial p}{\partial x}+\frac{\partial v}{\partial q}\frac{\partial q}{\partial x}+\frac{\partial v}{\partial r}\frac{\partial r}{\partial x}\ ,\ \frac{\partial v}{\partial y}=\frac{\partial v}{\partial p}\frac{\partial p}{\partial y}+\frac{\partial v}{\partial q}\frac{\partial q}{\partial y}+\frac{\partial v}{\partial r}\frac{\partial r}{\partial y}\ ,\ \frac{\partial v}{\partial z}=\frac{\partial v}{\partial p}\frac{\partial p}{\partial z}+\frac{\partial v}{\partial q}\frac{\partial q}{\partial z}+\frac{\partial v}{\partial r}\frac{\partial r}{\partial z}$$

20.



$u=f(s,t)$, $s=s(w,x,y,z)$, $t=t(w,x,y,z) \Rightarrow$

$$\frac{\partial u}{\partial w}=\frac{\partial u}{\partial s}\frac{\partial s}{\partial w}+\frac{\partial u}{\partial t}\frac{\partial t}{\partial w}\ ,\ \frac{\partial u}{\partial x}=\frac{\partial u}{\partial s}\frac{\partial s}{\partial x}+\frac{\partial u}{\partial t}\frac{\partial t}{\partial x}\ ,\ \frac{\partial u}{\partial y}=\frac{\partial u}{\partial s}\frac{\partial s}{\partial y}+\frac{\partial u}{\partial t}\frac{\partial t}{\partial y}\ ,\ \frac{\partial u}{\partial z}=\frac{\partial u}{\partial s}\frac{\partial s}{\partial z}+\frac{\partial u}{\partial t}\frac{\partial t}{\partial z}$$

21. $z=x^2+xy^3$, $x=uv^2+w^3$, $y=u+ve^w \Rightarrow \frac{\partial z}{\partial u}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial u}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial u}=\left(2x+y^3\right)\left(v^2\right)+\left(3xy^2\right)(1)$,

$\frac{\partial z}{\partial v}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial v}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial v}=\left(2x+y^3\right)(2uv)+\left(3xy^2\right)\left(e^w\right)$,
$\frac{\partial z}{\partial w}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial w}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial w}=\left(2x+y^3\right)\left(3w^2\right)+\left(3xy^2\right)\left(ve^w\right)$. When $u=2$, $v=1$, and $w=0$, we have $x=2$, $y=3$, so $\frac{\partial z}{\partial u}=(31)(1)+(54)(1)=85$, $\frac{\partial z}{\partial v}=(31)(4)+(54)(1)=178$, $\frac{\partial z}{\partial w}=(31)(0)+(54)(1)=54$.

22. $u=\left(r^2+s^2\right)^{1/2}$, $r=y+x\cos t$, $s=x+y\sin t\Rightarrow$
$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial r}\frac{\partial r}{\partial x}+\frac{\partial u}{\partial s}\frac{\partial s}{\partial x}=\frac{1}{2}\left(r^2+s^2\right)^{-1/2}(2r)(\cos t)+\frac{1}{2}\left(r^2+s^2\right)^{-1/2}(2s)(1)=(r\cos t+s)/\sqrt{r^2+s^2}$,
$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial r}\frac{\partial r}{\partial y}+\frac{\partial u}{\partial s}\frac{\partial s}{\partial y}=\frac{1}{2}\left(r^2+s^2\right)^{-1/2}(2r)(1)+\frac{1}{2}\left(r^2+s^2\right)^{-1/2}(2s)(\sin t)=(r+s\sin t)/\sqrt{r^2+s^2}$,
$\frac{\partial u}{\partial t}=\frac{\partial u}{\partial r}\frac{\partial r}{\partial t}+\frac{\partial u}{\partial s}\frac{\partial s}{\partial t}=\frac{1}{2}\left(r^2+s^2\right)^{-1/2}(2r)(-x\sin t)+\frac{1}{2}\left(r^2+s^2\right)^{-1/2}(2s)(y\cos t)=\frac{-rx\sin t+sy\cos t}{\sqrt{r^2+s^2}}$.
When $x=1$, $y=2$, and $t=0$ we have $r=3$ and $s=1$, so $\frac{\partial u}{\partial x}=\frac{4}{\sqrt{10}}$, $\frac{\partial u}{\partial y}=\frac{3}{\sqrt{10}}$, and $\frac{\partial u}{\partial t}=\frac{2}{\sqrt{10}}$.

23. $R=\ln\left(u^2+v^2+w^2\right)$, $u=x+2y$, $v=2x-y$, $w=2xy\Rightarrow$
$\frac{\partial R}{\partial x}=\frac{\partial R}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial R}{\partial v}\frac{\partial v}{\partial x}+\frac{\partial R}{\partial w}\frac{\partial w}{\partial x}=\frac{2u}{u^2+v^2+w^2}(1)+\frac{2v}{u^2+v^2+w^2}(2)+\frac{2w}{u^2+v^2+w^2}(2y)$
$=\frac{2u+4v+4wy}{u^2+v^2+w^2}$,

$\frac{\partial R}{\partial y}=\frac{\partial R}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial R}{\partial v}\frac{\partial v}{\partial y}+\frac{\partial R}{\partial w}\frac{\partial w}{\partial y}=\frac{2u}{u^2+v^2+w^2}(2)+\frac{2v}{u^2+v^2+w^2}(-1)+\frac{2w}{u^2+v^2+w^2}(2x)$
$=\frac{4u-2v+4wx}{u^2+v^2+w^2}$.

When $x=y=1$ we have $u=3$, $v=1$, and $w=2$, so $\frac{\partial R}{\partial x}=\frac{9}{7}$ and $\frac{\partial R}{\partial y}=\frac{9}{7}$.

24. $M=xe^{y-z^2}$, $x=2uv$, $y=u-v$, $z=u+v\Rightarrow$
$\frac{\partial M}{\partial u}=\frac{\partial M}{\partial x}\frac{\partial x}{\partial u}+\frac{\partial M}{\partial y}\frac{\partial y}{\partial u}+\frac{\partial M}{\partial z}\frac{\partial z}{\partial u}=e^{y-z^2}(2v)+xe^{y-z^2}(1)+x(-2z)e^{y-z^2}(1)$

$=e^{y-z^2}(2v+x-2xz)$,

$$\frac{\partial M}{\partial v}=\frac{\partial M}{\partial x}\frac{\partial x}{\partial v}+\frac{\partial M}{\partial y}\frac{\partial y}{\partial v}+\frac{\partial M}{\partial z}\frac{\partial z}{\partial v}=e^{y-z^2}(2u)+xe^{y-z^2}(-1)+x(-2z)e^{y-z^2}(1)$$

$=e^{y-z^2}(2u-x-2xz)$.

When $u=3$, $v=-1$ we have $x=-6$, $y=4$, and $z=2$, so $\frac{\partial M}{\partial u}=16$ and $\frac{\partial M}{\partial v}=36$.

25. $u=x^2+yz$, $x=pr\cos\theta$, $y=pr\sin\theta$, $z=p+r\Rightarrow$

$$\frac{\partial u}{\partial p}=\frac{\partial u}{\partial x}\frac{\partial x}{\partial p}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial p}+\frac{\partial u}{\partial z}\frac{\partial z}{\partial p}=(2x)(r\cos\theta)+(z)(r\sin\theta)+(y)(1)=2xr\cos\theta+zr\sin\theta+y ,$$

$$\frac{\partial u}{\partial r}=\frac{\partial u}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial r}+\frac{\partial u}{\partial z}\frac{\partial z}{\partial r}=(2x)(p\cos\theta)+(z)(p\sin\theta)+(y)(1)=2xp\cos\theta+zp\sin\theta+y ,$$

$$\frac{\partial u}{\partial \theta}=\frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta}+\frac{\partial u}{\partial z}\frac{\partial z}{\partial \theta}=(2x)(-pr\sin\theta)+(z)(pr\cos\theta)+(y)(0)=-2xpr\sin\theta+zpr\cos\theta .$$

When $p=2$, $r=3$, and $\theta=0$ we have $x=6$, $y=0$, and $z=5$, so $\frac{\partial u}{\partial p}=36$, $\frac{\partial u}{\partial r}=24$, and $\frac{\partial u}{\partial \theta}=30$.

26. $Y=w\tan^{-1}(uv)$, $u=r+s$, $v=s+t$, $w=t+r\Rightarrow$

$$\frac{\partial Y}{\partial r}=\frac{\partial Y}{\partial u}\frac{\partial u}{\partial r}+\frac{\partial Y}{\partial v}\frac{\partial v}{\partial r}+\frac{\partial Y}{\partial w}\frac{\partial w}{\partial r}=\frac{w}{1+(uv)^2}(v)(1)+\frac{w}{1+(uv)^2}(u)(0)+\tan^{-1}(uv)(1)$$

$$=\frac{vw}{1+u^2v^2}+\tan^{-1}(uv)$$

$$\frac{\partial Y}{\partial s}=\frac{\partial Y}{\partial u}\frac{\partial u}{\partial s}+\frac{\partial Y}{\partial v}\frac{\partial v}{\partial s}+\frac{\partial Y}{\partial w}\frac{\partial w}{\partial s}=\frac{wv}{1+u^2v^2}(1)+\frac{wu}{1+u^2v^2}(1)+\tan^{-1}(uv)(0)$$

$$=\frac{w(v+u)}{1+u^2v^2}$$

$$\frac{\partial Y}{\partial t}=\frac{\partial Y}{\partial u}\frac{\partial u}{\partial t}+\frac{\partial Y}{\partial v}\frac{\partial v}{\partial t}+\frac{\partial Y}{\partial w}\frac{\partial w}{\partial t}=\frac{wv}{1+u^2v^2}(0)+\frac{wu}{1+u^2v^2}(1)+\tan^{-1}(uv)(1)$$

$$=\frac{wu}{1+u^2v^2}+\tan^{-1}(uv)$$

When $r=1$, $s=0$, and $t=1$, we have $u=1$, $v=1$, and $w=2$, so $\frac{\partial Y}{\partial r}=1+\frac{\pi}{4}$, $\frac{\partial Y}{\partial s}=2$, and $\frac{\partial Y}{\partial t}=1+\frac{\pi}{4}$.

27. $\sqrt{xy}=1+x^2y$, so let $F(x,y)=(xy)^{1/2}-1-x^2y=0$. Then by Equation 6

$$\frac{dy}{dx}=-\frac{F_x}{F_y}=-\frac{\frac{1}{2}(xy)^{-1/2}(y)-2xy}{\frac{1}{2}(xy)^{-1/2}(x)-x^2}=-\frac{y-4xy\sqrt{xy}}{x-2x^2\sqrt{xy}}=\frac{4(xy)^{3/2}-y}{x-2x^2\sqrt{xy}} .$$

28. $y^5+x^2y^3=1+ye^{x^2}$, so let $F(x,y)=y^5+x^2y^3-1-ye^{x^2}=0$. Then

$$\frac{dy}{dx}=-\frac{F_x}{F_y}=-\frac{2xy^3-2xye^{x^2}}{5y^4+3x^2y^2-e^{x^2}}=\frac{2xye^{x^2}-2xy^3}{5y^4+3x^2y^2-e^{x^2}} .$$

29. $\cos(x-y)=xe^y$, so let $F(x,y)=\cos(x-y)-xe^y=0$.

Then $\frac{dy}{dx}=-\frac{F_x}{F_y}=-\frac{-\sin(x-y)-e^y}{-\sin(x-y)(-1)-xe^y}=\frac{\sin(x-y)+e^y}{\sin(x-y)-xe^y}$.

30. $\sin x+\cos y=\sin x\cos y$, so let $F(x,y)=\sin x+\cos y-\sin x\cos y=0$. Then

$$\frac{dy}{dx}=-\frac{F_x}{F_y}=-\frac{\cos x-\cos x\cos y}{-\sin y+\sin x\sin y}=\frac{\cos x(\cos y-1)}{\sin y(\sin x-1)} .$$

31. $x^2+y^2+z^2=3xyz$, so let $F(x,y,z)=x^2+y^2+z^2-3xyz=0$. Then by Equations 7

$$\frac{\partial z}{\partial x}=-\frac{F_x}{F_z}=-\frac{2x-3yz}{2z-3xy}=\frac{3yz-2x}{2z-3xy} \text{ and } \frac{\partial z}{\partial y}=-\frac{F_y}{F_z}=-\frac{2y-3xz}{2z-3xy}=\frac{3xz-2y}{2z-3xy} .$$

32. $xyz=\cos(x+y+z)$. Let $F(x,y,z)=xyz-\cos(x+y+z)=0$, so

$$\frac{\partial z}{\partial x}=-\frac{F_x}{F_z}=-\frac{yz+\sin(x+y+z)}{xy+\sin(x+y+z)} , \frac{\partial z}{\partial y}=-\frac{F_y}{F_z}=-\frac{xz+\sin(x+y+z)}{xy+\sin(x+y+z)} .$$

33. $x-z=\arctan(yz)$, so let $F(x,y,z)=x-z-\arctan(yz)=0$. Then $\frac{\partial z}{\partial x}=-\frac{F_x}{F_z}=-\frac{1}{-1-\frac{1}{1+(yz)^2}(y)}=\frac{1+y^2z^2}{1+y+y^2z^2}$

and $\frac{\partial z}{\partial y}=-\frac{F_y}{F_z}=-\frac{-\frac{1}{1+(yz)^2}(z)}{-1-\frac{1}{1+(yz)^2}(y)}=-\frac{\frac{z}{1+y^2z^2}}{\frac{1+y^2z^2+y}{1+y^2z^2}}=-\frac{z}{1+y+y^2z^2}$.

34. $yz=\ln(x+z)$, so let $F(x,y,z)=yz-\ln(x+z)=0$. Then $\frac{\partial z}{\partial x}=-\frac{F_x}{F_z}=-\frac{-\frac{1}{x+z}(1)}{y-\frac{1}{x+z}(1)}=\frac{1}{y(x+z)-1}$,

$\frac{\partial z}{\partial y}=-\frac{F_y}{F_z}=-\frac{z}{y-\frac{1}{x+z}}=-\frac{z(x+z)}{y(x+z)-1}$.

35. Since x and y are each functions of t , $T(x,y)$ is a function of t , so by the Chain Rule, $\frac{dT}{dt}=\frac{\partial T}{\partial x}\frac{dx}{dt}+\frac{\partial T}{\partial y}\frac{dy}{dt}$. After 3 seconds, $x=\sqrt{1+t}=\sqrt{1+3}=2$, $y=2+\frac{1}{3}t=2+\frac{1}{3}(3)=3$, $\frac{dx}{dt}=\frac{1}{2\sqrt{1+t}}=\frac{1}{2\sqrt{1+3}}=\frac{1}{4}$, and $\frac{dy}{dt}=\frac{1}{3}$. Then $\frac{dT}{dt}=T_x(2,3)\frac{dx}{dt}+T_y(2,3)\frac{dy}{dt}=4\left(\frac{1}{4}\right)+3\left(\frac{1}{3}\right)=2$. Thus the temperature is rising at a rate of $2°$ C/s .

36. **(a)** Since $\partial W/\partial T$ is negative, a rise in average temperature (while annual rainfall remains constant) causes a decrease in wheat production at the current production levels. Since $\partial W/\partial R$ is positive, an increase in annual rainfall (while the average temperature remains constant) causes an increase in wheat production.
(b) Since the average temperature is rising at a rate of $0.15°$ C/year , we know that $dT/dt=0.15$. Since rainfall is decreasing at a rate of 0.1 cm / year, we know $dR/dt=-0.1$. Then, by the Chain Rule, $\frac{dW}{dt}=\frac{\partial W}{\partial T}\frac{dT}{dt}+\frac{\partial W}{\partial R}\frac{dR}{dt}=(-2)(0.15)+(8)(-0.1)=-1.1$. Thus we estimate that wheat production will decrease at a rate of 1.1 units / year.

37. $C=1449.2+4.6T-0.055T^2+0.00029T^3+0.016D$, so $\frac{\partial C}{\partial T}=4.6-0.11T+0.00087T^2$ and $\frac{\partial C}{\partial D}=0.016$. According to the graph, the diver is experiencing a temperature of approximately $12.5°$ C at $t=20$ minutes, so $\frac{\partial C}{\partial T}=4.6-0.11(12.5)+0.00087(12.5)^2\approx 3.36$. By sketching tangent lines at $t=20$ to the graphs given, we estimate $\frac{dD}{dt}\approx\frac{1}{2}$ and $\frac{dT}{dt}\approx-\frac{1}{10}$. Then, by the Chain Rule,

$\frac{dC}{dt}=\frac{\partial C}{\partial T}\frac{dT}{dt}+\frac{\partial C}{\partial D}\frac{dD}{dt}\approx(3.36)\left(-\frac{1}{10}\right)+(0.016)\left(\frac{1}{2}\right)\approx-0.33$. Thus the speed of sound experienced by the diver is decreasing at a rate of approximately 0.33 m / s per minute.

38. $V=\pi r^2 h/3$, so $\frac{dV}{dt}=\frac{\partial V}{\partial r}\frac{dr}{dt}+\frac{\partial V}{\partial h}\frac{dh}{dt}=\frac{2\pi rh}{3}\,1.8+\frac{\pi r^2}{3}(-2.5)=20\,,160\pi-12\,,000\pi=8160\pi$ in^3/ s.

39. **(a)** $V=\ell wh$, so by the Chain Rule,

$$\frac{dV}{dt}=\frac{\partial V}{\partial \ell}\frac{d\ell}{dt}+\frac{\partial V}{\partial w}\frac{dw}{dt}+\frac{\partial V}{\partial h}\frac{dh}{dt}=wh\frac{d\ell}{dt}+\ell h\frac{dw}{dt}+\ell w\frac{dh}{dt}$$

$$=2\cdot 2\cdot 2+1\cdot 2\cdot 2+1\cdot 2\cdot(-3)=6\text{m}^3/\text{s}$$

(b) $S=2(\ell w+\ell h+wh)$, so by the Chain Rule,

$$\frac{dS}{dt}=\frac{\partial S}{\partial \ell}\frac{d\ell}{dt}+\frac{\partial S}{\partial w}\frac{dw}{dt}+\frac{\partial S}{\partial h}\frac{dh}{dt}=2(w+h)\frac{d\ell}{dt}+2(\ell+h)\frac{dw}{dt}+2(\ell+w)\frac{dh}{dt}$$

$$=2(2+2)2+2(1+2)2+2(1+2)(-3)=10\text{m}^2/\text{s}$$

(c) $L^2=\ell^2+w^2+h^2\Rightarrow 2L\frac{dL}{dt}=2\ell\frac{d\ell}{dt}+2w\frac{dw}{dt}+2h\frac{dh}{dt}=2(1)(2)+2(2)(2)+2(2)(-3)=0\Rightarrow dL/dt=0$ m / s.

40. $I=\frac{V}{R}\Rightarrow$

$$\frac{dI}{dt}=\frac{\partial I}{\partial V}\frac{dV}{dt}+\frac{\partial I}{\partial R}\frac{dR}{dt}=\frac{1}{R}\frac{dV}{dt}-\frac{V}{R^2}\frac{dR}{dt}=\frac{1}{R}\frac{dV}{dt}-\frac{I}{R}\frac{dR}{dt}$$

$$=\frac{1}{400}(-0.01)-\frac{0.08}{400}(0.03)=-0.000031\text{ A / s}$$

41. $\frac{dP}{dt}=0.05$, $\frac{dT}{dt}=0.15$, $V=8.31\frac{T}{P}$ and $\frac{dV}{dt}=\frac{8.31}{P}\frac{dT}{dt}-8.31\frac{T}{P^2}\frac{dP}{dt}$. Thus when $P=20$ and $T=320$, $\frac{dV}{dt}=8.31\left[\frac{0.15}{20}-\frac{(0.05)(320)}{400}\right]\approx-0.27$ L / s.

42. Let x and y be the respective distances of car A and car B from the intersection and let z be the distance between the two cars. Then $dx/dt=-90$, $dy/dt=-80$ and $z^2=x^2+y^2$. When $x=0.3$ and $y=0.4$, $z=\sqrt{0.25}=0.5$ and $2z(dz/dt)=2x(dx/dt)+2y(dy/dt)$ or $dz/dt=0.6(-90)+0.8(-80)=-118$ km / h.

43. **(a)** By the Chain Rule, $\frac{\partial z}{\partial r}=\frac{\partial z}{\partial x}\cos\theta+\frac{\partial z}{\partial y}\sin\theta$, $\frac{\partial z}{\partial \theta}=\frac{\partial z}{\partial x}(-r\sin\theta)+\frac{\partial z}{\partial y}r\cos\theta$.

(b) $\left(\frac{\partial z}{\partial r}\right)^2=\left(\frac{\partial z}{\partial x}\right)^2\cos^2\theta+2\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}\cos\theta\sin\theta+\left(\frac{\partial z}{\partial y}\right)^2\sin^2\theta$,

$\left(\frac{\partial z}{\partial \theta}\right)^2=\left(\frac{\partial z}{\partial x}\right)^2r^2\sin^2\theta-2\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}r^2\cos\theta\sin\theta+\left(\frac{\partial z}{\partial y}\right)^2r^2\cos^2\theta$. Thus

$\left(\frac{\partial z}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2=\left[\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2\right](\cos^2\theta+\sin^2\theta)=\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2$.

44. By the Chain Rule, $\frac{\partial u}{\partial s}=\frac{\partial u}{\partial x}e^s\cos t+\frac{\partial u}{\partial y}e^s\sin t$, $\frac{\partial u}{\partial t}=\frac{\partial u}{\partial x}\left(-e^s\sin t\right)+\frac{\partial u}{\partial y}e^s\cos t$. Then

$\left(\frac{\partial u}{\partial s}\right)^2=\left(\frac{\partial u}{\partial x}\right)^2e^{2s}\cos^2t+2\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}e^{2s}\cos t\sin t+\left(\frac{\partial u}{\partial y}\right)^2e^{2s}\sin^2t$ and

$\left(\frac{\partial u}{\partial t}\right)^2=\left(\frac{\partial u}{\partial x}\right)^2e^{2s}\sin^2t-2\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}e^{2s}\cos t\sin t+\left(\frac{\partial u}{\partial y}\right)^2e^{2s}\sin^2t$. Thus

$\left[\left(\frac{\partial u}{\partial s}\right)^2+\left(\frac{\partial u}{\partial t}\right)^2\right]e^{-2s}=\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2$.

45. Let $u=x-y$. Then $\frac{\partial z}{\partial x}=\frac{dz}{du}\frac{\partial u}{\partial x}=\frac{dz}{du}$ and $\frac{\partial z}{\partial y}=\frac{dz}{du}(-1)$. Thus $\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}=0$.

46. $\frac{\partial z}{\partial s}=\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}$ and $\frac{\partial z}{\partial t}=\frac{\partial z}{\partial x}-\frac{\partial z}{\partial y}$. Thus $\frac{\partial z}{\partial s}\frac{\partial z}{\partial t}=\left(\frac{\partial z}{\partial x}\right)^2-\left(\frac{\partial z}{\partial y}\right)^2$.

47. Let $u=x+at$, $v=x-at$. Then $z=f(u)+g(v)$, so $\partial z/\partial u=f'(u)$ and $\partial z/\partial v=g'(v)$.

Thus $\frac{\partial z}{\partial t}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial t}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial t}=af'(u)-ag'(v)$ and

$\frac{\partial^2 z}{\partial t^2}=a\frac{\partial}{\partial t}\left[f'(u)-g'(v)\right]=a\left(\frac{df'(u)}{du}\frac{\partial u}{\partial t}-\frac{dg'(v)}{dv}\frac{\partial v}{\partial t}\right)=a^2f''(u)+a^2g''(v)$. Similarly

$\frac{\partial z}{\partial x}=f'(u)+g'(v)$ and $\frac{\partial^2 z}{\partial x^2}=f''(u)+g''(v)$. Thus $\frac{\partial^2 z}{\partial t^2}=a^2\frac{\partial^2 z}{\partial x^2}$.

48. By the Chain Rule, $\frac{\partial u}{\partial s}=e^s\cos t\frac{\partial u}{\partial x}+e^s\sin t\frac{\partial u}{\partial y}$ and $\frac{\partial u}{\partial t}=-e^s\sin t\frac{\partial u}{\partial x}+e^s\cos t\frac{\partial u}{\partial y}$. Then

$\frac{\partial^2 u}{\partial s^2}=e^s\cos t\frac{\partial u}{\partial x}+e^s\cos t\frac{\partial}{\partial s}\left(\frac{\partial u}{\partial x}\right)+e^s\sin t\frac{\partial u}{\partial y}+e^s\sin t\frac{\partial}{\partial s}\left(\frac{\partial u}{\partial y}\right)$. But

$\frac{\partial}{\partial s}\left(\frac{\partial u}{\partial x}\right)=\frac{\partial^2 u}{\partial x^2}\frac{\partial x}{\partial s}+\frac{\partial^2 u}{\partial y\partial x}\frac{\partial y}{\partial s}=e^s\cos t\frac{\partial^2 u}{\partial x^2}+e^s\sin t\frac{\partial^2 u}{\partial y\partial x}$ and

$\frac{\partial}{\partial s}\left(\frac{\partial u}{\partial y}\right)=\frac{\partial^2 u}{\partial y^2}\frac{\partial y}{\partial s}+\frac{\partial^2 u}{\partial x\partial y}\frac{\partial x}{\partial s}=e^s\sin t\frac{\partial^2 u}{\partial y^2}+e^s\cos t\frac{\partial^2 u}{\partial x\partial y}$. Also, by continuity of the partials,

$\frac{\partial^2 u}{\partial x\partial y}=\frac{\partial^2 u}{\partial y\partial x}$. Thus

$$\begin{aligned}\frac{\partial^2 u}{\partial s^2} &= e^s\cos t\frac{\partial u}{\partial x}+e^s\cos t\left(e^s\cos t\frac{\partial^2 u}{\partial x^2}+e^s\sin t\frac{\partial^2 u}{\partial x\partial y}\right)+e^s\sin t\frac{\partial u}{\partial y}\\ &\quad+e^s\sin t\left(e^s\sin t\frac{\partial^2 u}{\partial y^2}+e^s\cos t\frac{\partial^2 u}{\partial x\partial y}\right)\\ &= e^s\cos t\frac{\partial u}{\partial x}+e^s\sin t\frac{\partial u}{\partial y}+e^{2s}\cos^2 t\frac{\partial^2 u}{\partial x^2}+2e^{2s}\cos t\sin t\frac{\partial^2 u}{\partial x\partial y}+e^{2s}\sin^2 t\frac{\partial^2 u}{\partial y^2}\end{aligned}$$

Similarly

$$\begin{aligned}\frac{\partial^2 u}{\partial t^2} &= -e^s\cos t\frac{\partial u}{\partial x}-e^s\sin t\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}\right)-e^s\sin t\frac{\partial u}{\partial y}+e^s\cos t\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial y}\right)\\ &= -e^s\cos t\frac{\partial u}{\partial x}-e^s\sin t\left(-e^s\sin t\frac{\partial^2 u}{\partial x^2}+e^s\cos t\frac{\partial^2 u}{\partial x\partial y}\right)\\ &\quad- e^s\sin t\frac{\partial u}{\partial y}+e^s\cos t\left(e^s\cos t\frac{\partial^2 u}{\partial y^2}-e^s\sin t\frac{\partial^2 u}{\partial x\partial y}\right)\\ &= -e^s\cos t\frac{\partial u}{\partial x}-e^s\sin t\frac{\partial u}{\partial y}+e^{2s}\sin^2 t\frac{\partial^2 u}{\partial x^2}-2e^{2s}\cos t\sin t\frac{\partial^2 u}{\partial x\partial y}+e^{2s}\cos^2 t\frac{\partial^2 u}{\partial y^2}\end{aligned}$$

Thus $e^{-2s}\left(\frac{\partial^2 u}{\partial s^2}+\frac{\partial^2 u}{\partial t^2}\right)=(\cos^2 t+\sin^2 t)\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}$, as desired.

49. $\frac{\partial z}{\partial s}=\frac{\partial z}{\partial x}2s+\frac{\partial z}{\partial y}2r$. Then

$$\frac{\partial^2 z}{\partial r\,\partial s} = \frac{\partial}{\partial r}\left(\frac{\partial z}{\partial x}2s\right)+\frac{\partial}{\partial r}\left(\frac{\partial z}{\partial y}2r\right)$$

$$= \frac{\partial^2 z}{\partial x^2} \frac{\partial x}{\partial r} 2s + \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) \frac{\partial y}{\partial r} 2s + \frac{\partial z}{\partial x} \frac{\partial}{\partial r} 2s + \frac{\partial^2 z}{\partial y^2} \frac{\partial y}{\partial r} 2r + \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) \frac{\partial x}{\partial r} 2r + \frac{\partial z}{\partial y} 2$$

$$= 4rs \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y \partial x} 4s^2 + 0 + 4rs \frac{\partial^2 z}{\partial y^2} + \frac{\partial^2 z}{\partial x \partial y} 4r^2 + 2 \frac{\partial z}{\partial y}$$

By the continuity of the partials, $\frac{\partial^2 z}{\partial r \partial s} = 4rs \frac{\partial^2 z}{\partial x^2} + 4rs \frac{\partial^2 z}{\partial y^2} + (4r^2 + 4s^2) \frac{\partial^2 z}{\partial x \partial y} + 2 \frac{\partial z}{\partial y}$.

50. By the Chain Rule,

(a) $\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x} \cos\theta + \frac{\partial z}{\partial y} \sin\theta$

(b) $\frac{\partial z}{\partial \theta} = -\frac{\partial z}{\partial x} r\sin\theta + \frac{\partial z}{\partial y} r\cos\theta$

(c)

$$\frac{\partial^2 z}{\partial r \partial \theta} = \frac{\partial^2 z}{\partial \theta\, \partial r} = \frac{\partial}{\partial \theta}\left(\frac{\partial z}{\partial x} \cos\theta + \frac{\partial z}{\partial y} \sin\theta\right)$$

$$= -\sin\theta \frac{\partial z}{\partial x} + \cos\theta \frac{\partial}{\partial \theta}\left(\frac{\partial z}{\partial x}\right) + \cos\theta \frac{\partial z}{\partial y} + \sin\theta \frac{\partial}{\partial \theta}\left(\frac{\partial z}{\partial y}\right)$$

$$= -\sin\theta \frac{\partial z}{\partial x} + \cos\theta \left(\frac{\partial^2 z}{\partial x^2} \frac{\partial x}{\partial \theta} + \frac{\partial^2 z}{\partial y \partial x} \frac{\partial y}{\partial \theta}\right) + \cos\theta \frac{\partial z}{\partial y} + \sin\theta \frac{\partial^2 z}{\partial y^2} \frac{\partial y}{\partial \theta} + \frac{\partial^2 z}{\partial x \partial y} \frac{\partial x}{\partial \theta}$$

$$= -\sin\theta \frac{\partial z}{\partial x} + \cos\theta \left(-r\sin\theta \frac{\partial^2 z}{\partial x^2} + r\cos\theta \frac{\partial^2 z}{\partial y \partial x}\right) + \cos\theta \frac{\partial z}{\partial y}$$

$$+ \sin\theta \left(r\cos\theta \frac{\partial^2 z}{\partial y^2} - r\sin\theta \frac{\partial^2 z}{\partial x \partial y}\right)$$

$$= -\sin\theta \frac{\partial z}{\partial x} - r\cos\theta \sin\theta \frac{\partial^2 z}{\partial x^2} + r\cos^2\theta \frac{\partial^2 z}{\partial y \partial x} + \cos\theta \frac{\partial z}{\partial y}$$

$$+ r\cos\theta \sin\theta \frac{\partial^2 z}{\partial y^2} - r\sin^2\theta \frac{\partial^2 z}{\partial y \partial x}$$

$$= \cos\theta \frac{\partial z}{\partial y} - \sin\theta \frac{\partial z}{\partial x} + r\cos\theta \sin\theta \left(\frac{\partial^2 z}{\partial y^2} - \frac{\partial^2 z}{\partial x^2}\right) + r\left(\cos^2\theta - \sin^2\theta\right) \frac{\partial^2 z}{\partial y \partial x}$$

51. $\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\cos\theta + \frac{\partial z}{\partial y}\sin\theta$ and $\frac{\partial z}{\partial \theta} = -\frac{\partial z}{\partial x} r\sin\theta + \frac{\partial z}{\partial y} r\cos\theta$. Then

$$\frac{\partial^2 z}{\partial r^2} = \cos\theta\left(\frac{\partial^2 z}{\partial x^2}\cos\theta + \frac{\partial^2 z}{\partial y\partial x}\sin\theta\right) + \sin\theta\left(\frac{\partial^2 z}{\partial y^2}\sin\theta + \frac{\partial^2 z}{\partial x\partial y}\cos\theta\right)$$

$$= \cos^2\theta\,\frac{\partial^2 z}{\partial x^2} + 2\cos\theta\sin\theta\,\frac{\partial^2 z}{\partial x\partial y} + \sin^2\theta\,\frac{\partial^2 z}{\partial y^2}$$

and

$$\frac{\partial^2 z}{\partial \theta^2} = -r\cos\theta\,\frac{\partial z}{\partial x} + (-r\sin\theta)\left(\frac{\partial^2 z}{\partial x^2}(-r\sin\theta) + \frac{\partial^2 z}{\partial y\partial x} r\cos\theta\right)$$

$$-r\sin\theta\,\frac{\partial z}{\partial y} + r\cos\theta\left(\frac{\partial^2 z}{\partial y^2} r\cos\theta + \frac{\partial^2 z}{\partial x\partial y}(-r\sin\theta)\right)$$

$$= -r\cos\theta\,\frac{\partial z}{\partial x} - r\sin\theta\,\frac{\partial z}{\partial y} + r^2\sin^2\theta\,\frac{\partial^2 z}{\partial x^2} - 2r^2\cos\theta\sin\theta\,\frac{\partial^2 z}{\partial x\partial y} + r^2\cos^2\theta\,\frac{\partial^2 z}{\partial y^2}$$

Thus

$$\frac{\partial^2 z}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 z}{\partial \theta^2} + \frac{1}{r}\frac{\partial z}{\partial r} = \left(\cos^2\theta + \sin^2\theta\right)\frac{\partial^2 z}{\partial x^2} + \left(\sin^2\theta + \cos^2\theta\right)\frac{\partial^2 z}{\partial y^2} - \frac{1}{r}\cos\theta\,\frac{\partial z}{\partial x}$$

$$-\frac{1}{r}\sin\theta\,\frac{\partial z}{\partial y} + \frac{1}{r}\left(\cos\theta\,\frac{\partial z}{\partial x} + \sin\theta\,\frac{\partial z}{\partial y}\right)$$

$$= \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} \quad \text{as desired.}$$

52. **(a)** $\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$. Then

$$\frac{\partial^2 z}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial z}{\partial x}\frac{\partial x}{\partial t}\right) + \frac{\partial}{\partial t}\left(\frac{\partial z}{\partial y}\frac{\partial y}{\partial t}\right)$$

$$= \frac{\partial}{\partial t}\left(\frac{\partial z}{\partial x}\right)\frac{\partial x}{\partial t} + \frac{\partial^2 x}{\partial t^2}\frac{\partial z}{\partial x} + \frac{\partial}{\partial t}\left(\frac{\partial z}{\partial y}\right)\frac{\partial y}{\partial t} + \frac{\partial^2 y}{\partial t^2}\frac{\partial z}{\partial y}$$

$$= \frac{\partial^2 z}{\partial x^2}\left(\frac{\partial x}{\partial t}\right)^2 + \frac{\partial^2 z}{\partial y \partial x}\frac{\partial x}{\partial t}\frac{\partial y}{\partial t} + \frac{\partial^2 x}{\partial t^2}\frac{\partial z}{\partial x} + \frac{\partial^2 z}{\partial y^2}\left(\frac{\partial y}{\partial t}\right)^2 + \frac{\partial^2 z}{\partial x \partial y}\frac{\partial y}{\partial t}\frac{\partial x}{\partial t} + \frac{\partial^2 y}{\partial t^2}\frac{\partial z}{\partial y}$$

$$= \frac{\partial^2 z}{\partial x^2}\left(\frac{\partial x}{\partial t}\right)^2 + 2\frac{\partial^2 z}{\partial x \partial y}\frac{\partial x}{\partial t}\frac{\partial y}{\partial t} + \frac{\partial^2 z}{\partial y^2}\left(\frac{\partial y}{\partial t}\right)^2 + \frac{\partial^2 x}{\partial t^2}\frac{\partial z}{\partial x} + \frac{\partial^2 y}{\partial t^2}\frac{\partial z}{\partial y}$$

(b)

$$\frac{\partial^2 z}{\partial s \partial t} = \frac{\partial}{\partial s}\left(\frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}\right)$$

$$= \left(\frac{\partial^2 z}{\partial x^2}\frac{\partial x}{\partial s} + \frac{\partial^2 z}{\partial y \partial x}\frac{\partial y}{\partial s}\right)\frac{\partial x}{\partial t} + \frac{\partial z}{\partial x}\frac{\partial^2 x}{\partial s \partial t} + \left(\frac{\partial^2 z}{\partial y^2}\frac{\partial y}{\partial s} + \frac{\partial^2 z}{\partial x \partial y}\frac{\partial x}{\partial s}\right)\frac{\partial y}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial^2 y}{\partial s \partial t}$$

$$= \frac{\partial^2 z}{\partial x^2}\frac{\partial x}{\partial s}\frac{\partial x}{\partial t} + \frac{\partial^2 z}{\partial x \partial y}\left(\frac{\partial y}{\partial s}\frac{\partial x}{\partial t} + \frac{\partial y}{\partial t}\frac{\partial x}{\partial s}\right) + \frac{\partial z}{\partial x}\frac{\partial^2 x}{\partial s \partial t} + \frac{\partial z}{\partial y}\frac{\partial^2 y}{\partial s \partial t} + \frac{\partial^2 z}{\partial y^2}\frac{\partial y}{\partial s}\frac{\partial y}{\partial t}$$

53. **(a)** Since f is a polynomial, it has continuous second–order partial derivatives, and

$$f(tx,ty) = (tx)^2(ty)+2(tx)(ty)^2+5(ty)^3 = t^3x^2y+2t^3xy^2+5t^3y^3$$
$$= t^3\left(x^2y+2xy^2+5y^3\right) = t^3 f(x,y)$$

Thus, f is homogeneous of degree 3.

(b) Differentiating both sides of $f(tx,ty)=t^n f(x,y)$ with respect to t using the Chain Rule, we get

$$\frac{\partial}{\partial t} f(tx,ty) = \frac{\partial}{\partial t}\left[t^n f(x,y)\right] \Leftrightarrow$$

$$\frac{\partial}{\partial (tx)} f(tx,ty)\cdot\frac{\partial (tx)}{\partial t} + \frac{\partial}{\partial (ty)} f(tx,ty)\cdot\frac{\partial (ty)}{\partial t} = x\frac{\partial}{\partial (tx)} f(tx,ty) + y\frac{\partial}{\partial (ty)} f(tx,ty) = nt^{n-1} f(x,y) .$$

Setting $t=1$: $x\frac{\partial}{\partial x} f(x,y) + y\frac{\partial}{\partial y} f(x,y) = nf(x,y)$.

54. Differentiating both sides of $f(tx,ty)=t^n f(x,y)$ with respect to t using the Chain Rule, we get $\frac{\partial}{\partial (tx)} f(tx,ty)\cdot\frac{\partial (tx)}{\partial t} + \frac{\partial}{\partial (ty)} f(tx,ty)\cdot\frac{\partial (ty)}{\partial t} = x\frac{\partial}{\partial (tx)} f(tx,ty) + y\frac{\partial}{\partial (ty)} f(tx,ty) = nt^{n-1} f(x,y)$ and differentiating again with respect to t gives

$$x\left[\frac{\partial^2}{\partial (tx)^2} f(tx,ty)\cdot\frac{\partial (tx)}{\partial t} + \frac{\partial^2}{\partial (ty)\partial (tx)} f(tx,ty)\cdot\frac{\partial (ty)}{\partial t}\right]$$

$$+\ y\left[\ \frac{\partial^2}{\partial(tx)\partial(ty)}\ f(\text{tx,ty})\cdot\ \frac{\partial(tx)}{\partial t}+\frac{\partial^2}{\partial(ty)^2}\ f(\text{tx,ty})\cdot\ \frac{\partial(ty)}{\partial t}\ \right]=n(n-1)t^{n-1}f(\text{x,y})\ .$$

Setting $t=1$ and using the fact that $f_{yx}=f_{xy}$, we have $x^2f_{xx}+2xyf_{xy}+y^2f_{yy}=n(n-1)f(\text{x,y})$.

55. Differentiating both sides of $f(\text{tx,ty})=t^nf(\text{x,y})$ with respect to x using the Chain Rule, we get

$$\frac{\partial}{\partial x}\ f(\text{tx,ty})=\frac{\partial}{\partial x}\left[t^nf(\text{x,y})\right]\Leftrightarrow$$

$$\frac{\partial}{\partial(tx)}\ f(\text{tx,ty})\cdot\ \frac{\partial(tx)}{\partial x}+\frac{\partial}{\partial(ty)}\ f(\text{tx,ty})\cdot\ \frac{\partial(ty)}{\partial x}=t^n\ \frac{\partial}{\partial x}\ f(\text{x,y})\Leftrightarrow tf_x(\text{tx,ty})=t^nf_x(\text{x,y})\ .$$

Thus $f_x(\text{tx,ty})=t^{n-1}f_x(\text{x,y})$.

56. $F(\text{x,y,z})=0$ is assumed to define z as a function of x and y , that is, $z=f(\text{x,y})$. So by (7), $\frac{\partial z}{\partial x}=-\frac{F_x}{F_z}$ since $F_z\neq 0$. Similarly, it is assumed that $F(\text{x,y,z})=0$ defines x as a function of y and z , that is $x=h(\text{x,z})$. Then $F(h(\text{y,z}),\text{y,z})=0$ and by the Chain Rule, $F_x\ \frac{\partial x}{\partial y}+F_y\ \frac{\partial y}{\partial y}+F_z\ \frac{\partial z}{\partial y}=0$. But $\frac{\partial z}{\partial y}=0$ and $\frac{\partial y}{\partial y}=1$, so $F_x\ \frac{\partial x}{\partial y}+F_y=0\Rightarrow\ \frac{\partial x}{\partial y}=-\frac{F_y}{F_x}$. A similar calculation shows that $\frac{\partial y}{\partial z}=-\frac{F_z}{F_y}$. Thus

$$\frac{\partial z}{\partial x}\ \frac{\partial x}{\partial y}\ \frac{\partial y}{\partial z}=\left(-\frac{F_x}{F_z}\right)\left(-\frac{F_y}{F_x}\right)\left(-\frac{F_z}{F_y}\right)=-1\ .$$

1. First we draw a line passing through Raleigh and the eye of the hurricane. We can approximate the directional derivative at Raleigh in the direction of the eye of the hurricane by the average rate of change of pressure between the points where this line intersects the contour lines closest to Raleigh. In the direction of the eye of the hurricane, the pressure changes from 996 millibars to 992 millibars. We estimate the distance between these two points to be approximately 40 miles, so the rate of change of pressure in the direction given is approximately $\frac{992-996}{40}=-0.1$ millibar / mi.

2. First we draw a line passing through Muskegon and Ludington. We approximate the directional derivative at Muskegon in the direction of Ludington by the average rate of change of snowfall between the points where the line
intersects the contour lines closest to Muskegon. In the direction of Ludington, the snowfall changes from 60 to 70 inches. We estimate the distance between these two points to be approximately 28 miles, so the rate of change of annual snowfall in the direction given is approximately $\frac{70-60}{28}\approx 0.36$ in / mi.

3. $D_{\boldsymbol{u}}f(-20,30)=\nabla f(-20,30)\cdot \boldsymbol{u}=f_T(-20,30)\left(\frac{1}{\sqrt{2}}\right)+f_v(-20,30)\left(\frac{1}{\sqrt{2}}\right)$.

$f_T(-20,30)=\lim_{h\to 0}\frac{f(-20+h,30)-f(-20,30)}{h}$, so we can approximate $f_T(-20,30)$ by considering $h=\pm 5$ and using the values given in the table: $f_T(-20,30)\approx\frac{f(-15,30)-f(-20,30)}{5}=\frac{-26-(-33)}{5}=1.4$,

$f_T(-20,30)\approx\frac{f(-25,30)-f(-20,30)}{-5}=\frac{-39-(-33)}{-5}=1.2$. Averaging these values gives $f_T(-20,30)\approx 1.3$.

Similarly, $f_v(-20,30)=\lim_{h\to 0}\frac{f(-20,30+h)-f(-20,30)}{h}$, so we can approximate $f_v(-20,30)$ with $h=\pm 10$:

$f_v(-20,30)\approx\frac{f(-20,40)-f(-20,30)}{10}=\frac{-34-(-33)}{10}=-0.1$,

$f_v(-20,30)\approx\frac{f(-20,20)-f(-20,30)}{-10}=\frac{-30-(-33)}{-10}=-0.3$. Averaging these values gives $f_v(-20,30)\approx -0.2$. Then $D_{\boldsymbol{u}}f(-20,30)\approx 1.3\left(\frac{1}{\sqrt{2}}\right)+(-0.2)\left(\frac{1}{\sqrt{2}}\right)\approx 0.778$.

4. $f(x,y)=x^2y^3-y^4\Rightarrow f_x(x,y)=2xy^3$ and $f_y(x,y)=3x^2y^2-4y^3$. If $\boldsymbol{u}$ is a unit vector in the direction of $\theta=\frac{\pi}{4}$, then from Equation 6,

$D_{\boldsymbol{u}}f(2,1)=f_x(2,1)\cos\left(\frac{\pi}{4}\right)+f_y(2,1)\sin\left(\frac{\pi}{4}\right)=4\cdot\frac{\sqrt{2}}{2}+8\cdot\frac{\sqrt{2}}{2}=6\sqrt{2}$.

5. $f(x,y)=\sqrt{5x-4y} \Rightarrow f_x(x,y)=\frac{1}{2}(5x-4y)^{-1/2}(5)=\frac{5}{2\sqrt{5x-4y}}$ and $f_y(x,y)=\frac{1}{2}(5x-4y)^{-1/2}(-4)=-\frac{2}{\sqrt{5x-4y}}$

. If $\boldsymbol{u}$ is a unit vector in the direction of $\theta=-\frac{\pi}{6}$, then from Equation 6,

$D_{\boldsymbol{u}}f(4,1)=f_x(4,1)\cos\left(-\frac{\pi}{6}\right)+f_y(4,1)\sin\left(-\frac{\pi}{6}\right)=\frac{5}{8}\cdot\frac{\sqrt{3}}{2}+\left(-\frac{1}{2}\right)\left(-\frac{1}{2}\right)=\frac{5\sqrt{3}}{16}+\frac{1}{4}$.

6. $f(x,y)=x\sin(xy) \Rightarrow f_x(x,y)=x\cos(xy)\cdot y+\sin(xy)=xy\cos(xy)+\sin(xy)$ and

$f_y(x,y)=x\cos(xy)\cdot x=x^2\cos(xy)$. If $\boldsymbol{u}$ is a unit vector in the direction of $\theta=\frac{\pi}{3}$, then from Equation 6,

$D_{\boldsymbol{u}}f(2,0)=f_x(2,0)\cos\frac{\pi}{3}+f_y(2,0)\sin\frac{\pi}{3}=0+4\left(\frac{\sqrt{3}}{2}\right)=2\sqrt{3}$.

7. $f(x,y)=5xy^2-4x^3y$

(a) $\nabla f(x,y)=\left\langle f_x(x,y),f_y(x,y)\right\rangle=\left\langle 5y^2-12x^2y,10xy-4x^3\right\rangle$

(b) $\nabla f(1,2)=\left\langle 5(2)^2-12(1)^2(2),10(1)(2)-4(1)^3\right\rangle=\langle -4,16\rangle$

(c) By Equation 9, $D_{\boldsymbol{u}}f(1,2)=\nabla f(1,2)\cdot \boldsymbol{u}=\langle -4,16\rangle\cdot\left\langle \frac{5}{13},\frac{12}{13}\right\rangle=(-4)\left(\frac{5}{13}\right)+(16)\left(\frac{12}{13}\right)=\frac{172}{13}$

.

8. **(a)** $\nabla f(x,y)=\left\langle f_x(x,y),f_y(x,y)\right\rangle=\langle y/x,\ln x\rangle$

(b) $\nabla f(1,-3)=\left\langle \frac{-3}{1},\ln 1\right\rangle=\langle -3,0\rangle$

(c) By Equation 9, $D_{\boldsymbol{u}}f(1,-3)=\nabla f(1,-3)\cdot \boldsymbol{u}=\langle -3,0\rangle\cdot\left\langle -\frac{4}{5},\frac{3}{5}\right\rangle=\frac{12}{5}$.

9. $f(x,y,z)=xe^{2yz}$

(a) $\nabla f(x,y,z)=\left\langle f_x(x,y,z),f_y(x,y,z),f_z(x,y,z)\right\rangle=\left\langle e^{2yz},2xze^{2yz},2xye^{2yz}\right\rangle$

(b) $\nabla f(3,0,2)=\langle 1,12,0\rangle$

(c) By Equation 14, $D_{\boldsymbol{u}}f(3,0,2)=\nabla f(3,0,2)\cdot \boldsymbol{u}=\langle 1,12,0\rangle\cdot\left\langle \frac{2}{3},-\frac{2}{3},\frac{1}{3}\right\rangle=\frac{2}{3}-\frac{24}{3}+0=-\frac{22}{3}$.

10.

$f(\mathrm{x,y,z})=\sqrt{x+yz}=(x+yz)^{1/2}$

(a)

$$\nabla f(\mathrm{x,y,z}) = \left\langle \frac{1}{2}(x+yz)^{-1/2}(1), \frac{1}{2}(x+yz)^{-1/2}(z), \frac{1}{2}(x+yz)^{-1/2}(y) \right\rangle$$

$$=\left\langle 1/\left(2\sqrt{x+yz}\right), \mathrm{z}/\left(2\sqrt{x+yz}\right), \mathrm{y}/\left(2\sqrt{x+yz}\right)\right\rangle$$

(b) $\nabla f(1,3,1)=\left\langle \frac{1}{4}, \frac{1}{4}, \frac{3}{4} \right\rangle$

(c) $D_{\boldsymbol{u}} f(1,3,1)=\nabla f(1,3,1)\cdot \boldsymbol{u}=\left\langle \frac{1}{4}, \frac{1}{4}, \frac{3}{4} \right\rangle \cdot \left\langle \frac{2}{7}, \frac{3}{7}, \frac{6}{7} \right\rangle = \frac{2}{28}+\frac{3}{28}+\frac{18}{28}=\frac{23}{28}$.

11. $f(\mathrm{x,y})=1+2x\sqrt{y} \Rightarrow \nabla f(\mathrm{x,y})=\left\langle 2\sqrt{y}, 2x\cdot \frac{1}{2}y^{-1/2} \right\rangle=\left\langle 2\sqrt{y}, \mathrm{x}/\sqrt{y}\right\rangle$, $\nabla f(3,4)=\left\langle 4, \frac{3}{2} \right\rangle$, and a unit vector in the direction of $\boldsymbol{v}$ is $\boldsymbol{u}=\frac{1}{\sqrt{4^2+(-3)^2}}\langle 4,-3\rangle=\left\langle \frac{4}{5}, -\frac{3}{5} \right\rangle$, so

$D_{\boldsymbol{u}} f(3,4)=\nabla f(3,4)\cdot \boldsymbol{u}=\left\langle 4, \frac{3}{2} \right\rangle \cdot \left\langle \frac{4}{5}, -\frac{3}{5} \right\rangle=\frac{23}{10}$.

12. $f(\mathrm{x,y})=\ln\left(x^2+y^2\right) \Rightarrow \nabla f(\mathrm{x,y})=\left\langle \frac{2x}{x^2+y^2}, \frac{2y}{x^2+y^2} \right\rangle$, $\nabla f(2,1)=\left\langle \frac{4}{5}, \frac{2}{5} \right\rangle$, and a unit vector in the direction of $\boldsymbol{v}=\langle -1,2\rangle$ is $\boldsymbol{u}=\frac{1}{\sqrt{1+4}}\langle -1,2\rangle=\left\langle -\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right\rangle$, so

$D_{\boldsymbol{u}} f(2,1)=\nabla f(2,1)\cdot \boldsymbol{u}=\left\langle \frac{4}{5}, \frac{2}{5} \right\rangle \cdot \left\langle -\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right\rangle=-\frac{4}{5\sqrt{5}}+\frac{4}{5\sqrt{5}}=0$.

13. $g(\mathrm{s,t})=s^2e^t \Rightarrow \nabla g(\mathrm{s,t})=2se^t\boldsymbol{i}+s^2e^t\boldsymbol{j}$, $\nabla g(2,0)=4\boldsymbol{i}+4\boldsymbol{j}$, and a unit vector in the direction of $\boldsymbol{v}$ is $\boldsymbol{u}=\frac{1}{\sqrt{2}}(\boldsymbol{i}+\boldsymbol{j})$, so $D_{\boldsymbol{u}} g(2,0)=\nabla g(2,0)\cdot \boldsymbol{u}=(4\boldsymbol{i}+4\boldsymbol{j})\cdot \frac{1}{\sqrt{2}}(\boldsymbol{i}+\boldsymbol{j})=\frac{8}{\sqrt{2}}=4\sqrt{2}$.

14. $g(\mathrm{r},\theta)=e^{-r}\sin\theta \Rightarrow \nabla g(\mathrm{r},\theta)=\left(-e^{-r}\sin\theta\right)\boldsymbol{i}+\left(e^{-r}\cos\theta\right)\boldsymbol{j}$, $\nabla g\left(0,\frac{\pi}{3}\right)=-\frac{\sqrt{3}}{2}\boldsymbol{i}+\frac{1}{2}\boldsymbol{j}$, and a unit vector in the direction of $\boldsymbol{v}$ is

$\boldsymbol{u}=\frac{1}{\sqrt{13}}\ (3\boldsymbol{i}-2\boldsymbol{j})$, so

$D_{\boldsymbol{u}}g\left(0,\frac{\pi}{3}\right)=\nabla g\left(0,\frac{\pi}{3}\right)\cdot\boldsymbol{u}=\left(-\frac{\sqrt{3}}{2}\ \boldsymbol{i}+\frac{1}{2}\ \boldsymbol{j}\right)\cdot\frac{1}{\sqrt{13}}\ (3\boldsymbol{i}-2\boldsymbol{j})=-\frac{3\sqrt{3}}{2\sqrt{13}}-\frac{1}{\sqrt{13}}=-\frac{3\sqrt{3}+2}{2\sqrt{13}}$.

15. $f(x,y,z)=\sqrt{x^2+y^2+z^2}\Rightarrow\nabla f(x,y,z)=\left\langle\frac{x}{\sqrt{x^2+y^2+z^2}},\frac{y}{\sqrt{x^2+y^2+z^2}},\frac{z}{\sqrt{x^2+y^2+z^2}}\right\rangle$,

$\nabla f(1,2,-2)=\left\langle\frac{1}{3},\frac{2}{3},-\frac{2}{3}\right\rangle$, and a unit vector in the direction of $\boldsymbol{v}$ is

$\boldsymbol{u}=\frac{1}{9}\ \langle-6,6,-3\rangle=\left\langle-\frac{2}{3},\frac{2}{3},-\frac{1}{3}\right\rangle$, so

$D_{\boldsymbol{u}}f(1,2,-2)=\nabla f(1,2,-2)\cdot\boldsymbol{u}=\left\langle\frac{1}{3},\frac{2}{3},-\frac{2}{3}\right\rangle\cdot\left\langle-\frac{2}{3},\frac{2}{3},-\frac{1}{3}\right\rangle=\frac{4}{9}$.

16. $f(x,y,z)=\frac{x}{y+z}\Rightarrow\nabla f(x,y,z)=\left\langle\frac{1}{y+z},-\frac{x}{(y+z)^2},-\frac{x}{(y+z)^2}\right\rangle$, $\nabla f(4,1,1)=\left\langle\frac{1}{2},-1,-1\right\rangle$, and a unit vector in the direction of $\boldsymbol{v}$ is $\boldsymbol{u}=\frac{1}{\sqrt{14}}\ \langle1,2,3\rangle$, so

$D_{\boldsymbol{u}}f(4,1,1)=\nabla f(4,1,1)\cdot\boldsymbol{u}=\left\langle\frac{1}{2},-1,-1\right\rangle\cdot\frac{1}{\sqrt{14}}\ \langle1,2,3\rangle=-\frac{9}{2\sqrt{14}}$.

17. $g(x,y,z)=(x+2y+3z)^{3/2}\Rightarrow$

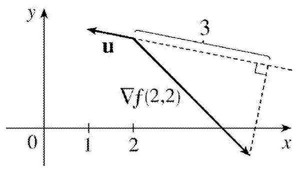
$\nabla g(x,y,z)=\left\langle\frac{3}{2}\,(x+2y+3z)^{1/2}(1),\frac{3}{2}\,(x+2y+3z)^{1/2}(2),\frac{3}{2}\,(x+2y+3z)^{1/2}(3)\right\rangle$

$=\left\langle\frac{3}{2}\sqrt{x+2y+3z},3\sqrt{x+2y+3z},\frac{9}{2}\sqrt{x+2y+3z}\right\rangle$, $\nabla g(1,1,2)=\left\langle\frac{9}{2},9,\frac{27}{2}\right\rangle$,

and a unit vector in the direction of $\boldsymbol{v}=2\boldsymbol{j}-\boldsymbol{k}$ is $\boldsymbol{u}=\frac{2}{\sqrt{5}}\ \boldsymbol{j}-\frac{1}{\sqrt{5}}\ \boldsymbol{k}$, so

$D_{\boldsymbol{u}}g(1,1,2)=\left\langle\frac{9}{2},9,\frac{27}{2}\right\rangle\cdot\left\langle0,\frac{2}{\sqrt{5}},-\frac{1}{\sqrt{5}}\right\rangle=\frac{18}{\sqrt{5}}-\frac{27}{2\sqrt{5}}=\frac{9}{2\sqrt{5}}$.

18. $D_{\boldsymbol{u}}f(2,2)=\nabla f(2,2)\cdot \boldsymbol{u}$, the scalar projection of $\nabla f(2,2)$ onto $\boldsymbol{u}$, so we draw a perpendicular from the tip of $\nabla f(2,2)$ to the line containing $\boldsymbol{u}$. We can use the point (2,2) to determine the scale of the axes, and we estimate the length of



the projection to be approximately 3.0 units. Since the angle between $\nabla f(2,2)$ and $\boldsymbol{u}$ is greater than $90°$, the scalar projection is negative. Thus $D_{\boldsymbol{u}}f(2,2)\approx -3$.

19. $f(x,y)=\sqrt{xy}\Rightarrow \nabla f(x,y)=\left\langle \frac{1}{2}(xy)^{-1/2}(y),\frac{1}{2}(xy)^{-1/2}(x)\right\rangle =\left\langle \frac{y}{2\sqrt{xy}},\frac{x}{2\sqrt{xy}}\right\rangle$, so $\nabla f(2,8)=\left\langle 1,\frac{1}{4}\right\rangle$. The unit vector in the direction of $\overrightarrow{PQ}=\langle 5-2,4-8\rangle=\langle 3,-4\rangle$ is $\boldsymbol{u}=\left\langle \frac{3}{5},-\frac{4}{5}\right\rangle$, so $D_{\boldsymbol{u}}f(2,8)=\nabla f(2,8)\cdot \boldsymbol{u}=\left\langle 1,\frac{1}{4}\right\rangle\cdot\left\langle \frac{3}{5},-\frac{4}{5}\right\rangle=\frac{2}{5}$.

20. $f(x,y,z)=x^2+y^2+z^2\Rightarrow \nabla f(x,y,z)=\langle 2x,2y,2z\rangle$, so $\nabla f(2,1,3)=\langle 4,2,6\rangle$. The unit vector in the direction of $\overrightarrow{PO}=\langle -2,-1,-3\rangle$ is $\boldsymbol{u}=\frac{1}{\sqrt{14}}\langle -2,-1,-3\rangle$, so $D_{\boldsymbol{u}}f(2,1,3)=\nabla f(2,1,3)\cdot \boldsymbol{u}=\langle 4,2,6\rangle\cdot\frac{1}{\sqrt{14}}\langle -2,-1,-3\rangle=-\frac{28}{\sqrt{14}}=-2\sqrt{14}$.

21. $f(x,y)=y^2/x=y^2x^{-1}\Rightarrow \nabla f(x,y)=\left\langle -y^2x^{-2},2yx^{-1}\right\rangle=\left\langle -y^2/x^2,2y/x\right\rangle$.
$\nabla f(2,4)=\langle -4,4\rangle$, or equivalently $\langle -1,1\rangle$, is the direction of maximum rate of change, and the maximum rate is $|\nabla f(2,4)|=\sqrt{16+16}=4\sqrt{2}$.

22. $f(p,q)=qe^{-p}+pe^{-q}\Rightarrow \nabla f(p,q)=\left\langle -qe^{-p}+e^{-q},e^{-p}-pe^{-q}\right\rangle$.
$\nabla f(0,0)=\langle 1,1\rangle$ is the direction of maximum rate of change and the maximum rate is $|\nabla f(0,0)|=\sqrt{2}$.

23. $f(x,y)=\sin(xy)\Rightarrow \nabla f(x,y)=\langle y\cos(xy),x\cos(xy)\rangle$, $\nabla f(1,0)=\langle 0,1\rangle$. Thus the maximum rate of change is $|\nabla f(1,0)|=1$ in the direction $\langle 0,1\rangle$.

24. $f(x,y,z)=x^2y^3z^4 \Rightarrow \nabla f(x,y,z)=\langle 2xy^3z^4, 3x^2y^2z^4, 4x^2y^3z^3\rangle$, $\nabla f(1,1,1)=\langle 2,3,4\rangle$. Thus the maximum rate of change is $|\nabla f(1,1,1)|=\sqrt{29}$ in the direction $\langle 2,3,4\rangle$.

25. $f(x,y,z)=\ln(xy^2z^3) \Rightarrow \nabla f(x,y,z)=\left\langle \frac{y^2z^3}{xy^2z^3}, \frac{2xyz^3}{xy^2z^3}, \frac{3xy^2z^2}{xy^2z^3} \right\rangle = \left\langle \frac{1}{x}, \frac{2}{y}, \frac{3}{z} \right\rangle$.

$\nabla f(1,-2,-3)=\langle 1,-1,-1\rangle$ is the direction of maximum rate of change and the maximum rate is $|\nabla f(1,-2,-3)|=\sqrt{3}$.

26. $f(x,y,z)=\tan(x+2y+3z) \Rightarrow \nabla f(x,y,z)=\langle \sec^2(x+2y+3z)(1), \sec^2(x+2y+3z)(2), \sec^2(x+2y+3z)(3)\rangle$.

$\nabla f(-5,1,1)=\langle \sec^2(0), 2\sec^2(0), 3\sec^2(0)\rangle=\langle 1,2,3\rangle$ is the direction of maximum rate of change and the maximum rate is $|\nabla f(-5,1,1)|=\sqrt{14}$.

27. **(a)** As in the proof of Theorem 15, $D_{\boldsymbol{u}}f=|\nabla f|\cos\theta$. Since the minimum value of $\cos\theta$ is -1 occurring when $\theta=\pi$, the minimum value of $D_{\boldsymbol{u}}f$ is $-|\nabla f|$ occurring when $\theta=\pi$, that is when $\boldsymbol{u}$ is in the opposite direction of ∇f (assuming $\nabla f\neq\mathbf{0}$).

(b) $f(x,y)=x^4y-x^2y^3 \Rightarrow \nabla f(x,y)=\langle 4x^3y-2xy^3, x^4-3x^2y^2\rangle$, so f decreases fastest at the point (2,–3) in the direction $-\nabla f(2,-3)=-\langle 12,-92\rangle=\langle -12,92\rangle$.

28. $f(x,y)=x^2+\sin xy \Rightarrow f_x(x,y)=2x+y\cos xy$, $f_y(x,y)=x\cos xy$ and $f_x(1,0)=2(1)+(0)\cos 0=2$, $f_y(1,0)=(1)\cos 0=1$. If $\boldsymbol{u}$ is a unit vector which makes an angle θ with the positive x –axis, then $D_{\boldsymbol{u}}f(1,0)=f_x(1,0)\cos\theta+f_y(1,0)\sin\theta=2\cos\theta+\sin\theta$. We want $D_{\boldsymbol{u}}f(1,0)=1$, so $2\cos\theta+\sin\theta=1 \Rightarrow$

$\sin\theta=1-2\cos\theta \Rightarrow \sin^2\theta=(1-2\cos\theta)^2 \Rightarrow 1-\cos^2\theta=1-4\cos\theta+4\cos^2\theta \Rightarrow 5\cos^2\theta-4\cos\theta=0 \Rightarrow$
$\cos\theta(5\cos\theta-4)=0 \Rightarrow$
$\cos\theta=0$ or $\cos\theta=\frac{4}{5} \Rightarrow \theta=\frac{\pi}{2}$ or $\theta=2\pi-\cos^{-1}\left(\frac{4}{5}\right)\approx 5.64$.

29. The direction of fastest change is $\nabla f(x,y)=(2x-2)\boldsymbol{i}+(2y-4)\boldsymbol{j}$, so we need to find all points (x,y) where $\nabla f(x,y)$ is parallel to $\boldsymbol{i}+\boldsymbol{j} \Leftrightarrow (2x-2)\boldsymbol{i}+(2y-4)\boldsymbol{j}= k(\boldsymbol{i}+\boldsymbol{j}) \Leftrightarrow k=2x-2$ and $k=2y-4$. Then $2x-2=2y-4 \Rightarrow y=x+1$, so the direction of fastest change is $\boldsymbol{i}+\boldsymbol{j}$ at all points on the line $y=x+1$.

30. The fisherman is traveling in the direction $\langle -80,-60\rangle$. A unit vector in this direction is

$\boldsymbol{u}=\frac{1}{100}\left\langle -80,-60\right\rangle =\left\langle -\frac{4}{5},-\frac{3}{5}\right\rangle$, and if the depth of the lake is given by $f(x,y)=200+0.02x^2-0.001y^3$, then $\nabla f(x,y)=\left\langle 0.04x,-0.003y^2\right\rangle$.

$D_{\boldsymbol{u}}f(80,60)=\nabla f(80,60)\cdot \boldsymbol{u}=\left\langle 3.2,-10.8\right\rangle \cdot \left\langle -\frac{4}{5},-\frac{3}{5}\right\rangle =3.92$. Since $D_{\boldsymbol{u}}f(80,60)$ is positive, the depth of the lake is increasing near (80,60) in the direction toward the buoy.

31. $T=\frac{k}{\sqrt{x^2+y^2+z^2}}$ and $120=T(1,2,2)=\frac{k}{3}$ so $k=360$.

(a) $\boldsymbol{u}=\frac{\left\langle 1,-1,1\right\rangle}{\sqrt{3}}$,

$$D_{\boldsymbol{u}}T(1,2,2) = \nabla T(1,2,2)\cdot \boldsymbol{u}=\left[-360\left(x^2+y^2+z^2\right)^{-3/2}\left\langle x,y,z\right\rangle\right]_{(1,2,2)}\cdot \boldsymbol{u}$$

$$= -\frac{40}{3}\left\langle 1,2,2\right\rangle \cdot \frac{1}{\sqrt{3}}\left\langle 1,-1,1\right\rangle =-\frac{40}{3\sqrt{3}}$$

(b) From (a), $\nabla T=-360\left(x^2+y^2+z^2\right)^{-3/2}\left\langle x,y,z\right\rangle$, and since $\left\langle x,y,z\right\rangle$ is the position vector of the point (x,y,z) , the vector $-\left\langle x,y,z\right\rangle$, and thus ∇T , always points toward the origin.

32. $\nabla T=-400e^{-x^2-3y^2-9z^2}\left\langle x,3y,9z\right\rangle$

(a) $\boldsymbol{u}=\frac{1}{\sqrt{6}}\left\langle 1,-2,1\right\rangle$, $\nabla T(2,-1,2)=-400e^{-43}\left\langle 2,-3,18\right\rangle$ and

$D_{\boldsymbol{u}}T(2,-1,2)=\left(-\frac{400e^{-43}}{\sqrt{6}}\right)(26)=-\frac{5200\sqrt{6}}{3e^{43}}$ □/ m.

(b) $\nabla T(2,-1,2)=400e^{-43}\left\langle -2,3,-18\right\rangle$ or equivalently $\left\langle -2,3,-18\right\rangle$.

(c) $|\nabla T|=400e^{-x^2-3y^2-9z^2}\sqrt{x^2+9y^2+8z^2}$ °C/m is the maximum rate of increase. At $(2,-1,2)$ the maximum rate of increase is $400e^{-43}\sqrt{337}$ °C/m.

33. $\nabla V(x,y,z)=\left\langle 10x-3y+yz,xz-3x,xy\right\rangle$, $\nabla V(3,4,5)=\left\langle 38,6,12\right\rangle$

(a) $D_{\boldsymbol{u}}V(3,4,5)=\left\langle 38,6,12\right\rangle \cdot \frac{1}{\sqrt{3}}\left\langle 1,1,-1\right\rangle =\frac{32}{\sqrt{3}}$

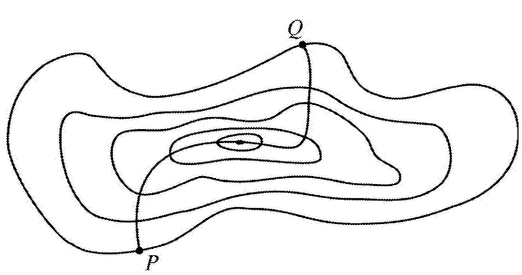
(b) $\nabla V(3,4,5)=\left\langle 38,6,12\right\rangle$ or equivalently $\left\langle 19,3,6\right\rangle$.

(c) $|\nabla V(3,4,5)|=\sqrt{38^2+6^2+12^2}=\sqrt{1624}=2\sqrt{406}$

34. $z=f(x,y)=1000-0.01x^2-0.02y^2 \Rightarrow \nabla f(x,y)=\langle -0.02x,-0.04y\rangle$ and $\nabla f(50,80)=\langle -1,-3.2\rangle$
(a) Due south is in the direction of the unit vector $\boldsymbol{u}=-\boldsymbol{j}$ and
$D_{\boldsymbol{u}}f(50,80)=\nabla f(50,80)\cdot\langle 0,-1\rangle=\langle -1,-3.2\rangle\cdot\langle 0,-1\rangle=0+3.2=3.2$. Thus, if you walk due south from $(50,80,847)$ you will ascend at a rate of 3.2 vertical meters per horizontal meter.
(b) Northwest is in the direction of the unit vector $\boldsymbol{u}=\frac{1}{\sqrt{2}}\langle -1,1\rangle$ and
$D_{\boldsymbol{u}}f(50,80)=\nabla f(50,80)\cdot\frac{1}{\sqrt{2}}\langle -1,1\rangle=\langle -1,-3.2\rangle\cdot\frac{1}{\sqrt{2}}\langle -1,1\rangle=-\frac{2.2}{\sqrt{2}}\approx -1.56$. Thus, if you walk northwest from $(50,80,847)$ you will descend at a rate of approximately 1.56 vertical meters per horizontal meter.
(c) $\nabla f(50,80)=\langle -1,-3.2\rangle$ is the direction of largest slope with a rate of ascent $|\nabla f(50,80)|=\sqrt{11.24}\approx 3.35$. The angle above the horizontal in which the path begins is given by $\tan\theta\approx 3.35\Rightarrow\theta\approx\tan^{-1}(3.35)\approx 73.4^\circ$.

35. A unit vector in the direction of $\overrightarrow{AB}$ is $\boldsymbol{i}$ and a unit vector in the direction of $\overrightarrow{AC}$ is $\boldsymbol{j}$. Thus $D_{\overrightarrow{AB}}f(1,3)=f_x(1,3)=3$ and $D_{\overrightarrow{AC}}f(1,3)=f_y(1,3)=26$. Therefore $\nabla f(1,3)=\langle f_x(1,3),f_y(1,3)\rangle=\langle 3,26\rangle$, and by definition, $D_{\overrightarrow{AD}}f(1,3)=\nabla f\cdot\boldsymbol{u}$ where $\boldsymbol{u}$ is a unit vector in the direction of $\overrightarrow{AD}$, which is $\left\langle \frac{5}{13},\frac{12}{13}\right\rangle$. Therefore, $D_{\overrightarrow{AD}}f(1,3)=\langle 3,26\rangle\cdot\left\langle \frac{5}{13},\frac{12}{13}\right\rangle=3\cdot\frac{5}{13}+26\cdot\frac{12}{13}=\frac{327}{13}$.

36. The curve of steepest ascent is perpendicular to all of the contour lines.



37. **(a)**

$$\nabla(au+bv)=\left\langle \frac{\partial(au+bv)}{\partial x},\frac{\partial(au+bv)}{\partial y}\right\rangle=\left\langle a\frac{\partial u}{\partial x}+b\frac{\partial v}{\partial x},a\frac{\partial u}{\partial y}+b\frac{\partial v}{\partial y}\right\rangle$$

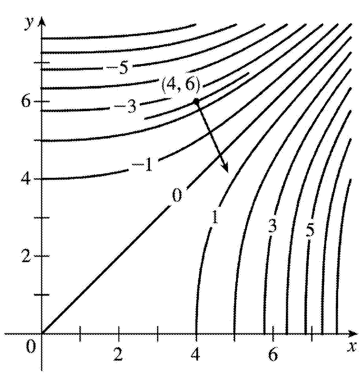
$$=a\left\langle \frac{\partial u}{\partial x},\frac{\partial u}{\partial y}\right\rangle +b\left\langle \frac{\partial v}{\partial x},\frac{\partial v}{\partial y}\right\rangle =a\nabla u+b\nabla v$$

(b) $\nabla(uv)=\left\langle v\frac{\partial u}{\partial x}+u\frac{\partial v}{\partial x},v\frac{\partial u}{\partial y}+u\frac{\partial v}{\partial y}\right\rangle =v\left\langle \frac{\partial u}{\partial x},\frac{\partial u}{\partial y}\right\rangle +u\left\langle \frac{\partial v}{\partial x},\frac{\partial v}{\partial y}\right\rangle =v\nabla u+u\nabla v$

(c) $\nabla\left(\frac{u}{v}\right)=\left\langle \frac{v\frac{\partial u}{\partial x}-u\frac{\partial v}{\partial x}}{v^2},\frac{v\frac{\partial u}{\partial y}-u\frac{\partial v}{\partial y}}{v^2}\right\rangle =\frac{v\left\langle \frac{\partial u}{\partial x},\frac{\partial u}{\partial y}\right\rangle -u\left\langle \frac{\partial v}{\partial x},\frac{\partial v}{\partial y}\right\rangle}{v^2}=\frac{v\nabla u-u\nabla v}{v^2}$

(d) $\nabla u^n=\left\langle \frac{\partial(u^n)}{\partial x},\frac{\partial(u^n)}{\partial y}\right\rangle =\left\langle nu^{n-1}\frac{\partial u}{\partial x},nu^{n-1}\frac{\partial u}{\partial y}\right\rangle =nu^{n-1}\nabla u$.

38. If we place the initial point of the gradient vector $\nabla f(4,6)$ at $(4,6)$, the vector is perpendicular to the level curve of f that includes $(4,6)$, so we sketch a portion of the level curve through $(4,6)$ (using the nearby level curves as a
guideline) and draw a line perpendicular to the curve at $(4,6)$. The gradient vector is parallel to this line, pointing in the direction of increasing function values, and with length equal to the maximum value of the directional derivative of f at $(4,6)$. We can estimate this length by finding the average rate of change in the direction of the gradient. The line intersects the contour lines corresponding to -2 and -3 with an estimated distance of 0.5 units. Thus the rate of change is approxi–
mately $\frac{-2-(-3)}{0.5}=2$, and we sketch the gradient vector with length 2 .



39. Let $F(x,y,z)=x^2+2y^2+3z^2$. Then $x^2+2y^2+3z^2=21$ is a level surface of F . $F_x(x,y,z)=2x\Rightarrow$
$F_x(4,-1,1)=8$, $F_y(x,y,z)=4y\Rightarrow F_y(4,-1,1)=-4$, and $F_z(x,y,z)=6z\Rightarrow F_z(4,-1,1)=6$.
(a) Equation 19 gives an equation of the tangent plane at $(4,-1,1)$ as $8(x-4)-4[y-(-1)]+6(z-1)=0$ or $4x-2y+3z=21$.

(b) By Equation 20, the normal line has symmetric equations
$\frac{x-4}{8}=\frac{y+1}{-4}=\frac{z-1}{6}$ or $\frac{x-4}{4}=\frac{y+1}{-2}=\frac{z-1}{3}$.

40. Let $F(x,y,z)=y^2+z^2-x$. Then $x=y^2+z^2-2$ is the level surface $F(x,y,z)=2$.
$F_x(x,y,z)=-1 \Rightarrow F_x(-1,1,0)=-1$, $F_y(x,y,z)=2y \Rightarrow F_y(-1,1,0)=2$,
and $F_z(x,y,z)=2z \Rightarrow F_z(-1,1,0)=0$.
(a) An equation of the tangent plane is $-1(x+1)+2(y-1)+0(z-0)=0$ or $-x+2y=3$.
(b) The normal line has symmetric equations $\frac{x+1}{-1}=\frac{y-1}{2}$, $z=0$.

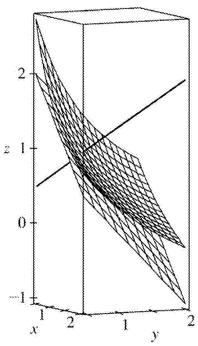
41. Let $F(x,y,z)=x^2-2y^2+z^2+yz$. Then $x^2-2y^2+z^2+yz=2$ is a level surface of F and
$\nabla F(x,y,z)=\langle 2x,-4y+z,2z+y\rangle$.
(a) $\nabla F(2,1,-1)=\langle 4,-5,-1\rangle$ is a normal vector for the tangent plane at $(2,1,-1)$, so an equation of the tangent plane is $4(x-2)-5(y-1)-1(z+1)=0$ or $4x-5y-z=4$.
(b) The normal line has direction $\langle 4,-5,-1\rangle$, so parametric equations are $x=2+4t$, $y=1-5t$, $z=-1-t$,
and symmetric equations are $\frac{x-2}{4}=\frac{y-1}{-5}=\frac{z+1}{-1}$.

42. Let $F(x,y,z)=x-z-4\arctan(yz)$. Then $x-z=4\arctan(yz)$ is the level surface $F(x,y,z)=0$, and
$\nabla F(x,y,z)=\left\langle 1,-\frac{4z}{1+y^2z^2},-1-\frac{4y}{1+y^2z^2}\right\rangle$.
(a) $\nabla F(1+\pi,1,1)=\langle 1,-2,-3\rangle$ and an equation of the tangent plane is $1(x-(1+\pi))-2(y-1)-3(z-1)=0$ or $x-2y-3z=-4+\pi$.
(b) The normal line has direction $\langle 1,-2,-3\rangle$, so parametric equations are $x=1+\pi+t$, $y=1-2t$, $z=1-3t$,
and symmetric equations are $x-1-\pi=\frac{y-1}{-2}=\frac{z-1}{-3}$.

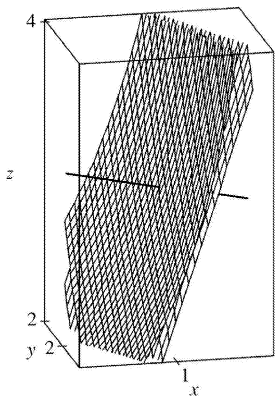
43. $F(x,y,z)=-z+xe^y\cos z \Rightarrow \nabla F(x,y,z)=\langle e^y\cos z,xe^y\cos z,-1-xe^y\sin z\rangle$, $\nabla F(1,0,0)=\langle 1,1,-1\rangle$
(a) $1(x-1)+1(y-0)-1(z-0)=0$ or $x+y-z=1$
(b) $x-1=y=-z$

44. $F(x,y,z)=yz-\ln(x+z) \Rightarrow \nabla F(x,y,z)=\left\langle -\frac{1}{x+z},z,y-\frac{1}{x+z}\right\rangle$, $\nabla F(0,0,1)=\langle -1,1,-1\rangle$.
(a) $(-1)(x-0)+(1)(y-0)-1(z-1)=0$ or $x-y+z=1$.
(b) Parametric equations are $x=-t$, $y=t$, $z=1-t$ and symmetric equations are $\frac{x}{-1}=\frac{y}{1}=\frac{z-1}{-1}$ or
$-x=y=1-z$.

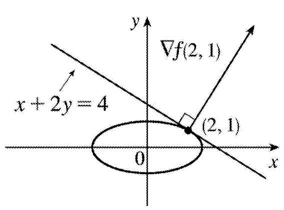
45. $F(x,y,z)=xy+yz+zx$, $\nabla F(x,y,z)=\langle y+z,x+z,y+x\rangle$, $\nabla F(1,1,1)=\langle 2,2,2\rangle$, so an equation of the tangent plane is $2x+2y+2z=6$ or $x+y+z=3$, and the normal line is given by $x-1=y-1=z-1$ or $x=y=z$.



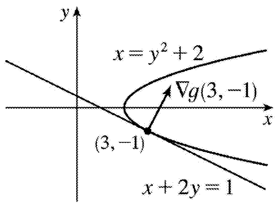
46. $F(x,y,z)=xyz$, $\nabla F(x,y,z)=\langle yz,xz,yx\rangle$, $\nabla F(1,2,3)=\langle 6,3,2\rangle$, so an equation of the tangent plane is $6x+3y+2z=18$, and the normal line is given by $\dfrac{x-1}{6}=\dfrac{y-2}{3}=\dfrac{z-3}{2}$ or $x=1+6t$, $y=2+3t$, $z=3+2t$.



47. $\nabla f(x,y)=\langle 2x,8y\rangle$, $\nabla f(2,1)=\langle 4,8\rangle$. The tangent line has equation $\nabla f(2,1)\cdot\langle x-2,y-1\rangle=0\Rightarrow$ $4(x-2)+8(y-1)=0$, which simplifies to $x+2y=4$.



48. $\nabla g(x,y)=\langle 1,-2y\rangle$, $\nabla g(3,-1)=\langle 1,2\rangle$. The tangent line has equation $\nabla g(3,-1)\cdot\langle x-3,y+1\rangle=0\Rightarrow$ $1(x-3)+2(y+1)=0$, which simplifies to $x+2y=1$.



49. $\nabla F(x_0,y_0,z_0)=\left\langle \frac{2x_0}{a^2},\frac{2y_0}{b^2},\frac{2z_0}{c^2}\right\rangle$. Thus an equation of the tangent plane at (x_0,y_0,z_0) is $\frac{2x_0}{a^2}x+\frac{2y_0}{b^2}y+\frac{2z_0}{c^2}z=2\left(\frac{x_0^2}{a^2}+\frac{y_0^2}{b^2}+\frac{z_0^2}{c^2}\right)=2(1)=2$ since (x_0,y_0,z_0) is a point on the ellipsoid. Hence $\frac{x_0}{a^2}x+\frac{y_0}{b^2}y+\frac{z_0}{c^2}z=1$ is an equation of the tangent plane.

50. $\nabla F(x_0,y_0,z_0)=\left\langle \frac{2x_0}{a^2},\frac{2y_0}{b^2},\frac{-2z_0}{c^2}\right\rangle$, so an equation of the tangent plane at (x_0,y_0,z_0) is $\frac{2x_0}{a^2}x+\frac{2y_0}{b^2}y-\frac{2z_0}{c^2}z=2\left(\frac{x_0^2}{a^2}+\frac{y_0^2}{b^2}-\frac{z_0^2}{c^2}\right)=2$ or $\frac{x_0}{a^2}x+\frac{y_0}{b^2}y-\frac{z_0}{c^2}z=1$.

51. $\nabla F(x_0,y_0,z_0)=\left\langle \frac{2x_0}{a^2},\frac{2y_0}{b^2},\frac{-1}{c}\right\rangle$, so an equation of the tangent plane is $\frac{2x_0}{a^2}x+\frac{2y_0}{b^2}y-\frac{1}{c}z=\frac{2x_0^2}{a^2}+\frac{2y_0^2}{b^2}-\frac{z_0}{c}$ or $\frac{2x_0}{a^2}x+\frac{2y_0}{b^2}y=\frac{z}{c}+2\left(\frac{x_0^2}{a^2}+\frac{y_0^2}{b^2}\right)-\frac{z_0}{c}$. But $\frac{z_0}{c}=\frac{x_0^2}{a^2}+\frac{y_0^2}{b^2}$, so the equation can be written as $\frac{2x_0}{a^2}x+\frac{2y_0}{b^2}y=\frac{z+z_0}{c}$.

52. Since $\nabla f(x_0,y_0,z_0)=\langle 2x_0,4y_0,6z_0\rangle$ and $\langle 3,-1,3\rangle$ are both normal vectors to the surface at (x_0,y_0,z_0), we need $\langle 2x_0,4y_0,6z_0\rangle=c\langle 3,-1,3\rangle$ or $\langle x_0,2y_0,3z_0\rangle=k\langle 3,-1,3\rangle$. Thus $x_0=3k$, $y_0=-\frac{1}{2}k$ and

$z_0=k$. But $x_0^2+2y_0^2+3z_0^2=1$ or $\left(9+\frac{1}{2}+3\right)k^2=1$, so $k=\pm\frac{\sqrt{2}}{5}$ and there are two such points: $\left(\pm\frac{3\sqrt{2}}{5},\mp\frac{1}{5\sqrt{2}},\pm\frac{\sqrt{2}}{5}\right)$.

53. $\nabla f\left(x_0,y_0,z_0\right)=\left\langle 2x_0,-2y_0,4z_0\right\rangle$ and the given line has direction numbers 2,4,6 , so $\left\langle 2x_0,-2y_0,4z_0\right\rangle=k\langle 2,4,6\rangle$ or $x_0=k$, $y_0=-2k$ and $z_0=\frac{3}{2}k$. But $x_0^2-y_0^2+2z_0^2=1$ or $\left(1-4+\frac{9}{2}\right)k^2=1$, so $k=\pm\sqrt{\frac{2}{3}}=\pm\frac{\sqrt{6}}{3}$ and there are two such points: $\left(\pm\frac{\sqrt{6}}{3},\mp\frac{2\sqrt{6}}{3},\pm\frac{\sqrt{6}}{2}\right)$.

54. First note that the point $(1,1,2)$ is on both surfaces. For the ellipsoid, an equation of the tangent plane at $(1,1,2)$ is $6x+4y+4z=18$ or $3x+2y+2z=9$, and for the sphere, an equation of the tangent plane at $(1,1,2)$ is $(2-8)x+(2-6)y+(4-8)z=-18$ or $-6x-4y-4z=-18$ or $3x+2y+2z=9$. Since these tangent planes are the same, the surfaces are tangent to each other at the point $(1,1,2)$.

55. Let $\left(x_0,y_0,z_0\right)$ be a point on the cone . Then an equation of the tangent plane to the cone at this point is $2x_0x+2y_0y-2z_0z=2\left(x_0^2+y_0^2-z_0^2\right)$. But $x_0^2+y_0^2=z_0^2$ so the tangent plane is given by $x_0x+y_0y-z_0z=0$, a plane which always contains the origin.

56. Let $\left(x_0,y_0,z_0\right)$ be a point on the sphere. Then the normal line is given by $\frac{x-x_0}{2x_0}=\frac{y-y_0}{2y_0}=\frac{z-z_0}{2z_0}$. For the center $(0,0,0)$ to be on the line, we need $-\frac{x_0}{2x_0}=-\frac{y_0}{2y_0}=-\frac{z_0}{2z_0}$ or equivalently $1=1=1$, which is true.

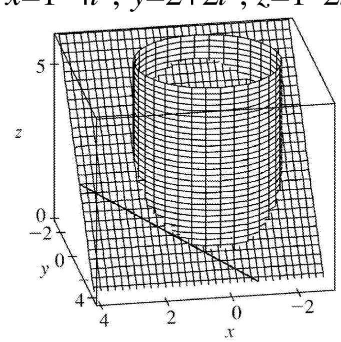
57. Let $\left(x_0,y_0,z_0\right)$ be a point on the surface. Then an equation of the tangent plane at the point is $\frac{x}{2\sqrt{x_0}}+\frac{y}{2\sqrt{y_0}}+\frac{z}{2\sqrt{z_0}}=\frac{\sqrt{x_0}+\sqrt{y_0}+\sqrt{z_0}}{2}$. But $\sqrt{x_0}+\sqrt{y_0}+\sqrt{z_0}=\sqrt{c}$, so the equation is $\frac{x}{\sqrt{x_0}}+\frac{y}{\sqrt{y_0}}+\frac{z}{\sqrt{z_0}}=\sqrt{c}$. The x –, y –, and z –intercepts are $\sqrt{cx_0}$, $\sqrt{cy_0}$ and $\sqrt{cz_0}$ respectively. (The x –intercept is found by setting $y=z=0$ and solving the resulting equation for x , and the y – and z –intercepts are found similarly.) So the sum of the intercepts is

$\sqrt{c}\left(\sqrt{x_0}+\sqrt{y_0}+\sqrt{z_0}\right)=c$, a constant.

58. Here the equation of the tangent plane to the point $\left(x_0,y_0,z_0\right)$ is $y_0z_0x+x_0z_0y+x_0y_0z=3x_0y_0z_0$ or $\frac{x}{3x_0}+\frac{y}{3y_0}+\frac{z}{3z_0}=1$. Then the x –, y – , and z –intercepts are $3x_0$, $3y_0$ and $3z_0$ respectively, and their product is $27x_0y_0z_0=27c^3$, a constant.

59. If $f(x,y,z)=z-x^2-y^2$ and $g(x,y,z)=4x^2+y^2+z^2$, then the tangent line is perpendicular to both ∇f and ∇g at $(-1,1,2)$. The vector $\boldsymbol{v}=\nabla f\times\nabla g$ will therefore be parallel to the tangent line. We have: $\nabla f(x,y,z)=\langle -2x,-2y,1\rangle\Rightarrow\nabla f(-1,1,2)=\langle 2,-2,1\rangle$, and $\nabla g(x,y,z)=\langle 8x,2y,2z\rangle\Rightarrow\nabla g(-1,1,2)=\langle -8,2,4\rangle$. Hence $\boldsymbol{v}=\nabla f\times\nabla g=\begin{vmatrix}\boldsymbol{i}&\boldsymbol{j}&\boldsymbol{k}\\2&-2&1\\-8&2&4\end{vmatrix}=-10\boldsymbol{i}-16\boldsymbol{j}-12\boldsymbol{k}$. Parametric equations are: $x=-1-10t$, $y=1-16t$, $z=2-12t$.

60. **(a)** Let $f(x,y,z)=y+z$ and $g(x,y,z)=x^2+y^2$. Then the required tangent line is perpendicular to both ∇f and ∇g at $(1,2,1)$ and the vector $\boldsymbol{v}=\nabla f\times\nabla g$ is parallel to the tangent line. We have $\nabla f(x,y,z)=\langle 0,1,1\rangle\Rightarrow\nabla f(1,2,1)=\langle 0,1,1\rangle$, and $\nabla g(x,y,z)=\langle 2x,2y,0\rangle\Rightarrow\nabla g(1,2,1)=\langle 2,4,0\rangle$. Hence $\boldsymbol{v}=\nabla f\times\nabla g=\begin{vmatrix}\boldsymbol{i}&\boldsymbol{j}&\boldsymbol{k}\\0&1&1\\2&4&0\end{vmatrix}=-4\boldsymbol{i}+2\boldsymbol{j}-2\boldsymbol{k}$. So parametric equations of the desired tangent line are $x=1-4t$, $y=2+2t$, $z=1-2t$.

(b) 

61. **(a)** The direction of the normal line of F is given by ∇F , and that of G by ∇G . Assuming that $\nabla F\neq 0\neq\nabla G$, the two normal lines are perpendicular at P if $\nabla F\cdot\nabla G=0$ at $P\Leftrightarrow$ $\langle\partial F/\partial x,\partial F/\partial y,\partial F/\partial z\rangle\cdot\langle\partial G/\partial x,\partial G/\partial y,\partial G/\partial z\rangle=0$ at $P\Leftrightarrow F_xG_x+F_yG_y+F_zG_z=0$ at P .

(b) Here $F=x^2+y^2-z^2$ and $G=x^2+y^2+z^2-r^2$, so $\nabla F\cdot\nabla G=\langle 2x,2y,-2z\rangle\cdot\langle 2x,2y,2z\rangle=4x^2+4y^2-4z^2=4F=0$

, since the point $\langle x,y,z\rangle$ lies on the graph of $F=0$. To see that this is true without using calculus, note that $G=0$ is the equation of a sphere centered at the origin and $F=0$ is the equation of a right circular cone with vertex at the origin (which is generated by lines through the origin). At any point of intersection, the sphere's normal line (which passes through the origin) lies on the cone, and thus is perpendicular to the cone's normal line. So the surfaces with equations $F=0$ and $G=0$ are everywhere orthogonal.

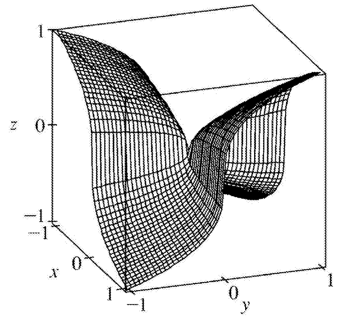
62. **(a)** The function $f(x,y)=(xy)^{1/3}$ is continuous on R^2 since it is a composition of a polynomial and the cube root function, both of which are continuous. (See the text just after Example 15.2.8 .)

$$f_x(0,0)=\lim_{h\to 0}\frac{f(0+h,0)-f(0,0)}{h}=\lim_{h\to 0}\frac{(h\cdot 0)^{1/3}-0}{h}=0\ ,$$

$$f_y(0,0)=\lim_{h\to 0}\frac{f(0,0+h)-f(0,0)}{h}=\lim_{h\to 0}\frac{(0\cdot h)^{1/3}-0}{h}=0$$. Therefore, $f_x(0,0)$ and $f_y(0,0)$ do exist and are equal to 0 . Now let $\boldsymbol{u}$ be any unit vector other than $\boldsymbol{i}$ and $\boldsymbol{j}$ (these correspond to f_x and f_y respectively.) Then $\boldsymbol{u}=a\boldsymbol{i}+b\boldsymbol{j}$ where $a\neq 0$ and $b\neq 0$. Thus

$$D_{\boldsymbol{u}}f(0,0)=\lim_{h\to 0}\frac{f(0+ha,0+hb)-f(0,0)}{h}=\lim_{h\to 0}\frac{\sqrt[3]{(ha)(hb)}}{h}=\lim_{h\to 0}\frac{\sqrt[3]{ab}}{h^{1/3}}$$ and this limit does not exist, so $D_{\boldsymbol{u}}f(0,0)$ does not exist.

(b)



Notice that if we start at the origin and proceed in the direction of the x – or y –axis, then the graph is flat. But if we proceed in any other direction, then the graph is extremely steep.

63. Let $\boldsymbol{u}=\langle a,b\rangle$ and $\boldsymbol{v}=\langle c,d\rangle$. Then we know that at the given point, $D_{\boldsymbol{u}}f=\nabla f\cdot \boldsymbol{u}=af_x+bf_y$ and $D_{\boldsymbol{v}}f=\nabla f\cdot \boldsymbol{v}=cf_x+df_y$. But these are just two linear equations in the two unknowns f_x and f_y , and since $\boldsymbol{u}$ and $\boldsymbol{v}$ are not parallel, we can solve the equations to find $\nabla f=\langle f_x,f_y\rangle$ at the given point. In fact, $\nabla f=\left\langle \frac{dD_{\boldsymbol{u}}f-bD_{\boldsymbol{v}}f}{ad-bc},\frac{aD_{\boldsymbol{v}}f-cD_{\boldsymbol{u}}f}{ad-bc}\right\rangle$.

64. Since $z=f(x,y)$ is differentiable at $\mathbf{x}_0=\left(x_0,y_0\right)$, by Definition 15.4.7 we have $\Delta z=f_x\left(x_0,y_0\right)\Delta x+f_y\left(x_0,y_0\right)\Delta y+\varepsilon_1\Delta x+\varepsilon_2\Delta y$ where $\varepsilon_1,\varepsilon_2\rightarrow 0$ as $(\Delta x,\Delta y)\rightarrow(0,0)$. Now $\Delta z=f(\mathbf{x})-f(\mathbf{x}_0)$, $\langle\Delta x,\Delta y\rangle=\mathbf{x}-\mathbf{x}_0$ so $(\Delta x,\Delta y)\rightarrow(0,0)$ is equivalent to $\mathbf{x}\rightarrow\mathbf{x}_0$ and $\left\langle f_x\left(x_0,y_0\right),f_y\left(x_0,y_0\right)\right\rangle=\nabla f(\mathbf{x}_0)$. Substituting into (15.4.7) gives $f(\mathbf{x})-f(\mathbf{x}_0)=\nabla f(\mathbf{x}_0)\cdot(\mathbf{x}-\mathbf{x}_0)+\left\langle\varepsilon_1,\varepsilon_2\right\rangle\cdot\langle\Delta x,\Delta y\rangle$ or $\left\langle\varepsilon_1,\varepsilon_2\right\rangle\cdot\left(\mathbf{x}-\mathbf{x}_0\right)=f(\mathbf{x})-f(\mathbf{x}_0)-\nabla f(\mathbf{x}_0)\cdot(\mathbf{x}-\mathbf{x}_0)$, and so $\dfrac{f(\mathbf{x})-f(\mathbf{x}_0)-\nabla f(\mathbf{x}_0)\cdot(\mathbf{x}-\mathbf{x}_0)}{\left|\mathbf{x}-\mathbf{x}_0\right|}=\dfrac{\left\langle\varepsilon_1,\varepsilon_2\right\rangle\cdot(\mathbf{x}-\mathbf{x}_0)}{\left|\mathbf{x}-\mathbf{x}_0\right|}$. But $\dfrac{\mathbf{x}-\mathbf{x}_0}{\left|\mathbf{x}-\mathbf{x}_0\right|}$ is a unit vector so $\lim\limits_{\mathbf{x}\rightarrow\mathbf{x}_0}\dfrac{\left\langle\varepsilon_1,\varepsilon_2\right\rangle\cdot(\mathbf{x}-\mathbf{x}_0)}{\left|\mathbf{x}-\mathbf{x}_0\right|}=0$ since $\varepsilon_1,\varepsilon_2\rightarrow 0$ as $\mathbf{x}\rightarrow\mathbf{x}_0$. Hence $\lim\limits_{\mathbf{x}\rightarrow\mathbf{x}_0}\dfrac{f(\mathbf{x})-f(\mathbf{x}_0)-\nabla f\left(\mathbf{x}_0\right)\cdot(\mathbf{x}-\mathbf{x}_0)}{\left|\mathbf{x}-\mathbf{x}_0\right|}=0$.

1. **(a)** First we compute $D(1,1)=f_{xx}(1,1)f_{yy}(1,1)-[f_{xy}(1,1)]^2=(4)(2)-(1)^2=7$. Since $D(1,1)>0$ and $f_{xx}(1,1)>0$, f has a local minimum at (1,1) by the Second Derivatives Test.

(b) $D(1,1)=f_{xx}(1,1)f_{yy}(1,1)-[f_{xy}(1,1)]^2=(4)(2)-(3)^2=-1$. Since $D(1,1)<0$, f has a saddle point at (1,1) by the Second Derivatives Test.

2. **(a)** $D=g_{xx}(0,2)g_{yy}(0,2)-[g_{xy}(0,2)]^2=(-1)(1)-(6)^2=-37$. Since $D<0$, g has a saddle point at (0,2) by the Second Derivatives Test.

(b) $D=g_{xx}(0,2)g_{yy}(0,2)-[g_{xy}(0,2)]^2=(-1)(-8)-(2)^2=4$. Since $D>0$ and $g_{xx}(0,2)<0$, g has a local maximum at (0,2) by the Second Derivatives Test.

(c) $D=g_{xx}(0,2)g_{yy}(0,2)-[g_{xy}(0,2)]^2=(4)(9)-(6)^2=0$. In this case the Second Derivatives Test gives no information about g at the point (0,2) .

3. In the figure, a point at approximately $(1,1)$ is enclosed by level curves which are oval in shape and indicate that as we move away from the point in any direction the values of f are increasing. Hence we would expect a local minimum at or near $(1,1)$. The level curves near $(0,0)$ resemble hyperbolas, and as we move away from the origin, the values of f increase in some directions and decrease in others, so we would expect to find a saddle point there.

To verify our predictions, we have $f(x,y)=4+x^3+y^3-3xy\Rightarrow f_x(x,y)=3x^2-3y$, $f_y(x,y)=3y^2-3x$. We have critical points where these partial derivatives are equal to 0 : $3x^2-3y=0$, $3y^2-3x=0$. Substituting $y=x^2$ from the first equation into the second equation gives $3\left(x^2\right)^2-3x=0\Rightarrow 3x(x^3-1)=0\Rightarrow x=0$ or $x=1$. Then we have two critical points, $(0,0)$ and $(1,1)$. The second partial derivatives are $f_{xx}(x,y)=6x$, $f_{xy}(x,y)=-3$, and $f_{yy}(x,y)=6y$, so

$D(x,y)=f_{xx}(x,y)f_{yy}(x,y)-[f_{xy}(x,y)]^2=(6x)(6y)-(-3)^2=36xy-9$. Then $D(0,0)=36(0)(0)-9=-9$, and $D(1,1)=36(1)(1)-9=27$. Since $D(0,0)<0$, f has a saddle point at (0,0) by the Second Derivatives Test. Since $D(1,1)>0$ and $f_{xx}(1,1)>0$, f has a local minimum at (1,1) .

4. In the figure, points at approximately (–1,1) and (–1,–1) are enclosed by oval–shaped level curves which indicate that as we move away from either point in any direction, the values of f are increasing. Hence we would expect local minima at or near $(-1,\pm 1)$. Similarly, the point (1,0) appears to be enclosed by oval–shaped level curves which indicate that as we move away from the point in any direction the values of f are decreasing, so we should have a local maximum there. We also show hyperbola–shaped level curves near the points $(-1,0)$, $(1,1)$, and $(1,-1)$. The values of f increase along some paths leaving these points and decrease in others, so we should have a saddle

point at each of these points.

To confirm our predictions, we have $f(x,y)=3x-x^3-2y^2+y^4 \Rightarrow f_x(x,y)=3-3x^2$, $f_y(x,y)=-4y+4y^3$.

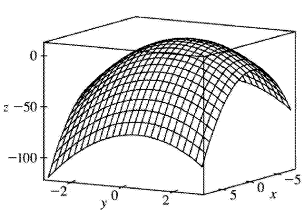
Setting these partial derivatives equal to 0 , we have $3-3x^2=0 \Rightarrow x=\pm 1$ and $-4y+4y^3=0 \Rightarrow y\left(y^2-1\right)=0 \Rightarrow y=0,\pm 1$. So our critical points are $(\pm 1,0)$, $(\pm 1,\pm 1)$. The second partial derivatives are $f_{xx}(x,y)=-6x$, $f_{xy}(x,y)=0$, and $f_{yy}(x,y)=12y^2-4$, so

$D(x,y)=f_{xx}(x,y)\,f_{yy}(x,y)-[f_{xy}(x,y)]^2=(-6x)\left(12y^2-4\right)-(0)^2=-72xy^2+24x$. We use the Second Derivatives Test to classify the 6 critical points:

Critical Point	D	f_{xx}	Conclusion
$(1,0)$	24	-6	$D>0, f_{xx}<0 \Rightarrow f$ has a local maximum at $(1,0)$
$(1,1)$	-48		$D<0 \Rightarrow f$ has a saddle point at $(1,1)$
$(1,-1)$	-48		$D<0 \Rightarrow f$ has a saddle point at $(1,-1)$
$(-1,0)$	-24		$D<0 \Rightarrow f$ has a saddle point at $(-1,0)$
$(-1,1)$	48	6	$D>0, f_{xx}>0 \Rightarrow f$ has a local minimum at $(-1,1)$
$(-1,-1)$	48	6	$D>0, f_{xx}>0 \Rightarrow f$ has a local minimum at $(-1,-1)$

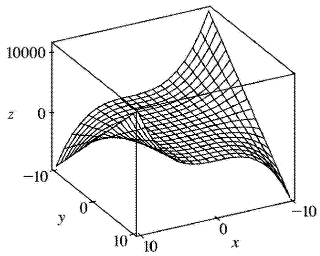
5. $f(x,y)=9-2x+4y-x^2-4y^2 \Rightarrow f_x=-2-2x$, $f_y=4-8y$, $f_{xx}=-2$, $f_{xy}=0$, $f_{yy}=-8$. Then $f_x=0$ and $f_y=0$ imply $x=-1$ and $y=\frac{1}{2}$, and the only critical point is $\left(-1,\frac{1}{2}\right)$.

$D(x,y)=f_{xx}f_{yy}-\left(f_{xy}\right)^2=(-2)\,(-8)-0^2=16$, and since $D\left(-1,\frac{1}{2}\right)=16>0$ and $f_{xx}\left(-1,\frac{1}{2}\right)=-2<0$, $f\left(-1,\frac{1}{2}\right)=11$ is a local maximum by the Second Derivatives Test.

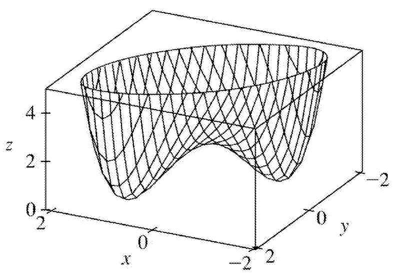


6. $f(x,y)=x^3y+12x^2-8y \Rightarrow f_x=3x^2y+24x$,

$f_y=x^3-8$, $f_{xx}=6xy+24$, $f_{xy}=3x^2$, $f_{yy}=0$. Then $f_y=0$ implies $x=2$, and substitution into $f_x=0$ gives $12y+48=0 \Rightarrow y=-4$. Thus, the only critical point is $(2,-4)$. $D(2,-4)=(-24)(0)-12^2=-144<0$, so $(2,-4)$ is a saddle point.

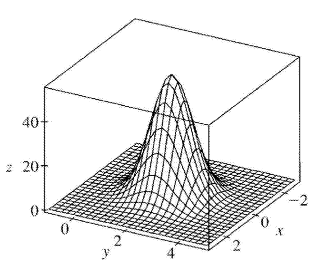


7. $f(x,y)=x^4+y^4-4xy+2 \Rightarrow f_x=4x^3-4y$, $f_y=4y^3-4x$, $f_{xx}=12x^2$, $f_{xy}=-4$, $f_{yy}=12y^2$. Then $f_x=0$ implies $y=x^3$, and substitution into $f_y=0 \Rightarrow x=y^3$ gives $x^9-x=0 \Rightarrow x(x^8-1)=0 \Rightarrow x=0$ or $x=\pm 1$. Thus the critical points are $(0,0)$, $(1,1)$, and $(-1,-1)$. Now $D(0,0)=0\cdot 0-(-4)^2=-16<0$, so $(0,0)$ is a saddle point. $D(1,1)=(12)(12)-(-4)^2>0$ and $f_{xx}(1,1)=12>0$, so

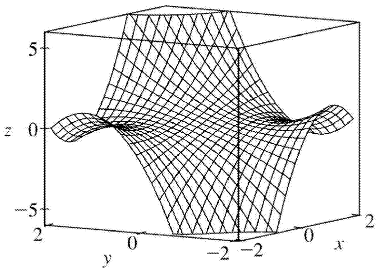


$f(1,1)=0$ is a local minimum. $D(-1,-1)=(12)(12)-(-4)^2>0$ and $f_{xx}=(-1,-1)=12>0$, so $f(-1,-1)=0$ is also a local minimum.

8. $f(x,y)=e^{4y-x^2-y^2} \Rightarrow f_x=-2xe^{4y-x^2-y^2}$, $f_y=(4-2y)e^{4y-x^2-y^2}$, $f_{xx}=(4x^2-2)e^{4y-x^2-y^2}$, $f_{xy}=-2x(4-2y)e^{4y-x^2-y^2}$, $f_{yy}=(4y^2-16y+14)e^{4y-x^2-y^2}$. Then $f_x=0$ and $f_y=0$ implies $x=0$ and $y=2$, so the only critical point is $(0,2)$. $D(0,2)=(-2e^4)(-2e^4)-0^2=4e^8>0$ and $f_{xx}(0,2)=-2e^4<0$, so $f(0,2)=e^4$ is a local maximum.

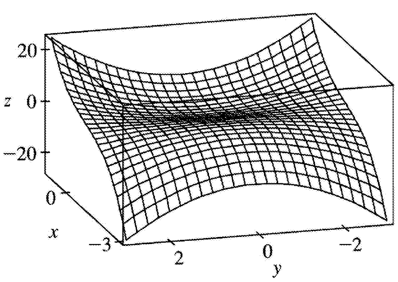


9. $f(x,y)=(1+xy)(x+y)=x+y+x^2y+xy^2 \Rightarrow f_x=1+2xy+y^2$, $f_y=1+x^2+2xy$, $f_{xx}=2y$, $f_{xy}=2x+2y$, $f_{yy}=2x$. Then $f_x=0$ implies $1+2xy+y^2=0$ and $f_y=0$ implies $1+x^2+2xy=0$. Subtracting the second equation from the first gives $y^2-x^2=0 \Rightarrow y=\pm x$, but if $y=x$ then $1+2xy+y^2=0 \Rightarrow 1+3x^2=0$ which has no real solution. If $y=-x$ then $1+2xy+y^2=0 \Rightarrow 1-x^2=0 \Rightarrow x=\pm 1$,

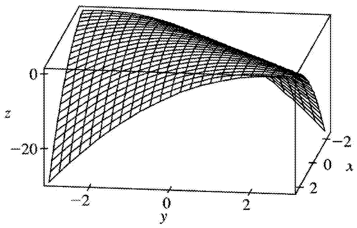


so critical points are $(1,-1)$ and $(-1,1)$. $D(1,-1)=(-2)(2)-0<0$ and $D(-1,1)=(2)(-2)-0<0$, so $(-1,1)$ and $(1,-1)$ are saddle points.

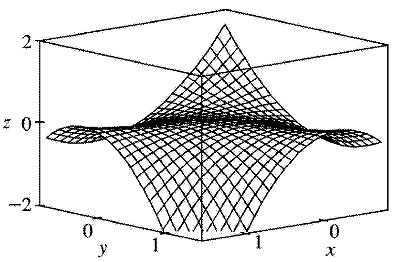
10. $f(x,y)=2x^3+xy^2+5x^2+y^2 \Rightarrow f_x=6x^2+y^2+10x$, $f_y=2xy+2y$, $f_{xx}=12x+10$, $f_{yy}=2x+2$, $f_{xy}=2y$. Then $f_y=0$ implies $y=0$ or $x=-1$. Substituting into $f_x=0$ gives the critical points $(0,0)$, $\left(-\frac{5}{3},0\right)$, $(-1,\pm 2)$. Now $D(0,0)=20>0$ and $f_{xx}(0,0)=10>0$, so $f(0,0)=0$ is a local minimum. Also $f_{xx}\left(-\frac{5}{3},0\right)<0$, $D\left(-\frac{5}{3},0\right)>0$, and $D(-1,\pm 2)<0$. Hence $f\left(-\frac{5}{3},0\right)=\frac{125}{27}$ is a local maximum while $(-1,\pm 2)$ are saddle points.



11. $f(x,y)=1+2xy-x^2-y^2 \Rightarrow f_x=2y-2x$, $f_y=2x-2y$, $f_{xx}=f_{yy}=-2$, $f_{xy}=2$. Then $f_x=0$ and $f_y=0$ implies $x=y$ so the critical points are all points of the form $\left(x_0,x_0\right)$. But $D\left(x_0,x_0\right)=4-4=0$ so the Second Derivatives Test gives no information. However $1+2xy-x^2-y^2=1-(x-y)^2$ and $1-(x-y)^2\leq 1$ for all (x,y) , with equality if and only if $x=y$. Thus $f\left(x_0,x_0\right)=1$ are local maxima.

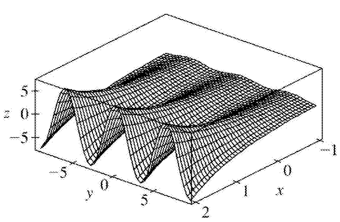


12. $f(x,y)=xy(1-x-y) \Rightarrow f_x=y-2xy-y^2$, $f_y=x-x^2-2xy$, $f_{xx}=-2y$, $f_{yy}=-2x$, $f_{xy}=1-2x-2y$. Then $f_x=0$ implies $y=0$ or $y=1-2x$. Substituting $y=0$ into $f_y=0$ gives $x=0$ or $x=1$ and substituting $y=1-2x$ into $f_y=0$ gives $3x^2-x=0$ so $x=0$ or $\frac{1}{3}$. Thus the critical points are $(0,0)$, $(1,0)$, $(0,1)$ and $\left(\frac{1}{3},\frac{1}{3}\right)$.



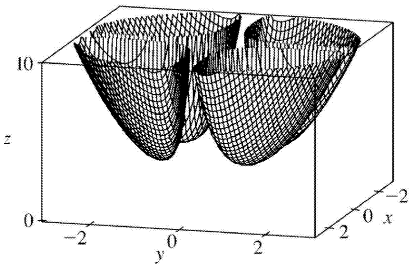
$D(0,0)=D(1,0)=D(0,1)=-1$ while $D\left(\frac{1}{3},\frac{1}{3}\right)=\frac{1}{3}$ and $f_{xx}\left(\frac{1}{3},\frac{1}{3}\right)=-\frac{2}{3}<0$. Thus $(0,0)$, $(1,0)$ and $(0,1)$ are saddle points, and $f\left(\frac{1}{3},\frac{1}{3}\right)=\frac{1}{27}$ is a local maximum.

13. $f(x,y)=e^x\cos y \Rightarrow f_x=e^x\cos y$, $f_y=-e^x\sin y$. Now $f_x=0$ implies $\cos y=0$ or $y=\frac{\pi}{2}+n\pi$ for n an integer. But $\sin\left(\frac{\pi}{2}+n\pi\right)\neq 0$, so there are no critical points.



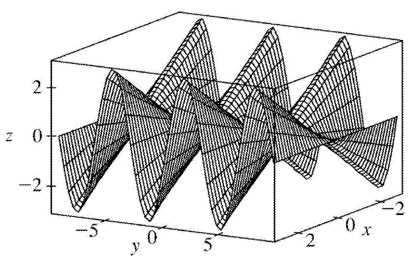
14. $f(x,y)=x^2+y^2+\frac{1}{x^2y^2} \Rightarrow f_x=2x-2x^{-3}y^{-2}$, $f_y=2y-2x^{-2}y^{-3}$, $f_{xx}=2+6x^{-4}y^{-2}$, $f_{yy}=2+6x^{-2}y^{-4}$, $f_{xy}=4x^{-3}y^{-3}$. Then $f_x=0$ implies $2x^4y^2-2=0$ or $x^4y^2=1$ or $y^2=x^{-4}$. Note that neither x nor y can be zero. Now $f_y=0$ implies $2x^2y^4-2=0$, and with $y^2=x^{-4}$ this implies $2x^{-6}-2=0$ or $x^6=1$. Thus $x=\pm 1$ and

if $x=1$, $y=\pm 1$;

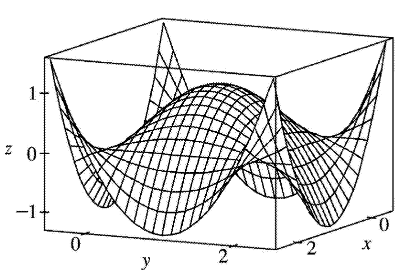


if $x=-1$, $y=\pm 1$. So the critical points are $(1,1)$, $(1,-1)$, $(-1,1)$ and $(-1,-1)$. Now $D(\pm 1,\pm 1)=D(\pm 1,\mp 1)=64-16>0$ and $f_{xx}>0$ always, so $f(\pm 1,\pm 1)=f(\pm 1,\mp 1)=3$ are local minima.

15. $f(x,y)=x\sin y \Rightarrow f_x=\sin y$, $f_y=x\cos y$, $f_{xx}=0$, $f_{yy}=-x\sin y$ and $f_{xy}=\cos y$. Then $f_x=0$ if and only if $y=n\pi$, n an integer, and substituting into $f_y=0$ requires $x=0$ for each of these y –values. Thus the critical points are $(0,n\pi)$, n an integer. But $D(0,n\pi)=-\cos^2(n\pi)<0$ so each critical point is a saddle point.



16. $f(x,y)=\left(2x-x^2\right)\left(2y-y^2\right) \Rightarrow f_x=(2-2x)\left(2y-y^2\right)$, $f_y=\left(2x-x^2\right)(2-2y)$, $f_{xx}=-2\left(2y-y^2\right)$, $f_{yy}=-2\left(2x-x^2\right)$ and $f_{xy}=(2-2x)(2-2y)$. Then $f_x=0$ implies $x=1$ or $y=0$ or $y=2$ and when $x=1$, $f_y=0$ implies $y=1$, when $y=0$, $f_y=0$ implies $x=0$ or $x=2$ and when $y=2$, $f_y=0$ implies $x=0$ or $x=2$. Thus the critical points are $(1,1)$, $(0,0)$, $(2,0)$,



$(0,2)$ and $(2,2)$. Now $D(0,0)=D(2,0)=D(0,2)=D(2,2)=-16$ so these critical points are saddle points, and $D(1,1)=4$ with $f_{xx}(1,1)=-2$, so $f(1,1)=1$ is a local maximum.

17. $f(x,y)=\left(x^2+y^2\right)e^{y^2-x^2} \Rightarrow f_x=\left(x^2+y^2\right)e^{y^2-x^2}(-2x)+2xe^{y^2-x^2}=2xe^{y^2-x^2}\left(1-x^2-y^2\right)$,

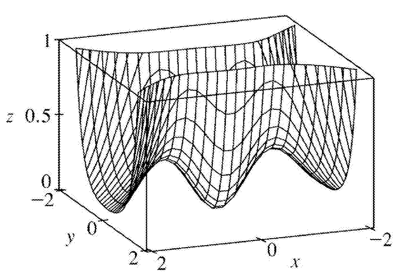
$f_y=\left(x^2+y^2\right)e^{y^2-x^2}(2y)+2ye^{y^2-x^2}=2ye^{y^2-x^2}\left(1+x^2+y^2\right)$,

$$f_{xx}=2xe^{y^2-x^2}(-2x)+\left(1-x^2-y^2\right)\left(2x\left(-2xe^{y^2-x^2}\right)+2e^{y^2-x^2}\right)$$
$$=2e^{y^2-x^2}\left(\left(1-x^2-y^2\right)\left(1-2x^2\right)-2x^2\right) ,$$

$f_{xy}=2xe^{y^2-x^2}(-2y)+2x(2y)e^{y^2-x^2}\left(1-x^2-y^2\right)=-4xye^{y^2-x^2}\left(x^2+y^2\right)$,

$$f_{yy}=2ye^{y^2-x^2}(2y)+\left(1+x^2+y^2\right)\left(2y\left(2ye^{y^2-x^2}\right)+2e^{y^2-x^2}\right)$$
$$=2e^{y^2-x^2}\left(\left(1+x^2+y^2\right)\left(1+2y^2\right)+2y^2\right) .$$

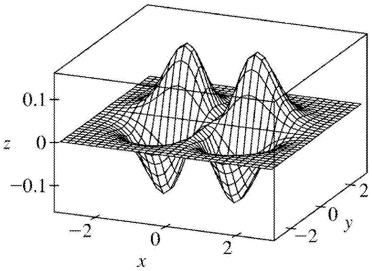
$f_y=0$ implies $y=0$, and substituting into $f_x=0$ gives $2xe^{-x^2}(1-x^2)=0 \Rightarrow x=0$ or $x=\pm 1$.
Thus the critical points are $(0,0)$ and $(\pm 1,0)$. $D(0,0)=(2)(2)-0>0$ and $f_{xx}(0,0)=2>0$, so $f(0,0)=0$ is a local minimum. $D(\pm 1,0)=\left(-4e^{-1}\right)\left(4e^{-1}\right)-0<0$ so $(\pm 1,0)$ are saddle points.



18. $f(x,y)=x^2ye^{-x^2-y^2} \Rightarrow f_x=x^2ye^{-x^2-y^2}(-2x)+2xye^{-x^2-y^2}=2xy\left(1-x^2\right)e^{-x^2-y^2}$,

$f_y=x^2ye^{-x^2-y^2}(-2y)+x^2e^{-x^2-y^2}=x^2\left(1-2y^2\right)e^{-x^2-y^2}$, $f_{xx}=2y\left(2x^4-5x^2+1\right)e^{-x^2-y^2}$,

$f_{xy}=2x\left(1-x^2\right)\left(1-2y^2\right)e^{-x^2-y^2}$, $f_{yy}=2x^2y\left(2y^2-3\right)e^{-x^2-y^2}$.

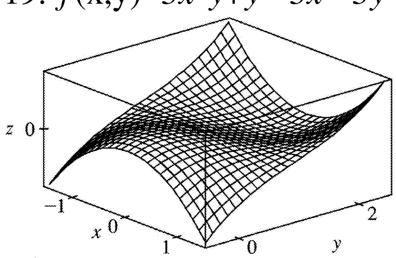


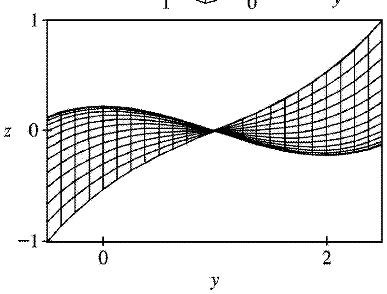
$f_x=0$ implies $x=0$, $y=0$, or $x=\pm 1$. If $x=0$ then $f_y=0$ for any y –value, so all points of the form $(0,y)$ are critical points. If $y=0$ then $f_y=0 \Rightarrow x^2 e^{-x^2}=0 \Rightarrow x=0$, so $(0,0)$ (already included above) is a critical point. If $x=\pm 1$ then $\left(1-2y^2\right)e^{-1-y^2}=0 \Rightarrow y=\pm\dfrac{1}{\sqrt{2}}$, so $\left(1,\pm\dfrac{1}{\sqrt{2}}\right)$ and $\left(-1,\pm\dfrac{1}{\sqrt{2}}\right)$ are critical points. $D(0,y)=0$, so the Second Derivatives Test gives no information. However, if $y>0$ then $x^2 ye^{-x^2-y^2}\geq 0$ with equality only when $x=0$, so we have local minimum values $f(0,y)=0$, $y>0$. Similarly, if $y<0$ then $x^2 ye^{-x^2-y^2}\leq 0$ with equality when $x=0$ so $f(0,y)=0$, $y<0$ are local maximum values, and $(0,0)$ is a saddle point.

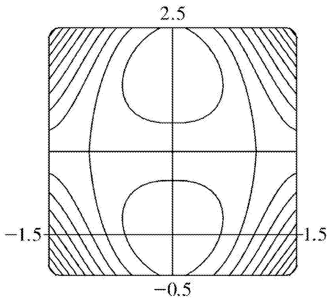
$D\left(\pm 1,\dfrac{1}{\sqrt{2}}\right)=8e^{-3}>0$, $f_{xx}\left(\pm 1,\dfrac{1}{\sqrt{2}}\right)=-2\sqrt{2}\,e^{-3/2}<0$ and

$D\left(\pm 1,-\dfrac{1}{\sqrt{2}}\right)=8e^{-3}>0$, $f_{xx}\left(\pm 1,-\dfrac{1}{\sqrt{2}}\right)=2\sqrt{2}\,e^{-3/2}>0$, so $f\left(\pm 1,\dfrac{1}{\sqrt{2}}\right)=\dfrac{1}{\sqrt{2}}\,e^{-3/2}$ are local maximum points while $f\left(\pm 1,-\dfrac{1}{\sqrt{2}}\right)=-\dfrac{1}{\sqrt{2}}\,e^{-3/2}$ are local minimum points.

19. $f(x,y)=3x^2 y+y^3-3x^2-3y^2+2$



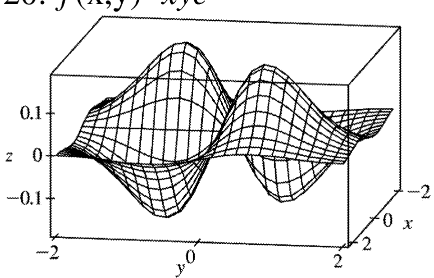


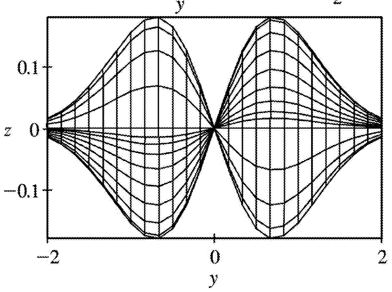


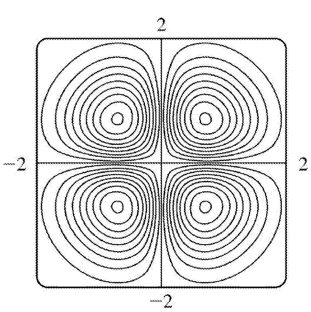
From the graphs, it appears that f has a local maximum $f(0,0)\approx 2$ and a local minimum $f(0,2)\approx -2$. There appear to be saddle points near $(\pm 1,1)$.

$f_x=6xy-6x$, $f_y=3x^2+3y^2-6y$. Then $f_x=0$ implies $x=0$ or $y=1$ and when $x=0$, $f_y=0$ implies $y=0$ or $y=2$; when $y=1$, $f_y=0$ implies $x^2=1$ or $x=\pm 1$. Thus the critical points are $(0,0)$, $(0,2)$, $(\pm 1,1)$. Now $f_{xx}=6y-6$, $f_{yy}=6y-6$ and $f_{xy}=6x$, so $D(0,0)=D(0,2)=36>0$ while $D(\pm 1,1)=-36<0$ and $f_{xx}(0,0)=-6$, $f_{xx}(0,2)=6$. Hence $(\pm 1,1)$ are saddle points while $f(0,0)=2$ is a local maximum and $f(0,2)=-2$ is a local minimum.

20. $f(x,y)=xye^{-x^2-y^2}$

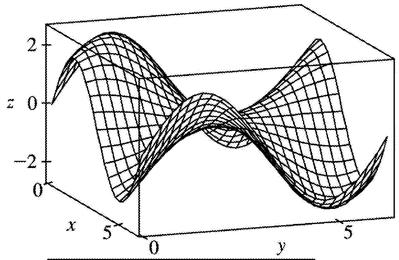


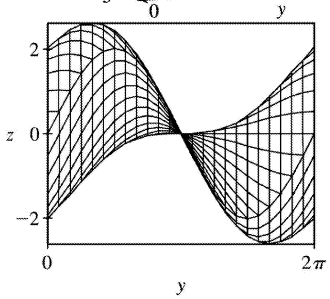


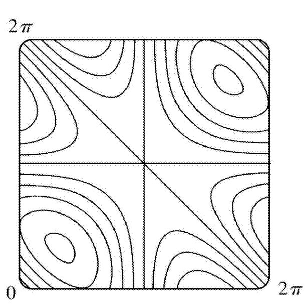


There appear to be local maxima of about $f(\pm 0.7,\pm 0.7)\approx 0.18$ and local minima of about $f(\pm 0.7,\mp 0.7)\approx -0.18$. Also, there seems to be a saddle point at the origin.

$f_x = ye^{-x^2-y^2}\left(1-2x^2\right)$, $f_y = xe^{-x^2-y^2}\left(1-2y^2\right)$, $f_{xx} = 2xye^{-x^2-y^2}\left(2x^2-3\right)$, $f_{yy} = 2xye^{-x^2-y^2}\left(2y^2-3\right)$,
$f_{xy} = \left(1-2x^2\right)e^{-x^2-y^2}\left(1-2y^2\right)$. Then $f_x = 0$ implies $y=0$ or $x = \pm\frac{1}{\sqrt{2}}$.
Substituting these values into $f_y = 0$ gives the critical points $(0,0)$, $\left(\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}\right)$,
$\left(-\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}\right)$. Then
$D(\mathrm{x,y}) = e^{2\left(-x^2-y^2\right)}\left[4x^2y^2\left(2x^2-3\right)\left(2y^2-3\right) - \left(1-2x^2\right)^2\left(1-2y^2\right)^2\right]$, so $D(0,0) = -1$, while
$D\left(\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}\right) > 0$ and $D\left(-\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}\right) > 0$. But $f_{xx}\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) < 0$, $f_{xx}\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) > 0$
, $f_{xx}\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) > 0$ and $f_{xx}\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) < 0$. Hence $(0,0)$ is a saddle point;
$f\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = f\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = -\frac{1}{2e}$ are local minima and
$f\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = f\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = \frac{1}{2e}$ are local maxima.
21. $f(\mathrm{x,y}) = \sin x + \sin y + \sin(x+y)$, $0 \le x \le 2\pi$, $0 \le y \le 2\pi$

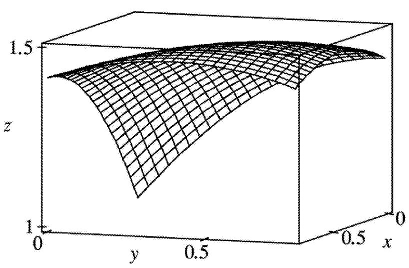


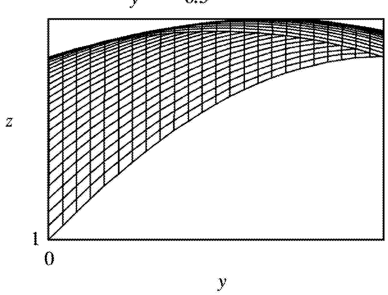


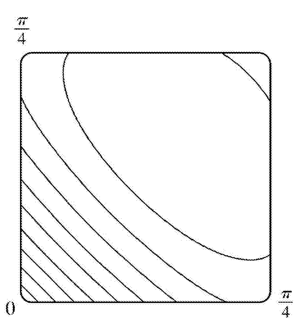


From the graphs it appears that f has a local maximum at about $(1,1)$ with value approximately 2.6 , a local minimum at about $(5,5)$ with value approximately -2.6 , and a saddle point at about $(3,3)$. $f_x=\cos x+\cos(x+y)$, $f_y=\cos y+\cos(x+y)$, $f_{xx}=-\sin x-\sin(x+y)$, $f_{yy}=-\sin y-\sin(x+y)$, $f_{xy}=-\sin(x+y)$. Setting $f_x=0$ and $f_y=0$ and subtracting gives $\cos x-\cos y=0$ or $\cos x=\cos y$. Thus $x=y$ or $x=2\pi-y$. If $x=y$, $f_x=0$ becomes $\cos x+\cos 2x=0$ or $2\cos^2 x+\cos x-1=0$, a quadratic in $\cos x$. Thus $\cos x=-1$ or $\frac{1}{2}$ and $x=\pi$, $\frac{\pi}{3}$, or $\frac{5\pi}{3}$, yielding the critical points (π,π) , $\left(\frac{\pi}{3},\frac{\pi}{3}\right)$ and $\left(\frac{5\pi}{3},\frac{5\pi}{3}\right)$. Similarly if $x=2\pi-y$, $f_x=0$ becomes $(\cos x)+1=0$ and the resulting critical point is (π,π) . Now $D(x,y)=\sin x\sin y+\sin x\sin(x+y)+\sin y\sin(x+y)$. So $D(\pi,\pi)=0$ and the Second Derivatives Test doesn't apply. $D\left(\frac{\pi}{3},\frac{\pi}{3}\right)=\frac{9}{4}>0$ and $f_{xx}\left(\frac{\pi}{3},\frac{\pi}{3}\right)<0$ so $f\left(\frac{\pi}{3},\frac{\pi}{3}\right)=\frac{3\sqrt{3}}{2}$ is a local maximum while $D\left(\frac{5\pi}{3},\frac{5\pi}{3}\right)=\frac{9}{4}>0$ and $f_{xx}\left(\frac{5\pi}{3},\frac{5\pi}{3}\right)>0$, so $f\left(\frac{5\pi}{3},\frac{5\pi}{3}\right)=-\frac{3\sqrt{3}}{2}$ is a local minimum.

22. $f(x,y)=\sin x+\sin y+\cos(x+y)$, $0\leq x\leq\frac{\pi}{4}$, $0\leq y\leq\frac{\pi}{4}$



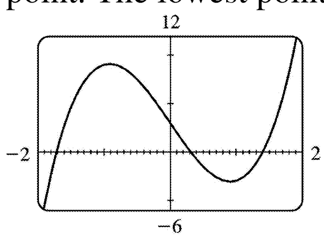


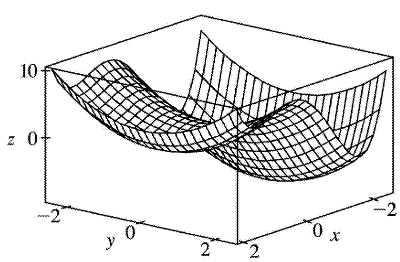


From the graphs, it seems that f has a local maximum at about $(0.5,0.5)$.

$f_x = \cos x - \sin(x+y)$, $f_y = \cos y - \sin(x+y)$, $f_{xx} = -\sin x - \cos(x+y)$, $f_{yy} = -\sin y - \cos(x+y)$, $f_{xy} = -\cos(x+y)$. Setting $f_x = 0$ and $f_y = 0$ and subtracting gives $\cos x = \cos y$. Thus $x = y$. Substituting $x = y$ into $f_x = 0$ gives $\cos x - \sin 2x = 0$ or $\cos x(1 - 2\sin x) = 0$. But $\cos x \neq 0$ for $0 \leq x \leq \frac{\pi}{4}$ and $1 - 2\sin x = 0$ implies $x = \frac{\pi}{6}$, so the only critical point is $\left(\frac{\pi}{6}, \frac{\pi}{6}\right)$. Here $f_{xx}\left(\frac{\pi}{6}, \frac{\pi}{6}\right) = -1 < 0$ and $D\left(\frac{\pi}{6}, \frac{\pi}{6}\right) = (-1)^2 - \frac{1}{4} > 0$. Thus $f\left(\frac{\pi}{6}, \frac{\pi}{6}\right) = \frac{3}{2}$ is a local maximum.

23. $f(x,y) = x^4 - 5x^2 + y^2 + 3x + 2 \Rightarrow f_x(x,y) = 4x^3 - 10x + 3$ and $f_y(x,y) = 2y$. $f_y = 0 \Rightarrow y = 0$, and the graph of f_x shows that the roots of $f_x = 0$ are approximately $x = -1.714$, 0.312 and 1.402 . (Alternatively, we could have used a calculator or a CAS to find these roots.) So to three decimal places, the critical points are $(-1.714, 0)$, $(1.402, 0)$, and $(0.312, 0)$. Now since $f_{xx} = 12x^2 - 10$, $f_{xy} = 0$, $f_{yy} = 2$, and $D = 24x^2 - 20$, we have $D(-1.714,0) > 0$, $f_{xx}(-1.714,0) > 0$, $D(1.402,0) > 0$, $f_{xx}(1.402,0) > 0$, and $D(0.312,0) < 0$. Therefore $f(-1.714,0) \approx -9.200$ and $f(1.402,0) \approx 0.242$ are local minima, and (0.312,0) is a saddle point. The lowest point on the graph is approximately $(-1.714, 0, -9.200)$.



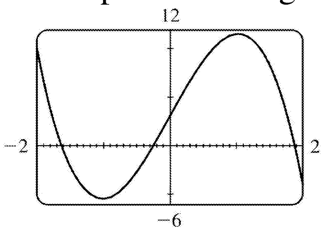


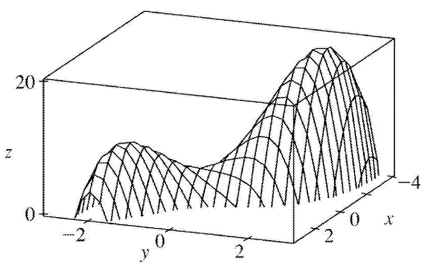
24. $f(x,y) = 5 - 10xy - 4x^2 + 3y - y^4 \Rightarrow f_x(x,y) = -10y - 8x$, $f_y(x,y) = -10x + 3 - 4y^3$.

Now $f_x = 0 \Rightarrow x = -\frac{5}{4}y$, so using a graph, we find solutions to $0 = f_y\left(-\frac{5}{4}y, y\right) = -10\left(-\frac{5}{4}y\right) + 3 - 4y^3 = -4y^3 + \frac{25}{2}y + 3$. (Alternatively, we could have found the roots of $f_x = f_y = 0$ directly, using a calculator or a CAS.) To three decimal places, the solutions are $y \approx 1.877$, -0.245

and -1.633 , so f has critical points at approximately $(-2.347,1.877)$, $(0.306,-0.245)$, and $(2.041,-1.633)$.

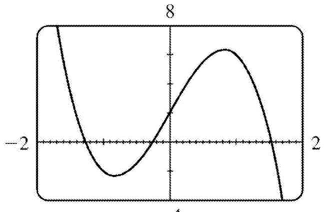
Now since $f_{xx}=-8$, $f_{xy}=-10$, $f_{yy}=-12y^2$, and $D=96y^2-100$, we have $D(-2.347,1.877)>0$, $D(0.306,-0.245)<0$, and $D(2.041,-1.633)>0$. Therefore, since $f_{xx}<0$ everywhere, $f(-2.347,1.877)\approx 20.238$ and $f(2.041,-1.633)\approx 9.657$ are local maxima, and $(0.306,-0.245)$ is a saddle point. The highest point on the graph is approximately $(-2.347,1.877,20.238)$.

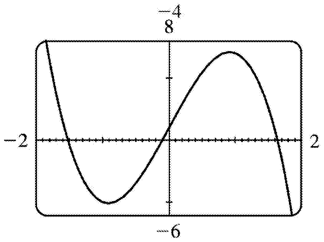


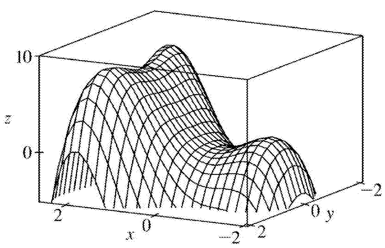


25. $f(x,y)=2x+4x^2-y^2+2xy^2-x^4-y^4 \Rightarrow f_x(x,y)=2+8x+2y^2-4x^3$, $f_y(x,y)=-2y+4xy-4y^3$. Now $f_y=0\Leftrightarrow$ $2y\left(2y^2-2x+1\right)=0\Leftrightarrow y=0$ or $y^2=x-\frac{1}{2}$.

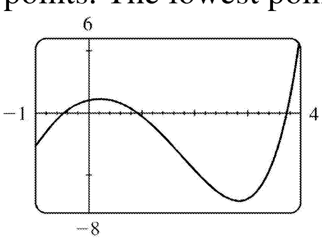
The first of these implies that $f_x=-4x^3+8x+2$, and the second implies that $f_x=2+8x+2\left(x-\frac{1}{2}\right)-4x^3=-4x^3+10x+1$. From the graphs, we see that the first possibility for f_x has roots at approximately -1.267 , -0.259 , and 1.526 , and the second has a root at approximately 1.629 (the negative roots do not give critical points, since $y^2=x-\frac{1}{2}$ must be positive). So to three decimal places, f has critical points at $(-1.267,0)$, $(-0.259,0)$, $(1.526,0)$, and $(1.629,\pm 1.063)$. Now since $f_{xx}=8-12x^2$, $f_{xy}=4y$, $f_{yy}=4x-12y^2$, and $D=(8-12x^2)(4x-12y^2)-16y^2$, we have $D(-1.267,0)>0$, $f_{xx}(-1.267,0)>0$, $D(-0.259,0)<0$, $D(1.526,0)<0$, $D(1.629,\pm 1.063)>0$, and $f_{xx}(1.629,\pm 1.063)<0$. Therefore, to three decimal places, $f(-1.267,0)\approx 1.310$ and $f(1.629,\pm 1.063)\approx 8.105$ are local maxima, and $(-0.259,0)$ and $(1.526,0)$ are saddle points. The highest points on the graph are approximately $(1.629,\pm 1.063,8.105)$.

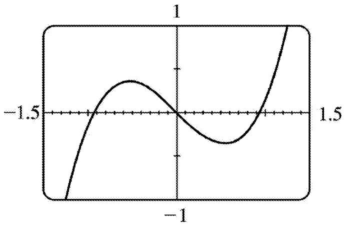


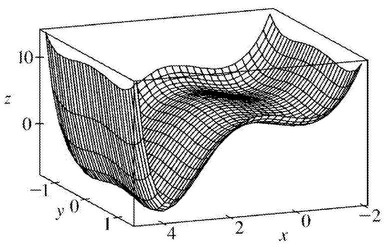




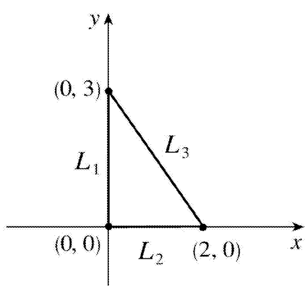
26. $f(x,y)=e^x+y^4-x^3+4\cos y \Rightarrow f_x(x,y)=e^x-3x^2$ and $f_y(x,y)=4y^3-4\sin y$. From the graphs, we see that to three decimal places, $f_x=0$ when $x\approx-0.459$, 0.910 , or 3.733 , and $f_y=0$ when $y\approx 0$ or ± 0.929 . (Alternatively, we could have used a calculator or a CAS to find the roots of $f_x=0$ and $f_y=0$.) So, to three decimal places, f has critical points at $(-0.459,0)$, $(-0.459,\pm 0.929)$, $(0.910,0)$, $(0.910,\pm 0.929)$, $(3.733,0)$, and $(3.733,\pm 0.929)$. Now $f_{xx}=e^x-6x$, $f_{xy}=0$, $f_{yy}=12y^2-4\cos y$, and $D=\left(e^x-6x\right)\left(12y^2-4\cos y\right)$. Therefore $D(-0.459,0)<0$, $D(-0.459,\pm 0.929)>0$, $f_{xx}(-0.459,\pm 0.929)>0$, $D(0.910,0)>0$, $f_{xx}(0.910,0)<0$, $D(0.910,\pm 0.929)<0$, $D(3.733,0)<0$, $D(3.733,\pm 0.929)>0$, and $f_{xx}(3.733,\pm 0.929)>0$. So $f(-0.459,\pm 0.929)\approx 3.868$ and $f(3.733,\pm 0.929)\approx-7.077$ are local minima, $f(0.910,0)\approx 5.731$ is a local maximum, and $(-0.459,0)$, $(0.910,\pm 0.929)$, and $(3.733,0)$ are saddle points. The lowest points on the graph are approximately $(3.733,\pm 0.929,-7.077)$.



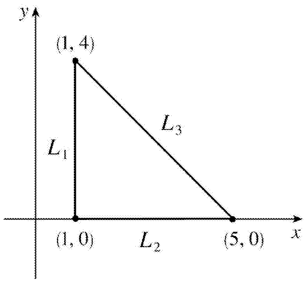




27. Since f is a polynomial it is continuous on D, so an absolute maximum and minimum exist. Here $f_x=4$, $f_y=-5$ so there are no critical points inside D. Thus the absolute extrema must both occur on the boundary. Along L_1, $x=0$ and $f(0,y)=1-5y$ for $0\leq y\leq 3$, a decreasing function in y, so the maximum value is $f(0,0)=1$ and the minimum value is $f(0,3)=-14$. Along L_2, $y=0$ and $f(x,0)=1+4x$ for $0\leq x\leq 2$, an increasing
function in x, so the minimum value is $f(0,0)=1$ and the maximum value is $f(2,0)=9$. Along L_3, $y=-\frac{3}{2}x+3$ and $f\left(x,-\frac{3}{2}x+3\right)=\frac{23}{2}x-14$ for $0\leq x\leq 2$, an increasing function in x, so the minimum value is $f(0,3)=-14$ and the maximum value is $f(2,0)=9$. Thus the absolue maximum of f on D is $f(2,0)=9$ and the absolute minimum is $f(0,3)=-14$.



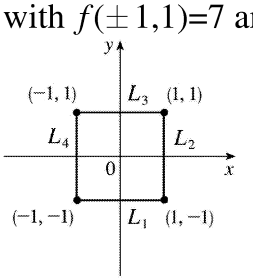
28. Since f is a polynomial it is continuous on D, so an absolute maximum and minimum exist. $f_x=y-1$, $f_y=x-2$, and setting $f_x=f_y=0$ gives $(2,1)$ as the only critical point, where $f(2,1)=1$. Along L_1 : $x=1$ and $f(1,y)=2-y$ for $0\leq y\leq 4$, a decreasing function in y, so the maximum value is $f(1,0)=2$ and the minimum value is $f(1,4)=-2$. Along L_2 : $y=0$ and $f(x,0)=3-x$ for $1\leq x\leq 5$, a decreasing function in x, so the maximum value is $f(1,0)=2$ and the minimum value is



$f(5,0)=-2$. Along L_3 , $y=5-x$ and $f(x,5-x)=-x^2+6x-7=-(x-3)^2+2$ for $1\leq x\leq 5$, which has a maximum at $x=3$ where $f(3,2)=2$ and a minimum at both $x=1$ and $x=5$, where $f(1,4)=f(5,0)=-2$. Thus the absolute maximum of f on D is $f(1,0)=f(3,2)=2$ and the absolute minimum is $f(1,4)=f(5,0)=-2$.

29. $f_x(x,y)=2x+2xy$, $f_y(x,y)=2y+x^2$, and setting $f_x=f_y=0$ gives $(0,0)$ as the only critical point in D , with $f(0,0)=4$.
On L_1 : $y=-1$, $f(x,-1)=5$, a constant.
On L_2 : $x=1$, $f(1,y)=y^2+y+5$, a quadratic in y which attains its maximum at $(1,1)$, $f(1,1)=7$ and its minimum at $\left(1,-\frac{1}{2}\right)$, $f\left(1,-\frac{1}{2}\right)=\frac{19}{4}$.
On L_3 : $f(x,1)=2x^2+5$ which attains its maximum at $(-1,1)$ and $(1,1)$ with $f(\pm 1,1)=7$ and its minimum at $(0,1)$, $f(0,1)=5$.
On L_4 : $f(-1,y)=y^2+y+5$ with maximum at $(-1,1)$, $f(-1,1)=7$ and

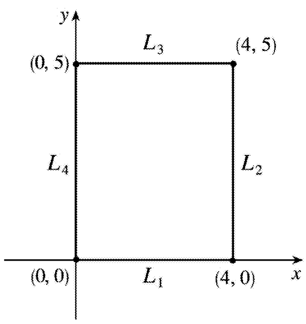
minimum at $\left(-1,-\frac{1}{2}\right)$, $f\left(-1,-\frac{1}{2}\right)=\frac{19}{4}$. Thus the absolute maximum is attained at both $(\pm 1,1)$ with $f(\pm 1,1)=7$ and the absolute minimum on D is attained at $(0,0)$ with $f(0,0)=4$.



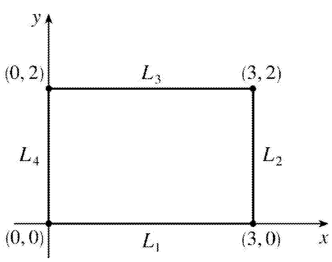
30. $f_x(x,y)=4-2x$ and $f_y(x,y)=6-2y$, so the only critical point is $(2,3)$ (which is in D) where $f(2,3)=13$. Along L_1 : $y=0$, so $f(x,0)=4x-x^2=-(x-2)^2+4$, $0\leq x\leq 4$, which has a maximum value when $x=2$ where $f(2,0)=4$ and a minimum value both when $x=0$ and $x=4$, where $f(0,0)=f(4,0)=0$.

Along L_2 : $x=4$, so $f(4,y)=6y-y^2=-(y-3)^2+9$, $0\le y\le 5$, which has a maximum value when $y=3$ where $f(4,3)=9$ and a minimum value when $y=0$ where $f(4,0)=0$. Along L_3 : $y=5$, so $f(x,5)=-x^2+4x+5=-(x-2)^2+9$, $0\le x\le 4$, which has a maximum value when $x=2$ where $f(2,5)=9$ and a minimum value both when $x=0$ and $x=4$, where $f(0,5)=f(4,5)=5$.

Along L_4 : $x=0$, so $f(0,y)=6y-y^2=-(y-3)^2+9$, $0\le y\le 5$, which has a maximum value when $y=3$ where $f(0,3)=9$ and a minimum value when $y=0$ where $f(0,0)=0$. Thus the absolute maximum is $f(2,3)=13$ and the absolute minimum is attained at both $(0,0)$ and $(4,0)$, where $f(0,0)=f(4,0)=0$.

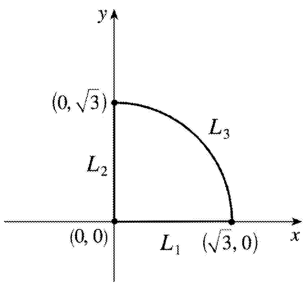


31. $f(x,y)=x^4+y^4-4xy+2$ is a polynomial and hence continuous on D , so it has an absolute maximum and minimum on D . In Exercise 7, we found the critical points of f ; only $(1,1)$ with $f(1,1)=0$ is inside D . On L_1 : $y=0$, $f(x,0)=x^4+2$, $0\le x\le 3$, a polynomial in x which attains its maximum at $x=3$, $f(3,0)=83$, and its minimum at $x=0$, $f(0,0)=2$. On L_2 : $x=3$, $f(3,y)=y^4-12y+83$, $0\le y\le 2$, a polynomial in y



which attains its minimum at $y=\sqrt[3]{3}$, $f\left(3,\sqrt[3]{3}\right)=83-9\sqrt[3]{3}\approx 70.0$, and its maximum at $y=0$, $f(3,0)=83$. On L_3 : $y=2$, $f(x,2)=x^4-8x+18$, $0\le x\le 3$, a polynomial in x which attains its minimum at $x=\sqrt[3]{2}$, $f\left(\sqrt[3]{2},2\right)=18-6\sqrt[3]{2}\approx 10.4$, and its maximum at $x=3, f(3,2)=75$. On L_4 : $x=0$, $f(0,y)=y^4+2$, $0\le y\le 2$, a polynomial in y which attains its maximum at $y=2$, $f(0,2)=18$, and its minimum at $y=0$, $f(0,0)=2$. Thus the absolute maximum of f on D is $f(3,0)=83$ and the absolute minimum is $f(1,1)=0$.

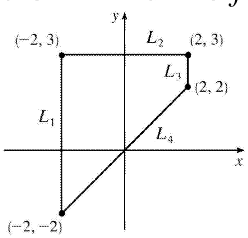
32. $f_x=y^2$ and $f_y=2xy$, and since $f_x=0 \Leftrightarrow y=0$, there are no critical points in the interior of D . Along L_1 , $y=0$ and $f(\text{x},0)=0$. Along L_2,$x=0$ and $f(0,\text{y})=0$. Along L_3 , $y=\sqrt{3-x^2}$, so let $g(x)=$ $f\left(\text{x},\sqrt{3-x^2}\right)=3x-x^3$ for $0\leq x\leq\sqrt{3}$. Then $g'(x)=3-3x^2=0\Leftrightarrow x=1$. The maximum value is $f\left(1,\sqrt{2}\right)=2$ and the minimum occurs both at $x=0$ and $x=\sqrt{3}$ where $f\left(0,\sqrt{3}\right)=f\left(\sqrt{3},0\right)=0$. Thus the absolute maximum of f
on D is $f\left(1,\sqrt{2}\right)=2$, and the absolute minimum is 0 which occurs at all points along L_1 and L_2 .



33. $f_x(\text{x,y})=6x^2$ and $f_y(\text{x,y})=4y^3$. And so $f_x=0$ and $f_y=0$ only occur when $x=y=0$. Hence, the only critical point inside the disk is at $x=y=0$ where $f(0,0)=0$. Now on the circle $x^2+y^2=1$, $y^2=1-x^2$ so let $g(x)=f(\text{x,y})=2x^3+\left(1-x^2\right)^2=x^4+2x^3-2x^2+1$, $-1\leq x\leq 1$. Then $g'(x)=4x^3+6x^2-4x=0\Rightarrow x=0$, -2 , or $\frac{1}{2}$.
$f(0,\pm 1)=g(0)=1$, $f\left(\frac{1}{2},\pm\frac{\sqrt{3}}{2}\right)=g\left(\frac{1}{2}\right)=\frac{13}{16}$, and $(-2,-3)$ is not in D . Checking the endpoints, we get $f(-1,0)=g(-1)=-2$ and $f(1,0)=g(1)=2$. Thus the absolute maximum and minimum of f on D are $f(1,0)=2$ and $f(-1,0)=-2$.
Another method: On the boundary $x^2+y^2=1$ we can write $x=\cos\theta$, $y=\sin\theta$, so $f(\cos\theta,\sin\theta)=2\cos^3\theta+\sin^4\theta$, $0\leq\theta\leq 2\pi$.

34. $f_x(\text{x,y})=3x^2-3$ and $f_y(\text{x,y})=-3y^2+12$ and the critical points are $(1,2)$, $(1,-2)$, $(-1,2)$, and $(-1,-2)$. But only $(1,2)$ and $(-1,2)$ are in D and $f(1,2)=14$, $f(-1,2)=18$. Along L_1 : $x=-2$ and $f(-2,\text{y})=-2-y^3+12y$, $-2\leq y\leq 3$, which has a maximum at $y=2$ where $f(-2,2)=14$ and a minimum at $y=-2$ where $f(-2,-2)=-18$. Along L_2 : $x=2$ and $f(2,\text{y})=2-y^3+12y$, $2\leq y\leq 3$, which has a maximum at
$y=2$ where $f(2,2)=18$ and a minimum at $y=3$ where $f(2,3)=11$. Along L_3 : $y=3$ and $f(\text{x},3)=x^3-3x+9$, $-2\leq x\leq 2$, which has a maximum at $x=-1$ and $x=2$ where $f(-1,3)=f(2,3)=11$ and a minimum at $x=1$

and $x=-2$ where $f(1,3)=f(-2,3)=7$.
Along L_4 : $y=x$ and $f(x,x)=9x$, $-2\le x\le 2$, which has a maximum at $x=2$ where $f(2,2)=18$ and a minimum at $x=-2$ where $f(-2,-2)=-18$. So the absolute maximum value of f on D is $f(2,2)=18$ and the minimum is $f(-2,-2)=-18$.



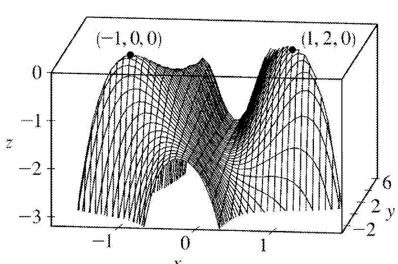
35. $f(x,y)=-\left(x^2-1\right)^2-\left(x^2y-x-1\right)^2 \Rightarrow f_x(x,y)=-2\left(x^2-1\right)(2x)-2\left(x^2y-x-1\right)(2xy-1)$ and $f_y(x,y)=-2\left(x^2y-x-1\right)x^2$. Setting $f_y(x,y)=0$ gives either $x=0$ or $x^2y-x-1=0$. There are no critical points for $x=0$, since $f_x(0,y)=-2$, so we set $x^2y-x-1=0 \Leftrightarrow y=\dfrac{x+1}{x^2}$ ($x\neq 0$), so

$$f_x\left(x,\frac{x+1}{x^2}\right)=-2\left(x^2-1\right)(2x)-2\left(x^2\,\frac{x+1}{x^2}-x-1\right)\left(2x\,\frac{x+1}{x^2}-1\right)=-4x\left(x^2-1\right)$$. Therefore

$f_x(x,y)=f_y(x,y)=0$ at the points $(1,2)$ and $(-1,0)$. To classify these critical points, we calculate $f_{xx}(x,y)=-12x^2-12x^2y^2+12xy+4y+2$, $f_{yy}(x,y)=-2x^4$, and $f_{xy}(x,y)=-8x^3y+6x^2+4x$.
In order to use the Second Derivatives Test we calculate

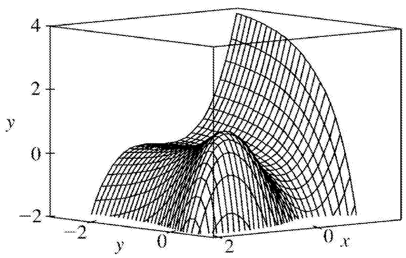
$$\begin{aligned} D(-1,0) &= f_{xx}(-1,0)\,f_{yy}(-1,0)-[f_{xy}(-1,0)]^2 \\ &= 16>0 , \end{aligned}$$

$f_{xx}(-1,0)=-10<0$, $D(1,2)=16>0$, and $f_{xx}(1,2)=-26<0$, so both $(-1,0)$ and $(1,2)$ give local maxima.



36. $f(x,y)=3xe^y-x^3-e^{3y}$ is differentiable everywhere, so the requirement for critical points is that (1) $f_x=3e^y-3x^2=0$ and (2) $f_y=3xe^y-3e^{3y}=0$. From (1) we obtain $e^y=x^2$, and then (2) gives $3x^3-3x^6=0\Rightarrow$

$x=1$ or 0 , but only $x=1$ is valid, since $x=0$ makes (1) impossible. So substituting $x=1$ into (1) gives $y=0$, and the only critical point is $(1,0)$.



The Second Derivatives Test shows that this gives a local maximum, since $D(1,0)=\left[-6x\left(3xe^{y}-9e^{3y}\right)-\left(3e^{y}\right)^{2}\right]_{(1,0)}=27>0$ and $f_{xx}(1,0)=[-6x]_{(1,0)}=-6<0$. But $f(1,0)=1$ is not an absolute maximum because, for instance, $f(-3,0)=17$. This can also be seen from the graph.

37. Let d be the distance from $(2,1,-1)$ to any point (x,y,z) on the plane $x+y-z=1$, so $d=\sqrt{(x-2)^2+(y-1)^2+(z+1)^2}$ where $z=x+y-1$, and we minimize $d^2=f(x,y)=(x-2)^2+(y-1)^2+(x+y)^2$. Then $f_x(x,y)=2(x-2)+2(x+y)=4x+2y-4$, $f_y(x,y)=2(y-1)+2(x+y)=2x+4y-2$. Solving $4x+2y-4=0$ and $2x+4y-2=0$ simultaneously gives $x=1$, $y=0$. An absolute minimum exists (since there is a minimum distance from the point to the plane) and it must occur at a critical point, so the shortest distance occurs for $x=1$, $y=0$ for which $d=\sqrt{(1-2)^2+(0-1)^2+(1+0)^2}=\sqrt{3}$.

38. Here the distance d from a point on the plane to the point $(1,2,3)$ is $d=\sqrt{(x-1)^2+(y-2)^2+(z-3)^2}$, where $z=4-x+y$. We can minimize $d^2=f(x,y)=(x-1)^2+(y-2)^2+(1-x+y)^2$, so $f_x(x,y)=2(x-1)+2(1-x+y)(-1)=4x-2y-4$ and $f_y(x,y)=2(y-2)+2(1-x+y)=4y-2x-2$. Solving $4x-2y-4=0$ and $4y-2x-2=0$ simultaneously gives $x=\frac{5}{3}$ and $y=\frac{4}{3}$, so the only critical point is $\left(\frac{5}{3},\frac{4}{3}\right)$. This point must correspond to the minimum distance, so the point on the plane closest to $(1,2,3)$ is $\left(\frac{5}{3},\frac{4}{3},\frac{11}{3}\right)$.

39. Minimize $d^2=x^2+y^2+z^2=x^2+y^2+xy+1$. Then $f_x=2x+y$, $f_y=2y+x$ so the critical point is $(0,0)$ and $D(0,0)=4-1>0$ with $f_{xx}(0,0)=2$ so this is a minimum. Thus $z^2=1$ or $z=\pm 1$ and the points on the surface are $(0,0,\pm 1)$.

40. Since $z=1/\left(x^2y^2\right)$ on the surface, we minimize $d^2=x^2+y^2+z^2=x^2+y^2+x^{-4}y^{-4}=f(\mathrm{x,y})$. $f_x=2x-\dfrac{4}{x^5y^4}$, $f_y=2y-\dfrac{4}{x^4y^5}$, so the critical points occur when $2x=\dfrac{4}{x^5y^4}$ and $2y=\dfrac{4}{x^4y^5}$ or $x^6y^4=2=x^4y^6$, so $x^2=y^2\Rightarrow$ $x=\pm y$ and $x^{10}=2\Rightarrow x=\pm 2^{1/10}$, $y=\pm 2^{1/10}$. The four critical points are $(\pm 2^{1/10},\pm 2^{1/10})$. The absolute minimum must occur at these points (there is no maximum since the surface is infinite in extent). Thus the points on the surface closest to the origin are $(\pm 2^{1/10},\pm 2^{1/10},2^{-2/5})$.

41. $x+y+z=100$, so maximize $f(\mathrm{x,y})=xy(100-x-y)$. $f_x=100y-2xy-y^2$, $f_y=100x-x^2-2xy$, $f_{xx}=-2y$, $f_{yy}=-2x$, $f_{xy}=100-2x-2y$. Then $f_x=0$ implies $y=0$ or $y=100-2x$. Substituting $y=0$ into $f_y=0$ gives $x=0$ or $x=100$ and substituting $y=100-2x$ into $f_y=0$ gives $3x^2-100x=0$ so $x=0$ or $\dfrac{100}{3}$. Thus the critical points are $(0,0)$, $(100,0)$, $(0,100)$ and $\left(\dfrac{100}{3},\dfrac{100}{3}\right)$.
$D(0,0)=D(100,0)=D(0,100)=-10,000$ while $D\left(\dfrac{100}{3},\dfrac{100}{3}\right)=\dfrac{10,000}{3}$ and $f_{xx}\left(\dfrac{100}{3},\dfrac{100}{3}\right)=-\dfrac{200}{3}<0$. Thus $(0,0)$, $(100,0)$ and $(0,100)$ are saddle points whereas $f\left(\dfrac{100}{3},\dfrac{100}{3}\right)$ is a local maximum. Thus the numbers are $x=y=z=\dfrac{100}{3}$.

42. Maximize $f(\mathrm{x,y})=x^ay^b(100-x-y)^c$.
$f_x=ax^{a-1}y^b(100-x-y)^c-cx^ay^b(100-x-y)^{c-1}=x^{a-1}y^b(100-x-y)^{c-1}[a(100-x-y)-cx]$
and $f_y=x^ay^{b-1}(100-x-y)^{c-1}[b(100-x-y)-cy]$. Since x , y and z are all positive, the only critical point occurs when $x=a\dfrac{100-y}{a+c}$ and $y=\dfrac{100b}{a+b+c}$. Thus the point is $\left(\dfrac{100a}{a+b+c},\dfrac{100b}{a+b+c}\right)$ and the numbers are $x=\dfrac{100a}{a+b+c}$, $y=\dfrac{100b}{a+b+c}$, $z=\dfrac{100c}{a+b+c}$.

43. Maximize $f(\mathrm{x,y})=xy\left(36-9x^2-36y^2\right)^{1/2}/2$ with $(\mathrm{x,y,z})$ in first octant. Then
$f_x=\dfrac{y\left(36-9x^2-36y^2\right)^{1/2}}{2}+\dfrac{-9x^2y\left(36-9x^2-36y^2\right)^{-1/2}}{2}=\dfrac{\left(36y-18x^2y-36y^3\right)}{2\left(36-9x^2-36y^2\right)^{1/2}}$ and

$f_y = \dfrac{36x - 9x^3 - 72xy^2}{2\left(36 - 9x^2 - 36y^2\right)^{1/2}}$. Setting $f_x = 0$ gives $y = 0$ or $y^2 = \dfrac{2 - x^2}{2}$ but $y > 0$, so only the latter solution applies. Substituting this y into $f_y = 0$ gives $x^2 = \dfrac{4}{3}$ or $x = \dfrac{2}{\sqrt{3}}$, $y = \dfrac{1}{\sqrt{3}}$ and then $z^2 = (36 - 12 - 12)/4 = 3$. The fact that this gives a maximum volume follows from the geometry. This maximum volume is $V = (2x)(2y)(2z) = 8\left(\dfrac{2}{\sqrt{3}}\right)\left(\dfrac{1}{\sqrt{3}}\right)(\sqrt{3}) = \dfrac{16}{\sqrt{3}}$.

44. Here maximize $f(\mathrm{x,y}) = xy\,\dfrac{\left(a^2b^2c^2 - b^2c^2x^2 - a^2c^2y^2\right)^{1/2}}{a^2b^2}$. Then

$f_x = yc^2\,\dfrac{a^2b^2 - 2b^2x^2 - a^2y^2}{a^2b^2\left(a^2b^2c^2 - b^2c^2x^2 - a^2c^2y^2\right)^{1/2}}$ and $f_y = xc^2\,\dfrac{a^2b^2 - 2a^2y^2 - b^2x^2}{a^2b^2\left(a^2b^2c^2 - b^2c^2x^2 - a^2c^2y^2\right)^{1/2}}$. Then $f_x = 0$ (with x , $y > 0$) implies $y^2 = \dfrac{a^2b^2 - 2b^2x^2}{a^2}$ and substituting into $f_y = 0$ implies $3b^2x^2 = a^2b^2$ or $x = \dfrac{1}{\sqrt{3}}\,a$, $y = \dfrac{1}{\sqrt{3}}\,b$ and then $z = \dfrac{1}{\sqrt{3}}\,c$. Thus the maximum volume of such a rectangle is $V = (2x)(2y)(2z) = \dfrac{8}{3\sqrt{3}}\,abc$.

45. Maximize $f(\mathrm{x,y}) = \dfrac{xy}{3}\,(6 - x - 2y)$, then the maximum volume is $V = xyz$. $f_x = \dfrac{1}{3}\left(6y - 2xy - y^2\right) = \dfrac{1}{3}\,y(6 - 2x - 2y)$ and $f_y = \dfrac{1}{3}\,x(6 - x - 4y)$. Setting $f_x = 0$ and $f_y = 0$ gives the critical point $(2,1)$ which geometrically must yield a maximum. Thus the volume of the largest such box is $V = (2)(1)\left(\dfrac{2}{3}\right) = \dfrac{4}{3}$.

46. Surface area $= 2(xy + xz + yz) = 64\ \mathrm{cm}^2$, so $xy + xz + yz = 32$ or $z = \dfrac{32 - xy}{x + y}$. Maximize the volume $f(\mathrm{x,y}) = xy\,\dfrac{32 - xy}{x + y}$. Then $f_x = \dfrac{32y^2 - 2xy^3 - x^2y^2}{(x + y)^2} = y^2\,\dfrac{32 - 2xy - x^2}{(x + y)^2}$ and $f_y = x^2\,\dfrac{32 - 2xy - y^2}{(x + y)^2}$. Setting $f_x = 0$ implies $y = \dfrac{32 - x^2}{2x}$ and substituting into $f_y = 0$ gives $32\left(4x^2\right) - \left(32 - x^2\right)\left(4x^2\right) - \left(32 - x^2\right)^2 = 0$ or $3x^4 + 64x^2 - (32)^2 = 0$. Thus $x^2 = \dfrac{64}{6}$ or $x = \dfrac{8}{\sqrt{6}}$, $y = \dfrac{64/3}{16/\sqrt{6}} = \dfrac{8}{\sqrt{6}}$ and $z = \dfrac{8}{\sqrt{6}}$. Thus the box is a cube

with edge length $\frac{8}{\sqrt{6}}$ cm.

47. Let the dimensions be x , y , and z ; then $4x+4y+4z=c$ and the volume is $V=xyz=xy\left(\frac{1}{4}c-x-y\right)=\frac{1}{4}cxy-x^2y-xy^2$, $x>0$, $y>0$. Then $V_x=\frac{1}{4}cy-2xy-y^2$ and $V_y=\frac{1}{4}cx-x^2-2xy$, so $V_x=0=V_y$ when $2x+y=\frac{1}{4}c$ and $x+2y=\frac{1}{4}c$. Solving, we get $x=\frac{1}{12}c$, $y=\frac{1}{12}c$ and $z=\frac{1}{4}c-x-y=\frac{1}{12}c$. From the geometrical nature of the problem, this critical point must give an absolute maximum. Thus the box is a cube with edge length $\frac{1}{12}c$.

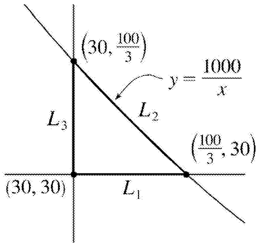
48. The cost equals $5xy+2(xz+yz)$ and $xyz=V$, so $C(\mathrm{x,y})=5xy+2V(x+y)/(xy)=5xy+2V\left(x^{-1}+y^{-1}\right)$. Then $C_x=5y-2Vx^{-2}$, $C_y=5x-2Vy^{-2}$, $f_x=0$ implies $y=2V/\left(5x^2\right)$, $f_y=0$ implies $x=\sqrt[3]{\frac{2}{5}V}=y$. Thus the dimensions of the aquarium which minimize the cost are $x=y=\sqrt[3]{\frac{2}{5}V}$ units, $z=V^{1/3}\left(\frac{5}{2}\right)^{2/3}$.

49. Let the dimensions be x , y and z , then minimize $xy+2(xz+yz)$ if $xyz=32\ ,000\ \mathrm{m}^3$. Then $f(\mathrm{x,y})=xy+[64]\ ,000(x+y)/xy]=xy+64\ ,000(x^{-1}+y^{-1})$, $f_x=y-64\ ,000x^{-2}$, $f_y=x-64\ ,000y^{-2}$. And $f_x=0$ implies $y=64\ ,000/x^2$; substituting into $f_y=0$ implies $x^3=64\ ,000$ or $x=40$ and then $y=40$. Now $D(\mathrm{x,y})=[(2)(64]\ ,000)]^2x^{-3}y^{-3}-1>0$ for $(40,40)$ and $f_{xx}(40,40)>0$ so this is indeed a minimum. Thus the dimensions of the box are $x=y=40$ cm, $z=20$ cm.

50. Let x be the length of the north and south walls, y the length of the east and west walls, and z the height of the building. The heat loss is given by $h=10(2yz)+8(2xz)+1(xy)+5(xy)=6xy+16xz+20\mathrm{yz}$. The volume is $4000\ \mathrm{m}^3$, so $xyz=4000$, and we substitute $z=\frac{4000}{xy}$ to obtain the heat loss function $h(\mathrm{x,y})=6xy+80\ ,000/x+64\ ,000/y$.

(a) Since $z=\frac{4000}{xy}\geq 4$, $xy\leq 1000\Rightarrow y\leq 1000/x$.

Also $x\geq 30$ and $y\geq 30$, so the domain of h is $D=\{(\mathrm{x,y})|x\geq 30,30\leq y\leq 1000/x\}$.



(b) $h(x,y)=6xy+80,000x^{-1}+64,000y^{-1} \Rightarrow h_x=6y-80,000x^{-2}$, $h_y=6x-64,000y^{-2}$.

$h_x=0$ implies $6x^2y=80,000 \Rightarrow y=\dfrac{80,000}{6x^2}$ and substituting into $h_y=0$ gives $6x=64,000\left(\dfrac{6x^2}{80,000}\right)^2$

$\Rightarrow x^3=\dfrac{80,000^2}{6\cdot 64,000}=\dfrac{50,000}{3}$, so $x=\sqrt[3]{\dfrac{50,000}{3}}=10\sqrt[3]{\dfrac{50}{3}} \Rightarrow y=\dfrac{80}{\sqrt[3]{60}}$,

and the only critical point of h is $\left(10\sqrt[3]{\dfrac{50}{3}}, \dfrac{80}{\sqrt[3]{60}}\right)\approx(25.54,20.43)$ which is not in D. Next we check the boundary of D. On L_1: $y=30$, $h(x,30)=180x+80,000/x+6400/3$, $30\le x\le\dfrac{100}{3}$. Since $h'(x,30)=180-80,000/x^2>0$ for $30\le x\le\dfrac{100}{3}$, $h(x,30)$ is an increasing function with minimum $h(30,30)=10,200$ and maximum $h\left(\dfrac{100}{3},30\right)\approx 10,533$. On L_2: $y=1000/x$, $h(x,1000/x)=6000+64x+80,000/x$, $30\le x\le\dfrac{100}{3}$. Since $h'(x,1000/x)=64-80,000/x^2<0$ for $30\le x\le\dfrac{100}{3}$, $h(x,1000/x)$ is a decreasing function with minimum $h\left(\dfrac{100}{3},30\right)\approx 10,533$ and maximum $h\left(30,\dfrac{100}{3}\right)\approx 10,587$. On L_3: $x=30$, $h(30,y)=180y+64,000/y+8000/3$, $30\le y\le\dfrac{100}{3}$.

$h'(30,y)=180-64,000/y^2>0$ for $30\le y\le\dfrac{100}{3}$, so $h(30,y)$ is an increasing function of y with minimum $h(30,30)=10,200$ and maximum $h\left(30,\dfrac{100}{3}\right)\approx 10,587$. Thus the absolute minimum of h is $h(30,30)=10,200$, and the dimensions of the building that minimize heat loss are walls 30 m in length and height $\dfrac{4000}{30^2}=\dfrac{40}{9}\approx 4.44$ m.

(c) From part (b), the only critical point of h, which gives a local (and absolute) minimum, is approximately $h(25.54,20.43)\approx 9396$. So a building of volume 4000 m^2 with dimensions $x\approx 25.54$ m, $y\approx 20.43$ m,

$z\approx \frac{4000}{(25.54)(20.43)} \approx 7.67$ m has the least amount of heat loss.

51. Let x,y , z be the dimensions of the rectangular box. Then the volume of the box is xyz and $L=\sqrt{x^2+y^2+z^2} \Rightarrow L^2=x^2+y^2+z^2 \Rightarrow z=\sqrt{L^2-x^2-y^2}$. Substituting, we have volume $V(\text{x,y})=xy\sqrt{L^2-x^2-y^2}$, x,y>0 .

$V_x=xy\cdot \frac{1}{2}\left(L^2-x^2-y^2\right)^{-1/2}(-2x)+y\sqrt{L^2-x^2-y^2}=y\sqrt{L^2-x^2-y^2}-\frac{x^2y}{\sqrt{L^2-x^2-y^2}}$,

$V_y=x\sqrt{L^2-x^2-y^2}-\frac{xy^2}{\sqrt{L^2-x^2-y^2}}$,

$V_x=0$ implies $y\left(L^2-x^2-y^2\right)=x^2y \Rightarrow y\left(L^2-2x^2-y^2\right)=0 \Rightarrow 2x^2+y^2=L^2$ (since $y>0$), and $V_y=0$ implies $x\left(L^2-x^2-y^2\right)=xy^2 \Rightarrow x\left(L^2-x^2-2y^2\right)=0 \Rightarrow x^2+2y^2=L^2$ (since $x>0$). Substituting $y^2=L^2-2x^2$ into $x^2+2y^2=L^2$ gives $x^2+2L^2-4x^2=L^2 \Rightarrow 3x^2=L^2 \Rightarrow x=L/\sqrt{3}$ (since $x>0$) and then $y=\sqrt{L^2-2\left(L/\sqrt{3}\right)^2}=L/\sqrt{3}$. So the only critical point is $\left(L/\sqrt{3},L/\sqrt{3}\right)$ which, from the geometrical nature of the problem, must give an absolute maximum. Thus the maximum volume is $V\left(L/\sqrt{3},L/\sqrt{3}\right)=\left(L/\sqrt{3}\right)^2\sqrt{L^2-\left(L/\sqrt{3}\right)^2-\left(L/\sqrt{3}\right)^2}=L^3/\left(3\sqrt{3}\right)$ cubic units.

52. Since $p+q+r=1$ we can substitute $p=1-r-q$ into P giving $P=P(\text{q,r})=2(1-r-q)q+2(1-r-q)r+2rq=2q-2q^2+2r-2r^2-2rq$. Since p , q and r represent proportions and $p+q+r=1$, we know $q\geq 0$, $r\geq 0$, and $q+r\leq 1$. Thus, we want to find the absolute maximum of the continuous function $P(\text{q,r})$ on the closed set D enclosed by the lines $q=0$, $r=0$, and $q+r=1$. To find any critical points, we set the partial derivatives equal to zero: $P_q(\text{q,r})=2-4q-2r=0$ and $P_r(\text{q,r})=2-4r-2q=0$. The first equation gives $r=1-2q$, and substituting into the second equation we have $2-4(1-2q)-2q=0 \Rightarrow q=\frac{1}{3}$. Then we have one critical point, $\left(\frac{1}{3},\frac{1}{3}\right)$, where $P\left(\frac{1}{3},\frac{1}{3}\right)=\frac{2}{3}$. Next we find the maximum values of P on the boundary of D which consists of three line segments. For the segment given by $r=0$, $0\leq q\leq 1$, $P(\text{q,r})=P(\text{q,0})=2q-2q^2$, $0\leq q\leq 1$. This represents a parabola with maximum value $P\left(\frac{1}{2},0\right)=\frac{1}{2}$. On the segment $q=0$, $0\leq r\leq 1$ we have $P(0,\text{r})=2r-2r^2$, $0\leq r\leq 1$. This represents a parabola with maximum value $P\left(0,\frac{1}{2}\right)=\frac{1}{2}$. Finally, on the segment $q+r=1$, $0\leq q\leq 1$,

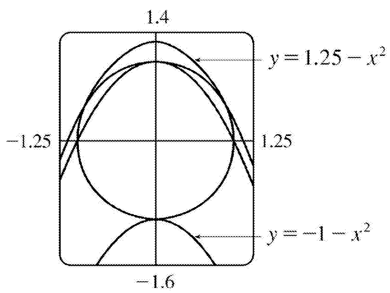
$P(q,r)=P(q,1-q)=2q-2q^2$, $0\le q\le 1$ which has a maximum value of $P\left(\frac{1}{2},\frac{1}{2}\right)=\frac{1}{2}$. Comparing these values with the value of P at the critical point, we see that the absolute maximum value of $P(q,r)$ on D is $\frac{2}{3}$.

53. Note that here the variables are m and b , and $f(m,b)=\sum_{i=1}^{n}\left[y_i-\left(mx_i+b\right)\right]^2$. Then $f_m=\sum_{i=1}^{n}-2x_i\left[y_i-\left(mx_i+b\right)\right]=0$ implies $\sum_{i=1}^{n}\left(x_iy_i-mx_i^2-bx_i\right)=0$ or $\sum_{i=1}^{n}x_iy_i=m\sum_{i=1}^{n}x_i^2+b\sum_{i=1}^{n}x_i$ and $f_b=\sum_{i=1}^{n}-2\left[y_i-\left(mx_i+b\right)\right]=0$ implies $\sum_{i=1}^{n}y_i=m\sum_{i=1}^{n}x_i+\sum_{i=1}^{n}b=m\left(\sum_{i=1}^{n}x_i\right)+nb$. Thus we have the two desired equations. Now $f_{mm}=\sum_{i=1}^{n}2x_i^2$, $f_{bb}=\sum_{i=1}^{n}2=2n$ and $f_{mb}=\sum_{i=1}^{n}2x_i$. And $f_{mm}(m,b)>0$ always and $D(m,b)=4n\left(\sum_{i=1}^{n}x_i^2\right)-4\left(\sum_{i=1}^{n}x_i\right)^2=4\left[n\left(\sum_{i=1}^{n}x_i^2\right)-\left(\sum_{i=1}^{n}x_i\right)^2\right]>0$ always so the solutions of these two equations do indeed minimize $\sum_{i=1}^{n}d_i^2$.

54. Any such plane must cut out a tetrahedron in the first octant. We need to minimize the volume of the tetrahedron that passes through the point $(1,2,3)$. Writing the equation of the plane as $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$, the volume of the tetrahedron is given by $V=\frac{abc}{6}$. But $(1,2,3)$ must lie on the plane, so we need $\frac{1}{a}+\frac{2}{b}+\frac{3}{c}=1$ (*) and thus can think of c as a function of a and b . Then $V_a=\frac{b}{6}\left(c+a\frac{\partial c}{\partial a}\right)$ and $V_b=\frac{a}{6}\left(c+b\frac{\partial c}{\partial b}\right)$. Differentiating (*) with respect to a we get $-a^{-2}-3c^{-2}\frac{\partial c}{\partial a}=0\Rightarrow\frac{\partial c}{\partial a}=\frac{-c^2}{3a^2}$, and differentiating (*) with respect to b gives $-2b^{-2}-3c^{-2}\frac{\partial c}{\partial b}=0\Rightarrow$ $\frac{\partial c}{\partial b}=\frac{-2c^2}{3b^2}$. Then $V_a=\frac{b}{6}\left(c+a\frac{-c^2}{3a^2}\right)=0\Rightarrow c=3a$, and $V_b=\frac{a}{6}\left(c+b\frac{-2c^2}{3b^2}\right)=0\Rightarrow c=\frac{3}{2}b$. Thus $3a=\frac{3}{2}b$ or $b=2a$. Putting these into (*) gives $\frac{3}{a}=1$ or $a=3$ and then $b=6$, $c=9$. Thus the equation of the required plane is $\frac{x}{3}+\frac{y}{6}+\frac{z}{9}=1$ or $6x+3y+2z=18$.

1. At the extreme values of f, the level curves of f just touch the curve $g(x,y)=8$ with a common tangent line. (See Figure 1 and the accompanying discussion.) We can observe several such occurrences on the contour map, but the level curve $f(x,y)=c$ with the largest value of c which still intersects the curve $g(x,y)=8$ is approximately $c=59$, and the smallest value of c corresponding to a level curve which intersects $g(x,y)=8$ appears to be $c=30$. Thus we estimate the maximum value of f subject to the constraint $g(x,y)=8$ to be about 59 and the minimum to be 30.

2. **(a)** The values $c=\pm 1$ and $c=1.25$ seem to give curves which are tangent to the circle. These values represent possible extreme values of the function x^2+y subject to the constraint $x^2+y^2=1$.



(b) $\nabla f=\langle 2x,1\rangle$, $\lambda\nabla g=\langle 2\lambda x,2\lambda y\rangle$. So $2x=2\lambda x\Rightarrow$ either $\lambda=1$ or $x=0$. If $\lambda=1$, then $y=\frac{1}{2}$ and so $x=\pm\frac{\sqrt{3}}{2}$ (from the constraint). If $x=0$, then $y=\pm 1$. Therefore f has possible extreme values at the points $(0,\pm 1)$ and $\left(\pm\frac{\sqrt{3}}{2},\frac{1}{2}\right)$. We calculate $f\left(\pm\frac{\sqrt{3}}{2},\frac{1}{2}\right)=\frac{5}{4}$ (the maximum value), $f(0,1)=1$, and $f(0,-1)=-1$ (the minimum value). These are our answers from (a).

3. $f(x,y)=x^2-y^2$, $g(x,y)=x^2+y^2=1\Rightarrow\nabla f=\langle 2x,-2y\rangle$, $\lambda\nabla g=\langle 2\lambda x,2\lambda y\rangle$. Then $2x=2\lambda x$ implies $x=0$ or $\lambda=1$. If $x=0$, then $x^2+y^2=1$ implies $y=\pm 1$ and if $\lambda=1$, then $-2y=2\lambda y$ implies $y=0$ and thus $x=\pm 1$. Thus the possible points for the extreme values of f are $(\pm 1,0)$, $(0,\pm 1)$. But $f(\pm 1,0)=1$ while $f(0,\pm 1)=-1$ so the maximum value of f on $x^2+y^2=1$ is $f(\pm 1,0)=1$ and the minimum value is $f(0,\pm 1)=-1$.

4. $f(x,y)=4x+6y$, $g(x,y)=x^2+y^2=13\Rightarrow\nabla f=\langle 4,6\rangle$, $\lambda\nabla g=\langle 2\lambda x,2\lambda y\rangle$. Then $2\lambda x=4$ and $2\lambda y=6$ imply $x=\frac{2}{\lambda}$ and $y=\frac{3}{\lambda}$. But $13=x^2+y^2=\left(\frac{2}{\lambda}\right)^2+\left(\frac{3}{\lambda}\right)^2\Rightarrow 13=\frac{13}{\lambda^2}\Rightarrow\lambda=\pm 1$, so f has possible extreme values at the points $(2,3)$, $(-2,-3)$. We compute $f(2,3)=26$ and $f(-2,-3)=-26$, so the maximum value of f on $x^2+y^2=13$ is $f(2,3)=26$ and the minimum value is $f(-2,-3)=-26$.

5. $f(x,y)=x^2y$, $g(x,y)=x^2+2y^2=6\Rightarrow \nabla f=\left\langle 2xy,x^2\right\rangle$, $\lambda\nabla g=\left\langle 2\lambda x,4\lambda y\right\rangle$. Then $2xy=2\lambda x$ implies $x=0$ or $\lambda=y$. If $x=0$, then $x^2=4\lambda y$ implies $\lambda=0$ or $y=0$. However, if $y=0$ then $g(x,y)=0$, a contradiction. So $\lambda=0$ and then $g(x,y)=6\Rightarrow y=\pm\sqrt{3}$. If $\lambda=y$, then $x^2=4\lambda y$ implies $x^2=4y^2$, and so $g(x,y)=6\Rightarrow$ $4y^2+2y^2=6\Rightarrow y^2=1\Rightarrow y=\pm 1$. Thus f has possible extreme values at the points $\left(0,\pm\sqrt{3}\right)$, $(\pm 2,1)$, and $(\pm 2,-1)$. After evaluating f at these points, we find the maximum value to be $f(\pm 2,1)=4$ and the minimum to be $f(\pm 2,-1)=-4$.

6. $f(x,y)=x^2+y^2$, $g(x,y)=x^4+y^4=1\Rightarrow \nabla f=\left\langle 2x,2y\right\rangle$, $\lambda\nabla g=\left\langle 4\lambda x^3,4\lambda y^3\right\rangle$. Then $x=2\lambda x^3$ implies $x=0$ or $\lambda=\frac{1}{2x^2}$. If $x=0$, then $x^4+y^4=1$ implies $y=\pm 1$. But $y=2\lambda y^3$ implies $y=0$ so $x=\pm 1$ or $\lambda=\frac{1}{2y^2}$ and $x^2=y^2$ and $2x^4=1$ so $x=\pm\frac{1}{\sqrt[4]{2}}$. Hence the possible points are $(0,\pm 1)$, $(\pm 1,0)$, $\left(\pm\frac{1}{\sqrt{[4]2}},\pm\frac{1}{\sqrt[4]{2}}\right)$, with the maximum value of f on $x^4+y^4=1$ being $f\left(\pm\frac{1}{\sqrt[4]{2}},\pm\frac{1}{\sqrt[4]{2}}\right)=\frac{2}{\sqrt{2}}=\sqrt{2}$ and the minimum value being $f(0,\pm 1)=f(\pm 1,0)=1$.

7. $f(x,y,z)=2x+6y+10z$, $g(x,y,z)=x^2+y^2+z^2=35\Rightarrow \nabla f=\left\langle 2,6,10\right\rangle$, $\lambda\nabla g=\left\langle 2\lambda x,2\lambda y,2\lambda z\right\rangle$. Then $2\lambda x=2$, $2\lambda y=6$, $2\lambda z=10$ imply $x=\frac{1}{\lambda}$, $y=\frac{3}{\lambda}$, and $z=\frac{5}{\lambda}$. But $35=x^2+y^2+z^2=\left(\frac{1}{\lambda}\right)^2+\left(\frac{3}{\lambda}\right)^2+\left(\frac{5}{\lambda}\right)^2\Rightarrow$ $35=\frac{35}{\lambda^2}\Rightarrow\lambda=\pm 1$, so f has possible extreme values

at the points $(1,3,5)$, $(-1,-3,-5)$. The maximum value of f on $x^2+y^2+z^2=35$ is $f(1,3,5)=70$, and the minimum is $f(-1,-3,-5)=-70$.

8. $f(x,y,z)=8x-4z$, $g(x,y,z)=x^2+10y^2+z^2=5\Rightarrow \nabla f=\left\langle 8,0,-4\right\rangle$, $\lambda\nabla g=\left\langle 2\lambda x,20\lambda y,2\lambda z\right\rangle$. Then $2\lambda x=8$, $20\lambda y=0$, $2\lambda z=-4$ imply $x=\frac{4}{\lambda}$, $y=0$, and $z=-\frac{2}{\lambda}$. But $5=x^2+10y^2+z^2=\left(\frac{4}{\lambda}\right)^2+10(0)^2+\left(-\frac{2}{\lambda}\right)^2\Rightarrow$ $5=\frac{20}{\lambda^2}\Rightarrow\lambda=\pm 2$, so f has possible extreme values at the points $(2,0,-1)$, $(-2,0,1)$. The maximum of f on $x^2+10y^2+z^2=5$ is $f(2,0,-1)=20$, and the minimum is $f(-2,0,1)=-20$.

9. $f(x,y,z)=xyz$, $g(x,y,z)=x^2+2y^2+3z^2=6\Rightarrow \nabla f=\left\langle yz,xz,xy\right\rangle$, $\lambda\nabla g=\left\langle 2\lambda x,4\lambda y,6\lambda z\right\rangle$. Then $\nabla f=\lambda\nabla g$ implies $\lambda=(yz)/(2x)=(xz)/(4y)=(xy)/(6z)$ or $x^2=2y^2$ and $z^2=\frac{2}{3}y^2$. Thus $x^2+2y^2+3z^2=6$ implies $6y^2=6$ or

$y=\pm 1$. Then the possible points are $\left(\sqrt{2},\pm 1,\sqrt{\frac{2}{3}}\right)$, $\left(\sqrt{2},\pm 1,-\sqrt{\frac{2}{3}}\right)$, $\left(-\sqrt{2},\pm 1,\sqrt{\frac{2}{3}}\right)$, $\left(-\sqrt{2},\pm 1,-\sqrt{\frac{2}{3}}\right)$. The maximum value of f on the ellipsoid is $\frac{2}{\sqrt{3}}$, occurring when all coordinates are positive or exactly two are negative and the minimum is $-\frac{2}{\sqrt{3}}$ occurring when 1 or 3 of the coordinates are negative.

10. $f(x,y,z)=x^2y^2z^2$, $g(x,y,z)=x^2+y^2+z^2=1\Rightarrow \nabla f=\left\langle 2xy^2z^2,2yx^2z^2,2zx^2y^2\right\rangle$, $\lambda\nabla g=\langle 2\lambda x,2\lambda y,2\lambda z\rangle$. Then $\nabla f=\lambda\nabla g$ implies (1) $\lambda=y^2z^2=x^2z^2=x^2y^2$ and $\lambda\neq 0$, or (2) $\lambda=0$ and one or two (but not three) of the coordinates are 0 . If (1) then $x^2=y^2=z^2=\frac{1}{3}$. The minimum value of f on the sphere occurs in case (2) with a value of 0 and the maximum value is $\frac{1}{27}$ which arises from all the points from (1), that is, the points $\left(\pm\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)$, $\left(\pm\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)$, $\left(\pm\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}}\right)$.

11. $f(x,y,z)=x^2+y^2+z^2$, $g(x,y,z)=x^4+y^4+z^4=1\Rightarrow$
$\nabla f=\langle 2x,2y,2z\rangle$, $\lambda\nabla g=\left\langle 4\lambda x^3,4\lambda y^3,4\lambda z^3\right\rangle$.
Case 1: If $x\neq 0$, $y\neq 0$ and $z\neq 0$, then $\nabla f=\lambda\nabla g$ implies $\lambda=1/\left(2x^2\right)=1/\left(2y^2\right)=1/\left(2z^2\right)$ or $x^2=y^2=z^2$ and $3x^4=1$ or $x=\pm\frac{1}{\sqrt[4]{3}}$ giving the points $\left(\pm\frac{1}{\sqrt[4]{3}},\frac{1}{\sqrt[4]{3}},\frac{1}{\sqrt[4]{3}}\right)$, $\left(\pm\frac{1}{\sqrt[4]{3}},-\frac{1}{\sqrt[4]{3}},\frac{1}{\sqrt[4]{3}}\right)$, $\left(\pm\frac{1}{\sqrt[4]{3}},\frac{1}{\sqrt[4]{3}},-\frac{1}{\sqrt[4]{3}}\right)$, $\left(\pm\frac{1}{\sqrt[4]{3}},-\frac{1}{\sqrt[4]{3}},-\frac{1}{\sqrt[4]{3}}\right)$ all with an f –value of $\sqrt{3}$. *Case 2:* If one of the variables equals zero and the other two are not zero, then the squares of the two nonzero coordinates are equal with common value $\frac{1}{\sqrt{2}}$ and corresponding f value of $\sqrt{2}$. *Case 3:* If exactly two of the variables are zero, then the third variable has value ± 1 with the corresponding f value of 1 . Thus on $x^4+y^4+z^4=1$, the maximum value of f is $\sqrt{3}$ and the minimum value is 1 .

12. $f(x,y,z)=x^4+y^4+z^4$, $g(x,y,z)=x^2+y^2+z^2=1\Rightarrow$
$\nabla f=\left\langle 4x^3,4y^3,4z^3\right\rangle$, $\lambda\nabla g=\langle 2\lambda x,2\lambda y,2\lambda z\rangle$.
Case 1: If $x\neq 0$, $y\neq 0$ and $z\neq 0$ then $\nabla f=\lambda\nabla g$ implies $\lambda=2x^2=2y^2=2z^2$ or $x^2=y^2=z^2=\frac{1}{3}$ yielding 8 points each with an f –value of $\frac{1}{3}$.
Case 2:

If one of the variables is 0 and the other two are not, then the squares of the two nonzero coordinates are equal with common value $\frac{1}{2}$ and the corresponding f –value is $\frac{1}{2}$.
Case 3: If exactly two of the variables are 0 , then the third variable has value ± 1 with corresponding f –value of 1 . Thus on $x^2+y^2+z^2=1$, the maximum value of f is 1 and the minimum value is $\frac{1}{3}$.

13. $f(x,y,z,t)=x+y+z+t$, $g(x,y,z,t)=x^2+y^2+z^2+t^2=1 \Rightarrow \langle 1,1,1,1\rangle=\langle 2\lambda x,2\lambda y,2\lambda z,2\lambda t\rangle$, so $\lambda=1/(2x)=1/(2y)=1/(2z)=1/(2t)$ and $x=y=z=t$. But $x^2+y^2+z^2+t^2=1$, so the possible points are $\left(\pm\frac{1}{2},\pm\frac{1}{2},\pm\frac{1}{2},\pm\frac{1}{2}\right)$. Thus the maximum value of f is $f\left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)=2$ and the minimum value is $f\left(-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right)=-2$.
14. $f\left(x_1,x_2,\ldots,x_n\right)=x_1+x_2+\cdots+x_n$, $g\left(x_1,x_2,\ldots,x_n\right)=x_1^2+x_2^2+\cdots+x_n^2=1\Rightarrow$ $\langle 1,1,\ldots,1\rangle=\left\langle 2\lambda x_1,2\lambda x_2,\ldots,2\lambda x_n\right\rangle$, so $\lambda=1/\left(2x_1\right)=1/\left(2x_2\right)=\cdots=1/\left(2x_n\right)$ and $x_1=x_2=\cdots=x_n$. But $x_1^2+x_2^2+\cdots+x_n^2=1$, so $x_i=\pm 1/\sqrt{n}$ for $i=1$, … , n . Thus the maximum value of f is $f\left(1/\sqrt{n},1/\sqrt{n},\ldots,1/\sqrt{n}\right)=\sqrt{n}$ and the minimum value is $f\left(-1/\sqrt{n},-1/\sqrt{n},\ldots,-1/\sqrt{n}\right)=-\sqrt{n}$.
15. $f(x,y,z)=x+2y$, $g(x,y,z)=x+y+z=1$, $h(x,y,z)=y^2+z^2=4\Rightarrow \nabla f=\langle 1,2,0\rangle$, $\lambda\nabla g=\langle \lambda,\lambda,\lambda\rangle$ and $\mu\nabla h=\langle 0,2\mu y,2\mu z\rangle$. Then $1=\lambda$, $2=\lambda+2\mu y$ and $0=\lambda+2\mu z$ so $\mu y=\frac{1}{2}=-\mu z$ or $y=1/(2\mu)$, $z=-1/(2\mu)$. Thus $x+y+z=1$ implies $x=1$ and $y^2+z^2=4$ implies $\mu=\pm\frac{1}{2\sqrt{2}}$. Then the possible points are $\left(1,\pm\sqrt{2},\mp\sqrt{2}\right)$ and the maximum value is $f\left(1,\sqrt{2},-\sqrt{2}\right)=1+2\sqrt{2}$ and the minimum value is $f\left(1,-\sqrt{2},\sqrt{2}\right)=1-2\sqrt{2}$.
16. $f(x,y,z)=3x-y-3z$, $g(x,y,z)=x+y-z=0$, $h(x,y,z)=x^2+2z^2=1\Rightarrow \nabla f=\langle 3,-1,-3\rangle$, $\lambda\nabla g=\langle \lambda,\lambda,-\lambda\rangle$, $\mu\nabla h=(2\mu x,0,4\mu z)$. Then $3=\lambda+2\mu x$, $-1=\lambda$ and $-3=-\lambda+4\mu z$, so $\lambda=-1$, $\mu z=-1$, $\mu x=2$. Thus $h(x,y,z)=1$ implies $\frac{4}{\mu^2}+2\left(\frac{1}{\mu^2}\right)=1$ or $\mu=\pm\sqrt{6}$, so $z=\mp\frac{1}{\sqrt{6}}$; $x=\pm\frac{2}{\sqrt{6}}$; and $g(x,y,z)=0$ implies $y=\mp\frac{3}{\sqrt{6}}$. Hence the maximum of f subject to the constraints is $f\left(\frac{\sqrt{6}}{3},-\frac{\sqrt{6}}{2},-\frac{\sqrt{6}}{6}\right)=2\sqrt{6}$ and the minimum is $f\left(-\frac{\sqrt{6}}{3},\frac{\sqrt{6}}{2},\frac{\sqrt{6}}{6}\right)=-2\sqrt{6}$.

17. $f(x,y,z)=yz+xy, g(x,y,z)=xy=1$, $h(x,y,z)=y^2+z^2=1\Rightarrow \nabla f=\langle y,x+z,y\rangle$, $\lambda\nabla g=\langle \lambda y,\lambda x,0\rangle$,

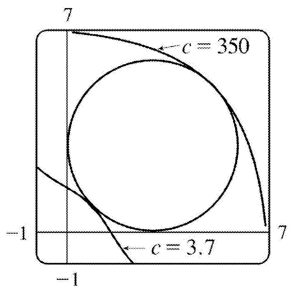
$\mu\nabla h=\langle 0,2\mu y,2\mu z\rangle$. Then $y=\lambda y$ implies $\lambda=1$, $x+z=\lambda x+2\mu y$ and $y=2\mu z$. Thus $\mu=z/(2y)=y/(2y)$ or $y^2=z^2$, and so $y^2+z^2=1$ implies $y=\pm\frac{1}{\sqrt{2}}$, $z=\pm\frac{1}{\sqrt{2}}$. Then $xy=1$ implies $x=\pm\sqrt{2}$ and the possible points are $\left(\pm\sqrt{2},\pm\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)$, $\left(\pm\sqrt{2},\pm\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\right)$. Hence the maximum of f subject to the constraints is $f\left(\pm\sqrt{2},\pm\frac{1}{\sqrt{2}},\pm\frac{1}{\sqrt{2}}\right)=\frac{3}{2}$ and the minimum is $f\left(\pm\sqrt{2},\pm\frac{1}{\sqrt{2}},\mp\frac{1}{\sqrt{2}}\right)=\frac{1}{2}$.
Note: Since $xy=1$ is one of the constraints we could have solved the problem by solving $f(y,z)=yz+1$ subject to $y^2+z^2=1$.

18. $f(x,y)=2x^2+3y^2-4x-5\Rightarrow\nabla f=\langle 4x-4,6y\rangle=\langle 0,0\rangle\Rightarrow x=1$, $y=0$. Thus $(1,0)$ is the only critical point of f , and it lies in the region $x^2+y^2<16$. On the boundary, $g(x,y)=x^2+y^2=16\Rightarrow\lambda\nabla g=\langle 2\lambda x,2\lambda y\rangle$, so $6y=2\lambda y\Rightarrow$ either $y=0$ or $\lambda=3$. If $y=0$, then $x=\pm 4$; if $\lambda=3$, then $4x-4=2\lambda x\Rightarrow x=-2$ and $y=\pm 2\sqrt{3}$. Now $f(1,0)=-7$, $f(4,0)=11$, $f(-4,0)=43$, and $f\left(-2,\pm 2\sqrt{3}\right)=47$. Thus the maximum value of $f(x,y)$ on the disk $x^2+y^2\le 16$ is $f\left(-2,\pm 2\sqrt{3}\right)=47$, and the minimum value is $f(1,0)=-7$.

19. $f(x,y)=e^{-xy}$. For the interior of the region, we find the critical points: $f_x=-ye^{-xy}$, $f_y=-xe^{-xy}$, so the only critical point is $(0,0)$, and $f(0,0)=1$. For the boundary, we use Lagrange multipliers. $g(x,y)=x^2+4y^2=1\Rightarrow\lambda\nabla g=\langle 2\lambda x,8\lambda y\rangle$, so setting $\nabla f=\lambda\nabla g$ we get $-ye^{-xy}=2\lambda x$ and $-xe^{-xy}=8\lambda y$. The first of these gives $e^{-xy}=-2\lambda x/y$, and then the second gives $-x(-2\lambda x/y)=8\lambda y\Rightarrow x^2=4y^2$. Solving this last equation with the constraint $x^2+4y^2=1$ gives $x=\pm\frac{1}{\sqrt{2}}$ and $y=\pm\frac{1}{2\sqrt{2}}$. Now $f\left(\pm\frac{1}{\sqrt{2}},\mp\frac{1}{2\sqrt{2}}\right)=e^{1/4}\approx 1.284$ and $f\left(\pm\frac{1}{\sqrt{2}},\pm\frac{1}{2\sqrt{2}}\right)=e^{-1/4}\approx 0.779$. The former are the maxima on the region and the latter are the minima.

20. **(a)** The graphs of $f(x,y)=3.7$ and $f(x,y)=350$ seem to be tangent to the circle, and so 3.7 and 350 are the approximate minimum and maximum values of the function $f(x,y)$ subject to the constraint $(x-3)^2+(y-3)^2=9$.
(b) Let $g(x,y)=(x-3)^2+(y-3)^2$. We calculate $f_x(x,y)=3x^2+3y$, $f_y(x,y)=3y^2+3x$, $g_x(x,y)=2x-6$, and $g_y(x,y)=2y-6$,
and use a CAS to search for solutions to the equations $g(x,y)=(x-3)^2+(y-3)^2=9$, $f_x=\lambda g_x$, and $f_y=\lambda g_y$ .The solutions are $(x,y)=\left(3-\frac{3}{2}\sqrt{2},3-\frac{3}{2}\sqrt{2}\right)\approx(0.879,0.879)$ and

$(x,y)=\left(3+\frac{3}{2}\sqrt{2},3+\frac{3}{2}\sqrt{2}\right)\approx(5.121,5.121)$. These give
$f\left(3-\frac{3}{2}\sqrt{2},3-\frac{3}{2}\sqrt{2}\right)=\frac{351}{2}-\frac{243}{2}\sqrt{2}\approx 3.673$ and
$f\left(3+\frac{3}{2}\sqrt{2},3+\frac{3}{2}\sqrt{2}\right)=\frac{351}{2}+\frac{243}{2}\sqrt{2}\approx 347.33$, in accordance with part (a).



21. $P(L,K)=bL^{\alpha}K^{1-\alpha}$, $g(L,K)=mL+nK=p\Rightarrow\nabla P=\left\langle\alpha bL^{\alpha-1}K^{1-\alpha},(1-\alpha)bL^{\alpha}K^{-\alpha}\right\rangle$, $\lambda\nabla g=\langle\lambda m,\lambda n\rangle$. Then $\alpha b(K/L)^{1-\alpha}=\lambda m$ and $(1-\alpha)b(L/K)^{\alpha}=\lambda n$ and $mL+nK=p$, so $\alpha b(K/L)^{1-\alpha}/m=(1-\alpha)b(L/K)^{\alpha}/n$ or $n\alpha/[m(1-\alpha)]=(L/K)^{\alpha}(L/K)^{1-\alpha}$ or $L=Kn\alpha/[m(1-\alpha)]$. Substituting into $mL+nK=p$ gives $K=(1-\alpha)p/n$ and $L=\alpha p/m$ for the maximum production.

22. $C(L,K)=mL+nK$, $g(L,K)=bL^{\alpha}K^{1-\alpha}=Q\Rightarrow\nabla C=\langle m,n\rangle$, $\lambda\nabla g=\left\langle\lambda\alpha bL^{\alpha-1}K^{1-\alpha},\lambda(1-\alpha)bL^{\alpha}K^{-\alpha}\right\rangle$. Then $\frac{m}{\alpha b}\left(\frac{L}{K}\right)^{1-\alpha}=\frac{n}{(1-\alpha)b}\left(\frac{K}{L}\right)^{\alpha}$ and
$bL^{\alpha}K^{1-\alpha}=Q\Rightarrow\frac{n\alpha}{m(1-\alpha)}=\left(\frac{L}{K}\right)^{1-\alpha}\left(\frac{L}{K}\right)^{\alpha}\Rightarrow L=\frac{Kn\alpha}{m(1-\alpha)}$ and so $b\left[\frac{Kn\alpha}{m(1-\alpha)}\right]^{\alpha}K^{1-\alpha}=Q$.
Hence $K=\frac{Q}{b(n\alpha/[m(1-\alpha)])^{\alpha}}=\frac{Qm^{\alpha}(1-\alpha)^{\alpha}}{bn^{\alpha}\alpha^{\alpha}}$ and $L=\frac{Qm^{\alpha-1}(1-\alpha)^{\alpha-1}}{bn^{\alpha-1}\alpha^{\alpha-1}}=\frac{Qn^{1-\alpha}\alpha^{1-\alpha}}{bm^{1-\alpha}(1-\alpha)^{1-\alpha}}$
minimizes cost.

23. Let the sides of the rectangle be x and y . Then $f(x,y)=xy$, $g(x,y)=2x+2y=p\Rightarrow\nabla f(x,y)=\langle y,x\rangle$, $\lambda\nabla g=\langle 2\lambda,2\lambda\rangle$. Then $\lambda=\frac{1}{2}y=\frac{1}{2}x$ implies $x=y$ and the rectangle with maximum area is a square with side length $\frac{1}{4}p$.

24. Let $f(x,y,z)=s(s-x)(s-y)(s-z)$, $g(x,y,z)=x+y+z$. Then $\nabla f=\langle -s(s-y)(s-z),-s(s-x)(s-z),-s(s-x)(s-y)\rangle$, $\lambda\nabla g=\langle\lambda,\lambda,\lambda\rangle$. \ Thus (1) $(s-y)(s-z)=(s-x)(s-z)$ and (2) $(s-x)(s-z)=(s-x)(s-y)$. (1) implies $x=y$ while (2) implies $y=z$, so $x=y=z=p/3$ and the triangle with maximum area is equilateral.

25. Let $f(\text{x,y,z})=d^2=(x-2)^2+(y-1)^2+(z+1)^2$, then we want to minimize f subject to the constraint $g(\text{x,y,z})=x+y-z=1$. $\nabla f=\lambda\nabla g\Rightarrow\langle 2(x-2),2(y-1),2(z+1)\rangle=\lambda\langle 1,1,-1\rangle$, so $x=(\lambda+4)/2$, $y=(\lambda+2)/2$, $z=-(\lambda+2)/2$. Substituting into the constraint equation gives $\frac{\lambda+4}{2}+\frac{\lambda+2}{2}+\frac{\lambda+2}{2}=1\Rightarrow 3\lambda+8=2\Rightarrow\lambda=-2$, so $x=1$, $y=0$, and $z=0$. This must correspond to a minimum, so the shortest distance is $d=\sqrt{(1-2)^2+(0-1)^2+(0+1)^2}=\sqrt{3}$.

26. Let $f(\text{x,y,z})=d^2=(x-1)^2+(y-2)^2+(z-3)^2$, then we want to minimize f subject to the constraint $g(\text{x,y,z})=x-y+z=4$. $\nabla f=\lambda\nabla g\Rightarrow\langle 2(x-1),2(y-2),2(z-3)\rangle=\lambda\langle 1,-1,1\rangle$, so $x=(\lambda+2)/2$, $y=(4-\lambda)/2$, $z=(\lambda+6)/2$. Substituting into the constraint equation gives $\frac{\lambda+2}{2}-\frac{4-\lambda}{2}+\frac{\lambda+6}{2}=4\Rightarrow\lambda=\frac{4}{3}$, so $x=\frac{5}{3}$, $y=\frac{4}{3}$, and $z=\frac{11}{3}$. This must correspond to a minimum, so the point on the plane closest to the point $(1,2,3)$ is $\left(\frac{5}{3},\frac{4}{3},\frac{11}{3}\right)$.

27. $f(\text{x,y,z})=x^2+y^2+z^2$, $g(\text{x,y,z})=z^2-xy-1=0\Rightarrow\nabla f=\langle 2\text{x},2\text{y},2z\rangle=\lambda\nabla g=\langle -\lambda\text{y},-\lambda\text{x},2\lambda z\rangle$. Then $2z=2\lambda z$ implies $z=0$ or $\lambda=1$. If $z=0$ then $g(\text{x,y,z})=1$ implies $xy=-1$ or $x=-1/y$. Thus $2x=-\lambda y$ and $2y=-\lambda x$ imply $\lambda=2/y^2=2y^2$ or $y=\pm 1$, $x=\pm 1$. If $\lambda=1$, then $2x=-y$ and $2y=-x$ imply $x=y=0$, so $z=\pm 1$. Hence the possible points are $(\pm 1,\mp 1,0)$, $(0,0,\pm 1)$ and the minimum value of f is $f(0,0,\pm 1)=1$, so the points closest to the origin are $(0,0,\pm 1)$.

28. $f(\text{x,y,z})=x^2+y^2+z^2$, $g(\text{x,y,z})=x^2y^2z=1\Rightarrow\nabla f=\langle 2\text{x},2\text{y},2z\rangle=\lambda\nabla g=\left\langle 2\lambda xy^2\text{z},2\lambda x^2\text{yz},\lambda x^2y^2\right\rangle$. Then $\lambda y^2z=1$, $\lambda x^2z=1$ and $\lambda x^2y^2=2z$ so $y^2z=x^2z$ and $x=\pm y$. Also $2z/1=\lambda x^2y^2/\left(\lambda x^2z\right)$ so $2z^2=y^2$ and $y=\pm\sqrt{2}\,z$. But $x^2y^2z=1$ implies $z>0$ and $4z^5=1$. Thus the points are $(\pm 2^{1/10},\pm 2^{1/10},2^{-2/5})$, and the minimum distance is attained at each of these.

29. $f(\text{x,y,z})=xyz$, $g(\text{x,y,z})=x+y+z=100\Rightarrow\nabla f=\langle \text{yz,xz,xy}\rangle=\lambda\nabla g=\langle\lambda,\lambda,\lambda\rangle$. Then $\lambda=yz=xz=xy$ implies $x=y=z=\frac{100}{3}$.

30. $f(\text{x,y,z})=x^ay^bz^c$, $g(\text{x,y,z})=x+y+z=100\Rightarrow$
$\nabla f=\left\langle ax^{a-1}y^bz^c,\text{b}x^ay^{b-1}z^c,\text{c}x^ay^bz^{c-1}\right\rangle=\lambda\nabla g=\langle\lambda,\lambda,\lambda\rangle$. Then $\lambda=ax^{a-1}y^bz^c=bx^ay^{b-1}z^c=cx^ay^bz^{c-1}$ or $ayz=bxz=cxy$. Thus $x=\frac{ay}{b}$, $z=\frac{cy}{b}$, and $\frac{ay}{b}+y+\frac{cy}{b}=100$ implies that $y=\frac{100b}{a+b+c}$, $x=\frac{100a}{a+b+c}$ and

$z=\dfrac{100c}{a+b+c}$ gives the maximum.

31. If the dimensions are $2x$, $2y$ and $2z$, then $f(x,y,z)=8xyz$ and $g(x,y,z)=9x^2+36y^2+4z^2=36\Rightarrow$ $\nabla f=\langle 8yz,8xz,8xy\rangle=\lambda\nabla g=\langle 18\lambda x,72\lambda y,8\lambda z\rangle$. Thus $18\lambda x=8yz$, $72\lambda y=8xz$, $8\lambda z=8xy$ so $x^2=4y^2$, $z^2=9y^2$ and $36y^2+36y^2+36y^2=36$ or $y=\dfrac{1}{\sqrt{3}}$ ($y>0$). Thus the volume of the largest such box is $8\left(\dfrac{1}{\sqrt{3}}\right)\left(\dfrac{2}{\sqrt{3}}\right)\left(\dfrac{3}{\sqrt{3}}\right)=16\sqrt{3}$.

32. $f(x,y,z)=8xyz$, $g(x,y,z)=b^2c^2x+a^2c^2y^2+a^2b^2z^2=a^2b^2c^2\Rightarrow$ $\nabla f=\langle 8yz,8xz,8xy\rangle=\lambda\nabla g=\left\langle 2\lambda b^2c^2x,2\lambda a^2c^2y,2\lambda a^2b^2z\right\rangle$. Then $4yz=\lambda b^2c^2x$, $4xz=\lambda a^2c^2y$, $4xy=\lambda a^2b^2z$ imply $\lambda=\dfrac{4yz}{b^2c^2x}=\dfrac{4xz}{a^2c^2y}=\dfrac{4xy}{a^2b^2z}$ or $\dfrac{y}{b^2x}=\dfrac{x}{a^2y}$ and $\dfrac{z}{c^2y}=\dfrac{y}{b^2z}$. Thus $x=\dfrac{ay}{b}$, $z=\dfrac{cy}{b}$, and $a^2c^2y^2+c^2a^2y^2+a^2c^2y^2=a^2b^2c^2$, or $y=\dfrac{b}{\sqrt{3}}$, $x=\dfrac{a}{\sqrt{3}}$, $z=\dfrac{c}{\sqrt{3}}$ and the volume is $\dfrac{8}{3\sqrt{3}}\,abc$.

33. $f(x,y,z)=xyz$, $g(x,y,z)=x+2y+3z=6\Rightarrow\nabla f=\langle yz,xz,xy\rangle=\lambda\nabla g=\langle\lambda,2\lambda,3\lambda\rangle$. Then $\lambda=yz=\dfrac{1}{2}xz=\dfrac{1}{3}xy$ implies $x=2y$, $z=\dfrac{2}{3}y$. But $2y+2y+2y=6$ so $y=1$, $x=2$, $z=\dfrac{2}{3}$ and the volume is $V=\dfrac{4}{3}$.

34. $f(x,y,z)=xyz$, $g(x,y,z)=xy+yz+xz=32\Rightarrow\nabla f=\langle yz,xz,xy\rangle=\lambda\nabla g=\langle\lambda(y+z),\lambda(x+z),\lambda(x+y)\rangle$. Then (1) $\lambda(y+z)=yz$, (2) $\lambda(x+z)=xz$ and (3) $\lambda(x+y)=xy$. And (1) minus (2) implies $\lambda(y-x)=z(y-x)$ so $x=y$ or $\lambda=z$. If $\lambda=z$, then (1) implies $z(y+z)=yz$ or $z=0$ which is false. Thus $x=y$. Similarly (2) minus (3) implies $\lambda(z-y)=x(z-y)$ so $y=z$ or $\lambda=x$. As above, $\lambda\neq x$, so $x=y=z$ and $3x^2=32$ or $x=y=z=\dfrac{8}{\sqrt{6}}$ cm.

35. $f(x,y,z)=xyz$, $g(x,y,z)=4(x+y+z)=c\Rightarrow\nabla f=\langle yz,xz,xy\rangle$, $\lambda\nabla g=\langle 4\lambda,4\lambda,4\lambda\rangle$. Thus $4\lambda=yz=xz=xy$ or $x=y=z=\dfrac{1}{12}c$ are the dimensions giving the maximum volume.

36. $C(x,y,z)=5xy+2xz+2yz$, $g(x,y,z)=xyz=V\Rightarrow$ $\nabla C=\langle 5y+2z,5x+2z,2x+2y\rangle=\lambda\nabla g=\langle\lambda yz,\lambda xz,\lambda xy\rangle$. Then (1) $\lambda yz=5y+2z$, (2) $\lambda xz=5x+2z$, (3) $\lambda xy=2(x+y)$ and (4) $xyz=V$. Now (1) – (2) implies $\lambda z(y-x)=5(y-x)$, so $x=y$ or $\lambda=5/z$, but z can't be 0 , so $x=y$. Then twice (2) minus five times (3) together with $x=y$ implies $\lambda y(2x-5y)=2(2z-5y)$ which gives $z=\dfrac{5}{2}y$. Hence $\dfrac{5}{2}y^3=V$ and the dimensions which minimize cost are $x=y=\sqrt[3]{\dfrac{2}{5}V}$ units,

$z=V^{1/3}\left(\frac{5}{2}\right)^{2/3}$ units.

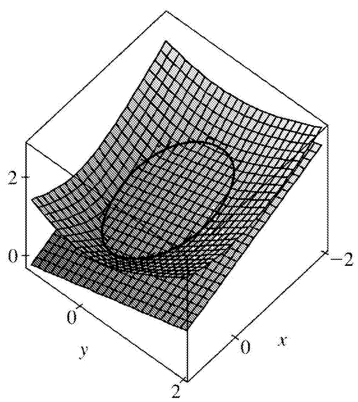
37. If the dimensions of the box are given by x , y , and z , then we need to find the maximum value of $f(x,y,z)=xyz$ ($x,y,z>0$) subject to the constraint $L=\sqrt{x^2+y^2+z^2}$ or $g(x,y,z)=x^2+y^2+z^2=L^2$. $\nabla f=\lambda\nabla g\Rightarrow\langle yz,xz,xy\rangle=\lambda\langle 2x,2y,2z\rangle$, so $yz=2\lambda x\Rightarrow\lambda=\frac{yz}{2x}$, $xz=2\lambda y\Rightarrow\lambda=\frac{xz}{2y}$, and $xy=2\lambda z\Rightarrow\lambda=\frac{xy}{2z}$. Thus $\lambda=\frac{yz}{2x}=\frac{xz}{2y}\Rightarrow x^2=y^2\Rightarrow x=y$ and $\lambda=\frac{yz}{2x}=\frac{xy}{2z}\Rightarrow x=z$. Substituting into the constraint equation gives $x^2+x^2+x^2=L^2\Rightarrow x^2=L^2/3\Rightarrow x=L/\sqrt{3}=y=z$ and the maximum volume is $\left(L/\sqrt{3}\right)^3=L^3/\left(3\sqrt{3}\right)$.

38. Let the dimensions of the box be x , y , and z , so its volume is $f(x,y,z)=xyz$, its surface area is $g(x,y,z)=xy+yz+xz=750$ and its total edge length is $h(x,y,z)=x+y+z=50$. Then $\nabla f=\langle yz,xz,xy\rangle=\lambda\nabla g+\mu\nabla h=\langle\lambda(y+z),\lambda(x+z),\lambda(x+y)\rangle+\langle\mu,\mu,\mu\rangle$. So (1) $yz=\lambda(y+z)+\mu$, (2) $xz=\lambda(x+z)+\mu$, and (3) $xy=\lambda(x+y)+\mu$. Notice that the box can't be a cube or else $x=y=z=\frac{50}{3}$ but then $xy+yz+xz=\frac{2500}{3}\neq 750$. Assume x is the distinct side, that is, $x\neq y$, $x\neq z$. Then (1) minus (2) implies $z(y-x)=\lambda(y-x)$ or $\lambda=z$, and (1) minus (3) implies $y(z-x)=\lambda(z-x)$ or $\lambda=y$. So $y=z=\lambda$ and $x+y+z=50$ implies $x=50-2\lambda$; also $xy+yz+xz=750$ implies $x(2\lambda)+\lambda^2=750$. Hence $50-2\lambda=\frac{750-\lambda^2}{2\lambda}$ or $3\lambda^2-100\lambda+750=0$ and $\lambda=\frac{50\pm 5\sqrt{10}}{3}$, giving the points $\left(\frac{1}{3}\left(50\mp 10\sqrt{10}\right),\frac{1}{3}\left(50\pm 5\sqrt{10}\right),\frac{1}{3}\left(50\pm 5\sqrt{10}\right)\right)$. Thus the minimum of f is $f\left(\frac{1}{3}\left(50-10\sqrt{3}\right),\frac{1}{3}\left(50+5\sqrt{10}\right),\frac{1}{3}\left(50+5\sqrt{10}\right)\right)=\frac{1}{27}\left(87{,}500-2500\sqrt{10}\right)$, and its maximum is $f\left(\frac{1}{3}\left(50+10\sqrt{10}\right),\frac{1}{3}\left(50-5\sqrt{10}\right),\frac{1}{3}\left(50-5\sqrt{10}\right)\right)=\frac{1}{27}\left(87{,}500+2500\sqrt{10}\right)$.
Note: If either y or z is the distinct side, then symmetry gives the same result.

39. We need to find the extreme values of $f(x,y,z)=x^2+y^2+z^2$ subject to the two constraints $g(x,y,z)=x+y+2z=2$ and $h(x,y,z)=x^2+y^2-z=0$. $\nabla f=\langle 2x,2y,2z\rangle$, $\lambda\nabla g=\langle\lambda,\lambda,2\lambda\rangle$ and $\mu\nabla h=\langle 2\mu x,2\mu y,-\mu\rangle$. Thus we need (1) $2x=\lambda+2\mu x$, (2) $2y=\lambda+2\mu y$, (3) $2z=2\lambda-\mu$, (4) $x+y+2z=2$, and (5) $x^2+y^2-z=0$. From (1) and (2), $2(x-y)=2\mu(x-y)$, so if $x\neq y$, $\mu=1$. Putting this in (3) gives $2z=2\lambda-1$ or $\lambda=z+\frac{1}{2}$, but putting $\mu=1$ into (1) says $\lambda=0$. Hence $z+\frac{1}{2}=0$ or $z=-\frac{1}{2}$. Then (4) and (5)

become $x+y-3=0$ and $x^2+y^2+\frac{1}{2}=0$. The last equation cannot be true, so this case gives no solution. So we must have $x=y$. Then (4) and (5) become $2x+2z=2$ and $2x^2-z=0$ which imply $z=1-x$ and $z=2x^2$. Thus $2x^2=1-x$ or $2x^2+x-1=(2x-1)(x+1)=0$ so $x=\frac{1}{2}$ or $x=-1$. The two points to check are $\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)$ and $(-1,-1,2)$: $f\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)=\frac{3}{4}$ and $f(-1,-1,2)=6$. Thus $\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)$ is the point on the ellipse nearest the origin and $(-1,-1,2)$ is the one farthest from the origin.

40. **(a)** Parametric equations for the ellipse are easiest to determine using cylindrical coordinates. The cone is given by $z=r$, and the plane is $4r\cos\theta-3r\sin\theta+8z=5$. Substituting $z=r$ into the plane equation gives $4r\cos\theta-3r\sin\theta+8r=5 \Rightarrow r=\dfrac{5}{4\cos\theta-3\sin\theta+8}$. Since $z=r$ on the ellipse, parametric equations (in cylindrical coordinates) are $\theta=t$, $r=z=\dfrac{5}{4\cos t-3\sin t+8}$, $0\le t\le 2\pi$.



(b) We need to find the extreme values of $f(x,y,z)=z$ subject to the two constraints $g(x,y,z)=4x-3y+8z=5$ and $h(x,y,z)=x^2+y^2-z^2=0$. $\nabla f=\lambda\nabla g+\mu\nabla h \Rightarrow$ $\langle 0,0,1\rangle=\lambda\langle 4,-3,8\rangle+\mu\langle 2x,2y,-2z\rangle$, so we need (1) $4\lambda+2\mu x=0 \Rightarrow x=-\dfrac{2\lambda}{\mu}$, (2) $-3\lambda+2\mu y=0 \Rightarrow y=\dfrac{3\lambda}{2\mu}$, (3) $8\lambda-2\mu z=1 \Rightarrow z=\dfrac{8\lambda-1}{2\mu}$, (4) $4x-3y+8z=5$, and (5) $x^2+y^2=z^2$. Substituting (1), (2), and (3) into (4) gives $4\left(-\dfrac{2\lambda}{\mu}\right)-3\left(\dfrac{3\lambda}{2\mu}\right)+8\left(\dfrac{8\lambda-1}{2\mu}\right)=5 \Rightarrow \mu=\dfrac{39\lambda-8}{10}$ and into (5) gives $\left(-\dfrac{2\lambda}{\mu}\right)^2+\left(\dfrac{3\lambda}{2\mu}\right)^2=\left(\dfrac{8\lambda-1}{2\mu}\right)^2 \Rightarrow 16\lambda^2+9\lambda^2=(8\lambda-1)^2 \Rightarrow 39\lambda^2-16\lambda+1=0 \Rightarrow$ $\lambda=\dfrac{1}{13}$ or $\lambda=\dfrac{1}{3}$. If $\lambda=\dfrac{1}{13}$ then $\mu=-\dfrac{1}{2}$ and $x=\dfrac{4}{13}$, $y=-\dfrac{3}{13}$, $z=\dfrac{5}{13}$. If $\lambda=\dfrac{1}{3}$ then $\mu=\dfrac{1}{2}$ and $x=-\dfrac{4}{3}$, $y=1$, $z=\dfrac{5}{3}$. Thus the highest point on the ellipse is $\left(-\dfrac{4}{3},1,\dfrac{5}{3}\right)$ and the lowest point is

$\left(\frac{4}{13}, -\frac{3}{13}, \frac{5}{13} \right)$.

41. $f(x,y,z)=ye^{x-z}$, $g(x,y,z)=9x^2+4y^2+36z^2=36$, $h(x,y,z)=xy+yz=1$.
$\nabla f=\lambda\nabla g+\mu\nabla h \Rightarrow \left\langle ye^{x-z}, e^{x-z}, -ye^{x-z} \right\rangle = \lambda \left\langle 18x,8y,72z \right\rangle + \mu \left\langle y,x+z,y \right\rangle$, so $ye^{x-z}=18\lambda x+\mu y$, $e^{x-z}=8\lambda y+\mu(x+z)$, $-ye^{x-z}=72\lambda z+\mu y$, $9x^2+4y^2+36z^2=36$, $xy+yz=1$. Using a CAS to solve these 5 equations simultaneously for x , y , z , λ , and μ (in Maple, use the allvalues command), we get 4 real –valued solutions:

$x\approx 0.222444$, $y\approx -2.157012$, $z\approx -0.686049$, $\lambda\approx -0.200401$, $\mu\approx 2.108584$
$x\approx -1.951921$, $y\approx -0.545867$, $z\approx 0.119973$, $\lambda\approx 0.003141$, $\mu\approx -0.076238$
$x\approx 0.155142$, $y\approx 0.904622$, $z\approx 0.950293$, $\lambda\approx -0.012447$, $\mu\approx 0.489938$
$x\approx 1.138731$, $y\approx 1.768057$, $z\approx -0.573138$, $\lambda\approx 0.317141$, $\mu\approx 1.862675$

Substituting these values into f gives $f(0.222444,-2.157012,-0.686049)\approx -5.3506$, $f(-1.951921,-0.545867,0.119973)\approx -0.0688$, $f(0.155142,0.904622,0.950293)\approx 0.4084$, $f(1.138731,1.768057,-0.573138)\approx 9.7938$. Thus the maximum is approximately 9.7938 , and the mininum is approximately –5.3506 .

42. $f(x,y,z)=x+y+z$, $g(x,y,z)=x^2-y^2-z=0$, $h(x,y,z)=x^2+z^2=4$.
$\nabla f=\lambda\nabla g+\mu\nabla h \Rightarrow \left\langle 1,1,1 \right\rangle = \lambda \left\langle 2x,-2y,-1 \right\rangle + \mu \left\langle 2x,0,2z \right\rangle$, so $1=2\lambda x+2\mu x$, $1=-2\lambda y$, $1=-\lambda+2\mu z$, $x^2-y^2=z$, $x^2+z^2=4$. Using a CAS to solve these 5 equations simultaneously for x , y , z , λ , and μ , we get 4 real–valued solutions:

$x\approx -1.652878$, $y\approx -1.964194$, $z\approx -1.126052$, $\lambda\approx 0.254557$, $\mu\approx -0.557060$
$x\approx -1.502800$, $y\approx 0.968872$, $z\approx 1.319694$, $\lambda\approx -0.516064$, $\mu\approx 0.183352$
$x\approx -0.992513$, $y\approx 1.649677$, $z\approx -1.736352$, $\lambda\approx -0.303090$, $\mu\approx -0.200682$
$x\approx 1.895178$, $y\approx 1.718347$, $z\approx 0.638984$, $\lambda\approx -0.290977$, $\mu\approx 0.554805$

Substituting these values into f gives $f(-1.652878,-1.964194,-1.126052)\approx -4.7431$, $f(-1.502800,0.968872,1.319694)\approx 0.7858$, $f(-0.992513,1.649677,-1.736352)\approx -1.0792$, $f(1.895178,1.718347,0.638984)\approx 4.2525$. Thus the maximum is approximately 4.2525 , and the mininum is approximately –4.7431 .

43. **(a)** We wish to maximize $f\left(x_1,x_2,\ldots,x_n\right)=\sqrt[n]{x_1x_2\cdots x_n}$ subject to $g\left(x_1,x_2,\ldots,x_n\right)=x_1+x_2+\cdots+x_n=c$ and $x_i>0$. ∇f

$=\left\langle \frac{1}{n}\left(x_1x_2\cdots x_n\right)^{\frac{1}{n}-1}\left(x_2\cdots x_n\right), \frac{1}{n}\left(x_1x_2\cdots x_n\right)^{\frac{1}{n}-1}\left(x_1x_3\cdots x_n\right),\ldots, \frac{1}{n}\left(x_1x_2\cdots x_n\right)^{\frac{1}{n}-1}\left(x_1\cdots x_{n-1}\right)\right.$

and

$\lambda\nabla g=\left\langle \lambda,\lambda,\ldots,\lambda\right\rangle$, so we need to solve the system of equations

$$\frac{1}{n}\left(x_1x_2\cdots x_n\right)^{\frac{1}{n}-1}\left(x_2\cdots x_n\right)=\lambda \quad\Rightarrow\quad x_1^{1/n}x_2^{1/n}\cdots x_n^{1/n}=n\lambda x_1$$

$$\frac{1}{n}\left(x_1x_2\cdots x_n\right)^{\frac{1}{n}-1}\left(x_1x_3\cdots x_n\right)=\lambda \quad\Rightarrow\quad x_1^{1/n}x_2^{1/n}\cdots x_n^{1/n}=n\lambda x_2$$

$$\cdots$$

$$\cdots$$

$$\cdots$$

$$\frac{1}{n}\left(x_1x_2\cdots x_n\right)^{\frac{1}{n}-1}\left(x_1\cdots x_{n-1}\right)=\lambda \quad\Rightarrow\quad x_1^{1/n}x_2^{1/n}\cdots x_n^{1/n}=n\lambda x_n$$

This implies $n\lambda x_1=n\lambda x_2=\cdots=n\lambda x_n$. Note $\lambda\neq 0$, otherwise we can't have all $x_i>0$. Thus $x_1=x_2=\cdots=x_n$. But $x_1+x_2+\cdots+x_n=c\Rightarrow nx_1=c\Rightarrow x_1=\frac{c}{n}=x_2=x_3=\cdots=x_n$. Then the only point where f can have an extreme value is $\left(\frac{c}{n},\frac{c}{n},\ldots,\frac{c}{n}\right)$. Since we can choose values for $\left(x_1,x_2,\ldots,x_n\right)$ that make f as close to zero (but not equal) as we like, f has no minimum value. Thus the maximum value is $f\left(\frac{c}{n},\frac{c}{n},\ldots,\frac{c}{n}\right)=\sqrt[n]{\frac{c}{n}\cdot\frac{c}{n}\cdot\cdots\cdot\frac{c}{n}}=\frac{c}{n}$.

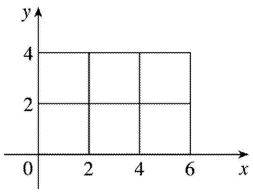
(b) From part (a), $\frac{c}{n}$ is the maximum value of f . Thus $f\left(x_1,x_2,\ldots,x_n\right)=\sqrt[n]{x_1x_2\cdots x_n}\leq\frac{c}{n}$. But $x_1+x_2+\cdots+x_n=c$, so $\sqrt[n]{x_1x_2\cdots x_n}\leq\frac{x_1+x_2+\cdots+x_n}{n}$. These two means are equal when f attains its maximum value $\frac{c}{n}$, but this can occur only at the point $\left(\frac{c}{n},\frac{c}{n},\ldots,\frac{c}{n}\right)$ we found in part (a). So the means are equal only when $x_1=x_2=x_3=\cdots=x_n=\frac{c}{n}$.

44. **(a)** Let $f\left(x_1,\ldots,x_n,y_1,\ldots,y_n\right)=\Sigma_{i=1}^{n}x_iy_i$, $g\left(x_1,\ldots,x_n\right)=\Sigma_{i=1}^{n}x_i^2$, and $h\left(x_1,\ldots,x_n\right)=\Sigma_{i=1}^{n}y_i^2$. Then $\nabla f=\nabla\Sigma_{i=1}^{n}x_iy_i=\left\langle y_1,y_2,\ldots,y_n,x_1,x_2,\ldots,x_n\right\rangle$, $\nabla g=\nabla\Sigma_{i=1}^{n}x_i^2=\left\langle 2x_1,2x_2,\ldots,2x_n,0,0,\ldots,0\right\rangle$ and

$\nabla h=\nabla \sum_{i=1}^{n} y_i^2=\left\langle 0,0,\ldots,0,2y_1,2y_2,\ldots,2y_n \right\rangle$. So $\nabla f=\lambda\nabla g+\mu\nabla h \Leftrightarrow y_i=2\lambda x_i$ and $x_i=2\mu y_i$, $1\le i\le n$.
Then $1=\sum_{i=1}^{n} y_i^2=\sum_{i=1}^{n} 4\lambda^2 x_i^2=4\lambda^2\sum_{i=1}^{n} x_i^2=4\lambda^2 \Rightarrow \lambda=\pm\frac{1}{2}$.
If $\lambda=\frac{1}{2}$ then $y_i=2\left(\frac{1}{2}\right)x_i=x_i$, $1\le i\le n$. Thus $\sum_{i=1}^{n} x_i y_i=\sum_{i=1}^{n} x_i^2=1$. Similarly if $\lambda=-\frac{1}{2}$ we get $y_i=-x_i$ and $\sum_{i=1}^{n} x_i y_i=-1$. Similarly we get $\mu=\pm\frac{1}{2}$ giving $y_i=\pm x_i$, $1\le i\le n$, and $\sum_{i=1}^{n} x_i y_i=\pm 1$. Thus the maximum value of $\sum_{i=1}^{n} x_i y_i$ is 1 .

(b) Here we assume $\sum_{i=1}^{n} a_i^2\ne 0$ and $\sum_{i=1}^{n} b_i^2\ne 0$. (If $\sum_{i=1}^{n} a_i^2=0$, then each $a_i=0$ and so the inequality is trivially true.) $x_i=\frac{a_i}{\sqrt{\sum a_j^2}} \Rightarrow \sum x_i^2=\frac{\sum a_i^2}{\sum a_j^2}=1$, and $y_i=\frac{b_i}{\sqrt{\sum b_j^2}} \Rightarrow \sum y_i^2=\frac{\sum b_i^2}{\sum b_j^2}=1$. Therefore, from (a), $\sum x_i y_i=\sum \frac{a_i b_i}{\sqrt{\sum a_j^2}\sqrt{\sum b_j^2}} \le 1 \Leftrightarrow \sum a_i b_i \le \sqrt{\sum a_j^2}\sqrt{\sum b_j^2}$.

1. **(a)** The subrectangles are shown in the figure.
The surface is the graph of $f(x,y)=xy$ and $\Delta A=4$,
so we estimate

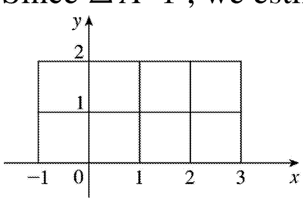


$$V \approx \sum_{i=1}^{3}\sum_{j=1}^{2} f\left(x_i, y_j\right)\Delta A$$
$$=f(2,2)\Delta A+f(2,4)\Delta A+f(4,2)\Delta A+f(4,4)\Delta A+f(6,2)\Delta A+f(6,4)\Delta A$$
$$=4(4)+8(4)+8(4)+16(4)+12(4)+24(4)=288$$

(b)

$$V \approx \sum_{i=1}^{3}\sum_{j=1}^{2} f\left(\overline{x}_i, \overline{y}_j\right)\Delta A$$
$$=f(1,1)\Delta A+f(1,3)\Delta A+f(3,1)\Delta A+f(3,3)\Delta A+f(5,1)\Delta A+f(5,3)\Delta A$$
$$=1(4)+3(4)+3(4)+9(4)+5(4)+15(4)=144$$

2. The subrectangles are shown in the figure.
Since $\Delta A=1$, we estimate

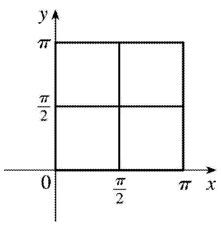


$$\iint_R \left(y^2-2x^2\right)dA \approx \sum_{i=1}^{4}\sum_{j=1}^{2} f\left(x_{ij}^*, y_{ij}^*\right)\Delta A$$
$$=f(-1,1)\Delta A+f(-1,2)\Delta A+f(0,1)\Delta A+f(0,2)\Delta A$$
$$+f(1,1)\Delta A+f(1,2)\Delta A+f(2,1)\Delta A+f(2,2)\Delta A$$
$$=-1(1)+2(1)+1(1)+4(1)-1(1)+2(1)-7(1)-4(1)=-4$$

3. **(a)** The subrectangles are shown in the figure. Since $\Delta A=\pi^2/4$, we estimate

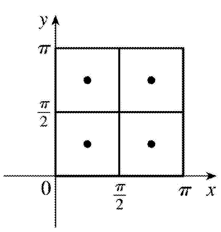
$$\iint_R \sin(x+y)\,dA \approx \sum_{i=1}^{2}\sum_{j=1}^{2} f\left(x_{ij}^*, y_{ij}^*\right)\Delta A$$
$$=f(0,0)\Delta A+f\left(0,\frac{\pi}{2}\right)\Delta A+f\left(\frac{\pi}{2},0\right)\Delta A+f\left(\frac{\pi}{2},\frac{\pi}{2}\right)\Delta A$$

$$=0\left(\frac{\pi^2}{4}\right)+1\left(\frac{\pi^2}{4}\right)+1\left(\frac{\pi^2}{4}\right)+0\left(\frac{\pi^2}{4}\right)=\frac{\pi^2}{2}\approx 4.935$$



(b)

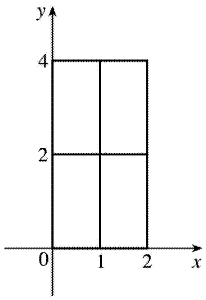
$$\iint_R \sin(x+y)\,dA \approx \sum_{i=1}^{2}\sum_{j=1}^{2} f(\overline{x}_i,\overline{y}_j)\,\Delta A$$
$$=f\left(\frac{\pi}{4},\frac{\pi}{4}\right)\Delta A+f\left(\frac{\pi}{4},\frac{3\pi}{4}\right)\Delta A$$
$$+f\left(\frac{3\pi}{4},\frac{\pi}{4}\right)\Delta A+f\left(\frac{3\pi}{4},\frac{3\pi}{4}\right)\Delta A$$
$$=1\left(\frac{\pi^2}{4}\right)+0\left(\frac{\pi^2}{4}\right)+0\left(\frac{\pi^2}{4}\right)+(-1)\left(\frac{\pi^2}{4}\right)=0$$



4. **(a)** The subrectangles are shown in the figure.

The surface is the graph of $f(x,y)=x+2y^2$ and $\Delta A=2$, so we estimate

$$V=\iint_R (x+2y^2)\,dA \approx \sum_{i=1}^{2}\sum_{j=1}^{2} f\left(x_{ij}^*,y_{ij}^*\right)\Delta A$$
$$=f(1,0)\,\Delta A+f(1,2)\,\Delta A+f(2,0)\,\Delta A+f(2,2)\,\Delta A$$
$$=1(2)+9(2)+2(2)+10(2)=44$$

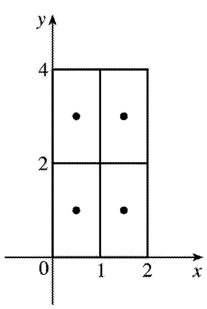


(b)

$$V=\iint_R (x+2y^2)\,dA \approx \sum_{i=1}^{2}\sum_{j=1}^{2} f(\bar{x}_i,\bar{y}_j)\,\Delta A$$

$$=f\left(\frac{1}{2},1\right)\Delta A+f\left(\frac{1}{2},3\right)\Delta A+f\left(\frac{3}{2},1\right)\Delta A+f\left(\frac{3}{2},3\right)\Delta A$$

$$=\frac{5}{2}(2)+\frac{37}{2}(2)+\frac{7}{2}(2)+\frac{39}{2}(2)=88$$



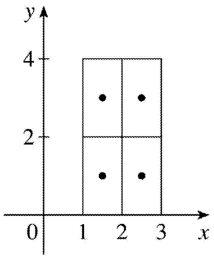
5. **(a)** Each subrectangle and its midpoint are shown in the figure. The area of each subrectangle is $\Delta A=2$, so we evaluate f at each midpoint and estimate

$$\iint_R f(x,y)\,dA \approx \sum_{i=1}^{2}\sum_{j=1}^{2} f\left(\bar{x}_i,\bar{y}_j\right)\Delta A$$

$$=f(1.5,1)\,\Delta A+f(1.5,3)\,\Delta A$$

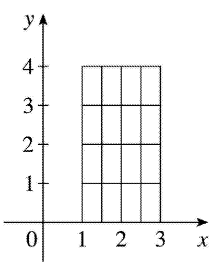
$$+f(2.5,1)\,\Delta A+f(2.5,3)\,\Delta A$$

$$=1(2)+(-8)(2)+5(2)+(-1)(2)=-6$$



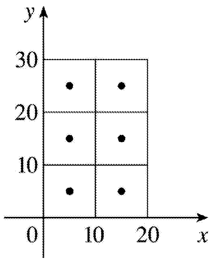
(b) The subrectangles are shown in the figure. In each subrectangle, the sample point farthest from the origin is the upper right corner, and the area of each subrectangle is $\Delta A=\frac{1}{2}$. Thus we estimate

$$\iint_R f(x,y)\,dA \approx \sum_{i=1}^{4}\sum_{j=1}^{4} f\left(x_i,y_j\right)\Delta A$$
$$=f(1.5,1)\Delta A+f(1.5,2)\Delta A+f(1.5,3)\Delta A+f(1.5,4)\Delta A$$
$$+f(2,1)\Delta A+f(2,2)\Delta A+f(2,3)\Delta A+f(2,4)\Delta A$$
$$+f(2.5,1)\Delta A+f(2.5,2)\Delta A+f(2.5,3)\Delta A+f(2.5,4)\Delta A$$
$$+f(3,1)\Delta A+f(3,2)\Delta A+f(3,3)\Delta A+f(3,4)\Delta A$$



$$=1\left(\frac{1}{2}\right)+(-4)\left(\frac{1}{2}\right)+(-8)\left(\frac{1}{2}\right)+(-6)\left(\frac{1}{2}\right)+3\left(\frac{1}{2}\right)+0\left(\frac{1}{2}\right)+(-5)\left(\frac{1}{2}\right)+(-8)\left(\frac{1}{2}\right)$$
$$+5\left(\frac{1}{2}\right)+3\left(\frac{1}{2}\right)+(-1)\left(\frac{1}{2}\right)+(-4)\left(\frac{1}{2}\right)+8\left(\frac{1}{2}\right)+6\left(\frac{1}{2}\right)+3\left(\frac{1}{2}\right)+0\left(\frac{1}{2}\right)$$
$$=-3.5$$

6. To approximate the volume, let R be the planar region corresponding to the surface of the water in the pool, and place R on coordinate axes so that x and y correspond to the dimensions given. Then we define $f(x,y)$ to be the depth of the water at (x,y) , so the volume of water in the pool is the volume of the solid that lies above the rectangle $R=[0,20]\times[0,30]$ and below the graph of $f(x,y)$. We can estimate this volume using the Midpoint Rule with $m=2$ and $n=3$, so $\Delta A=100$. Each subrectangle with its midpoint is shown in the figure. Then



$$\begin{aligned} V &\approx \sum_{i=1}^{2}\sum_{j=1}^{3} f\left(\overline{x}_i,\overline{y}_j\right)\Delta A \\ &= \Delta A\left[f(5,5)+f(5,15)+f(5,25)+f(15,5)+f(15,15)+f(15,25)\right] \\ &= 100\,(3+7+10+3+5+8)=3600 \end{aligned}$$

Thus, we estimate that the pool contains 3600 cubic feet of water.
Alternatively, we can approximate the volume with a Riemann sum where $m=4$, $n=6$ and the sample points are taken to be, for example, the upper right corner of each subrectangle. Then $\Delta A=25$ and

$$\begin{aligned} V &\approx \sum_{i=1}^{4}\sum_{j=1}^{6} f\left(x_i,y_j\right)\Delta A \\ &= 25\,[3+4+7+8+10+8+4+6+8+10+12+10+3+4 \\ &\quad +5+6+8+7+2+2+2+3+4+4] \\ &= 25(140)=3500 \end{aligned}$$

So we estimate that the pool contains 3500 ft^3 of water.

7. The values of $f(x,y)=\sqrt{52-x^2-y^2}$ get smaller as we move farther from the origin, so on any of the subrectangles in the problem, the function will have its largest value at the lower left corner of the subrectangle and its smallest value at the upper right corner, and any other value will lie between these two. So using these subrectangles we have $U<V<L$. (Note that this is true no matter how R is divided into subrectangles.)

8. From the level curves we see that $f\left(\frac{1}{2},\frac{1}{2}\right)\approx 11$. So, using the Midpoint Rule with only one subrectangle, we get $\iint_R f(x,y)\,dA\approx 1\cdot f\left(\frac{1}{2},\frac{1}{2}\right)\approx 11$. Dividing R into four squares of equal size, we get

$$\iint_R f(x,y)\,dA\approx \frac{1}{4}\left[f\left(\frac{1}{4},\frac{1}{4}\right)+f\left(\frac{1}{4},\frac{3}{4}\right)+f\left(\frac{3}{4},\frac{1}{4}\right)+f\left(\frac{3}{4},\frac{3}{4}\right)\right]\approx\frac{1}{4}(11+13+9.5+11)\approx 11$$

.

Using sixteen squares we get the same result. So $\iint_R f(x,y)\,dA\approx 11$.

9. **(a)** With $m=n=2$, we have $\Delta A=4$. Using the contour map to estimate the value of f at the center of each subrectangle, we have

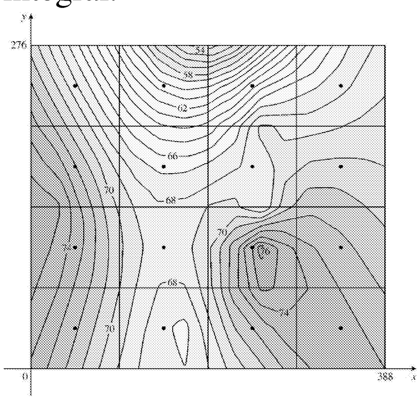
$$\iint_R f(x,y)\,dA \approx \sum_{i=1}^{2}\sum_{j=1}^{2} f\left(\overline{x}_i,\overline{y}_j\right)\Delta A=\Delta A\left[f(1,1)+f(1,3)+f(3,1)+f(3,3)\right]$$
$$\approx 4(27+4+14+17)=248$$

(b) $f_{\text{ave}}=\dfrac{1}{A(R)}\iint_R f(x,y)\,dA\approx\dfrac{1}{16}(248)=15.5$

10. As in Example 4, we place the origin at the southwest corner of the state. Then $R=[0,388]\times[0,276]$ (in miles) is the rectangle corresponding to Colorado and we define $f(x,y)$ to be the temperature at the location (x,y) . The average temperature is given by

$$f_{\text{ave}}=\frac{1}{A(R)}\iint_R f(x,y)\,dA=\frac{1}{388\cdot 276}\iint_R f(x,y)\,dA$$

We can use the Midpoint Rule with $m=n=4$ to give a reasonable estimate of the value of the double integral.



Thus, we divide R into 16 regions of equal size, as shown in the figure, with the center of each subrectangle indicated. The area of each subrectangle is $\Delta A=\dfrac{388}{4}\cdot\dfrac{276}{4}=6693$, so using the contour map to estimate the function values at each midpoint, we have

$$\iint_R f(x,y)\,dA \approx \sum_{i=1}^{4}\sum_{j=1}^{4} f\left(\overline{x}_i,\overline{y}_j\right)\Delta A$$
$$\approx \Delta A$$
$$+72.0+74.9+68.4+63.7+73.2+72.3+70.3+67.7]$$
$$=6693(1111.5)$$

Therefore,

$f_{\text{ave}} \approx \dfrac{6693 \cdot 1111.5}{388 \cdot 276} \approx 69.5$, so the average temperature in Colorado on May 1, 1996, was approximately $69.5°$.
Alternatively, we can use the Midpoint Rule with $m=n=2$ which is easier computationally but will most likely be less accurate since we have fewer subrectangles. In this case, $\Delta A = \dfrac{388}{2} \cdot \dfrac{276}{2} = 26,772$ and we can use the same grid to estimate the function values at the midpoints of the four subrectangles. Then

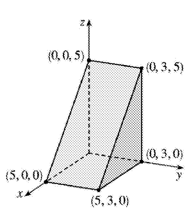
$$\begin{aligned}\iint_R f(x,y)\,dA &\approx \sum_{i=1}^{2}\sum_{j=1}^{2} f\left(\bar{x}_i,\bar{y}_j\right)\Delta A \approx 26,772[70.0+66.5+74.3+68.5] \\ &= 26,772 \cdot 279.3\end{aligned}$$

and $f_{\text{ave}} \approx \dfrac{26,772 \cdot 279.3}{388 \cdot 276} \approx 69.8°$.

11. $z=3>0$, so we can interpret the integral as the volume of the solid S that lies below the plane $z=3$ and above the rectangle $[-2,2]\times[1,6]$. S is a rectangular solid, thus $\iint_R 3\,dA = 4 \cdot 5 \cdot 3 = 60$.

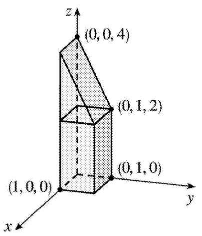
12. $z=5-x \geq 0$ for $0 \leq x \leq 5$, so we can interpret the integral as the volume of the solid S that lies below the plane $z=5-x$ and above the rectangle $[0,5]\times[0,3]$. S is a triangular cylinder whose volume is $3\,(\text{area of triangle}) = 3\left(\frac{1}{2}\cdot 5 \cdot 5\right) = 37.5$. Thus,

$$\iint_R (5-x)\,dA = 37.5$$

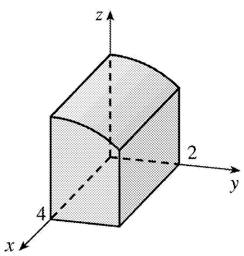


13. $z=f(x,y)=4-2y \geq 0$ for $0 \leq y \leq 1$. Thus the integral represents the volume of that part of the rectangular solid $[0,1]\times[0,1]\times[0,4]$ which lies below the plane $z=4-2y$.
So

$$\iint_R (4-2y)\,dA = (1)(1)(2) + \frac{1}{2}(1)(1)(2) = 3$$



14. Here $z=\sqrt{9-y^2}$, so $z^2+y^2=9$, $z\geq 0$. Thus the integral represents the volume of the top half of the part of the circular cylinder $z^2+y^2=9$ that lies above the rectangle $[0,4]\times[0,2]$.



15. To calculate the estimates using a programmable calculator, we can use an algorithm similar to that of Exercise 5.1.7 [ET 5.1.7]. In Maple, we can define the function $f(x,y)=e^{-x^2-y^2}$ (calling it f), load the student package, and then use the command
middlesum(middlesum(f,x=0..1,m),
y=0..1,m);

to get the estimate with $n=m^2$ squares of equal size. Mathematica has no special Riemann sum command, but we can define f and then use nested Sum commands to calculate the estimates.

n	estimate
1	0.6065
4	0.5694
16	0.5606
64	0.5585
256	0.5579
1024	0.5578

16.

n	estimate
1	0.9922

4	0.9262
16	0.8797

n	estimate
64	0.8660
256	0.8625
1024	0.8616

17. If we divide R into mn subrectangles, $\iint_R k\,dA \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f\left(x_{ij}^*, y_{ij}^*\right) \Delta A$ for any choice of sample points $\left(x_{ij}^*, y_{ij}^*\right)$. But $f\left(x_{ij}^*, y_{ij}^*\right)=k$ always and $\sum_{i=1}^{m} \sum_{j=1}^{n} \Delta A=$ area of $R=(b-a)(d-c)$. Thus, no matter how we choose the sample points, $\sum_{i=1}^{m} \sum_{j=1}^{n} f\left(x_{ij}^*, y_{ij}^*\right) \Delta A = k \sum_{i=1}^{m} \sum_{j=1}^{n} \Delta A = k(b-a)(d-c)$ and so

$$\begin{aligned} \iint_R k\,dA &= \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f\left(x_{ij}^*, y_{ij}^*\right) \Delta A = \lim_{m,n\to\infty} k \sum_{i=1}^{m} \sum_{j=1}^{n} \Delta A \\ &= \lim_{m,n\to\infty} k(b-a)(d-c) = k(b-a)(d-c) \end{aligned}$$

18. On R , $0 \le x+y \le 2 < \pi$ and $\sin\theta \ge 0$ for $0 \le \theta \le \pi$. Thus $f(x,y)=\sin(x+y) \ge 0$ for all $(x,y) \in R$. Since $0 \le \sin(x+y) \le 1$, Property (9) gives $\iint_R 0\,dA \le \iint_R \sin(x+y)\,dA \le \iint_R 1\,dA$, so by Exercise 17 we have $0 \le \iint_R \sin(x+y)\,dA \le 1$.

1. $\int_0^3 \left(2x+3x^2 y\right) dx = \left[x^2+x^3 y\right]_{x=0}^{x=3} = (9+27y)-(0+0) = 9+27y$,

$\int_0^4 \left(2x+3x^2 y\right) dy = \left[2xy+3x^2 \frac{y^2}{2}\right]_{y=0}^{y=4} = \left(8x+3x^2 \cdot \frac{16}{2}\right)-(0+0) = 8x+24x^2$

2. $\int_0^3 \frac{y}{x+2}\, dx = \left[y\ln|x+2|\right]_{x=0}^{x=3} = y\ln 5 - y\ln 2 = y\ln\frac{5}{2}$,

$\int_0^4 \frac{y}{x+2}\, dy = \frac{1}{x+2}\left[\frac{y^2}{2}\right]_{y=0}^{y=4} = \frac{1}{x+2}\left(\frac{16}{2}-0\right) = \frac{8}{x+2}$

3. $\int_1^3\int_0^1 (1+4xy)\,dx\,dy = \int_1^3 \left[x+2x^2 y\right]_{x=0}^{x=1} dy = \int_1^3 (1+2y)\,dy = \left[y+y^2\right]_1^3 = (3+9)-(1+1) = 10$

4.

$\int_2^4\int_{-1}^1 \left(x^2+y^2\right) dy\,dx = \int_2^4 \left[x^2 y+\frac{1}{3}y^3\right]_{y=-1}^{y=1} dx = \int_2^4 \left[\left(x^2+\frac{1}{3}\right)-\left(-x^2-\frac{1}{3}\right)\right] dx$

$= \int_2^4 \left(2x^2+\frac{2}{3}\right) dx = \left[\frac{2}{3}x^3+\frac{2}{3}x\right]_2^4 = \left(\frac{128}{3}+\frac{8}{3}\right)-\left(\frac{16}{3}+\frac{4}{3}\right) = \frac{116}{3}$

5. $\int_0^2\int_0^{\pi/2} x\sin y\,dy\,dx = \int_0^2 x\,dx \int_0^{\pi/2} \sin y\,dy$ [as in Example 5] $= \left[\frac{x^2}{2}\right]_0^2 \left[-\cos y\right]_0^{\pi/2} = (2-0)(0+1) = 2$.

6.

$\int_1^4\int_0^2 \left(x+\sqrt{y}\right) dx\,dy = \int_1^4 \left[\frac{1}{2}x^2+x\sqrt{y}\right]_{x=0}^{x=2} dy = \int_1^4 \left(2+2\sqrt{y}\right) dy$

$= \left[2y+2\cdot\frac{2}{3}y^{3/2}\right]_1^4 = \left(8+\frac{4}{3}\cdot 8\right)-\left(2+\frac{4}{3}\right) = \frac{46}{3}$

7.

$\int_0^2\int_0^1 (2x+y)^8\,dx\,dy = \int_0^2 \left[\frac{1}{2}\frac{(2x+y)^9}{9}\right]_{x=0}^{x=1} dy$

$$=\frac{1}{18}\int_0^2[(2+y)^9-(0+y)^9]\,dy=\frac{1}{18}\left[\frac{(2+y)^{10}}{10}-\frac{y^{10}}{10}\right]_0^2$$

$$=\frac{1}{180}[(4^{10}-2^{10})-(2^{10}-0^{10})]=\frac{1{,}046{,}528}{180}=\frac{261{,}632}{45}$$

8.

$$\int_0^1\int_1^2\frac{xe^x}{y}\,dy\,dx=\int_0^1 xe^x\,dx\int_1^2\frac{1}{y}\,dy \text{ [as in Example 5]}$$

$$=\left[xe^x-e^x\right]_0^1\left[\ln|y|\right]_1^2 \text{ [by integrating by parts]}$$

$$=[(e-e)-(0-1)](\ln 2-0)=\ln 2$$

9.

$$\int_1^4\int_1^2\left(\frac{x}{y}+\frac{y}{x}\right)dy\,dx=\int_1^4\left[x\ln|y|+\frac{1}{x}\cdot\frac{1}{2}y^2\right]_{y=1}^{y=2}dx=\int_1^4\left(x\ln 2+\frac{3}{2x}\right)dx$$

$$=\left[\frac{1}{2}x^2\ln 2+\frac{3}{2}\ln|x|\right]_1^4=8\ln 2+\frac{3}{2}\ln 4-\frac{1}{2}\ln 2$$

$$=\frac{15}{2}\ln 2+3\ln 4^{1/2}=\frac{21}{2}\ln 2$$

10.

$$\int_1^2\int_0^1(x+y)^{-2}\,dx\,dy=\int_1^2\left[-(x+y)^{-1}\right]_{x=0}^{x=1}dy=\int_1^2\left[y^{-1}-(1+y)^{-1}\right]dy$$

$$=\left[\ln y-\ln(1+y)\right]_1^2=\ln 2-\ln 3-0+\ln 2=\ln\frac{4}{3}$$

11.

$$\int_0^{\ln 2}\int_0^{\ln 5}e^{2x-y}\,dx\,dy=\left(\int_0^{\ln 5}e^{2x}\,dx\right)\left(\int_0^{\ln 2}e^{-y}\,dy\right)=\left[\frac{1}{2}e^{2x}\right]_0^{\ln 5}\left[-e^{-y}\right]_0^{\ln 2}$$

$$=\left(\frac{25}{2}-\frac{1}{2}\right)\left(-\frac{1}{2}+1\right)=6$$

12.

$$\int_0^1\int_0^1 \frac{xy}{\sqrt{x^2+y^2+1}}\,dy\,dx = \int_0^1\left[x\sqrt{x^2+y^2+1}\right]_{y=0}^{y=1}dx = \int_0^1 x\left(\sqrt{x^2+2}-\sqrt{x^2+1}\right)dx$$

$$=\frac{1}{3}\left[(x^2+2)^{3/2}-(x^2+1)^{3/2}\right]_0^1=\frac{1}{3}\left[(3^{3/2}-2^{3/2})-(2^{3/2}-1)\right]$$

$$=\frac{1}{3}\left(3\sqrt{3}-4\sqrt{2}+1\right)$$

13.

$$\iint_R (6x^2y^3-5y^4)\,dA = \int_0^3\int_0^1 (6x^2y^3-5y^4)\,dy\,dx = \int_0^3\left[\frac{3}{2}x^2y^4-y^5\right]_{y=0}^{y=1}dx$$

$$=\int_0^3\left(\frac{3}{2}x^2-1\right)dx=\left[\frac{1}{2}x^3-x\right]_0^3=\frac{27}{2}-3=\frac{21}{2}$$

14.

$$\iint_R \cos(x+2y)\,dA = \int_0^{\pi}\int_0^{\pi/2}\cos(x+2y)\,dy\,dx$$

$$=\int_0^{\pi}\left[\frac{1}{2}\sin(x+2y)\right]_{y=0}^{y=\pi/2}dx=\frac{1}{2}\int_0^{\pi}(\sin(x+\pi)-\sin x)\,dx$$

$$=\frac{1}{2}\left[-\cos(x+\pi)+\cos x\right]_0^{\pi}=\frac{1}{2}\left[-\cos 2\pi+\cos\pi-(-\cos\pi+\cos 0)\right]$$

$$=\frac{1}{2}(-1-1-(1+1))=-2$$

15.

$$\iint_R \frac{xy^2}{x^2+1}\,dA = \int_0^1\int_{-3}^3 \frac{xy^2}{x^2+1}\,dy\,dx = \int_0^1\frac{x}{x^2+1}\,dx\int_{-3}^3 y^2\,dy$$

$$=\left[\frac{1}{2}\ln(x^2+1)\right]_0^1\left[\frac{1}{3}y^3\right]_{-3}^3=\frac{1}{2}(\ln 2-\ln 1)\cdot\frac{1}{3}(27+27)=9\ln 2$$

16.

$$\iint_R \frac{1+x^2}{1+y^2}\,dA = \int_0^1\int_0^1\frac{1+x^2}{1+y^2}\,dy\,dx=\int_0^1(1+x^2)\,dx\int_0^1\frac{1}{1+y^2}\,dy$$

$$=\left[x+\frac{1}{3}x^3\right]_0^1\left[\tan^{-1}y\right]_0^1=\left(1+\frac{1}{3}-0\right)\left(\frac{\pi}{4}-0\right)=\frac{\pi}{3}$$

17. $\int_0^{\pi/6}\int_0^{\pi/3} x\sin(x+y)\,dy\,dx$

$=\int_0^{\pi/6}\left[-x\cos(x+y)\right]_{y=0}^{y=\pi/3}dx=\int_0^{\pi/6}\left[x\cos x-x\cos\left(x+\frac{\pi}{3}\right)\right]dx$

$=x\left[\sin x-\sin\left(x+\frac{\pi}{3}\right)\right]_0^{\pi/6}-\int_0^{\pi/6}\left[\sin x-\sin\left(x+\frac{\pi}{3}\right)\right]dx$

[by integrating by parts separately for each term]

$=\frac{\pi}{6}\left[\frac{1}{2}-1\right]-\left[-\cos x+\cos\left(x+\frac{\pi}{3}\right)\right]_0^{\pi/6}=-\frac{\pi}{12}-\left[-\frac{\sqrt{3}}{2}+0-\left(-1+\frac{1}{2}\right)\right]$

$=\frac{\sqrt{3}-1}{2}-\frac{\pi}{12}$

18. $\iint_R \frac{x}{1+xy}\,dA=\int_0^1\int_0^1 \frac{x}{1+xy}\,dy\,dx$

$=\int_0^1\left[\ln(1+xy)\right]_{y=0}^{y=1}dx=\int_0^1\left[\ln(1+x)-\ln 1\right]dx$

$=\int_0^1\ln(1+x)\,dx=\left[(1+x)\ln(1+x)-x\right]_0^1$

[by integrating by parts]

$=(2\ln 2-1)-(\ln 1-0)=2\ln 2-1$

19.

$\iint_R xye^{x^2y}dA=\int_0^2\int_0^1 xye^{x^2y}dx\,dy=\int_0^2\left[\frac{1}{2}e^{x^2y}\right]_{x=0}^{x=1}dy=\frac{1}{2}\int_0^2(e^y-1)\,dy$

$$=\frac{1}{2}\left[e^y-y\right]_0^2=\frac{1}{2}\left[(e^2-2)-(1-0)\right]=\frac{1}{2}(e^2-3)$$

20. $\int_0^1\int_1^2 \frac{x}{x^2+y^2}\,dx\,dy=\int_0^1\left[\frac{1}{2}\ln(x^2+y^2)\right]_{x=1}^{x=2}dy=\frac{1}{2}\int_0^1\left[\ln(4+y^2)-\ln(1+y^2)\right]dy$

To evaluate the first term, we integrate by parts with $u=\ln(4+y^2)\Rightarrow$

$du=\dfrac{2y}{4+y^2}\,dy$ and

$dv=dy \Rightarrow v=y$. Then

$$\int \ln(4+y^2)dy = y\ln(4+y^2)-\int \frac{2y^2}{4+y^2}dy = y\ln(4+y^2)-\int\left(2-\frac{8}{4+y^2}\right)dy$$

$$= y\ln(4+y^2)-2y+8\cdot\frac{1}{2}\tan^{-1}\left(\frac{y}{2}\right) = y\ln(4+y^2)-2y+4\tan^{-1}\left(\frac{y}{2}\right)$$

Similarly, $\int \ln(1+y^2)dy = y\ln(1+y^2)-2y+2\tan^{-1}y$. Thus,

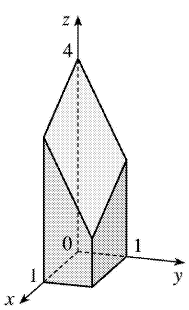
$$\int_0^1\int_1^2 \frac{x}{x^2+y^2}\,dx\,dy = \frac{1}{2}\int_0^1\left[\ln(4+y^2)-\ln(1+y^2)\right]dy$$

$$= \frac{1}{2}\left[y\ln(4+y^2)-2y+4\tan^{-1}\left(\frac{y}{2}\right)-y\ln(1+y^2)+2y-2\tan^{-1}y\right]_0^1$$

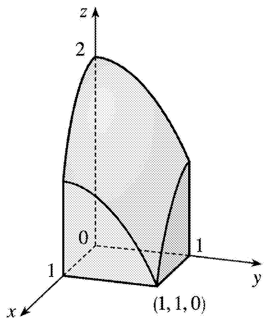
$$= \frac{1}{2}\left[\left(\ln 5+4\tan^{-1}\left(\frac{1}{2}\right)-\ln 2-2\tan^{-1}1\right)-0\right]$$

$$= \frac{1}{2}\left[\ln 5-\ln 2+4\tan^{-1}\left(\frac{1}{2}\right)-2\left(\frac{\pi}{4}\right)\right] = \frac{1}{2}\ln\frac{5}{2}+2\tan^{-1}\left(\frac{1}{2}\right)-\frac{\pi}{4}$$

21. $z=f(x,y)=4-x-2y\geq 0$ for $0\leq x\leq 1$ and $0\leq y\leq 1$.
So the solid is the region in the first octant which lies below the plane $z=4-x-2y$ and above $[0,1]\times[0,1]$.



22. $z=2-x^2-y^2\geq 0$ for $0\leq x\leq 1$ and $0\leq y\leq 1$. So the solid is the region in the first octant which lies below the circular paraboloid $z=2-x^2-y^2$ and above $[0,1]\times[0,1]$.



23.

$$V=\iint_R (12-3x-2y)\,dA=\int_{-2}^{3}\int_0^1 (12-3x-2y)\,dx\,dy=\int_{-2}^{3}\left[12x-\frac{3}{2}x^2-2xy\right]_{x=0}^{x=1}dy$$

$$=\int_{-2}^{3}\left(\frac{21}{2}-2y\right)dy=\left[\frac{21}{2}y-y^2\right]_{-2}^{3}=\frac{95}{2}$$

24.

$$V=\iint_R (4+x^2-y^2)\,dA=\int_{-1}^{1}\int_0^2 (4+x^2-y^2)\,dy\,dx=\int_{-1}^{1}\left[4y+x^2y-\frac{1}{3}y^3\right]_{y=0}^{y=2}dx$$

$$=\int_{-1}^{1}\left(2x^2+\frac{16}{3}\right)dx=\left[\frac{2}{3}x^3+\frac{16}{3}x\right]_{-1}^{1}=\frac{2}{3}+\frac{16}{3}+\frac{2}{3}+\frac{16}{3}=12$$

25.

$$V=\int_{-2}^{2}\int_{-1}^{1}\left(1-\frac{1}{4}x^2-\frac{1}{9}y^2\right)dx\,dy=4\int_0^2\int_0^1\left(1-\frac{1}{4}x^2-\frac{1}{9}y^2\right)dx\,dy$$

$$=4\int_0^2\left[x-\frac{1}{12}x^3-\frac{1}{9}y^2x\right]_{x=0}^{x=1}dy=4\int_0^2\left(\frac{11}{12}-\frac{1}{9}y^2\right)dy=4\left[\frac{11}{12}y-\frac{1}{27}y^3\right]_0^2=4\cdot\frac{83}{54}=\frac{166}{27}$$

26.

$$V=\int_{-1}^{1}\int_0^{\pi}(1+e^x\sin y)\,dy\,dx=\int_{-1}^{1}\left[y-e^x\cos y\right]_{y=0}^{y=\pi}dx=\int_{-1}^{1}(\pi+e^x-0+e^x)dx$$

$$=\int_{-1}^{1}(\pi+2e^x)dx=\left[\pi x+2e^x\right]_{-1}^{1}=2\pi+2e-\frac{2}{e}$$

27. Here we need the volume of the solid lying under the surface $z=x\sqrt{x^2+y}$ and above the square $R=[0,1]\times[0,1]$ in the xy –plane.

$$V \int_0^1\int_0^1 x\sqrt{x^2+y}\,dx\,dy=\int_0^1 \frac{1}{3}\left[(x^2+y)^{3/2}\right]_{x=0}^{x=1}dy=\frac{1}{3}\int_0^1\left[(1+y)^{3/2}-y^{3/2}\right]dy$$

$$=\frac{1}{3}\cdot\frac{2}{5}\left[(1+y)^{5/2}-y^{5/2}\right]_0^1=\frac{4}{15}\left(2\sqrt{2}-1\right)$$

28. Here we need the volume of the solid lying under the surface $z=1+(x-1)^2+4y^2$ and above the rectangle $R=[0,3]\times[0,2]$ in the xy –plane.

$$V=\int_0^3\int_0^2\left[1+(x-1)^2+4y^2\right]dy\,dx=\int_0^3\left[y+(x-1)^2y+\frac{4}{3}y^3\right]_{y=0}^{y=2}dx$$

$$=\int_0^3\left[2+2(x-1)^2+\frac{32}{3}\right]dx=\left[\frac{38}{3}x+\frac{2}{3}(x-1)^3\right]_0^3=44$$

29. In the first octant, $z\geq 0\Rightarrow y\leq 3$, so

$$V=\int_0^3\int_0^2(9-y^2)\,dx\,dy=\int_0^3\left[9x-y^2x\right]_{x=0}^{x=2}dy=\int_0^3(18-2y^2)\,dy=\left[18y-\frac{2}{3}y^3\right]_0^3=36$$

30. **(a)** Here we need the volume of the solid lying under the surface $z=6-xy$ and above the rectangle $R=[-2,2]\times[0,3]$ in the xy –plane.

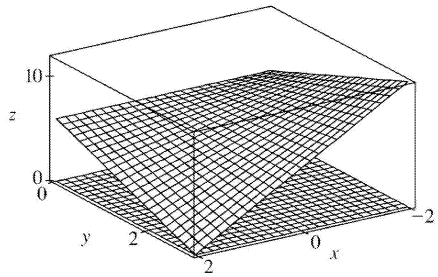
$$V=\int_{-2}^2\int_0^3(6-xy)\,dy\,dx$$

$$=\int_{-2}^2\left[6y-\frac{1}{2}xy^2\right]_{y=0}^{y=3}dx$$

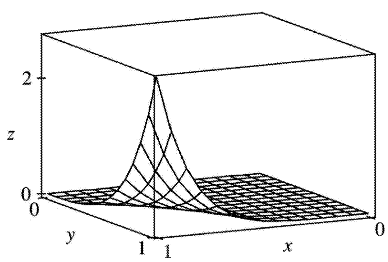
$$=\int_{-2}^2\left(18-\frac{9}{2}x\right)dx$$

$$=\left[18x-\frac{9}{4}x^2\right]_{-2}^2=72$$

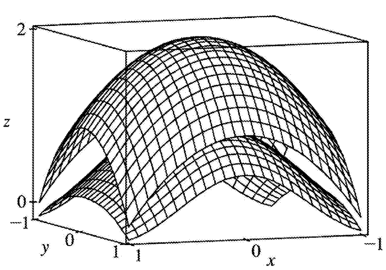
(b) The solid occupies the region between the two surfaces shown.



31. In Maple, we can calculate the integral by defining the integrand as f and then using the command int(int(f,x=0..1),y=0..1);.
In Mathematica, we can use the command Integrate[Integrate[f,{x,0,1}], {y,0,1}]. We find that $\iint_R x^5 y^3 e^{xy}\, dA = 21e - 57 \approx 0.0839$. We can use plot3d (in Maple) or Plot3d (in Mathematica) to graph the function.



32. In Maple, we can calculate the integral by defining f:=exp(–x^2)*cos(x^2+y^2); and g:=2–x^2–y^2; and then using the command evalf(int(int(g–f,x=–1..1),y=–1..1),5);. In Mathematica, we can use the command N[Integrate[Integrate[f,{x,0,1}],{y,0,1}],5].



In each of these commands, the 5 indicates that we want only five significant digits; this speeds up the calculation considerably. We find that $\iint_R \left[(2 - x^2 - y^2) - \left(e^{-x^2} \cos(x^2 + y^2) \right) \right] dA \approx 3.0271$. We can use the plot3d command (in Maple) or Plot3d (in Mathematica) to graph both functions on the same screen.

33. R is the rectangle $[-1,1] \times [0,5]$. Thus, $A(R) = 2 \cdot 5 = 10$ and

$$f_{\text{ave}} = \frac{1}{A(R)}\iint_R f(x,y)\,dA = \frac{1}{10}\int_0^5\int_{-1}^1 x^2 y\,dx\,dy = \frac{1}{10}\int_0^5 \left[\frac{1}{3}x^3 y\right]_{x=-1}^{x=1} dy = \frac{1}{10}\int_0^5 \frac{2}{3}y\,dy$$

$$= \frac{1}{10}\left[\frac{1}{3}y^2\right]_0^5 = \frac{5}{6}$$

34. $A(R)=4\cdot 1=4$, so

$$f_{\text{ave}} = \frac{1}{A(R)}\iint_R f(x,y)\,dA = \frac{1}{4}\int_0^4\int_0^1 e^y\sqrt{x+e^y}\,dy\,dx = \frac{1}{4}\int_0^4\left[\frac{2}{3}(x+e^y)^{3/2}\right]_{y=0}^{y=1} dx$$

$$= \frac{1}{4}\cdot\frac{2}{3}\int_0^4\left[(x+e)^{3/2}-(x+1)^{3/2}\right]dx = \frac{1}{6}\left[\frac{2}{5}(x+e)^{5/2}-\frac{2}{5}(x+1)^{5/2}\right]_0^4$$

$$= \frac{1}{6}\cdot\frac{2}{5}\left[(4+e)^{5/2}-5^{5/2}-e^{5/2}+1\right] = \frac{1}{15}\left[(4+e)^{5/2}-e^{5/2}-5^{5/2}+1\right]\approx 3.327$$

35. Let $f(x,y)=\dfrac{x-y}{(x+y)^3}$. Then a CAS gives $\int_0^1\int_0^1 f(x,y)\,dy\,dx=\frac{1}{2}$ and $\int_0^1\int_0^1 f(x,y)\,dx\,dy=-\frac{1}{2}$.
To explain the seeming violation of Fubini's Theorem, note that f has an infinite discontinuity at $(0,0)$ and thus does not satisfy the conditions of Fubini's Theorem. In fact, both iterated integrals involve improper integrals which diverge at their lower limits of integration.

36. **(a)** Loosely speaking, Fubini's Theorem says that the order of integration of a function of two variables does not affect the value of the double integral, while Clairaut's Theorem says that the order of differentiation of such a function does not affect the value of the second–order derivative. Also, both theorems require continuity (though Fubini's allows a finite number of smooth curves to contain discontinuities).
(b) To find g_{xy} , we first hold y constant and use the single–variable Fundamental Theorem of Calculus, Part 1: $g_x=\dfrac{d}{dx}g(x,y)=\dfrac{d}{dx}\int_a^x\left(\int_c^y f(s,t)\,dt\right)ds=\int_c^y f(x,t)\,dt$. Now we use the Fundamental Theorem again: $g_{xy}=\dfrac{d}{dy}\int_c^y f(x,t)\,dt=f(x,y)$.

To find g_{yx} , we first use Fubini's Theorem to find that $\int_a^x\int_c^y f(s,t)\,dt\,ds=\int_c^y\int_a^x f(s,t)\,dt\,ds$, and then use the Fundamental Theorem twice, as above, to get $g_{yx}=f(x,y)$. So $g_{xy}=g_{yx}=f(x,y)$.

1.

$$\int_0^1\int_0^{x^2}(x+2y)\,dy\,dx=\int_0^1\Big[xy+y^2\Big]_{y=0}^{y=x^2}dx=\int_0^1\Big[x(x^2)+(x^2)^2-0-0\Big]dx$$

$$=\int_0^1(x^3+x^4)\,dx=\left[\frac{1}{4}x^4+\frac{1}{5}x^5\right]_0^1=\frac{9}{20}$$

2.

$$\int_1^2\int_y^2 xy\,dx\,dy=\int_1^2\left[\frac{1}{2}x^2y\right]_{x=y}^{x=2}dy=\int_1^2\frac{1}{2}y(4-y^2)\,dy=\frac{1}{2}\int_1^2(4y-y^3)\,dy$$

$$=\frac{1}{2}\left[2y^2-\frac{1}{4}y^4\right]_1^2=\frac{1}{2}\left(8-4-2+\frac{1}{4}\right)=\frac{9}{8}$$

3.

$$\int_0^1\int_y^{e^y}\sqrt{x}\,dx\,dy=\int_0^1\left[\frac{2}{3}x^{3/2}\right]_{x=y}^{x=e^y}dy=\frac{2}{3}\int_0^1(e^{3y/2}-y^{3/2})\,dy=\frac{2}{3}\left[\frac{2}{3}e^{3y/2}-\frac{2}{5}y^{5/2}\right]_0^1$$

$$=\frac{2}{3}\left(\frac{2}{3}e^{3/2}-\frac{2}{5}-\frac{2}{3}e^0+0\right)=\frac{4}{9}e^{3/2}-\frac{32}{45}$$

4.

$$\int_0^1\int_x^{2-x}(x^2-y)\,dy\,dx=\int_0^1\left[x^2y-\frac{1}{2}y^2\right]_{y=x}^{y=2-x}dx=\int_0^1\left[x^2(2-x)-\frac{1}{2}(2-x)^2-x^2(x)+\frac{1}{2}x^2\right]dx$$

$$=\int_0^1(-2x^3+2x^2+2x-2)\,dx=\left[-\frac{1}{2}x^4+\frac{2}{3}x^3+x^2-2x\right]_0^1=-\frac{5}{6}$$

5.

$$\int_0^{\pi/2}\int_0^{\cos\theta}e^{\sin\theta}\,dr\,d\theta=\int_0^{\pi/2}\Big[re^{\sin\theta}\Big]_{r=0}^{r=\cos\theta}d\theta=\int_0^{\pi/2}(\cos\theta)e^{\sin\theta}\,d\theta=e^{\sin\theta}\Big]_0^{\pi/2}$$

$$=e^{\sin(\pi/2)}-e^0=e-1$$

6.

$$\int_0^1\int_0^v \sqrt{1-v^2}\,du\,dv = \int_0^1 \Big[u\sqrt{1-v^2}\,\Big]_{u=0}^{u=v} dv = \int_0^1 v\sqrt{1-v^2}\,dv = \left. -\frac{1}{3}(1-v^2)^{3/2}\right]_0^1$$

$$= -\frac{1}{3}(0-1) = \frac{1}{3}$$

7.

$$\iint_D x^3y^2\,dA = \int_0^2\int_{-x}^{x} x^3y^2\,dy\,dx = \int_0^2\left[\frac{1}{3}x^3y^3\right]_{y=-x}^{y=x} dx = \frac{1}{3}\int_0^2 2x^6\,dx$$

$$= \frac{2}{3}\left[\frac{1}{7}x^7\right]_0^2 = \frac{2}{21}\left[2^7-0\right] = \frac{256}{21}$$

8.

$$\iint_D \frac{4y}{x^3+2}\,dA = \int_1^2\int_0^{2x} \frac{4y}{x^3+2}\,dy\,dx = \int_1^2\left[\frac{2y^2}{x^3+2}\right]_{y=0}^{y=2x} dx = \int_1^2 \frac{8x^2}{x^3+2}\,dx$$

$$= \left.\frac{8}{3}\ln\left|x^3+2\right|\,\right]_1^2 = \frac{8}{3}(\ln 10-\ln 3) = \frac{8}{3}\ln\frac{10}{3}$$

9.

$$\int_0^1\int_0^{\sqrt{x}} \frac{2y}{x^2+1}\,dy\,dx = \int_0^1\left[\frac{y^2}{x^2+1}\right]_{y=0}^{y=\sqrt{x}} dx = \int_0^1 \frac{x}{x^2+1}\,dx$$

$$= \left.\frac{1}{2}\ln\left|x^2+1\right|\,\right]_0^1 = \frac{1}{2}(\ln 2-\ln 1) = \frac{1}{2}\ln 2$$

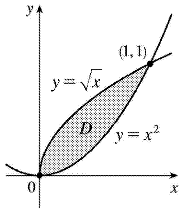
10. $\displaystyle\int_0^1\int_0^y e^{y^2}\,dx\,dy = \int_0^1\left[xe^{y^2}\right]_{x=0}^{x=y} dy = \int_0^1 ye^{y^2}\,dy = \left.\frac{1}{2}e^{y^2}\right]_0^1 = \frac{1}{2}(e-1)$

11. $\displaystyle\int_1^2\int_y^{y^3} e^{x/y}\,dx\,dy = \int_1^2\left[ye^{x/y}\right]_{x=y}^{x=y^3} dy = \int_1^2\left(ye^{y^2}-ey\right)dy = \left[\frac{1}{2}e^{y^2}-\frac{1}{2}ey^2\right]_1^2 = \frac{1}{2}(e^4-4e)$

12. $\displaystyle\int_0^1\int_0^y x\sqrt{y^2-x^2}\,dx\,dy = \int_0^1\left[-\frac{1}{3}(y^2-x^2)^{3/2}\right]_{x=0}^{x=y} dy = \frac{1}{3}\int_0^1 y^3\,dy = \left.\frac{1}{3}\cdot\frac{1}{4}y^4\right]_0^1 = \frac{1}{12}$

13. $\int_0^1 \int_0^{x^2} x\cos y\,dy\,dx = \int_0^1 [x\sin y]_{y=0}^{y=x^2} dx = \int_0^1 x\sin x^2\,dx = \left. -\frac{1}{2}\cos x^2 \right]_0^1 = \frac{1}{2}(1-\cos 1)$

14.

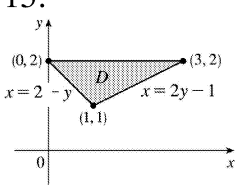


$$\int_0^1 \int_{x^2}^{\sqrt{x}} (x+y)\,dy\,dx = \int_0^1 \left[xy + \frac{1}{2}y^2 \right]_{y=x^2}^{y=\sqrt{x}} dx$$

$$= \int_0^1 \left(x^{3/2} + \frac{1}{2}x - x^3 - \frac{1}{2}x^4 \right) dx$$

$$= \left[\frac{2}{5}x^{5/2} + \frac{1}{4}x^2 - \frac{1}{4}x^4 - \frac{1}{10}x^5 \right]_0^1 = \frac{3}{10}$$

15.



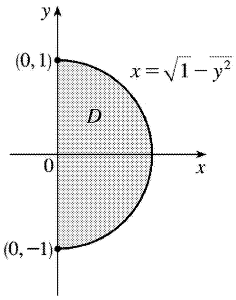
$$\int_1^2 \int_{2-y}^{2y-1} y^3\,dx\,dy = \int_1^2 \left[xy^3 \right]_{x=2-y}^{x=2y-1} dy$$

$$= \int_1^2 [(2y-1)-(2-y)]\,y^3\,dy$$

$$= \int_1^2 (3y^4 - 3y^3)\,dy = \left[\frac{3}{5}y^5 - \frac{3}{4}y^4 \right]_1^2$$

$$= \frac{96}{5} - 12 - \frac{3}{5} + \frac{3}{4} = \frac{147}{20}$$

16.



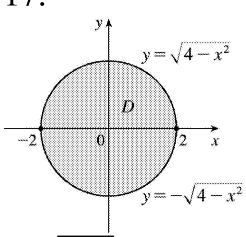
$$\iint_D xy^2\,dA=\int_{-1}^{1}\int_{0}^{\sqrt{1-y^2}} xy^2\,dx\,dy$$

$$=\int_{-1}^{1} y^2\left[\frac{1}{2}x^2\right]_{x=0}^{x=\sqrt{1-y^2}}dy=\frac{1}{2}\int_{-1}^{1} y^2(1-y^2)\,dy$$

$$=\frac{1}{2}\int_{-1}^{1}(y^2-y^4)\,dy=\frac{1}{2}\left[\frac{1}{3}y^3-\frac{1}{5}y^5\right]_{-1}^{1}$$

$$=\frac{1}{2}\left(\frac{1}{3}-\frac{1}{5}+\frac{1}{3}-\frac{1}{5}\right)=\frac{2}{15}$$

17.



$$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}(2x-y)\,dy\,dx$$

$$=\int_{-2}^{2}\left[2xy-\frac{1}{2}y^2\right]_{y=-\sqrt{4-x^2}}^{y=\sqrt{4-x^2}}dx$$

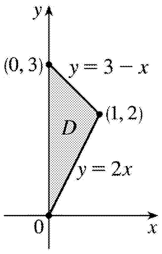
$$=\int_{-2}^{2}\left[2x\sqrt{4-x^2}-\frac{1}{2}\left(4-x^2\right)+2x\sqrt{4-x^2}+\frac{1}{2}\left(4-x^2\right)\right]dx$$

$$=\int_{-2}^{2}4x\sqrt{4-x^2}\,dx=\left.-\frac{4}{3}\left(4-x^2\right)^{3/2}\right]_{-2}^{2}=0$$

(Or, note that $4x\sqrt{4-x^2}$ is an odd function, so

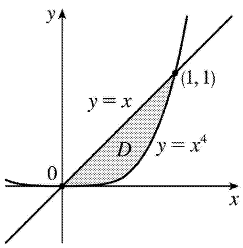
$\int_{-2}^{2} 4x\sqrt{4-x^2}\,dx=0$.)

18.



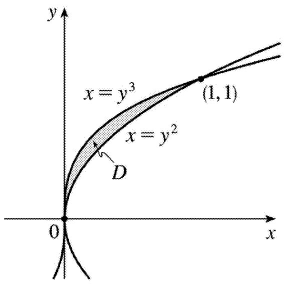
$$\iint_D xy\,dA = \int_0^1 \int_{2x}^{3-x} 2xy\,dy\,dx = \int_0^1 \left[xy^2\right]_{y=2x}^{y=3-x} dx$$
$$= \int_0^1 x[(3-x)^2-(2x)^2]\,dx$$
$$= \int_0^1 (-3x^3-6x^2+9x)\,dx$$
$$= \left[-\frac{3}{4}x^4-2x^3+\frac{9}{2}x^2\right]_0^1 = -\frac{3}{4}-2+\frac{9}{2}=\frac{7}{4}$$

19.



$$V = \int_0^1 \int_{x^4}^{x} (x+2y)\,dy\,dx$$
$$= \int_0^1 \left[xy+y^2\right]_{y=x^4}^{y=x} dx = \int_0^1 (2x^2-x^5-x^8)\,dx$$
$$= \left[\frac{2}{3}x^3-\frac{1}{6}x^6-\frac{1}{9}x^9\right]_0^1 = \frac{2}{3}-\frac{1}{6}-\frac{1}{9}=\frac{7}{18}$$

20.

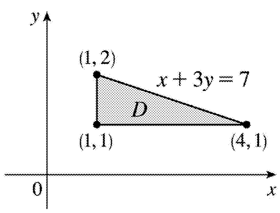


$$V = \int_0^1 \int_{y^3}^{y^2} (2x+y^2)\,dx\,dy$$

$$= \int_0^1 \Big[x^2 + xy^2 \Big]_{x=y^3}^{x=y^2} dy = \int_0^1 (2y^4 - y^6 - y^5)\,dy$$

$$= \left[\frac{2}{5} y^5 - \frac{1}{7} y^7 - \frac{1}{6} y^6 \right]_0^1 = \frac{19}{210}$$

21.

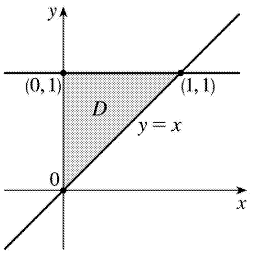


$$V = \int_1^2 \int_1^{7-3y} xy\,dx\,dy = \int_1^2 \left[\frac{1}{2} x^2 y \right]_{x=1}^{x=7-3y} dy$$

$$= \frac{1}{2} \int_1^2 (48y - 42y^2 + 9y^3)\,dy$$

$$= \frac{1}{2} \left[24y^2 - 14y^3 + \frac{9}{4} y^4 \right]_1^2 = \frac{31}{8}$$

22.

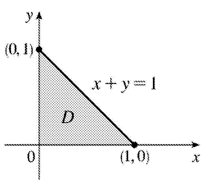


$$V=\int_0^1\int_x^1 (x^2+3y^2)\,dy\,dx$$

$$=\int_0^1\left[x^2y+y^3\right]_{y=x}^{y=1}dx=\int_0^1(x^2+1-2x^3)\,dx$$

$$=\left[\frac{1}{3}x^3+x-\frac{1}{2}x^4\right]_0^1=\frac{5}{6}$$

23.



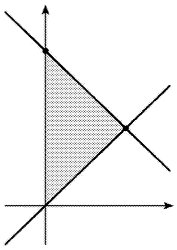
$$V=\int_0^1\int_0^{1-x}(1-x-y)\,dy\,dx$$

$$=\int_0^1\left[y-xy-\frac{y^2}{2}\right]_{y=0}^{y=1-x}dx$$

$$=\int_0^1\left[(1-x)^2-\frac{1}{2}(1-x)^2\right]dx$$

$$=\int_0^1\frac{1}{2}(1-x)^2\,dx=\left[-\frac{1}{6}(1-x)^3\right]_0^1=\frac{1}{6}$$

24.

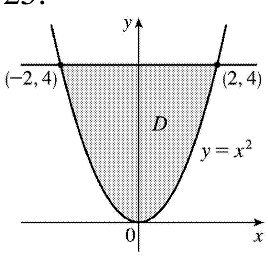


$$V = \int_0^1 \int_x^{2-x} x\,dy\,dx$$

$$= \int_0^1 x[\,y\,]_{y=x}^{y=2-x}\,dx = \int_0^1 (2x - 2x^2)\,dx$$

$$= \left[\,x^2 - \frac{2}{3}x^3\,\right]_0^1 = \frac{1}{3}$$

25.

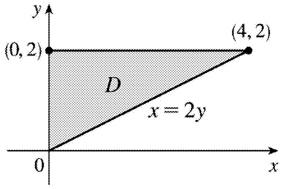


$$V = \int_{-2}^{2} \int_{x^2}^{4} x^2\,dy\,dx$$

$$= \int_{-2}^{2} x^2[y]_{y=x^2}^{y=4}\,dx = \int_{-2}^{2} (4x^2 - x^4)\,dx$$

$$= \left[\,\frac{4}{3}x^3 - \frac{1}{5}x^5\,\right]_{-2}^{2} = \frac{32}{3} - \frac{32}{5} + \frac{32}{3} - \frac{32}{5} = \frac{128}{15}$$

26.

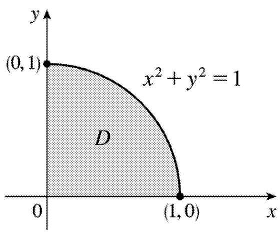


$$V = \int_0^2 \int_0^{2y} \sqrt{4-y^2}\, dx\,dy$$

$$= \int_0^2 \left[x\sqrt{4-y^2} \right]_{x=0}^{x=2y} dy = \int_0^2 2y\sqrt{4-y^2}\, dy$$

$$= \left[-\frac{2}{3}\left(4-y^2\right)^{3/2} \right]_0^2 = 0 + \frac{16}{3} = \frac{16}{3}$$

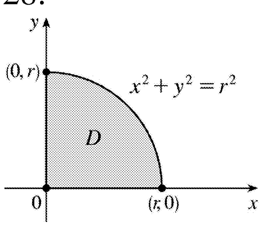
27.



$$V = \int_0^1 \int_0^{\sqrt{1-x^2}} y\,dy\,dx = \int_0^1 \left[\frac{y^2}{2} \right]_{y=0}^{y=\sqrt{1-x^2}} dx$$

$$= \int_0^1 \frac{1-x^2}{2}\, dx = \frac{1}{2}\left[x - \frac{1}{3}x^3 \right]_0^1 = \frac{1}{3}$$

28.



By symmetry, the desired volume V is 8 times the volume V_1 in the first octant. Now

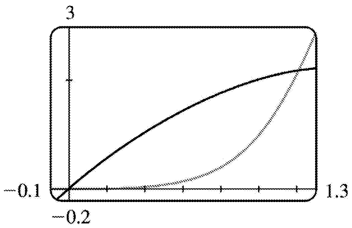
$$V_1 = \int_0^r \int_0^{\sqrt{r^2-y^2}} \sqrt{r^2-y^2}\, dx\,dy$$

$$= \int_0^r \left[x\sqrt{r^2-y^2} \right]_{x=0}^{x=\sqrt{r^2-y^2}} dy$$

$$= \int_0^r (r^2 - y^2)\,dy = \left[r^2 y - \frac{1}{3} y^3 \right]_0^r = \frac{2}{3} r^3$$

Thus $V = \frac{16}{3} r^3$.

29.



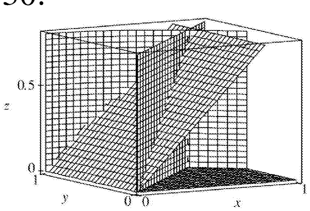
From the graph, it appears that the two curves intersect at $x=0$ and at $x \approx 1.213$. Thus the desired integral is

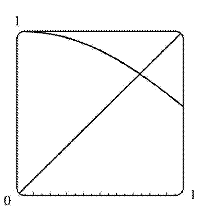
$$\iint_D x\,dA \approx \int_0^{1.213} \int_{x^4}^{3x-x^2} x\,dy\,dx = \int_0^{1.213} [xy]_{y=x^4}^{y=3x-x^2} dx$$

$$= \int_0^{1.213} (3x^2 - x^3 - x^5)\,dx = \left[x^3 - \frac{1}{4} x^4 - \frac{1}{6} x^6 \right]_0^{1.213}$$

$$\approx 0.713$$

30.





The desired solid is shown in the first graph. From the second graph, we estimate that $y=\cos x$ intersects $y=x$ at $x \approx 0.7391$. Therefore the volume of the solid is

$$V \approx \int_0^{0.7391} \int_x^{\cos x} z\,dy\,dx = \int_0^{0.7391} \int_x^{\cos x} x\,dy\,dx$$

$$= \int_0^{0.7391} [xy]_{y=x}^{y=\cos x} dx = \int_0^{0.7391} (x\cos x - x^2)\,dx$$

$$= \left[\cos x + x\sin x - \frac{1}{3}x^3 \right]_0^{0.7391} \approx 0.1024$$

Note: There is a different solid which can also be construed to satisfy the conditions stated in the exercise. This is the solid bounded by all of the given surfaces, as well as the plane $y=0$. In case you calculated the volume of this solid and want to check your work, its volume is

$$V \approx \int_0^{0.7391} \int_0^{x} x\,dy\,dx + \int_{0.7391}^{\pi/2} \int_0^{\cos x} x\,dy\,dx \approx 0.4684 .$$

31. The two bounding curves $y=1-x^2$ and $y=x^2-1$ intersect at $(\pm 1,0)$ with $1-x^2 \geq x^2-1$ on $[-1,1]$. Within this region, the plane $z=2x+2y+10$ is above the plane $z=2-x-y$, so

$$V = \int_{-1}^{1} \int_{x^2-1}^{1-x^2} (2x+2y+10)\,dy\,dx - \int_{-1}^{1} \int_{x^2-1}^{1-x^2} (2-x-y)\,dy\,dx$$

$$= \int_{-1}^{1} \int_{x^2-1}^{1-x^2} (2x+2y+10-(2-x-y))\,dy\,dx = \int_{-1}^{1} \int_{x^2-1}^{1-x^2} (3x+3y+8)\,dy\,dx$$

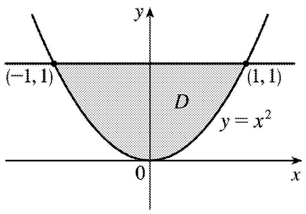
$$= \int_{-1}^{1} \left[3xy + \frac{3}{2}y^2 + 8y \right]_{y=x^2-1}^{y=1-x^2} dx$$

$$= \int_{-1}^{1} \left[3x(1-x^2) + \frac{3}{2}(1-x^2)^2 + 8(1-x^2) - 3x(x^2-1) - \frac{3}{2}(x^2-1)^2 - 8(x^2-1) \right] dx$$

$$= \int_{-1}^{1} (-6x^3 - 16x^2 + 6x + 16)\,dx = \left[-\frac{3}{2}x^4 - \frac{16}{3}x^3 + 3x^2 + 16x \right]_{-1}^{1}$$

$$= -\frac{3}{2} - \frac{16}{3} + 3 + 16 + \frac{3}{2} - \frac{16}{3} - 3 + 16 = \frac{64}{3}$$

32. The two planes intersect in the line $y=1$, $z=3$, so the region of integration is the plane region enclosed by the parabola $y=x^2$ and the line $y=1$. We have $2+y \geq 3y$ for $0 \leq y \leq 1$, so the solid region is bounded above by $z=2+y$ and bounded below by $z=3y$.



$$V=\int_{-1}^{1}\int_{x^2}^{1}(2+y)\,dy\,dx-\int_{-1}^{1}\int_{x^2}^{1}(3y)\,dy\,dx=\int_{-1}^{1}\int_{x^2}^{1}(2+y-3y)\,dy\,dx=\int_{-1}^{1}\int_{x^2}^{1}(2-2y)\,dy\,dx$$

$$=\int_{-1}^{1}\left[2y-y^2\right]_{y=x^2}^{y=1}dx=\int_{-1}^{1}(1-2x^2+x^4)\,dx=\left.x-\frac{2}{3}x^3+\frac{1}{5}x^5\right]_{-1}^{1}=\frac{16}{15}$$

33. The two bounding curves $y=x^3-x$ and $y=x^2+x$ intersect at the origin and at $x=2$, with $x^2+x>x^3-x$ on $(0,2)$. Using a CAS, we find that the volume is

$$V=\int_0^2\int_{x^3-x}^{x^2+x} z\,dy\,dx=\int_0^2\int_{x^3-x}^{x^2+x}(x^3y^4+xy^2)\,dy\,dx=\frac{13{,}984{,}735{,}616}{14{,}549{,}535}$$

34. For $|x|\le 1$ and $|y|\le 1$, $2x^2+y^2<8-x^2-2y^2$. Also, the cylinder is described by the inequalities $-1\le x\le 1$, $-\sqrt{1-x^2}\le y\le\sqrt{1-x^2}$. So the volume is given by

$$V=\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\left[(8-x^2-2y^2)-(2x^2+y^2)\right]dy\,dx=\frac{13\pi}{2}$$

[using a CAS]

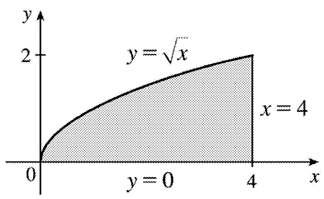
35. The two surfaces intersect in the circle $x^2+y^2=1$, $z=0$ and the region of integration is the disk D : $x^2+y^2\le 1$. Using a CAS, the volume is $\iint_D(1-x^2-y^2)\,dA=\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}(1-x^2-y^2)\,dy\,dx=\frac{\pi}{2}$.

36. The projection onto the xy –plane of the intersection of the two surfaces is the circle $x^2+y^2=2y\Rightarrow$ $x^2+y^2-2y=0\Rightarrow x^2+(y-1)^2=1$, so the region of integration is given by $-1\le x\le 1$,

$1-\sqrt{1-x^2}\le y\le 1+\sqrt{1-x^2}$. In this region, $2y\ge x^2+y^2$ so, using a CAS, the volume is

$$V=\int_{-1}^{1}\int_{1-\sqrt{1-x^2}}^{1+\sqrt{1-x^2}}[2y-(x^2+y^2)]\,dy\,dx=\frac{\pi}{2}\ .$$

37.



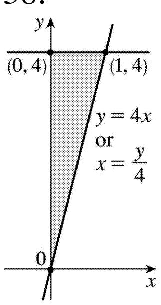
Because the region of integration is

$$D=\{(x,y)\mid 0\le y\le\sqrt{x},0\le x\le 4\}$$
$$=\{(x,y)\mid y^2\le x\le 4,0\le y\le 2\}$$

we have

$$\int_0^4\int_0^{\sqrt{x}} f(x,y)\,dy\,dx=\iint_D f(x,y)\,dA=\int_0^2\int_{y^2}^4 f(x,y)\,dx\,dy\ .$$

38.



Because the region of integration is

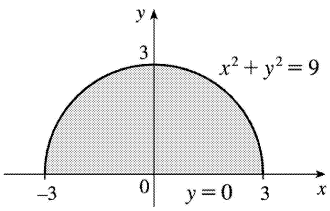
$$D=\{(x,y)\mid 4x\le y\le 4,0\le x\le 1\}$$
$$=\left\{(x,y)\mid 0\le x\le\frac{y}{4},0\le y\le 4\right\}$$

we have

$$\int_0^1\int_{4x}^4 f(x,y)\,dy\,dx=\iint_D f(x,y)\,dA$$

$$=\int_0^4 \int_0^{y/4} f(x,y)\,dx\,dy$$

39.



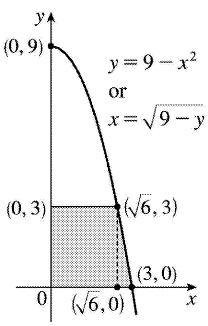
Because the region of integration is

$$D = \left\{ (x,y) \mid -\sqrt{9-y^2} \le x \le \sqrt{9-y^2}, 0 \le y \le 3 \right\}$$
$$= \left\{ (x,y) \mid 0 \le y \le \sqrt{9-x^2}, -3 \le x \le 3 \right\}$$

we have

$$\int_0^3 \int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} f(x,y)\,dx\,dy = \iint_D f(x,y)\,dA = \int_{-3}^3 \int_0^{\sqrt{9-x^2}} f(x,y)\,dy\,dx$$

40.



To reverse the order, we must break the region into two separate type I regions. Because the region of integration is

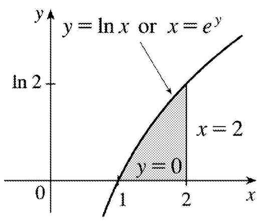
$$D = \left\{ (x,y) \mid 0 \le x \le \sqrt{9-y}, 0 \le y \le 3 \right\}$$
$$= \left\{ (x,y) \mid 0 \le y \le 3, 0 \le x \le \sqrt{6} \right\} \cup \left\{ (x,y) \mid 0 \le y \le 9-x^2, \sqrt{6} \le x \le 3 \right\}$$

we have

$$\int_0^3 \int_0^{\sqrt{9-y}} f(x,y)\,dx\,dy = \iint_D f(x,y)\,dA$$

$$= \int_0^{\sqrt{6}} \int_0^3 f(x,y)\,dy\,dx + \int_{\sqrt{6}}^3 \int_0^{9-x^2} f(x,y)\,dy\,dx$$

41.



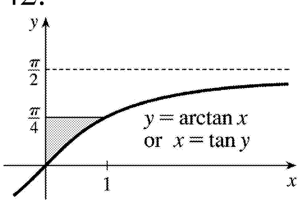
Because the region of integration is

$$D = \{(x,y) \mid 0 \le y \le \ln x, 1 \le x \le 2\}$$
$$= \{(x,y) \mid e^y \le x \le 2, 0 \le y \le \ln 2\}$$

we have

$$\int_1^2 \int_0^{\ln x} f(x,y)\,dy\,dx = \iint_D f(x,y)\,dA$$
$$= \int_0^{\ln 2} \int_{e^y}^2 f(x,y)\,dx\,dy$$

42.



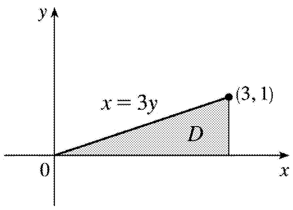
Because the region of integration is

$$D = \left\{ (x,y) \mid \arctan x \le y \le \frac{\pi}{4}, 0 \le x \le 1 \right\}$$
$$= \left\{ (x,y) \mid 0 \le x \le \tan y, 0 \le y \le \frac{\pi}{4} \right\}$$

we have

$$\int_0^1 \int_{\arctan x}^{\pi/4} f(x,y)\,dy\,dx = \iint_D f(x,y)\,dA$$
$$= \int_0^{\pi/4} \int_0^{\tan y} f(x,y)\,dx\,dy$$

43.

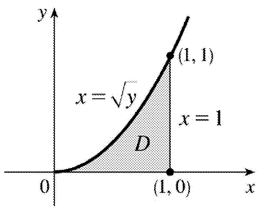


$$\int_0^1 \int_{3y}^3 e^{x^2}\, dx\, dy = \int_0^3 \int_0^{x/3} e^{x^2}\, dy\, dx$$

$$= \int_0^3 \left[e^{x^2} y \right]_{y=0}^{y=x/3} dx = \int_0^3 \left(\frac{x}{3} \right) e^{x^2}\, dx$$

$$= \frac{1}{6}\, e^{x^2} \Big]_0^3 = \frac{e^9 - 1}{6}$$

44.

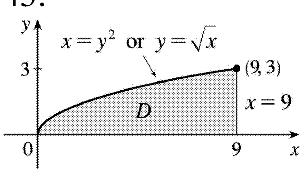


$$\int_0^1 \int_{\sqrt{y}}^1 \sqrt{x^3+1}\, dx\, dy = \int_0^1 \int_0^{x^2} \sqrt{x^3+1}\, dy\, dx$$

$$= \int_0^1 \left[\sqrt{x^3+1}\; y \right]_{y=0}^{y=x^2} dx = \int_0^1 x^2 \sqrt{x^3+1}\, dx$$

$$= \frac{2}{9} (x^3+1)^{3/2} \Big]_0^1 = \frac{2}{9} \left(2^{3/2} - 1 \right)$$

45.

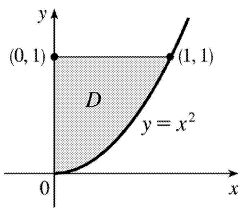


$$\int_0^3 \int_{y^2}^9 y \cos x^2\, dx\, dy = \int_0^9 \int_0^{\sqrt{x}} y \cos x^2\, dy\, dx$$

$$=\int_0^9 \cos x^2 \left[\frac{y^2}{2} \right]_{y=0}^{y=\sqrt{x}} dx = \int_0^9 \frac{1}{2} x\cos x^2 \, dx$$

$$= \left. \frac{1}{4} \sin x^2 \right]_0^9 = \frac{1}{4} \sin 81$$

46.



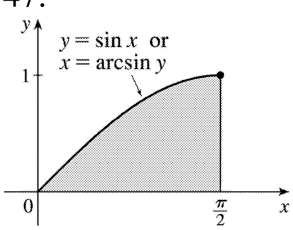
$$\int_0^1 \int_{x^2}^1 x^3 \sin (y^3) \, dy dx = \int_0^1 \int_0^{\sqrt{y}} x^3 \sin (y^3) \, dx dy$$

$$= \int_0^1 \left[\frac{x^4}{4} \sin (y^3) \right]_{x=0}^{x=\sqrt{y}} dy$$

$$= \int_0^1 \frac{1}{4} y^2 \sin (y^3) \, dy$$

$$= \left. -\frac{1}{12} \cos (y^3) \right]_0^1 = \frac{1}{12} (1-\cos 1)$$

47.



$$\int_0^1 \int_{\arcsin y}^{\pi/2} \cos x \sqrt{1+\cos^2 x} \, dx dy$$

$$= \int_0^{\pi/2} \int_0^{\sin x} \cos x \sqrt{1+\cos^2 x} \, dy dx$$

$$= \int_0^{\pi/2} \cos x \sqrt{1+\cos^2 x} \, [y]_{y=0}^{y=\sin x} dx$$

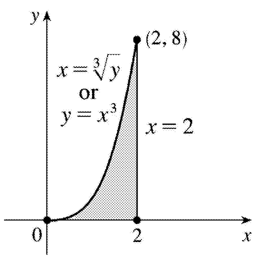
$$= \int_0^{\pi/2} \cos x \sqrt{1+\cos^2 x} \sin x\, dx$$

$[\text{Let } u=\cos x, du=-\sin x\,dx, dx=du/(-\sin x)]$

$$= \int_1^0 -u\sqrt{1+u^2}\, du = \left. -\frac{1}{3}\left(1+u^2\right)^{3/2}\right]_1^0$$

$$= \frac{1}{3}\left(\sqrt{8}-1\right) = \frac{1}{3}\left(2\sqrt{2}-1\right)$$

48.



$$\int_0^8 \int_{\sqrt[3]{y}}^2 e^{x^4}\, dx\, dy = \int_0^2 \int_0^{x^3} e^{x^4}\, dy\, dx$$

$$= \int_0^2 e^{x^4}\,[y]_{y=0}^{y=x^3}\, dx = \int_0^2 x^3 e^{x^4}\, dx$$

$$= \left. \frac{1}{4} e^{x^4}\right]_0^2 = \frac{1}{4}(e^{16}-1)$$

49.

$$D = \{(x,y) \mid 0 \le x \le 1, -x+1 \le y \le 1\} \cup \{(x,y) \mid -1 \le x \le 0, x+1 \le y \le 1\}$$
$$\cup \{(x,y) \mid 0 \le x \le 1, -1 \le y \le x-1\} \cup \{(x,y) \mid -1 \le x \le 0, -1 \le y \le -x-1\}\ ,$$

all type I.

$$\iint_D x^2\, dA = \int_0^1 \int_{1-x}^1 x^2\, dy\, dx + \int_{-1}^0 \int_{x+1}^1 x^2\, dy\, dx + \int_0^1 \int_{-1}^{x-1} x^2\, dy\, dx + \int_{-1}^0 \int_{-1}^{-x-1} x^2\, dy\, dx$$

$$= 4\int_0^1 \int_{1-x}^1 x^2\, dy\, dx [\text{ by symmetry of the regions and because } f(x,y)=x^2 \ge 0]$$

$$= 4\int_0^1 x^3\, dx = 4\left[\frac{1}{4}x^4\right]_0^1 = 1$$

50.

$$D=\left\{(x,y)\mid -1\le x\le 0, -1\le y\le 1+x^2\right\}\cup\left\{(x,y)\mid 0\le x\le 1, \sqrt{x}\le y\le 1+x^2\right\}$$
$$\cup\left\{(x,y)\mid 0\le x\le 1, -1\le y\le -\sqrt{x}\right\},$$

all type I.

$$\iint_D xy\,dA = \int_{-1}^{0}\int_{-1}^{1+x^2} xy\,dy\,dx+\int_{0}^{1}\int_{\sqrt{x}}^{1+x^2} xy\,dy\,dx+\int_{0}^{1}\int_{-1}^{-\sqrt{x}} xy\,dy\,dx$$
$$=\int_{-1}^{0}\left[\frac{1}{2}xy^2\right]_{y=-1}^{y=1+x^2}dx+\int_{0}^{1}\left[\frac{1}{2}xy^2\right]_{y=\sqrt{x}}^{y=1+x^2}dx+\int_{0}^{1}\left[\frac{1}{2}xy^2\right]_{y=-1}^{y=-\sqrt{x}}dx$$
$$=\int_{-1}^{0}\left(x^3+\frac{1}{2}x^5\right)dx+\int_{0}^{1}\frac{1}{2}(x^5+2x^3-x^2+x)\,dx+\int_{0}^{1}\frac{1}{2}(x^2-x)\,dx$$
$$=\left[\frac{1}{4}x^4+\frac{1}{12}x^6\right]_{-1}^{0}+\frac{1}{2}\left[\frac{1}{6}x^6+\frac{1}{2}x^4-\frac{1}{3}x^3+\frac{1}{2}x^2\right]_{0}^{1}+\frac{1}{2}\left[\frac{1}{3}x^3-\frac{1}{2}x^2\right]_{0}^{1}$$
$$=-\frac{1}{3}+\frac{5}{12}-\frac{1}{12}=0$$

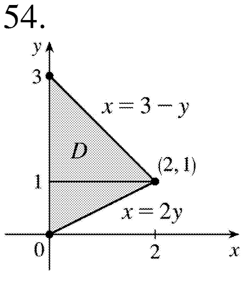
51. For $D=[0,1]\times[0,1]$, $0\le\sqrt{x^3+y^3}\le\sqrt{2}$ and $A(D)=1$, so $0\le\iint_D\sqrt{x^3+y^3}\,dA\le\sqrt{2}$.

52. Since $D=\left\{(x,y)\mid x^2+y^2\le\frac{1}{4}\right\}$, $1=e^0\le e^{x^2+y^2}\le e^{1/4}$ and $A(D)=\frac{\pi}{4}$, so $\frac{\pi}{4}\le\iint_D e^{x^2+y^2}\,dA\le(e^{1/4})\frac{\pi}{4}$

.

53. Since $m\le f(x,y)\le M$, $\iint_D m\,dA\le\iint_D f(x,y)\,dA\le\iint_D M\,dA$ by (8) $\Rightarrow$
$m\iint_D 1\,dA\le\iint_D f(x,y)\,dA\le M\iint_D 1\,dA$ by (7) $\Rightarrow mA(D)\le\iint_D f(x,y)\,dA\le MA(D)$ by (10).

54.



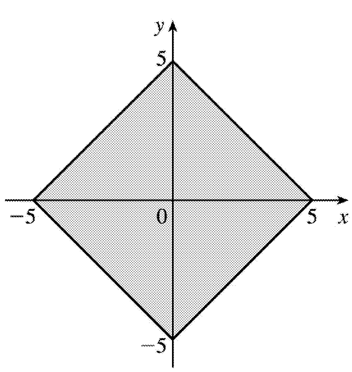
$$\iint_D f(x,y)\,dA = \int_0^1 \int_0^{2y} f(x,y)\,dx\,dy + \int_1^3 \int_0^{3-y} f(x,y)\,dx\,dy$$
$$= \int_0^2 \int_{x/2}^{3-x} f(x,y)\,dy\,dx$$

55. $\iint_D (x^2 \tan x + y^3 + 4)\, dA = \iint_D x^2 \tan x\, dA + \iint_D y^3\, dA + \iint_D 4\, dA$. But $x^2 \tan x$ is an odd function of x and D is symmetric with respect to the y –axis, so $\iint_D x^2 \tan x\, dA = 0$. Similarly, y^3 is an odd function of y and D is symmetric with respect to the x –axis, so $\iint_D y^3\, dA = 0$. Thus

$$\iint_D (x^2 \tan x + y^3 + 4)\, dA = 4\iint_D dA = 4(\text{area of } D) = 4\cdot \pi \left(\sqrt{2}\right)^2 = 8\pi$$

56. First, 0in0.19in] The region D , shown in the figure, is symmetric with respect to the y –axis and $3x$ is an odd function of x , so $\iint_D 3x\, dA = 0$. Similarly, $4y$ is an odd function of y and D is symmetric with respect to the x –axis, so $\iint_D 4y\, dA = 0$. Then

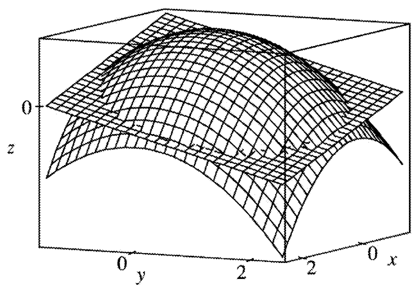
$$\iint_D (2-3x+4y)\, dA = \iint_D 2\, dA = 2\iint_D dA$$
$$= 2(\text{area of } D) = 2(50)$$
$$= 100$$



57. Since $\sqrt{1-x^2-y^2} \geq 0$, we can interpret $\iint_D \sqrt{1-x^2-y^2}\, dA$ as the volume of the solid that lies below

the graph of $z=\sqrt{1-x^2-y^2}$ and above the region D in the xy –plane. $z=\sqrt{1-x^2-y^2}$ is equivalent to $x^2+y^2+z^2=1$, $z\geq 0$ which meets the xy –plane in the circle $x^2+y^2=1$, the boundary of D . Thus, the solid is an upper hemisphere of radius 1 which has volume $\frac{1}{2}\left[\frac{4}{3}\pi(1)^3\right]=\frac{2}{3}\pi$.

58. To find the equations of the boundary curves, we require that the z –values of the two surfaces be the same. In Maple, we use the command solve(4–x^2–y^2=1–x–y,y);
and in Mathematica, we use Solve[4–x^2–y^ 2==1–x–y,y] . We find that the curves have equations $y=\frac{1\pm\sqrt{13+4x-4x^2}}{2}$.



To find the two points of intersection of these curves, we use the CAS to solve $13+4x-4x^2=0$, finding that $x=\frac{1\pm\sqrt{14}}{2}$. So, using the CAS to evaluate the integral, the volume of intersection is

$$V=\int_{(1-\sqrt{14})/2}^{(1+\sqrt{14})/2}\int_{\left(1-\sqrt{13+4x-4x^2}\right)/2}^{\left(1+\sqrt{13+4x-4x^2}\right)/2}[(4-x^2-y^2)-(1-x-y)]\,dy\,dx=\frac{49\pi}{8} .$$

1. The region R is more easily described by polar coordinates: $R=\{(r,\theta)|0\le r\le 2, 0\le\theta\le 2\pi\}$. Thus $\iint_R f(x,y)\,dA=\int_0^{2\pi}\int_0^2 f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$.

2. The region R is more easily described by rectangular coordinates: $R=\{(x,y)|0\le x\le 2, 0\le y\le 2-x\}$. Thus $\iint_R f(x,y)\,dA=\int_0^2\int_0^{2-x} f(x,y)\,dy\,dx$.

3. The region R is more easily described by rectangular coordinates: $R=\{(x,y)|-2\le x\le 2, x\le y\le 2\}$. Thus $\iint_R f(x,y)\,dA=\int_{-2}^2\int_x^2 f(x,y)\,dy\,dx$.

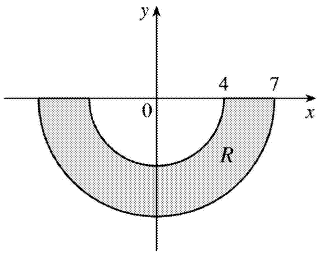
4. The region R is more easily described by polar coordinates: $R=\left\{(r,\theta)|1\le r\le 3, 0\le\theta\le\frac{\pi}{2}\right\}$. Thus $\iint_R f(x,y)\,dA=\int_0^{\pi/2}\int_1^3 f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$.

5. The region R is more easily described by polar coordinates: $R=\{(r,\theta)|2\le r\le 5, 0\le\theta\le 2\pi\}$. Thus $\iint_R f(x,y)\,dA=\int_0^{2\pi}\int_2^5 f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$.

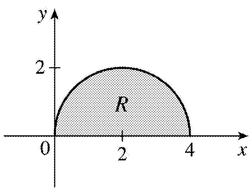
6. The region R is more easily described by polar coordinates: $R=\left\{(r,\theta)|0\le r\le 2\sqrt{2}, \frac{\pi}{4}\le\theta\le\frac{5\pi}{4}\right\}$. Thus $\iint_R f(x,y)\,dA=\int_{\pi/4}^{5\pi/4}\int_0^{2\sqrt{2}} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$.

7. The integral $\int_\pi^{2\pi}\int_4^7 r\,dr\,d\theta$ represents the area of the region $R=\{(r,\theta)|4\le r\le 7, \pi\le\theta\le 2\pi\}$, the lower half of a ring.

$$\int_\pi^{2\pi}\int_4^7 r\,dr\,d\theta = \left(\int_\pi^{2\pi} d\theta\right)\left(\int_4^7 r\,dr\right)$$
$$= [\theta]_\pi^{2\pi}\left[\frac{1}{2}r^2\right]_4^7=\pi\cdot\frac{1}{2}(49-16)=\frac{33\pi}{2}$$



8. The integral $\int_0^{\pi/2} \int_0^{4\cos\theta} r\,dr\,d\theta$ represents the area of the region $R=\{(r,\theta)\,|\,0\le r\le 4\cos\theta, 0\le\theta\le\pi/2\}$. Since $r=4\cos\theta \Leftrightarrow r^2=4r\cos\theta \Leftrightarrow x^2+y^2=4x \Leftrightarrow (x-2)^2+y^2=4$, R is the portion in the first quadrant of a circle of radius 2 with center $(2,0)$.



$$\int_0^{\pi/2} \int_0^{4\cos\theta} r\,dr\,d\theta = \int_0^{\pi/2} \left[\frac{1}{2} r^2 \right]_{r=0}^{r=4\cos\theta} d\theta = \int_0^{\pi/2} 8\cos^2\theta\, d\theta$$

$$= \int_0^{\pi/2} 4(1+\cos 2\theta)\,d\theta = 4\left[\theta + \frac{1}{2}\sin 2\theta \right]_0^{\pi/2} = 2\pi$$

9. The disk D can be described in polar coordinates as $D=\{(r,\theta)\,|\,0\le r\le 3, 0\le\theta\le 2\pi\}$. Then

$$\iint_D xy\,dA = \int_0^{2\pi}\int_0^3 (r\cos\theta)(r\sin\theta)\,r\,dr\,d\theta = \left(\int_0^{2\pi} \sin\theta\cos\theta\, d\theta \right)\left(\int_0^3 r^3\, dr \right)$$

$$= \left[\frac{1}{2}\sin^2\theta \right]_0^{2\pi} \left[\frac{1}{4} r^4 \right]_0^3 = 0$$

10.

$$\iint_R (x+y)\,dA = \int_{\pi/2}^{3\pi/2}\int_1^2 (r\cos\theta + r\sin\theta)\,r\,dr\,d\theta = \int_{\pi/2}^{3\pi/2}\int_1^2 r^2(\cos\theta + \sin\theta\, dr\,d\theta$$

$$= \left(\int_{\pi/2}^{3\pi/2} (\cos\theta+\sin\theta\, d\theta \right)\left(\int_1^2 r^2\, dr \right) = [\sin\theta - \cos\theta]_{\pi/2}^{3\pi/2} \left[\frac{1}{3} r^3 \right]_1^2$$

$$= (-1-0-1+0)\left(\frac{8}{3} - \frac{1}{3} \right) = -\frac{14}{3}$$

11.

$$\iint_R \cos(x^2+y^2)\,dA = \int_0^{\pi}\int_0^3 \cos(r^2)\,r\,dr\,d\theta = \left(\int_0^{\pi} d\theta\right)\left(\int_0^3 r\cos(r^2)\,dr\right)$$

$$=[\theta]_0^{\pi}\left[\frac{1}{2}\sin(r^2)\right]_0^3 = \pi\cdot\frac{1}{2}(\sin 9-\sin 0) = \frac{\pi}{2}\sin 9$$

12.

$$\iint_R \sqrt{4-x^2-y^2}\,dA = \int_{-\pi/2}^{\pi/2}\int_0^2 \sqrt{4-r^2}\,r\,dr\,d\theta = \left(\int_{-\pi/2}^{\pi/2} d\theta\right)\left(\int_0^2 r\sqrt{4-r^2}\,dr\right)$$

$$=[\theta]_{-\pi/2}^{\pi/2}\left[-\frac{1}{2}\cdot\frac{2}{3}(4-r^2)^{3/2}\right]_0^2 = \left(\frac{\pi}{2}+\frac{\pi}{2}\right)\left(-\frac{1}{3}(0-4^{3/2})\right) = \frac{8}{3}\pi$$

13.

$$\iint_D e^{-x^2-y^2}\,dA = \int_{-\pi/2}^{\pi/2}\int_0^2 e^{-r^2}\,r\,dr\,d\theta = \left(\int_{-\pi/2}^{\pi/2} d\theta\right)\left(\int_0^2 re^{-r^2}\,dr\right)$$

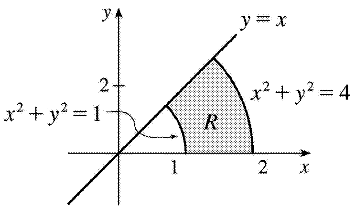
$$=[\theta]_{-\pi/2}^{\pi/2}\left[-\frac{1}{2}e^{-r^2}\right]_0^2 = \pi\left(-\frac{1}{2}\right)(e^{-4}-e^0) = \frac{\pi}{2}(1-e^{-4})$$

14. $\iint_R ye^x\,dA = \int_0^{\pi/2}\int_0^5 (r\sin\theta)e^{r\cos\theta}\,r\,dr\,d\theta = \int_0^5\int_0^{\pi/2} r^2\sin\theta\, e^{r\cos\theta}\,d\theta\,dr$. First we integrate $\int_0^{\pi/2} r^2\sin\theta\, e^{r\cos\theta}\,d\theta$: Let $u=r\cos\theta \Rightarrow du=-r\sin\theta\,d\theta$, and $\int_0^{\pi/2} r^2\sin\theta\, e^{r\cos\theta}\,d\theta = \int_{u=r}^{u=0} -r\,e^u\,du = -r[e^0-e^r] = re^r-r$. Then $\int_0^5\int_0^{\pi/2} r^2\sin\theta\, e^{r\cos\theta}\,d\theta\,dr = \int_0^5 (re^r-r)\,dr = \left[re^r-e^r-\frac{1}{2}r^2\right]_0^5 = 4e^5-\frac{23}{2}$, where we integrated by parts in the first term.

15. R is the region shown in the figure, and can be described by $R=\{(r,\theta)\mid 0\le\theta\le\pi/4, 1\le r\le 2\}$.

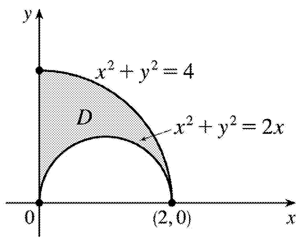
Thus $\iint_R \arctan(y/x)\,dA = \int_0^{\pi/4}\int_1^2 \arctan(\tan\theta)\,r\,dr\,d\theta$

since $y/x=\tan\theta$. Also, $\arctan(\tan\theta)=\theta$ for $0\le\theta\le\pi/4$,
so the integral becomes



$$\int_0^{\pi/4}\int_1^2 \theta\, r\, dr\, d\theta = \int_0^{\pi/4} \theta\, d\theta \int_1^2 r\, dr = \left[\frac{1}{2}\theta^2\right]_0^{\pi/4}\left[\frac{1}{2}r^2\right]_1^2 = \frac{\pi^2}{32}\cdot\frac{3}{2} = \frac{3}{64}\pi^2 .$$

16.



$$\iint_D x\,dA = \iint_{\substack{x^2+y^2\le 4\\ x\ge 0, y\ge 0}} x\,dA - \iint_{\substack{(x-1)^2+y^2\le 1\\ y\ge 0}} x\,dA$$

$$= \int_0^{\pi/2}\int_0^2 r^2\cos\theta\, dr\, d\theta - \int_0^{\pi/2}\int_0^{2\cos\theta} r^2\cos\theta\, dr\, d\theta$$

$$= \int_0^{\pi/2} \frac{1}{3}(8\cos\theta)\,d\theta - \int_0^{\pi/2}\frac{1}{3}(8\cos^4\theta)\,d\theta$$

$$= \frac{8}{3} - \frac{8}{12}\left[\cos^3\theta\sin\theta + \frac{3}{2}(\theta + \sin\theta\cos\theta)\right]_0^{\pi/2}$$

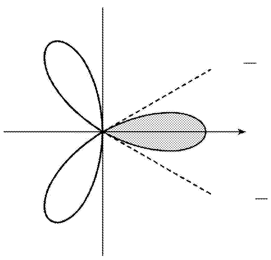
$$= \frac{8}{3} - \frac{2}{3}\left[0 + \frac{3}{2}\left(\frac{\pi}{2}\right)\right] = \frac{16-3\pi}{6}$$

17. One loop is given by the region
$D=\{(\mathrm{r},\theta)\,|-\pi/6\le\theta\le\pi/6, 0\le r$□$3\theta\}$, so the area is

$$\iint_D dA = \int_{-\pi/6}^{\pi/6}\int_0^{\cos 3\theta} r\, dr\, d\theta = \int_{-\pi/6}^{\pi/6}\left[\frac{1}{2}r^2\right]_{r=0}^{r=\cos 3\theta} d\theta$$

$$= \int_{-\pi/6}^{\pi/6}\frac{1}{2}\cos^2 3\theta\, d\theta = 2\int_0^{\pi/6}\frac{1}{2}\left(\frac{1+\cos 6\theta}{2}\right)d\theta$$

$$=\frac{1}{2}\left[\theta+\frac{1}{6}\sin 6\theta\right]_0^{\pi/6}=\frac{\pi}{12}$$



18. $D=\{(r,\theta)\mid 0\le\theta\le 2\pi, 0\le r\le 4+3\cos\theta\}$, so

$$A(D)=\iint_D dA=\int_0^{2\pi}\int_0^{4+3\cos\theta} r\,dr\,d\theta=\int_0^{2\pi}\left[\frac{1}{2}r^2\right]_{r=0}^{r=4+3\cos\theta}d\theta=\frac{1}{2}\int_0^{2\pi}(4+3\cos\theta)^2 d\theta$$

$$=\frac{1}{2}\int_0^{2\pi}(16+24\cos\theta+9\cos^2\theta)d\theta=\frac{1}{2}\int_0^{2\pi}\left(16+24\cos\theta+9\cdot\frac{1+\cos 2\theta}{2}\right)d\theta$$

$$=\frac{1}{2}\left[16\theta+24\sin\theta+\frac{9}{2}\theta+\frac{9}{4}\sin 2\theta\right]_0^{2\pi}=\frac{41}{2}\pi$$

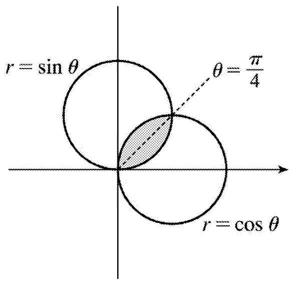
19. By symmetry,

$$A=2\int_0^{\pi/4}\int_0^{\sin\theta} r\,dr\,d\theta=2\int_0^{\pi/4}\left[\frac{1}{2}r^2\right]_{r=0}^{r=\sin\theta}d\theta$$

$$=\int_0^{\pi/4}\sin^2\theta\,d\theta=\int_0^{\pi/4}\frac{1}{2}(1-\cos 2\theta)d\theta$$

$$=\frac{1}{2}\left[\theta-\frac{1}{2}\sin 2\theta\right]_0^{\pi/4}$$

$$=\frac{1}{2}\left[\frac{\pi}{4}-\frac{1}{2}\sin\frac{\pi}{2}-0+\frac{1}{2}\sin 0\right]=\frac{1}{8}(\pi-2)$$



20. $2 = 4\sin\theta$ implies that $\theta = \frac{\pi}{6}$ or $\frac{5\pi}{6}$, so

$$A = \int_{\pi/6}^{5\pi/6}\int_{2}^{4\sin\theta} r\,dr\,d\theta = \int_{\pi/6}^{5\pi/6}\left[\frac{1}{2}r^2\right]_{r=2}^{r=4\sin\theta} d\theta = \int_{\pi/6}^{5\pi/6}(8\sin^2\theta - 2)\,d\theta$$

$$= \int_{\pi/6}^{5\pi/6}[4(1-\cos 2\theta)-2]\,d\theta = [2\theta - 2\sin 2\theta]_{\pi/6}^{5\pi/6} = \frac{4\pi}{3} + 2\sqrt{3}\ .$$

21. $$V = \iint_{x^2+y^2\le 9}(x^2+y^2)\,dA = \int_0^{2\pi}\int_0^3 (r^2)\,r\,dr\,d\theta = \int_0^{2\pi} d\theta \int_0^3 r^3\,dr = [\theta]_0^{2\pi}\left[\frac{1}{4}r^4\right]_0^3 = 2\pi\left(\frac{81}{4}\right) = \frac{81\pi}{2}$$

22. The sphere $x^2+y^2+z^2=16$ intersects the xy –plane in the circle $x^2+y^2=16$, so

$$V = 2\iint_{4\le x^2+y^2\le 16}\sqrt{16-x^2-y^2}\,dA \text{ [by symmetry]}$$

$$= 2\int_0^{2\pi}\int_2^4\sqrt{16-r^2}\,r\,dr\,d\theta = 2\int_0^{2\pi}d\theta\int_2^4 r(16-r^2)^{1/2}\,dr$$

$$= 2[\theta]_0^{2\pi}\left[-\frac{1}{3}(16-r^2)^{3/2}\right]_2^4 = -\frac{2}{3}(2\pi)(0-12^{3/2}) = \frac{4\pi}{3}\left(12\sqrt{12}\right) = 32\sqrt{3}\,\pi$$

23. By symmetry,

$$V = 2\iint_{x^2+y^2\le a^2}\sqrt{a^2-x^2-y^2}\,dA = 2\int_0^{2\pi}\int_0^a\sqrt{a^2-r^2}\,r\,dr\,d\theta = 2\int_0^{2\pi}d\theta\int_0^a r\sqrt{a^2-r^2}\,dr$$

$$= 2[\theta]_0^{2\pi}\left[-\frac{1}{3}(a^2-r^2)^{3/2}\right]_0^a = 2(2\pi)\left(0+\frac{1}{3}a^3\right) = \frac{4\pi}{3}a^3$$

24. The paraboloid $z=10-3x^2-3y^2$ intersects the plane $z=4$ when $4=10-3x^2-3y^2$ or $x^2+y^2=2$. So

$$V=\iint_{x^2+y^2\le 2}\left[(10-3x^2-3y^2)-4\right]dA=\int_0^{2\pi}\int_0^{\sqrt{2}}(6-3r^2)\,r\,dr\,d\theta$$

$$=\int_0^{2\pi}d\theta\int_0^{\sqrt{2}}(6r-3r^3)\,dr=\left[\theta\right]_0^{2\pi}\left[3r^2-\frac{3}{4}r^4\right]_0^{\sqrt{2}}=6\pi$$

25. The cone $z=\sqrt{x^2+y^2}$ intersects the sphere $x^2+y^2+z^2=1$ when $x^2+y^2+\left(\sqrt{x^2+y^2}\right)^2=1$ or $x^2+y^2=\frac{1}{2}$. So

$$V=\iint_{x^2+y^2\le 1/2}\left(\sqrt{1-x^2-y^2}-\sqrt{x^2+y^2}\right)dA=\int_0^{2\pi}\int_0^{1/\sqrt{2}}\left(\sqrt{1-r^2}-r\right)r\,dr\,d\theta$$

$$=\int_0^{2\pi}d\theta\int_0^{1/\sqrt{2}}\left(r\sqrt{1-r^2}-r^2\right)dr=\left[\theta\right]_0^{2\pi}\left[-\frac{1}{3}(1-r^2)^{3/2}-\frac{1}{3}r^3\right]_0^{1/\sqrt{2}}$$

$$=2\pi\left(-\frac{1}{3}\right)\left(\frac{1}{\sqrt{2}}-1\right)=\frac{\pi}{3}\left(2-\sqrt{2}\right)$$

26. The two paraboloids intersect when $3x^2+3y^2=4-x^2-y^2$ or $x^2+y^2=1$. So

$$V=\iint_{x^2+y^2\le 1}\left[(4-x^2-y^2)-3(x^2+y^2)\right]dA=\int_0^{2\pi}\int_0^{1}4(1-r^2)\,r\,dr\,d\theta$$

$$=\int_0^{2\pi}d\theta\int_0^{1}(4r-4r^3)\,dr=\left[\theta\right]_0^{2\pi}\left[2r^2-r^4\right]_0^{1}=2\pi$$

27. The given solid is the region inside the cylinder $x^2+y^2=4$ between the surfaces $z=\sqrt{64-4x^2-4y^2}$ and $z=-\sqrt{64-4x^2-4y^2}$. So

$$V=\iint_{x^2+y^2\le 4}\left[\sqrt{64-4x^2-4y^2}-\left(-\sqrt{64-4x^2-4y^2}\right)\right]dA$$

$$=\iint_{x^2+y^2\le 4}2\sqrt{64-4x^2-4y^2}\,dA=4\int_0^{2\pi}\int_0^{2}\sqrt{16-r^2}\,r\,dr\,d\theta$$

$$= 4\int_0^{2\pi} d\theta \int_0^2 r\sqrt{16-r^2}\,dr = 4[\,\theta\,]_0^{2\pi}\left[-\frac{1}{3}(16-r^2)^{3/2}\right]_0^2$$

$$= 8\pi\left(-\frac{1}{3}\right)(12^{3/2}-16^{2/3}) = \frac{8\pi}{3}\left(64-24\sqrt{3}\right)$$

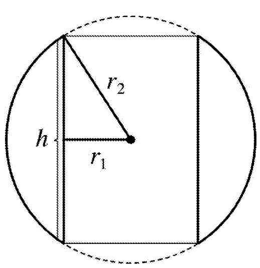
28. **(a)** Here the region in the *xy* –plane is the annular region $r_1^2 \le x^2+y^2 \le r_2^2$ and the desired volume is twice that above the *xy* –plane. Hence

$$V = 2\iint_{r_1^2 \le x^2+y^2 \le r_2^2} \sqrt{r_2^2-x^2-y^2}\,dA = 2\int_0^{2\pi}\int_{r_1}^{r_2}\sqrt{r_2^2-r^2}\,r\,dr\,d\theta$$

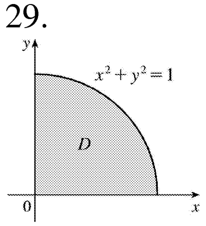
$$= 2\int_0^{2\pi} d\theta \int_{r_1}^{r_2}\sqrt{r_2^2-r^2}\,r\,dr = \frac{4\pi}{3}\left[-(r_2^2-r^2)^{3/2}\right]_{r_1}^{r_2} = \frac{4\pi}{3}(r_2^2-r_1^2)^{3/2}$$

(b) A cross–sectional cut is shown in the figure. So $r_2^2 = \left(\frac{1}{2}h\right)^2 + r_1^2$ or $\frac{1}{4}h^2 = r_2^2 - r_1^2$.

Thus the volume in terms of *h* is $V = \frac{4\pi}{3}\left(\frac{1}{4}h^2\right)^{3/2} = \frac{\pi}{6}h^3$.



29.

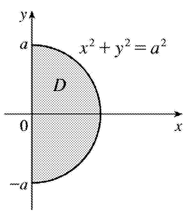


$$\int_0^1 \int_0^{\sqrt{1-x^2}} e^{x^2+y^2}\,dy\,dx = \int_0^{\pi/2}\int_0^1 e^{r^2}\,r\,dr\,d\theta$$

$$= \int_0^{\pi/2} d\theta \int_0^1 r e^{r^2} \, dr$$

$$= [\,\theta\,]_0^{\pi/2} \left[\frac{1}{2} e^{r^2} \right]_0^1 = \frac{1}{4} \pi (e-1)$$

30.

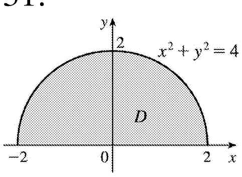


$$\int_{-\pi/2}^{\pi/2} \int_0^a (r^2)^{3/2} r \, dr \, d\theta = \int_{-\pi/2}^{\pi/2} d\theta \int_0^a r^4 \, dr$$

$$= [\,\theta\,]_{-\pi/2}^{\pi/2} \left[\frac{1}{5} r^5 \right]_0^a$$

$$= \frac{1}{5} \pi a^5$$

31.



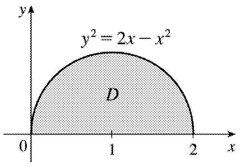
$$\int_0^{\pi} \int_0^2 (r\cos\theta)^2 (r\sin\theta)^2 r \, dr \, d\theta = \int_0^{\pi} (\sin\theta\cos\theta)^2 d\theta \int_0^2 r^5 \, dr$$

$$= \int_0^{\pi} \left(\frac{1}{2} \sin 2\theta \right)^2 d\theta \int_0^2 r^5 \, dr$$

$$= \frac{1}{4} \left[\frac{1}{2}\theta - \frac{1}{8} \sin 4\theta \right]_0^{\pi} \left[\frac{1}{6} r^6 \right]_0^2$$

$$= \frac{1}{4} \left(\frac{\pi}{2} \right) \left(\frac{64}{6} \right) = \frac{4\pi}{3}$$

32.



$$\int_0^{\pi/2}\int_0^{2\cos\theta} r^2\,dr\,d\theta = \int_0^{\pi/2}\left[\frac{1}{3}r^3\right]_{r=0}^{r=2\cos\theta} d\theta$$

$$= \int_0^{\pi/2}\left(\frac{8}{3}\cos^3\theta\right)d\theta$$

$$= \frac{8}{3}\left[\sin\theta - \frac{1}{3}\sin^3\theta\right]_0^{\pi/2} = \frac{16}{9}$$

33. The surface of the water in the pool is a circular disk D with radius 20 ft. If we place D on coordinate axes with the origin at the center of D and define $f(x,y)$ to be the depth of the water at (x,y) , then the volume of water in the pool is the volume of the solid that lies above $D=\left\{(x,y)|x^2+y^2\leq 400\right\}$ and below the graph of $f(x,y)$. We can associate north with the positive y –direction, so we are given that the depth is constant in the x –direction and the depth increases linearly in the y –direction from $f(0,-20)=2$ to $f(0,20)=7$. The trace in the yz –plane is a line segment from $(0,-20,2)$ to $(0,20,7)$. The slope of this line is $\frac{7-2}{20-(-20)}=\frac{1}{8}$, so an equation of the line is $z-7=\frac{1}{8}(y-20)\Rightarrow z=\frac{1}{8}y+\frac{9}{2}$. Since $f(x,y)$ is independent of x , $f(x,y)=\frac{1}{8}y+\frac{9}{2}$. Thus the volume is given by $\iint_D f(x,y)\,dA$, which is most conveniently evaluated using polar coordinates. Then $D=\{(r,\theta)|0\leq r\leq 20, 0\leq\theta\leq 2\pi\}$ and substituting $x=r\cos\theta$, $y=r\sin\theta$ the integral becomes

$$\int_0^{2\pi}\int_0^{20}\left(\frac{1}{8}r\sin\theta+\frac{9}{2}\right)r\,dr\,d\theta = \int_0^{2\pi}\left[\frac{1}{24}r^3\sin\theta+\frac{9}{4}r^2\right]_{r=0}^{r=20}d\theta = \int_0^{2\pi}\left(\frac{1000}{3}\sin\theta+900\right)d\theta$$

$$= \left[-\frac{1000}{3}\cos\theta+900\theta\right]_0^{2\pi} = 1800\pi$$

Thus the pool contains $1800\pi\approx 5655$ ft^3 of water.

34. **(a)** The total amount of water supplied each hour to the region within R feet of the sprinkler is

$$V = \int_0^{2\pi}\int_0^R e^{-r}r\,dr\,d\theta = \int_0^{2\pi}d\theta\int_0^R re^{-r}dr = [\theta]_0^{2\pi}\left[-re^{-r}-e^{-r}\right]_0^R$$

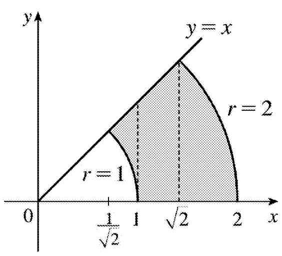
$$= 2\pi[-Re^{-R}-e^{-R}+0+1] = 2\pi(1-Re^{-R}-e^{-R})\text{ft}^3$$

(b) The average amount of water per hour per square foot supplied to the region within R feet of the sprinkler is $\dfrac{V}{\text{areaofregion}}=\dfrac{V}{\pi R^2}=\dfrac{2\left(1-Re^{-R}-e^{-R}\right)}{R^2}$ ft ^{3} (per hour per square foot). See the definition of the average value of a function on page 1022 [ET 986].

35. $\displaystyle\int_{1/\sqrt{2}}^{1}\int_{\sqrt{1-x^2}}^{x} xy\,dy\,dx+\int_{1}^{\sqrt{2}}\int_{0}^{x} xy\,dy\,dx+\int_{\sqrt{2}}^{2}\int_{0}^{\sqrt{4-x^2}} xy\,dy\,dx$

$\displaystyle =\int_0^{\pi/4}\int_1^2 r^3\cos\theta\sin\theta\,dr\,d\theta=\int_0^{\pi/4}\left[\frac{r^4}{4}\cos\theta\sin\theta\right]_{r=1}^{r=2}d\theta$

$\displaystyle =\frac{15}{4}\int_0^{\pi/4}\sin\theta\cos\theta\,d\theta=\frac{15}{4}\left[\frac{\sin^2\theta}{2}\right]_0^{\pi/4}=\frac{15}{16}$



36. **(a)** $\displaystyle\iint_{D_a} e^{-(x^2+y^2)}\,dA=\int_0^{2\pi}\int_0^a re^{-r^2}\,dr\,d\theta=2\pi\left[-\frac{1}{2}e^{-r^2}\right]_0^a=\pi\left(1-e^{-a^2}\right)$ for each a. Then $\displaystyle\lim_{a\to\infty}\pi\left(1-e^{-a^2}\right)=\pi$ since $e^{-a^2}\to 0$ as $a\to\infty$. Hence $\displaystyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dA=\pi$.

(b) $\displaystyle\iint_{S_a} e^{-(x^2+y^2)}\,dA=\int_{-a}^{a}\int_{-a}^{a} e^{-x^2}e^{-y^2}\,dx\,dy=\left(\int_{-a}^{a} e^{-x^2}\,dx\right)\left(\int_{-a}^{a} e^{-y^2}\,dy\right)$ for each a.

Then, from (a), $\displaystyle\pi=\iint_{R^2}-(x^2+y^2)\,dA$, so

$\displaystyle\pi=\lim_{a\to\infty}\iint_{S_a} e^{-(x^2+y^2)}\,dA=\lim_{a\to\infty}\left(\int_{-a}^{a} e^{-x^2}\,dx\right)\left(\int_{-a}^{a} e^{-y^2}\,dy\right)=\left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2}\,dy\right).$

To evaluate $\displaystyle\lim_{a\to\infty}\left(\int_{-a}^{a} e^{-x^2}\,dx\right)\left(\int_{-a}^{a} e^{-y^2}\,dy\right)$, we are using the fact that these integrals are bounded.

This is true since on $[-1,1]$, $0<e^{-x^2}\le 1$ while on $(-\infty,-1)$, $0<e^{-x^2}\le e^{x}$ and on $(1,\infty)$, $0<e^{-x^2}<e^{-x}$.
Hence $0\le \int_{-\infty}^{\infty} e^{-x^2}dx\le \int_{-\infty}^{-1} e^{x}dx+\int_{-1}^{1} dx+\int_{1}^{\infty} e^{-x}dx=2(e^{-1}+1)$.

(c) Since $\left(\int_{-\infty}^{\infty} e^{-x^2}dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2}dy\right)=\pi$ and y can be replaced by x , $\left(\int_{-\infty}^{\infty} e^{-x^2}dx\right)^2=\pi$ implies that $\int_{-\infty}^{\infty} e^{-x^2}dx=\pm\sqrt{\pi}$. But $e^{-x^2}\ge 0$ for all x , so $\int_{-\infty}^{\infty} e^{-x^2}dx=\sqrt{\pi}$.

(d) Letting $t=\sqrt{2}\,x$, $\int_{-\infty}^{\infty} e^{-x^2}dx=\int_{-\infty}^{\infty} \frac{1}{\sqrt{2}}\left(e^{-t^2/2}\right)dt$, so that $\sqrt{\pi}=\frac{1}{\sqrt{2}}\int_{-\infty}^{\infty} e^{-t^2/2}dt$ or $\int_{-\infty}^{\infty} e^{-t^2/2}dt=\sqrt{2\pi}$.

37. **(a)** We integrate by parts with $u=x$ and $dv=xe^{-x^2}dx$. Then $du=dx$ and $v=-\frac{1}{2}e^{-x^2}$, so

$$\int_0^{\infty} x^2e^{-x^2}dx = \lim_{t\to\infty}\int_0^t x^2e^{-x^2}dx=\lim_{t\to\infty}\left(-\frac{1}{2}xe^{-x^2}\Big]_0^t+\int_0^t \frac{1}{2}e^{-x^2}dx\right)$$

$$= \lim_{t\to\infty}\left(-\frac{1}{2}te^{-t^2}\right)+\frac{1}{2}\int_0^{\infty} e^{-x^2}dx=0+\frac{1}{2}\int_0^{\infty} e^{-x^2}dx \text{ [by l'Hospital's Rule]}$$

$$= \frac{1}{4}\int_{-\infty}^{\infty} e^{-x^2}dx \text{ [since } e^{-x^2} \text{ is an even function]}$$

$$= \frac{1}{4}\sqrt{\pi} \text{ [by Exercise 36(c)]}$$

(b) Let $u=\sqrt{x}$. Then $u^2=x\Rightarrow dx=2u\,du\Rightarrow$

$$\int_0^{\infty}\sqrt{x}\,e^{-x}dx = \lim_{t\to\infty}\int_0^t\sqrt{x}\,e^{-x}dx=\lim_{t\to\infty}\int_0^{\sqrt{t}} ue^{-u^2}2u\,du=2\int_0^{\infty} u^2e^{-u^2}du$$

$$= 2\left(\frac{1}{4}\sqrt{\pi}\right) \text{ [by part(a)]}=\frac{1}{2}\sqrt{\pi}$$

1.

$$Q=\iint_D \sigma(\mathrm{x,y})\,dA=\int_1^3\int_0^2(2xy+y^2)\,dy\,dx=\int_1^3\left[xy^2+\frac{1}{3}y^3\right]_{y=0}^{y=2}dx$$

$$=\int_1^3\left(4x+\frac{8}{3}\right)dx=\left[2x^2+\frac{8}{3}x\right]_1^3=16+\frac{16}{3}=\frac{64}{3}\text{ C}$$

2.

$$Q=\iint_D \sigma(\mathrm{x,y})\,dA=\iint_D(x+y+x^2+y^2)\,dA=\int_0^{2\pi}\int_0^2(r\cos\theta+r\sin\theta+r^2)\,r\,dr\,d\theta$$

$$=\int_0^{2\pi}\int_0^2 dr\,d\theta=\int_0^{2\pi}\left[\frac{1}{3}r^3(\cos\theta+\sin\theta)+\frac{1}{4}r^4\right]_{r=0}^{r=2}d\theta$$

$$=\int_0^{2\pi}\left[\frac{8}{3}(\cos\theta+\sin\theta)+4\right]d\theta=\left[\frac{8}{3}(\sin\theta-\cos\theta)+4\theta\right]_0^{2\pi}=8\pi\text{C}$$

3. $$m=\iint_D \rho(\mathrm{x,y})\,dA=\int_0^2\int_{-1}^1 xy^2\,dy\,dx=\int_0^2 x\,dx\int_{-1}^1 y^2\,dy=\left[\frac{1}{2}x^2\right]_0^2\left[\frac{1}{3}y^3\right]_{-1}^1=2\cdot\frac{2}{3}=\frac{4}{3}\ ,$$

$$\bar{\mathrm{x}}=\frac{1}{m}\iint_D x\rho(\mathrm{x,y})\,dA=\frac{3}{4}\int_0^2\int_{-1}^1 x^2y^2\,dy\,dx=\frac{3}{4}\int_0^2 x^2\,dx\int_{-1}^1 y^2\,dy=\frac{3}{4}\left[\frac{1}{3}x^3\right]_0^2\left[\frac{1}{3}y^3\right]_{-1}^1=\frac{3}{4}\cdot\frac{8}{3}\cdot\frac{2}{3}=\frac{4}{3}\ ,$$

$$\bar{\mathrm{y}}=\frac{1}{m}\iint_D y\rho(\mathrm{x,y})\,dA=\frac{3}{4}\int_0^2\int_{-1}^1 xy^3\,dy\,dx=\frac{3}{4}\int_0^2 x\,dx\int_{-1}^1 y^3\,dy=\frac{3}{4}\left[\frac{1}{2}x^2\right]_0^2\left[\frac{1}{4}y^4\right]_{-1}^1=\frac{3}{4}\cdot 2\cdot 0=0\ .$$

Hence, $(\bar{\mathrm{x}},\bar{\mathrm{y}})=\left(\frac{4}{3},0\right)$.

4. $$m=\iint_D \rho(\mathrm{x,y})\,dA=\int_0^a\int_0^b cxy\,dy\,dx=c\int_0^a x\,dx\int_0^b y\,dy=c\left[\frac{1}{2}x^2\right]_0^a\left[\frac{1}{2}y^2\right]_0^b=\frac{1}{4}a^2b^2c\ ,$$

$$M_y=\iint_D x\rho(\mathrm{x,y})\,dA=\int_0^a\int_0^b cx^2y\,dy\,dx=c\int_0^a x^2\,dx\int_0^b y\,dy=c\left[\frac{1}{3}x^3\right]_0^a\left[\frac{1}{2}y^2\right]_0^b=\frac{1}{6}a^3b^2c\ ,\text{ and}$$

$$M_x=\iint_D y\rho(\mathrm{x,y})\,dA=\int_0^a\int_0^b cxy^2\,dy\,dx=c\int_0^a x\,dx\int_0^b y^2\,dy=c\left[\frac{1}{2}x^2\right]_0^a\left[\frac{1}{3}y^3\right]_0^b=\frac{1}{6}a^2b^3c\ .$$

Hence, $(\bar{\mathrm{x}},\bar{\mathrm{y}})=\left(\frac{M_y}{m},\frac{M_x}{m}\right)=\left(\frac{2}{3}\mathrm{a},\frac{2}{3}b\right)$.

5.

$$m=\int_0^2\int_{x/2}^{3-x}(x+y)\,dy\,dx=\int_0^2\left[xy+\frac{1}{2}y^2\right]_{y=x/2}^{y=3-x}dx=\int_0^2\left[x\left(3-\frac{3}{2}x\right)+\frac{1}{2}(3-x)^2-\frac{1}{8}x^2\right]dx$$

$$=\int_0^2\left(-\frac{9}{8}x^2+\frac{9}{2}\right)dx=\left[-\frac{9}{8}\left(\frac{1}{3}x^3\right)+\frac{9}{2}x\right]_0^2=6\ ,$$

$$M_y=\int_0^2\int_{x/2}^{3-x}(x^2+xy)\,dy\,dx=\int_0^2\left[x^2y+\frac{1}{2}xy^2\right]_{y=x/2}^{y=3-x}dx=\int_0^2\left(\frac{9}{2}x-\frac{9}{8}x^3\right)dx=\frac{9}{2}$$, and

$$M_x=\int_0^2\int_{x/2}^{3-y}(xy+y^2)\,dy\,dx=\int_0^2\left[\frac{1}{2}xy^2+\frac{1}{3}y^3\right]_{y=x/2}^{y=3-x}dx=\int_0^2\left(9-\frac{9}{2}x\right)dx=9$$. Hence $m=6$,

$$(\overline{x},\overline{y})=\left(\frac{M_y}{m},\frac{M_x}{m}\right)=\left(\frac{3}{4},\frac{3}{2}\right)\ .$$

6. $$m=\int_0^1\int_y^{4-3y}x\,dx\,dy=\int_0^1\left[\frac{1}{2}(4-3y)^2-\frac{1}{2}y^2\right]dy=\left[-\frac{1}{18}(4-3y)^3-\frac{1}{6}y^3\right]_0^1=\frac{10}{3}\ ,$$

$$M_y=\int_0^1\int_y^{4-3y}x^2\,dx\,dy=\int_0^1\left[\frac{1}{3}(4-3y)^3-\frac{1}{3}y^3\right]dy=\left[-\frac{1}{36}(4-3y)^4-\frac{1}{12}y^4\right]_0^1=7\ ,$$

$$M_x=\int_0^1\int_y^{4-3y}xy\,dx\,dy=\int_0^1\left[\frac{1}{2}y(4-3y)^2-\frac{1}{2}y^3\right]dy=\int_0^1(8y-12y^2+4y^3)\,dy=1\ .$$

Hence $m=\frac{10}{3}$, $(\overline{x},\overline{y})=(2.1,0.3)$.

7. $$m=\int_0^1\int_0^{e^x}y\,dy\,dx=\int_0^1\left[\frac{1}{2}y^2\right]_{y=0}^{y=e^x}dx=\frac{1}{2}\int_0^1e^{2x}dx=\left.\frac{1}{4}e^{2x}\right]_0^1=\frac{1}{4}(e^2-1)\ ,$$

$$M_y=\int_0^1\int_0^{e^x}xy\,dy\,dx=\frac{1}{2}\int_0^1xe^{2x}dx=\frac{1}{2}\left[\frac{1}{2}xe^{2x}-\frac{1}{4}e^{2x}\right]_0^1=\frac{1}{8}(e^2+1)$$, and

$$M_x=\int_0^1\int_0^{e^x}y^2\,dy\,dx=\int_0^1\left[\frac{1}{3}y^3\right]_{y=0}^{y=e^x}dx=\frac{1}{3}\int_0^1e^{3x}dx=\frac{1}{3}\left[\frac{1}{3}e^{3x}\right]_0^1=\frac{1}{9}(e^3-1)\ .$$

Hence $m=\frac{1}{4}(e^2-1)$, $$(\overline{x},\overline{y})=\left(\frac{\frac{1}{8}(e^2+1)}{\frac{1}{4}(e^2-1)},\frac{\frac{1}{9}(e^3-1)}{\frac{1}{4}(e^2-1)}\right)=\left(\frac{e^2+1}{2(e^2-1)},\frac{4(e^3-1)}{9(e^2-1)}\right)\ .$$

8.

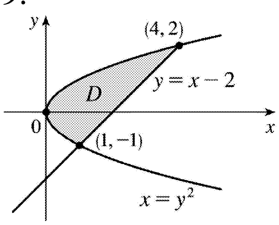
$m=\int_0^1\int_0^{\sqrt{x}} x\,dy\,dx=\int_0^1 x\,[y]_{y=0}^{y=\sqrt{x}}\,dx=\int_0^1 x^{3/2}\,dx=\left.\frac{2}{5}x^{5/2}\right]_0^1=\frac{2}{5}$,

$M_y=\int_0^1\int_0^{\sqrt{x}} x^2\,dy\,dx=\int_0^1 x\,[y]_{y=0}^{y=\sqrt{x}}\,dx=\int_0^1 x^{5/2}\,dx=\left.\frac{2}{7}x^{7/2}\right]_0^1=\frac{2}{7}$, and

$M_x=\int_0^1\int_0^{\sqrt{x}} yx\,dy\,dx=\int_0^1 x\left[\frac{1}{2}y^2\right]_{y=0}^{y=\sqrt{x}}dx=\frac{1}{2}\int_0^1 x^2\,dx=\frac{1}{2}\left[\frac{1}{3}x^3\right]_0^1=\frac{1}{6}$.

Hence $m=\frac{2}{5}$, $(\overline{x},\overline{y})=\left(\frac{2/7}{2/5},\frac{1/6}{2/5}\right)=\left(\frac{5}{7},\frac{5}{12}\right)$.

9.



$m=\int_{-1}^{2}\int_{y^2}^{y+2} 3\,dx\,dy=\int_{-1}^{2}(3y+6-3y^2)\,dy=\frac{27}{2}$,

$M_y=\int_{-1}^{2}\int_{y^2}^{y+2} 3x\,dx\,dy=\int_{-1}^{2}\frac{3}{2}\left[(y+2)^2-y^4\right]dy$

$=\left[\frac{1}{2}(y+2)^3-\frac{3}{10}y^5\right]_{-1}^{2}=\frac{108}{5}$

and

$M_x=\int_{-1}^{2}\int_{y^2}^{y+2} 3y\,dx\,dy=\int_{-1}^{2}(3y^2+6y-3y^3)\,dy$

$=\left[y^3+3y^2-\frac{3}{4}y^4\right]_{-1}^{2}=\frac{27}{4}$

Hence $m=\frac{27}{2}$, $(\overline{x},\overline{y})=\left(\frac{8}{5},\frac{1}{2}\right)$.

10. $m=\int_0^{\pi/2}\int_0^{\cos x} x\,dy\,dx=\int_0^{\pi/2} x\cos x\,dx=[x\sin x+\cos x]_0^{\pi/2}=\frac{\pi}{2}-1$,

$M_y=\int_0^{\pi/2}\int_0^{\cos x} x^2\,dy\,dx=\int_0^{\pi/2} x^2\cos x\,dx=\left[x^2\sin x+2x\cos x-2\sin x\right]_0^{\pi/2}=\frac{\pi^2}{4}-2$, and

$$M_x=\int_0^{\pi/2}\int_0^{\cos x} xy\,dy\,dx=\int_0^{\pi/2}\frac{1}{2}x\cos^2 x\,dx=\frac{1}{2}\left[\frac{1}{4}x^2+\frac{1}{4}x\sin 2x+\frac{1}{8}\cos 2x\right]_0^{\pi/2}=\frac{\pi^2}{32}-\frac{1}{8}.$$

Hence $m=\dfrac{\pi-2}{2}$, $(\overline{x},\overline{y})=\left(\dfrac{\pi^2-8}{2(\pi-2)},\dfrac{\pi+2}{16}\right)$.

11. $\rho(x,y)=ky=kr\sin\theta$, $m=\int_0^{\pi/2}\int_0^1 kr^2\sin\theta\,dr\,d\theta=\frac{1}{3}k\int_0^{\pi/2}\sin\theta\,d\theta=\frac{1}{3}k\left[-\cos\theta\right]_0^{\pi/2}=\frac{1}{3}k$,

$$M_y=\int_0^{\pi/2}\int_0^1 kr^3\sin\theta\cos\theta\,dr\,d\theta=\frac{1}{4}k\int_0^{\pi/2}\sin\theta\cos\theta\,d\theta=\frac{1}{8}k\left[-\cos 2\theta\right]_0^{\pi/2}=\frac{1}{8}k,$$

$$M_x=\int_0^{\pi/2}\int_0^1 kr^3\sin^2\theta\,dr\,d\theta=\frac{1}{4}k\int_0^{\pi/2}\sin^2\theta\,d\theta=\frac{1}{8}k\left[\theta+\sin 2\theta\right]_0^{\pi/2}=\frac{\pi}{16}k.$$

Hence $(\overline{x},\overline{y})=\left(\dfrac{3}{8},\dfrac{3\pi}{16}\right)$.

12. $\rho(x,y)=k(x^2+y^2)=kr^2$, $m=\int_0^{\pi/2}\int_0^1 kr^3\,dr\,d\theta=\frac{\pi}{8}k$,

$$M_y=\int_0^{\pi/2}\int_0^1 kr^4\cos\theta\,dr\,d\theta=\frac{1}{5}k\int_0^{\pi/2}\cos\theta\,d\theta=\frac{1}{5}k\left[\sin\theta\right]_0^{\pi/2}=\frac{1}{5}k,$$

$$M_x=\int_0^{\pi/2}\int_0^1 kr^4\sin\theta\,dr\,d\theta=\frac{1}{5}k\int_0^{\pi/2}\sin\theta\,d\theta=\frac{1}{5}k\left[-\cos\theta\right]_0^{\pi/2}=\frac{1}{5}k.$$

Hence $(\overline{x},\overline{y})=\left(\dfrac{8}{5\pi},\dfrac{8}{5\pi}\right)$.

13. Placing the vertex opposite the hypotenuse at $(0,0)$, $\rho(x,y)=k(x^2+y^2)$. Then

$$m=\int_0^a\int_0^{a-x}k\left(x^2+y^2\right)dy\,dx=k\int_0^a\left[ax^2-x^3+\frac{1}{3}(a-x)^3\right]dx$$

$$=k\left[\frac{1}{3}ax^3-\frac{1}{4}x^4-\frac{1}{12}(a-x)^4\right]_0^a=\frac{1}{6}ka^4$$

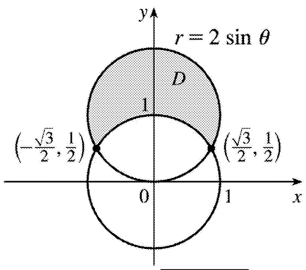
By symmetry,

$$M_y=M_x=\int_0^a\int_0^{a-x}ky(x^2+y^2)\,dy\,dx=k\int_0^a\left[\frac{1}{2}(a-x)^2x^2+\frac{1}{4}(a-x)^4\right]dx$$

$$=k\left[\frac{1}{6}a^2x^3-\frac{1}{4}ax^4+\frac{1}{10}x^5-\frac{1}{20}(a-x)^5\right]_0^a=\frac{1}{15}ka^5$$

Hence $(\bar{x},\bar{y})=\left(\frac{2}{5}a,\frac{2}{5}a\right)$.

14.



$\rho(x,y)=k/\sqrt{x^2+y^2}=k/r$,

$$m=\int_{\pi/6}^{5\pi/6}\int_{1}^{2\sin\theta}\frac{k}{r}\,r\,dr\,d\theta=k\int_{\pi/6}^{5\pi/6}[(2\sin\theta)-1]\,d\theta$$

$$=k\left[-2\cos\theta-\theta\right]_{\pi/6}^{5\pi/6}=2k\left(\sqrt{3}-\frac{\pi}{3}\right)$$

By symmetry of D and $f(x)=x$, $M_y=0$, and

$$M_x=\int_{\pi/6}^{5\pi/6}\int_{1}^{2\sin\theta}kr\sin\theta\,dr\,d\theta=\frac{1}{2}k\int_{\pi/6}^{5\pi/6}(4\sin^3\theta-\sin\theta)\,d\theta$$

$$=\frac{1}{2}k\left[-3\cos\theta+\frac{4}{3}\cos^3\theta\right]_{\pi/6}^{5\pi/6}=\sqrt{3}\,k$$

Hence $(\bar{x},\bar{y})=\left(0,\frac{3\sqrt{3}}{2(3\sqrt{3}-\pi)}\right)$.

15.

$$I_x=\iint_D y^2\rho(x,y)\,dA=\int_0^1\int_0^{e^x}y^2\cdot y\,dy\,dx=\int_0^1\left[\frac{1}{4}y^4\right]_{y=0}^{y=e^x}dx=\frac{1}{4}\int_0^1 e^{4x}\,dx$$

$$=\frac{1}{4}\left[\frac{1}{4}e^{4x}\right]_0^1=\frac{1}{16}(e^4-1)\ ,$$

$$I_y=\iint_D x^2\rho(x,y)\,dA=\int_0^1\int_0^{e^x}x^2y\,dy\,dx=\int_0^1 x^2\left[\frac{1}{2}y^2\right]_{y=0}^{y=e^x}dx=\frac{1}{2}\int_0^1 x^2e^{2x}\,dx$$

$$=\frac{1}{2}\left[\left(\frac{1}{2}x^2-\frac{1}{2}x+\frac{1}{4}\right)e^{2x}\right]_0^1 \text{ [integrate by parts twice]}$$
$$=\frac{1}{8}(e^2-1),$$

and $I_0=I_x+I_y=\frac{1}{16}(e^4-1)+\frac{1}{8}(e^2-1)=\frac{1}{16}(e^4+2e^2-3)$.

16. $I_x=\int_0^{\pi/2}\int_0^1(r^2\sin^2\theta)(kr^2)r\,dr\,d\theta=\frac{1}{6}k\int_0^{\pi/2}\sin^2\theta\,d\theta=\frac{1}{6}k\left[\frac{1}{4}(2\theta-\sin 2\theta)\right]_0^{\pi/2}=\frac{\pi}{24}k$,
$I_y=\int_0^{\pi/2}\int_0^1(r^2\cos^2\theta)(kr^2)r\,dr\,d\theta=\frac{1}{6}k\int_0^{\pi/6}\cos^2\theta\,d\theta=\frac{1}{6}k\left[\frac{1}{4}(2\theta+\sin 2\theta)\right]_0^{\pi/2}=\frac{\pi}{24}k$, and
$I_0=I_x+I_y=\frac{\pi}{12}k$.

17. $I_x=\int_{-1}^{2}\int_{y^2}^{y+2}3y^2\,dx\,dy=\int_{-1}^{2}(3y^3+6y^2-3y^4)dy=\left[\frac{3}{4}y^4+2y^3-\frac{3}{5}y^5\right]_{-1}^{2}=\frac{189}{20}$,
$I_y=\int_{-1}^{2}\int_{y^2}^{y+2}3x^2\,dx\,dy=\int_{-1}^{2}\left[(y+2)^3-y^6\right]dy=\left[\frac{1}{4}(y+2)^4-\frac{1}{7}y^7\right]_{-1}^{2}=\frac{1269}{28}$, and $I_0=I_x+I_y=\frac{1917}{35}$.

18. If we find the moments of inertia about the x – and y –axes, we can determine in which direction rotation will be more difficult. (See the explanation following Example 4.) The moment of inertia about the x –axis is given by

$$I_x=\iint_D y^2\rho(\text{x,y})dA=\int_0^2\int_0^2 y^2(1+0.1x)dy\,dx=\int_0^2(1+0.1x)\left[\frac{1}{3}y^3\right]_{y=0}^{y=2}dx$$
$$=\frac{8}{3}\int_0^2(1+0.1x)dx=\frac{8}{3}\left[x+0.1\cdot\frac{1}{2}x^2\right]_0^2=\frac{8}{3}(2.2)\approx 5.87$$

Similarly, the moment of inertia about the y –axis is given by

$$I_y=\iint_D x^2\rho(\text{x,y})dA=\int_0^2\int_0^2 x^2(1+0.1x)dy\,dx=\int_0^2 x^2(1+0.1x)\left[y\right]_{y=0}^{y=2}dx$$
$$=2\int_0^2(x^2+0.1x^3)dx=2\left[\frac{1}{3}x^3+0.1\cdot\frac{1}{4}x^4\right]_0^2=2\left(\frac{8}{3}+0.4\right)\approx 6.13$$

Since

$I_y > I_x$, more force is required to rotate the fan blade about the y –axis.

19. Using a CAS, we find □ . Then $\bar{x} = \frac{1}{m} \iint_D x\rho(x,y)\,dA = \frac{8}{\pi^2} \int_0^{\pi} \int_0^{\sin x} x^2 y\,dy\,dx = \frac{2\pi}{3} - \frac{1}{\pi}$ and

$\bar{y} = \frac{1}{m} \iint_D y\rho(x,y)\,dA = \frac{8}{\pi^2} \int_0^{\pi} \int_0^{\sin x} xy^2\,dy\,dx = \frac{16}{9\pi}$, so $(\bar{x},\bar{y}) = \left(\frac{2\pi}{3} - \frac{1}{\pi}, \frac{16}{9\pi} \right)$.

The moments of inertia are $I_x = \iint_D y^2 \rho(x,y)\,dA = \int_0^{\pi} \int_0^{\sin x} xy^3\,dy\,dx = \frac{3\pi^2}{64}$,

$I_y = \iint_D x^2 \rho(x,y)\,dA = \int_0^{\pi} \int_0^{\sin x} x^3 y\,dy\,dx = \frac{\pi^2}{16}(\pi^2 - 3)$, and $I_0 = I_x + I_y = \frac{\pi^2}{64}(4\pi^2 - 9)$.

20. Using a CAS, we find $m = \iint_D \sqrt{x^2+y^2}\,dA = \int_0^{2\pi} \int_0^{1+\cos\theta} r^2\,dr\,d\theta = \frac{5}{3}\pi$,

$\bar{x} = \frac{1}{m} \iint_D x\sqrt{x^2+y^2}\,dA = \frac{3}{5\pi} \int_0^{2\pi} \int_0^{1+\cos\theta} r^3 \cos\theta\,dr\,d\theta = \frac{21}{20}$ and

$\bar{y} = \frac{1}{m} \iint_D y\sqrt{x^2+y^2}\,dA = \frac{3}{5\pi} \int_0^{2\pi} \int_0^{1+\cos\theta} r^3 \sin\theta\,dr\,d\theta = 0$, so $(\bar{x},\bar{y}) = \left(\frac{21}{20}, 0 \right)$.

The moments of inertia are $I_x = \iint_D y^2\sqrt{x^2+y^2}\,dA = \int_0^{2\pi} \int_0^{1+\cos\theta} r^4 \sin^2\theta\,dr\,d\theta = \frac{33}{40}\pi$,

$I_y = \iint_D x^2\sqrt{x^2+y^2}\,dA = \int_0^{2\pi} \int_0^{1+\cos\theta} r^4 \cos^2\theta\,dr\,d\theta = \frac{93}{40}\pi$, and $I_0 = I_x + I_y = \frac{63}{20}\pi$.

21. $I_x = \int_0^a \int_0^a \rho y^2\,dx\,dy = \rho \int_0^a dx \int_0^a y^2\,dy = \rho\,[x]_0^a \left[\frac{1}{3} y^3 \right]_0^a = \rho a \left(\frac{1}{3} a^3 \right) = \frac{1}{3} \rho a^4 = I_y$ by symmetry, and $m = \rho a^2$

since the lamina is homogeneous. Hence $\bar{\bar{x}}^2 = \frac{I_y}{m} \Rightarrow \bar{\bar{x}} = \left[\left(\frac{1}{3} \rho a^4 \right) / (\rho a^2) \right]^{1/2} = \frac{1}{\sqrt{3}} a$ and

$\bar{\bar{y}}^2 = \frac{I_x}{m} \Rightarrow \bar{\bar{y}} = \frac{1}{\sqrt{3}} a$.

22.

$$m=\int_0^{\pi}\int_0^{\sin x}\rho\,dy\,dx=\rho\int_0^{\pi}\sin x\,dx=\rho\left[-\cos x\right]_0^{\pi}=2\rho\ ,$$

$$I_x=\int_0^{\pi}\int_0^{\sin x}\rho y^2\,dy\,dx=\frac{1}{3}\rho\int_0^{\pi}\sin^3 x\,dx=\frac{1}{3}\rho\int_0^{\pi}(1-\cos^2 x)\sin x\,dx$$

$$=\frac{1}{3}\rho\left[-\cos\theta+\frac{1}{3}\cos^3\theta\right]_0^{\pi}=\frac{4}{9}\rho\ ,$$

$$I_y=\int_0^{\pi}\int_0^{\sin x}\rho x^2\,dy\,dx=\rho\int_0^{\pi}x^2\sin x\,dx$$

$$=\rho\left[-x^2\cos x+2x\sin x+2\cos x\right]_0^{\pi}$$

$$=\rho(\pi^2-4)\ .$$

Then $\overline{\overline{y}}^2=\frac{I_x}{m}=\frac{2}{9}$, so $\overline{\overline{y}}=\frac{\sqrt{2}}{3}$ and $\overline{\overline{x}}^2=\frac{I_y}{m}=\frac{\pi^2-4}{2}$, so $\overline{\overline{x}}=\sqrt{\frac{\pi^2-4}{2}}$.

23. **(a)** $f(x,y)$ is a joint density function, so we know $\iint_{R^2} f(x,y)\,dA=1$. Since $f(x,y)=0$ outside the rectangle $[0,1]\times[0,2]$, we can say

$$\iint_{R^2} f(x,y)\,dA=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(x,y)\,dy\,dx=\int_0^1\int_0^2 Cx(1+y)\,dy\,dx$$

$$=C\int_0^1 x\left[y+\frac{1}{2}y^2\right]_{y=0}^{y=2}dx=C\int_0^1 4x\,dx=C\left[2x^2\right]_0^1=2C$$

Then $2C=1\Rightarrow C=\frac{1}{2}$.

(b)

$$P(X\le 1,Y\le 1)=\int_{-\infty}^{1}\int_{-\infty}^{1}f(x,y)\,dy\,dx=\int_0^1\int_0^1\frac{1}{2}x(1+y)\,dy\,dx$$

$$=\int_0^1\frac{1}{2}x\left[y+\frac{1}{2}y^2\right]_{y=0}^{y=1}dx=\int_0^1\frac{1}{2}x\left(\frac{3}{2}\right)dx=\frac{3}{4}\left[\frac{1}{2}x^2\right]_0^1=\frac{3}{8}\ \text{or } 0.375$$

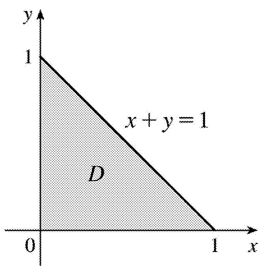
(c) $P(X+Y\le 1)=P((X,Y)\in D)$ where D is the triangular region shown in the figure. Thus

$$P(X+Y\le 1)=\iint_D f(x,y)\,dA=\int_0^1\int_0^{1-x}\frac{1}{2}x(1+y)\,dy\,dx$$

$$=\int_0^1\frac{1}{2}x\left[y+\frac{1}{2}y^2\right]_{y=0}^{y=1-x}dx=\int_0^1\frac{1}{2}x\left(\frac{1}{2}x^2-2x+\frac{3}{2}\right)dx$$

$$=\frac{1}{4}\int_0^1\left(x^3-4x^2+3x\right)dx=\frac{1}{4}\left[\frac{x^4}{4}-4\frac{x^3}{3}+3\frac{x^2}{2}\right]_0^1$$

$$=\frac{5}{48}\approx 0.1042$$



24. **(a)** $f(x,y)\ge 0$, so f is a joint density function if $\iint_{R^2} f(x,y)\,dA=1$. Here, $f(x,y)=0$ outside the square $[0,1]\times[0,1]$, so $\iint_{R^2} f(x,y)\,dA=\int_0^1\int_0^1 4xy\,dy\,dx=\int_0^1\left[2xy^2\right]_{y=0}^{y=1}dx=\int_0^1 2x\,dx=\left.x^2\right]_0^1=1$.

Thus, $f(x,y)$ is a joint density function.

(b)

(a) No restriction is placed on Y , so

$$P\left(X\ge\frac{1}{2}\right)=\int_{1/2}^{\infty}\int_{-\infty}^{\infty}f(x,y)\,dy\,dx=\int_{1/2}^{1}\int_0^1 4xy\,dy\,dx$$

$$=\int_{1/2}^{1}\left[2xy^2\right]_{y=0}^{y=1}dx=\int_{1/2}^{1}2x\,dx=\left.x^2\right]_{1/2}^{1}=\frac{3}{4}$$

(b) No restriction is placed on Y , so

$$P\left(X\ge\frac{1}{2}\right)=\int_{1/2}^{\infty}\int_{-\infty}^{\infty}f(x,y)\,dy\,dx=\int_{1/2}^{1}\int_0^1 4xy\,dy\,dx$$

$$=\int_{1/2}^{1}\left[2xy^2\right]_{y=0}^{y=1}dx=\int_{1/2}^{1}2x\,dx=\left.x^2\right]_{1/2}^{1}=\frac{3}{4}$$

(c)

$$P\left(X\geq \frac{1}{2},Y\leq \frac{1}{2}\right)=\int_{1/2}^{\infty}\int_{-\infty}^{1/2} f(x,y)\,dy\,dx=\int_{1/2}^{1}\int_{0}^{1/2} 4xy\,dy\,dx$$

$$=\int_{1/2}^{1}\left[2xy^2\right]_{y=0}^{y=1/2}dx=\int_{1/2}^{1}\frac{1}{2}x\,dx=\left.\frac{1}{2}\cdot\frac{1}{2}x^2\right]_{1/2}^{1}=\frac{3}{16}$$

(d)

$$P\left(X\geq \frac{1}{2},Y\leq \frac{1}{2}\right)=\int_{1/2}^{\infty}\int_{-\infty}^{1/2} f(x,y)\,dy\,dx=\int_{1/2}^{1}\int_{0}^{1/2} 4xy\,dy\,dx$$

$$=\int_{1/2}^{1}\left[2xy^2\right]_{y=0}^{y=1/2}dx=\int_{1/2}^{1}\frac{1}{2}x\,dx=\left.\frac{1}{2}\cdot\frac{1}{2}x^2\right]_{1/2}^{1}=\frac{3}{16}$$

(c) The expected value of X is given by $\mu_1=\iint_{R^2} x f(x,y)\,dA=\int_0^1\int_0^1 x(4xy)\,dy\,dx=\int_0^1 2x^2\left[y^2\right]_{y=0}^{y=1}dx$

$$=2\int_0^1 x^2\,dx=2\left[\frac{1}{3}x^3\right]_0^1=\frac{2}{3}$$

The expected value of Y is $\mu_2=\iint_{R^2} y f(x,y)\,dA=\int_0^1\int_0^1 y(4xy)\,dy\,dx=\int_0^1 4x\left[\frac{1}{3}y^3\right]_{y=0}^{y=1}dx$

$$=\frac{4}{3}\int_0^1 x\,dx=\frac{4}{3}\left[\frac{1}{2}x^2\right]_0^1=\frac{2}{3}$$

25. **(a)** $f(x,y)\geq 0$, so f is a joint density function if $\iint_{R^2} f(x,y)\,dA=1$. Here, $f(x,y)=0$ outside the first quadrant, so

$$\iint_{R^2} f(x,y)\,dA = \int_0^{\infty}\int_0^{\infty} 0.1e^{-(0.5x+0.2y)}\,dy\,dx=0.1\int_0^{\infty}\int_0^{\infty} e^{-0.5x}e^{-0.2y}\,dy\,dx$$

$$= 0.1\int_0^{\infty} e^{-0.5x}dx\int_0^{\infty} e^{-0.2y}dy=0.1\lim_{t\to\infty}\int_0^{t} e^{-0.5x}dx\lim_{t\to\infty}\int_0^{t} e^{-0.2y}dy$$

$$= 0.1\lim_{t\to\infty}\left[-2e^{-0.5x}\right]_0^t\lim_{t\to\infty}\left[-5e^{-0.2y}\right]_0^t$$

$$= 0.1\lim_{t\to\infty}\left[-2(e^{-0.5t}-1)\right]\lim_{t\to\infty}\left[-5(e^{-0.2t}-1)\right]$$

$$= (0.1)\cdot(-2)(0-1)\cdot(-5)(0-1)=1$$

Thus $f(x,y)$ is a joint density function.

(b)

(a) No restriction is placed on X, so

$$P(Y \geq 1)=\int_{-\infty}^{\infty}\int_{1}^{\infty} f(x,y)\,dy\,dx=\int_{0}^{\infty}\int_{1}^{\infty} 0.1e^{-(0.5x+0.2y)}\,dy\,dx$$

$$= 0.1\int_{0}^{\infty} e^{-0.5x}dx \int_{1}^{\infty} e^{-0.2y}dy=0.1\lim_{t\to\infty}\int_{0}^{t} e^{-0.5x}dx\lim_{t\to\infty}\int_{1}^{t} e^{-0.2y}dy$$

$$= 0.1\lim_{t\to\infty}\left[-2e^{-0.5x}\right]_0^t\lim_{t\to\infty}\left[-5e^{-0.2y}\right]_1^t$$

$$= 0.1\lim_{t\to\infty}\left[-2(e^{-0.5t}-1)\right]\lim_{t\to\infty}\left[-5(e^{-0.2t}-e^{-0.2})\right]$$

$$= (0.1)\cdot(-2)(0-1)\cdot(-5)(0-e^{-0.2})=e^{-0.2}\approx 0.8187$$

(b) No restriction is placed on X, so

$$P(Y \geq 1)=\int_{-\infty}^{\infty}\int_{1}^{\infty} f(x,y)\,dy\,dx=\int_{0}^{\infty}\int_{1}^{\infty} 0.1e^{-(0.5x+0.2y)}\,dy\,dx$$

$$= 0.1\int_{0}^{\infty} e^{-0.5x}dx \int_{1}^{\infty} e^{-0.2y}dy=0.1\lim_{t\to\infty}\int_{0}^{t} e^{-0.5x}dx\lim_{t\to\infty}\int_{1}^{t} e^{-0.2y}dy$$

$$= 0.1\lim_{t\to\infty}\left[-2e^{-0.5x}\right]_0^t\lim_{t\to\infty}\left[-5e^{-0.2y}\right]_1^t$$

$$= 0.1\lim_{t\to\infty}\left[-2(e^{-0.5t}-1)\right]\lim_{t\to\infty}\left[-5(e^{-0.2t}-e^{-0.2})\right]$$

$$= (0.1)\cdot(-2)(0-1)\cdot(-5)(0-e^{-0.2})=e^{-0.2}\approx 0.8187$$

(c)

$$P(X\leq 2,Y\leq 4) =\int_{-\infty}^{2}\int_{-\infty}^{4} f(x,y)\,dy\,dx=\int_{0}^{2}\int_{0}^{4}0.1e^{-(0.5x+0.2y)}\,dy\,dx$$

$$=0.1\int_{0}^{2} e^{-0.5x}dx\int_{0}^{4} e^{-0.2y}dy=0.1\left[-2e^{-0.5x}\right]_0^2\left[-5e^{-0.2y}\right]_0^4$$

$$=(0.1)\cdot(-2)(e^{-1}-1)\cdot(-5)(e^{-0.8}-1)$$

$$=(e^{-1}-1)(e^{-0.8}-1)=1+e^{-1.8}-e^{-0.8}-e^{-1}\approx 0.3481$$

(d)

$$P(X\leq 2,Y\leq 4) = \int_{-\infty}^{2}\int_{-\infty}^{4} f(x,y)\,dy\,dx=\int_0^2\int_0^4 0.1e^{-(0.5x+0.2y)}\,dy\,dx$$
$$=0.1\int_0^2 e^{-0.5x}dx\int_0^4 e^{-0.2y}dy=0.1\left[-2e^{-0.5x}\right]_0^2\left[-5e^{-0.2y}\right]_0^4$$
$$=(0.1)\cdot(-2)(e^{-1}-1)\cdot(-5)(e^{-0.8}-1)$$
$$=(e^{-1}-1)(e^{-0.8}-1)=1+e^{-1.8}-e^{-0.8}-e^{-1}\approx 0.3481$$

(c) The expected value of X is given by

$$\mu_1 = \iint_{R^2} x f(x,y)\,dA=\int_0^{\infty}\int_0^{\infty} x\left[0.1e^{-(0.5x+0.2y)}\right]dy\,dx$$
$$= 0.1\int_0^{\infty} xe^{-0.5x}dx\int_0^{\infty} e^{-0.2y}dy=0.1\lim_{t\to\infty}\int_0^t xe^{-0.5x}dx\lim_{t\to\infty}\int_0^t e^{-0.2y}dy$$

To evaluate the first integral, we integrate by parts with $u=x$ and $dv=e^{-0.5x}dx$ (or we can use Formula 96
in the Table of Integrals):
$\int xe^{-0.5x}dx=-2xe^{-0.5x}-\int -2e^{-0.5x}dx=-2xe^{-0.5x}-4e^{-0.5x}=-2(x+2)e^{-0.5x}$. Thus

$$\mu_1=0.1\lim_{t\to\infty}\left[-2(x+2)e^{-0.5x}\right]_0^t\lim_{t\to\infty}\left[-5e^{-0.2y}\right]_0^t$$
$$=0.1\lim_{t\to\infty}(-2)\left[(t+2)e^{-0.5t}-2\right]\lim_{t\to\infty}(-5)\left[e^{-0.2t}-1\right]$$
$$=0.1(-2)\left(\lim_{t\to\infty}\frac{t+2}{e^{0.5t}}-2\right)(-5)(-1)=2 \text{ [by l'Hospital's Rule]}$$

The expected value of Y is given by

$$\mu_2\&=\&\iint_{R^2} y f(x,y)\,dA=\int_0^{\infty}\int_0^{\infty} y\left[0.1e^{-(0.5+0.2y)}\right]dy\,dx$$
$$=0.1\int_0^{\infty} e^{-0.5x}dx\int_0^{\infty} ye^{-0.2y}dy=0.1\lim_{t\to\infty}\int_0^t e^{-0.5x}dx\lim_{t\to\infty}\int_0^t ye^{-0.2y}dy$$

To evaluate the second integral, we integrate by parts with $u=y$ and $dv=e^{-0.2y}dy$ (or again we can use Formula 96 in the Table of Integrals) which gives $\int ye^{-0.2y}dy=-5ye^{-0.2y}+\int 5e^{-0.2y}dy=-5(y+5)e^{-0.2y}$. Then

$$\mu_2 = 0.1 \lim_{t\to\infty} \left[-2e^{-0.5x}\right]_0^t \lim_{t\to\infty} \left[-5(y+5)e^{-0.2y}\right]_0^t$$
$$= 0.1 \lim_{t\to\infty} \left[-2(e^{-0.5t}-1)\right] \lim_{t\to\infty} \left(-5\left[(t+5)e^{-0.2t}-5\right]\right)$$
$$= 0.1(-2)(-1)\cdot(-5)\left(\lim_{t\to\infty} \frac{t+5}{e^{0.2t}} - 5\right) = 5 \text{ [by l'Hospital's Rule]}$$

26. **(a)** Each lamp has exponential density function

$$f(t) = \begin{cases} 0 & \text{if } t<0 \\ \frac{1}{1000} e^{-t/1000} & \text{if } t \geq 0 \end{cases}$$

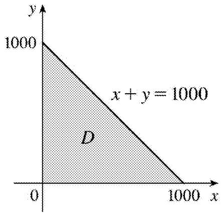
If X and Y are the lifetimes of the individual bulbs, then X and Y are independent, so the joint density function is the product of the individual density functions:

$$f(x,y) = \begin{cases} 10^{-6} e^{-(x+y)/1000} & \text{if } x\geq 0, y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

The probability that both of the bulbs fail within 1000 hours is

$$P(X \leq 1000, Y \leq 1000) = \int_{-\infty}^{1000} \int_{-\infty}^{1000} f(x,y)\,dy\,dx$$
$$= \int_0^{1000} \int_0^{1000} 10^{-6} e^{-(x+y)/1000}\,dy\,dx$$
$$= 10^{-6} \int_0^{1000} e^{-x/1000}\,dx \int_0^{1000} e^{-y/1000}\,dy$$
$$= 10^{-6}\left[-1000e^{-x/1000}\right]_0^{1000} \left[-1000e^{-y/1000}\right]_0^{1000}$$
$$= \left(e^{-1}-1\right)^2 \approx 0.3996$$

(b) Now we are asked for the probability that the combined lifetimes of both bulbs is 1000 hours or less. Thus we want to find $P(X+Y \leq 1000)$, or equivalently $P\left((X,Y) \in D\right)$ where D is the triangular region shown in the figure.
Then



$$P(X+Y \le 1000) = \iint_D f(x,y)\,dA = \int_0^{1000}\int_0^{1000-x} 10^{-6} e^{-(x+y)/1000}\,dy\,dx$$

$$= 10^{-6}\int_0^{1000}\left[-1000e^{-(x+y)/1000}\right]_{y=0}^{y=1000-x} dx$$

$$= -10^{-3}\int_0^{1000}\left(e^{-1} - e^{-x/1000}\right) dx$$

$$= -10^{-3}\left[e^{-1}x + 1000e^{-x/1000}\right]_0^{1000} = 1 - 2e^{-1} \approx 0.2642$$

27. **(a)** The random variables X and Y are normally distributed with $\mu_1 = 45$, $\mu_2 = 20$, $\sigma_1 = 0.5$, and $\sigma_2 = 0.1$.

The individual density functions for X and Y , then, are $f_1(x) = \frac{1}{0.5\sqrt{2\pi}}\, e^{-(x-45)^2/0.5}$ and $f_2(y) = \frac{1}{0.1\sqrt{2\pi}}\, e^{-(y-20)^2/0.02}$. Since X and Y are independent, the joint density function is the product $f(x,y)$ & = & $f_1(x)f_2(y) = \frac{1}{0.5\sqrt{2\pi}}\, e^{-(x-45)^2/0.5}\, \frac{1}{0.1\sqrt{2\pi}}\, e^{-(y-20)^2/0.02}$

& = & $\frac{10}{\pi}\, e^{-2(x-45)^2 - 50(y-20)^2}$ Then

$P(40$ & $\le$ & $X \le 50, 20 \le Y \le 25) = \int_{40}^{50}\int_{20}^{25} f(x,y)\,dy\,dx$

& = & $\frac{10}{\pi}\int_{40}^{50}\int_{20}^{25} e^{-2(x-45)^2 - 50(y-20)^2}\,dy\,dx$ Using a CAS or calculator to evaluate the integral, we get $P(40 \le X \le 50\ ,\ 20 \le Y \le 25) \approx 0.500$.

(b) $P(4(X-45)^2 + 100(Y-20)^2 \le 2) = \iint_D \frac{10}{\pi}\, e^{-2(x-45)^2 - 50(y-20)^2}\,dA$, where D is the region enclosed by the ellipse

$4(x-45)^2+100(y-20)^2=2$. Solving for y gives $y=20\pm\frac{1}{10}\sqrt{2-4(x-45)^2}$, the upper and lower halves of the ellipse, and these two halves meet where $y=20 \Rightarrow 4(x-45)^2=2 \Rightarrow x=45\pm\frac{1}{\sqrt{2}}$. Thus

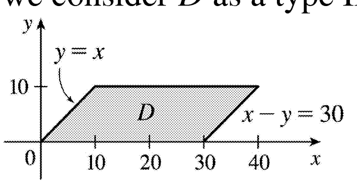
$$\iint_D \frac{10}{\pi} e^{-2(x-45)^2-50(y-20)^2} dA=\frac{10}{\pi}\int_{45-1/\sqrt{2}}^{45+1/\sqrt{2}}\int_{20-\frac{1}{10}\sqrt{2-4(x-45)^2}}^{20+\frac{1}{10}\sqrt{2-4(x-45)^2}} e^{-2(x-45)^2-50(y-20)^2}\, dy\, dx .$$

Using a CAS or calculator to evaluate the integral, we get $P(4(X-45)^2+100(Y-20)^2\le 2)\approx 0.632$.

28. Because X and Y are independent, the joint density function for Xavier's and Yolanda's arrival times is the product of the individual density functions:

$$f(x,y)=f_1(x)f_2(y)=\begin{cases} \frac{1}{50}e^{-x}y & \text{if } x\ge 0, 0\le y\le 10 \\ 0 & \text{otherwise} \end{cases}$$

Since Xavier won't wait for Yolanda, they won't meet unless $X\ge Y$. Additionally, Yolanda will wait up to half an
hour but no longer, so they won't meet unless $X-Y\le 30$. Thus the probability that they meet is $P((X,Y)\in D)$ where D is the parallelogram shown in the figure. The integral is simpler to evaluate if we consider D as a type II region, so



$$P((X,Y)\in D)=\iint_D f(x,y)\,dx\,dy=\int_0^{10}\int_y^{y+30}\frac{1}{50}e^{-x}y\,dx\,dy=\frac{1}{50}\int_0^{10} y\left[-e^{-x}\right]_{x=y}^{x=y+30} dy$$

$$=\frac{1}{50}\int_0^{10} y(-e^{-(y+30)}+e^{-y})dy=\frac{1}{50}(1-e^{-30})\int_0^{10} ye^{-y}dy$$

By integration by parts (or Formula 96 in the Table of Integrals), this is $\frac{1}{50}(1-e^{-30})\left[-(y+1)e^{-y}\right]_0^{10}=\frac{1}{50}(1-e^{-30})(1-11e^{-10})\approx 0.020$. Thus there is only about a 2% chance they will meet. Such is student life!

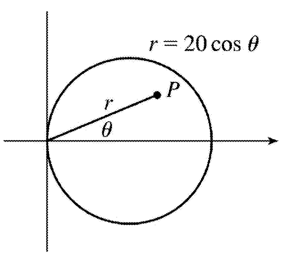
29. **(a)** If $f(P,A)$ is the probability that an individual at A will be infected by an individual at P , and $k\,dA$ is the number of infected individuals in an element of area dA , then $f(P,A)k\,dA$ is the number of

infections that should result from exposure of the individual at A to infected people in the element of area dA . Integration over D gives the number of infections of the person at A due to all the infected people in D . In rectangular coordinates (with the origin at the city's center), the exposure of a person at A is

$$E=\iint_D kf(\mathrm{P,A})\,dA=k\iint_D \frac{20-d(\mathrm{P,A})}{20}\,dA=k\iint_D\left[1-\frac{\sqrt{(x-x_0)^2+(y-y_0)^2}}{20}\right]dx\,dy\,.$$

(b) If $A=(0,0)$, then

$$\begin{aligned}E &= k\iint_D\left[1-\frac{1}{20}\sqrt{x^2+y^2}\right]dx\,dy\\ &= k\int_0^{2\pi}\int_0^{10}\left(1-\frac{r}{20}\right)r\,dr\,d\theta=2\pi k\left[\frac{r^2}{2}-\frac{r^3}{60}\right]_0^{10}\\ &= 2\pi k\left(50-\frac{50}{3}\right)=\frac{200}{3}\pi k\approx 209k\end{aligned}$$



For A at the edge of the city, it is convenient to use a polar coordinate system centered at A . Then the polar equation for the circular boundary of the city becomes $r=20\cos\theta$ instead of $r=10$, and the distance from A to a point P in the city is again r (see the figure). So

$$\begin{aligned}E &= k\int_{-\pi/2}^{\pi/2}\int_0^{20\cos\theta}\left(1-\frac{r}{20}\right)r\,dr\,d\theta=k\int_{-\pi/2}^{\pi/2}\left[\frac{r^2}{2}-\frac{r^3}{60}\right]_{r=0}^{r=20\cos\theta}d\theta\\ &= k\int_{-\pi/2}^{\pi/2}\left(200\cos^2\theta-\frac{400}{3}\cos^3\theta\right)d\theta=200k\int_{-\pi/2}^{\pi/2}\left[\frac{1}{2}+\frac{1}{2}\cos 2\theta-\frac{2}{3}\left(1-\sin^2\theta\right)\cos\theta\right]d\theta\\ &= 200k\left[\frac{1}{2}\theta+\frac{1}{4}\sin 2\theta-\frac{2}{3}\sin\theta+\frac{2}{3}\cdot\frac{1}{3}\sin^3\theta\right]_{-\pi/2}^{\pi/2}=200k\left[\frac{\pi}{4}+0-\frac{2}{3}+\frac{2}{9}+\frac{\pi}{4}+0-\frac{2}{3}+\frac{2}{9}\right]\\ &= 200k\left(\frac{\pi}{2}-\frac{8}{9}\right)\approx 136k\end{aligned}$$

Therefore the risk of infection is much lower at the edge of the city than in the middle, so it is better to live at the edge.

1. Here $z=f(x,y)=2+3x+4y$ and D is the rectangle $[0,5]\times[1,4]$, so by Formula 2 the area of the surface is

$$A(S) = \iint_D \sqrt{[f_x(x,y)]^2+[f_y(x,y)]^2+1}\, dA=\iint_D \sqrt{3^2+4^2+1}\, dA=\sqrt{26}\iint_D dA$$
$$= \sqrt{26}\, A(D)=\sqrt{26}\,(5)(3)=15\sqrt{26}$$

2. $z=f(x,y)=10-2x-5y$ and D is the disk $x^2+y^2\le 9$, so by Formula 2

$$A(S) = \iint_D \sqrt{(-2)^2+(-5)^2+1}\, dA=\sqrt{30}\iint_D dA=\sqrt{30}\, A(D)$$
$$= \sqrt{30}\,(\pi\cdot 3^2)=9\sqrt{30}\,\pi$$

3. $z=f(x,y)=6-3x-2y$ which intersects the xy –plane in the line $3x+2y=6$, so D is the triangular region given by $\left\{ (x,y) \;\middle|\; 0\le x\le 2, 0\le y\le 3-\frac{3}{2}x \right\}$. Thus

$$A(S) = \iint_D \sqrt{(-3)^2+(-2)^2+1}\, dA=\sqrt{14}\iint_D dA=\sqrt{14}\, A(D)$$
$$= \sqrt{14}\left(\frac{1}{2}\cdot 2\cdot 3\right)=3\sqrt{14}$$

4. $z=f(x,y)=1+3x+2y^2$ with $0\le x\le 2y$, $0\le y\le 1$. Thus by Formula 2,

$$A(S) =\iint_D \sqrt{(3)^2+(4y)^2+1}\, dA=\int_0^1\int_0^{2y} \sqrt{10+16y^2}\, dx\,dy=\int_0^1 \sqrt{10+16y^2}\,[x]_{x=0}^{x=2y}\, dy$$
$$=\int_0^1 2y\sqrt{10+16y^2}\, dy= \left. 2\cdot\frac{1}{32}\cdot\frac{2}{3}(10+16y^2)^{3/2}\right]_0^1=\frac{1}{24}(26^{3/2}-10^{3/2})$$

5. $y^2+z^2=9\Rightarrow z=\sqrt{9-y^2}$. $f_x=0$, $f_y=-y(9-y^2)^{-1/2}\Rightarrow$

$$A(S) =\int_0^4\int_0^2 \sqrt{0^2+[-y(9-y^2)^{-1/2}]^2+1}\, dy\,dx=\int_0^4\int_0^2 \sqrt{\frac{y^2}{9-y^2}+1}\, dy\,dx$$
$$=\int_0^4\int_0^2 \frac{3}{\sqrt{9-y^2}}\, dy\,dx=3\int_0^4 \left[\sin^{-1}\frac{y}{3}\right]_{y=0}^{y=2} dx=3\left[\left(\sin^{-1}\left(\frac{2}{3}\right)\right)x\right]_0^4=12\sin^{-1}\left(\frac{2}{3}\right)$$

6. $z=f(x,y)=4-x^2-y^2$ and D is the projection of the paraboloid $z=4-x^2-y^2$ onto the xy –plane, that is, $D=\left\{(x,y) \mid x^2+y^2\le 4\right\}$. So $f_x=-2x$, $f_y=-2y\Rightarrow$

$$A(S)=\iint_D \sqrt{(-2x)^2+(-2y)^2+1}\,dA=\iint_D \sqrt{4(x^2+y^2)+1}\,dA=\int_0^{2\pi}\int_0^2 \sqrt{4r^2+1}\,r\,dr\,d\theta$$

$$=\int_0^{2\pi}\left[\frac{1}{12}(4r^2+1)^{3/2}\right]_{r=0}^{r=2}d\theta=\int_0^{2\pi}\frac{1}{12}\left(17\sqrt{17}-1\right)d\theta=\frac{\pi}{6}\left(17\sqrt{17}-1\right)$$

7. $z=f(x,y)=y^2-x^2$ with $1\le x^2+y^2\le 4$. Then

$$A(S)=\iint_D \sqrt{1+4x^2+4y^2}\,dA=\int_0^{2\pi}\int_1^2 \sqrt{1+4r^2}\,r\,dr\,d\theta=\int_0^{2\pi}d\theta\int_1^2 r\sqrt{1+4r^2}\,dr$$

$$=\left[\theta\right]_0^{2\pi}\left[\frac{1}{12}(1+4r^2)^{3/2}\right]_1^2=\frac{\pi}{6}\left(17\sqrt{17}-5\sqrt{5}\right)$$

8. $z=f(x,y)=\frac{2}{3}(x^{3/2}+y^{3/2})$ and $D=\{(x,y)\mid 0\le x\le 1, 0\le y\le 1\}$. Then $f_x=x^{1/2}$, $f_y=y^{1/2}$ and

$$A(S)=\iint_D \sqrt{\left(\sqrt{x}\right)^2+\left(\sqrt{y}\right)^2+1}\,dA=\int_0^1\int_0^1 \sqrt{x+y+1}\,dy\,dx$$

$$=\int_0^1\left[\frac{2}{3}(x+y+1)^{3/2}\right]_{y=0}^{y=1}dx=\frac{2}{3}\int_0^1\left[(x+2)^{3/2}-(x+1)^{3/2}\right]dx$$

$$=\frac{2}{3}\left[\frac{2}{5}(x+2)^{5/2}-\frac{2}{5}(x+1)^{5/2}\right]_0^1=\frac{4}{15}(3^{5/2}-2^{5/2}-2^{5/2}+1)$$

$$=\frac{4}{15}(3^{5/2}-2^{7/2}+1)$$

9. $z=f(x,y)=xy$ with $0\le x^2+y^2\le 1$, so $f_x=y$, $f_y=x\Rightarrow$

$$A(S)=\iint_D \sqrt{y^2+x^2+1}\,dA=\int_0^{2\pi}\int_0^1 \sqrt{r^2+1}\,r\,dr\,d\theta=\int_0^{2\pi}\left[\frac{1}{3}(r^2+1)^{3/2}\right]_{r=0}^{r=1}d\theta$$

$$=\int_0^{2\pi}\frac{1}{3}\left(2\sqrt{2}-1\right)d\theta=\frac{2\pi}{3}\left(2\sqrt{2}-1\right)$$

10. Given the sphere $x^2+y^2+z^2=4$, when $z=1$, we get $x^2+y^2=3$ so $D=\left\{ (x,y) | x^2+y^2 \le 3 \right\}$ and $z=f(x,y)=\sqrt{4-x^2-y^2}$. Thus

$$A(S)=\iint_D \sqrt{[(-x)(4-x^2-y^2)^{-1/2}]^2+[(-y)(4-x^2-y^2)^{-1/2}]^2+1}\; dA$$

$$=\int_0^{2\pi}\int_0^{\sqrt{3}} \sqrt{\frac{r^2}{4-r^2}+1}\; r\,dr\,d\theta=\int_0^{2\pi}\int_0^{\sqrt{3}} \frac{2r}{\sqrt{4-r^2}}\,dr\,d\theta$$

$$=\int_0^{2\pi}\left[-2(4-r^2)^{1/2}\right]_{r=0}^{r=\sqrt{3}} d\theta=\int_0^{2\pi}(-2+4)\,d\theta=2\theta\,\Big]_0^{2\pi}=4\pi$$

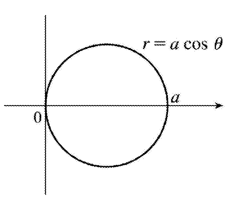
11. $z=\sqrt{a^2-x^2-y^2}$, $z_x=-x(a^2-x^2-y^2)^{-1/2}$, $z_y=-y(a^2-x^2-y^2)^{-1/2}$,

$$A(S)=\iint_D \sqrt{\frac{x^2+y^2}{a^2-x^2-y^2}+1}\; dA$$

$$=\int_{-\pi/2}^{\pi/2}\int_0^{a\cos\theta} \sqrt{\frac{r^2}{a^2-r^2}+1}\; r\,dr\,d\theta$$

$$=\int_{-\pi/2}^{\pi/2}\int_0^{a\cos\theta} \frac{ar}{\sqrt{a^2-r^2}}\,dr\,d\theta$$

$$=\int_{-\pi/2}^{\pi/2}\left[-a\sqrt{a^2-r^2}\right]_{r=0}^{r=a\cos\theta} d\theta$$



$$=\int_{-\pi/2}^{\pi/2}-a\left(\sqrt{a^2-a^2\cos^2\theta}-a\right)d\theta=2a^2\int_0^{\pi/2}\left(1-\sqrt{1-\cos^2\theta}\right)d\theta$$

$$=2a^2\int_0^{\pi/2} d\theta - 2a^2\int_0^{\pi/2}\sqrt{\sin^2\theta}\ d\theta = a^2\pi - 2a^2\int_0^{\pi/2}\sin\theta\ d\theta = a^2(\pi-2)$$

12. To find the region D : $z=x^2+y^2$ implies $z+z^2=4z$ or $z^2-3z=0$. Thus $z=0$ or $z=3$ are the planes where the surfaces intersect. But $x^2+y^2+z^2=4z$ implies $x^2+y^2+(z-2)^2=4$, so $z=3$ intersects the upper hemisphere. Thus $(z-2)^2=4-x^2-y^2$ or $z=2+\sqrt{4-x^2-y^2}$. Therefore D is the region inside the circle $x^2+y^2+(3-2)^2=4$, that is, $D=\left\{ (x,y)\,|\,x^2+y^2\le 3\right\}$.

$$A(S)=\iint_D \sqrt{1+[(-x)(4-x^2-y^2)^{-1/2}]^2+[(-y)(4-x^2-y^2)^{-1/2}]^2}\ dA$$

$$=\int_0^{2\pi}\int_0^{\sqrt{3}}\sqrt{1+\frac{r^2}{4-r^2}}\ r\,dr\,d\theta=\int_0^{2\pi}\int_0^{\sqrt{3}}\frac{2r\,dr}{\sqrt{4-r^2}}\ d\theta=\int_0^{2\pi}\left[-2(4-r^2)^{1/2}\right]_{r=0}^{r=\sqrt{3}} d\theta$$

$$=\int_0^{2\pi}(-2+4)\,d\theta = 2\theta\,\Big]_0^{2\pi}=4\pi$$

13. $z=f(x,y)=e^{-x^2-y^2}$, $f_x=-2xe^{-x^2-y^2}$, $f_y=-2ye^{-x^2-y^2}$. Then

$$A(S)=\iint_{x^2+y^2\le 4}\sqrt{(-2xe^{-x^2-y^2})^2+(-2ye^{-x^2-y^2})^2+1}\ dA=\iint_{x^2+y^2\le 4}\sqrt{4(x^2+y^2)e^{-2(x^2+y^2)}+1}\ dA\ .$$

Converting to polar coordinates we have

$$A(S)=\int_0^{2\pi}\int_0^{2}\sqrt{4r^2e^{-2r^2}+1}\ r\,dr\,d\theta=\int_0^{2\pi}d\theta\int_0^{2}r\sqrt{4r^2e^{-2r^2}+1}\ dr$$

$$=2\pi\int_0^{2}r\sqrt{4r^2e^{-2r^2}+1}\ dr\approx 13.9783 \text{ using a calculator.}$$

14. $z=f(x,y)=\cos(x^2+y^2)$, $f_x=-2x\sin(x^2+y^2)$, $f_y=-2y\sin(x^2+y^2)$.

$$A(S)=\iint_{x^2+y^2\le 1}\sqrt{4x^2\sin^2(x^2+y^2)+4y^2\sin^2(x^2+y^2)+1}\ dA$$

$$=\iint_{x^2+y^2\le 1}\sqrt{4(x^2+y^2)\sin^2(x^2+y^2)+1}\ dA$$

Converting to polar coordinates gives

$$A(S)=\int_0^{2\pi}\int_0^1 \sqrt{4r^2\sin^2(r^2)+1}\; r\,dr\,d\theta=\int_0^{2\pi} d\theta \int_0^1 r\sqrt{4r^2\sin^2(r^2)+1}\;dr$$

$$=2\pi\int_0^1 r\sqrt{4r^2\sin^2(r^2)+1}\;dr\approx 4.1073 \text{ using a calculator.}$$

15. **(a)** The midpoints of the four squares are $\left(\frac{1}{4},\frac{1}{4}\right)$, $\left(\frac{1}{4},\frac{3}{4}\right)$, $\left(\frac{3}{4},\frac{1}{4}\right)$, and $\left(\frac{3}{4},\frac{3}{4}\right)$.
Here $f(\mathrm{x,y})=x^2+y^2$, so the Midpoint Rule gives

$$A(S)=\iint_D \sqrt{[f_x(\mathrm{x,y})]^2+[f_y(\mathrm{x,y})]^2+1}\;dA=\iint_D \sqrt{(2x)^2+(2y)^2+1}\;dA$$

$$\approx \frac{1}{4}\left(\sqrt{\left[2\left(\frac{1}{4}\right)\right]^2+\left[2\left(\frac{1}{4}\right)\right]^2+1}+\sqrt{\left[2\left(\frac{1}{4}\right)\right]^2+\left[2\left(\frac{3}{4}\right)\right]^2+1}\right.$$

$$\left.+\sqrt{\left[2\left(\frac{3}{4}\right)\right]^2+\left[2\left(\frac{1}{4}\right)\right]^2+1}+\sqrt{\left[2\left(\frac{3}{4}\right)\right]^2+\left[2\left(\frac{3}{4}\right)\right]^2+1}\right)$$

$$=\frac{1}{4}\left(\sqrt{\frac{3}{2}}+2\sqrt{\frac{7}{2}}+\sqrt{\frac{11}{2}}\right)\approx 1.8279$$

(b) A CAS estimates the integral to be $A(S)=\iint_D \sqrt{1+(2x)^2+(2y)^2}\;dA=\int_0^1\int_0^1 \sqrt{1+4x^2+4y^2}\;dy\,dx\approx 1.8616$.
This agrees with the Midpoint estimate only in the first decimal place.

16. **(a)** With $m=n=2$ we have four squares with midpoints $\left(\frac{1}{2},\frac{1}{2}\right)$, $\left(\frac{1}{2},\frac{3}{2}\right)$, $\left(\frac{3}{2},\frac{1}{2}\right)$, and $\left(\frac{3}{2},\frac{3}{2}\right)$. Since $z=xy+x^2+y^2$, the Midpoint Rule gives

$$A(S)=\iint_D \sqrt{[f_x(\mathrm{x,y})]^2+[f_y(\mathrm{x,y})]^2+1}\;dA=\iint_D \sqrt{(y+2x)^2+(x+2y)^2+1}\;dA$$

$$\approx 1\left(\sqrt{\left(\frac{3}{2}\right)^2+\left(\frac{3}{2}\right)^2+1}+\sqrt{\left(\frac{5}{2}\right)^2+\left(\frac{7}{2}\right)^2+1}+\sqrt{\left(\frac{7}{2}\right)^2+\left(\frac{5}{2}\right)^2+1}+\sqrt{\left(\frac{9}{2}\right)^2+\left(\frac{9}{2}\right.}\right.$$

$$=\frac{\sqrt{22}}{2}+\frac{\sqrt{78}}{2}+\frac{\sqrt{78}}{2}+\frac{\sqrt{166}}{2}\approx 17.619$$

(b) Using a CAS, we have $A(S)=\iint_D \sqrt{(y+2x)^2+(x+2y)^2+1}\, dA=\int_0^2\int_0^2 \sqrt{1+(y+2x)^2+(x+2y)^2}\, dy\,dx \approx 17.7165$. This is within about 0.1 of the Midpoint Rule estimate.

17. $z=1+2x+3y+4y^2$, so

$$A(S)=\iint_D \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\, dA=\int_1^4\int_0^1 \sqrt{1+4+(3+8y)^2}\, dy\,dx$$

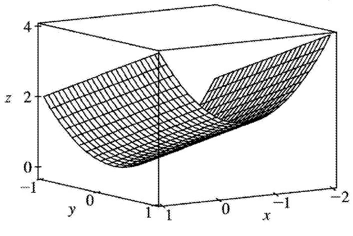
$$=\int_1^4\int_0^1 \sqrt{14+48y+64y^2}\, dy\,dx .$$

Using a CAS, we have

$$\int_1^4\int_0^1 \sqrt{14+48y+64y^2}\, dy\,dx=\frac{45}{8}\sqrt{14}+\frac{15}{16}\ln\left(11\sqrt{5}+3\sqrt{14}\sqrt{5}\right)-\frac{15}{16}\ln\left(3\sqrt{5}+\sqrt{14}\sqrt{5}\right)$$

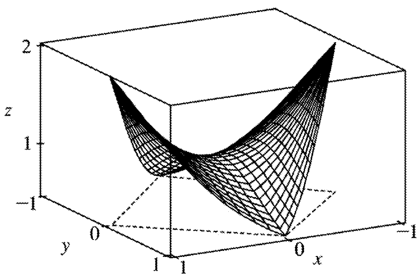
or $\frac{45}{8}\sqrt{14}+\frac{15}{16}\ln\frac{11\sqrt{5}+3\sqrt{70}}{3\sqrt{5}+\sqrt{70}}$.

18. $f(\mathrm{x,y})=1+x+y+x^2 \Rightarrow f_x=1+2x$, $f_y=1$. We use a CAS to calculate the integral $A(S)=\int_{-2}^1\int_{-1}^1 \sqrt{f_x^2+f_y^2+1}\, dy\,dx=\int_{-2}^1\int_{-1}^1 \sqrt{(1+2x)^2+2}\, dy\,dx=2\int_{-2}^1 \sqrt{4x^2+4x+3}\, dx$ and find that $A(S)=3\sqrt{11}+2\sinh^{-1}\left(\frac{3\sqrt{2}}{2}\right)$ or $A(S)=3\sqrt{11}+\ln\left(10+3\sqrt{11}\right)$.



19. $f(\mathrm{x,y})=1+x^2y^2 \Rightarrow f_x=2xy^2$, $f_y=2x^2y$. We use a CAS (with precision reduced to five significant digits, to speed up the calculation) to estimate the integral $A(S)=\int_{-1}^1\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{f_x^2+f_y^2+1}\, dy\,dx=\int_{-1}^1\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{4x^2y^4+4x^4y^2+1}\, dy\,dx$, and find that $A(S)\approx 3.3213$.

20. Let $f(x,y)=\frac{1+x^2}{1+y^2}$. Then $f_x=\frac{2x}{1+y^2}$, $f_y=\left(1+x^2\right)\left[-\frac{2y}{\left(1+y^2\right)^2}\right]=-\frac{2y\left(1+x^2\right)}{\left(1+y^2\right)^2}$. We use a CAS to estimate $\int_{-1}^{1}\int_{-(1-|x|)}^{1-|x|}\sqrt{f_x^2+f_y^2+1}\,dy\,dx\approx 2.6959$. In order to graph only the part of the surface above the square, we use $-(1-|x|)\le y\le 1-|x|$ as the y –range in our plot command.



21. Here $z=f(x,y)=ax+by+c$, $f_x(x,y)=a$, $f_y(x,y)=b$, so
$A(S)=\iint_D\sqrt{a^2+b^2+1}\,dA=\sqrt{a^2+b^2+1}\iint_D dA=\sqrt{a^2+b^2+1}\,A(D)$.

22. Let S be the upper hemisphere. Then $z=f(x,y)=\sqrt{a^2-x^2-y^2}$, so

$$A(S)=\iint_D\sqrt{[-x(a^2-x^2-y^2)^{-1/2}]^2+[-y(a^2-x^2-y^2)^{-1/2}]^2+1}\,dA$$

$$=\iint_D\sqrt{\frac{x^2+y^2}{a^2-x^2-y^2}+1}\,dA=\int_0^{2\pi}\int_0^t\sqrt{\frac{r^2}{a^2-r^2}+1}\,r\,dr\,d\theta$$

$$=\lim_{t\to a^-}\int_0^{2\pi}\int_0^t\frac{ar}{\sqrt{a^2-r^2}}\,dr\,d\theta=2\pi\lim_{t\to a^-}\left[-a\sqrt{a^2-r^2}\right]_0^t=2\pi\lim_{t\to a^-}-a\left[\sqrt{a^2-t^2}-a\right]$$

$=2\pi(-a)(-a)=2\pi a^2$. Thus the surface area of the entire sphere is $4\pi a^2$.

23. If we project the surface onto the xz –plane, then the surface lies ‘‘above’’ the disk $x^2+z^2\le 25$ in the xz –plane.

We have $y=f(x,z)=x^2+z^2$ and, adapting Formula 2, the area of the surface is

$$A(S)=\iint_{x^2+z^2\le 25}\sqrt{[f_x(x,z)]^2+[f_z(x,z)]^2+1}\,dA=\iint_{x^2+z^2\le 25}\sqrt{4x^2+4z^2+1}\,dA$$

Converting to polar coordinates $x=r\cos\theta$, $z=r\sin\theta$ we have

$$A(S)=\int_0^{2\pi}\int_0^5\sqrt{4r^2+1}\; r\,dr\,d\theta=\int_0^{2\pi}d\theta\int_0^5 r(4r^2+1)^{1/2}dr=[\theta]_0^{2\pi}\left[\frac{1}{12}(4r^2+1)^{3/2}\right]_0^5$$

$$=\frac{\pi}{6}\left(101\sqrt{101}-1\right)$$

24. First we find the area of the face of the surface that intersects the positive y –axis. As in Exercise 23 , we can project the face onto the xz –plane, so the surface lies ''above'' the disk $x^2+z^2\leq 1$. Then $z=f(x,z)=\sqrt{1-z^2}$ and the area is

$$A(S)=\iint_{x^2+z^2\leq 1}\sqrt{[f_x(x,z)]^2+[f_z(x,z)]^2+1}\;dA=\iint_{x^2+z^2\leq 1}\sqrt{0+\left(\frac{-z}{\sqrt{1-z^2}}\right)^2+1}\;dA$$

$$=\iint_{x^2+z^2\leq 1}\sqrt{\frac{z^2}{1-z^2}+1}\;dA=\int_{-1}^{1}\int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}}\frac{1}{\sqrt{1-z^2}}\,dx\,dz$$

$$=4\int_0^1\int_0^{\sqrt{1-z^2}}\frac{1}{\sqrt{1-z^2}}\,dx\,dz \text{ [by the symmetry of the surface]}$$

This integral is improper (when $z=1$), so

$$A(S)=\lim_{t\to 1^-}4\int_0^t\int_0^{\sqrt{1-z^2}}\frac{1}{\sqrt{1-z^2}}\,dx\,dz=\lim_{t\to 1^-}4\int_0^t\frac{\sqrt{1-z^2}}{\sqrt{1-z^2}}\,dz=\lim_{t\to 1^-}4\int_0^t dz=\lim_{t\to 1^-}4t=4 .$$

Since the complete surface consists of four congruent faces, the total surface area is 4(4)=16 .

1.

$$\iiint_B xyz^2\,dV = \int_0^1\int_{-1}^2\int_0^3 xyz^2\,dz\,dx\,dy = \int_0^1\int_{-1}^2 xy\left[\frac{1}{3}z^3\right]_{z=0}^{z=3}dx\,dy = \int_0^1\int_{-1}^2 9xy\,dx\,dy$$

$$= \int_0^1\left[\frac{9}{2}x^2y\right]_{x=-1}^{x=2}dy = \int_0^1 \frac{27}{2}\,y\,dy = \left.\frac{27}{4}y^2\right]_0^1 = \frac{27}{4}$$

2. There are six different possible orders of integration.

$$\iiint_E (xz-y^3)\,dV = \int_{-1}^1\int_0^2\int_0^1 (xz-y^3)\,dz\,dy\,dx = \int_{-1}^1\int_0^2\left[\frac{1}{2}xz^2-y^3z\right]_{z=0}^{z=1}dy\,dx$$

$$= \int_{-1}^1\int_0^2\left(\frac{1}{2}x-y^3\right)dy\,dx = \int_{-1}^1\left[\frac{1}{2}xy-\frac{1}{4}y^4\right]_{y=0}^{y=2}dx$$

$$= \int_{-1}^1 (x-4)\,dx = \left[\frac{1}{2}x^2-4x\right]_{-1}^1 = -8$$

$$\iiint_E (xz-y^3)\,dV = \int_0^2\int_{-1}^1\int_0^1 (xz-y^3)\,dz\,dx\,dy = \int_0^2\int_{-1}^1\left[\frac{1}{2}xz^2-y^3z\right]_{z=0}^{z=1}dx\,dy$$

$$= \int_0^2\int_{-1}^1\left(\frac{1}{2}x-y^3\right)dx\,dy = \int_0^2\left[\frac{1}{4}x^2-xy^3\right]_{x=-1}^{x=1}dy$$

$$= \int_0^2 -2y^3\,dy = \left.-\frac{1}{2}y^4\right]_0^2 = -8$$

$$\iiint_E (xz-y^3)\,dV = \int_{-1}^1\int_0^1\int_0^2 (xz-y^3)\,dy\,dz\,dx = \int_{-1}^1\int_0^1\left[xyz-\frac{1}{4}y^4\right]_{y=0}^{y=2}dz\,dx$$

$$= \int_{-1}^1\int_0^1 (2xz-4)\,dz\,dx = \int_{-1}^1\left[xz^2-4z\right]_{z=0}^{z=1}dx$$

$$= \int_{-1}^1 (x-4)\,dx = \left[\frac{1}{2}x^2-4x\right]_{-1}^1 = -8$$

$$\iiint_E (xz-y^3)\,dV = \int_0^1\int_{-1}^1\int_0^2 (xz-y^3)\,dy\,dx\,dz = \int_0^1\int_{-1}^1\left[xyz-\frac{1}{4}y^4\right]_{y=0}^{y=2}dx\,dz$$

$$=\int_0^1\int_{-1}^1 (2xz-4)\,dx\,dz=\int_0^1 \Big[x^2z-4x\Big]_{x=-1}^{x=1}dz$$

$$=\int_0^1 -8\,dz= -8z\Big]_0^1=-8$$

$$\iiint_E (xz-y^3)\,dV =\int_0^2\int_0^1\int_{-1}^1 (xz-y^3)\,dx\,dz\,dy=\int_0^2\int_0^1 \left[\frac{1}{2}x^2z-xy^3\right]_{x=-1}^{x=1}dz\,dy$$

$$=\int_0^2\int_0^1 -2y^3\,dz\,dy=\int_0^2 \Big[-2y^3z\Big]_{z=0}^{z=1}dy=\int_0^2 -2y^3\,dy= -\frac{1}{2}y^4\Big]_0^2=-8$$

$$\iiint_E \left(xz-y^3\right)dV =\int_0^1\int_0^2\int_{-1}^1 (xz-y^3)\,dx\,dy\,dz=\int_0^1\int_0^2 \left[\frac{1}{2}x^2z-xy^3\right]_{x=-1}^{x=1}dy\,dz$$

$$=\int_0^1\int_0^2 -2y^3\,dy\,dz=\int_0^1 \left[-\frac{1}{2}y^4\right]_{y=0}^{y=2}dz=\int_0^1 -8\,dz= -8z\Big]_0^1=-8$$

3.

$$\int_0^1\int_0^z\int_0^{x+z} 6xz\,dy\,dx\,dz =\int_0^1\int_0^z \big[6xyz\big]_{y=0}^{y=x+z}dx\,dz=\int_0^1\int_0^z 6xz(x+z)\,dx\,dz$$

$$=\int_0^1 \Big[2x^3z+3x^2z^2\Big]_{x=0}^{x=z}dz=\int_0^1 (2z^4+3z^4)\,dz=\int_0^1 5z^4\,dz= z^5\Big]_0^1=1$$

4.

$$\int_0^1\int_x^{2x}\int_0^y 2xyz\,dz\,dy\,dx =\int_0^1\int_x^{2x} \Big[xyz^2\Big]_{z=0}^{z=y}dy\,dx=\int_0^1\int_x^{2x} xy^3\,dy\,dx$$

$$=\int_0^1 \left[\frac{1}{4}xy^4\right]_{y=x}^{y=2x}dx=\int_0^1 \frac{15}{4}x^5\,dx= \frac{5}{8}x^6\Big]_0^1=\frac{5}{8}$$

5.

$$\int_0^3\int_0^1\int_0^{\sqrt{1-z^2}} ze^y\,dx\,dz\,dy = \int_0^3\int_0^1\left[xze^y\right]_{x=0}^{x=\sqrt{1-z^2}}dz\,dy = \int_0^3\int_0^1 ze^y\sqrt{1-z^2}\,dz\,dy$$

$$= \int_0^3\left[-\frac{1}{3}(1-z^2)^{3/2}e^y\right]_{z=0}^{z=1}dy = \int_0^3\frac{1}{3}e^y\,dy = \left.\frac{1}{3}e^y\right]_0^3 = \frac{1}{3}(e^3-1)$$

6.

$$\int_0^1\int_0^z\int_0^y ze^{-y^2}\,dx\,dy\,dz = \int_0^1\int_0^z\left[xze^{-y^2}\right]_{x=0}^{x=y}dy\,dz = \int_0^1\int_0^z yze^{-y^2}\,dy\,dz = \int_0^1\left[-\frac{1}{2}ze^{-y^2}\right]_{y=0}^{y=z}dz$$

$$= \int_0^1 -\frac{1}{2}z\left(e^{-z^2}-1\right)dz = \frac{1}{2}\int_0^1\left(z-ze^{-z^2}\right)dz$$

$$= \frac{1}{2}\left[\frac{1}{2}z^2+\frac{1}{2}e^{-z^2}\right]_0^1 = \frac{1}{4}(1+e^{-1}-0-1) = \frac{1}{4e}$$

7.

$$\iiint_E 2x\,dV = \int_0^2\int_0^{\sqrt{4-y^2}}\int_0^y 2x\,dz\,dx\,dy = \int_0^2\int_0^{\sqrt{4-y^2}}\left[2xz\right]_{z=0}^{z=y}dx\,dy = \int_0^2\int_0^{\sqrt{4-y^2}} 2xy\,dx\,dy$$

$$= \int_0^2\left[x^2y\right]_{x=0}^{x=\sqrt{4-y^2}}dy = \int_0^2(4-y^2)y\,dy = \left[2y^2-\frac{1}{4}y^4\right]_0^2 = 4$$

8.

$$\iiint_E yz\cos(x^5)\,dV = \int_0^1\int_0^x\int_x^{2x} yz\cos(x^5)\,dz\,dy\,dx = \int_0^1\int_0^x\left[\frac{1}{2}yz^2\cos(x^5)\right]_{z=x}^{z=2x}dy\,dx$$

$$= \frac{1}{2}\int_0^1\int_0^x 3x^2y\cos(x^5)\,dy\,dx = \frac{1}{2}\int_0^1\left[\frac{3}{2}x^2y^2\cos(x^5)\right]_{y=0}^{y=x}dx$$

$$= \frac{3}{4}\int_0^1 x^4\cos(x^5)\,dx = \frac{3}{4}\left[\frac{1}{5}\sin(x^5)\right]_0^1 = \frac{3}{20}(\sin 1-\sin 0) = \frac{3}{20}\sin 1$$

9. Here $E=\left\{(x,y,z)\,|\,0\le x\le 1, 0\le y\le\sqrt{x}, 0\le z\le 1+x+y\right\}$, so

$$\iiint_E 6xy\,dV = \int_0^1 \int_0^{\sqrt{x}} \int_0^{1+x+y} 6xy\,dz\,dy\,dx = \int_0^1 \int_0^{\sqrt{x}} \left[6xyz\right]_{z=0}^{z=1+x+y} dy\,dx$$

$$= \int_0^1 \int_0^{\sqrt{x}} 6xy(1+x+y)\,dy\,dx = \int_0^1 \left[3xy^2 + 3x^2y^2 + 2xy^3\right]_{y=0}^{y=\sqrt{x}} dx$$

$$= \int_0^1 (3x^2 + 3x^3 + 2x^{5/2})\,dx = \left[x^3 + \frac{3}{4}x^4 + \frac{4}{7}x^{7/2}\right]_0^1 = \frac{65}{28}$$

10. Here E is the region in the first octant that lies below the plane $2x+2y+z=4$ (and above the region in the xy –plane bounded by the lines $x=0$, $y=0$, $x+y=2$). So

$$\iiint_E y\,dV = \int_0^2 \int_0^{2-x} \int_0^{4-2x-2y} y\,dz\,dy\,dx = \int_0^2 \int_0^{2-x} y(4-2x-2y)\,dy\,dx$$

$$= \int_0^2 \int_0^{2-x} (4y - 2xy - 2y^2)\,dy\,dx = \int_0^2 \left[2y^2 - xy^2 - \frac{2}{3}y^3\right]_{y=0}^{y=2-x} dx$$

$$= \int_0^2 \left[2(2-x)^2 - x(2-x)^2 - \frac{2}{3}(2-x)^3\right] dx$$

$$= \int_0^2 \left[(2-x)(2-x)^2 - \frac{2}{3}(2-x)^3\right] dx = \frac{1}{3}\int_0^2 (2-x)^3\,dx$$

$$= \frac{1}{3}\left[-\frac{1}{4}(2-x)^4\right]_0^2 = -\frac{1}{12}(0-16) = \frac{4}{3}$$

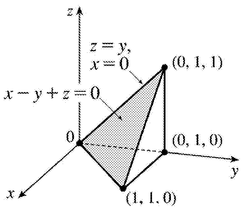
11. Here E is the region that lies below the plane with x –, y –, and z –intercepts 1 , 2 , and 3 respectively, that is, below the plane $2z+6x+3y=6$ and above the region in the xy –plane bounded by the lines $x=0$, $y=0$ and $6x+3y=6$. So

$$\iiint_E xy\,dV = \int_0^1 \int_0^{2-2x} \int_0^{3-3x-3y/2} xy\,dz\,dy\,dx = \int_0^1 \int_0^{2-2x} \left(3xy - 3x^2y - \frac{3}{2}xy^2\right) dy\,dx$$

$$= \int_0^1 \left[\frac{3}{2}xy^2 - \frac{3}{2}x^2y^2 - \frac{1}{2}xy^3\right]_{y=0}^{y=2-2x} dx = \int_0^1 (2x - 6x^2 + 6x^3 - 2x^4)\,dx$$

$$= \left[x^2 - 2x^3 + \frac{3}{2}x^4 - \frac{2}{5}x^5\right]_0^1 = \frac{1}{10} .$$

12.

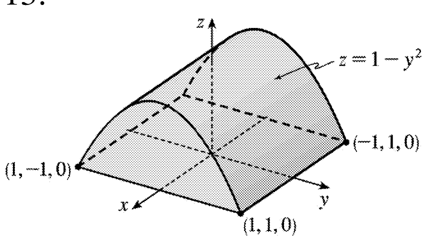


$$\int_0^1\int_0^y\int_0^{y-z} xz\,dx\,dz\,dy = \int_0^1\int_0^y \frac{1}{2}(y-z)^2 z\,dz\,dy$$

$$= \frac{1}{2}\int_0^1 \left[\frac{1}{2}y^2z^2 - \frac{2}{3}yz^3 + \frac{1}{4}z^4\right]_{z=0}^{z=y} dy$$

$$= \frac{1}{24}\int_0^1 y^4\,dy = \frac{1}{24}\left[\frac{1}{5}y^5\right]_0^1 = \frac{1}{120}$$

13.



E is the region below the parabolic cylinder $z=1-y^2$ and above the square $[-1,1]\times[-1,1]$ in the xy – plane.

$$\iiint_E x^2e^y\,dV = \int_{-1}^1\int_{-1}^1\int_0^{1-y^2} x^2e^y\,dz\,dy\,dx$$

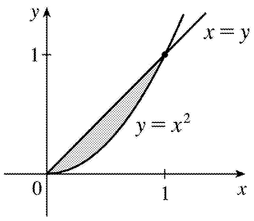
$$= \int_{-1}^1\int_{-1}^1 x^2e^y(1-y^2)\,dy\,dx$$

$$= \int_{-1}^1 x^2\,dx \int_{-1}^1 (e^y - y^2e^y)\,dy$$

$$= \left[\frac{1}{3}x^3\right]_{-1}^1 \left[e^y - (y^2-2y+2)e^y\right]_{-1}^1$$

$$= \frac{1}{3}(2)[e-e-e^{-1}+5e^{-1}] = \frac{8}{3e}$$

14.



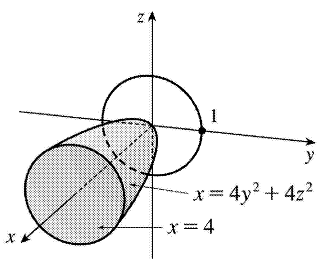
E is the solid above the region shown in the xy –plane and below the plane $z=x$. Thus,

$$\iiint_E (x+2y)\,dV=\int_0^1\int_{x^2}^{x}\int_0^x (x+2y)\,dz\,dy\,dx$$

$$=\int_0^1\int_{x^2}^{x}(x^2+2yx)\,dy\,dx=\int_0^1\left[x^2y+xy^2\right]_{y=x^2}^{y=x}dx$$

$$=\int_0^1(2x^3-x^4-x^5)\,dx=\left[\frac{1}{2}x^4-\frac{1}{5}x^5-\frac{1}{6}x^6\right]_0^1=\frac{2}{15}$$

15.



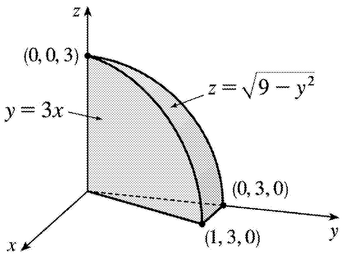
The projection E on the yz –plane is the disk $y^2+z^2\le 1$. Using polar coordinates $y=r\cos\theta$ and $z=r\sin\theta$, we get

$$\iiint_E x\,dV=\iint_D\left[\int_{4y^2+4z^2}^{4}x\,dx\right]dA$$

$$=\frac{1}{2}\iint_D\left[4^2-(4y^2+4z^2)^2\right]dA=8\int_0^{2\pi}\int_0^1(1-r^4)\,r\,dr\,d\theta$$

$$=8\int_0^{2\pi}d\theta\int_0^1(r-r^5)\,dr=8(2\pi)\left[\frac{1}{2}r^2-\frac{1}{6}r^6\right]_0^1=\frac{16\pi}{3}$$

16.



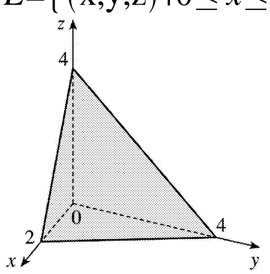
$$\int_0^1\int_{3x}^3\int_0^{\sqrt{9-y^2}} z\,dz\,dy\,dx=\int_0^1\int_{3x}^3 \frac{1}{2}(9-y^2)\,dy\,dx$$

$$=\int_0^1\left[\frac{9}{2}y-\frac{1}{6}y^3\right]_{y=3x}^{y=3}dx$$

$$=\int_0^1\left[9-\frac{27}{2}x+\frac{9}{2}x^3\right]dx$$

$$=\left[9x-\frac{27}{4}x^2+\frac{9}{8}x^4\right]_0^1=\frac{27}{8}$$

17. The plane $2x+y+z=4$ intersects the xy –plane when $2x+y+0=4\Rightarrow y=4-2x$, so $E=\{(x,y,z)\mid 0\le x\le 2, 0\le y\le 4-2x, 0\le z\le 4-2x-y\}$ and



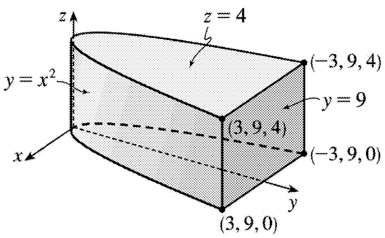
$$V=\int_0^2\int_0^{4-2x}\int_0^{4-2x-y} dz\,dy\,dx=\int_0^2\int_0^{4-2x}(4-2x-y)\,dy\,dx$$

$$=\int_0^2\left[4y-2xy-\frac{1}{2}y^2\right]_{y=0}^{y=4-2x}dx$$

$$=\int_0^2\left[4(4-2x)-2x(4-2x)-\frac{1}{2}(4-2x)^2\right]dx$$

$$=\int_0^2 (2x^2-8x+8)\,dx=\left[\frac{2}{3}x^3-4x^2+8x\right]_0^2=\frac{16}{3}$$

18.



$$V=\iiint_E dV=\int_{-3}^{3}\int_{x^2}^{9}\int_0^4 dz\,dy\,dx$$

$$=4\int_{-3}^{3}\int_{x^2}^{9} dy\,dx=4\int_{-3}^{3}\left(9-x^2\right)dx$$

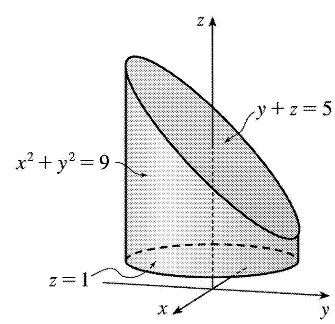
$$=4\left[9x-\frac{1}{3}x^3\right]_{-3}^{3}=4(27-9+27-9)=144$$

19.

$$V=\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_1^{5-y} dz\,dy\,dx=\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}(5-y-1)\,dy\,dx=\int_{-3}^{3}\left[4y-\frac{1}{2}y^2\right]_{y=-\sqrt{9-x^2}}^{y=\sqrt{9-x^2}}dx$$

$$=\int_{-3}^{3} 8\sqrt{9-x^2}\,dx=8\left[\frac{x}{2}\sqrt{9-x^2}+\frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right)\right]_{-3}^{3}$$

[using trigonometric substitutio
or Formula 30 in the Table of



$$=8\left[\frac{9}{2}\sin^{-1}(1)-\frac{9}{2}\sin^{-1}(-1)\right]=36\left(\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)\right)=36\pi$$

Alternatively, use polar coordinates to evaluate the double integral:

$$\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}(4-y)\,dy\,dx=\int_0^{2\pi}\int_0^3(4-r\sin\theta)\,r\,dr\,d\theta$$

$$=\int_0^{2\pi}\left[2r^2-\frac{1}{3}r^3\sin\theta\right]_{r=0}^{r=3}d\theta$$

$$=\int_0^{2\pi}(18-9\sin\theta)\,d\theta$$

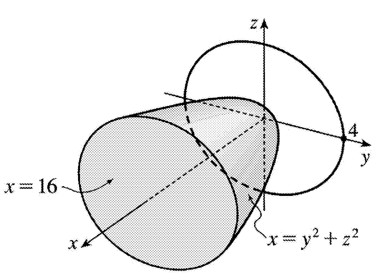
$$=\left.18\theta+9\cos\theta\right]_0^{2\pi}=36\pi$$

20. The paraboloid $x=y^2+z^2$ intersects the plane $x=16$ in the circle $y^2+z^2=16$, $x=16$.
Thus, $E=\left\{(x,y,z)\,|\,y^2+z^2\le x\le 16, y^2+z^2\le 16\right\}$.
Let $D=\left\{(y,z)\,|\,y^2+z^2\le 16\right\}$. Then using polar coordinates $y=r\cos\theta$ and $z=r\sin\theta$, we have

$$V=\iint_D\left(\int_{y^2+z^2}^{16}dx\right)dA=\iint_D(16-(y^2+z^2))\,dA$$

$$=\int_0^{2\pi}\int_0^4(16-r^2)\,r\,dr\,d\theta=\int_0^{2\pi}d\theta\int_0^4(16r-r^3)\,dr$$

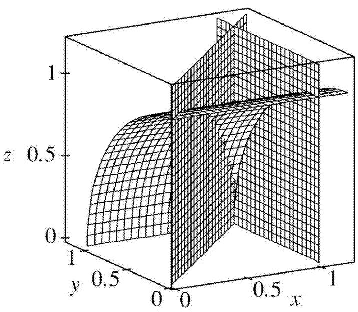
$$=\left[\theta\right]_0^{2\pi}\left[8r^2-\frac{1}{4}r^4\right]_0^4=2\pi(128-64)=128\pi$$



21. **(a)** The wedge can be described as the region
$D=\left\{(x,y,z)\,|\,y^2+z^2\le 1, 0\le x\le 1, 0\le y\le x\right\}=\left\{(x,y,z)\,|\,0\le x\le 1, 0\le y\le x, 0\le z\le\sqrt{1-y^2}\right\}$

So the integral expressing the volume of the wedge is $\iiint_D dV=\int_0^1\int_0^x\int_0^{\sqrt{1-y^2}}dz\,dy\,dx$.

(b) A CAS gives $\int_0^1 \int_0^x \int_0^{\sqrt{1-y^2}} dz\,dy\,dx = \frac{\pi}{4} - \frac{1}{3}$.
(Or use Formulas 30 and 87 from the Table of Integrals.)



22. **(a)** Note that $\Delta V = \left(\frac{1}{2}\right)^3 = \frac{1}{8}$, so the Midpoint Rule gives

$$\iiint_B \mathrm{f(x,y,z)\,dV} \approx \frac{1}{8}\left[f\left(\frac{1}{4},\frac{1}{4},\frac{1}{4}\right)+f\left(\frac{1}{4},\frac{1}{4},\frac{3}{4}\right)+f\left(\frac{1}{4},\frac{3}{4},\frac{1}{4}\right)+f\left(\frac{3}{4},\frac{1}{4},\frac{1}{4}\right)\right.$$
$$\left.+\ f\left(\frac{1}{4},\frac{3}{4},\frac{3}{4}\right)+f\left(\frac{3}{4},\frac{1}{4},\frac{3}{4}\right)+f\left(\frac{3}{4},\frac{3}{4},\frac{1}{4}\right)+f\left(\frac{3}{4},\frac{3}{4},\frac{3}{4}\right)\right]$$
$$=\frac{1}{8}\left[e^{-3(1/4)^2}+3e^{-2(1/4)^2-(3/4)^2}+3e^{-(1/4)^2-2(3/4)^2}+e^{-3(3/4)^2}\right]\approx 0.42968$$

(b) A CAS estimates the integral to be $\iiint_B e^{-x^2-y^2-z^2}\,dV \approx 0.42$. The estimate in part (a) is correct to one decimal place, and is larger than the actual value of the integral.

23. Here $f(\mathrm{x,y,z}) = \frac{1}{\ln(1+x+y+z)}$ and $\Delta V = 2\cdot 4\cdot 2 = 16$, so the Midpoint Rule gives

$$\begin{aligned}\iiint_B f(\mathrm{x,y,z})\,dV &\approx \Sigma_{i=1}^{l}\Sigma_{j=1}^{m}\Sigma_{k=1}^{n} f\left(\bar{\mathrm{x}}_i,\bar{\mathrm{y}}_j,\bar{\mathrm{z}}_k\right)\Delta V\\ &= 16[f(1,2,1)+f(1,2,3)+f(1,6,1)+f(1,6,3)\\ &\quad +\ f(3,2,1)+f(3,2,3)+f(3,6,1)+f(3,6,3)]\\ &= 16\left[\frac{1}{\ln 5}+\frac{1}{\ln 7}+\frac{1}{\ln 9}+\frac{1}{\ln 11}+\frac{1}{\ln 7}+\frac{1}{\ln 9}+\frac{1}{\ln 11}+\frac{1}{\ln 13}\right]\approx 60.533\end{aligned}$$

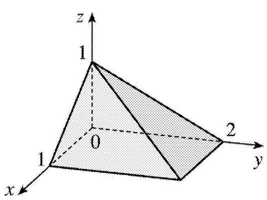
24. Here $f(x,y,z)=\sin(xy^2z^3)$ and $\Delta V=2\cdot 1\cdot \frac{1}{2}=1$, so the Midpoint Rule gives

$$\iiint_B f(x,y,z)\,dV \approx \sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f\left(\bar{x}_i,\bar{y}_j,\bar{z}_k\right)\Delta V$$

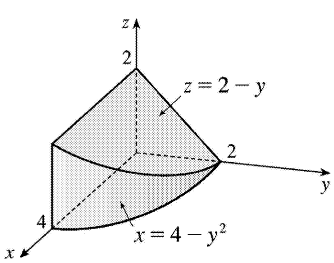
$$= 1\left[f\left(1,\frac{1}{2},\frac{1}{4}\right)+f\left(1,\frac{1}{2},\frac{3}{4}\right)+f\left(1,\frac{3}{2},\frac{1}{4}\right)+f\left(1,\frac{3}{2},\frac{3}{4}\right) + f\left(3,\frac{1}{2},\frac{1}{4}\right)+f\left(3,\frac{1}{2},\frac{3}{4}\right)+f\left(3,\frac{3}{2},\frac{1}{4}\right)+f\left(3,\frac{3}{2},\frac{3}{4}\right)\right]$$

$$= \sin\frac{1}{256}+\sin\frac{27}{256}+\sin\frac{9}{256}+\sin\frac{243}{256}+\sin\frac{3}{256}+\sin\frac{81}{256}+\sin\frac{27}{256}+\sin\frac{729}{256}\approx 1.675$$

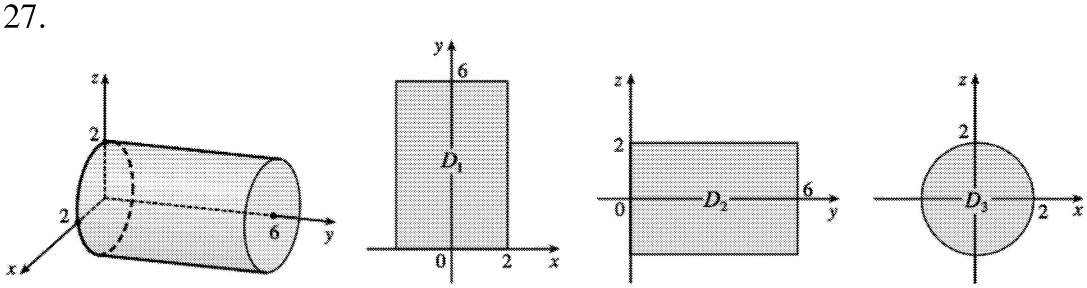
25. $E=\{(x,y,z)\,|\,0\le x\le 1, 0\le z\le 1-x, 0\le y\le 2-2z\}$,
the solid bounded by the three coordinate planes and the planes $z=1-x$, $y=2-2z$.



26. $E=\left\{(x,y,z)\,|\,0\le y\le 2, 0\le z\le 2-y, 0\le x\le 4-y^2\right\}$,
the solid bounded by the three coordinate planes, the plane $z=2-y$, and the cylindrical surface $x=4-y^2$.



27.



If D_1 , D_2 , D_3 are the projections of E on the xy –, yz –, and xz –planes, then

$D_1 = \{(x,y)|-2\le x\le 2, 0\le y\le 6\}$

$D_2 = \{(y,z)|-2\le z\le 2, 0\le y\le 6\}$

$D_3 = \left\{(x,z)|x^2+z^2\le 4\right\}$

Therefore

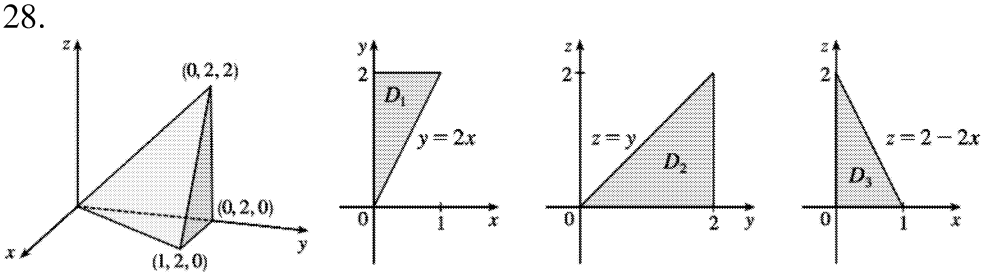
$$E = \left\{(x,y,z)|-\sqrt{4-x^2}\le z\le \sqrt{4-x^2}, -2\le x\le 2, 0\le y\le 6\right\}$$
$$= \left\{(x,y,z)|-\sqrt{4-z^2}\le x\le \sqrt{4-z^2}, -2\le z\le 2, 0\le y\le 6\right\}$$

$$\iiint_E f(x,y,z)\,dV=\int_{-2}^{2}\int_{0}^{6}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} f(x,y,z)\,dz\,dy\,dx=\int_{0}^{6}\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} f(x,y,z)\,dz\,dx\,dy$$

$$=\int_{0}^{6}\int_{-2}^{2}\int_{-\sqrt{4-z^2}}^{\sqrt{4-z^2}} f(x,y,z)\,dx\,dz\,dy=\int_{-2}^{2}\int_{0}^{6}\int_{-\sqrt{4-z^2}}^{\sqrt{4-z^2}} f(x,y,z)\,dx\,dy\,dz$$

$$=\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{0}^{6} f(x,y,z)\,dy\,dz\,dx=\int_{-2}^{2}\int_{-\sqrt{4-z^2}}^{\sqrt{4-z^2}}\int_{0}^{6} f(x,y,z)\,dy\,dx\,dz$$

28.



If D_1, D_2, and D_3 are the projections of E on the xy–, yz–, and xz–planes, then

$$D_1=\{(x,y)|0\le x\le 1, 2x\le y\le 2\}=\{(x,y)|0\le y\le 2, 0\le x\le y/2\},$$

$D_2 = \{(y,z)|0\le y\le 2, 0\le z\le y\}=\{(y,z)|0\le z\le 2, z\le y\le 2\}$, and

$D_3 = \{(x,z)|0\le x\le 1, 0\le z\le 2-2x\}=\{(x,z)|0\le z\le 2, 0\le x\le (2-z)/2\}$

Therefore

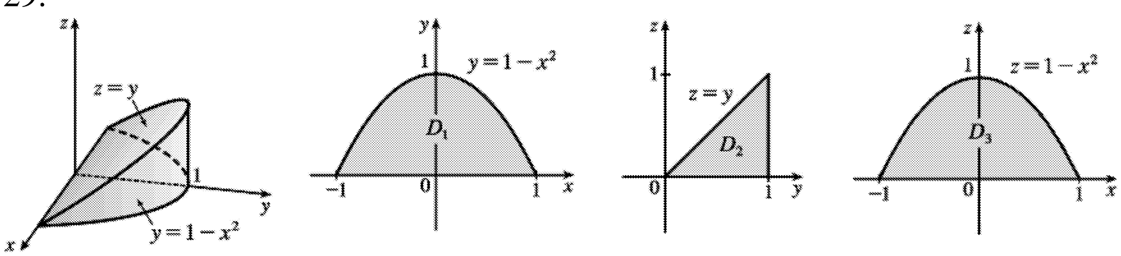
$$E = \{(x,y,z)|0\le x\le 1, 2x\le y\le 2, 0\le z\le y-2x\}$$
$$= \{(x,y,z)|0\le y\le 2, 0\le x\le y/2, 0\le z\le y-2x\}$$

$= \{(x,y,z)|0\le y\le 2, 0\le z\le y, 0\le x\le (y-z)/2\}$
$= \{(x,y,z)|0\le z\le 2, z\le y\le 2, 0\le x\le (y-z)/2\}$
$= \{(x,y,z)|0\le x\le 1, 0\le z\le 2-2x, z+2x\le y\le 2\}$
$= \{(x,y,z)|0\le z\le 2, 0\le x\le (2-z)/2, z+2x\le y\le 2\}$

Then

$$\iiint_E f(x,y,z)\,dV = \int_0^1\int_{2x}^2\int_0^{y-2x} f(x,y,z)\,dz\,dy\,dx$$
$$= \int_0^2\int_0^{y/2}\int_0^{y-2x} f(x,y,z)\,dz\,dx\,dy$$
$$= \int_0^2\int_0^y\int_0^{(y-z)/2} f(x,y,z)\,dx\,dz\,dy$$
$$= \int_0^2\int_z^2\int_0^{(y-z)/2} f(x,y,z)\,dx\,dy\,dz$$
$$= \int_0^1\int_0^{2-2x}\int_{z+2x}^2 f(x,y,z)\,dy\,dz\,dx$$
$$= \int_0^2\int_0^{(2-z)/2}\int_{z+2x}^2 f(x,y,z)\,dy\,dx\,dz$$

29.



If D_1, D_2, and D_3 are the projections of E on the xy–, yz–, and xz–planes, then

$D_1 = \{(x,y)|-1\le x\le 1, 0\le y\le 1-x^2\} = \{(x,y)|0\le y\le 1, -\sqrt{1-y}\le x\le \sqrt{1-y}\}$,

$D_2 = \{(y,z)|0\le y\le 1, 0\le z\le y\} = \{(y,z)|0\le z\le 1, z\le y\le 1\}$, and

$D_3 = \{(x,z)|-1\le x\le 1, 0\le z\le 1-x^2\} = \{(x,z)|0\le z\le 1, -\sqrt{1-z}\le x\le \sqrt{1-z}\}$

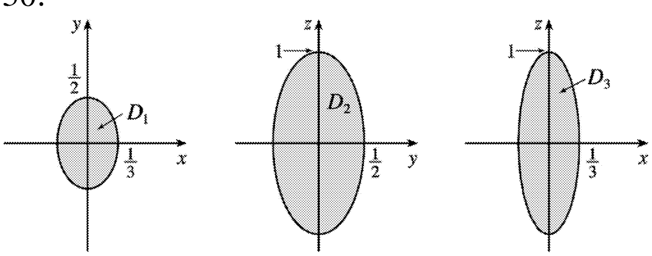
Therefore

$$
\begin{aligned}
E&=\left\{ (x,y,z) \mid -1\le x\le 1, 0\le y\le 1-x^2, 0\le z\le y\right\}\\
&=\left\{ (x,y,z) \mid 0\le y\le 1, -\sqrt{1-y}\le x\le \sqrt{1-y}, 0\le z\le y\right\}\\
&=\left\{ (x,y,z) \mid 0\le y\le 1, 0\le z\le y, -\sqrt{1-y}\le x\le \sqrt{1-y}\right\}\\
&=\left\{ (x,y,z) \mid 0\le z\le 1, z\le y\le 1, -\sqrt{1-y}\le x\le \sqrt{1-y}\right\}\\
&=\left\{ (x,y,z) \mid -1\le x\le 1, 0\le z\le 1-x^2, z\le y\le 1-x^2\right\}\\
&=\left\{ (x,y,z) \mid 0\le z\le 1, -\sqrt{1-z}\le x\le \sqrt{1-z}, z\le y\le 1-x^2\right\}
\end{aligned}
$$

Then

$$
\begin{aligned}
\iiint_E f(x,y,z)\,dV &= \int_{-1}^{1}\int_{0}^{1-x^2}\int_{0}^{y} f(x,y,z)\,dz\,dy\,dx = \int_{0}^{1}\int_{-\sqrt{1-y}}^{\sqrt{1-y}}\int_{0}^{y} f(x,y,z)\,dz\,dx\,dy\\
&= \int_{0}^{1}\int_{0}^{y}\int_{-\sqrt{1-y}}^{\sqrt{1-y}} f(x,y,z)\,dx\,dz\,dy = \int_{0}^{1}\int_{z}^{1}\int_{-\sqrt{1-y}}^{\sqrt{1-y}} f(x,y,z)\,dx\,dy\,dz\\
&= \int_{-1}^{1}\int_{0}^{1-x^2}\int_{z}^{1-x^2} f(x,y,z)\,dy\,dz\,dx = \int_{0}^{1}\int_{-\sqrt{1-z}}^{\sqrt{1-z}}\int_{z}^{1-x^2} f(x,y,z)\,dy\,dx\,dz
\end{aligned}
$$

30.



If D_1, D_2 and D_3 are the projections of E on the xy−, yz−, and xz−planes, then $D_1=\left\{ (x,y) \mid 9x^2+4y^2\le 1\right\}$, $D_2=\left\{ (y,z) \mid 4y^2+z^2\le 1\right\}$, $D_3=\left\{ (x,z) \mid 9x^2+z^2\le 1\right\}$. Therefore

$$
\begin{aligned}
\iiint_E f(x,y,z)\,dV &= \int_{-1/3}^{1/3}\int_{-\sqrt{1-9x^2}/2}^{\sqrt{1-9x^2}/2}\int_{-\sqrt{1-9x^2-4y^2}}^{\sqrt{1-9x^2-4y^2}} f(x,y,z)\,dz\,dy\,dx\\
&= \int_{-1/2}^{1/2}\int_{-\sqrt{1-4y^2}/3}^{\sqrt{1-4y^2}/3}\int_{-\sqrt{1-9x^2-4y^2}}^{\sqrt{1-9x^2-4y^2}} f(x,y,z)\,dz\,dx\,dy
\end{aligned}
$$

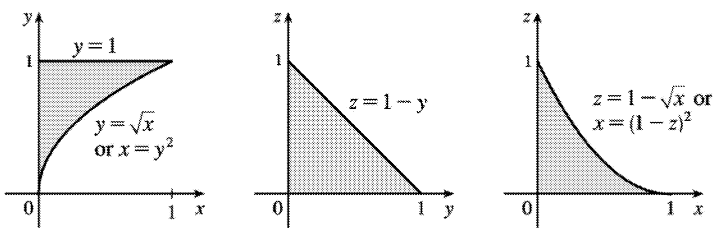
$$= \int_{-1/2}^{1/2} \int_{-\sqrt{1-4y^2}}^{\sqrt{1-4y^2}} \int_{-\sqrt{1-4y^2-z^2}/3}^{\sqrt{1-4y^2-z^2}/3} f(\mathrm{x,y,z})\,dx\,dz\,dy$$

$$= \int_{-1}^{1} \int_{-\sqrt{1-z^2}/2}^{\sqrt{1-z^2}/2} \int_{-\sqrt{1-4y^2-z^2}/3}^{\sqrt{1-4y^2-z^2}/3} f(\mathrm{x,y,z})\,dx\,dy\,dz$$

$$= \int_{-1/3}^{1/3} \int_{-\sqrt{1-9x^2}}^{\sqrt{1-9x^2}} \int_{-\sqrt{1-9x^2-z^2}/2}^{\sqrt{1-9x^2-z^2}/2} f(\mathrm{x,y,z})\,dy\,dz\,dx$$

$$= \int_{-1}^{1} \int_{-\sqrt{1-z^2}/3}^{\sqrt{1-z^2}/3} \int_{-\sqrt{1-9x^2-z^2}/2}^{\sqrt{1-9x^2-z^2}/2} f(\mathrm{x,y,z})\,dy\,dx\,dz$$

31.



The diagrams show the projections of E on the xy –, yz –, and xz –planes. Therefore

$$\int_0^1 \int_{\sqrt{x}}^1 \int_0^{1-y} f(\mathrm{x,y,z})\,dz\,dy\,dx = \int_0^1 \int_0^{y^2} \int_0^{1-y} f(\mathrm{x,y,z})\,dz\,dx\,dy$$

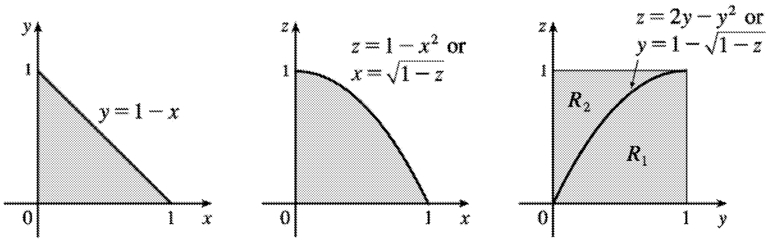
$$= \int_0^1 \int_0^{1-z} \int_0^{y^2} f(\mathrm{x,y,z})\,dx\,dy\,dz$$

$$= \int_0^1 \int_0^{1-y} \int_0^{y^2} f(\mathrm{x,y,z})\,dx\,dz\,dy$$

$$= \int_0^1 \int_0^{1-\sqrt{x}} \int_{\sqrt{x}}^{1-z} f(\mathrm{x,y,z})\,dy\,dz\,dx$$

$$= \int_0^1 \int_0^{(1-z)^2} \int_{\sqrt{x}}^{1-z} f(\mathrm{x,y,z})\,dy\,dx\,dz$$

32.



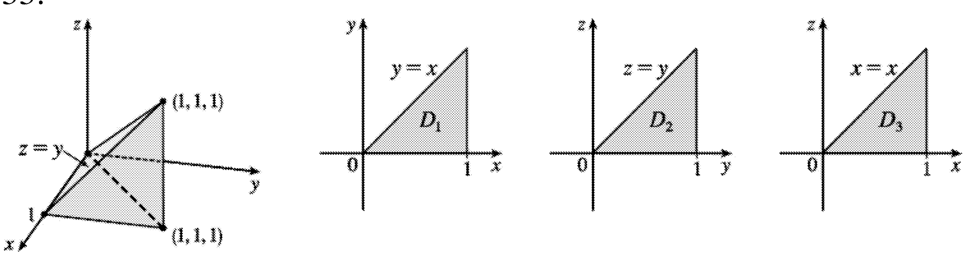
The projections of E onto the xy – and xz –planes are as in the first two diagrams and so

$$\int_0^1\int_0^{1-x^2}\int_0^{1-x} f(x,y,z)\,dy\,dz\,dx=\int_0^1\int_0^{\sqrt{1-z}}\int_0^{1-x} f(x,y,z)\,dy\,dx\,dz=\int_0^1\int_0^{1-y}\int_0^{1-x^2} f(x,y,z)\,dz\,dx\,dy=\int_0^1\int_0^{1-x}\int_0^{1-x^2} f(x,y,z)\,dz\,dy\,dx$$

Now the surface $z=1-x^2$ intersects the plane $y=1-x$ in a curve whose projection in the yz –plane is $z=1-(1-y)^2$ or $z=2y-y^2$. So we must split up the projection of E on the yz –plane into two regions as in the third diagram. For (y,z) in R_1 , $0\le x\le 1-y$ and for (y,z) in R_2 , $0\le x\le\sqrt{1-z}$, and so the given integral is also equal to

$$\int_0^1\int_0^{1-\sqrt{1-z}}\int_0^{\sqrt{1-z}} f(x,y,z)\,dx\,dy\,dz+\int_0^1\int_{1-\sqrt{1-z}}^{1}\int_0^{1-y} f(x,y,z)\,dx\,dy\,dz=\int_0^1\int_0^{2y-y^2}\int_0^{1-y} f(x,y,z)\,dx\,dz\,dy+\int_0^1\int_{2y-y^2}^{1}\int_0^{\sqrt{1-z}} f(x,y,z)\,dx\,dz$$

33.



$$\int_0^1\int_y^1\int_0^y f(x,y,z)\,dz\,dx\,dy=\iiint_E f(x,y,z)\,dV \text{ where } E=\{(x,y,z)\mid 0\le z\le y,\ y\le x\le 1,\ 0\le y\le 1\}\ .$$

If D_1 , D_2 , and D_3 are the projections of E on the xy –, yz – and xz –planes then

$D_1=\{(x,y)\mid 0\le y\le 1, y\le x\le 1\}=\{(x,y)\mid 0\le x\le 1, 0\le y\le x\}$,

$D_2=\{(y,z)\mid 0\le y\le 1, 0\le z\le y\}=\{(y,z)\mid 0\le z\le 1, z\le y\le 1\}$, and

$D_3=\{(x,z)\mid 0\le x\le 1, 0\le z\le x\}=\{(x,z)\mid 0\le z\le 1, z\le x\le 1\}$.

Thus we also have

$$\begin{aligned} E &=\{(x,y,z)\mid 0\le x\le 1, 0\le y\le x, 0\le z\le y\}=\{(x,y,z)\mid 0\le y\le 1, 0\le z\le y, y\le x\le 1\}\\ &=\{(x,y,z)\mid 0\le z\le 1, z\le y\le 1, y\le x\le 1\}=\{(x,y,z)\mid 0\le x\le 1, 0\le z\le x, z\le y\le x\}\\ &=\{(x,y,z)\mid 0\le z\le 1, z\le x\le 1, z\le y\le x\}. \end{aligned}$$

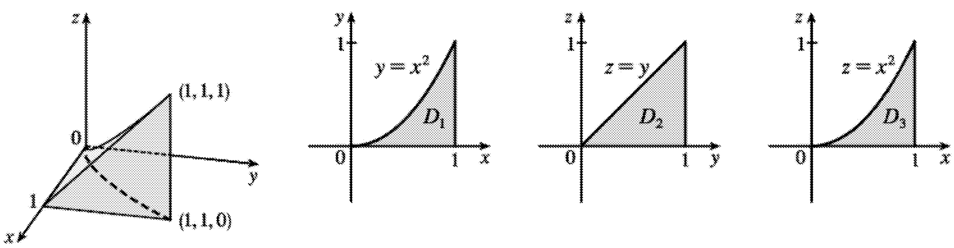
Then

$$\int_0^1\int_y^1\int_0^y f(x,y,z)\,dz\,dx\,dy=\int_0^1\int_0^x\int_0^y f(x,y,z)\,dz\,dy\,dx=\int_0^1\int_0^y\int_y^1 f(x,y,z)\,dx\,dz\,dy$$

$$=\int_0^1\int_z^1\int_y^1 f(x,y,z)\,dx\,dy\,dz=\int_0^1\int_0^x\int_z^x f(x,y,z)\,dy\,dz\,dx$$

$$=\int_0^1\int_z^1\int_z^x f(x,y,z)\,dy\,dx\,dz$$

34.



$\int_0^1\int_0^{x^2}\int_0^y f(x,y,z)\,dz\,dy\,dx=\iiint_E f(x,y,z)\,dV$ where $E=\left\{(x,y,z)\,|\,0\le x\le 1, 0\le y\le x^2, 0\le z\le y\right\}$. If D_1 , D_2 , D_3 are the projections of E on the

xy –, yz –, and xz –planes, then $D_1=\left\{(x,y)\,|\,0\le x\le 1, 0\le y\le x^2\right\}=\left\{(x,y)\,|\,0\le y\le 1, \sqrt{y}\le x\le 1\right\}$,

$D_2=\{(y,z)\,|\,0\le y\le 1, 0\le z\le y\}=\{(y,z)\,|\,0\le z\le 1, z\le y\le 1\}$,

$D_3=\left\{(x,z)\,|\,0\le x\le 1, 0\le z\le x^2\right\}=\left\{(x,z)\,|\,0\le z\le 1, \sqrt{z}\le x\le 1\right\}$. Thus we also have

$$\begin{aligned}E&=\left\{(x,y,z)\,|\,0\le y\le 1, \sqrt{y}\le x\le 1, 0\le z\le y\right\}\\&=\left\{(x,y,z)\,|\,0\le y\le 1, 0\le z\le y, \sqrt{y}\le x\le 1\right\}\\&=\left\{(x,y,z)\,|\,0\le z\le 1, z\le y\le 1, \sqrt{y}\le x\le 1\right\}\\&=\left\{(x,y,z)\,|\,0\le x\le 1, 0\le z\le x^2, z\le y\le x^2\right\}\\&=\left\{(x,y,z)\,|\,0\le z\le 1, \sqrt{z}\le x\le 1, z\le y\le x^2\right\}\end{aligned}$$

Then

$$\int_0^1\int_0^{x^2}\int_0^y f(x,y,z)\,dz\,dy\,dx=\int_0^1\int_{\sqrt{y}}^1\int_0^y f(x,y,z)\,dz\,dx\,dy$$

$$= \int_0^1 \int_0^y \int_{\sqrt{y}}^1 f(x,y,z)\,dx\,dz\,dy$$

$$= \int_0^1 \int_z^1 \int_{\sqrt{y}}^1 f(x,y,z)\,dx\,dy\,dz$$

$$= \int_0^1 \int_0^{x^2} \int_z^{x^2} f(x,y,z)\,dy\,dz\,dx$$

$$= \int_0^1 \int_{\sqrt{z}}^1 \int_z^{x^2} f(x,y,z)\,dy\,dx\,dz$$

35.

$$m = \iiint_E \rho(x,y,z)\,dV = \int_0^1 \int_0^{\sqrt{x}} \int_0^{1+x+y} 2\,dz\,dy\,dx$$

$$= \int_0^1 \int_0^{\sqrt{x}} 2(1+x+y)\,dy\,dx = \int_0^1 \left[2y+2xy+y^2\right]_{y=0}^{y=\sqrt{x}} dx$$

$$= \int_0^1 \left(2\sqrt{x}+2x^{3/2}+x\right) dx = \left[\frac{4}{3}x^{3/2}+\frac{4}{5}x^{5/2}+\frac{1}{2}x^2\right]_0^1 = \frac{79}{30}$$

$$M_{yz} = \iiint_E x\rho(x,y,z)\,dV = \int_0^1 \int_0^{\sqrt{x}} \int_0^{1+x+y} 2x\,dz\,dy\,dx$$

$$= \int_0^1 \int_0^{\sqrt{x}} 2x(1+x+y)\,dy\,dx = \int_0^1 \left[2xy+2x^2y+xy^2\right]_{y=0}^{y=\sqrt{x}} dx$$

$$= \int_0^1 (2x^{3/2}+2x^{5/2}+x^2)\,dx = \left[\frac{4}{5}x^{5/2}+\frac{4}{7}x^{7/2}+\frac{1}{3}x^3\right]_0^1 = \frac{179}{105}$$

$$M_{xz} = \iiint_E y\rho(x,y,z)\,dV = \int_0^1 \int_0^{\sqrt{x}} \int_0^{1+x+y} 2y\,dz\,dy\,dx$$

$$= \int_0^1 \int_0^{\sqrt{x}} 2y(1+x+y)\,dy\,dx = \int_0^1 \left[y^2+xy^2+\frac{2}{3}y^3\right]_{y=0}^{y=\sqrt{x}} dx$$

$$=\int_0^1\left(x+x^2+\frac{2}{3}x^{3/2}\right)dx=\left[\frac{1}{2}x^2+\frac{1}{3}x^3+\frac{4}{15}x^{5/2}\right]_0^1=\frac{11}{10}$$

$$M_{xy}=\iiint_E z\rho(\mathrm{x,y,z})\,dV=\int_0^1\int_0^{\sqrt{x}}\int_0^{1+x+y}2z\,dz\,dy\,dx$$

$$=\int_0^1\int_0^{\sqrt{x}}\left[z^2\right]_{z=0}^{z=1+x+y}dy\,dx=\int_0^1\int_0^{\sqrt{x}}(1+x+y)^2\,dy\,dx$$

$$=\int_0^1\int_0^{\sqrt{x}}(1+2x+2y+2xy+x^2+y^2)\,dy\,dx$$

$$=\int_0^1\left[y+2xy+y^2+xy^2+x^2y+\frac{1}{3}y^3\right]_{y=0}^{y=\sqrt{x}}dx=\int_0^1\left(\sqrt{x}+\frac{7}{3}x^{3/2}+x+x^2+x^{5/2}\right)dx$$

$$=\left[\frac{2}{3}x^{3/2}+\frac{14}{15}x^{5/2}+\frac{1}{2}x^2+\frac{1}{3}x^3+\frac{2}{7}x^{7/2}\right]_0^1=\frac{571}{210}$$

Thus the mass is $\frac{79}{30}$ and the center of mass is $(\overline{\mathrm{x}},\overline{\mathrm{y}},\overline{\mathrm{z}})=\left(\frac{M_{yz}}{m},\frac{M_{xz}}{m},\frac{M_{xy}}{m}\right)=\left(\frac{358}{553},\frac{33}{79},\frac{571}{553}\right)$.

36.

$$m=\int_{-1}^1\int_0^{1-y^2}\int_0^{1-z}4\,dx\,dz\,dy=4\int_{-1}^1\int_0^{1-y^2}(1-z)\,dz\,dy$$

$$=4\int_{-1}^1\left[z-\frac{1}{2}z^2\right]_{z=0}^{z=1-y^2}dy=2\int_{-1}^1(1-y^4)\,dy=\frac{16}{5}\ ,$$

$$M_{yz}=\int_{-1}^1\int_0^{1-y^2}\int_0^{1-z}4x\,dx\,dz\,dy=2\int_{-1}^1\int_0^{1-y^2}(1-z)^2\,dz\,dy=2\int_{-1}^1\left[-\frac{1}{3}(1-z)^3\right]_{z=0}^{z=1-y^2}dy$$

$$=\frac{2}{3}\int_{-1}^1\left(1-y^6\right)dy=\left(\frac{4}{3}\right)\left(\frac{6}{7}\right)=\frac{24}{21}$$

$$M_{xz}=\int_{-1}^1\int_0^{1-y^2}\int_0^{1-z}4y\,dx\,dz\,dy=\int_{-1}^1\int_0^{1-y^2}4y(1-z)\,dz\,dy$$

$$=\int_{-1}^{1}\left[4y(1-y^2)-2y(1-y^2)^2\right]dy=\int_{-1}^{1}(2y-2y^5)dy=0 \text{ [the integrand is odd]}$$

$$M_{xy}=\int_{-1}^{1}\int_{0}^{1-y^2}\int_{0}^{1-z}4z\,dx\,dz\,dy=\int_{-1}^{1}\int_{0}^{1-y^2}(4z-4z^2)dz\,dy=2\int_{-1}^{1}\left[(1-y^2)^2-\frac{2}{3}(1-y^2)^3\right]dy$$

$$=2\int_{-1}^{1}\left[\frac{1}{3}-y^4+\frac{2}{3}y^6\right]dy=\left[\frac{4}{3}y-\frac{4}{5}y^5+\frac{8}{21}y^7\right]_0^1=\frac{96}{105}=\frac{32}{35}$$

Thus, $(\bar{x},\bar{y},\bar{z})=\left(\frac{5}{14},0,\frac{2}{7}\right)$

37.

$$m=\int_0^a\int_0^a\int_0^a(x^2+y^2+z^2)dx\,dy\,dz=\int_0^a\int_0^a\left[\frac{1}{3}x^3+xy^2+xz^2\right]_{x=0}^{x=a}dy\,dz$$

$$=\int_0^a\int_0^a\left(\frac{1}{3}a^3+ay^2+az^2\right)dy\,dz=\int_0^a\left[\frac{1}{3}a^3y+\frac{1}{3}ay^3+ayz^2\right]_{y=0}^{y=a}dz$$

$$=\int_0^a\left(\frac{2}{3}a^4+a^2z^2\right)dz=\left[\frac{2}{3}a^4z+\frac{1}{3}a^2z^3\right]_0^a=\frac{2}{3}a^5+\frac{1}{3}a^5=a^5$$

$$M_{yz}=\int_0^a\int_0^a\int_0^a\left[x^3+x(y^2+z^2)\right]dx\,dy\,dz=\int_0^a\int_0^a\left[\frac{1}{4}a^4+\frac{1}{2}a^2(y^2+z^2)\right]dy\,dz$$

$$=\int_0^a\left(\frac{1}{4}a^5+\frac{1}{6}a^5+\frac{1}{2}a^3z^2\right)dz=\frac{1}{4}a^6+\frac{1}{3}a^6=\frac{7}{12}a^6$$

$=M_{xz}=M_{xy}$ by symmetry of E and $\rho(x,y,z)$

Hence $(\bar{x},\bar{y},\bar{z})=\left(\frac{7}{12}a,\frac{7}{12}a,\frac{7}{12}a\right)$.

38.

$$m=\int_0^1\int_0^{1-x}\int_0^{1-x-y}y\,dz\,dy\,dx=\int_0^1\int_0^{1-x}\left[(1-x)y-y^2\right]dy\,dx$$

$$=\int_0^1 \left[\frac{1}{2}(1-x)^3-\frac{1}{3}(1-x)^3 \right] dx=\frac{1}{6}\int_0^1 (1-x)^3 dx=\frac{1}{24}$$

$$M_{yz}=\int_0^1\int_0^{1-x}\int_0^{1-x-y} xy\,dz\,dy\,dx=\int_0^1\int_0^{1-x}\left[(x-x^2)y-xy^2\right]dy\,dx$$

$$=\int_0^1 \left[\frac{1}{2}x(1-x)^3-\frac{1}{3}x(1-x)^3 \right] dx=\frac{1}{6}\int_0^1 \left(x-3x^2+3x^3-x^4\right)dx$$

$$=\frac{1}{6}\left(\frac{1}{2}-1+\frac{3}{4}-\frac{1}{5}\right)=\frac{1}{120}$$

$$M_{xz}=\int_0^1\int_0^{1-x}\int_0^{1-x-y} y^2\,dz\,dy\,dx=\int_0^1\int_0^{1-x}\left[(1-x)y^2-y^3\right]dy\,dx$$

$$=\int_0^1 \left[\frac{1}{3}(1-x)^4-\frac{1}{4}(1-x)^4 \right] dx=\frac{1}{12}\left[-\frac{1}{5}(1-x)^5\right]_0^1=\frac{1}{60}$$

$$M_{xy}=\int_0^1\int_0^{1-x}\int_0^{1-x-y} yz\,dz\,dy\,dx=\int_0^1\int_0^{1-x}\left[\frac{1}{2}y(1-x-y)^2\right]dy\,dx$$

$$=\frac{1}{2}\int_0^1\int_0^{1-x}\left[(1-x)^2y-2(1-x)y^2+y^3\right]dy\,dx$$

$$=\frac{1}{2}\int_0^1 \left[\frac{1}{2}(1-x)^4-\frac{2}{3}(1-x)^4+\frac{1}{4}(1-x)^4 \right] dx$$

$$=\frac{1}{24}\int_0^1 (1-x)^4 dx=-\frac{1}{24}\left[\frac{1}{5}(1-x)^5\right]_0^1=\frac{1}{120}$$

Hence $(\overline{x},\overline{y},\overline{z})=\left(\frac{1}{5},\frac{2}{5},\frac{1}{5}\right)$.

39. **(a)** $m=\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_1^{5-y}\sqrt{x^2+y^2}\,dz\,dy\,dx$

(b) $(\overline{x},\overline{y},\overline{z})=\left(\frac{M_{yz}}{m},\frac{M_{xz}}{m},\frac{M_{xy}}{m}\right)$ where

$M_{yz}=\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_{1}^{5-y} x\sqrt{x^2+y^2}\,dzdydx$, $M_{xz}=\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_{1}^{5-y} y\sqrt{x^2+y^2}\,dzdydx$, and

$M_{xy}=\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_{1}^{5-y} z\sqrt{x^2+y^2}\,dzdydx$.

(c) $I_z=\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_{1}^{5-y} (x^2+y^2)\sqrt{x^2+y^2}\,dzdydx=\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_{1}^{5-y} (x^2+y^2)^{3/2}dzdydx$

40. **(a)** $m=\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\int_{0}^{\sqrt{1-x^2-y^2}} \sqrt{x^2+y^2+z^2}\,dzdxdy$

(b) $(\bar{x},\bar{y},\bar{z})$ where $\bar{x}=m^{-1}\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\int_{0}^{\sqrt{1-x^2-y^2}} x\sqrt{x^2+y^2+z^2}\,dzdxdy$,

$\bar{y}=m^{-1}\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\int_{0}^{\sqrt{1-x^2-y^2}} y\sqrt{x^2+y^2+z^2}\,dzdxdy$, $\bar{z}=m^{-1}\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\int_{0}^{\sqrt{1-x^2-y^2}} z\sqrt{x^2+y^2+z^2}\,dzdxdy$

(c) $I_z=\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\int_{0}^{\sqrt{1-x^2-y^2}} (x^2+y^2)(1+x+y+z)\,dzdxdy$

41. **(a)** $m=\int_{0}^{1}\int_{0}^{\sqrt{1-x^2}}\int_{0}^{y}(1+x+y+z)\,dzdydx=\frac{3\pi}{32}+\frac{11}{24}$

(b)

$(\bar{x},\bar{y},\bar{z})=\Bigg(m^{-1}\int_{0}^{1}\int_{0}^{\sqrt{1-x^2}}\int_{0}^{y}x(1+x+y+z)\,dzdy\,\mathrm{dx},$

$m^{-1}\int_{0}^{1}\int_{0}^{\sqrt{1-x^2}}\int_{0}^{y}y(1+x+y+z)\,dzdy\,\mathrm{dx},$

$$m^{-1}\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^y z(1+x+y+z)\,dz\,dy\,dx\Bigg)$$

$$=\left(\frac{28}{9\pi+44},\frac{30\pi+128}{45\pi+220},\frac{45\pi+208}{135\pi+660}\right)$$

(c) $I_z=\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^y (x^2+y^2)(1+x+y+z)\,dz\,dy\,dx=\dfrac{68+15\pi}{240}$

42. **(a)** $m=\int_0^1\int_{3x}^3\int_0^{\sqrt{9-y^2}}(x^2+y^2)\,dz\,dy\,dx=\dfrac{56}{5}=11.2$

(b) $(\bar{x},\bar{y},\bar{z})$ where $\bar{x}=m^{-1}\int_0^1\int_{3x}^3\int_0^{\sqrt{9-y^2}}x(x^2+y^2)\,dz\,dy\,dx\approx 0.375$,

$\bar{y}=m^{-1}\int_0^1\int_{3x}^3\int_0^{\sqrt{9-y^2}}y(x^2+y^2)\,dz\,dy\,dx=\dfrac{45\pi}{64}\approx 2.209$, $\bar{z}=m^{-1}\int_0^1\int_{3x}^3\int_0^{\sqrt{9-y^2}}z(x^2+y^2)\,dz\,dy\,dx=\dfrac{15}{16}=0.9375$.

(c) $I_z=\int_0^1\int_{3x}^3\int_0^{\sqrt{9-y^2}}(x^2+y^2)^2\,dz\,dy\,dx=\dfrac{10{,}464}{175}\approx 59.79$

43.

$$I_x=\int_0^L\int_0^L\int_0^L k(y^2+z^2)\,dz\,dy\,dx=k\int_0^L\int_0^L\left(Ly^2+\frac{1}{3}L^3\right)dy\,dx=k\int_0^L\frac{2}{3}L^4\,dx=\frac{2}{3}kL^5\ .$$

By symmetry, $I_x=I_y=I_z=\dfrac{2}{3}kL^5$.

44. Let k be the density. Then

$$I_x=\int_{-c/2}^{c/2}\int_{-b/2}^{b/2}\int_{-a/2}^{a/2}k(y^2+z^2)\,dx\,dy\,dz=ka\int_{-c/2}^{c/2}\int_{-b/2}^{b/2}(y^2+z^2)\,dy\,dz$$

$$=ak\int_{-c/2}^{c/2}\left[\frac{1}{3}y^3+z^2y\right]_{y=-b/2}^{y=b/2}dz=ak\int_{-c/2}^{c/2}\left(\frac{1}{12}b^3+bz^2\right)dz=ak\left[\frac{1}{12}b^3z+\frac{1}{3}bz^3\right]_{-c/2}^{c/2}$$

$$= ak\left(\frac{1}{12}b^3c+\frac{1}{12}bc^3\right)=\frac{1}{12}kabc(b^2+c^2)$$

By symmetry, $I_y=\frac{1}{12}kabc(a^2+c^2)$ and $I_z=\frac{1}{12}kabc(a^2+b^2)$.

45. **(a)** $f(x,y,z)$ is a joint density function, so we know $\iiint_{R^3} f(x,y,z)dV=1$. Here we have

$$\begin{aligned}\iiint_{R^3} f(x,y,z)dV &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y,z)dzdydx=\int_0^2\int_0^2\int_0^2 Cxyzdzdydx\\ &= C\int_0^2 xdx\int_0^2 ydy\int_0^2 zdz=C\left[\frac{x^2}{2}\right]_0^2\left[\frac{y^2}{2}\right]_0^2\left[\frac{z^2}{2}\right]_0^2\\ &= 8C\end{aligned}$$

Then we must have $8C=1\Rightarrow C=\frac{1}{8}$.

(b)

$$\begin{aligned}P(X\leq 1,Y\leq 1,Z\leq 1) &= \int_{-\infty}^{1}\int_{-\infty}^{1}\int_{-\infty}^{1} f(x,y,z)dzdydx\\ &=\int_0^1\int_0^1\int_0^1 \frac{1}{8}xyzdzdydx=\frac{1}{8}\int_0^1 xdx\int_0^1 ydy\int_0^1 zdz\\ &=\frac{1}{8}\left[\frac{x^2}{2}\right]_0^1\left[\frac{y^2}{2}\right]_0^1\left[\frac{z^2}{2}\right]_0^1=\frac{1}{8}\left(\frac{1}{2}\right)^3=\frac{1}{64}\end{aligned}$$

(c) $P(X+Y+Z\leq 1)=P((X,Y,Z)\in E)$ where E is the solid region in the first octant bounded by the coordinate planes and the plane $x+y+z=1$. The plane $x+y+z=1$ meets the xy –plane in the line $x+y=1$, so we have

$$\begin{aligned}P(X+Y+Z\leq 1) &= \iiint_E f(x,y,z)dV=\int_0^1\int_0^{1-x}\int_0^{1-x-y}\frac{1}{8}xyzdzdydx\\ &= \frac{1}{8}\int_0^1\int_0^{1-x} xy\left[\frac{1}{2}z^2\right]_{z=0}^{z=1-x-y}dydx\\ &= \frac{1}{16}\int_0^1\int_0^{1-x} xy(1-x-y)^2dydx\end{aligned}$$

$$= \frac{1}{16}\int_0^1\int_0^{1-x}\left[(x^3-2x^2+x)y+(2x^2-2x)y^2+xy^3\right]dy\,dx$$

$$= \frac{1}{16}\int_0^1\left[(x^3-2x^2+x)\frac{1}{2}y^2+(2x^2-2x)\frac{1}{3}y^3+x\left(\frac{1}{4}y^4\right)\right]_{y=0}^{y=1-x}dx$$

$$= \frac{1}{192}\int_0^1(x-4x^2+6x^3-4x^4+x^5)dx=\frac{1}{192}\left(\frac{1}{30}\right)=\frac{1}{5760}$$

46. **(a)** $f(x,y,z)$ is a joint density function, so we know $\iiint_{R^3} f(x,y,z)\,dV=1$. Here we have

$$\iiint_{R^3} f(x,y,z)\,dV = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y,z)\,dz\,dy\,dx$$

$$= \int_0^{\infty}\int_0^{\infty}\int_0^{\infty} Ce^{-(0.5x+0.2y+0.1z)}dz\,dy\,dx$$

$$= C\int_0^{\infty} e^{-0.5x}dx\int_0^{\infty} e^{-0.2y}dy\int_0^{\infty} e^{-0.1z}dz$$

$$= C\lim_{t\to\infty}\int_0^t e^{-0.5x}dx\lim_{t\to\infty}\int_0^t e^{-0.2y}dy\lim_{t\to\infty}\int_0^t e^{-0.1z}dz$$

$$= C\lim_{t\to\infty}\left[-2e^{-0.5x}\right]_0^t\lim_{t\to\infty}\left[-5e^{-0.2y}\right]_0^t\lim_{t\to\infty}\left[-10e^{-0.1z}\right]_0^t$$

$$= C\lim_{t\to\infty}\left[-2(e^{-0.5t}-1)\right]\lim_{t\to\infty}\left[-5(e^{-0.2t}-1)\right]\lim_{t\to\infty}\left[-10(e^{-0.1t}-1)\right]$$

$$= C\cdot(-2)(0-1)\cdot(-5)(0-1)\cdot(-10)(0-1)=100C$$

So we must have $100C=1\Rightarrow C=\frac{1}{100}$.

(b) We have no restriction on Z , so

$$P(X\le 1,Y\le 1) = \int_{-\infty}^{1}\int_{-\infty}^{1}\int_{-\infty}^{\infty} f(x,y,z)\,dz\,dy\,dx$$

$$=\int_0^1\int_0^1\int_0^{\infty}\frac{1}{100}e^{-(0.5x+0.2y+0.1z)}dz\,dy\,dx$$

$$= \frac{1}{100}\int_0^1 e^{-0.5x}dx\int_0^1 e^{-0.2y}dy\int_0^\infty e^{-0.1z}dz$$

$$= \frac{1}{100}\left[-2e^{-0.5x}\right]_0^1\left[-5e^{-0.2y}\right]_0^1\lim_{t\to\infty}\left[-10e^{-0.1z}\right]_0^t \text{ [by part (a)]}$$

$$= \frac{1}{100}(2-2e^{-0.5})(5-5e^{-0.2})(10)=(1-e^{-0.5})(1-e^{-0.2})\approx 0.07132$$

(c)

$$P(X\le 1,\mathrm{Y}\le 1,\mathrm{Z}\le 1) = \int_{-\infty}^1\int_{-\infty}^1\int_{-\infty}^1 f(\mathrm{x,y,z})\,dz\,dy\,dx$$

$$=\int_0^1\int_0^1\int_0^1 \frac{1}{100}e^{-(0.5x+0.2y+0.1z)}dz\,dy\,dx$$

$$= \frac{1}{100}\int_0^1 e^{-0.5x}dx\int_0^1 e^{-0.2y}dy\int_0^1 e^{-0.1z}dz$$

$$= \frac{1}{100}\left[-2e^{-0.5x}\right]_0^1\left[-5e^{-0.2y}\right]_0^1\left[-10e^{-0.1z}\right]_0^1$$

$$=(1-e^{-0.5})(1-e^{-0.2})(1-e^{-0.1})\approx 0.006787$$

47. $V(E)=L^3$,

$$f_{\text{ave}} = \frac{1}{L^3}\int_0^L\int_0^L\int_0^L xyz\,dx\,dy\,dz= \frac{1}{L^3}\int_0^L x\,dx\int_0^L y\,dy\int_0^L z\,dz$$

$$= \frac{1}{L^3}\left[\frac{x^2}{2}\right]_0^L\left[\frac{y^2}{2}\right]_0^L\left[\frac{z^2}{2}\right]_0^L= \frac{1}{L^3}\frac{L^2}{2}\frac{L^2}{2}\frac{L^2}{2} = \frac{L^3}{8}$$

48.

$$V(E) =\int_{-1}^1\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_0^{1-x^2-y^2} dz\,dy\,dx=\int_{-1}^1\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}(1-x^2-y^2)\,dy\,dx$$

$$=\int_0^{2\pi}\int_0^1(1-r^2)r\,dr\,d\theta=\int_0^{2\pi}d\theta\int_0^1(r-r^3)dr= 2\pi\left(\frac{r^2}{2}-\frac{r^4}{4}\right)\Bigg]_0^1= \frac{\pi}{2} .$$

Then

$$f_{\text{ave}} = \frac{1}{\pi/2}\iiint_E (x^2z+y^2z)\,dV = \frac{2}{\pi}\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{0}^{1-x^2-y^2}(x^2+y^2)\,z\,dz\,dy\,dx$$

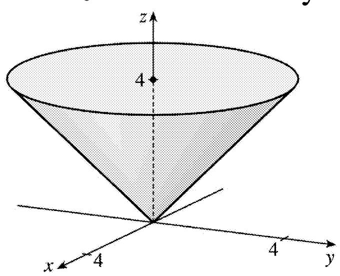
$$= \frac{2}{\pi}\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}(x^2+y^2)\cdot\frac{1}{2}(1-x^2-y^2)^2\,dy\,dx = \frac{1}{\pi}\int_0^{2\pi}\int_0^1 r^2(1-r^2)^2\,r\,dr\,d\theta$$

$$= \frac{1}{\pi}\int_0^{2\pi}d\theta\int_0^1(r^3-2r^5+r^7)\,dr = \frac{1}{\pi}(2\pi)\left[\frac{1}{4}r^4-\frac{1}{3}r^6+\frac{1}{8}r^8\right]_0^1$$

$$=2\left(\frac{1}{24}\right)=\frac{1}{12}$$

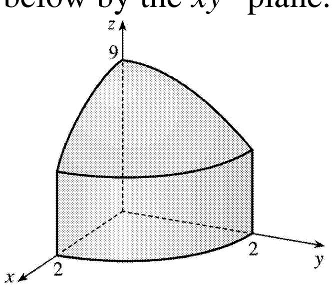
49. The triple integral will attain its maximum when the integrand $1-x^2-2y^2-3z^2$ is positive in the region E and negative everywhere else. For if E contains some region F where the integrand is negative, the integral could be increased by excluding F from E , and if E fails to contain some part G of the region where the integrand is positive, the integral could be increased by including G in E . So we require that $x^2+2y^2+3z^2\le 1$. This describes the region bounded by the ellipsoid $x^2+2y^2+3z^2=1$.

1. The region of integration is given in cylindrical coordinates by $E=\{(r,\theta,z)|0\le\theta\le 2\pi, 0\le r\le 4, r\le z\le 4\}$. This represents the solid region bounded below by the cone $z=r$ and above by the horizontal plane $z=4$.



$$\int_0^4\int_0^{2\pi}\int_r^4 r\,dz\,d\theta\,dr=\int_0^4\int_0^{2\pi}[rz]_{z=r}^{z=4}d\theta\,dr$$
$$=\int_0^4\int_0^{2\pi} r(4-r)\,d\theta\,dr$$
$$=\int_0^4(4r-r^2)\,dr\int_0^{2\pi}d\theta$$
$$=\left[2r^2-\frac{1}{3}r^3\right]_0^4[\theta]_0^{2\pi}$$
$$=\left(32-\frac{64}{3}\right)(2\pi)=\frac{64\pi}{3}$$

2. The region of integration is given in cylindrical coordinates by $E=\left\{(r,\theta,z)|0\le\theta\le\pi/2, 0\le r\le 2, 0\le z\le 9-r^2\right\}$. This represents the solid region in the first octant enclosed by the circular cylinder $r=2$, bounded above by $z=9-r^2$, a circular paraboloid, and bounded below by the xy –plane.



$$\int_0^{\pi/2}\int_0^2\int_0^{9-r^2} r\,dz\,dr\,d\theta=\int_0^{\pi/2}\int_0^2[rz]_{z=0}^{z=9-r^2}dr\,d\theta$$

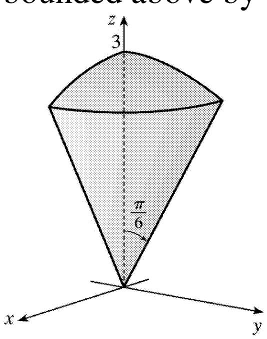
$$=\int_0^{\pi/2}\int_0^2 r(9-r^2)\,dr\,d\theta$$

$$=\int_0^{\pi/2} d\theta \int_0^2 (9r-r^3)\,dr$$

$$=[\theta]_0^{\pi/2}\left[\frac{9}{2}r^2-\frac{1}{4}r^4\right]_0^2$$

$$=\frac{\pi}{2}(18-4)=7\pi$$

3. The region of integration is given in spherical coordinates by $E=\{(\rho,\theta,\phi)\,|\,0\le\rho\le 3, 0\le\theta\le\pi/2, 0\le\phi\le\pi/6\}$. This represents the solid region in the first octant bounded above by the sphere $\rho=3$ and below by the cone $\phi=\pi/6$.



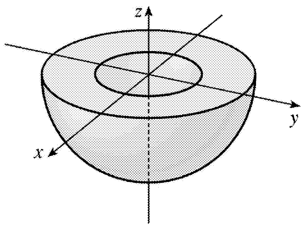
$$\int_0^{\pi/6}\int_0^{\pi/2}\int_0^3 \rho^2\sin\phi\,d\rho\,d\theta\,d\phi=\int_0^{\pi/6}\sin\phi\,d\phi\int_0^{\pi/2}d\theta\int_0^3\rho^2\,d\rho$$

$$=[-\cos\phi]_0^{\pi/6}[\theta]_0^{\pi/2}\left[\frac{1}{3}\rho^3\right]_0^3$$

$$=\left(1-\frac{\sqrt{3}}{2}\right)\left(\frac{\pi}{2}\right)(9)$$

$$=\frac{9\pi}{4}\left(2-\sqrt{3}\right)$$

4. The region of integration is given in spherical coordinates by $E=\{(\rho,\theta,\phi)\,|\,1\le\rho\le 2, 0\le\theta\le 2\pi, \pi/2\le\phi\le\pi\}$. This represents the solid region between the spheres $\rho=1$ and $\rho=2$ and below the xy –plane.



$$\int_0^{2\pi}\int_{\pi/2}^{\pi}\int_1^2 \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta = \int_0^{2\pi} d\theta \int_{\pi/2}^{\pi} \sin\phi\, d\phi \int_1^2 \rho^2 d\rho$$
$$= [\theta]_0^{2\pi} [-\cos\phi]_{\pi/2}^{\pi} \left[\frac{1}{3}\rho^3\right]_1^2$$
$$= 2\pi(1)\left(\frac{7}{3}\right) = \frac{14\pi}{3}$$

5. The solid E is most conveniently described if we use cylindrical coordinates:
$E = \left\{ (r,\theta,z) | 0 \le \theta \le \frac{\pi}{2}, 0 \le r \le 3, 0 \le z \le 2 \right\}$. Then
$$\iiint_E f(x,y,z)\,dV = \int_0^{\pi/2}\int_0^3\int_0^2 f(r\cos\theta, r\sin\theta, z)\, r\, dz\, dr\, d\theta \ .$$

6. The solid E is most conveniently described if we use spherical coordinates:
$E = \left\{ (\rho,\theta,\phi) | 1 \le \rho \le 2, \frac{\pi}{2} \le \theta \le 2\pi, 0 \le \phi \le \frac{\pi}{2} \right\}$. Then □ .

7. In cylindrical coordinates, E is given by $\{(r,\theta,z) | 0 \le \theta \le 2\pi, 0 \le r \le 4, -5 \le z \le 4\}$. So
$$\iiint_E \sqrt{x^2+y^2}\, dV = \int_0^{2\pi}\int_0^4\int_{-5}^4 \sqrt{r^2}\, r\, dz\, dr\, d\theta = \int_0^{2\pi} d\theta \int_0^4 r^2 dr \int_{-5}^4 dz$$
$$= [\theta]_0^{2\pi} \left[\frac{1}{3}r^3\right]_0^4 [z]_{-5}^4 = (2\pi)\left(\frac{64}{3}\right)(9) = 384\pi$$

8. The paraboloid $z = 1 - x^2 - y^2$ intersects the xy −plane in the circle $x^2 + y^2 = r^2 = 1$ or $r = 1$, so in cylindrical coordinates, E is given by $\left\{ (r,\theta,z) \,\middle|\, 0 \le \theta \le \frac{\pi}{2}, 0 \le r \le 1, 0 \le z \le 1 - r^2 \right\}$. Thus
$$\iiint_E (x^3 + xy^2)\, dV = \int_0^{\pi/2}\int_0^1\int_0^{1-r^2} (r^3\cos^3\theta + r^3\cos\theta\sin^2\theta)\, r\, dz\, dr\, d\theta$$

$$= \int_0^{\pi/2}\int_0^1\int_0^{1-r^2} r^4\cos\theta\, dz\,dr\,d\theta = \int_0^{\pi/2}\int_0^1 r^4\cos\theta\,[z]_{z=0}^{z=1-r^2}\,dr\,d\theta$$

$$= \int_0^{\pi/2}\int_0^1 r^4(1-r^2)\cos\theta\, dr\,d\theta = \int_0^{\pi/2}\cos\theta\left[\frac{1}{5}r^5-\frac{1}{7}r^7\right]_{r=0}^{r=1}d\theta$$

$$= \int_0^{\pi/2}\frac{2}{35}\cos\theta\,d\theta = \frac{2}{35}\,[\sin\theta]_0^{\pi/2} = \frac{2}{35}$$

9. In cylindrical coordinates E is bounded by the paraboloid $z=1+r^2$, the cylinder $r^2=5$ or $r=\sqrt{5}$, and the xy –plane, so E is given by $\{(r,\theta,z)\,|\,0\le\theta\le 2\pi, 0\le r\le\sqrt{5}, 0\le z\le 1+r^2\}$. Thus

$$\iiint_E e^z dV = \int_0^{2\pi}\int_0^{\sqrt{5}}\int_0^{1+r^2} e^z r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^{\sqrt{5}} r\left[e^z\right]_{z=0}^{z=1+r^2}dr\,d\theta = \int_0^{2\pi}\int_0^{\sqrt{5}} r(e^{1+r^2}-1)\,dr\,d\theta$$

$$= \int_0^{2\pi} d\theta\int_0^{\sqrt{5}}\left(re^{1+r^2}-r\right)dr = 2\pi\left[\frac{1}{2}e^{1+r^2}-\frac{1}{2}r^2\right]_0^{\sqrt{5}} = \pi(e^6-e-5)$$

10. In cylindrical coordinates E is bounded by the planes $z=0$, $z=r\cos\theta+r\sin\theta+3$ and the cylinders $r=2$ and $r=3$, so E is given by $\{(r,\theta,z)|0\le\theta\le 2\pi, 2\le r\le 3, 0\le z\le r\cos\theta+r\sin\theta+3\}$. Thus

$$\iiint_E x\,dV = \int_0^{2\pi}\int_2^3\int_0^{r\cos\theta+r\sin\theta+3}(r\cos\theta)\,r\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_2^3 (r^2\cos\theta)\,]_{z=0}^{z=r\cos\theta+r\sin\theta+3}\,dr\,d\theta = \int_0^{2\pi}\int_2^3 (r^2\cos\theta)(r\cos\theta+r\sin\theta+3)\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_2^3 (r^3(\cos^2\theta+\cos\theta\sin\theta)+3r^2\cos\theta)\,dr\,d\theta$$

$$= \int_0^{2\pi}\left[\frac{1}{4}r^4(\cos^2\theta+\cos\theta\sin\theta)+r^3\cos\theta\right]_{r=2}^{r=3}d\theta$$

$$= \int_0^{2\pi}\left[\left(\frac{81}{4}-\frac{16}{4}\right)(\cos^2\theta+\cos\theta\sin\theta)+(27-8)\cos\theta\right]d\theta$$

$$= \int_0^{2\pi}\left(\frac{65}{4}\left(\frac{1}{2}(1+\cos 2\theta)+\cos\theta\sin\theta\right)+19\cos\theta\right)d\theta$$

$$=\left[\frac{65}{8}\theta+\frac{65}{16}\sin 2\theta+\frac{65}{8}\sin^2\theta+19\sin\theta\right]_0^{2\pi}=\frac{65}{4}\pi$$

11. In cylindrical coordinates, E is bounded by the cylinder $r=1$, the plane $z=0$, and the cone $z=2r$. So $E=\{(r,\theta,z)\mid 0\le\theta\le 2\pi, 0\le r\le 1, 0\le z\le 2r\}$ and

$$\iiint_E x^2\,dV=\int_0^{2\pi}\int_0^1\int_0^{2r} r^2\cos^2\theta\, r\,dz\,dr\,d\theta=\int_0^{2\pi}\int_0^1\left[r^3\cos^2\theta\, z\right]_{z=0}^{z=2r}dr\,d\theta$$

$$=\int_0^{2\pi}\int_0^1 2r^4\cos^2\theta\,dr\,d\theta=\int_0^{2\pi}\left[\frac{2}{5}r^5\cos^2\theta\right]_{r=0}^{r=1}d\theta=\frac{2}{5}\int_0^{2\pi}\cos^2\theta\,d\theta$$

$$=\frac{2}{5}\int_0^{2\pi}\frac{1+\cos 2\theta}{2}\,d\theta=\frac{1}{5}\left[\theta+\frac{1}{2}\sin 2\theta\right]_0^{2\pi}=\frac{2\pi}{5}$$

12. In cylindrical coordinates E is the solid region within the cylinder $r=1$ bounded above and below by the sphere $r^2+z^2=4$, so $E=\left\{(r,\theta,z)\mid 0\le\theta\le 2\pi, 0\le r\le 1, -\sqrt{4-r^2}\le z\le\sqrt{4-r^2}\right\}$. Thus the volume is

$$\iiint_E dV=\int_0^{2\pi}\int_0^1\int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} r\,dz\,dr\,d\theta=\int_0^{2\pi}\int_0^1 2r\sqrt{4-r^2}\,dr\,d\theta$$

$$=\int_0^{2\pi}d\theta\int_0^1 2r\sqrt{4-r^2}\,dr=2\pi\left[-\frac{2}{3}(4-r^2)^{3/2}\right]_0^1=\frac{4}{3}\pi(8-3^{3/2})$$

13. **(a)** The paraboloids intersect when $x^2+y^2=36-3x^2-3y^2\Rightarrow x^2+y^2=9$, so the region of integration is $D=\left\{(x,y)\mid x^2+y^2\le 9\right\}$. Then, in cylindrical coordinates, $E=\left\{(r,\theta,z)\mid r^2\le z\le 36-3r^2, 0\le r\le 3, 0\le\theta\le 2\pi\right\}$ and

$$V=\int_0^{2\pi}\int_0^3\int_{r^2}^{36-3r^2} r\,dz\,dr\,d\theta=\int_0^{2\pi}\int_0^3\left(36r-4r^3\right)dr\,d\theta$$

$$=\int_0^{2\pi}\left[18r^2-r^4\right]_{r=0}^{r=3}d\theta=\int_0^{2\pi}81\,d\theta=162\pi$$

(b) For constant density K , $m=KV=162\pi K$ from part (a). Since the region is homogeneous and symmetric,

$M_{yz}=M_{xz}=0$ and

$$M_{xy}=\int_0^{2\pi}\int_0^3\int_{r^2}^{36-3r^2}(zK)r\,dz\,dr\,d\theta=K\int_0^{2\pi}\int_0^3 r\left[\frac{1}{2}z^2\right]_{z=r^2}^{z=36-3r^2}dr\,d\theta$$

$$=\frac{K}{2}\int_0^{2\pi}\int_0^3 r((36-3r^2)^2-r^4)\,dr\,d\theta=\frac{K}{2}\int_0^{2\pi}d\theta\int_0^3(8r^5-216r^3+1296r)\,dr$$

$$=\frac{K}{2}(2\pi)\left[\frac{8}{6}r^6-\frac{216}{4}r^4+\frac{1296}{2}r^2\right]_0^3=\pi K(2430)=2430\pi K$$

Thus $(\overline{x},\overline{y},\overline{z})=\left(\frac{M_{yz}}{m},\frac{M_{xz}}{m},\frac{M_{xy}}{m}\right)=\left(0,0,\frac{2430\pi K}{162\pi K}\right)=(0,0,15)$.

14. **(a)** $V=\int_{-\pi/2}^{\pi/2}\int_0^{a\cos\theta}\int_{-\sqrt{a^2-r^2}}^{\sqrt{a^2-r^2}}r\,dz\,dr\,d\theta$

$=4\int_0^{\pi/2}\int_0^{a\cos\theta}\int_0^{\sqrt{a^2-r^2}}r\,dz\,dr\,d\theta$

$=4\int_0^{\pi/2}\int_0^{a\cos\theta}r\sqrt{a^2-r^2}\,dr\,d\theta$

$=-\frac{4}{3}\int_0^{\pi/2}\left[\left(a^2-r^2\right)^{3/2}\right]_{r=0}^{r=a\cos\theta}d\theta$

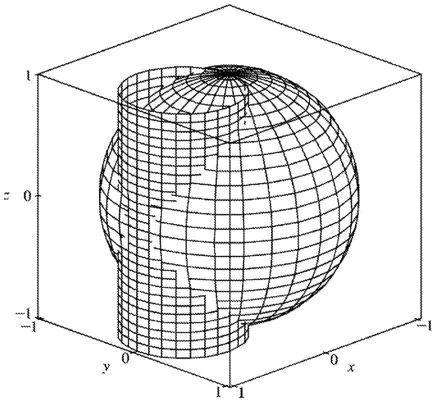
$=-\frac{4}{3}\int_0^{\pi/2}\left[\left(a^2-a^2\cos^2\theta\right)^{3/2}-a^3\right]d\theta$

$=-\frac{4}{3}\int_0^{\pi/2}\left[\left(a^2\sin^2\theta\right)^{3/2}-a^3\right]d\theta$

$=-\frac{4}{3}\int_0^{\pi/2}\left(a^3\sin^3\theta-a^3\right)d\theta$

$=-\frac{4a^3}{3}\int_0^{\pi/2}\left[\sin\theta\left(1-\cos^2\theta\right)-1\right]d\theta$

(b)



$=-\frac{4a^3}{3}\left[-\cos\theta+\frac{1}{3}\cos^3\theta-\theta\right]_0^{\pi/2}=-\frac{4a^3}{3}\left(-\frac{\pi}{2}+\frac{2}{3}\right)=\frac{2}{9}a^3(3\pi-4)$

To plot the cylinder and the sphere on the same screen in Maple, we can use the sequence of commands

sphere:=plot3d(1,theta=0..2*Pi,phi=0..Pi,coords=spherical):
cylinder:=plot3d(,
theta=0..2*Pi,z=−1..1,coords=cylindrical):
with(plots): display3d(sphere,cylinder});

In Mathematica, we can use

sphere==SphericalPlot3d[1,{theta,0,2Pi},{phi,0,Pi}],
cylinder=ParametricPlot3d[{Sin[theta],Cos[theta],z},
{theta,0,2Pi},{z,−1,1}]
Show[{sphere, cylinder}]

15. The paraboloid $z=4x^2+4y^2$ intersects the plane $z=a$ when $a=4x^2+4y^2$ or $x^2+y^2=\frac{1}{4}a$. So, in cylindrical coordinates, $E=\left\{(r,\theta,z)|0\le r\le\frac{1}{2}\sqrt{a},0\le\theta\le 2\pi,4r^2\le z\le a\right\}$. Thus

$$m=\int_0^{2\pi}\int_0^{\sqrt{a}/2}\int_{4r^2}^{a}Kr\,dz\,dr\,d\theta=K\int_0^{2\pi}\int_0^{\sqrt{a}/2}(ar-4r^3)\,dr\,d\theta$$

$$=K\int_0^{2\pi}\left[\frac{1}{2}ar^2-r^4\right]_{r=0}^{r=\sqrt{a}/2}d\theta=K\int_0^{2\pi}\frac{1}{16}a^2\,d\theta=\frac{1}{8}a^2\pi K$$

Since the region is homogeneous and symmetric, $M_{yz}=M_{xz}=0$ and

$$M_{xy} = \int_0^{2\pi}\int_0^{\sqrt{a}/2}\int_{4r^2}^{a} Krz\,dz\,dr\,d\theta = K\int_0^{2\pi}\int_0^{\sqrt{a}/2}\left(\frac{1}{2}a^2r - 8r^5\right)dr\,d\theta$$

$$= K\int_0^{2\pi}\left[\frac{1}{4}a^2r^2 - \frac{4}{3}r^6\right]_{r=0}^{r=\sqrt{a}/2} d\theta = K\int_0^{2\pi}\frac{1}{24}a^3\,d\theta = \frac{1}{12}a^3\pi K$$

Hence $(\overline{x},\overline{y},\overline{z}) = \left(0,0,\frac{2}{3}a\right)$.

16. Since density is proportional to the distance from the z –axis, we can say $\rho(x,y,z) = K\sqrt{x^2+y^2}$. Then

$$m = 2\int_0^{2\pi}\int_0^{a}\int_0^{\sqrt{a^2-r^2}} Kr^2\,dz\,dr\,d\theta = 2K\int_0^{2\pi}\int_0^{a} r^2\sqrt{a^2-r^2}\,dr\,d\theta$$

$$= 2K\int_0^{2\pi}\left[\frac{1}{8}r(2r^2-a^2)\sqrt{a^2-r^2} + \frac{1}{8}a^4\sin^{-1}(r/a)\right]_{r=0}^{r=a} d\theta$$

$$= 2K\int_0^{2\pi}\left[\left(\frac{1}{8}a^4\right)\left(\frac{\pi}{2}\right)\right]d\theta = \frac{1}{4}a^4\pi^2K .$$

17. In spherical coordinates, B is represented by $\{(\rho,\theta,\phi)\,|\,0\le\rho\le 1, 0\le\theta\le 2\pi, 0\le\phi\le\pi\}$. Thus

$$\iiint_B (x^2+y^2+z^2)\,dV = \int_0^{\pi}\int_0^{2\pi}\int_0^{1}(\rho^2)\rho^2\sin\phi\,d\rho\,d\theta\,d\phi = \int_0^{\pi}\sin\phi\,d\phi\int_0^{2\pi}d\theta\int_0^{1}\rho^4\,d\rho$$

$$= [-\cos\phi]_0^{\pi}\,[\theta]_0^{2\pi}\left[\frac{1}{5}\rho^5\right]_0^1 = (2)(2\pi)\left(\frac{1}{5}\right) = \frac{4\pi}{5}$$

18. In spherical coordinates, H is represented by $\left\{(\rho,\theta,\phi)\,\middle|\,0\le\rho\le 1, 0\le\theta\le 2\pi, 0\le\phi\le\frac{\pi}{2}\right\}$. Thus

$$\iiint_H (x^2+y^2)\,dV = \int_0^{2\pi}\int_0^{\pi/2}\int_0^{1}(\rho^2\sin^2\phi)\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \int_0^{2\pi}d\theta\int_0^{\pi/2}\sin^3\phi\,d\phi\int_0^{1}\rho^4\,d\rho$$

$$= [\theta]_0^{2\pi}\left[-\cos\phi + \frac{1}{3}\cos^3\phi\right]_0^{\pi/2}\left[\frac{1}{5}\rho^5\right]_0^1 = \frac{4\pi}{15}$$

19. In spherical coordinates, E is represented by

$\left\{ (\rho,\theta,\phi) \,\middle|\, 1\le\rho\le 2, 0\le\theta\le\frac{\pi}{2}, 0\le\phi\le\frac{\pi}{2} \right\}$. Thus

$$\iiint_E z\,dV = \int_0^{\pi/2}\int_0^{\pi/2}\int_1^2 (\rho\cos\phi)\rho^2\sin\phi\,d\rho\,d\theta\,d\phi$$

$$= \int_0^{\pi/2}\cos\phi\sin\phi\,d\phi\int_0^{\pi/2}d\theta\int_1^2\rho^3\,d\rho = \left[\frac{1}{2}\sin^2\phi\right]_0^{\pi/2}\left[\theta\right]_0^{\pi/2}\left[\frac{1}{4}\rho^4\right]_1^2$$

$$= \left(\frac{1}{2}\right)\left(\frac{\pi}{2}\right)\left(\frac{15}{4}\right) = \frac{15\pi}{16}$$

20.

$$\iiint_E e^{\sqrt{x^2+y^2+z^2}}\,dV = \int_0^{\pi/2}\int_0^{\pi/2}\int_0^3 e^{\sqrt{\rho^2}}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \int_0^{\pi/2}\int_0^{\pi/2}\int_0^3\rho^2 e^{\rho}\sin\phi\,d\rho\,d\phi\,d\theta$$

$$= \int_0^{\pi/2}d\theta\int_0^{\pi/2}\sin\phi\,d\phi\int_0^3\rho^2 e^{\rho}\,d\rho = \left[\theta\right]_0^{\pi/2}\left[-\cos\phi\right]_0^{\pi/2}\left[(\rho^2-2\rho+2)e^{\rho}\right]_0^3$$

[integrate by parts twice]

$$= \frac{\pi}{2}(0+1)(5e^3-2) = \frac{\pi}{2}(5e^3-2)$$

21.

$$\iiint_E x^2\,dV = \int_0^{\pi}\int_0^{\pi}\int_3^4(\rho\sin\phi\cos\theta)^2\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

$$= \int_0^{\pi}\cos^2\theta\,d\theta\int_0^{\pi}\sin^3\phi\,d\phi\int_3^4\rho^4\,d\rho$$

$$= \left[\frac{1}{2}\theta+\frac{1}{4}\sin 2\theta\right]_0^{\pi}\left[-\frac{1}{3}(2+\sin^2\phi)\cos\phi\right]_0^{\pi}\left[\frac{1}{5}\rho^5\right]_3^4$$

$$= \left(\frac{\pi}{2}\right)\left(\frac{2}{3}+\frac{2}{3}\right)\frac{1}{5}(4^5-3^5) = \frac{1562}{15}\pi$$

22.

$$\iiint_E xyz\,dV = \int_0^{\pi/3}\int_0^{2\pi}\int_2^4(\rho\sin\phi\cos\theta)(\rho\sin\phi\sin\theta)(\rho\cos\phi)\rho^2\sin\phi\,d\rho\,d\theta\,d\phi$$

$$= \int_0^{\pi/3} \sin^3\phi \cos\phi \, d\phi \int_0^{2\pi} \sin\theta \cos\theta \, d\theta \int_2^4 \rho^5 d\rho = \left[\frac{1}{4} \sin^4\phi \right]_0^{\pi/3} \left[\frac{1}{2} \sin^2\theta \right]_0^{2\pi} \left[\frac{1}{6} \rho^6 \right]_2^4 = 0$$

23. Since $\rho = 4\cos\phi$ implies $\rho^2 = 4\rho\cos\phi$, the equation is that of a sphere of radius 2 with center at $(0,0,2)$. Thus

$$V = \int_0^{2\pi} \int_0^{\pi/3} \int_0^{4\cos\phi} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = \int_0^{2\pi} \int_0^{\pi/3} \left[\frac{1}{3} \rho^3 \right]_{\rho=0}^{\rho=4\cos\phi} \sin\phi \, d\phi \, d\theta$$

$$= \int_0^{2\pi} \int_0^{\pi/3} \left(\frac{64}{3} \cos^3\phi \right) \sin\phi \, d\phi \, d\theta = \int_0^{2\pi} \left[-\frac{16}{3} \cos^4\phi \right]_{\phi=0}^{\phi=\pi/3} d\theta$$

$$= \int_0^{2\pi} -\frac{16}{3} \left(\frac{1}{16} - 1 \right) d\theta = 5\theta \Big]_0^{2\pi} = 10\pi$$

24. In spherical coordinates, the sphere $x^2 + y^2 + z^2 = 4$ is equivalent to $\rho = 2$ and the cone $z = \sqrt{x^2 + y^2}$ is represented by $\phi = \frac{\pi}{4}$. Thus, the solid is given by $\left\{ (\rho, \theta, \phi) \;\middle|\; 0 \le \rho \le 2, 0 \le \theta \le 2\pi, \frac{\pi}{4} \le \phi \le \frac{\pi}{2} \right\}$ and

$$V = \int_{\pi/4}^{\pi/2} \int_0^{2\pi} \int_0^2 \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi = \int_{\pi/4}^{\pi/2} \sin\phi \, d\phi \int_0^{2\pi} d\theta \int_0^2 \rho^2 d\rho$$

$$= [-\cos\phi]_{\pi/4}^{\pi/2} [\theta]_0^{2\pi} \left[\frac{1}{3} \rho^3 \right]_0^2 = \left(\frac{\sqrt{2}}{2} \right) (2\pi) \left(\frac{8}{3} \right) = \frac{8\sqrt{2}\,\pi}{3}$$

25. By the symmetry of the region, $M_{xy} = 0$ and $M_{yz} = 0$. Assuming constant density K,

$$m = \iiint_E KV = K \int_0^{\pi} \int_0^{\pi} \int_3^4 \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = K \int_0^{\pi} d\theta \int_0^{\pi} \sin\phi \, d\phi \int_3^4 \rho^2 d\rho$$

$$= K\pi [-\cos\phi]_0^{\pi} \left[\frac{1}{3} \rho^3 \right]_3^4 = 2K\pi \cdot \frac{37}{3} = \frac{74}{3} \pi K$$

and

$$M_{xz} = \iiint_E yK \, dV = K \int_0^{\pi} \int_0^{\pi} \int_3^4 (\rho \sin\phi \sin\theta) \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$=K\int_0^{\pi}\sin\theta\,d\theta\int_0^{\pi}\sin^2\phi\,d\phi\int_3^4\rho^3 d\rho$$

$$=K\left[-\cos\theta\right]_0^{\pi}\left[\frac{1}{2}\phi-\frac{1}{4}\sin 2\phi\right]_0^{\pi}\left[\frac{1}{4}\rho^4\right]_3^4$$

$$=K(2)\left(\frac{\pi}{2}\right)\frac{1}{4}(256-81)=\frac{175}{4}\pi K$$

Thus the centroid is $(\bar{x},\bar{y},\bar{z})=\left(\frac{M_{yz}}{m},\frac{M_{xz}}{m},\frac{M_{xy}}{m}\right)=\left(0,\frac{175\pi K/4}{74\pi K/3},0\right)=\left(0,\frac{525}{296},0\right)$.

26. **(a)** Placing the center of the base at $(0,0,0)$, $\rho(x,y,z)=K\sqrt{x^2+y^2+z^2}$ is the density function. So

$$m=\int_0^{2\pi}\int_0^{\pi/2}\int_0^{a}K\rho^3\sin\phi\,d\rho\,d\phi\,d\theta=K\int_0^{2\pi}d\theta\int_0^{\pi/2}\sin\phi\,d\phi\int_0^{a}\rho^3 d\rho$$

$$=K\left[\theta\right]_0^{2\pi}\left[-\cos\phi\right]_0^{\pi/2}\left[\frac{1}{4}\rho^4\right]_0^{a}=K(2\pi)(1)\left(\frac{1}{4}a^4\right)=\frac{1}{2}\pi Ka^4$$

(b) By the symmetry of the problem $M_{yz}=M_{xz}=0$. Then

$$M_{xy}=\int_0^{2\pi}\int_0^{\pi/2}\int_0^{a}K\rho^4\sin\phi\cos\phi\,d\rho\,d\phi\,d\theta=K\int_0^{2\pi}d\theta\int_0^{\pi/2}\sin\phi\cos\phi\,d\phi\int_0^{a}\rho^4 d\rho$$

$$=K\left[\theta\right]_0^{2\pi}\left[\frac{1}{2}\sin^2\phi\right]_0^{\pi/2}\left[\frac{1}{5}\rho^5\right]_0^{a}=K(2\pi)\left(\frac{1}{2}\right)\left(\frac{1}{5}a^5\right)=\frac{1}{5}\pi Ka^5$$

Hence $(\bar{x},\bar{y},\bar{z})=\left(0,0,\frac{2}{5}a\right)$.

(c)

$$I_z=\int_0^{2\pi}\int_0^{\pi/2}\int_0^{a}(K\rho^3\sin\phi)(\rho^2\sin^2\phi)\,d\rho\,d\phi\,d\theta=K\int_0^{2\pi}d\theta\int_0^{\pi/2}\sin^3\phi\,d\phi\int_0^{a}\rho^5 d\rho$$

$$=K\left[\theta\right]_0^{2\pi}\left[-\cos\phi+\frac{1}{3}\cos^3\phi\right]_0^{\pi/2}\left[\frac{1}{6}\rho^6\right]_0^{a}=K(2\pi)\left(\frac{2}{3}\right)\left(\frac{1}{6}a^6\right)=\frac{2}{9}\pi Ka^6$$

27. **(a)** The density function is $\rho(x,y,z)=K$, a constant, and by the symmetry of the problem $M_{xz}=M_{yz}=0$. Then $M_{xy}=\int_0^{2\pi}\int_0^{\pi/2}\int_0^{a}K\rho^3\sin\phi\cos\phi\,d\rho\,d\phi\,d\theta=\frac{1}{2}\pi Ka^4\int_0^{\pi/2}\sin\phi\cos\phi\,d\phi=\frac{1}{8}\pi Ka^4$. But

the mass is K (volume of the hemisphere) $= \frac{2}{3}\pi K a^3$, so the centroid is $\left(0,0,\frac{3}{8}a\right)$.

(b) Place the center of the base at $(0,0,0)$; the density function is $\rho(x,y,z)=K$. By symmetry, the moments of inertia about any two such diameters will be equal, so we just need to find I_x :

$$I_x = \int_0^{2\pi}\int_0^{\pi/2}\int_0^a \left(K\rho^2 \sin\phi\right)\rho^2\left(\sin^2\phi\sin^2\theta+\cos^2\phi\right)d\rho\, d\phi\, d\theta$$

$$= K\int_0^{2\pi}\int_0^{\pi/2}\left(\sin^3\phi\sin^2\theta+\sin\phi\cos^2\phi\right)\left(\frac{1}{5}a^5\right)d\phi\, d\theta$$

$$= \frac{1}{5}Ka^5\int_0^{2\pi}\left[\sin^2\theta\left(-\cos\phi+\frac{1}{3}\cos^3\phi\right)+\left(-\frac{1}{3}\cos^3\phi\right)\right]_{\phi=0}^{\phi=\pi/2} d\theta$$

$$= \frac{1}{5}Ka^5\int_0^{2\pi}\left[\frac{2}{3}\sin^2\theta+\frac{1}{3}\right]d\theta = \frac{1}{5}Ka^5\left[\frac{2}{3}\left(\frac{1}{2}\theta-\frac{1}{4}\sin 2\theta\right)+\frac{1}{3}\theta\right]_0^{2\pi}$$

$$= \frac{1}{5}Ka^5\left[\frac{2}{3}(\pi-0)+\frac{1}{3}(2\pi-0)\right] = \frac{4}{15}Ka^5\pi$$

28. Place the center of the base at $(0,0,0)$, then the density is $\rho(x,y,z)=Kz$, K a constant. Then

$$m = \int_0^{2\pi}\int_0^{\pi/2}\int_0^a (K\rho\cos\phi)\rho^2\sin\phi\, d\rho\, d\phi\, d\theta$$

$$= 2\pi K\int_0^{\pi/2}\cos\phi\sin\phi\cdot\frac{1}{4}a^4\, d\phi$$

$$= \frac{1}{2}\pi K a^4\left[-\frac{1}{4}\cos 2\phi\right]_0^{\pi/2} = \frac{\pi}{4}Ka^4$$

By the symmetry of the problem $M_{xz}=M_{yz}=0$, and

$$M_{xy} = \int_0^{2\pi}\int_0^{\pi/2}\int_0^a K\rho^4\cos^2\phi\sin\phi\, d\rho\, d\phi\, d\theta$$

$$= \frac{2}{5}\pi K a^5\int_0^{\pi/2}\cos^2\phi\sin\phi\, d\phi$$

$$= \frac{2}{5}\pi K a^5\left[-\frac{1}{3}\cos^3\theta\right]_0^{\pi/2} = \frac{2}{15}\pi K a^5$$

Hence $(\overline{x},\overline{y},\overline{z}) = \left(0,0,\frac{8}{15}a\right)$.

29. In spherical coordinates $z=\sqrt{x^2+y^2}$ becomes $\cos\phi=\sin\phi$ or $\phi=\frac{\pi}{4}$. Then

$$V=\int_0^{2\pi}\int_0^{\pi/4}\int_0^{1}\rho^2\sin\phi\ d\rho\, d\phi\, d\theta=\int_0^{2\pi}d\theta\int_0^{\pi/4}\sin\phi\ d\phi\int_0^{1}\rho^2 d\rho=\frac{1}{3}\pi\left(2-\sqrt{2}\right),$$

$$M_{xy}=\int_0^{2\pi}\int_0^{\pi/4}\int_0^{1}\rho^3\sin\phi\cos\phi\ d\rho\, d\phi\, d\theta=2\pi\left[-\frac{1}{4}\cos 2\phi\right]_0^{\pi/4}\left(\frac{1}{4}\right)=\frac{\pi}{8}$$ and by symmetry $M_{yz}=M_{xz}=0$

. Hence $\left(\overline{x},\overline{y},\overline{z}\right)=\left(0,0,\frac{3}{8\left(2-\sqrt{2}\right)}\right)$.

30. Place the center of the sphere at $(0,0,0)$, let the diameter of intersection be along the z –axis, one of the planes be the xz –plane and the other be the plane whose angle with the xz –plane is $\theta=\frac{\pi}{6}$. Then in spherical coordinates the volume is given by

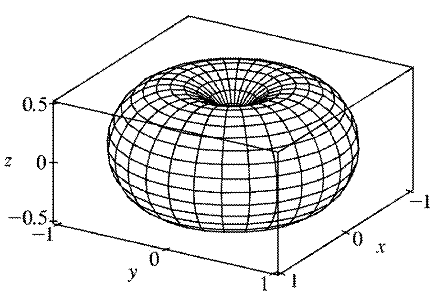
$$V=\int_0^{\pi/6}\int_0^{\pi}\int_0^{a}\rho^2\sin\phi\ d\rho\, d\phi\, d\theta=\int_0^{\pi/6}d\theta\int_0^{\pi}\sin\phi\ d\phi\int_0^{a}\rho^2 d\rho=\frac{\pi}{6}(2)\left(\frac{1}{3}a^3\right)=\frac{1}{9}\pi a^3 .$$

31. In cylindrical coordinates the paraboloid is given by $z=r^2$ and the plane by $z=2r\sin\theta$ and they intersect in the circle $r=2\sin\theta$. Then $\iiint_E z dV=\int_0^{\pi}\int_0^{2\sin\theta}\int_{r^2}^{2r\sin\theta} rz dz dr\, d\theta=\frac{5\pi}{6}$ [using a CAS].

32. **(a)** The region enclosed by the torus is $\{(\rho,\theta,\phi)\,|\,0\le\theta\le 2\pi, 0\le\phi\le\pi, 0\le\rho\le\sin\phi\}$, so its volume is

$$V=\int_0^{2\pi}\int_0^{\pi}\int_0^{\sin\phi}\rho^2\sin\phi\ d\rho\, d\phi\, d\theta=2\pi\int_0^{\pi}\frac{1}{3}\sin^4\phi\ d\phi=\frac{2}{3}\pi\left[\frac{3}{8}\phi-\frac{1}{4}\sin 2\phi+\frac{1}{16}\sin 4\phi\right]_0^{\pi}=\frac{1}{4}\pi^2 .$$

(b) In Maple, we can plot the torus using the plots[sphereplot] command, or with the coords=spherical option in a regular plot command. In Mathematica, use ParametricPlot3d.



33. The region E of integration is the region above the paraboloid $z=x^2+y^2$, or $z=r^2$, and below the

paraboloid $z=2-x^2-y^2$, or $z=2-r^2$. Also, we have $-1\le x\le 1$ with $-\sqrt{1-x^2}\le y\le\sqrt{1-x^2}$ which describes the unit circle in the xy –plane. Thus,

$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{x^2+y^2}^{2-x^2-y^2}(x^2+y^2)^{3/2}\,dz\,dy\,dx=\int_0^{2\pi}\int_0^1\int_{r^2}^{2-r^2}(r^2)^{3/2}\,r\,dz\,dr\,d\theta$$

$$=\int_0^{2\pi}\int_0^1\left[r^4z\right]_{z=r^2}^{z=2-r^2}dr\,d\theta=\int_0^{2\pi}\int_0^1(2r^4-r^6-r^6)\,dr\,d\theta=\int_0^{2\pi}\left(\frac{2}{5}-\frac{2}{7}\right)d\theta=\frac{8\pi}{35}$$

34. The region E of integration is the region above the paraboloid $z=x^2+y^2=r^2$ and below the cone $z=\sqrt{x^2+y^2}=r$. Also, we have $0\le y\le 1$, $0\le x\le\sqrt{1-y^2}$ which is equivalent to $0\le\theta\le\frac{\pi}{2}$, $0\le r\le 1$. Thus

$$\int_0^1\int_0^{\sqrt{1-y^2}}\int_{x^2+y^2}^{\sqrt{x^2+y^2}}xyz\,dz\,dx\,dy=\int_0^{\pi/2}\int_0^1\int_{r^2}^{r}r^2\cos\theta\sin\theta\,zr\,dz\,dr\,d\theta$$

$$=\frac{1}{2}\int_0^{\pi/2}\int_0^1 r^3\cos\theta\sin\theta\left[z^2\right]_{z=r^2}^{z=r}dr\,d\theta=\frac{1}{2}\int_0^{\pi/2}\int_0^1(r^5-r^7)\cos\theta\sin\theta\,dr\,d\theta$$

$$=\frac{1}{2}\int_0^{\pi/2}\left[\frac{1}{6}r^6-\frac{1}{8}r^8\right]_{r=0}^{r=1}\cos\theta\sin\theta\,d\theta=\frac{1}{2}\int_0^{\pi/2}\frac{1}{24}\cos\theta\sin\theta\,d\theta$$

$$=\frac{1}{48}\int_0^{\pi/2}\frac{1}{2}\sin 2\theta\,d\theta=\frac{1}{96}\left[-\frac{1}{2}\cos 2\theta\right]_0^{\pi/2}=\frac{1}{96}$$

35. The region of integration E is the top half of the sphere $x^2+y^2+z^2=9$. So

$$\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_0^{\sqrt{9-x^2-y^2}}z\sqrt{x^2+y^2+z^2}\,dz\,dy\,dx=\iiint_E z\sqrt{x^2+y^2+z^2}\,dV$$

$$=\int_0^{2\pi}\int_0^{\pi/2}\int_0^3\left(\rho^2\cos\phi\right)\left(\rho^2\sin\phi\right)d\rho\,d\phi\,d\theta=\int_0^{2\pi}d\theta\int_0^{\pi/2}\cos\phi\sin\phi\,d\phi\int_0^3\rho^4\,d\rho$$

$$=[\theta]_0^{2\pi}\left[\frac{1}{2}\sin^2\phi\right]_0^{\pi/2}\left[\frac{1}{5}\rho^5\right]_0^3=(2\pi)\left(\frac{1}{2}\right)\left(\frac{243}{5}\right)=\frac{243}{5}\pi$$

36. The region of integration E is the region above the cone $z=\sqrt{x^2+y^2}$ and below the sphere

$x^2+y^2+z^2=18$ in the first octant. Because E is in the first octant we have $0\le\theta\le\frac{\pi}{2}$. The cone has equation $\phi=\frac{\pi}{4}$ (as in Example 4) and so $0\le\phi\le\frac{\pi}{4}$. Also $0\le\rho\le\sqrt{18}=3\sqrt{2}$. So the integral becomes

$$\int_0^{\pi/2}\int_0^{\pi/4}\int_0^{3\sqrt{2}}\rho^4\sin\phi\,d\rho\,d\phi\,d\theta=\int_0^{\pi/2}d\theta\int_0^{\pi/4}\sin\phi\,d\phi\int_0^{3\sqrt{2}}\rho^4\,d\rho$$
$$=[\theta]_0^{\pi/2}[-\cos\phi]_0^{\pi/4}\left[\frac{1}{5}\rho^5\right]_0^{3\sqrt{2}}$$
$$=\left(\frac{\pi}{2}\right)\left(1-\frac{\sqrt{2}}{2}\right)\left(\frac{972\sqrt{2}}{5}\right)=486\pi\left(\frac{\sqrt{2}-1}{5}\right)$$

37. If E is the solid enclosed by the surface $\rho=1+\frac{1}{5}\sin 6\theta\sin 5\phi$, it can be described in spherical coordinates as $E=\left\{(\rho,\theta,\phi)\,|\,0\le\rho\le 1+\frac{1}{5}\sin 6\theta\sin 5\phi,0\le\theta\le 2\pi,0\le\phi\le\pi\right\}$. Its volume is given by $V(E)=\iiint_E dV=\int_0^{\pi}\int_0^{2\pi}\int_0^{1+(\sin 6\theta\sin 5\phi)/5}\rho^2\sin\phi\,d\rho\,d\theta\,d\phi=\frac{136\pi}{99}$.

38. The given integral is equal to

$\lim_{R\to\infty}\int_0^{2\pi}\int_0^{\pi}\int_0^{R}\rho e^{-\rho^2}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta=\lim_{R\to\infty}\left(\int_0^{2\pi}d\theta\right)\left(\int_0^{\pi}\sin\phi\,d\phi\right)\left(\int_0^{R}\rho^3e^{-\rho^2}\,d\rho\right)$. Now use integration

by parts with $u=\rho^2$, $dv=\rho e^{-\rho^2}\,d\rho$ to get

$$\lim_{R\to\infty}2\pi(2)\left(\left.\rho^2\left(-\frac{1}{2}\right)e^{-\rho^2}\right]_0^R-\int_0^R 2\rho\left(-\frac{1}{2}\right)e^{-\rho^2}\,d\rho\right)=\lim_{R\to\infty}4\pi\left(-\frac{1}{2}R^2e^{-R^2}+\left[-\frac{1}{2}e^{-\rho^2}\right]_0^R\right)$$
$$=4\pi\lim_{R\to\infty}\left[-\frac{1}{2}R^2e^{-R^2}-\frac{1}{2}e^{-R^2}+\frac{1}{2}\right]$$
$$=4\pi\left(\frac{1}{2}\right)=2\pi$$

(Note that $R^2e^{-R^2}\to 0$ as $R\to\infty$ by l'Hospital's Rule.)

39. **(a)** From the diagram, $z=r\cot\phi_0$ to

$z=\sqrt{a^2-r^2}$, $r=0$ to $r=a\sin\phi_0$ (or use $a^2-r^2=r^2\cot^2\phi_0$). Thus

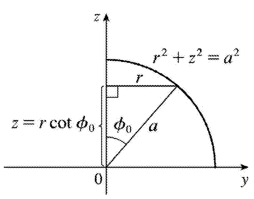
$$V=\int_0^{2\pi}\int_0^{a\sin\phi_0}\int_{r\cot\phi_0}^{\sqrt{a^2-r^2}} r\,dz\,dr\,d\theta$$

$$=2\pi\int_0^{a\sin\phi_0}\left(r\sqrt{a^2-r^2}-r^2\cot\phi_0\right)dr$$

$$=\frac{2\pi}{3}\left[-(a^2-r^2)^{3/2}-r^3\cot\phi_0\right]_0^{a\sin\phi_0}$$

$$=\frac{2\pi}{3}\left[-\left(a^2-a^2\sin^2\phi_0\right)^{3/2}-a^3\sin^3\phi_0\cot\phi_0+a^3\right]$$

$$=\frac{2}{3}\pi a^3\left[1-\left(\cos^3\phi_0+\sin^2\phi_0\cos\phi_0\right)\right]=\frac{2}{3}\pi a^3(1-\cos\phi_0)$$

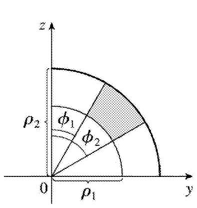


(b) The wedge in question is the shaded area rotated from $\theta=\theta_1$ to $\theta=\theta_2$. Letting

$V_{ij}=$ volume of the region bounded by the sphere of radius ρ_i

and the cone with angle ϕ_j ($\theta=\theta_1$ to θ_2)

and letting V be the volume of the wedge, we have



$$V=\left(V_{22}-V_{21}\right)-\left(V_{12}-V_{11}\right)$$

$$=\frac{1}{3}\left(\theta_2-\theta_1\right)\left[\rho_2^3\left(1-\cos\phi_2\right)-\rho_2^3\left(1-\cos\phi_1\right)-\rho_1^3\left(1-\cos\phi_2\right)+\rho_1^3\left(1-\cos\phi_1\right)\right]$$
$$=\frac{1}{3}\left(\theta_2-\theta_1\right)\left[\left(\rho_2^3-\rho_1^3\right)\left(1-\cos\phi_2\right)-\left(\rho_2^3-\rho_1^3\right)\left(1-\cos\phi_1\right)\right]$$
$$=\frac{1}{3}\left(\theta_2-\theta_1\right)\left[\left(\rho_2^3-\rho_1^3\right)\left(\cos\phi_1-\cos\phi_2\right)\right]$$

Or: Show that $V=\int_{\theta_1}^{\theta_2}\int_{\rho_1\sin\phi_1}^{\rho_2\sin\phi_2}\int_{r\cot\phi_2}^{r\cot\phi_1} r\,dz\,dr\,d\theta$.

(c) By the Mean Value Theorem with $f(\rho)=\rho^3$ there exists some $\tilde{\rho}$ with $\rho_1\le\tilde{\rho}\le\rho_2$ such that $f\left(\rho_2\right)-f\left(\rho_1\right)=f'\left(\tilde{\rho}\right)\left(\rho_2-\rho_1\right)$ or $\rho_1^3-\rho_2^3=3\tilde{\rho}^2\,\Delta\rho$. Similarly there exists $\tilde{\phi}$ with $\phi_1\le\tilde{\phi}\le\phi_2$ such that $\cos\phi_2-\cos\phi_1=\left(-\sin\tilde{\phi}\right)\Delta\phi$. Substituting into the result from (b) gives $\Delta V=(\tilde{\rho}^2\,\Delta\rho)(\theta_2-\theta_1)(\sin\tilde{\phi})\,\Delta\phi=\tilde{\rho}^2\,\sin\tilde{\phi}\,\Delta\rho\,\Delta\phi\,\Delta\theta$.

40. **(a)** The mountain comprises a solid conical region C . The work done in lifting a small volume of material ΔV with density $g(P)$ to a height $h(P)$ above sea level is $h(P)g(P)\Delta V$. Summing over the whole mountain we get $W=\iiint_C h(P)g(P)\,\Delta V$.

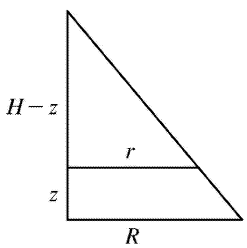
(b) Here C is a solid right circular cone with radius $R=62,000$ ft, height $H=12,400$ ft, and density $g(P)=200\text{ lb / ft}^3$ at all points P in C . We use cylindrical coordinates:

$$W=\int_0^{2\pi}\int_0^H\int_0^{R(1-z/H)} z\cdot 200r\,dr\,dz\,d\theta$$
$$=2\pi\int_0^H 200z\left[\frac{1}{2}r^2\right]_{r=0}^{r=R(1-z/H)}dz$$
$$=400\pi\int_0^H z\,\frac{R^2}{2}\left(1-\frac{z}{H}\right)^2 dz$$
$$=200\pi R^2\int_0^H\left(z-\frac{2z^2}{H}+\frac{z^3}{H^2}\right)dz$$

$$= 200\pi R^2 \left[\frac{z^2}{2} - \frac{2z^3}{3H} + \frac{z^4}{4H^2} \right]_0^H$$

$$= 200\pi R^2 \left(\frac{H^2}{2} - \frac{2H^2}{3} + \frac{H^2}{4} \right) = \frac{50}{3}\pi R^2 H^2$$

$$= \frac{50}{3}\pi (62{,}000)^2 (12{,}400)^2 \approx 3.1\times 10^{19} \text{ ft–lb}$$



$$\frac{r}{R} = \frac{H-z}{H} = 1 - \frac{z}{H}$$

1. $x=u+4v$, $y=3u-2v$.

The Jacobian is $\dfrac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}\partial x/\partial u & \partial x/\partial v\\ \partial y/\partial u & \partial y/\partial v\end{vmatrix}=\begin{vmatrix}1 & 4\\ 3 & -2\end{vmatrix}=1(-2)-4(3)=-14$.

2. $\dfrac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}\partial x/\partial u & \partial x/\partial v\\ \partial y/\partial u & \partial y/\partial v\end{vmatrix}=\begin{vmatrix}2u & -2v\\ 2u & 2v\end{vmatrix}=4uv-(-4uv)=8uv$

3.

$$\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}\dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v}\\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v}\end{vmatrix}=\begin{vmatrix}\dfrac{v}{(u+v)^2} & -\dfrac{u}{(u+v)^2}\\ -\dfrac{v}{(u-v)^2} & \dfrac{u}{(u-v)^2}\end{vmatrix}=\frac{uv}{(u+v)^2(u-v)^2}-\frac{uv}{(u+v)^2(u-v)^2}=0$$

4. $\dfrac{\partial(x,y)}{\partial(\alpha,\beta)}=\begin{vmatrix}\partial x/\partial\alpha & \partial x/\partial\beta\\ \partial y/\partial\alpha & \partial y/\partial\beta\end{vmatrix}=\begin{vmatrix}\sin\beta & \alpha\cos\beta\\ \cos\beta & -\alpha\sin\beta\end{vmatrix}=-\alpha\sin^2\beta-\alpha\cos^2\beta=-\alpha$

5.

$$\frac{\partial(x,y,z)}{\partial(u,v,w)}=\begin{vmatrix}\partial x/\partial u & \partial x/\partial v & \partial x/\partial w\\ \partial y/\partial u & \partial y/\partial v & \partial y/\partial w\\ \partial z/\partial u & \partial z/\partial v & \partial z/\partial w\end{vmatrix}=\begin{vmatrix}v & u & 0\\ 0 & w & v\\ w & 0 & u\end{vmatrix}$$

$$=v\begin{vmatrix}w & v\\ 0 & u\end{vmatrix}-u\begin{vmatrix}0 & v\\ w & u\end{vmatrix}+0\begin{vmatrix}0 & w\\ w & 0\end{vmatrix}=v(uw-0)-u(0-vw)=2uvw$$

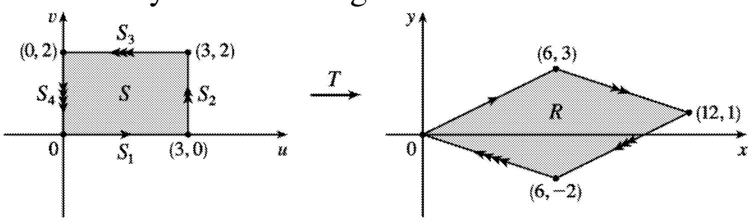
6.

$$\frac{\partial(x,y,z)}{\partial(u,v,w)}=\begin{vmatrix}e^{u-v} & -e^{u-v} & 0\\ e^{u+v} & e^{u+v} & 0\\ e^{u+v+w} & e^{u+v+w} & e^{u+v+w}\end{vmatrix}=e^{u+v+w}\begin{vmatrix}e^{u-v} & -e^{u-v}\\ e^{u+v} & e^{u+v}\end{vmatrix}$$

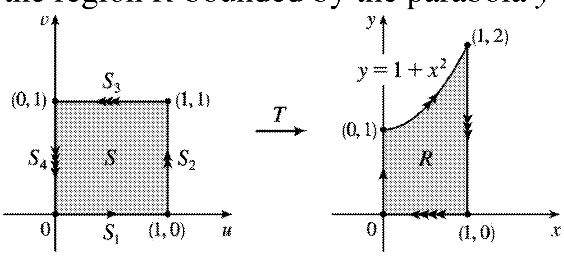
$$=e^{u+v+w}\left(e^{u-v}e^{u+v}+e^{u-v}e^{u+v}\right)=e^{u+v+w}\left(2e^{2u}\right)=2e^{3u+v+w}$$

7. The transformation maps the boundary of S to the boundary of the image R , so we first look at side S_1 in the uv –plane. S_1 is described by $v=0$ ($0\le u\le 3$), so $x=2u+3v=2u$ and $y=u-v=u$. Eliminating u , we have $x=2y$, $0\le x\le 6$. S_2 is the line segment $u=3$, $0\le v\le 2$, so $x=6+3v$ and $y=3-v$. Then $v=3-y\Rightarrow x=6+3(3-y)=15-3y$, $6\le x\le 12$. S_3 is the line segment $v=2$, $0\le u\le 3$, so $x=2u+6$ and $y=u-2$, giving $u=y+2\Rightarrow x=2y+10$, $6\le x\le 12$. Finally, S_4 is the segment $u=0$, $0\le v\le 2$, so $x=3v$ and $y=-v\Rightarrow x=-3y$, $0\le x\le 6$. The image of set S is the region R shown in the xy –plane, a parallelogram

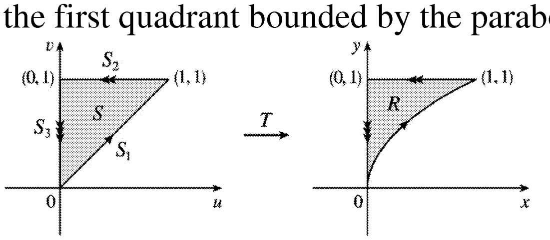
bounded by these four segments.



8. S_1 is the line segment $v=0$, $0\leq u\leq 1$, so $x=v=0$ and $y=u(1+v^2)=u$. Since $0\leq u\leq 1$, the image is the line segment $x=0$, $0\leq y\leq 1$. S_2 is the segment $u=1$, $0\leq v\leq 1$, so $x=v$ and $y=u(1+v^2)=1+x^2$. Thus the image is the portion of the parabola $y=1+x^2$ for $0\leq x\leq 1$. S_3 is the segment $v=1$, $0\leq u\leq 1$, so $x=1$ and $y=2u$. The image is the segment $x=1$, $0\leq y\leq 2$. S_4 is described by $u=0$, $0\leq v\leq 1$, so $0\leq x=v\leq 1$ and $y=u(1+v^2)=0$. The image is the line segment $y=0$, $0\leq x\leq 1$. Thus, the image of S is the region R bounded by the parabola $y=1+x^2$, the x –axis, and the lines $x=0$, $x=1$.

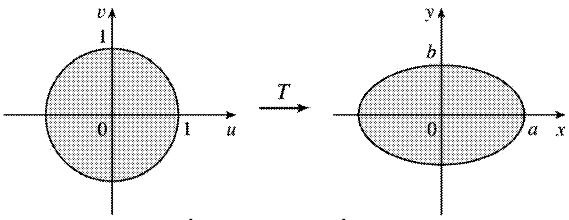


9. S_1 is the line segment $u=v$, $0\leq u\leq 1$, so $y=v=u$ and $x=u^2=y^2$. Since $0\leq u\leq 1$, the image is the portion of the parabola $x=y^2$, $0\leq y\leq 1$. S_2 is the segment $v=1$, $0\leq u\leq 1$, thus $y=v=1$ and $x=u^2$, so $0\leq x\leq 1$. The image is the line segment $y=1$, $0\leq x\leq 1$. S_3 is the segment $u=0$, $0\leq v\leq 1$, so $x=u^2=0$ and $y=v\Rightarrow 0\leq y\leq 1$. The image is the segment $x=0$, $0\leq y\leq 1$. Thus, the image of S is the region R in the first quadrant bounded by the parabola $x=y^2$, the y –axis, and the line $y=1$.



10. Substituting $u=\frac{x}{a}$, $v=\frac{y}{b}$ into $u^2+v^2\leq 1$ gives $\frac{x^2}{a^2}+\frac{y^2}{b^2}\leq 1$, so the image of $u^2+v^2\leq 1$ is the elliptical region

$\frac{x^2}{a^2}+\frac{y^2}{b^2}\leq 1$.



11. $\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix}=3$ and $x-3y=(2u+v)-3(u+2v)=-u-5v$. To find the region S in the uv –plane that corresponds to R we first find the corresponding boundary under the given transformation. The line through $(0,0)$ and $(2,1)$ is $y=\frac{1}{2}x$ which is the image of $u+2v=\frac{1}{2}(2u+v)\Rightarrow v=0$; the line through $(2,1)$ and $(1,2)$ is $x+y=3$ which is the image of $(2u+v)+(u+2v)=3\Rightarrow u+v=1$; the line through $(0,0)$ and $(1,2)$ is $y=2x$ which is the image of $u+2v=2(2u+v)\Rightarrow u=0$. Thus S is the triangle $0\leq v\leq 1-u$, $0\leq u\leq 1$ in the uv –plane and

$$\iint_R (x-3y)\,dA=\int_0^1\int_0^{1-u}(-u-5v)|3|\,dv\,du$$

$$=-3\int_0^1\left[uv+\frac{5}{2}v^2\right]_{v=0}^{v=1-u}du=-3\int_0^1\left(u-u^2+\frac{5}{2}(1-u)^2\right)du$$

$$=-3\left[\frac{1}{2}u^2-\frac{1}{3}u^3-\frac{5}{6}(1-u)^3\right]_0^1=-3\left(\frac{1}{2}-\frac{1}{3}+\frac{5}{6}\right)=-3$$

12. $\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix} 1/4 & 1/4 \\ -3/4 & 1/4 \end{vmatrix}=\frac{1}{4}$, $4x+8y=4\cdot\frac{1}{4}(u+v)+8\cdot\frac{1}{4}(v-3u)=3v-5u$. R is a parallelogram bounded by the lines $x-y=-4$, $x-y=4$, $3x+y=0$, $3x+y=8$. Since $u=x-y$ and $v=3x+y$, R is the image of the rectangle enclosed by the lines $u=-4$, $u=4$, $v=0$, and $v=8$. Thus

$$\iint_R (4x+8y)\,dA=\int_{-4}^4\int_0^8(3v-5u)\left|\frac{1}{4}\right|dv\,du=\frac{1}{4}\int_{-4}^4\left[\frac{3}{2}v^2-5uv\right]_{v=0}^{v=8}du$$

$$=\frac{1}{4}\int_{-4}^4(96-40u)\,du=\frac{1}{4}\left[96u-20u^2\right]_{-4}^4=192$$

13. $\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix}=6$, $x^2=4u^2$ and the planar ellipse $9x^2+4y^2\leq 36$ is the image of the disk $u^2+v^2\leq 1$. Thus

$$\iint_R x^2\,dA = \iint_{u^2+v^2\le 1} (4u^2)(6)\,du\,dv = \int_0^{2\pi}\int_0^1 (24r^2\cos^2\theta)\,r\,dr\,d\theta$$
$$= 24\int_0^{2\pi}\cos^2\theta\,d\theta\int_0^1 r^3\,dr = 24\left[\frac{1}{2}x+\frac{1}{4}\sin 2x\right]_0^{2\pi}\left[\frac{1}{4}r^4\right]_0^1$$
$$= 24(\pi)\left(\frac{1}{4}\right) = 6\pi$$

14. $\dfrac{\partial(\mathrm{x,y})}{\partial(\mathrm{u,v})} = \begin{vmatrix} \sqrt{2} & -\sqrt{2/3} \\ \sqrt{2} & \sqrt{2/3} \end{vmatrix} = \dfrac{4}{\sqrt{3}}$, $x^2-xy+y^2=2u^2+2v^2$ and the planar ellipse $x^2-xy+y^2\le 2$ is the image of the disk $u^2+v^2\le 1$. Thus

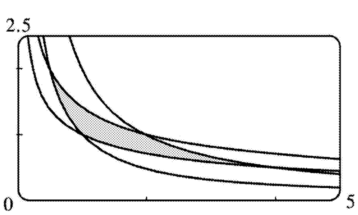
$$\iint_R (x^2-xy+y^2)\,dA = \iint_{u^2+v^2\le 1} (2u^2+2v^2)\left(\frac{4}{\sqrt{3}}\,du\,dv\right) = \int_0^{2\pi}\int_0^1 \frac{8}{\sqrt{3}}\,r^3\,dr\,d\theta = \frac{4\pi}{\sqrt{3}}$$

15. $\dfrac{\partial(\mathrm{x,y})}{\partial(\mathrm{u,v})} = \begin{vmatrix} 1/\mathrm{v} & -\mathrm{u}/\mathrm{v}^2 \\ 0 & 1 \end{vmatrix} = \dfrac{1}{v}$, $xy=u$, $y=x$ is the image of the parabola $v^2=u$, $y=3x$ is the image of the parabola $v^2=3u$, and the hyperbolas $xy=1$, $xy=3$ are the images of the lines $u=1$ and $u=3$ respectively. Thus

$$\iint_R xy\,dA = \int_1^3\int_{\sqrt{u}}^{\sqrt{3u}} u\left(\frac{1}{v}\right)dv\,du = \int_1^3 u\left(\ln\sqrt{3u}-\ln\sqrt{u}\right)du$$
$$= \int_1^3 u\ln\sqrt{3}\,du = 4\ln\sqrt{3} = 2\ln 3$$

16. Here $y=\dfrac{v}{u}$, $x=\dfrac{u^2}{v}$ so $\dfrac{\partial(\mathrm{x,y})}{\partial(\mathrm{u,v})} = \begin{vmatrix} 2\mathrm{u}/\mathrm{v} & -\mathrm{u}^2/\mathrm{v}^2 \\ -\mathrm{v}/\mathrm{u}^2 & 1/\mathrm{u} \end{vmatrix} = \dfrac{1}{v}$ and R is the image of the square with vertices $(1,1)$, $(2,1)$, $(2,2)$, and $(1,2)$. So

$$\iint_R y^2\,dA = \int_1^2\int_1^2 \frac{v^2}{u^2}\left(\frac{1}{v}\right)du\,dv = \int_1^2 \frac{v}{2}\,dv = \frac{3}{4}$$



17. **(a)** $\dfrac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc$ and since $u=\dfrac{x}{a}$, $v=\dfrac{y}{b}$, $w=\dfrac{z}{c}$ the solid enclosed by the ellipsoid is the image of the ball $u^2+v^2+w^2 \le 1$. So

$$\iiint_E dV = \iiint_{u^2+v^2+w^2 \le 1} abc\, du\, dv\, dw = (abc)(\text{ volume of the ball }) = \frac{4}{3}\pi abc$$

(b) If we approximate the surface of Earth by the ellipsoid $\dfrac{x^2}{6378^2}+\dfrac{y^2}{6378^2}+\dfrac{z^2}{6356^2}=1$, then we can estimate the volume of Earth by finding the volume of the solid E enclosed by the ellipsoid. From part (a), this is $\iiint_E dV = \frac{4}{3}\pi(6378)(6378)(6356) \approx 1.083 \times 10^{12}\text{ km}^3$.

18. $\dfrac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc$ and the solid enclosed by the ellipsoid is the image of the ball $u^2+v^2+w^2 \le 1$. Now $x^2 y = (a^2u^2)(bv)$, so

$$\begin{aligned}
\iiint_E x^2 y\, dV &= \iiint_{u^2+v^2+w^2 \le 1} (a^2bu^2v)(abc)\, du\, dv\, dw \\
&= \int_0^{2\pi}\int_0^{\pi}\int_0^1 (a^3b^2c)(\rho^2 \sin^2\phi \cos^2\theta)(\rho \sin\phi \sin\theta)\rho^2 \sin\phi\, d\rho\, d\phi\, d\theta \\
&= a^3b^2c\int_0^{2\pi}\int_0^{\pi}\int_0^1 (\rho^5 \sin^4\phi \cos^2\theta \sin\theta)\, d\rho\, d\phi\, d\theta \\
&= a^3b^2c\int_0^{2\pi} \cos^2\theta \sin\theta\, d\theta \int_0^{\pi} \sin^4\phi\, d\phi \int_0^1 \rho^5\, d\rho \\
&= 0 \text{ since } \int_0^{2\pi} \cos^2\theta \sin\theta\, d\theta = 0
\end{aligned}$$

19. Letting $u=x-2y$ and $v=3x-y$, we have $x=\frac{1}{5}(2v-u)$ and $y=\frac{1}{5}(v-3u)$. Then $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} -1/5 & 2/5 \\ -3/5 & 1/5 \end{vmatrix} = \dfrac{1}{5}$ and R is the image of the rectangle enclosed by the lines $u=0$, $u=4$, $v=1$, and $v=8$. Thus

$$\iint_R \frac{x-2y}{3x-y}\,dA=\int_0^4\int_1^8 \frac{u}{v}\left|\frac{1}{5}\right| dv\,du=\frac{1}{5}\int_0^4 u\,du\int_1^8 \frac{1}{v}\,dv=\frac{1}{5}\left[\frac{1}{2}u^2\right]_0^4[\ln|v|]_1^8=\frac{8}{5}\ln 8 .$$

20. Letting $u=x+y$ and $v=x-y$, we have $x=\frac{1}{2}(u+v)$ and $y=\frac{1}{2}(u-v)$. Then $\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{vmatrix}=-\frac{1}{2}$ and R is the image of the rectangle enclosed by the lines $u=0$, $u=3$, $v=0$, and $v=2$. Thus

$$\iint_R (x+y)e^{x^2-y^2}\,dA=\int_0^3\int_0^2 ue^{uv}\left|-\frac{1}{2}\right| dv\,du=\frac{1}{2}\int_0^3\left[e^{uv}\right]_{v=0}^{v=2}du=\frac{1}{2}\int_0^3(e^{2u}-1)\,du$$

$$=\frac{1}{2}\left[\frac{1}{2}e^{2u}-u\right]_0^3=\frac{1}{2}\left(\frac{1}{2}e^6-3-\frac{1}{2}\right)=\frac{1}{4}(e^6-7)$$

21. Letting $u=y-x$, $v=y+x$, we have $y=\frac{1}{2}(u+v)$, $x=\frac{1}{2}(v-u)$. Then $\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{vmatrix}=-\frac{1}{2}$ and R is the image of the trapezoidal region with vertices $(-1,1)$, $(-2,2)$, $(2,2)$, and $(1,1)$. Thus

$$\iint_R \cos\frac{y-x}{y+x}\,dA=\int_1^2\int_{-v}^{v}\cos\frac{u}{v}\left|-\frac{1}{2}\right| du\,dv=\frac{1}{2}\int_1^2\left[v\sin\frac{u}{v}\right]_{u=-v}^{u=v}dv$$

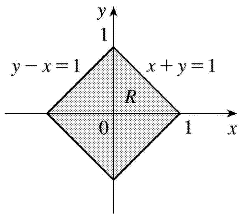
$$=\frac{1}{2}\int_1^2 2v\sin(1)\,dv=\frac{3}{2}\sin 1$$

22. Letting $u=3x$, $v=2y$, we have $9x^2+4y^2=u^2+v^2$, $x=\frac{1}{3}u$, and $y=\frac{1}{2}v$. Then $\frac{\partial(x,y)}{\partial(u,v)}=\frac{1}{6}$ and R is the image of the quarter–disk D given by $u^2+v^2\le 1$, $u\ge 0$, $v\ge 0$. Thus

$$\iint_R \sin(9x^2+4y^2)\,dA=\iint_D \frac{1}{6}\sin(u^2+v^2)\,du\,dv=\int_0^{\pi/2}\int_0^1\frac{1}{6}\sin(r^2)\,r\,dr\,d\theta$$

$$=\frac{\pi}{12}\left[-\frac{1}{2}\cos r^2\right]_0^1=\frac{\pi}{24}(1-\cos 1)$$

23. Let $u=x+y$ and $v=-x+y$. Then $u+v=2y\Rightarrow y=\frac{1}{2}(u+v)$ and $u-v=2x\Rightarrow x=\frac{1}{2}(u-v)$. $\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{vmatrix}=\frac{1}{2}$. Now $|u|=|x+y|\le|x|+|y|\le 1\Rightarrow -1\le u\le 1$, and $|v|=|-x+y|\le|x|+|y|\le 1\Rightarrow -1\le v\le 1$. R is the image of



the square region with vertices $(1,1)$, $(1,-1)$, $(-1,-1)$, and $(-1,1)$. So

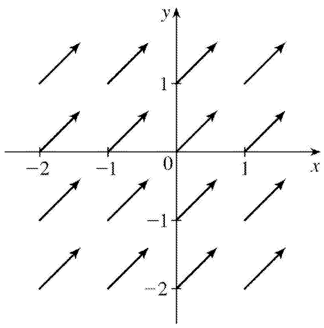
$$\iint_R e^{x+y}\,dA=\frac{1}{2}\int_{-1}^{1}\int_{-1}^{1} e^{u}\,du\,dv=\frac{1}{2}\left[e^{u}\right]_{-1}^{1}\left[v\right]_{-1}^{1}=e-e^{-1}.$$

24. Let $u=x+y$ and $v=y$, then $x=u-v$, $y=v$, $\dfrac{\partial(x,y)}{\partial(u,v)}=1$ and R is the image under T of the triangular region with vertices $(0,0)$, $(1,0)$ and $(1,1)$. Thus

$$\iint_R f(x+y)\,dA=\int_0^1\int_0^u (1)\,f(u)\,dv\,du=\int_0^1 f(u)\left[v\right]_{v=0}^{v=u}du=\int_0^1 uf(u)\,du$$ as desired.

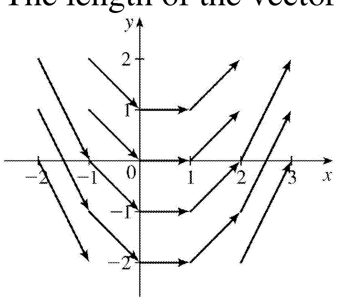
1. $\boldsymbol{F}(x,y)=\frac{1}{2}(\boldsymbol{i}+\boldsymbol{j})$

All vectors in this field are identical, with length $\frac{1}{\sqrt{2}}$ and direction parallel to the line $y=x$.



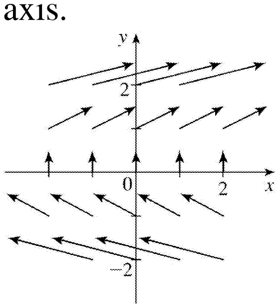
2. $\boldsymbol{F}(x, y)=\boldsymbol{i}+x\,\boldsymbol{j}$

The length of the vector $\boldsymbol{i}+x\,\boldsymbol{j}$ is $\sqrt{1+x^2}$. Vectors are tangent to parabolas opening about the y –axis.



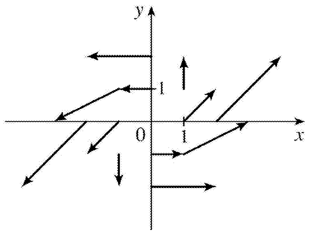
3. $\boldsymbol{F}(x, y)=y\boldsymbol{i}+\frac{1}{2}\,\boldsymbol{j}$.

The length of the vector $y\boldsymbol{i}+\frac{1}{2}\,\boldsymbol{j}$ is $\sqrt{y^2+\frac{1}{4}}$. Vectors are tangent to parabolas opening about the x –axis.



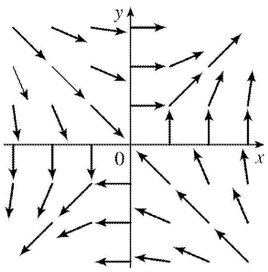
4. $\boldsymbol{F}(x, y)=(x-y)\boldsymbol{i}+x\,\boldsymbol{j}$

The length of the vector $(x-y)\boldsymbol{i}+x\,\boldsymbol{j}$ is $\sqrt{(x-y)^2+x^2}$. Vectors along the line $y=x$ are vertical.



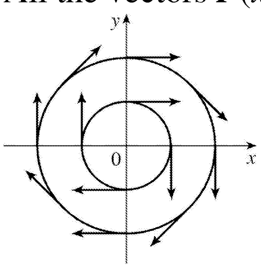
5. $\boldsymbol{F}(x, y)=\dfrac{y\boldsymbol{i}+x\boldsymbol{j}}{\sqrt{x^2+y^2}}$

The length of the vector $\dfrac{y\boldsymbol{i}+x\boldsymbol{j}}{\sqrt{x^2+y^2}}$ is 1.

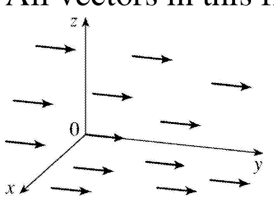


6. $\boldsymbol{F}(x, y)=\dfrac{y\boldsymbol{i}-x\boldsymbol{j}}{\sqrt{x^2+y^2}}$

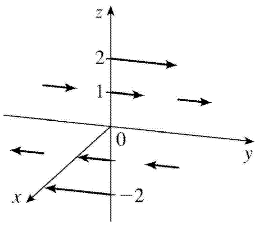
All the vectors $\boldsymbol{F}(x, y)$ are unit vectors tangent to circles centered at the origin with radius $\sqrt{x^2+y^2}$.



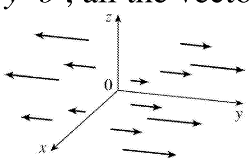
7. $\boldsymbol{F}(x, y, z)=\boldsymbol{j}$
All vectors in this field are parallel to the y –axis and have length 1 .



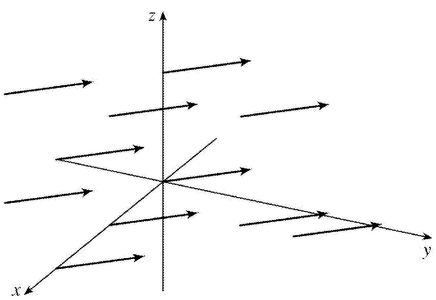
8. $\boldsymbol{F}(x, y, z)=z\boldsymbol{j}$
At each point (x, y, z) , $\boldsymbol{F}(x, y, z)$ is a vector of length $|z|$. For $z>0$, all point in the direction of the positive y –axis while for $z<0$, all are in the direction of the negative y –axis.



9. $\boldsymbol{F}(x, y, z)=y\,\boldsymbol{j}$
The length of $\boldsymbol{F}(x, y, z)$ is $|y|$. No vectors emanate from the xz –plane since $y=0$ there. In each plane $y=b$, all the vectors are identical.



10. $\boldsymbol{F}(x, y, z)=\boldsymbol{j}-\boldsymbol{i}$
All vectors in this field have length $\sqrt{2}$ and point in the same direction, parallel to the xy –plane.



11. $\boldsymbol{F}(x, y)=\langle y, x\rangle$ corresponds to graph II. In the first quadrant all the vectors have positive x – and y –components, in the second quadrant all vectors have positive x –components and negative y –components, in the third quadrant all vectors have negative x – and y –components, and in the fourth quadrant all vectors have negative x –components and positive y –components. In addition, the vectors get shorter as we approach the origin.

12. $\boldsymbol{F}(x, y)=\langle 1, \sin y\rangle$ corresponds to graph IV since the x –component of each vector is constant, the vectors are independent of x (vectors along horizontal lines are identical), and the vector field appears to repeat the same pattern vertically.

13. $\boldsymbol{F}(x, y)=\langle x-2, x+1\rangle$ corresponds to graph I since the vectors are independent of y (vectors along vertical lines are identical) and, as we move to the right, both the x – and the y –components get larger.

14. $\boldsymbol{F}(x, y)=\langle y, 1/x\rangle$ corresponds to graph III. As in Exercise 11, all the vectors in the first quadrant have positive x – and y –components, in the second quadrant all vectors have positive x –components and negative y –components, in the third quadrant all vectors have negative x – and y –components,

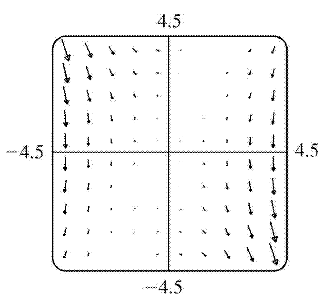
and in the fourth quadrant all vectors have negative x –components and positive y –components. Also, the vectors become longer as we approach the y –axis.

15. $\boldsymbol{F}(x, y, z)=\boldsymbol{i}+2\boldsymbol{j}+3\boldsymbol{k}$ corresponds to graph IV, since all vectors have identical length and direction.

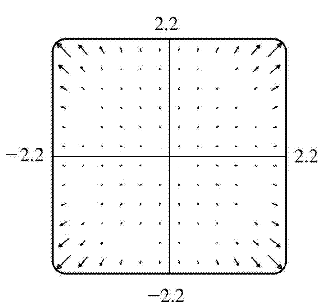
16. $\boldsymbol{F}(x, y, z)=\boldsymbol{i}+2\boldsymbol{j}+z\boldsymbol{k}$ corresponds to graph I, since the horizontal vector components remain constant, but the vectors above the xy –plane point generally upward while the vectors below the xy –plane point generally downward.

17. $\boldsymbol{F}(x, y, z)=x\boldsymbol{i}+y\boldsymbol{j}+3\boldsymbol{k}$ corresponds to graph III; the projection of each vector onto the xy –plane is $x\boldsymbol{i}+y\boldsymbol{j}$, which points away from the origin, and the vectors point generally upward because their z –components are all 3 .

18. $\boldsymbol{F}(x, y, z)=x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k}$ corresponds to graph II; each vector $\boldsymbol{F}(x, y, z)$ has the same length and direction as the position vector of the point (x, y, z) , and therefore the vectors all point directly away from the origin.

19. 

The vector field seems to have very short vectors near the line $y=2x$. For $\boldsymbol{F}(x, y)=\langle 0, 0\rangle$ we must have $y^2-2xy=0$ and $3xy-6x^2=0$. The first equation holds if $y=0$ or $y=2x$, and the second holds if $x=0$ or $y=2x$. So both equations hold [and thus $\boldsymbol{F}(x, y)=\boldsymbol{0}$] along the line $y=2x$.

20. 

From the graph, it appears that all of the vectors in the field lie on lines through the origin, and that the vectors have very small magnitudes near the circle $|\boldsymbol{x}|=2$ and near the origin. Note that $\boldsymbol{F}(\boldsymbol{x})=\boldsymbol{0}\Leftrightarrow r(r-2)=0\Leftrightarrow r=0$ or 2 , so as we suspected, $\boldsymbol{F}(\boldsymbol{x})=\boldsymbol{0}$ for $|x|=2$ and for $|x|=0$. Note that where $r^2-r<0$, the vectors point towards the origin, and where $r^2-r>0$, they point away from the origin.

21. $\nabla f(x, y)=f_x(x, y)\boldsymbol{i}+f_y(x, y)\boldsymbol{j}=\dfrac{1}{x+2y}\,\boldsymbol{i}+\dfrac{2}{x+2y}\,\boldsymbol{j}$

22. $\nabla f(x, y)=f_x(x, y)\boldsymbol{i}+f_y(x, y)\boldsymbol{j}=\left[x^{\alpha}(-\beta e^{-\beta x})+\alpha x^{\alpha-1}e^{-\beta x}\right]\boldsymbol{i}+0\,\boldsymbol{j}=(\alpha-\beta x)x^{\alpha-1}e^{-\beta x}\boldsymbol{i}$

23.

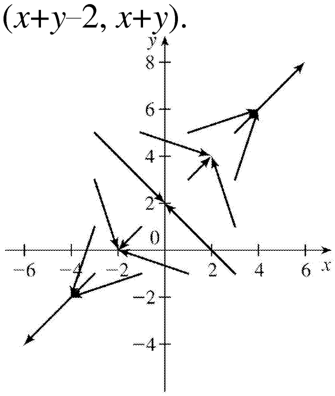
$$\begin{aligned}\nabla f(x, y, z) &= f_x(x, y, z)\boldsymbol{i}+f_y(x, y, z)\boldsymbol{j}+f_z(x, y, z)\boldsymbol{k}\\ &= \frac{x}{\sqrt{x^2+y^2+z^2}}\,\boldsymbol{i}+\frac{y}{\sqrt{x^2+y^2+z^2}}\,\boldsymbol{j}+\frac{z}{\sqrt{x^2+y^2+z^2}}\,\boldsymbol{k}\end{aligned}$$

24.

$$\begin{aligned}\nabla f(x, y, z) &= f_x(x, y, z)\boldsymbol{i}+f_y(x, y, z)\boldsymbol{j}+f_z(x, y, z)\boldsymbol{k}\\ &= \left(\cos\frac{y}{z}\right)\boldsymbol{i}-x\left(\sin\frac{y}{z}\right)\left(\frac{1}{z}\right)\boldsymbol{j}-x\left(\sin\frac{y}{z}\right)\left(-\frac{y}{z^2}\right)\boldsymbol{k}\\ &= \left(\cos\frac{y}{z}\right)\boldsymbol{i}-\frac{x}{z}\left(\sin\frac{y}{z}\right)\boldsymbol{j}+\frac{xy}{z^2}\left(\sin\frac{y}{z}\right)\boldsymbol{k}\end{aligned}$$

25. $f(x, y)=xy-2x \Rightarrow \nabla f(x, y)=(y-2)\boldsymbol{i}+x\boldsymbol{j}$.

The length of $\nabla f(x, y)$ is $\sqrt{(y-2)^2+x^2}$ and $\nabla f(x, y)$ terminates on the line $y=x+2$ at the point $(x+y-2, x+y)$.

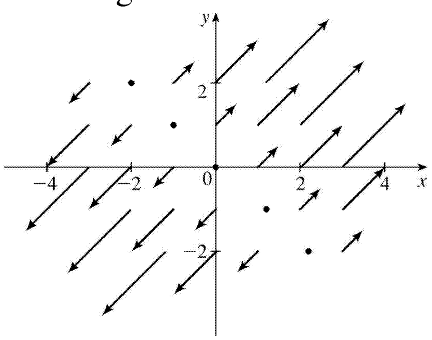


26. $f(x, y)=\dfrac{1}{4}(x+y)^2 \Rightarrow$

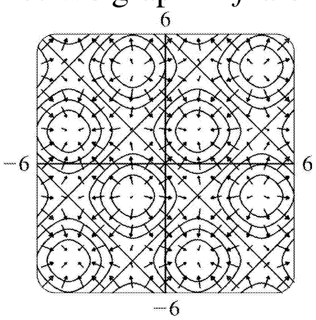
$\nabla f(x, y)=\dfrac{1}{2}(x+y)\boldsymbol{i}+\dfrac{1}{2}(x+y)\boldsymbol{j}$.

The length of $\nabla f(x, y)$ is

$\sqrt{\frac{1}{2}(x+y)^2} = \frac{1}{\sqrt{2}}|x+y|$. The vectors are perpendicular to the line $y=-x$ and point away from the line, with length that increases as the distance from the line $y=-x$ increases.

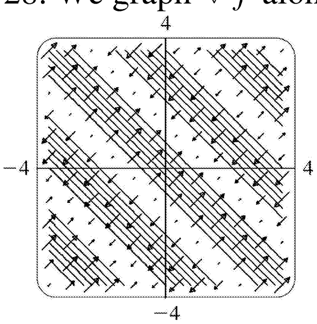


27. We graph ∇f along with a contour map of f .



The graph shows that the gradient vectors are perpendicular to the level curves. Also, the gradient vectors point in the direction in which f is increasing and are longer where the level curves are closer together.

28. We graph ∇f along with a contour map of f .



The graph shows that the gradient vectors are perpendicular to the level curves. Also, the gradient vectors point in the direction in which f is increasing and are longer where the level curves are closer together.

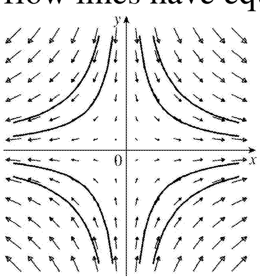
29. $f(x, y)=xy \Rightarrow \nabla f(x, y)=y\mathbf{i}+x\mathbf{j}$. In the first quadrant, both components of each vector are positive, while in the third quadrant both components are negative. However, in the second quadrant each vector's x –component is positive while its y –component is negative (and vice versa in the fourth quadrant). Thus, ∇f is graph IV.

30. $f(x, y)=x^2-y^2 \Rightarrow \nabla f(x, y)=2x\boldsymbol{i}-2y\boldsymbol{j}$. In the first quadrant, the x –component of each vector is positive while the y –component is negative. The other three quadrants are similar, where the x –component of each vector has the same sign as the x –value of its initial point, and the y –component has sign opposite that of the y –value of the initial point. Thus, ∇f is graph III.

31. $f(x, y)=x^2+y^2 \Rightarrow \nabla f(x, y)=2x\boldsymbol{i}+2y\boldsymbol{j}$. Thus, each vector $\nabla f(x, y)$ has the same direction and twice the length of the position vector of the point (x, y) , so the vectors all point directly away from the origin and their lengths increase as we move away from the origin. Hence, ∇f is graph II.

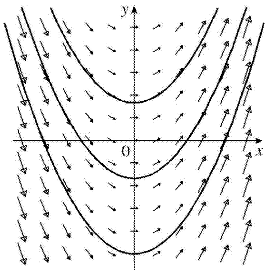
32. $f(x, y)=\sqrt{x^2+y^2} \Rightarrow \nabla f(x, y)=\frac{x}{\sqrt{x^2+y^2}}\boldsymbol{i}+\frac{y}{\sqrt{x^2+y^2}}\boldsymbol{j}$. Then $|\nabla f(x, y)|=\frac{1}{\sqrt{x^2+y^2}}\sqrt{x^2+y^2}=1$, so all vectors are unit vectors. In addition, each vector $\nabla f(x, y)$ has the same direction as the position vector of the point (x,y) , so the vectors all point directly away from the origin. Hence, ∇f is graph I.

33. **(a)** We sketch the vector field $\boldsymbol{F}(x, y)=x\boldsymbol{i}-y\boldsymbol{j}$ along with several approximate flow lines.The flow lines appear to be hyperbolas with shape similar to the graph of $y=\pm 1/x$, so we might guess that the flow lines have equations $y=C/x$.



(b) If $x=x(t)$ and $y=y(t)$ are parametric equations of a flow line, then the velocity vector of the flow line at the point (x, y) is $x'(t)\boldsymbol{i}+y'(t)\boldsymbol{j}$. Since the velocity vectors coincide with the vectors in the vector field, we have $x'(t)\boldsymbol{i}+y'(t)\boldsymbol{j}=x\boldsymbol{i}-y\boldsymbol{j} \Rightarrow dx/dt=x$, $dy/dt=-y$. To solve these differential equations, we know $dx/dt=x \Rightarrow dx/x=dt \Rightarrow \ln|x|=t+C \Rightarrow x=\pm e^{t+C}=Ae^t$ for some constant A , and $dy/dt=-y \Rightarrow dy/y=-dt \Rightarrow \ln|y|=-t+K \Rightarrow y=\pm e^{-t+K}=Be^{-t}$ for some constant B . Therefore $xy=Ae^tBe^{-t}=AB=$ constant. If the flow line passes through (1, 1) then (1)(1)= constant $=1 \Rightarrow xy=1 \Rightarrow y=1/x$, $x>0$.

34. **(a)** We sketch the vector field $\boldsymbol{F}(x, y)=\boldsymbol{i}+x\boldsymbol{j}$ along with several approximate flow lines.The flow lines appear to be parabolas.



(b) If $x=x(t)$ and $y=y(t)$ are parametric equations of a flow line, then the velocity vector of the flow line at the point (x, y) is $x'(t)\mathbf{i}+y'(t)\mathbf{j}$. Since the velocity vectors coincide with the vectors in the vector field, we have $x'(t)\mathbf{i}+y'(t)\mathbf{j}=\mathbf{i}+x\,\mathbf{j}\Rightarrow \frac{dx}{dt}=1$, $\frac{dy}{dt}=x$. Thus $\frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{x}{1}=x$.

(c) From part (b), $dy/dx=x$. Integrating, we have $y=\frac{1}{2}x^2+c$. Since the particle starts at the origin, we know (0, 0) is on the curve, so $0=0+c\Rightarrow c=0$ and the path the particle follows is $y=\frac{1}{2}x^2$.

1. $x=t^2$ and $y=t$, $0\le t\le 2$, so by Formula 3

$$\int_C y\,ds = \int_0^2 t\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt=\int_0^2 t\sqrt{(2t)^2+(1)^2}\,dt$$
$$= \int_0^2 t\sqrt{4t^2+1}\,dt= \left.\frac{1}{12}(4t^2+1)^{3/2}\right]_0^2=\frac{1}{12}(17\sqrt{17}-1)$$

2.

$$\int_C \frac{y}{x}\,ds =\int_{1/2}^1 \frac{t^3}{t^4}\sqrt{(4t^3)^2+(3t^2)^2}\,dt=\int_{1/2}^1 \frac{1}{t}\sqrt{16t^6+9t^4}\,dt=\int_{1/2}^1 t\sqrt{16t^2+9}\,dt$$
$$= \left.\frac{1}{48}(16t^2+9)^{3/2}\right]_{1/2}^1=\frac{1}{48}(25^{3/2}-13^{3/2})=\frac{1}{48}(125-13\sqrt{13})$$

3. Parametric equations for C are $x=4\cos t$, $y=4\sin t$, $-\frac{\pi}{2}\le t\le\frac{\pi}{2}$. Then

$$\int_C xy^4\,ds = \int_{-\pi/2}^{\pi/2}(4\cos t)(4\sin t)^4\sqrt{(-4\sin t)^2+(4\cos t)^2}\,dt$$
$$= \int_{-\pi/2}^{\pi/2}4^5\cos t\sin^4 t\sqrt{16(\sin^2 t+\cos^2 t)}\,dt$$
$$= 4^5\int_{-\pi/2}^{\pi/2}(\sin^4 t\cos t)(4)\,dt=(4)^6\left[\frac{1}{5}\sin^5 t\right]_{-\pi/2}^{\pi/2}=\frac{2\cdot 4^6}{5}=1638.4$$

4. Parametric equations for C are $x=1+3t$, $y=2+5t$, $0\le t\le 1$. Then

$$\int_C ye^x\,ds=\int_0^1(2+5t)e^{1+3t}\sqrt{3^2+5^2}\,dt=\sqrt{34}\int_0^1(2+5t)e^{1+3t}\,dt$$

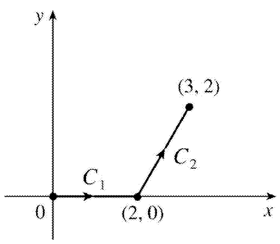
Integrating by parts with $u=2+5t\Rightarrow du=5\,dt$, $dv=e^{1+3t}\Rightarrow v=\frac{1}{3}e^{1+3t}dt$ gives

$$\int_C ye^x\,ds = \sqrt{34}\left[\frac{1}{3}(2+5t)e^{1+3t}-\frac{5}{9}e^{1+3t}\right]_0^1$$
$$= \sqrt{34}\left[\left(\frac{7}{3}-\frac{5}{9}\right)e^4-\left(\frac{2}{3}-\frac{5}{9}\right)e\right]=\frac{\sqrt{34}}{9}(16e^4-e)$$

5. If we choose x as the parameter, parametric equations for C are $x=x$, $y=x^2$ for $1\le x\le 3$ and

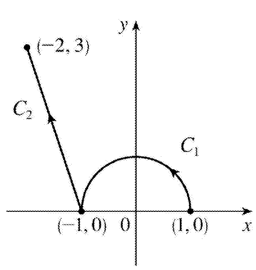
$$\int_C (xy+\ln x)\,dy = \int_1^3 (x\cdot x^2+\ln x)2x\,dx=\int_1^3 2(x^4+x\ln x)dx$$
$$= 2\left[\frac{1}{5}x^5+\frac{1}{2}x^2\ln x-\frac{1}{4}x^2\right]_1^3 \text{ (by integrating by parts in the second term)}$$
$$= 2\left(\frac{243}{5}+\frac{9}{2}\ln 3-\frac{9}{4}-\frac{1}{5}+\frac{1}{4}\right)=\frac{464}{5}+9\ln 3$$

6. Choosing y as the parameter, we have $x=e^y$, $y=y$, $0\le y\le 1$. Then
$\int_c xe^y\,dx=\int_0^1 e^y(e^y)e^y\,dy=\int_0^1 e^{3y}\,dy= \left.\frac{1}{3}e^{3y}\right]_0^1=\frac{1}{3}(e^3-1)$.



7.
$C=C_1+C_2$
On C_1 : $x=x$, $y=0\Rightarrow dy=0\,dx$, $0\le x\le 2$.
On C_2 : $x=x$, $y=2x-4\Rightarrow dy=2\,dx$, $2\le x\le 3$.
Then

$$\int_C xy\,dx+(x-y)\,dy =\int_{C_1} xy\,dx+(x-y)\,dy+\int_{C_2} xy\,dx+(x-y)\,dy$$
$$=\int_0^2 (0+0)\,dx+\int_2^3 dx$$
$$=\int_2^3 (2x^2-6x+8)\,dx=\frac{17}{3}$$



8.
$C=C_1+C_2$
On C_1 : $x=\cos t\Rightarrow dx=-\sin t\,dt$, $y=\sin t\Rightarrow$

$dy = \cos t\, dt$, $0 \le t \le \pi$.
On C_2 : $x = -1 - t \Rightarrow dx = -dt$, $y = 3t \Rightarrow$
$dy = 3\,dt$, $0 \le t \le 1$.

Then

$$\int_C \sin x\,dx + \cos y\,dy = \int_{C_1} \sin x\,dx + \cos y\,dy + \int_{C_2} \sin x\,dx + \cos y\,dy$$
$$= \int_0^{\pi} \sin(\cos t)(-\sin t\,dt) + \cos(\sin t)\cos t\,dt$$
$$+ \int_0^1 \sin(-1-t)(-dt) + \cos(3t)(3\,dt)$$
$$= \left[-\cos(\cos t) + \sin(\sin t)\right]_0^{\pi} + \left[-\cos(-1-t) + \sin(3t)\right]_0^1$$
$$= -\cos(\cos \pi) + \sin(\sin \pi) + \cos(\cos 0) - \sin(\sin 0)$$
$$-\cos(-2) + \sin(3) + \cos(-1) - \sin(0)$$
$$= -\cos(-1) + \sin 0 + \cos(1) - \sin 0 - \cos(-2) + \sin 3 + \cos(-1)$$
$$= -\cos 1 + \cos 1 - \cos 2 + \sin 3 + \cos 1 = \cos 1 - \cos 2 + \sin 3$$

where we have used the identity $\cos(-\theta) = \cos\theta$.

9. $x = 4\sin t$, $y = 4\cos t$, $z = 3t$, $0 \le t \le \frac{\pi}{2}$. Then by Formula 9,

$$\int_C xy^3\,ds = \int_0^{\pi/2} (4\sin t)(4\cos t)^3 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt$$
$$= \int_0^{\pi/2} 4^4 \cos^3 t \sin t \sqrt{(4\cos t)^2 + (-4\sin t)^2 + (3)^2}\,dt$$
$$= \int_0^{\pi/2} 256\cos^3 t \sin t \sqrt{16(\cos^2 t + \sin^2 t + 9)}\,dt$$
$$= 1280 \int_0^{\pi/2} \cos^3 t \sin t\,dt = -320\cos^4 t\Big]_0^{\pi/2} = 320$$

10. Parametric equations for C are $x = 4t$, $y = 6 - 5t$, $z = -1 + 6t$, $0 \le t \le 1$. Then

$$\int_C x^2 z\,ds = \int_0^1 (4t)^2(6t-1)\sqrt{4^2 + (-5)^2 + 6^2}\,dt = \sqrt{77}\int_0^1 (96t^3 - 16t^2)\,dt$$
$$= \sqrt{77}\left[96 \cdot \frac{t^4}{4} - 16 \cdot \frac{t^3}{3}\right]_0^1 = \frac{56}{3}\sqrt{77}$$

11. Parametric equations for C are $x=t$, $y=2t$, $z=3t$, $0\le t\le 1$. Then

$$\int_C xe^{yz}\,ds = \int_0^1 te^{(2t)(3t)}\sqrt{1^2+2^2+3^2}\,dt=\sqrt{14}\int_0^1 te^{6t^2}\,dt$$

$$= \sqrt{14}\left[\frac{1}{12}e^{6t^2}\right]_0^1=\frac{\sqrt{14}}{12}(e^6-1)$$

12. $\sqrt{(dx/dt)^2+(dy/dt)^2+(dz/dt)^2}=\sqrt{1^2+(2t)^2+(3t^2)^2}=\sqrt{1+4t^2+9t^4}$. Then

$$\int_C(2x+9z)\,ds=\int_0^1(2t+9t^3)\sqrt{1+4t^2+9t^4}\,dt \text{ [let } u=1+4t^2+9t^4\Rightarrow \frac{1}{4}du=(2t+9t^3)dt \text{]}$$

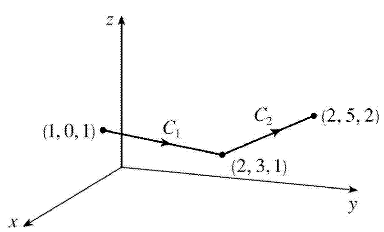
$$=\int_1^{14}\frac{1}{4}\sqrt{u}\,du=\left.\frac{1}{6}u^{3/2}\right]_1^{14}=\frac{1}{6}(14^{3/2}-1)$$

13. $\int_C x^2y\sqrt{z}\,dz=\int_0^1(t^3)^2(t)\sqrt{t^2}\cdot 2t\,dt=\int_0^1 2t^9\,dt=\left.\frac{1}{5}t^{10}\right]_0^1=\frac{1}{5}$

14.

$$\int_C zdx+xdy+ydz=\int_0^1 t^2\cdot 2t\,dt+t^2\cdot 3t^2\,dt+t^3\cdot 2t\,dt=\int_0^1(2t^3+5t^4)\,dt$$

$$=\left[\frac{1}{2}t^4+t^5\right]_0^1=\frac{1}{2}+1=\frac{3}{2}$$



15.

On C_1 : $x=1+t\Rightarrow dx=dt$, $y=3t\Rightarrow dy=3\,dt$, $z=1$
$\Rightarrow dz=0\,dt$, $0\le t\le 1$.
On C_2 : $x=2\Rightarrow dx=0\,dt$, $y=3+2t\Rightarrow$
$dy=2\,dt$, $z=1+t\Rightarrow dz=dt$, $0\le t\le 1$.

Then $\int_C(x+yz)\,dx+2x\,dy+xyz\,dz$

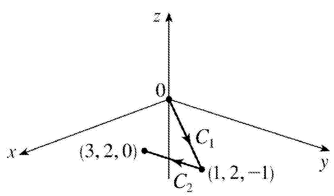
$$=\int_{C_1}(x+yz)\,dx+2x\,dy+xyz\,dz+\int_{C_2}(x+yz)\,dx+2x\,dy+xyz\,dz$$

$=\int_0^1 (1+t+(3t)(1))\,dt+2(1+t)\cdot 3\,dt+(1+t)(3t)(1)\cdot 0\,dt$

$+\int_0^1 (2+(3+2t)(1+t))\cdot 0\,dt+2(2)\cdot 2\,dt+(2)(3+2t)(1+t)\,dt$

$=\int_0^1 (10t+7)\,dt+\int_0^1 (4t^2+10t+14)\,dt$

$=\left[5t^2+7t\right]_0^1+\left[\frac{4}{3}t^3+5t^2+14t\right]_0^1=12+\frac{61}{3}=\frac{97}{3}$

16. 

On C_1 : $x=t \Rightarrow dx=dt$, $y=2t \Rightarrow dy=2\,dt$, $z=-t$
$\Rightarrow dz=-dt$, $0\le t\le 1$.
On C_2 : $x=1+2t \Rightarrow dx=2\,dt$, $y=2 \Rightarrow$
$dy=0\,dt$, $z=-1+t \Rightarrow dz=dt$, $0\le t\le 1$.

Then $\int_C x^2\,dx+y^2\,dy+z^2\,dz$
$=\int_{C_1} x^2\,dx+y^2\,dy+z^2\,dz+\int_{C_2} x^2\,dx+y^2\,dy+z^2\,dz$

$=\int_0^1 t^2\,dt+(2t)^2\cdot 2\,dt+(-t)^2(-dt)+\int_0^1 (1+2t)^2\cdot 2\,dt+2^2\cdot 0\,dt+(-1+t)^2\,dt$

$=\int_0^1 8t^2\,dt+\int_0^1 (9t^2+6t+3)\,dt=\left[\frac{8}{3}t^3\right]_0^1+\left[3t^3+3t^2+3t\right]_0^1=\frac{35}{3}$

17. **(a)** Along the line $x=-3$, the vectors of $\boldsymbol{F}$ have positive y –components, so since the path goes upward, the integrand $\boldsymbol{F}\cdot\boldsymbol{T}$ is always positive. Therefore $\int_{C_1} \boldsymbol{F}\cdot d\boldsymbol{r}=\int_{C_1} \boldsymbol{F}\cdot\boldsymbol{T}\,ds$ is positive.

(b) All of the (nonzero) field vectors along the circle with radius 3 are pointed in the clockwise direction, that is, opposite the direction to the path. So $\boldsymbol{F}\cdot\boldsymbol{T}$ is negative, and therefore $\int_{C_2} \boldsymbol{F}\cdot d\boldsymbol{r}=\int_{C_2} \boldsymbol{F}\cdot\boldsymbol{T}\,ds$ is negative.

18. Vectors starting on

C_1 point in roughly the same direction as C_1, so the tangential component $\boldsymbol{F}\cdot\boldsymbol{T}$ is positive. Then $\int_{C_1}\boldsymbol{F}\cdot d\boldsymbol{r}=\int_{C_1}\boldsymbol{F}\cdot\boldsymbol{T}\,ds$ is positive. On the other hand, no vectors starting on C_2 point in the same direction as C_2, while some vectors point in roughly the opposite direction, so we would expect $\int_{C_2}\boldsymbol{F}\cdot d\boldsymbol{r}=\int_{C_2}\boldsymbol{F}\cdot\boldsymbol{T}\,ds$ to be negative.

19. $\boldsymbol{r}(t)=t^2\boldsymbol{i}-t^3\boldsymbol{j}$, so $\boldsymbol{F}(\boldsymbol{r}(t))=(t^2)^2(-t^3)^3\boldsymbol{i}-(-t^3)\sqrt{t^2}\,\boldsymbol{j}=-t^{13}\boldsymbol{i}+t^4\boldsymbol{j}$ and $\boldsymbol{r}'(t)=2t\boldsymbol{i}-3t^2\boldsymbol{j}$.
Thus $\int_C\boldsymbol{F}\cdot d\boldsymbol{r}=\int_0^1\boldsymbol{F}(\boldsymbol{r}(t))\cdot\boldsymbol{r}'(t)dt=\int_0^1(-2t^{14}-3t^6)dt=\left[-\frac{2}{15}t^{15}-\frac{3}{7}t^7\right]_0^1=-\frac{59}{105}$.

20. $\boldsymbol{F}(\boldsymbol{r}(t))=(t^2)(t^3)\boldsymbol{i}+(t)(t^3)\boldsymbol{j}+(t)(t^2)\boldsymbol{k}=t^5\boldsymbol{i}+t^4\boldsymbol{j}+t^3\boldsymbol{k}$, $\boldsymbol{r}'(t)=\boldsymbol{i}+2t\boldsymbol{j}+3t^2\boldsymbol{k}$.
Thus $\int_C\boldsymbol{F}\cdot d\boldsymbol{r}=\int_0^2\boldsymbol{F}(\boldsymbol{r}(t))\cdot\boldsymbol{r}'(t)dt=\int_0^2(t^5+2t^5+3t^5)dt=t^6\Big]_0^2=64$.

21.

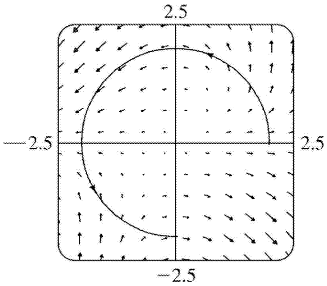
$$\int_C\boldsymbol{F}\cdot d\boldsymbol{r}=\int_0^1\left\langle\sin t^3,\cos(-t^2),\mathrm{t}^4\right\rangle\cdot\left\langle 3t^2,-2\mathrm{t},1\right\rangle dt$$
$$=\int_0^1(3t^2\sin t^3-2t\cos t^2+t^4)dt=\left[-\cos t^3-\sin t^2+\frac{1}{5}t^5\right]_0^1=\frac{6}{5}-\cos 1-\sin 1$$

22.

$$\int_C\boldsymbol{F}\cdot d\boldsymbol{r}=\int_0^\pi\left\langle\cos\mathrm{t},\sin\mathrm{t},-t\right\rangle\cdot\left\langle 1,\cos\mathrm{t},-\sin t\right\rangle dt=\int_0^\pi(\cos t+\sin t\cos t+t\sin t)dt$$
$$=\left[\sin t+\frac{1}{2}\sin^2 t+(\sin t-t\cos t)\right]_0^\pi=\pi$$

23. We graph $\boldsymbol{F}(x,y)=(x-y)\boldsymbol{i}+xy\boldsymbol{j}$ and the curve C. We see that most of the vectors starting on C point in roughly the same direction as C, so for these portions of C the tangential component $\boldsymbol{F}\cdot\boldsymbol{T}$ is positive. Although some vectors in the third quadrant which start on C point in roughly the opposite direction, and hence give negative tangential components, it seems reasonable that the effect of these portions of C is outweighed by the positive tangential components. Thus, we would expect $\int_C\boldsymbol{F}\cdot d\boldsymbol{r}=\int_C\boldsymbol{F}\cdot\boldsymbol{T}\,ds$ to be positive.

To verify, we evaluate $\int_C\boldsymbol{F}\cdot d\boldsymbol{r}$. The curve C can be represented by $\boldsymbol{r}(t)=2\cos t\,\boldsymbol{i}+2\sin t\,\boldsymbol{j}$, $0\le t\le\frac{3\pi}{2}$, so $\boldsymbol{F}(\boldsymbol{r}(t))=(2\cos t-2\sin t)\boldsymbol{i}+4\cos t\sin t\,\boldsymbol{j}$ and $\boldsymbol{r}'(t)=-2\sin t\,\boldsymbol{i}+2\cos t\,\boldsymbol{j}$. Then



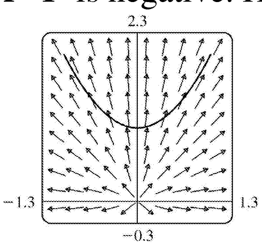
$$\int_C \boldsymbol{F}\cdot d\boldsymbol{r} = \int_0^{3\pi/2} \boldsymbol{F}(\boldsymbol{r}(t))\cdot \boldsymbol{r}'(t)\,dt$$

$$= \int_0^{3\pi/2} [-2\sin t(2\cos t - 2\sin t) + 2\cos t(4\cos t\sin t)]\,dt$$

$$= 4\int_0^{3\pi/2} (\sin^2 t - \sin t\cos t + 2\sin t\cos^2 t)\,dt$$

$$= 3\pi + \frac{2}{3} \quad \text{[using a CAS]}$$

24. We graph $\boldsymbol{F}(x, y) = \frac{x}{\sqrt{x^2+y^2}}\,\boldsymbol{i} + \frac{y}{\sqrt{x^2+y^2}}\,\boldsymbol{j}$ and the curve C. In the first quadrant, each vector starting on C points in roughly the same direction as C, so the tangential component $\boldsymbol{F}\cdot\boldsymbol{T}$ is positive. In the second quadrant, each vector starting on C points in roughly the direction opposite to C, so $\boldsymbol{F}\cdot\boldsymbol{T}$ is negative. Here, it appears that the tangential components in the first and second quadrants



counteract each other, so it seems reasonable to guess that $\int_C \boldsymbol{F}\cdot d\boldsymbol{r} = \int_C \boldsymbol{F}\cdot\boldsymbol{T}\,ds$ is zero. To verify, we evaluate $\int_C \boldsymbol{F}\cdot d\boldsymbol{r}$. The curve C can be represented by $\boldsymbol{r}(t) = t\,\boldsymbol{i} + (1+t^2)\,\boldsymbol{j}$, $-1 \le t \le 1$, so

$\boldsymbol{F}(\boldsymbol{r}(t)) = \frac{t}{\sqrt{t^2+(1+t^2)^2}}\,\boldsymbol{i} + \frac{1+t^2}{\sqrt{t^2+(1+t^2)^2}}\,\boldsymbol{j}$ and $\boldsymbol{r}'(t) = \boldsymbol{i} + 2t\,\boldsymbol{j}$. Then

$$\int_C \boldsymbol{F}\cdot d\boldsymbol{r} = \int_{-1}^{1} \boldsymbol{F}(\boldsymbol{r}(t))\cdot \boldsymbol{r}'(t)\,dt$$

$$= \int_{-1}^{1} \left(\frac{t}{\sqrt{t^2+(1+t^2)^2}} + \frac{2t(1+t^2)}{\sqrt{t^2+(1+t^2)^2}} \right) dt$$

$= \int_{-1}^{1} \frac{t\,(3+2t^2)}{\sqrt{t^4+3t^2+1}}\, dt=0$ [since the integrand is an odd function]

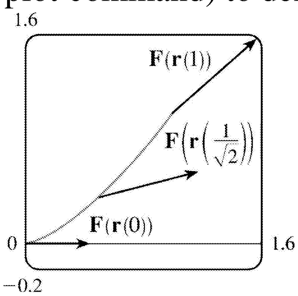
25. **(a)** $\int_C \boldsymbol{F}\cdot d\boldsymbol{r} = \int_0^1 \left\langle e^{t^2-1}, t^5 \right\rangle \cdot \left\langle 2t, 3t^2 \right\rangle dt = \int_0^1 \left(2te^{t^2-1} + 3t^7 \right) dt = \left[e^{t^2-1} + \frac{3}{8}t^8 \right]_0^1 = \frac{11}{8} - 1/e$

(b) $\boldsymbol{r}(0)=\boldsymbol{0}$, $\boldsymbol{F}(\boldsymbol{r}(0))=\left\langle e^{-1}, 0 \right\rangle$;

$\boldsymbol{r}\left(\frac{1}{\sqrt{2}} \right)=\left\langle \frac{1}{2}, \frac{1}{2\sqrt{2}} \right\rangle$, $\boldsymbol{F}\left(\boldsymbol{r}\left(\frac{1}{\sqrt{2}} \right) \right)=\left\langle e^{-1/2}, \frac{1}{4\sqrt{2}} \right\rangle$;

$\boldsymbol{r}(1)=\langle 1, 1 \rangle$, $\boldsymbol{F}(\boldsymbol{r}(1))=\langle 1, 1 \rangle$.

In order to generate the graph with Maple, we use the PLOT command (not to be confused with the plot command) to define each of the vectors. For example,

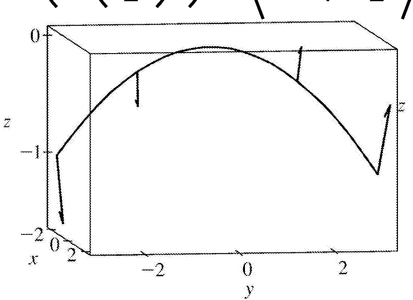


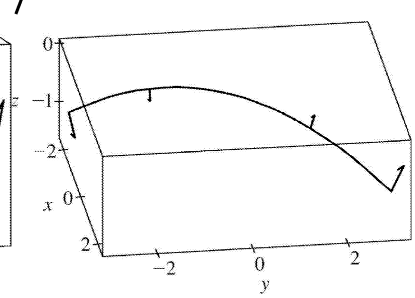
v1:=PLOT(CURVES([[0, 0], [evalf(1/exp(1)), 0]])); generates the vector from the vector field at the point (0, 0) (but without an arrowhead) and gives it the name v1. To show everything on the same screen, we use the display command. In Mathematica, we use ListPlot (with the PlotJoined – > True option) to generate the vectors, and then Show to show everything on the same screen.

26. **(a)** $\int_C \boldsymbol{F}\cdot d\boldsymbol{r} = \int_{-1}^{1} \left\langle 2t, t^2, 3t \right\rangle \cdot \langle 2, 3, -2t \rangle dt = \int_{-1}^{1} (4t+3t^2-6t^2) dt = \left[2t^2 - t^3 \right]_{-1}^{1} = -2$

(b) Now $\boldsymbol{F}(\boldsymbol{r}(t))=\left\langle 2t, t^2, 3t \right\rangle$, so $\boldsymbol{F}(\boldsymbol{r}(-1))=\langle -2, 1, -3 \rangle$, $\boldsymbol{F}\left(\boldsymbol{r}\left(-\frac{1}{2} \right) \right)=\left\langle -1, \frac{1}{4}, -\frac{3}{2} \right\rangle$,

$\boldsymbol{F}\left(\boldsymbol{r}\left(\frac{1}{2} \right) \right)=\left\langle 1, \frac{1}{4}, \frac{3}{2} \right\rangle$, and $\boldsymbol{F}(\boldsymbol{r}(1))=\langle 2, 1, 3 \rangle$.





27. The part of the astroid that lies in the quadrant is parametrized by $x=\cos^3 t$, $y=\sin^3 t$, $0\le t\le \frac{\pi}{2}$.
Now $\frac{dx}{dt}=3\cos^2 t(-\sin t)$ and $\frac{dy}{dt}=3\sin^2 t\cos t$, so

$$\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}=\sqrt{9\cos^4 t\sin^2 t+9\sin^4 t\cos^2 t}=3\cos t\sin t\sqrt{\cos^2 t+\sin^2 t}=3\cos t\sin t \ .$$

Therefore $\int_C x^3 y^5\,ds=\int_0^{\pi/2}\cos^9 t\sin^{15} t(3\cos t\sin t)\,dt=\frac{945}{16,777,216}\pi$.

28. We parametrize the line as $\mathbf{r}(t)=\langle 1,2,1\rangle+t\left[\langle 6,4,5\rangle-\langle 1,2,1\rangle\right]=(1+5t)\boldsymbol{i}+(2+2t)\boldsymbol{j}+(1+4t)\boldsymbol{k}$, $0\le t\le 1$. Using a CAS, we calculate

$$\int_C \boldsymbol{F}\cdot d\boldsymbol{r}=\int_0^1\left\langle (1+5t)^4 e^{2+2t}, \ln(1+4t), \sqrt{(2+2t)^2+(1+4t)^2}\right\rangle\cdot\langle 5,2,4\rangle\,dt$$

$$=\frac{5235e^4}{4}-\frac{6285e^2}{4}+\frac{9\sqrt{5}\sinh^{-1}\left(\frac{14}{3}\right)}{25}-\frac{9\sqrt{5}\sinh^{-1}\left(\frac{4}{3}\right)}{25}+\frac{5\ln 5}{2}+\frac{14\sqrt{41}}{5}-\frac{4\sqrt{5}}{5}-2$$

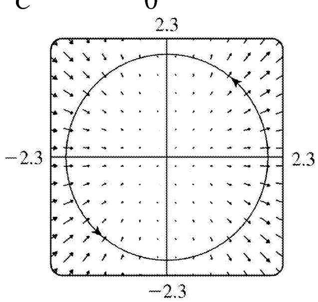
$$=\frac{5235e^4}{4}-\frac{6285e^2}{4}-\frac{18\sqrt{5}\ln 3}{25}+\frac{9\sqrt{5}\ln(14+\sqrt{205})}{25}+\frac{5\ln 5}{2}+\frac{14\sqrt{41}-4\sqrt{5}}{5}-2$$

The first answer is the one given by Maple. The two answers are equivalent by Equation 7.6.3 .

29. A calculator or CAS gives $\int_C x\sin y\,ds=\int_1^2\ln t\sin(e^{-t})\sqrt{(1/t)^2+(-e^{-t})^2}\,dt\approx 0.052$.

30. **(a)** We parametrize the circle C as $\boldsymbol{r}(t)=2\cos t\,\boldsymbol{i}+2\sin t\,\boldsymbol{j}$, $0\le t\le 2\pi$. So
$\boldsymbol{F}(\boldsymbol{r}(t))=\left\langle 4\cos^2 t, 4\cos t\sin t\right\rangle$, $\boldsymbol{r}'(t)=\langle -2\sin t, 2\cos t\rangle$, and
$W=\int_C \boldsymbol{F}\cdot d\boldsymbol{r}=\int_0^{2\pi}(-8\cos^2 t\sin t+8\cos^2 t\sin t)\,dt=0$.

(b)



From the graph, we see that all of the vectors in the field are perpendicular to the path. This indicates that the field does no work on the particle, since the field never pulls the particle in the direction in which it is going. In other words, at any point along C , $\boldsymbol{F}\cdot\boldsymbol{T}=0$, and so certainly $\int_C \boldsymbol{F}\cdot d\boldsymbol{r}=0$.

31. We use the parametrization $x=2\cos t$, $y=2\sin t$, $-\frac{\pi}{2}\le t\le\frac{\pi}{2}$. Then

$ds=\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt=\sqrt{(-2\sin t)^2+(2\cos t)^2}\,dt=2\,dt$, so $m=\int_C k\,ds=2k\int_{-\pi/2}^{\pi/2}dt=2k(\pi)$,

$\bar{x}=\frac{1}{2\pi k}\int_C xk\,ds=\frac{1}{2\pi}\int_{-\pi/2}^{\pi/2}(2\cos t)2\,dt=\frac{1}{2\pi}\left[4\sin t\right]_{-\pi/2}^{\pi/2}=\frac{4}{\pi}$, $\bar{y}=\frac{1}{2\pi k}\int_C yk\,ds=\frac{1}{2\pi}\int_{-\pi/2}^{\pi/2}(2\sin t)2\,dt=0$.

Hence $(\bar{x},\bar{y})=\left(\frac{4}{\pi},0\right)$.

32. We use the parametrization $x=r\cos t$, $y=r\sin t$, $0\le t\le\frac{\pi}{2}$. Then

$ds=\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt=\sqrt{(-r\sin t)^2+(r\cos t)^2}\,dt=r\,dt$, so

$m=\int_C(x+y)ds=\int_0^{\pi/2}(r\cos t+r\sin t)r\,dt=r^2\left[\sin t-\cos t\right]_0^{\pi/2}=2r^2$,

$$\bar{x}=\frac{1}{2r^2}\int_C x(x+y)\,ds=\frac{1}{2r^2}\int_0^{\pi/2}(r^2\cos^2 t+r^2\cos t\sin t)r\,dt=\frac{r}{2}\left[\frac{t}{2}+\frac{\sin 2t}{4}-\frac{\cos 2t}{4}\right]_0^{\pi/2}$$

$$=\frac{r(\pi+2)}{8}$$, and

$$\bar{y}=\frac{1}{2r^2}\int_C y(x+y)\,ds=\frac{1}{2r^2}\int_0^{\pi/2}(r^2\sin t\cos t+r^2\sin^2 t)r\,dt$$

$$=\frac{r}{2}\left[-\frac{\cos 2t}{4}+\frac{t}{2}-\frac{\sin 2t}{4}\right]_0^{\pi/2}=\frac{r(\pi+2)}{8}$$.

Therefore $(\bar{x},\bar{y})=\left(\frac{r(\pi+2)}{8},\frac{r(\pi+2)}{8}\right)$.

33. **(a)** $\bar{x}=\frac{1}{m}\int_C x\rho(x,y,z)\,ds$, $\bar{y}=\frac{1}{m}\int_C y\rho(x,y,z)\,ds$, $\bar{z}=\frac{1}{m}\int_C z\rho(x,y,z)\,ds$ where $m=\int_C\rho(x,y,z)\,ds$.

(b) $m=\int_C k\,ds=k\int_0^{2\pi}\sqrt{4\sin^2 t+4\cos^2 t+9}\,dt=k\sqrt{13}\int_0^{2\pi}dt=2\pi k\sqrt{13}$, $\bar{x}=\frac{1}{2\pi k\sqrt{13}}\int_0^{2\pi}k2\sqrt{13}\sin t\,dt=0$,

$\bar{y}=\frac{1}{2\pi k\sqrt{13}}\int_0^{2\pi}k2\sqrt{13}\cos t\,dt=0$, $\bar{z}=\frac{1}{2\pi k\sqrt{13}}\int_0^{2\pi}(k\sqrt{13})(3t)\,dt=\frac{3}{2\pi}(2\pi^2)=3\pi$. Hence

$(\bar{x}, \bar{y}, \bar{z})=(0, 0, 3\pi)$.

34.

$$m=\int_C (x^2+y^2+z^2)ds=\int_0^{2\pi}(t^2+1)\sqrt{(1)^2+(-\sin t)^2+(\cos t)^2}\,dt=\int_0^{2\pi}(t^2+1)\sqrt{2}\,dt$$
$$=\sqrt{2}\left(\frac{8}{3}\pi^3+2\pi\right),$$

$$\bar{x}=\frac{1}{\sqrt{2}\left(\frac{8}{3}\pi^3+2\pi\right)}\int_0^{2\pi}\sqrt{2}\,(t^3+t)\,dt=\frac{4\pi^4+2\pi^2}{\frac{8}{3}\pi^3+2\pi}=\frac{3\pi(2\pi^2+1)}{4\pi^2+3},$$
$$\bar{y}=\frac{3}{2\sqrt{2}\pi(4\pi^2+3)}\int_0^{2\pi}(\sqrt{2}\cos t)(t^2+1)\,dt=0 \text{ , and}$$
$$\bar{z}=\frac{3}{2\sqrt{2}\pi(4\pi^2+3)}\int_0^{2\pi}(\sqrt{2}\sin t)(t^2+1)\,dt=0 \text{ . Hence } (\bar{x}, \bar{y}, \bar{z})=\left(\frac{3\pi(2\pi^2+1)}{4\pi^2+3}, 0, 0\right).$$

35.

From Example 3, $\rho(x, y)=k(1-y)$, $x=\cos t$, $y=\sin t$, and $ds=dt$, $0\le t\le\pi\Rightarrow$

$$I_x=\int_C y^2\rho(x, y)ds=\int_0^{\pi}\sin^2 t\,dt=k\int_0^{\pi}(\sin^2 t-\sin^3 t)dt$$
$$=\frac{1}{2}k\int_0^{\pi}(1-\cos 2t)dt-k\int_0^{\pi}(1-\cos^2 t)\sin t\,dt \quad \text{[Let } u=t, du=-tdt-3pt \text{ in the second integral]}$$
$$=k\left[\frac{\pi}{2}+\int_1^{-1}(1-u^2)du\right]=k\left(\frac{\pi}{2}-\frac{4}{3}\right)$$
$$I_y=\int_C x^2\rho(x, y)ds=k\int_0^{\pi}\cos^2 t(1-\sin t)dt=\frac{k}{2}\int_0^{\pi}(1+\cos 2t)dt-k\int_0^{\pi}\cos^2 t\sin t\,dt$$
$$=k\left(\frac{\pi}{2}-\frac{2}{3}\right)$$, using the same substitution as above.

36.

The wire is given as $x=2\sin t$, $y=2\cos t$, $z=3t$, $0\le t\le 2\pi$ with $\rho(x, y, z)=k$. Then

$$ds=\sqrt{(2\cos t)^2+(-2\sin t)^2+3^2}=\sqrt{4(\cos^2 t+\sin^2 t)+9}=\sqrt{13} \text{ and}$$
$$I_x=\int_C (y^2+z^2)\rho(x, y, z)ds=\int_0^{2\pi}(4\cos^2 t+9t^2)(k)\sqrt{13}\,dt=\sqrt{13}\,k\left[4\left(\frac{1}{2}t+\frac{1}{4}\sin 2t\right)+3t^3\right]_0^{2\pi}$$

$=\sqrt{13}\,k(4\pi+24\pi^3)=4\sqrt{13}\,\pi k(1+6\pi^2)$

$I_y=\int_C(x^2+z^2)\rho(x,y,z)\,ds=\int_0^{2\pi}\left(4\sin^2 t+9t^2\right)(k)\sqrt{13}\,dt=\sqrt{13}\,k\left[4\left(\frac{1}{2}t-\frac{1}{4}\sin 2t\right)+3t^3\right]_0^{2\pi}$

$=\sqrt{13}\,k(4\pi+24\pi^3)=4\sqrt{13}\,\pi k(1+6\pi^2)$

$I_z=\int_C(x^2+y^2)\rho(x,y,z)\,ds=\int_0^{2\pi}(4\sin^2 t+4\cos^2 t)(k)\sqrt{13}\,dt=4\sqrt{13}\,k\int_0^{2\pi}dt=8\pi\sqrt{13}\,k$

37.

$W=\int_C \boldsymbol{F}\cdot d\boldsymbol{r}=\int_0^{2\pi}\langle t-\sin t,3-\cos t\rangle\cdot\langle 1-\cos t,\sin t\rangle\,dt$

$=\int_0^{2\pi}(t-t\cos t-\sin t+\sin t\cos t+3\sin t-\sin t\cos t)\,dt$

$=\int_0^{2\pi}(t-t\cos t+2\sin t)\,dt=\left[\frac{1}{2}t^2-(t\sin t+\cos t)-2\cos t\right]_0^{2\pi}$ [by integrating by parts in the second term]

$=2\pi^2$

38.

$x=x$, $y=x^2$, $-1\le x\le 2$,

$W=\int_{-1}^{2}\left\langle x\sin x^2,x^2\right\rangle\cdot\langle 1,2x\rangle\,dx=\int_{-1}^{2}(x\sin x^2+2x^3)\,dx=\left[-\frac{1}{2}\cos x^2+\frac{1}{2}x^4\right]_{-1}^{2}$

$=\frac{1}{2}(15+\cos 1-\cos 4)$

39.

$\boldsymbol{r}(t)=\langle 1+2t,4t,2t\rangle$, $0\le t\le 1$,

$W=\int_C \boldsymbol{F}\cdot d\boldsymbol{r}=\int_0^1\langle 6t,1+4t,1+6t\rangle\cdot\langle 2,4,2\rangle\,dt=\int_0^1(12t+4(1+4t)+2(1+6t))\,dt$

$=\int_0^1(40t+6)\,dt=\left[20t^2+6t\right]_0^1=26$

40.

$\boldsymbol{r}(t)=2\boldsymbol{i}+t\,\boldsymbol{j}+5t\,\boldsymbol{k}$, $0\le t\le 1$. Therefore

$W=\int_C \boldsymbol{F}\cdot d\boldsymbol{r}=\int_0^1\frac{K\langle 2,t,5t\rangle}{(4+26t^2)^{3/2}}\cdot\langle 0,1,5\rangle\,dt=K\int_0^1\frac{26t}{(4+26t^2)^{3/2}}\,dt$

$= K\left[-(4+26t^2)^{-1/2}\right]_0^1 = K\left(\frac{1}{2}-\frac{1}{\sqrt{30}}\right)$

41.
Let $\boldsymbol{F}=185\boldsymbol{k}$. To parametrize the staircase, let
$x=20\cos t$, $y=20\sin t$, $z=\frac{90}{6\pi}t=\frac{15}{\pi}t$, $0\le t\le 6\pi \Rightarrow$

$W = \int_C \boldsymbol{F}\cdot d\boldsymbol{r} = \int_0^{6\pi}\langle 0, 0, 185\rangle\cdot\left\langle -20\sin t, 20\cos t, \frac{15}{\pi}\right\rangle dt = (185)\frac{15}{\pi}\int_0^{6\pi} dt = (185)(90)$

$\approx 1.67\times 10^4$ ft–lb

42.
This time m is a function of t : $m=185-\frac{9}{6\pi}t=185-\frac{3}{2\pi}t$. So let $\boldsymbol{F}=\left(185-\frac{3}{2\pi}t\right)\boldsymbol{k}$. To parametrize the staircase, let $x=20\cos t$, $y=20\sin t$, $z=\frac{90}{6\pi}t=\frac{15}{\pi}t$, $0\le t\le 6\pi$. Therefore

$W = \int_C \boldsymbol{F}\cdot d\boldsymbol{r} = \int_0^{6\pi}\left\langle 0, 0, 185-\frac{3}{2\pi}t\right\rangle\cdot\left\langle -20\sin t, 20\cos t, \frac{15}{\pi}\right\rangle dt = \frac{15}{\pi}\int_0^{6\pi}\left(185-\frac{3}{2\pi}t\right)dt$

$= \frac{15}{\pi}\left[185t-\frac{3}{4\pi}t^2\right]_0^{6\pi} = 90\left(185-\frac{9}{2}\right)\approx 1.62\times 10^4$ ft–lb

43. **(a)** $\boldsymbol{r}(t)=\langle\cos t, \sin t\rangle$, $0\le t\le 2\pi$, and let $\boldsymbol{F}=\langle a, b\rangle$. Then

$W=\int_C \boldsymbol{F}\cdot d\boldsymbol{r} = \int_0^{2\pi}\langle a, b\rangle\cdot\langle -\sin t, \cos t\rangle\, dt$

$=\int_0^{2\pi}(-a\sin t+b\cos t)\,dt=[a\cos t+b\sin t]_0^{2\pi}$

$=a+0-a+0=0$

(b) Yes. $\boldsymbol{F}(x, y)=k\boldsymbol{x}=\langle kx, ky\rangle$ and

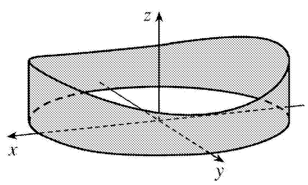
$W=\int_C \boldsymbol{F}\cdot d\boldsymbol{r} = \int_0^{2\pi}\langle k\cos t, k\sin t\rangle\cdot\langle -\sin t, \cos t\rangle\, dt$

$=\int_0^{2\pi}(-k\sin t\cos t+k\sin t\cos t)\,dt=\int_0^{2\pi}0\,dt=0$

44. Consider the base of the fence in the xy –plane, centered at the origin, with the height given by $z=h(x, y)$. The fence can be graphed using the parametric equations

$x=10\cos u$, $y=10\sin u$,

$$z = v\left[4+0.01((10\cos u)^2-(10\sin u)^2)\right]$$
$$= v(4+\cos^2 u-\sin^2 u)$$
$$= v(4+\cos 2u) ,\ 0\le u\le 2\pi ,\ 0\le v\le 1 .$$



The area of the fence is $\int_C h(x, y)\,ds$ where C , the base of the fence, is given by $x=10\cos t$, $y=10\sin t$, $0\le t\le 2\pi$. Then

$$\int_C h(x, y)\,ds = \int_0^{2\pi}\left[4+0.01((10\cos t)^2-(10\sin t)^2)\right]\sqrt{(-10\sin t)^2+(10\cos t)^2}\,dt$$
$$= \int_0^{2\pi}(4+\cos 2t)\sqrt{100}\,dt=10\left[4t+\frac{1}{2}\sin 2t\right]_0^{2\pi}$$
$$= 10(8\pi)=80\pi\,\text{m}^2$$

If we paint both sides of the fence, the total surface area to cover is 160π m^2 , and since 1 L of paint covers 100 m^2 , we require $\frac{160\pi}{100}=1.6\pi\approx 5.03$ L of paint.

45. The work done in moving the object is $\int_C \boldsymbol{F}\cdot d\boldsymbol{r}=\int_C \boldsymbol{F}\cdot\boldsymbol{T}\,ds$. We can approximate this integral by dividing C into 7 segments of equal length $\Delta s=2$ and approximating $\boldsymbol{F}\cdot\boldsymbol{T}$, that is, the tangential component of force, at a
point (x_i^*, y_i^*) on each segment. Since C is composed of straight line segments, $\boldsymbol{F}\cdot\boldsymbol{T}$ is the scalar projection of each force vector onto C . If we choose (x_i^*, y_i^*) to be the point on the segment closest to the origin, then the work done is
$\int_C \boldsymbol{F}\cdot\boldsymbol{T}\,ds\approx\sum_{i=1}^{7}\left[\boldsymbol{F}(x_i^*, y_i^*)\cdot\boldsymbol{T}(x_i^*, y_i^*)\right]\Delta s=[2+2+2+2+1+1+1](2)=22$. Thus, we estimate the work done to be approximately 22 J.

46. Use the orientation pictured in the figure. Then since $\boldsymbol{B}$ is tangent to any circle that lies in the plane perpendicular
to the wire, $\boldsymbol{B}=|\boldsymbol{B}|\boldsymbol{T}$ where $\boldsymbol{T}$ is the unit tangent to the circle C : $x=r\cos\theta$, $y=r\sin\theta$. Thus

$\boldsymbol{B} = |\boldsymbol{B}| \left\langle -\sin\theta, \cos\theta \right\rangle$. Then

$\int_C \boldsymbol{B} \cdot d\boldsymbol{r} = \int_0^{2\pi} |\boldsymbol{B}| \left\langle -\sin\theta, \cos\theta \right\rangle \cdot \left\langle -r\sin\theta, r\cos\theta \right\rangle d\theta = \int_0^{2\pi} |\boldsymbol{B}| r \, d\theta = 2\pi r |\boldsymbol{B}|$. (Note that $|\boldsymbol{B}|$ here is the magnitude of the field at a distance r from the wire's center). But by Ampere's Law $\int_C \boldsymbol{B} \cdot d\boldsymbol{r} = \mu_0 I$.

Hence $|\boldsymbol{B}| = \dfrac{\mu_0 I}{2\pi r}$.

1. C appears to be a smooth curve, and since ∇f is continuous, we know f is differentiable. Then Theorem 2 says that the value of $\int_C \nabla f \cdot d\mathbf{r}$ is simply the difference of the values of f at the terminal and initial points of C . From the graph, this is 50–10=40.

2. C is represented by the vector function $\mathbf{r}(t)=(t^2+1)\mathbf{i}+(t^3+t)\mathbf{j}$, $0\le t\le 1$, so $\mathbf{r}'(t)=2t\mathbf{i}+(3t^2+1)\mathbf{j}$. Since $3t^2+1\neq 0$, we have $\mathbf{r}'(t)\neq \mathbf{0}$, thus C is a smooth curve. ∇f is continuous, and hence f is differentiable, so by Theorem 2 we have $\int_C \nabla f \cdot d\mathbf{r}=f(\mathbf{r}(1))-f(\mathbf{r}(0))=f(2, 2)-f(1, 0)=9-3=6$.

3. $\partial(6x+5y)/\partial y=5=\partial(5x+4y)/\partial x$ and the domain of $\mathbf{F}$ is R^2 which is open and simply–connected, so by Theorem 6 $\mathbf{F}$ is conservative. Thus, there exists a function f such that $\nabla f=\mathbf{F}$, that is, $f_x(x, y)=6x+5y$ and $f_y(x, y)=5x+4y$. But $f_x(x, y)=6x+5y$ implies $f(x,y)=3x^2+5xy+g(y)$ and differentiating both sides of this equation with respect to y gives $f_y(x, y)=5x+g'(y)$. Thus $5x+4y=5x+g'(y)$ so $g'(y)=4y$ and $g(y)=2y^2+K$ where K is a constant. Hence $f(x, y)=3x^2+5xy+2y^2+K$ is a potential function for $\mathbf{F}$.

4. $\partial(x^3+4xy)/\partial y=4x$, $\partial(4xy-y^3)/\partial x=4y$. Since these are not equal, $\mathbf{F}$ is not conservative.

5. $\partial(xe^y)/\partial y=xe^y$,$\partial(ye^x)/\partial x=ye^x$. Since these are not equal, $\mathbf{F}$ is not conservative.

6. $\partial(e^y)/\partial y=e^y=\partial(xe^y)/\partial x$ and the domain of $\mathbf{F}$ is R^2 . Hence $\mathbf{F}$ is conservative so there exists a function f such that $\nabla f=\mathbf{F}$. Then $f_x(x, y)=e^y$ implies $f(x, y)=xe^y+g(y)$ and $f_y(x, y)=xe^y+g'(y)$. But $f_y(x, y)=xe^y$ so $g'(y)=0\Rightarrow g(y)=K$. Then $f(x, y)=xe^y+K$ is a potential function for $\mathbf{F}$.

7. $\partial(2x\cos y-y\cos x)/\partial y=-2x\sin y-\cos x=\partial(-x^2\sin y-\sin x)/\partial x$ and the domain of $\mathbf{F}$ is R^2 . Hence $\mathbf{F}$ is conservative so there exists a function f such that $\nabla f=\mathbf{F}$. Then $f_x(x, y)=2x\cos y-y\cos x$ implies $f(x, y)=x^2\cos y-y\sin x+g(y)$ and $f_y(x, y)=-x^2\sin y-\sin x+g'(y)$. But $f_y(x, y)=-x^2\sin y-\sin x$ so $g'(y)=0\Rightarrow g(y)=K$. Then $f(x, y)=x^2\cos y-y\sin x+K$ is a potential function for $\mathbf{F}$.

8. $\partial(1+2xy+\ln x)/\partial y=2x=\partial(x^2)/\partial x$ and the domain of $\mathbf{F}$ is $\{ (x, y) \mid x>0 \}$ which is open and simply–connected. Hence $\mathbf{F}$ is conservative, so there exists a function f such that $\nabla f=\mathbf{F}$. Then $f_x(x, y)=1+2xy+\ln x$ implies $f(x, y)=x+x^2y+x\ln x-x+g(y)$ and $f_y(x, y)=x^2+g'(y)$. But $f_y(x, y)=x^2$ so $g'(y)=0\Rightarrow g(y)=K$. Then $f(x,y)=x^2y+x\ln x+K$ is a potential function for $\mathbf{F}$.

9. $\partial(ye^x+\sin y)/\partial y=e^x+\cos y=\partial(e^x+x\cos y)/\partial x$ and the domain of $\boldsymbol{F}$ is R^2 . Hence $\boldsymbol{F}$ is conservative so there exists a function f such that $\nabla f=\boldsymbol{F}$. Then $f_x(x, y)=ye^x+\sin y$ implies $f(x, y)=ye^x+x\sin y+g(y)$ and $f_y(x, y)=e^x+x\cos y+g'(y)$. But $f_y(x, y)=e^x+x\cos y$ so $g(y)=K$ and $f(x, y)=ye^x+x\sin y+K$ is a potential function for $\boldsymbol{F}$.

10. $\dfrac{\partial(xy\cosh xy+\sinh xy)}{\partial y}=x^2y\sinh xy+x\cosh xy+x\cosh xy=x^2y\sinh xy+2x\cosh xy=\dfrac{\partial(x^2\cosh xy)}{\partial x}$ and the domain of $\boldsymbol{F}$ is R^2 . Thus $\boldsymbol{F}$ is conservative, so there exists a function f such that $\nabla f=\boldsymbol{F}$. Then $f_x(x, y)=xy\cosh xy+\sinh xy$ implies $f(x, y)=x\sinh xy+g(y)\Rightarrow f_y(x, y)=x^2\cosh xy+g'(y)$. But $f_y(x, y)=x^2\cosh xy$ so $g(y)=K$ and $f(x, y)=x\sinh xy+K$ is a potential function for $\boldsymbol{F}$.

11. **(a)** $\boldsymbol{F}$ has continuous first–order partial derivatives and $\dfrac{\partial}{\partial y}2xy=2x=\dfrac{\partial}{\partial x}(x^2)$ on R^2 , which is open and simply–connected. Thus, $\boldsymbol{F}$ is conservative by Theorem 6. Then we know that the line integral of $\boldsymbol{F}$ is independent of path; in particular, the value of $\int_C \boldsymbol{F}\cdot d\boldsymbol{r}$ depends only on the endpoints of C . Since all three curves have the same initial and terminal points, $\int_C \boldsymbol{F}\cdot d\boldsymbol{r}$ will have the same value for each curve.

(b) We first find a potential function f , so that $\nabla f=\boldsymbol{F}$. We know $f_x(x, y)=2xy$ and $f_y(x, y)=x^2$. Integrating $f_x(x, y)$ with respect to x , we have $f(x, y)=x^2y+g(y)$. Differentiating both sides with respect to y gives $f_y(x, y)=x^2+g'(y)$, so we must have $x^2+g'(y)=x^2\Rightarrow g'(y)=0\Rightarrow g(y)=K$, a constant. Thus $f(x, y)=x^2y+K$. All three curves start at $(1, 2)$ and end at $(3, 2)$, so by Theorem 2, $\int_C \boldsymbol{F}\cdot d\boldsymbol{r}=f(3, 2)-f(1, 2)=18-2=16$ for each curve.

12. **(a)** $f_x(x, y)=y$ implies $f(x, y)=xy+g(y)$ and $f_y(x, y)=x+g'(y)$. But $f_y(x, y)=x+2y$ so $g'(y)=2y\Rightarrow g(y)=y^2+K$. We can take $K=0$, so $f(x, y)=xy+y^2$.

(b) $\int_C \boldsymbol{F}\cdot d\boldsymbol{r}=f(2, 1)-f(0, 1)=3-1=2$.

13. **(a)** $f_x(x, y)=x^3y^4$ implies $f(x, y)=\frac{1}{4}x^4y^4+g(y)$ and $f_y(x, y)=x^4y^3+g'(y)$. But $f_y(x, y)=x^4y^3$ so $g'(y)=0\Rightarrow g(y)=K$, a constant. We can take $K=0$, so $f(x, y)=\frac{1}{4}x^4y^4$.

(b) The initial point of C is $\boldsymbol{r}(0)=(0, 1)$ and the terminal point is $\boldsymbol{r}(1)=(1, 2)$, so
$\int_C \boldsymbol{F}\cdot d\boldsymbol{r}=f(1, 2)-f(0, 1)=4-0=4$.

14. **(a)** $f_x(x, y)=y^2/(1+x^2)$ implies $f(x, y)=y^2\arctan x+g(y)\Rightarrow f_y(x, y)=2y\arctan x+g'(y)$. But $f_y(x, y)=2y\arctan x$ so $g'(y)=0\Rightarrow g(y)=K$. We can take $K=0$, so $f(x, y)=y^2\arctan x$.
(b) The initial point of C is $\boldsymbol{r}(0)=(0, 0)$ and the terminal point is $\boldsymbol{r}(1)=(1, 2)$, so
$\int_C \boldsymbol{F}\cdot d\boldsymbol{r}=f(1, 2)-f(0, 0)=4\arctan 1-0=4\cdot \frac{\pi}{4}=\pi$.

15. **(a)** $f_x(x, y, z)=yz$ implies $f(x, y, z)=xyz+g(y, z)$ and so $f_y(x, y, z)=xz+g_y(y, z)$. But $f_y(x, y, z)=xz$ so $g_y(y, z)=0\Rightarrow g(y, z)=h(z)$. Thus $f(x, y, z)=xyz+h(z)$ and $f_z(x, y, z)=xy+h'(z)$. But $f_z(x, y, z)=xy+2z$, so $h'(z)=2z\Rightarrow h(z)=z^2+K$. Hence $f(x, y, z)=xyz+z^2$ (taking $K=0$).
(b) $\int_C \boldsymbol{F}\cdot d\boldsymbol{r}=f(4, 6, 3)-f(1, 0, -2)=81-4=77$.

16. **(a)** $f_x(x, y, z)=2xz+y^2$ implies $f(x, y, z)=x^2z+xy^2+g(y, z)$ and so $f_y(x, y, z)=2xy+g_y(y, z)$. But $f_y(x, y, z)=2xy$ so $g_y(y, z)=0 \Rightarrow g(y, z)=h(z)$. Thus $f(x, y, z)=x^2z+xy^2+h(z)$ and $f_z(x, y, z)=x^2+h'(z)$. But $f_z(x, y, z)=x^2+3z^2$, so $h'(z)=3z^2\Rightarrow h(z)=z^3+K$. Hence $f(x, y, z)=x^2z+xy^2+z^3$ (taking $K=0$).
(b) $t=0$ corresponds to the point $(0, 1, -1)$ and $t=1$ corresponds to $(1, 2, 1)$, so
$\int_C \boldsymbol{F}\cdot d\boldsymbol{r}=f(1, 2, 1)-f(0, 1, -1)=6-(-1)=7$.

17. **(a)** $f_x(x, y, z)=y^2\cos z$ implies $f(x, y, z)=xy^2\cos z+g(y, z)$ and so $f_y(x, y, z)=2xy\cos z+g_y(y, z)$. But $f_y(x, y, z)=2xy\cos z$ so $g_y(y, z)=0\Rightarrow g(y, z)=h(z)$. Thus $f(x, y, z)=xy^2\cos z+h(z)$ and $f_z(x, y, z)=-xy^2\sin z+h'(z)$. But $f_z(x, y, z)=-xy^2\sin z$, so $h'(z)=0\Rightarrow h(z)=K$. Hence $f(x, y, z)=xy^2\cos z$ (taking $K=0$).
(b) $\boldsymbol{r}(0)=\langle 0, 0, 0\rangle$, $\boldsymbol{r}(\pi)=\langle \pi^2, 0, \pi\rangle$ so $\int_C \boldsymbol{F}\cdot d\boldsymbol{r}=f(\pi^2, 0, \pi)-f(0, 0, 0)=0-0=0$.

18. **(a)** $f_x(x, y, z)=e^y$ implies $f(x, y, z)=xe^y+g(y, z)$ and so $f_y(x, y, z)=xe^y+g_y(y, z)$. But $f_y(x, y, z)=xe^y$ so $g_y(y, z)=0\Rightarrow g(y, z)=h(z)$. Thus $f(x, y, z)=xe^y+h(z)$ and $f_z(x, y, z)=0+h'(z)$. But $f_z(x, y, z)=(z+1)e^z$, so

$h'(z)=(z+1)e^z \Rightarrow h(z)=ze^z+K$ (using integration by parts). Hence $f(x, y, z)=xe^y+ze^z$ (taking $K=0$).
(b) $\boldsymbol{r}(0)=\langle 0, 0, 0\rangle$, $\boldsymbol{r}(1)=\langle 1, 1, 1\rangle$ so $\int_C \boldsymbol{F}\cdot d\boldsymbol{r}=f(1, 1, 1)-f(0, 0, 0)=2e-0=2e$.

19. Here $\boldsymbol{F}(x, y)=\tan y\boldsymbol{i}+x\sec^2 y\boldsymbol{j}$. Then $f(x, y)=x\tan y$ is a potential function for $\boldsymbol{F}$, that is, $\nabla f=\boldsymbol{F}$ so $\boldsymbol{F}$ is conservative and thus its line integral is independent of path. Hence
$\int_C \tan y\,dx+x\sec^2 y\,dy=\int_C \boldsymbol{F}\cdot d\boldsymbol{r}=f\left(2, \frac{\pi}{4}\right)-f(1, 0)=2\tan\frac{\pi}{4}-\tan 0=2$.

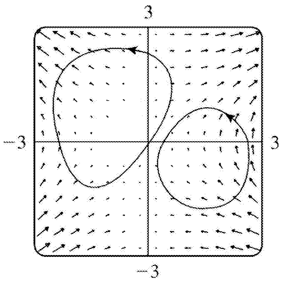
20. Here $\boldsymbol{F}(x, y)=(1-ye^{-x})\boldsymbol{i}+e^{-x}\boldsymbol{j}$. Then $f(x, y)=x+ye^{-x}$ is a potential function for $\boldsymbol{F}$, that is, $\nabla f=\boldsymbol{F}$ so $\boldsymbol{F}$ is conservative and thus its line integral is independent of path. Hence
$\int_C (1-ye^{-x})dx+e^{-x}dy=\int_C \boldsymbol{F}\cdot d\boldsymbol{r}=f(1, 2)-f(0, 1)=(1+2e^{-1})-1=2/e$.

21. $\boldsymbol{F}(x, y)=2y^{3/2}\boldsymbol{i}+3x\sqrt{y}\,\boldsymbol{j}$, $W=\int_C \boldsymbol{F}\cdot d\boldsymbol{r}$. Since $\partial(2y^{3/2})/\partial y=3\sqrt{y}=\partial(3x\sqrt{y})/\partial x$, there exists a function f such that $\nabla f=\boldsymbol{F}$. In fact, $f_x(x, y)=2y^{3/2} \Rightarrow f(x, y)=2xy^{3/2}+g(y) \Rightarrow f_y(x, y)=3xy^{1/2}+g'(y)$. But $f_y(x, y)=3x\sqrt{y}$ so $g'(y)=0$ or $g(y)=K$. We can take $K=0 \Rightarrow f(x, y)=2xy^{3/2}$. Thus
$W=\int_C \boldsymbol{F}\cdot d\boldsymbol{r}=f(2, 4)-f(1, 1)=2(2)(8)-2(1)=30$.

22. $\boldsymbol{F}(x, y)=\frac{y^2}{x^2}\boldsymbol{i}-\frac{2y}{x}\boldsymbol{j}$, $W=\int_C \boldsymbol{F}\cdot d\boldsymbol{r}$. Since $\frac{\partial}{\partial y}\left(\frac{y^2}{x^2}\right)=\frac{2y}{x^2}=\frac{\partial}{\partial x}\left(-\frac{2y}{x}\right)$, there exists a function f such that $\nabla f=\boldsymbol{F}$. In fact, $f_x=y^2/x^2 \Rightarrow f(x, y)=-y^2/x+g(y) \Rightarrow f_y=-2y/x+g'(y) \Rightarrow g'(y)=0$, so we can take $f(x, y)=-y^2/x$ as a potential function for $\boldsymbol{F}$. Thus
$W=\int_C \boldsymbol{F}\cdot d\boldsymbol{r}=f(4, -2)-f(1, 1)=-[(-2)^2/4]+(1/1)=0$.

23. We know that if the vector field (call it $\boldsymbol{F}$) is conservative, then around any closed path C , $\int_C \boldsymbol{F}\cdot d\boldsymbol{r}=0$. But take C to be some circle centered at the origin, oriented counterclockwise. All of the field vectors along C oppose motion along C , so the integral around C will be negative. Therefore the field is not conservative.

24.

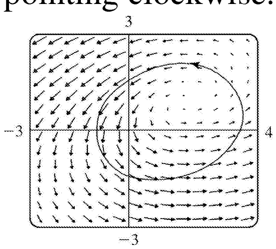


From the graph, it appears that $\boldsymbol{F}$ is conservative, since around all closed paths, the number and size of the field vectors pointing in directions similar to that of the path seem to be roughly the same as the number and size of the vectors pointing in the opposite direction. To check, we calculate

$$\frac{\partial}{\partial y}(2xy+\sin y)=2x+\cos y\ ,\ \frac{\partial}{\partial x}(x^2+x\cos y)=2x+\cos y$$

Thus $\boldsymbol{F}$ is conservative, by Theorem 6.

25. From the graph, it appears that $\boldsymbol{F}$ is not conservative. For example, any closed curve containing the point (2, 1) seems to have many field vectors pointing counterclockwise along it, and none pointing clockwise. So along this path the integral $\int \boldsymbol{F}\cdot d\boldsymbol{r}\neq 0$. To confirm our guess, we calculate



$$\frac{\partial}{\partial y}\left(\frac{x-2y}{\sqrt{1+x^2+y^2}}\right)=(x-2y)\left[\frac{-y}{(1+x^2+y^2)^{3/2}}\right]-\frac{2}{\sqrt{1+x^2+y^2}}=\frac{-2-2x^2-xy}{(1+x^2+y^2)^{3/2}}\ ,$$

$$\frac{\partial}{\partial x}\left(\frac{x-2}{\sqrt{1+x^2+y^2}}\right)=(x-2)\left[\frac{-x}{(1+x^2+y^2)^{3/2}}\right]+\frac{1}{\sqrt{1+x^2+y^2}}=\frac{1+y^2+2x}{(1+x^2+y^2)^{3/2}}\ .$$

These are not equal, so the field is not conservative, by Theorem 5.

26. $\nabla f(x,y)=\cos(x-2y)\boldsymbol{i}-2\cos(x-2y)\boldsymbol{j}$

(a) We use Theorem 2: $\int_{C_1}\boldsymbol{F}\cdot d\boldsymbol{r}=\int_{C_1}\nabla f\cdot d\boldsymbol{r}=f(\boldsymbol{r}(b))-f(\boldsymbol{r}(a))$ where C_1 starts at $t=a$ and ends at $t=b$.

So because $f(0,0)=\sin 0=0$ and $f(\pi,\pi)=\sin(\pi-2\pi)=0$, one possible curve C_1 is the straight line from (0, 0) to (π,π) ; that is, $\boldsymbol{r}(t)=\pi t\boldsymbol{i}+\pi t\boldsymbol{j}$, $0\leq t\leq 1$.

(b) From (a), $\int_{C_2}\boldsymbol{F}\cdot d\boldsymbol{r}=f(\boldsymbol{r}(b))-f(\boldsymbol{r}(a))$. So because $f(0,0)=\sin 0=0$ and $f\left(\frac{\pi}{2},0\right)=1$, one possible curve C_2 is $\boldsymbol{r}(t)=\frac{\pi}{2}t\boldsymbol{i}$, $0\leq t\leq 1$, the straight line from (0, 0) to $\left(\frac{\pi}{2},0\right)$.

27. Since $\boldsymbol{F}$ is conservative, there exists a function f such that $\boldsymbol{F}=\nabla f$, that is, $P=f_x$, $Q=f_y$, and $R=f_z$. Since P , Q and R have continuous first order partial derivatives, Clairaut's Theorem says that $\partial P/\partial y=f_{xy}=f_{yx}=\partial Q/\partial x$, $\partial P/\partial z=f_{xz}=f_{zx}=\partial R/\partial x$, and $\partial Q/\partial z=f_{yz}=f_{zy}=\partial R/\partial y$.

28. Here $\boldsymbol{F}(x, y, z)=y\boldsymbol{i}+x\boldsymbol{j}+xyz\boldsymbol{k}$. Then using the notation of Exercise 27, $\partial P/\partial z=0$ while $\partial R/\partial x=yz$. Since these aren't equal, $\boldsymbol{F}$ is not conservative. Thus by Theorem 4, the line integral of $\boldsymbol{F}$ is not independent of path.

29. $D=\{ (x, y) \mid x>0,\ y>0 \}$= the first quadrant (excluding the axes).
(a) D is open because around every point in D we can put a disk that lies in D .
(b) D is connected because the straight line segment joining any two points in D lies in D .
(c) D is simply–connected because it's connected and has no holes.

30. $D=\{ (x, y) \mid x\neq 0 \}$ consists of all points in the xy –plane except for those on the y –axis.
(a) D is open.
(b) Points on opposite sides of the y –axis cannot be joined by a path that lies in D , so D is not connected.
(c) D is not simply–connected because it is not connected.

31. $D=\{ (x, y) \mid 1<x^2+y^2<4 \}$= the annular region between the circles with center (0, 0) and radii 1 and 2 .
(a) D is open.
(b) D is connected.
(c) D is not simply–connected. For example, $x^2+y^2=(1.5)^2$ is simple and closed and lies within D but encloses points that are not in D . (Or we can say, D has a hole, so is not simply–connected.)

32. $D=\{ (x, y) \mid x^2+y^2\leq 1 \text{ or } 4\leq x^2+y^2\leq 9 \}$= the points on or inside the circle $x^2+y^2=1$, together with the points on or between the circles $x^2+y^2=4$ and $x^2+y^2=9$.
(a) D is not open because, for instance, no disk with center (0, 2) lies entirely within D .
(b) D is not connected because, for example, (0, 0) and (0, 2.5) lie in D but cannot be joined by a path that lies entirely in D .
(c) D is not simply–connected because, for example, $x^2+y^2=9$ is a simple closed curve in D but encloses points that are not in D .

33. **(a)** $P=-\dfrac{y}{x^2+y^2}$, $\dfrac{\partial P}{\partial y}=\dfrac{y^2-x^2}{(x^2+y^2)^2}$ and $Q=\dfrac{x}{x^2+y^2}$, $\dfrac{\partial Q}{\partial x}=\dfrac{y^2-x^2}{(x^2+y^2)^2}$. Thus $\dfrac{\partial P}{\partial y}=\dfrac{\partial Q}{\partial x}$.
(b) C_1 : $x=\cos t$, $y=\sin t$, $0\leq t\leq \pi$, C_2 : $x=\cos t$, $y=\sin t$, $t=2\pi$ to $t=\pi$. Then

$\int_{C_1} \boldsymbol{F}\cdot d\boldsymbol{r}=\int_0^{\pi} \frac{(-\sin t)(-\sin t)+(\cos t)(\cos t)}{\cos^2 t+\sin^2 t}\, dt=\int_0^{\pi} dt=\pi$ and $\int_{C_2} \boldsymbol{F}\cdot d\boldsymbol{r}=\int_{2\pi}^{\pi} dt=-\pi$

Since these aren't equal, the line integral of $\boldsymbol{F}$ isn't independent of path. (Or notice that $\int_{C_3} \boldsymbol{F}\cdot d\boldsymbol{r}=\int_0^{2\pi} dt=2\pi$ where C_3 is the circle $x^2+y^2=1$, and apply the contrapositive of Theorem 3). This doesn't contradict Theorem 6, since the domain of $\boldsymbol{F}$, which is R^2 except the origin, isn't simply–connected.

34. **(a)** Here $\boldsymbol{F}(\boldsymbol{r})=c\boldsymbol{r}/|\boldsymbol{r}|^3$ and $\boldsymbol{r}=x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k}$. Then $f(\boldsymbol{r})=-c/|\boldsymbol{r}|$ is a potential function for $\boldsymbol{F}$, that is, $\nabla f=\boldsymbol{F}$. (See the discussion of gradient fields in Section 17.1). Hence $\boldsymbol{F}$ is conservative and its line integral is independent of path. Let $P_1=(x_1, y_1, z_1)$ and $P_2=(x_2, y_2, z_2)$.

$$W=\int_C \boldsymbol{F}\cdot d\boldsymbol{r}=f(P_2)-f(P_1)=-\frac{c}{(x_2^2+y_2^2+z_2^2)^{1/2}}+\frac{c}{(x_1^2+y_1^2+z_1^2)^{1/2}}=c\left(\frac{1}{d_1}-\frac{1}{d_2}\right).$$

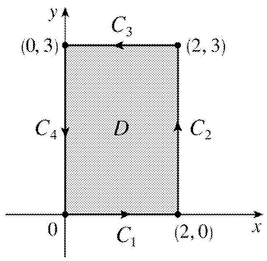
(b) In this case, $c=-(mMG)\Rightarrow$

$$\begin{aligned} W &= -mMG\left(\frac{1}{1.52\times 10^8}-\frac{1}{1.47\times 10^8}\right) \\ &= -(5.97\times 10^{24})(1.99\times 10^{30})(6.67\times 10^{-11})(-2.2377\times 10^{-10})\approx 1.77\times 10^{35}\text{J} \end{aligned}$$

(c) In this case, $c=\varepsilon qQ\Rightarrow$

$$W=\varepsilon qQ\left(\frac{1}{10^{-12}}-\frac{1}{5\times 10^{-13}}\right)=(8.985\times 10^{10})(1)(-1.6\times 10^{-19})(-10^{12})\approx 1.4\times 10^4\text{ J}.$$

1. **(a)**



C_1 : $x=t \Rightarrow dx=dt$, $y=0 \Rightarrow dy=0\,dt$, $0 \le t \le 2$.

C_2 : $x=2 \Rightarrow dx=0\,dt$, $y=t \Rightarrow dy=dt$, $0 \le t \le 3$.

C_3 : $x=2-t \Rightarrow dx=-dt$, $y=3 \Rightarrow dy=0\,dt$, $0 \le t \le 2$.

C_4 : $x=0 \Rightarrow dx=0\,dt$, $y=3-t \Rightarrow dy=-dt$, $0 \le t \le 3$.

Thus

$$\oint_C xy^2\,dx+x^3\,dy \quad \oint_{C_1+C_2+C_3+C_4} xy^2\,dx+x^3\,dy$$

$$=\int_0^2 0\,dt+\int_0^3 8\,dt+\int_0^2 -9(2-t)\,dt+\int_0^3 0\,dt$$

$$=0+24-18+0=6$$

(b)

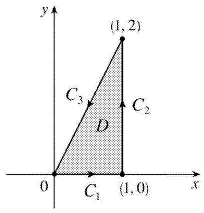
$$\oint_C xy^2\,dx+x^3\,dy=\iint_D \left[\frac{\partial}{\partial x}(x^3)-\frac{\partial}{\partial y}(xy^2) \right] dA=\int_0^2\int_0^3 (3x^2-2xy)\,dy\,dx$$

$$=\int_0^2 (9x^2-9x)\,dx=24-18=6$$

2. **(a)** $x=\cos t$, $y=\sin t$, $0 \le t \le 2\pi$. Then $\oint_C y\,dx-x\,dy=\int_0^{2\pi} [\sin t(-\sin t)-\cos t(\cos t)]\,dt=-\int_0^{2\pi} dt=-2\pi$.

(b) $\oint_C y\,dx-x\,dy=\iint_D \left[\frac{\partial}{\partial x}(-x)-\frac{\partial}{\partial y}(y) \right] dA=-2\iint_D dA=-2A(D)=-2\pi(1)^2=-2\pi$

3. **(a)**



$C_1 : x=t \Rightarrow dx=dt$, $y=0 \Rightarrow dy=0dt$, $0\leq t\leq 1$

$C_2 : x=1 \Rightarrow dx=0dt$, $y=t \Rightarrow dy=dt$, $0\leq t\leq 2$.

$C_3 : x=1-t \Rightarrow dx=-dt$, $y=2-2t \Rightarrow dy=-2dt$, $0\leq t\leq 1$.

Thus

$$\oint_C xy\,dx+x^2y^3\,dy = \oint_{C_1+C_2+C_3} xy\,dx+x^2y^3\,dy$$
$$=\int_0^1 0dt+\int_0^2 t^3dt+\int_0^1\Big[-(1-t)(2-2t)-2(1-t)^2(2-2t)^3\Big]dt$$
$$=0+\left[\frac{1}{4}t^4\right]_0^2+\left[\frac{2}{3}(1-t)^3+\frac{8}{3}(1-t)^6\right]_0^1=4-\frac{10}{3}=\frac{2}{3}$$

(b)

$$\oint_C xy\,dx+x^2y^3\,dy=\iint_D\left[\frac{\partial}{\partial x}(x^2y^3)-\frac{\partial}{\partial y}(xy)\right]dA=\int_0^1\int_0^{2x}(2xy^3-x)\,dy\,dx$$
$$=\int_0^1\left[\frac{1}{2}xy^4-xy\right]_{y=0}^{y=2x}dx=\int_0^1(8x^5-2x^2)dx=\frac{4}{3}-\frac{2}{3}=\frac{2}{3}$$

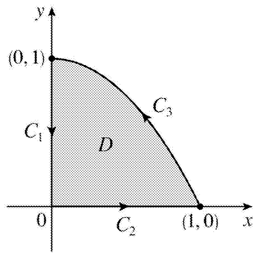
4. (a)

C_1:x=0 $\Rightarrow dx=0dt$, $y=1-t\Rightarrow dy=-dt$, $0\leq t\leq 1$

C_2:x=$t \Rightarrow dx=dt$, $y=0\Rightarrow dy=0dt$, $0\leq t\leq 1$

C_3:x=$1-t \Rightarrow dx=-dt$, $y=1-(1-t)^2=2t-t^2 \Rightarrow dy=(2-2t)dt$, $0\leq t\leq 1$

Thus



$$\oint_C x\,dx+y\,dy=\oint_{C_1+C_2+C_3} x\,dx+y\,dy$$
$$=\int_0^1 (0\,dt+(1-t)(-dt))+\int_0^1 (t\,dt+0\,dt)+\int_0^1 ((1-t)(-dt)+(2t-t^2)(2-2t)\,dt)$$
$$=\left[\frac{1}{2}t^2-t\right]_0^1+\left[\frac{1}{2}t^2\right]_0^1+\left[\frac{1}{2}t^4-2t^3+\frac{5}{2}t^2-t\right]_0^1$$
$$=-\frac{1}{2}+\frac{1}{2}+\left(\frac{1}{2}-2+\frac{5}{2}-1\right)=0$$

(b) $\oint_C x\,dx+y\,dy=\iint_D\left[\frac{\partial}{\partial x}(y)-\frac{\partial}{\partial y}(x)\right]dA=\iint_D 0\,dA=0$

5. We can parametrize C as $x=\cos\theta$, $y=\sin\theta$, $0\le\theta\le 2\pi$. Then the line integral is

$\oint_C P\,dx+Q\,dy=\int_0^{2\pi}\cos^4\theta\sin^5\theta\,(-\sin\theta)\,d\theta+\int_0^{2\pi}(-\cos^7\theta\sin^6\theta)\cos\theta\,d\theta=-\frac{29\pi}{1024}$, according to a CAS.

The double integral is

$$\iint_D\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dA=\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}(-7x^6y^6-5x^4y^4)\,dy\,dx=-\frac{29\pi}{1024} ,$$

verifying Green's Theorem in this case.

6.

Since $y=x^2$ along the first part of C and $y=x$ along the second part, the line integral is

$$\oint_C P\,dx+Q\,dy=\int_0^1\left[x^4\sin x+x^2\sin(x^2)(2x)\right]dx+\int_1^0(x^2\sin x+x^2\sin x)\,dx$$
$$=-16\cos 1-23\sin 1+28$$

according to a CAS. The double integral is

$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \int_0^1 \int_{x^2}^{x} (2x\sin y - 2y\sin x)\,dy\,dx = -16\cos 1 - 23\sin 1 + 28$$

7. The region D enclosed by C is $[0, 1] \times [0, 1]$, so

$$\int_C e^y dx + 2xe^y dy = \iint_D \left[\frac{\partial}{\partial x} (2xe^y) - \frac{\partial}{\partial y} (e^y) \right] dA = \int_0^1 \int_0^1 (2e^y - e^y)\,dy\,dx$$

$$= \int_0^1 dx \int_0^1 e^y dy = (1)(e^1 - e^0) = e - 1$$

8. The region D enclosed by C is given by $\{ (x, y) \mid 0 \le x \le 1,\ 3x \le y \le 3 \}$, so

$$\int_C x^2 y^2 dx + 4xy^3 dy = \iint_D \left[\frac{\partial}{\partial x} (4xy^3) - \frac{\partial}{\partial y} (x^2 y^2) \right] dA = \int_0^1 \int_{3x}^{3} (4y^3 - 2x^2 y)\,dy\,dx$$

$$= \int_0^1 \left[y^4 - x^2 y^2 \right]_{y=3x}^{y=3} dx = \int_0^1 (81 - 9x^2 - 72x^4)\,dx = 81 - 3 - \frac{72}{5} = \frac{318}{5}$$

9.

$$\int_C (y + e^{\sqrt{x}})dx + (2x + \cos y^2)dy = \iint_D \left[\frac{\partial}{\partial x} (2x + \cos y^2) - \frac{\partial}{\partial y} (y + e^{\sqrt{x}}) \right] dA$$

$$= \int_0^1 \int_{y^2}^{\sqrt{y}} (2 - 1)\,dx\,dy = \int_0^1 (y^{1/2} - y^2)\,dy = \frac{1}{3}$$

10.

$$\int_C xe^{-2x} dx + (x^4 + 2x^2 y^2)\,dy = \iint_D \left[\frac{\partial}{\partial x} (x^4 + 2x^2 y^2) - \frac{\partial}{\partial y} (xe^{-2x}) \right] dA = \iint_D (4x^3 + 4xy^2 - 0)\,dA$$

$$= 4\iint_D x(x^2 + y^2)\,dA = 4\int_0^{2\pi} \int_1^2 (r\cos\theta)(r^2)\,r\,dr\,d\theta$$

$$= 4\int_0^{2\pi} \cos\theta\,d\theta \int_1^2 r^4 dr = 4\left[\sin\theta\right]_0^{2\pi} \left[\frac{1}{5} r^5 \right]_1^2 = 0$$

11.

$$\int_C y^3 dx - x^3 dy = \iint_D \left[\frac{\partial}{\partial x}(-x^3) - \frac{\partial}{\partial y}(y^3) \right] dA = \iint_D (-3x^2 - 3y^2) dA = \int_0^{2\pi} \int_0^2 (-3r^2) r\,dr\,d\theta$$

$$= -3 \int_0^{2\pi} d\theta \int_0^2 r^3 dr = -3(2\pi)(4) = -24\pi$$

12. $\int_C \sin y\,dx + x\cos y\,dy = \iint_D \left[\frac{\partial}{\partial x}(x\cos y) - \frac{\partial}{\partial y}(\sin y) \right] dA = \iint_D (\cos y - \cos y) dA = \iint_D 0\,dA = 0$

13. $\boldsymbol{F}(x, y) = \left\langle \sqrt{x} + y^3, x^2 + \sqrt{y} \right\rangle$ and the region D enclosed by C is given by $\{ (x, y) \mid 0 \le x \le \pi, 0 \le y \le \sin x \}$. C is traversed clockwise, so $-C$ gives the positive orientation.

$$\int_C \boldsymbol{F} \cdot d\boldsymbol{r} = -\int_{-C} \left(\sqrt{x} + y^3\right) dx + \left(x^2 + \sqrt{y}\right) dy = -\iint_D \left[\frac{\partial}{\partial x}\left(x^2 + \sqrt{y}\right) - \frac{\partial}{\partial y}\left(\sqrt{x} + y^3\right) \right] dA$$

$$= -\int_0^{\pi} \int_0^{\sin x} (2x - 3y^2) dy\,dx = -\int_0^{\pi} \left[2xy - y^3 \right]_{y=0}^{y=\sin x} dx$$

$$= -\int_0^{\pi} (2x\sin x - \sin^3 x) dx = -\int_0^{\pi} (2x\sin x - (1 - \cos^2 x)\sin x) dx$$

$$= -\left[2\sin x - 2x\cos x + \cos x - \frac{1}{3}\cos^3 x \right]_0^{\pi}$$

$$= -\left(2\pi - 2 + \frac{2}{3} \right) = \frac{4}{3} - 2\pi$$

14. $\boldsymbol{F}(x, y) = \left\langle y^2 \cos x, x^2 + 2y\sin x \right\rangle$ and the region D enclosed by C is given by $\{ (x, y) \mid 0 \le x \le 2, 0 \le y \le 3x \}$. C is traversed clockwise, so $-C$ gives the positive orientation.

$$\int_C \boldsymbol{F} \cdot d\boldsymbol{r} = -\int_{-C} (y^2 \cos x) dx + (x^2 + 2y\sin x) dy$$

$$= -\iint_D \left[\frac{\partial}{\partial x}(x^2 + 2y\sin x) - \frac{\partial}{\partial y}(y^2 \cos x) \right] dA$$

$$= -\iint_D (2x + 2y\cos x - 2y\cos x) dA = -\int_0^2 \int_0^{3x} 2x\,dy\,dx$$

$$=-\int_0^2 2x\left[y\right]_{y=0}^{y=3x}dx=-\int_0^2 6x^2dx=\left.-2x^3\right]_0^2=-16$$

15. $\boldsymbol{F}(x, y)=\left\langle e^x+x^2y, e^y-xy^2\right\rangle$ and the region D enclosed by C is the disk $x^2+y^2\leq 25$. C is traversed clockwise, so $-C$ gives the positive orientation.

$$\int_C \boldsymbol{F}\cdot d\boldsymbol{r}=-\int_{-C}(e^x+x^2y)dx+(e^y-xy^2)dy$$

$$=-\iint_D\left[\frac{\partial}{\partial x}(e^y-xy^2)-\frac{\partial}{\partial y}(e^x+x^2y)\right]dA=-\iint_D(-y^2-x^2)dA$$

$$=\iint_D(x^2+y^2)dA=\int_0^{2\pi}\int_0^5(r^2)r\,dr\,d\theta=\int_0^{2\pi}d\theta\int_0^5 r^3dr=2\pi\left[\frac{1}{4}r^4\right]_0^5=\frac{625}{2}\pi$$

16. $\boldsymbol{F}(x, y)=\left\langle y-\ln(x^2+y^2), 2\tan^{-1}\left(\frac{y}{x}\right)\right\rangle$ and the region D enclosed by C is the disk with radius 1 centered at (2, 3). C is oriented positively, so

$$\int_C \boldsymbol{F}\cdot d\boldsymbol{r}=\int_C(y-\ln(x^2+y^2))dx+\left(2\tan^{-1}\left(\frac{y}{x}\right)\right)dy$$

$$=\iint_D\left[\frac{\partial}{\partial x}\left(2\tan^{-1}\left(\frac{y}{x}\right)\right)-\frac{\partial}{\partial y}(y-\ln(x^2+y^2))\right]dA$$

$$=\iint_D\left[2\left(\frac{-yx^{-2}}{1+(y/x)^2}\right)-\left(1-\frac{2y}{x^2+y^2}\right)\right]dA=\iint_D\left[-\frac{2y}{x^2+y^2}-1+\frac{2y}{x^2+y^2}\right]dA$$

$$=-\iint_D dA=-(\text{ area of } D)=-\pi$$

17. By Green's Theorem, $W=\int_C \boldsymbol{F}\cdot d\boldsymbol{r}=\int_C x(x+y)dx+xy^2dy=\iint_D(y^2-x)dy\,dx$ where C is the path described in the question and D is the triangle bounded by C . So

$$W=\int_0^1\int_0^{1-x}(y^2-x)dy\,dx=\int_0^1\left[\frac{1}{3}y^3-xy\right]_{y=0}^{y=1-x}dx=\int_0^1\left(\frac{1}{3}(1-x)^3-x(1-x)\right)dx$$

$$= \left[-\frac{1}{12}(1-x)^4 - \frac{1}{2}x^2 + \frac{1}{3}x^3 \right]_0^1 = \left(-\frac{1}{2} + \frac{1}{3} \right) - \left(-\frac{1}{12} \right) = -\frac{1}{12}$$

18. By Green's Theorem, $W = \int_C \boldsymbol{F} \cdot d\boldsymbol{r} = \int_C x\,dx + (x^3 + 3xy^2)\,dy = \iint_D (3x^2 + 3y^2 - 0)\,dA$, where D is the semicircular region bounded by C . Converting to polar coordinates, we have

$W = 3\int_0^2 \int_0^\pi r^2 \cdot r\,d\theta\,dr = 3\pi \left[\frac{1}{4} r^4 \right]_0^2 = 12\pi$.

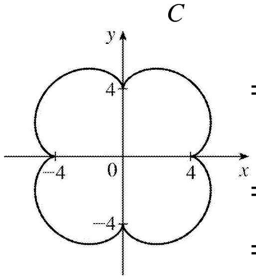
19. Let C_1 be the arch of the cycloid from (0, 0) to $(2\pi, 0)$, which corresponds to $0 \leq t \leq 2\pi$, and let C_2 be the segment from $(2\pi, 0)$ to (0, 0) , so C_2 is given by $x = 2\pi - t$, $y = 0$, $0 \leq t \leq 2\pi$. Then $C = C_1 \cup C_2$ is traversed
clockwise, so $-C$ is oriented positively. Thus $-C$ encloses the area under one arch of the cycloid and from (5) we have

$$A = -\oint_{-C} y\,dx = \int_{C_1} y\,dx + \int_{C_2} y\,dx = \int_0^{2\pi} (1 - \cos t)(1 - \cos t)\,dt + \int_0^{2\pi} 0(-dt)$$

$$= \int_0^{2\pi} (1 - 2\cos t + \cos^2 t)\,dt + 0 = \left[t - 2\sin t + \frac{1}{2}t + \frac{1}{4}\sin 2t \right]_0^{2\pi} = 3\pi$$

20.

$$A = \oint_C x\,dy = \int_0^{2\pi} (5\cos t - \cos 5t)(5\cos t - 5\cos 5t)\,dt$$



$$= \int_0^{2\pi} (25\cos^2 t - 30\cos t\cos 5t + 5\cos^2 5t)\,dt$$

$$= \left[25\left(\frac{1}{2}t + \frac{1}{4}\sin 2t \right) - 30\left(\frac{1}{8}\sin 4t + \frac{1}{12}\sin 6t \right) + 5\left(\frac{1}{2}t + \frac{1}{20}\sin 10t \right) \right]_0^{2\pi}$$

$$= 30\pi$$

21. **(a)** Using Equation 17.2.8 , we write parametric equations of the line segment as $x = (1 - t)x_1 + tx_2$, $y = (1 - t)y_1 + ty_2$, $0 \leq t \leq 1$. Then $dx = (x_2 - x_1)dt$ and $dy = (y_2 - y_1)dt$, so

$$\int_C x\,dy - y\,dx = \int_0^1 [(1 - t)x_1 + tx_2](y_2 - y_1)dt + [(1 - t)y_1 + ty_2](x_2 - x_1)dt$$

$$= \int_0^1 (x_1(y_2-y_1)-y_1(x_2-x_1)+t[(y_2-y_1)(x_2-x_1)-(x_2-x_1)(y_2-y_1)])dt$$

$$= \int_0^1 (x_1y_2-x_2y_1)dt=x_1y_2-x_2y_1$$

(b) We apply Green's Theorem to the path $C=C_1 \cup C_2 \cup \cdots \cup C_n$, where C_i is the line segment that joins (x_i, y_i) to (x_{i+1}, y_{i+1}) for $i=1, 2, \ldots, n-1$, and C_n is the line segment that joins (x_n, y_n) to (x_1, y_1). From (5), $\frac{1}{2}\int_C xdy-ydx=\iint_D dA$, where D is the polygon bounded by C. Therefore

$$\text{area of polygon } =A(D)=\iint_D dA=\frac{1}{2}\int_C xdy-ydx$$

$$=\frac{1}{2}\left(\int_{C_1} xdy-ydx+\int_{C_2} xdy-ydx+\ldots+\int_{C_{n-1}} xdy-ydx+\int_{C_n} xdy-ydx\right)$$

To evaluate these integrals we use the formula from (a) to get
$A(D)=[(x_1y_2-x_2y_1)+(x_2y_3-x_3y_2)+\ldots+(x_{n-1}y_n-x_ny_{n-1})+(x_ny_1-x_1y_n)]$.

(c)

$$A=\frac{1}{2}[(0\cdot 1-2\cdot 0)+(2\cdot 3-1\cdot 1)+(1\cdot 2-0\cdot 3)+(0\cdot 1-(-1)\cdot 2)+(-1\cdot 0-0\cdot 1)]$$

$$=\frac{1}{2}(0+5+2+2)=\frac{9}{2}$$

22. By Green's Theorem, $\frac{1}{2A}\oint_C x^2dy=\frac{1}{2A}\iint_D 2xdA=\frac{1}{A}\iint_D xdA=\bar{x}$ and

$-\frac{1}{2A}\oint_C y^2dx=-\frac{1}{2A}\iint_D(-2y)dA=\frac{1}{A}\iint_D ydA=\bar{y}$.

23. Here $A=\frac{1}{2}(1)(1)=\frac{1}{2}$ and $C=C_1+C_2+C_3$, where $C_1: x=x$, $y=0$, $0\le x\le 1$; $C_2: x=x$, $y=1-x$, $x=1$ to $x=0$; and $C_3: x=0$, $y=1$ to $y=0$. Then

$\bar{x}=\frac{1}{2A}\int_C x^2dy=\int_{C_1} x^2dy+\int_{C_2} x^2dy+\int_{C_3} x^2dy=0+\int_1^0 (x^2)(-dx)+0=\frac{1}{3}$. Similarly,

$\bar{y}=-\frac{1}{2A}\int_C y^2dx=\int_{C_1} y^2dx+\int_{C_2} y^2dx+\int_{C_3} y^2dx=0+\int_1^0 (1-x)^2(-dx)+0=\frac{1}{3}$.

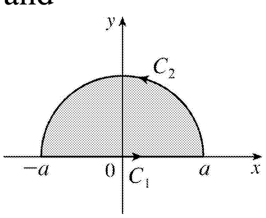
Therefore $(\bar{x},\bar{y})=\left(\frac{1}{3},\frac{1}{3}\right)$.

24. $A=\frac{\pi a^2}{2}$ so $\bar{x}=\frac{1}{\pi a^2}\oint_C x^2dy$ and $\bar{y}=-\frac{1}{\pi a^2}\oint_C y^2dx$.

Orienting the semicircular region as in the figure,

$$\bar{x}=\frac{1}{\pi a^2}\oint_{C_1+C_2} x^2dy$$
$$=\frac{1}{\pi a^2}\left[0+\int_0^\pi (a^2\cos^2 t)(a\cos t)dt\right]=0$$

and



$\bar{y}=-\frac{1}{\pi a^2}\left[\int_{-a}^a 0dx+\int_0^\pi (a^2\sin^2 t)(-a\sin t)dt\right]=\frac{a}{\pi}\int_0^\pi \sin^3 tdt=\frac{a}{\pi}\left[-\cos t+\frac{1}{3}(\cos^3 t)\right]_0^\pi=\frac{4a}{3\pi}$.

Thus $(\bar{x},\bar{y})=\left(0,\frac{4a}{3\pi}\right)$.

25. By Green's Theorem, $-\frac{1}{3}\rho\oint_C y^3dx=-\frac{1}{3}\rho\int\int_D(-3y^2)dA=\int\int_D y^2\rho dA=I_x$ and

$\frac{1}{3}\rho\oint_C x^3dy=\frac{1}{3}\rho\int\int_D(3x^2)dA=\int\int_D x^2\rho dA=I_y$.

26. By symmetry the moments of inertia about any two diameters are equal. Centering the disk at the origin, the moment of inertia about a diameter equals

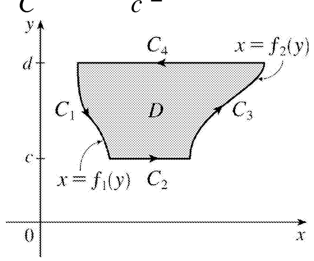
$$I_y = \frac{1}{3}\rho\oint_C x^3 dy = \frac{1}{3}\rho\int_0^{2\pi}(a^4\cos^4 t)dt = \frac{1}{3}a^4\rho\int_0^{2\pi}\left[\frac{3}{8}+\frac{1}{2}\cos 2t+\frac{1}{8}\cos 4t\right]dt$$

$$= \frac{1}{3}a^4\rho\cdot\frac{3(2\pi)}{8} = \frac{1}{4}\pi a^4\rho$$

27. Since C is a simple closed path which doesn't pass through or enclose the origin, there exists an open region that doesn't contain the origin but does contain D . Thus $P=-y/(x^2+y^2)$ and $Q=x/(x^2+y^2)$ have continuous partial derivatives on this open region containing D and we can apply Green's Theorem. But by Exercise 17.3.33(a) , $\partial P/\partial y=\partial Q/\partial x$, so $\oint_C \mathbf{F}\cdot d\mathbf{r}=\iint_D 0dA=0$.

28. We express D as a type II region: $D=\{ (x, y) \mid f_1(y)\leq x\leq f_2(y), c\leq y\leq d \}$ where f_1 and f_2 are continuous functions. Then $\iint_D \frac{\partial Q}{\partial x}\, dA=\int_c^d\int_{f_1(y)}^{f_2(y)} \frac{\partial Q}{\partial x}\, dxdy=\int_c^d\left[Q(f_2(y), y)-Q(f_1(y), y)\right]dy$ by the Fundamental Theorem of Calculus. But referring to the figure, $\oint_C Qdy=\oint_{C_1+C_2+C_3+C_4} Qdy$. Then $\int_{C_1} Qdy=\int_d^c Q(f_1(y), y)dy$, $\int_{C_2} Qdy=\int_{C_4} Qdy=0$, and $\int_{C_3} Qdy=\int_c^d Q(f_2(y), y)dy$. Hence

$$\oint_C Qdy=\int_c^d\left[Q(f_2(y), y)-Q(f_1(y), y)\right]dy=\iint_D(\partial Q/\partial x)dA.$$



29. Using the first part of (5), we have that $\iint_R dxdy=A(R)=\int_{\partial R} xdy$. But $x=g(u, v)$, and $dy=\frac{\partial h}{\partial u}\, du+\frac{\partial h}{\partial v}\, dv$, and we orient ∂S by taking the positive direction to be that which corresponds, under the mapping, to the positive direction along ∂R , so

$$\int_{\partial R} x\,dy = \int_{\partial S} g(u,v)\left(\frac{\partial h}{\partial u}\,du + \frac{\partial h}{\partial v}\,dv\right) = \int_{\partial S} g(u,v)\frac{\partial h}{\partial u}\,du + g(u,v)\frac{\partial h}{\partial v}\,dv$$

$$= \pm\iint_S \left[\frac{\partial}{\partial u}\left(g(u,v)\frac{\partial h}{\partial v}\right) - \frac{\partial}{\partial v}\left(g(u,v)\frac{\partial h}{\partial u}\right)\right] dA$$ [using Green's Theorem in the uv – plane]

$$= \pm\iint_S \left(\frac{\partial g}{\partial u}\frac{\partial h}{\partial v} + g(u,v)\frac{\partial^2 h}{\partial u\,\partial v} - \frac{\partial g}{\partial v}\frac{\partial h}{\partial u} - g(u,v)\frac{\partial^2 h}{\partial v\,\partial u}\right) dA$$

$$= \pm\iint_S \left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right) dA$$ [by the equality of mixed partials] $$= \pm\iint_S \frac{\partial(x,y)}{\partial(u,v)}\,du\,dv$$

The sign is chosen to be positive if the orientation that we gave to ∂S corresponds to the usual positive orientation, and it is negative otherwise. In either case, since $A(R)$ is positive, the sign chosen must be the same as the sign of $\frac{\partial(x,y)}{\partial(u,v)}$. Therefore $A(R) = \iint_R dx\,dy = \iint_S \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$.

1. **(a)**

$$\text{curl}\,\boldsymbol{F}=\nabla\times\boldsymbol{F}=\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ xyz & 0 & -x^2y\end{vmatrix}=(-x^2-0)\boldsymbol{i}-(-2xy-xy)\boldsymbol{j}+(0-xz)\boldsymbol{k}$$

$$=-x^2\boldsymbol{i}+3xy\boldsymbol{j}-xz\boldsymbol{k}$$

(b) $\text{div}\,\boldsymbol{F}=\nabla\cdot\boldsymbol{F}=\frac{\partial}{\partial x}(xyz)+\frac{\partial}{\partial y}(0)+\frac{\partial}{\partial z}(-x^2y)=yz+0+0=yz$

2. **(a)**

$$\text{curl}\boldsymbol{F}=\nabla\times\boldsymbol{F}=\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ x^2yz & xy^2z & xyz^2\end{vmatrix}=(xz^2-xy^2)\boldsymbol{i}-(yz^2-x^2y)\boldsymbol{j}+(y^2z-x^2z)\boldsymbol{k}$$

$$=x(z^2-y^2)\boldsymbol{i}+y(x^2-z^2)\boldsymbol{j}+z(y^2-x^2)\boldsymbol{k}$$

(b) $\text{div}\,\boldsymbol{F}=\nabla\cdot\boldsymbol{F}=\frac{\partial}{\partial x}(x^2yz)+\frac{\partial}{\partial y}(xy^2z)+\frac{\partial}{\partial z}(xyz^2)=2xyz+2xyz+2xyz=6xyz$

3. **(a)**

$$\text{curl}\boldsymbol{F}=\nabla\times\boldsymbol{F}=\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ 1 & x+yz & xy-\sqrt{z}\end{vmatrix}=(x-y)\boldsymbol{i}-(y-0)\boldsymbol{j}+(1-0)\boldsymbol{k}$$

$$=(x-y)\boldsymbol{i}-y\boldsymbol{j}+\boldsymbol{k}$$

(b) $\text{div}\,\boldsymbol{F}=\nabla\cdot\boldsymbol{F}=\frac{\partial}{\partial x}(1)+\frac{\partial}{\partial y}(x+yz)+\frac{\partial}{\partial z}(xy-\sqrt{z})=z-\frac{1}{2\sqrt{z}}$

4. **(a)**

$$\text{curl}\boldsymbol{F}=\nabla\times\boldsymbol{F}=\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ 0 & \cos xz & -\sin xy\end{vmatrix}$$

$$=(-x\cos xy+x\sin xz)\boldsymbol{i}-(-y\cos xy-0)\boldsymbol{j}+(-z\sin xz-0)\boldsymbol{k}$$

$$=x(\sin xz-\cos xy)\boldsymbol{i}+y\cos xy\,\boldsymbol{j}-z\sin xz\,\boldsymbol{k}$$

(b) $\text{div}\,\boldsymbol{F}=\nabla\cdot\boldsymbol{F}=\frac{\partial}{\partial x}(0)+\frac{\partial}{\partial y}(\cos xz)+\frac{\partial}{\partial z}(-\sin xy)=0+0+0=0$

5. **(a)** curl $\mathbf{F}=\nabla\times\mathbf{F}=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ e^x \sin y & e^x \cos y & z \end{vmatrix}=(0-0)\mathbf{i}-(0-0)\mathbf{j}+(e^x\cos y-e^x\cos y)\mathbf{k}=\mathbf{0}$

(b) div $\mathbf{F}=\nabla\cdot\mathbf{F}=\frac{\partial}{\partial x}(e^x\sin y)+\frac{\partial}{\partial y}(e^x\cos y)+\frac{\partial}{\partial z}(z)=e^x\sin y-e^x\sin y+1=1$

6. **(a)**

$$\text{curl }\mathbf{F}=\nabla\times\mathbf{F}=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \frac{x}{x^2+y^2+z^2} & \frac{y}{x^2+y^2+z^2} & \frac{z}{x^2+y^2+z^2} \end{vmatrix}$$

$$=\frac{1}{(x^2+y^2+z^2)^2}[(-2yz+2yz)\mathbf{i}-(-2xz+2xz)\mathbf{j}+(-2xy+2xy)\mathbf{k}]=\mathbf{0}$$

(b)

$$\text{div }\mathbf{F}=\nabla\cdot\mathbf{F}=\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2+z^2}\right)+\frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2+z^2}\right)+\frac{\partial}{\partial z}\left(\frac{z}{x^2+y^2+z^2}\right)$$

$$=\frac{x^2+y^2+z^2-2x^2}{(x^2+y^2+z^2)^2}+\frac{x^2+y^2+z^2-2y^2}{(x^2+y^2+z^2)^2}+\frac{x^2+y^2+z^2-2z^2}{(x^2+y^2+z^2)^2}$$

$$=\frac{x^2+y^2+z^2}{(x^2+y^2+z^2)^2}=\frac{1}{x^2+y^2+z^2}$$

7. **(a)**

$$\text{curl}\mathbf{F}=\nabla\times\mathbf{F}=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \ln x & \ln(xy) & \ln(xyz) \end{vmatrix}=\left(\frac{xz}{xyz}-0\right)\mathbf{i}-\left(\frac{yz}{xyz}-0\right)\mathbf{j}+\left(\frac{y}{xy}-0\right)\mathbf{k}$$

$$=\left\langle \frac{1}{y},-\frac{1}{x},\frac{1}{x}\right\rangle$$

(b) div $\mathbf{F}=\nabla\cdot\mathbf{F}=\frac{\partial}{\partial x}(\ln x)+\frac{\partial}{\partial y}(\ln(xy))+\frac{\partial}{\partial z}(\ln(xyz))=\frac{1}{x}+\frac{x}{xy}+\frac{xy}{xyz}=\frac{1}{x}+\frac{1}{y}+\frac{1}{z}$

8. **(a)**

$$\operatorname{curl}\boldsymbol{F} = \nabla \times \boldsymbol{F} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xe^{-y} & xz & ze^{y} \end{vmatrix} = (ze^{y}-x)\boldsymbol{i}-(0-0)\boldsymbol{j}+(z-xe^{-y}(-1))\boldsymbol{k}$$

$$=\left\langle ze^{y}-x,\ 0,\ z+xe^{-y}\right\rangle$$

(b) div $\boldsymbol{F}=\nabla\cdot\boldsymbol{F}=\frac{\partial}{\partial x}(xe^{-y})+\frac{\partial}{\partial y}(xz)+\frac{\partial}{\partial z}(ze^{y})=e^{-y}+0+e^{y}=e^{y}+e^{-y}$

9. If the vector field is $\boldsymbol{F}=P\boldsymbol{i}+Q\boldsymbol{j}+R\boldsymbol{k}$, then we know $R=0$. In addition, the x –component of each vector of $\boldsymbol{F}$ is 0 , so $P=0$, hence $\frac{\partial P}{\partial x}=\frac{\partial P}{\partial y}=\frac{\partial P}{\partial z}=\frac{\partial R}{\partial x}=\frac{\partial R}{\partial y}=\frac{\partial R}{\partial z}=0$. Q decreases as y increases, so $\frac{\partial Q}{\partial y}<0$, but Q doesn't change in the x – or z –directions, so $\frac{\partial Q}{\partial x}=\frac{\partial Q}{\partial z}=0$.

(a) div$\boldsymbol{F}=\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}=0+\frac{\partial Q}{\partial y}+0<0$

(b) curl$\boldsymbol{F}=\left(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}\right)\boldsymbol{i}+\left(\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x}\right)\boldsymbol{j}+\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)\boldsymbol{k}=(0-0)\boldsymbol{i}+(0-0)\boldsymbol{j}+(0-0)\boldsymbol{k}=\boldsymbol{0}$

10. If the vector field is $\boldsymbol{F}=P\boldsymbol{i}+Q\boldsymbol{j}+R\boldsymbol{k}$, then we know $R=0$. In addition, P and Q don't vary in the z –direction, so $\frac{\partial R}{\partial x}=\frac{\partial R}{\partial y}=\frac{\partial R}{\partial z}=\frac{\partial P}{\partial z}=\frac{\partial Q}{\partial z}=0$. As x increases, the x –component of each vector of $\boldsymbol{F}$ increases while the y –component remains constant, so $\frac{\partial P}{\partial x}>0$ and $\frac{\partial Q}{\partial x}=0$. Similarly, as y increases, the y –component of each vector increases while the x –component remains constant, so $\frac{\partial Q}{\partial y}>0$ and $\frac{\partial P}{\partial y}=0$.

(a) div$\boldsymbol{F}=\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}=\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+0>0$

(b) curl$\boldsymbol{F}=\left(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}\right)\boldsymbol{i}+\left(\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x}\right)\boldsymbol{j}+\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)\boldsymbol{k}=(0-0)\boldsymbol{i}+(0-0)\boldsymbol{j}+(0-0)\boldsymbol{k}=\boldsymbol{0}$

11. If the vector field is $\boldsymbol{F}=P\boldsymbol{i}+Q\boldsymbol{j}+R\boldsymbol{k}$, then we know $R=0$. In addition, the y –component of each vector of $\boldsymbol{F}$ is 0 , so $Q=0$, hence $\frac{\partial Q}{\partial x}=\frac{\partial Q}{\partial y}=\frac{\partial Q}{\partial z}=\frac{\partial R}{\partial x}=\frac{\partial R}{\partial y}=\frac{\partial R}{\partial z}=0$. P increases as y increases, so $\frac{\partial P}{\partial y}>0$, but P doesn't change in the x – or z –directions, so $\frac{\partial P}{\partial x}=\frac{\partial P}{\partial z}=0$.

(a) $\operatorname{div}\boldsymbol{F}=\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}=0+0+0=0$

(b)

$$\operatorname{curl}\boldsymbol{F}=\left(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}\right)\boldsymbol{i}+\left(\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x}\right)\boldsymbol{j}+\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)\boldsymbol{k}$$

$$=(0-0)\boldsymbol{i}+(0-0)\boldsymbol{j}+\left(0-\frac{\partial P}{\partial y}\right)\boldsymbol{k}=-\frac{\partial P}{\partial y}\boldsymbol{k}$$

Since $\frac{\partial P}{\partial y}>0$, $-\frac{\partial P}{\partial y}\boldsymbol{k}$ is a vector pointing in the negative z –direction.

12. **(a)** $\operatorname{curl} f=\nabla\times f$ is meaningless because f is a scalar field.
(b) $\operatorname{grad} f$ is a vector field.
(c) $\operatorname{div}\boldsymbol{F}$ is a scalar field.
(d) $\operatorname{curl}(\operatorname{grad} f)$ is a vector field.
(e) $\operatorname{grad}\boldsymbol{F}$ is meaningless because $\boldsymbol{F}$ is not a scalar field.
(f) $\operatorname{grad}(\operatorname{div}\boldsymbol{F})$ is a vector field.
(g) $\operatorname{div}(\operatorname{grad} f)$ is a scalar field.
(h) $\operatorname{grad}(\operatorname{div} f)$ is meaningless because f is a scalar field.
(i) $\operatorname{curl}(\operatorname{curl}\boldsymbol{F})$ is a vector field.
(j) $\operatorname{div}(\operatorname{div}\boldsymbol{F})$ is meaningless because $\operatorname{div}\boldsymbol{F}$ is a scalar field.
(k) $(\operatorname{grad} f)\times(\operatorname{div}\boldsymbol{F})$ is meaningless because $\operatorname{div}\boldsymbol{F}$ is a scalar field.
(l) $\operatorname{div}(\operatorname{curl}(\operatorname{grad} f))$ is a scalar field.

13. $\operatorname{curl}\boldsymbol{F}=\nabla\times\boldsymbol{F}=\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ yz & xz & xy\end{vmatrix}=(x-x)\boldsymbol{i}-(y-y)\boldsymbol{j}+(z-z)\boldsymbol{k}=\boldsymbol{0}$

and $\boldsymbol{F}$ is defined on all of R^3 with component functions which have continuous partial derivatives, so by Theorem 4, $\boldsymbol{F}$ is conservative. Thus, there exists a function f such that $\boldsymbol{F}=\nabla f$. Then $f_x(x, y, z)=yz$ implies $f(x, y, z)=xyz+g(y, z)$ and $f_y(x, y, z)=xz+g_y(y, z)$. But $f_y(x, y, z)=xz$, so $g(y, z)=h(z)$ and $f(x, y, z)=xyz+h(z)$. Thus $f_z(x, y, z)=xy+h'(z)$ but $f_z(x, y, z)=xy$ so $h(z)=K$, a constant. Hence a potential function for $\boldsymbol{F}$ is $f(x, y, z)=xyz+K$.

14. $\operatorname{curl}\boldsymbol{F}=\nabla\times\boldsymbol{F}=\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ 3z^2 & \cos y & 2xz\end{vmatrix}=(0-0)\boldsymbol{i}-(2z-6z)\boldsymbol{j}+(0-0)\boldsymbol{k}=4z\boldsymbol{j}\neq\boldsymbol{0}$,

so $\boldsymbol{F}$ is not conservative.

15. curl $\boldsymbol{F}=\nabla\times\boldsymbol{F}=\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ 2xy & x^2+2yz & y^2\end{vmatrix}=(2y-2y)\boldsymbol{i}-(0-0)\boldsymbol{j}+(2x-2x)\boldsymbol{k}=\boldsymbol{0}$, $\boldsymbol{F}$ is defined on all of R^3 , and the partial derivatives of the component functions are continuous, so $\boldsymbol{F}$ is conservative. Thus there exists a function f such that $\nabla f=\boldsymbol{F}$. Then $f_x(x, y, z)=2xy$ implies $f(x, y, z)=x^2y+g(y, z)$ and $f_y(x, y, z)=x^2+g_y(y, z)$. But $f_y(x, y, z)=x^2+2yz$, so $g(y, z)=y^2z+h(z)$ and $f(x, y, z)=x^2y+y^2z+h(z)$. Thus $f_z(x, y, z)=y^2+h'(z)$ but $f_z(x, y, z)=y^2$ so $h(z)=K$ and $f(x, y, z)=x^2y+y^2z+K$.

16. curl $\boldsymbol{F}=\nabla\times\boldsymbol{F}=\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ e^z & 1 & xe^z\end{vmatrix}=(0-0)\boldsymbol{i}-(e^z-e^z)\boldsymbol{j}+(0-0)\boldsymbol{k}=\boldsymbol{0}$ and $\boldsymbol{F}$ is defined on all of R^3 with component functions that have continuous partial deriatives, so $\boldsymbol{F}$ is conservative. Thus there exists a function f such that $\nabla f=\boldsymbol{F}$. Then $f_x(x, y, z)=e^z$ implies $f(x, y, z)=xe^z+g(y, z)\Rightarrow$ $f_y(x, y, z)=g_y(y, z)$. But $f_y(x, y, z)=1$, so $g(y, z)=y+h(z)$ and $f(x, y, z)=xe^z+y+h(z)$. Thus $f_z(x, y, z)=xe^z+h'(z)$ but $f_z(x, y, z)=xe^z$, so $h(z)=K$, a constant. Hence a potential function for $\boldsymbol{F}$ is $f(x, y, z)=xe^z+y+K$.

17. curl $\boldsymbol{F}=\nabla\times\boldsymbol{F}=\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ ye^{-x} & e^{-x} & 2z\end{vmatrix}=(0-0)\boldsymbol{i}-(0-0)\boldsymbol{j}+(-e^{-x}-e^{-x})\boldsymbol{k}=-2e^{-x}\boldsymbol{k}\neq\boldsymbol{0}$, so $\boldsymbol{F}$ is not conservative.

18.

$$\text{curl}\boldsymbol{F}=\nabla\times\boldsymbol{F}=\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ y\cos xy & x\cos xy & -\sin z\end{vmatrix}$$

$$=(0-0)\boldsymbol{i}-(0-0)\boldsymbol{j}+[(-xy\sin xy+\cos xy)-(-xy\sin xy+\cos xy)]\boldsymbol{k}=0$$

$\boldsymbol{F}$ is defined on all of R^3 , and the partial derivatives of the component functions are continuous, so $\boldsymbol{F}$ is conservative. Thus there exists a function f such that $\nabla f=\boldsymbol{F}$. Then $f_x(x, y, z)=y\cos xy$ implies $f(x, y, z)=\sin xy+g(y, z)\Rightarrow f_y(x, y, z)=x\cos xy+g_y(y, z)$. But $f_y(x, y, z)=x\cos xy$, so $g(y, z)=h(z)$ and $f(x, y, z)=\sin xy+h(z)$. Thus $f_z(x, y, z)=h'(z)$ but $f_z(x, y, z)=-\sin z$ so $h(z)=\cos z+K$ and a potential function for $\boldsymbol{F}$ is $f(x, y, z)=\sin xy+\cos z+K$.

19. No. Assume there is such a $\boldsymbol{G}$. Then $\operatorname{div}(\operatorname{curl}\boldsymbol{G})=y^2+z^2+x^2\neq 0$, which contradicts Theorem 11.

20. No. Assume there is such a $\boldsymbol{G}$. Then $\operatorname{div}(\operatorname{curl}\boldsymbol{G})=xz\neq 0$ which contradicts Theorem 11.

21. $\operatorname{curl}\boldsymbol{F}=\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ f(x) & g(y) & h(z) \end{vmatrix}=(0-0)\boldsymbol{i}+(0-0)\boldsymbol{j}+(0-0)\boldsymbol{k}=\boldsymbol{0}$.
Hence $\boldsymbol{F}=f(x)\boldsymbol{i}+g(y)\boldsymbol{j}+h(z)\boldsymbol{k}$ is irrotational.

22. $\operatorname{div}\boldsymbol{F}=\dfrac{\partial(f(y, z))}{\partial x}+\dfrac{\partial(g(x, z))}{\partial y}+\dfrac{\partial(h(x, y))}{\partial z}=0$ so $\boldsymbol{F}$ is incompressible.

23.

$$\begin{aligned}\operatorname{div}(\boldsymbol{F}+\boldsymbol{G}) &= \frac{\partial(P_1+P_2)}{\partial x}+\frac{\partial(Q_1+Q_2)}{\partial y}+\frac{\partial(R_1+R_2)}{\partial z} \\ &=\left(\frac{\partial P_1}{\partial x}+\frac{\partial Q_1}{\partial y}+\frac{\partial R_1}{\partial z}\right)+\left(\frac{\partial P_2}{\partial x}+\frac{\partial Q_2}{\partial y}+\frac{\partial R_3}{\partial z}\right)=\operatorname{div}\boldsymbol{F}+\operatorname{div}\boldsymbol{G}\end{aligned}$$

24.

$$\begin{aligned}\operatorname{curl}\boldsymbol{F}+\operatorname{curl}\boldsymbol{G} &= \left[\left(\frac{\partial R_1}{\partial y}-\frac{\partial Q_1}{\partial z}\right)\boldsymbol{i}+\left(\frac{\partial P_1}{\partial z}-\frac{\partial R_1}{\partial x}\right)\boldsymbol{j}+\left(\frac{\partial Q_1}{\partial x}-\frac{\partial P_1}{\partial y}\right)\boldsymbol{k}\right] \\ &\quad+\left[\left(\frac{\partial R_2}{\partial y}-\frac{\partial Q_2}{\partial z}\right)\boldsymbol{i}+\left(\frac{\partial P_2}{\partial z}-\frac{\partial R_2}{\partial x}\right)\boldsymbol{j}+\left(\frac{\partial Q_2}{\partial x}-\frac{\partial P_2}{\partial y}\right)\boldsymbol{k}\right] \\ &=\left[\frac{\partial(R_1+R_2)}{\partial y}-\frac{\partial(Q_1+Q_2)}{\partial z}\right]\boldsymbol{i}+\left[\frac{\partial(P_1+P_2)}{\partial z}-\frac{\partial(R_1+R_2)}{\partial x}\right]\boldsymbol{j} \\ &\quad+\left[\frac{\partial(Q_1+Q_2)}{\partial x}-\frac{\partial(P_1+P_2)}{\partial y}\right]\boldsymbol{k}=\operatorname{curl}(\boldsymbol{F}+\boldsymbol{G})\end{aligned}$$

25.

$$\begin{aligned}\operatorname{div}(f\boldsymbol{F}) &= \frac{\partial(fP_1)}{\partial x}+\frac{\partial(fQ_1)}{\partial y}+\frac{\partial(fR_1)}{\partial z} \\ &=\left(f\frac{\partial P_1}{\partial x}+P_1\frac{\partial f}{\partial x}\right)+\left(f\frac{\partial Q_1}{\partial y}+Q_1\frac{\partial f}{\partial y}\right)+\left(f\frac{\partial R_1}{\partial z}+R_1\frac{\partial f}{\partial z}\right)\end{aligned}$$

$$=f\left(\frac{\partial P_1}{\partial x}+\frac{\partial Q_1}{\partial y}+\frac{\partial R_1}{\partial z}\right)+\left\langle P_1,Q_1,R_1\right\rangle\cdot\left\langle\frac{\partial f}{\partial x},\frac{\partial f}{\partial y},\frac{\partial f}{\partial z}\right\rangle=f\operatorname{div}\boldsymbol{F}+\boldsymbol{F}\cdot\nabla f$$

26.

$$\operatorname{curl}(f\boldsymbol{F})=\left[\frac{\partial(fR_1)}{\partial y}-\frac{\partial(fQ_1)}{\partial z}\right]\boldsymbol{i}+\left[\frac{\partial(fP_1)}{\partial z}-\frac{\partial(fR_1)}{\partial x}\right]\boldsymbol{j}+\left[\frac{\partial(fQ_1)}{\partial x}-\frac{\partial(fP_1)}{\partial y}\right]\boldsymbol{k}$$

$$=\left[f\frac{\partial R_1}{\partial y}+R_1\frac{\partial f}{\partial y}-f\frac{\partial Q_1}{\partial z}-Q_1\frac{\partial f}{\partial z}\right]\boldsymbol{i}+\left[f\frac{\partial P_1}{\partial z}+P_1\frac{\partial f}{\partial z}-f\frac{\partial R_1}{\partial x}-R_1\frac{\partial f}{\partial x}\right]\boldsymbol{j}$$

$$+\left[f\frac{\partial Q_1}{\partial x}+Q_1\frac{\partial f}{\partial x}-f\frac{\partial P_1}{\partial y}-P_1\frac{\partial f}{\partial y}\right]\boldsymbol{k}$$

$$=f\left[\frac{\partial R_1}{\partial y}-\frac{\partial Q_1}{\partial z}\right]\boldsymbol{i}+f\left[\frac{\partial P_1}{\partial z}-\frac{\partial R_1}{\partial x}\right]\boldsymbol{j}+f\left[\frac{\partial Q_1}{\partial x}-\frac{\partial P_1}{\partial y}\right]\boldsymbol{k}$$

$$+\left[R_1\frac{\partial f}{\partial y}-Q_1\frac{\partial f}{\partial z}\right]\boldsymbol{i}+\left[P_1\frac{\partial f}{\partial z}-R_1\frac{\partial f}{\partial x}\right]\boldsymbol{j}+\left[Q_1\frac{\partial f}{\partial x}-P_1\frac{\partial f}{\partial y}\right]\boldsymbol{k}$$

$$=f\operatorname{curl}\boldsymbol{F}+(\nabla f)\times\boldsymbol{F}$$

27.

$$\operatorname{div}(\boldsymbol{F})\times\boldsymbol{G}=\nabla\cdot(\boldsymbol{F}\times\boldsymbol{G})=\begin{vmatrix}\partial/\partial x & \partial/\partial y & \partial/\partial z\\ P_1 & Q_1 & R_1\\ P_2 & Q_2 & R_2\end{vmatrix}=\frac{\partial}{\partial x}\begin{vmatrix}Q_1 & R_1\\ Q_2 & R_2\end{vmatrix}-\frac{\partial}{\partial y}\begin{vmatrix}P_1 & R_1\\ P_2 & R_2\end{vmatrix}+\frac{\partial}{\partial z}\begin{vmatrix}P_1 & Q_1\\ P_2 & Q_2\end{vmatrix}$$

$$=\left[Q_1\frac{\partial R_2}{\partial x}+R_2\frac{\partial Q_1}{\partial x}-Q_2\frac{\partial R_1}{\partial x}-R_1\frac{\partial Q_2}{\partial x}\right]$$

$$-\left[P_1\frac{\partial R_2}{\partial y}+R_2\frac{\partial P_1}{\partial y}-P_2\frac{\partial R_1}{\partial y}-R_1\frac{\partial P_2}{\partial y}\right]$$

$$+\left[P_1\frac{\partial Q_2}{\partial z}+Q_2\frac{\partial P_1}{\partial z}-P_2\frac{\partial Q_1}{\partial z}-Q_1\frac{\partial P_2}{\partial z}\right]$$

$$=\left[P_2\left(\frac{\partial R_1}{\partial y}-\frac{\partial Q_1}{\partial z}\right)+Q_2\left(\frac{\partial P_1}{\partial z}-\frac{\partial R_1}{\partial x}\right)+R_2\left(\frac{\partial Q_1}{\partial x}-\frac{\partial P_1}{\partial y}\right)\right]$$

$$-\left[P_1\left(\frac{\partial R_2}{\partial y}-\frac{\partial Q_2}{\partial z}\right)+Q_1\left(\frac{\partial P_2}{\partial z}-\frac{\partial R_2}{\partial x}\right)+R_1\left(\frac{\partial Q_2}{\partial x}-\frac{\partial P_2}{\partial y}\right)\right]$$

$$=\boldsymbol{G}\cdot\operatorname{curl}\boldsymbol{F}-\boldsymbol{F}\cdot\operatorname{curl}\boldsymbol{G}$$

28. $\operatorname{div}(\nabla f \times \nabla g) = \nabla g \cdot \operatorname{curl}(\nabla f) - \nabla f \cdot \operatorname{curl}(\nabla g) = 0$ (by Theorem 3)

29.

$$\operatorname{curl}\operatorname{curl}\boldsymbol{F} = \nabla\times(\nabla\times\boldsymbol{F}) = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \partial R_1/\partial y - \partial Q_1/\partial z & \partial P_1/\partial z - \partial R_1/\partial x & \partial Q_1/\partial x - \partial P_1/\partial y \end{vmatrix}$$

$$= \left(\frac{\partial^2 Q_1}{\partial y \partial x} - \frac{\partial^2 P_1}{\partial y^2} - \frac{\partial^2 P_1}{\partial z^2} + \frac{\partial^2 R_1}{\partial z \partial x} \right)\boldsymbol{i} + \left(\frac{\partial^2 R_1}{\partial z \partial y} - \frac{\partial^2 Q_1}{\partial z^2} - \frac{\partial^2 Q_1}{\partial x^2} + \frac{\partial^2 P_1}{\partial x \partial y} \right)\boldsymbol{j}$$
$$+ \left(\frac{\partial^2 P_1}{\partial x \partial z} - \frac{\partial^2 R_1}{\partial x^2} - \frac{\partial^2 R_1}{\partial y^2} + \frac{\partial^2 Q_1}{\partial y \partial z} \right)\boldsymbol{k}$$

Now let's consider $\operatorname{grad}\operatorname{div}\boldsymbol{F} - \nabla^2\boldsymbol{F}$ and compare with the above.
(Note that $\nabla^2\boldsymbol{F}$ is defined on page 1130).

$$\operatorname{grad}\operatorname{div}\boldsymbol{F} - \nabla^2\boldsymbol{F} = \left[\left(\frac{\partial^2 P_1}{\partial x^2} + \frac{\partial^2 Q_1}{\partial x \partial y} + \frac{\partial^2 R_1}{\partial x \partial z} \right)\boldsymbol{i} + \left(\frac{\partial^2 P_1}{\partial y \partial x} + \frac{\partial^2 Q_1}{\partial y^2} + \frac{\partial^2 R_1}{\partial y \partial z} \right)\boldsymbol{j} \right.$$
$$\left. + \left(\frac{\partial^2 P_1}{\partial z \partial x} + \frac{\partial^2 Q_1}{\partial z \partial y} + \frac{\partial^2 R_1}{\partial z^2} \right)\boldsymbol{k} \right]$$
$$- \left[\left(\frac{\partial^2 P_1}{\partial x^2} + \frac{\partial^2 P_1}{\partial y^2} + \frac{\partial^2 P_1}{\partial z^2} \right)\boldsymbol{i} + \left(\frac{\partial^2 Q_1}{\partial x^2} + \frac{\partial^2 Q_1}{\partial y^2} + \frac{\partial^2 Q_1}{\partial z^2} \right)\boldsymbol{j} \right.$$
$$\left. + \left(\frac{\partial^2 R_1}{\partial x^2} + \frac{\partial^2 R_1}{\partial y^2} + \frac{\partial^2 R_1}{\partial z^2} \right)\boldsymbol{k} \right]$$
$$= \left(\frac{\partial^2 Q_1}{\partial x \partial y} + \frac{\partial^2 R_1}{\partial x \partial z} - \frac{\partial^2 P_1}{\partial y^2} - \frac{\partial^2 P_1}{\partial z^2} \right)\boldsymbol{i} + \left(\frac{\partial^2 P_1}{\partial y \partial x} + \frac{\partial^2 R_1}{\partial y \partial z} - \frac{\partial^2 Q_1}{\partial x^2} - \frac{\partial^2 Q_1}{\partial z^2} \right)\boldsymbol{j}$$
$$+ \left(\frac{\partial^2 P_1}{\partial z \partial x} + \frac{\partial^2 Q_1}{\partial z \partial y} - \frac{\partial^2 R_1}{\partial x^2} - \frac{\partial^2 R_2}{\partial y^2} \right)\boldsymbol{k}$$

Then applying Clairaut's Theorem to reverse the order of differentiation in the second partial derivatives as needed and comparing, we have curl curl$\boldsymbol{F}$=grad div$\boldsymbol{F}-\nabla^2\boldsymbol{F}$ as desired.

30. **(a)** $\nabla\cdot\boldsymbol{r}=\left(\frac{\partial}{\partial x}\boldsymbol{i}+\frac{\partial}{\partial y}\boldsymbol{j}+\frac{\partial}{\partial z}\boldsymbol{k}\right)\cdot(x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k})=1+1+1=3$

(b)

$$\begin{aligned}\nabla\cdot(r\boldsymbol{r})&=\nabla\cdot\sqrt{x^2+y^2+z^2}\,(x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k})\\&=\left(\frac{x^2}{\sqrt{x^2+y^2+z^2}}+\sqrt{x^2+y^2+z^2}\right)+\left(\frac{y^2}{\sqrt{x^2+y^2+z^2}}+\sqrt{x^2+y^2+z^2}\right)\\&\quad+\left(\frac{z^2}{\sqrt{x^2+y^2+z^2}}+\sqrt{x^2+y^2+z^2}\right)\\&=\frac{1}{\sqrt{x^2+y^2+z^2}}(4x^2+4y^2+4z^2)=4\sqrt{x^2+y^2+z^2}=4r\end{aligned}$$

Another method:

By Exercise 25, $\nabla\cdot(r\boldsymbol{r})=\mathrm{div}(r\boldsymbol{r})=r\,\mathrm{div}\boldsymbol{r}+\boldsymbol{r}\cdot\nabla r=3r+\boldsymbol{r}\cdot\frac{\boldsymbol{r}}{r}=4r$.

(c)

$$\begin{aligned}\nabla^2r^3&=\nabla^2(x^2+y^2+z^2)^{3/2}\\&=\frac{\partial}{\partial x}\left[\frac{3}{2}(x^2+y^2+z^2)^{1/2}(2x)\right]+\frac{\partial}{\partial y}\left[\frac{3}{2}(x^2+y^2+z^2)^{1/2}(2y)\right]\\&\quad+\frac{\partial}{\partial z}\left[\frac{3}{2}(x^2+y^2+z^2)^{1/2}(2z)\right]\\&=3\left[\frac{1}{2}(x^2+y^2+z^2)^{-1/2}(2x)(x)+(x^2+y^2+z^2)^{1/2}\right]\\&\quad+3\left[\frac{1}{2}(x^2+y^2+z^2)^{-1/2}(2y)(y)+(x^2+y^2+z^2)^{1/2}\right]\\&\quad+3\left[\frac{1}{2}(x^2+y^2+z^2)^{-1/2}(2z)(z)+(x^2+y^2+z^2)^{1/2}\right]\\&=3(x^2+y^2+z^2)^{-1/2}(4x^2+4y^2+4z^2)=12(x^2+y^2+z^2)^{1/2}\\&=12r\end{aligned}$$

Another method: $\frac{\partial}{\partial x}(x^2+y^2+z^2)^{3/2}=3x\sqrt{x^2+y^2+z^2}\Rightarrow\nabla r^3=3r(x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k})=3r\boldsymbol{r}$, so $\nabla^2r^3=\nabla\cdot\nabla r^3=\nabla\cdot(3r\boldsymbol{r})=3(4r)=12r$ by part (b).

31. **(a)**

$$\nabla r = \nabla \sqrt{x^2+y^2+z^2} = \frac{x}{\sqrt{x^2+y^2+z^2}}\,\boldsymbol{i} + \frac{y}{\sqrt{x^2+y^2+z^2}}\,\boldsymbol{j} + \frac{z}{\sqrt{x^2+y^2+z^2}}\,\boldsymbol{k}$$

$$= \frac{x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k}}{\sqrt{x^2+y^2+z^2}} = \frac{\boldsymbol{r}}{r}$$

(b)

$$\nabla \times \boldsymbol{r} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}(z) - \frac{\partial}{\partial z}(y)\right]\boldsymbol{i} + \left[\frac{\partial}{\partial z}(x) - \frac{\partial}{\partial x}(z)\right]\boldsymbol{j} + \left[\frac{\partial}{\partial x}(y) - \frac{\partial}{\partial y}(x)\right]\boldsymbol{k} = \mathbf{0}$$

(c)

$$\nabla\left(\frac{1}{r}\right) = \nabla\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right)$$

$$= \frac{-\frac{1}{2\sqrt{x^2+y^2+z^2}}(2x)}{x^2+y^2+z^2}\,\boldsymbol{i} - \frac{\frac{1}{2\sqrt{x^2+y^2+z^2}}(2y)}{x^2+y^2+z^2}\,\boldsymbol{j} - \frac{\frac{1}{2\sqrt{x^2+y^2+z^2}}(2z)}{x^2+y^2+z^2}\,\boldsymbol{k}$$

$$= -\frac{x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k}}{(x^2+y^2+z^2)^{3/2}} = -\frac{\boldsymbol{r}}{r^3}$$

(d)

$$\nabla \ln r = \nabla \ln\left(x^2+y^2+z^2\right)^{1/2} = \frac{1}{2}\nabla \ln\left(x^2+y^2+z^2\right)$$

$$= \frac{x}{x^2+y^2+z^2}\,\boldsymbol{i} + \frac{y}{x^2+y^2+z^2}\,\boldsymbol{j} + \frac{z}{x^2+y^2+z^2}\,\boldsymbol{k} = \frac{x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k}}{x^2+y^2+z^2} = \frac{\boldsymbol{r}}{r^2}$$

32. $\boldsymbol{r} = x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k} \Rightarrow r = |\boldsymbol{r}| = \sqrt{x^2+y^2+z^2}$, so

$$\boldsymbol{F} = \frac{\boldsymbol{r}}{r^p} = \frac{x}{(x^2+y^2+z^2)^{p/2}}\,\boldsymbol{i} + \frac{y}{(x^2+y^2+z^2)^{p/2}}\,\boldsymbol{j} + \frac{z}{(x^2+y^2+z^2)^{p/2}}\,\boldsymbol{k}$$

Then $\frac{\partial}{\partial x}\frac{x}{(x^2+y^2+z^2)^{p/2}}=\frac{(x^2+y^2+z^2)-px^2}{(x^2+y^2+z^2)^{1+p/2}}=\frac{r^2-px^2}{r^{p+2}}$. Similarly,

$\frac{\partial}{\partial y}\frac{y}{(x^2+y^2+z^2)^{p/2}}=\frac{r^2-py^2}{r^{p+2}}$ and $\frac{\partial}{\partial z}\frac{z}{(x^2+y^2+z^2)^{p/2}}=\frac{r^2-pz^2}{r^{p+2}}$. Thus

$$\operatorname{div}\boldsymbol{F}=\nabla\cdot\boldsymbol{F}=\frac{r^2-px^2}{r^{p+2}}+\frac{r^2-py^2}{r^{p+2}}+\frac{r^2-pz^2}{r^{p+2}}=\frac{3r^2-px^2-py^2-pz^2}{r^{p+2}}$$

$$=\frac{3r^2-p(x^2+y^2+z^2)}{r^{p+2}}=\frac{3r^2-pr^2}{r^{p+2}}=\frac{3-p}{r^p}$$

Consequently, if $p=3$ we have $\operatorname{div}\boldsymbol{F}=0$.

33. By (13), $\oint_C f(\nabla g)\cdot\boldsymbol{n}ds=\iint_D \operatorname{div}(f\nabla g)dA=\iint_D[f\operatorname{div}(\nabla g)+\nabla g\cdot\nabla f]dA$ by Exercise 25. But

$\operatorname{div}(\nabla g)=\nabla^2 g$. Hence $\iint_D f\nabla^2 g dA=\oint_C f(\nabla g)\cdot\boldsymbol{n}ds-\iint_D\nabla g\cdot\nabla f dA$.

34. By Exercise 33, $\iint_D f\nabla^2 g dA=\oint_C f(\nabla g)\cdot\boldsymbol{n}ds-\iint_D\nabla g\cdot\nabla f dA$ and

$\iint_D g\nabla^2 f dA=\oint_C g(\nabla f)\cdot\boldsymbol{n}ds-\iint_D\nabla f\cdot\nabla g dA$. Hence

$$\iint_D(f\nabla^2 g-g\nabla^2 f)dA=\oint_C[f(\nabla g)\cdot\boldsymbol{n}-g(\nabla f)\cdot\boldsymbol{n}]ds+\iint_D(\nabla f\cdot\nabla g-\nabla g\cdot\nabla f)dA$$

$$=\oint_C[f\nabla g-g\nabla f]\cdot\boldsymbol{n}ds$$

35. **(a)** We know that $\omega=v/d$, and from the diagram $\sin\theta=d/r\Rightarrow v=d\omega=(\sin\theta)r\omega=|\boldsymbol{w}\times\boldsymbol{r}|$. But $\boldsymbol{v}$ is perpendicular to both $\boldsymbol{w}$ and $\boldsymbol{r}$, so that $\boldsymbol{v}=\boldsymbol{w}\times\boldsymbol{r}$.

(b) From (a), $\boldsymbol{v}=\boldsymbol{w}\times\boldsymbol{r}=\begin{vmatrix}\boldsymbol{i}&\boldsymbol{j}&\boldsymbol{k}\\0&0&\omega\\x&y&z\end{vmatrix}=(0\cdot z-\omega y)\boldsymbol{i}+(\omega x-0\cdot z)\boldsymbol{j}+(0\cdot y-x\cdot 0)\boldsymbol{k}=-\omega y\boldsymbol{i}+\omega x\boldsymbol{j}$

(c)

$$\operatorname{curl}\boldsymbol{v}=\nabla\times\boldsymbol{v}=\begin{vmatrix}\boldsymbol{i}&\boldsymbol{j}&\boldsymbol{k}\\\partial/\partial x&\partial/\partial y&\partial/\partial y\\-\omega y&\omega x&0\end{vmatrix}$$

$$=\left[\frac{\partial}{\partial y}(0)-\frac{\partial}{\partial z}(\omega x)\right]\boldsymbol{i}+\left[\frac{\partial}{\partial z}(-\omega y)-\frac{\partial}{\partial x}(0)\right]\boldsymbol{j}+\left[\frac{\partial}{\partial x}(\omega x)-\frac{\partial}{\partial y}(-\omega y)\right]\boldsymbol{k}$$
$$=[\omega-(-\omega)]\boldsymbol{k}=2\omega\,\boldsymbol{k}=2\boldsymbol{w}$$

36. Let $\boldsymbol{H}=\left\langle h_1, h_2, h_3\right\rangle$ and $\boldsymbol{E}=\left\langle E_1, E_2, E_3\right\rangle$.

(a)

$$\nabla\times(\nabla\times\boldsymbol{E})=\nabla\times(\text{curl}\boldsymbol{E})=\nabla\times\left(-\frac{1}{c}\frac{\partial\boldsymbol{H}}{\partial t}\right)=-\frac{1}{c}\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ \partial h_1/\partial t & \partial h_2/\partial t & \partial h_3/\partial t\end{vmatrix}$$
$$=-\frac{1}{c}\left[\left(\frac{\partial^2 h_3}{\partial y\partial t}-\frac{\partial^2 h_2}{\partial z\partial t}\right)\boldsymbol{i}+\left(\frac{\partial^2 h_1}{\partial z\partial t}-\frac{\partial^2 h_3}{\partial x\partial t}\right)\boldsymbol{j}+\left(\frac{\partial^2 h_2}{\partial x\partial t}-\frac{\partial^2 h_1}{\partial y\partial t}\right)\boldsymbol{k}\right]$$
$$=-\frac{1}{c}\frac{\partial}{\partial t}\left[\left(\frac{\partial h_3}{\partial y}-\frac{\partial h_2}{\partial z}\right)\boldsymbol{i}+\left(\frac{\partial h_1}{\partial z}-\frac{\partial h_3}{\partial x}\right)\boldsymbol{j}+\left(\frac{\partial h_2}{\partial x}-\frac{\partial h_1}{\partial y}\right)\boldsymbol{k}\right]$$

[assuming that the partial derivatives are continuous so that the order of differentiation does not matter]

$$=-\frac{1}{c}\frac{\partial}{\partial t}\text{curl}\boldsymbol{H}=-\frac{1}{c}\frac{\partial}{\partial t}\left(\frac{1}{c}\frac{\partial\boldsymbol{E}}{\partial t}\right)=-\frac{1}{c^2}\frac{\partial^2\boldsymbol{E}}{\partial t^2}$$

(b)

$$\nabla\times(\nabla\times\boldsymbol{H})=\nabla\times(\text{curl}\boldsymbol{H})=\nabla\times\left(\frac{1}{c}\frac{\partial\boldsymbol{E}}{\partial t}\right)=\frac{1}{c}\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ \partial E_1/\partial t & \partial E_2/\partial t & \partial E_3/\partial t\end{vmatrix}$$
$$=\frac{1}{c}\left[\left(\frac{\partial^2 E_3}{\partial y\partial t}-\frac{\partial^2 E_2}{\partial z\partial t}\right)\boldsymbol{i}+\left(\frac{\partial^2 E_1}{\partial z\partial t}-\frac{\partial^2 E_3}{\partial x\partial t}\right)\boldsymbol{j}+\left(\frac{\partial^2 E_2}{\partial x\partial t}-\frac{\partial^2 E_1}{\partial y\partial t}\right)\boldsymbol{k}\right]$$
$$=\frac{1}{c}\frac{\partial}{\partial t}\left[\left(\frac{\partial E_3}{\partial y}-\frac{\partial E_2}{\partial z}\right)\boldsymbol{i}+\left(\frac{\partial E_1}{\partial z}-\frac{\partial E_3}{\partial x}\right)\boldsymbol{j}+\left(\frac{\partial E_2}{\partial x}-\frac{\partial E_1}{\partial y}\right)\boldsymbol{k}\right]$$

[assuming that the partial derivatives are continuous so that the order of differentiation does not matter]

$$=\frac{1}{c}\frac{\partial}{\partial t}\text{curl}\boldsymbol{E}=\frac{1}{c}\frac{\partial}{\partial t}\left(-\frac{1}{c}\frac{\partial\boldsymbol{H}}{\partial t}\right)=-\frac{1}{c^2}\frac{\partial^2\boldsymbol{H}}{\partial t^2}$$

(c) Using Exercise 29, we have that curl curl$\boldsymbol{E}$=grad div$\boldsymbol{E}-\nabla^2\boldsymbol{E}\Rightarrow$

$\nabla^2 \boldsymbol{E}=\text{grad div}\boldsymbol{E}-\text{curl curl}\boldsymbol{E}=\text{grad}\,0+\frac{1}{c^2}\frac{\partial^2 \boldsymbol{E}}{\partial t^2}=\frac{1}{c^2}\frac{\partial^2 \boldsymbol{E}}{\partial t^2}$.

(d) As in part (c), $\nabla^2 \boldsymbol{H}=\text{grad div}\boldsymbol{H}-\text{curl curl}\boldsymbol{H}=\text{grad}\,0+\frac{1}{c^2}\frac{\partial^2 \boldsymbol{H}}{\partial t^2}=\frac{1}{c^2}\frac{\partial^2 \boldsymbol{H}}{\partial t^2}$.

37. For any continuous function f on R^3, define a vector field $\boldsymbol{G}(x, y, z)=\langle g(x, y, z), 0, 0\rangle$ where $g(x, y, z)=\int_0^x f(t, y, z)\,dt$. Then

div $\boldsymbol{G}=\frac{\partial}{\partial x}(g(x, y, z))+\frac{\partial}{\partial y}(0)+\frac{\partial}{\partial z}(0)=\frac{\partial}{\partial x}\int_0^x f(t, y, z)dt=f(x, y, z)$ by the Fundamental Theorem of Calculus. Thus every continuous function f on R^3 is the divergence of some vector field.

1. $\boldsymbol{r}(u, v)=u\cos v\,\boldsymbol{i}+u\sin v\,\boldsymbol{j}+u^2\boldsymbol{k}$, so the corresponding parametric equations for the surface are $x=u\cos v$, $y=u\sin v$, $z=u^2$. For any point (x, y, z) on the surface, we have $x^2+y^2=u^2\cos^2 v+u^2\sin^2 v=u^2=z$. Since no restrictions are placed on the parameters, the surface is $z=x^2+y^2$, which we recognize as a circular paraboloid opening upward whose axis is the z –axis.

2. $\boldsymbol{r}(u, v)=(1+2u)\boldsymbol{i}+(-u+3v)\boldsymbol{j}+(2+4u+5v)\boldsymbol{k}=\langle 1, 0, 2\rangle+u\langle 2,-1,4\rangle+v\langle 0, 3, 5\rangle$. From Example 3, we recognize this as a vector equation of a plane through the point (1, 0, 2) and containing vectors $\boldsymbol{a}=\langle 2, -1, 4\rangle$ and $\boldsymbol{b}=\langle 0, 3, 5\rangle$. If we wish to find a more conventional equation for the plane, a normal vector to the plane is

$$\boldsymbol{a}\times\boldsymbol{b}=\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 2 & -1 & 4 \\ 0 & 3 & 5 \end{vmatrix}=-17\boldsymbol{i}-10\boldsymbol{j}+6\boldsymbol{k}$$

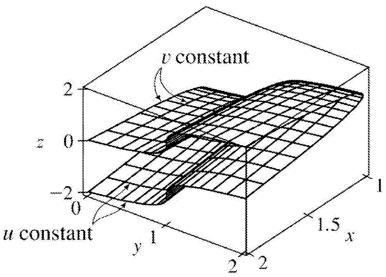
and an equation of the plane is $-17(x-1)-10(y-0)+6(z-2)=0$ or $-17x-10y+6z=-5$.

3. $\boldsymbol{r}(x, \theta)=\langle x, \cos\theta, \sin\theta\rangle$, so the corresponding parametric equations for the surface are $x=x$, $y=\cos\theta$, $z=\sin\theta$. For any point (x, y, z) on the surface, we have $y^2+z^2=\cos^2\theta+\sin^2\theta=1$, so any vertical trace in $x=k$ is the circle $y^2+z^2=1$, $x=k$. Since $x=x$ with no restriction, the surface is a circular cylinder with radius 1 whose axis is the x –axis.

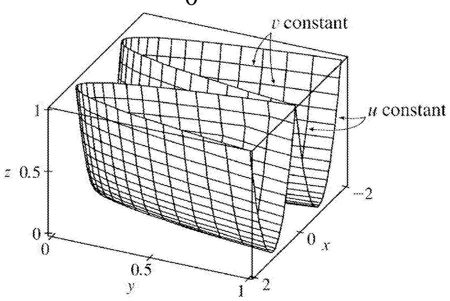
4. $\boldsymbol{r}(x, \theta)=\langle x, x\cos\theta, x\sin\theta\rangle$, so the corresponding parametric equations for the surface are $x=x$, $y=x\cos\theta$, $z=x\sin\theta$. For any point (x, y, z) on the surface, we have $y^2+z^2=x^2\cos^2\theta+x^2\sin^2\theta=x^2$. With $x=x$ and no restrictions on the parameters, the surface is $x^2=y^2+z^2$, which we recognize as a circular cone whose axis is the x –axis.

5. $\boldsymbol{r}(u, v)=\langle u^2+1, v^3+1, u+v\rangle$, $-1\le u\le 1$, $-1\le v\le 1$.

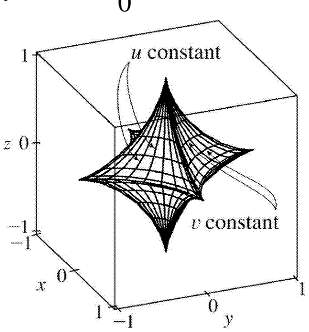
The surface has parametric equations $x=u^2+1$, $y=v^3+1$, $z=u+v$, $-1\le u\le 1$, $-1\le v\le 1$. If we keep u constant at u_0 , $x=u_0^2+1$, a constant, so the corresponding grid curves must be the curves parallel to the yz –plane. If v is constant, we have $y=v_0^3+1$, a constant, so these grid curves are the curves parallel to the xz –plane.



6. $\boldsymbol{r}(u, v)=\left\langle u+v, u^2, v^2\right\rangle$, $-1\leq u\leq 1$, $-1\leq v\leq 1$.
The surface has parametric equations $x=u+v$, $y=u^2$, $z=v^2$, $-1\leq u\leq 1$, $-1\leq v\leq 1$. If $u=u_0$ is constant, $y=u_0^2=$ constant, so the corresponding grid curves are the curves parallel to the xz –plane. If $v=v_0$ is constant, $z=v_0^2=$ constant, so the corresponding grid curves are the curves parallel to the xy –plane.

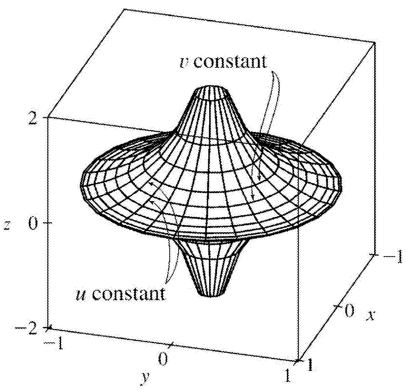


7. $\boldsymbol{r}(u, v)=\left\langle \cos^3 u\cos^3 v, \sin^3 u\cos^3 v, \sin^3 v\right\rangle$. The surface has parametric equations $x=\cos^3 u\cos^3 v$, $y=\sin^3 u\cos^3 v$, $z=\sin^3 v$, $0\leq u\leq \pi$, $0\leq v\leq 2\pi$. Note that if $v=v_0$ is constant then $z=\sin^3 v_0$ is constant, so the corresponding grid curves must be the curves parallel to the xy –plane. The vertically oriented grid curves, then, correspond to $u=u_0$ being held constant, giving $x=\cos^3 u_0\cos^3 v$, $y=\sin^3 u_0\cos^3 v$, $z=\sin^3 v$. These curves lie in vertical planes that contain the z –axis.

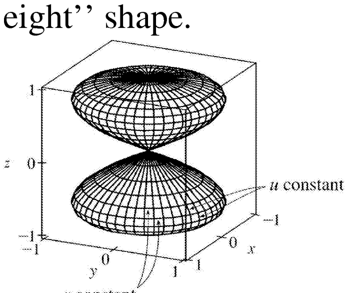


8. $\boldsymbol{r}(u, v)=\left\langle \cos u\sin v, \sin u\sin v, \cos v+\ln \tan (v/2)\right\rangle$.
The surface has parametric equations $x=\cos u\sin v$, $y=\sin u\sin v$, $z=\cos v+\ln \tan (v/2)$, $0\leq u\leq 2\pi$,

$0.1 \le v \le 6.2$. Note that if $v = v_0$ is constant, the parametric equations become $x = \cos u \sin v_0$, $y = \sin u \sin v_0$, $z = \cos v_0 + \ln \tan (v_0/2)$ which represent a circle of radius $\sin v_0$ in the plane $z = \cos v_0 + \ln \tan (v_0/2)$. So the circular grid curves we see lying horizontally are the grid curves with v constant. The vertically oriented grid curves correspond to $u = u_0$ being held constant, giving $x = \cos u_0 \sin v$, $y = \sin u_0 \sin v$, $z = \cos v + \ln \tan (v/2)$. These curves lie in vertical planes that contain the z –axis.

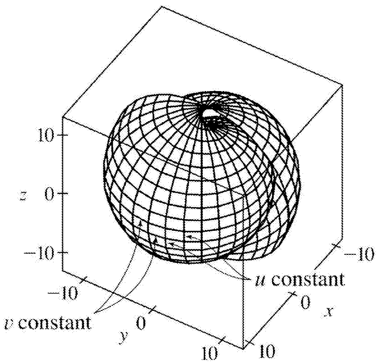


9. $x = \cos u \sin 2v$, $y = \sin u \sin 2v$, $z = \sin v$.
The complete graph of the surface is given by the parametric domain $0 \le u \le \pi, 0 \le v \le 2\pi$. Note that if $v = v_0$ is constant, the parametric equations become $x = \cos u \sin 2v_0$, $y = \sin u \sin 2v_0$, $z = \sin v_0$ which represent a circle of radius $\sin 2v_0$ in the plane $z = \sin v_0$. So the circular grid curves we see lying horizontally are the grid curves which have v constant. The vertical grid curves, then, correspond to $u = u_0$ being held constant, giving $x = \cos u_0 \sin 2v$ and $y = \sin u_0 \sin 2v$ with $z = \sin v$ which has a ''figure–eight'' shape.



10. $x = u \sin u \cos v$, $y = u \cos u \cos v$, $z = u \sin v$.
We graph the portion of the surface with parametric domain $0 \le u \le 4\pi, 0 \le v \le 2\pi$. Note that if $v = v_0$ is constant, the parametric equations become $x = u \sin u \cos v_0$, $y = u \cos u \cos v_0$, $z = u \sin v_0$. The equations for x and y show that the projections onto the xy –plane give a spiral shape, so the corresponding grid curves are the almost–horizontal spiral curves we see. The vertical grid curves, which look approximately circular, correspond to $u = u_0$ being held constant, giving $x = u_0 \sin u_0 \cos v$,

$y=u_0\cos u_0\cos v$, $z=u_0\sin v$.



11. $\boldsymbol{r}(u, v)=\cos v\boldsymbol{i}+\sin v\boldsymbol{j}+u\boldsymbol{k}$. The parametric equations for the surface are $x=\cos v$, $y=\sin v$, $z=u$. Then $x^2+y^2=\cos^2 v+\sin^2 v=1$ and $z=u$ with no restriction on u , so we have a circular cylinder, graph IV. The grid curves with u constant are the horizontal circles we see in the plane $z=u$. If v is constant, both x and y are constant with z free to vary, so the corresponding grid curves are the lines on the cylinder parallel to the z –axis.

12. $\boldsymbol{r}(u, v)=u\cos v\boldsymbol{i}+u\sin v\boldsymbol{j}+u\boldsymbol{k}$. The parametric equations for the surface are $x=u\cos v$, $y=u\sin v$, $z=u$. Then $x^2+y^2=u^2\cos^2 v+u^2\sin^2 v=u^2=z^2$, which represents the equation of a cone with axis the z –axis, graph V. The grid curves with u constant are the horizontal circles we see, corresponding to the equations $x^2+y^2=u^2$ in the plane $z=u$. If v is constant, x, y, z are each scalar multiples of u , corresponding to the straight line grid curves through the origin.

13. $\boldsymbol{r}(u, v)=u\cos v\boldsymbol{i}+u\sin v\boldsymbol{j}+v\boldsymbol{k}$. The parametric equations for the surface are $x=u\cos v, y=u\sin v, z=v$. We look at the grid curves first; if we fix v , then x and y parametrize a straight line in the plane $z=v$ which intersects the z –axis. If u is held constant, the projection onto the xy –plane is circular; with $z=v$, each grid curve is a helix. The surface is a spiraling ramp, graph I.

14. $x=u^3$, $y=u\sin v$, $z=u\cos v$. Then $y^2+z^2=u^2\sin^2 v+u^2\cos^2 v=u^2$, so if u is held constant, each grid curve is a circle of radius u in the plane $x=u^3$. The graph then must be graph III. If v is held constant, so $v=v_0$, we have $y=u\sin v_0$ and $z=u\cos v_0$. Then $y=(\tan v_0)z$, so the grid curves we see running lengthwise along the surface in the planes $y=kz$ correspond to keeping v constant.

15. $x=(u-\sin u)\cos v$, $y=(1-\cos u)\sin v$, $z=u$. If u is held constant, x and y give an equation of an ellipse in the plane $z=u$, thus the grid curves are horizontally oriented ellipses. Note that when $u=0$, the ‘‘ellipse’’ is the single point $(0, 0, 0)$, and when $u=\pi$, we have $y=0$ while x ranges from $-\pi$ to π , a line segment parallel to the x –axis in the plane $z=\pi$. This is the upper ‘‘seam’’ we see in graph II. When v is held constant, $z=u$ is free to vary, so the corresponding grid curves are the curves we see running up and down along the surface.

16. $x=(1-u)(3+\cos v)\cos 4\pi u$, $y=(1-u)(3+\cos v)\sin 4\pi u$, $z=3u+(1-u)\sin v$. These equations correspond to graph VI: when $u=0$, then $x=3+\cos v$, $y=0$, and $z=\sin v$, which are equations of a circle with radius 1 in the xz –plane centered at $(3, 0, 0)$. When $u=\frac{1}{2}$, then $x=\frac{3}{2}+\frac{1}{2}\cos v$, $y=0$, and $z=\frac{3}{2}+\frac{1}{2}\sin v$, which are equations of a circle with radius $\frac{1}{2}$ in the xz –plane centered at $\left(\frac{3}{2}, 0, \frac{3}{2}\right)$. When $u=1$, then $x=y=0$ and $z=3$, giving the topmost point shown in the graph. This suggests that the grid curves with u constant are the vertically oriented circles visible on the surface. The spiralling grid curves correspond to keeping v constant.

17. From Example 3, parametric equations for the plane through the point $(1, 2, -3)$ that contains the vectors $\boldsymbol{a}=\langle 1, 1, -1\rangle$ and $\boldsymbol{b}=\langle 1, -1, 1\rangle$ are $x=1+u(1)+v(1)=1+u+v$, $y=2+u(1)+v(-1)=2+u-v$, $z=-3+u(-1)+v(1)=-3-u+v$.

18. Solving the equation for z gives $z^2=1-2x^2-4y^2 \Rightarrow z=-\sqrt{1-2x^2-4y^2}$ (since we want the lower half of the ellipsoid). If we let x and y be the parameters, parametric equations are $x=x, y=y, z=-\sqrt{1-2x^2-4y^2}$.
Alternate solution : The equation can be rewritten as $\frac{x^2}{(1/\sqrt{2})^2}+\frac{y^2}{(1/2)^2}+z^2=1$, and if we let $x=\frac{1}{\sqrt{2}}u\cos v$ and $y=\frac{1}{2}u\sin v$, then $z=-\sqrt{1-2x^2-4y^2}=-\sqrt{1-u^2\cos^2 v-u^2\sin^2 v}=-\sqrt{1-u^2}$, where $0\le u\le 1$ and $0\le v\le 2\pi$.

19. Solving the equation for y gives $y^2=1-x^2+z^2 \Rightarrow y=\sqrt{1-x^2+z^2}$. (We choose the positive root since we want the part of the hyperboloid that corresponds to $y\ge 0$). If we let x and z be the parameters, parametric equations are $x=x$, $z=z$, $y=\sqrt{1-x^2+z^2}$.

20. $x=4-y^2-2z^2$, $y=y$, $z=z$ where $y^2+2z^2\le 4$ since $x\ge 0$. Then the associated vector equation is $\boldsymbol{r}(y, z)=(4-y^2-2z^2)\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k}$.

21. Since the cone intersects the sphere in the circle $x^2+y^2=2$, $z=\sqrt{2}$ and we want the portion of the sphere above this, we can parametrize the surface as $x=x$, $y=y$, $z=\sqrt{4-x^2-y^2}$ where $x^2+y^2\le 2$.
Alternate solution: Using spherical coordinates, $x=2\sin\phi\cos\theta$, $y=2\sin\phi\sin\theta$, $z=2\cos\phi$ where $0\le\phi\le\frac{\pi}{4}$ and $0\le\theta\le 2\pi$.

22. In spherical coordinates, parametric equations are $x=4\sin\phi\cos\theta$, $y=4\sin\phi\sin\theta$, $z=4\cos\phi$. The intersection of the sphere with the plane $z=2$ corresponds to $z=4\cos\phi=2 \Rightarrow \cos\phi=\frac{1}{2} \Rightarrow \phi=\frac{\pi}{3}$. By symmetry, the intersection of the sphere with the plane $z=-2$ corresponds to $\phi=\pi-\frac{\pi}{3}=\frac{2\pi}{3}$. Thus the surface is described by $0\le\theta\le 2\pi$, $\frac{\pi}{3}\le\phi\le\frac{2\pi}{3}$.

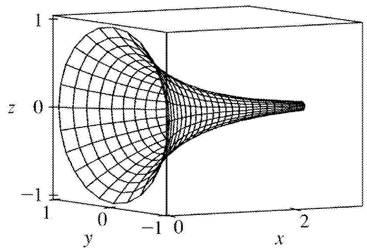
23. Parametric equations are $x=x$, $y=4\cos\theta$, $z=4\sin\theta$, $0\le x\le 5$, $0\le\theta\le 2\pi$.

24. Using x and y as the parameters, $x=x$, $y=y$, $z=x+3$ where $0\le x^2+y^2\le 1$. Also, since the plane intersects the cylinder in an ellipse, the surface is a planar ellipse in the plane $z=x+3$. Thus, parametrizing with respect to s and θ , we have $x=s\cos\theta$, $y=s\sin\theta$, $z=3+s\cos\theta$ where $0\le s\le 1$ and $0\le\theta\le 2\pi$.

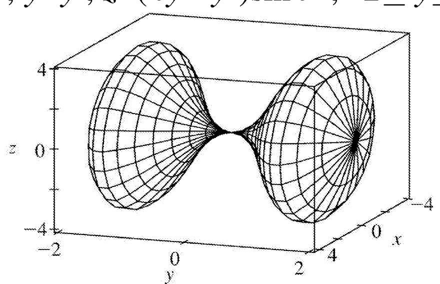
25. The surface appears to be a portion of a circular cylinder of radius 3 with axis the x –axis. An equation of the cylinder is $y^2+z^2=9$, and we can impose the restrictions $0\le x\le 5$, $y\le 0$ to obtain the portion shown.
To graph the surface on a CAS, we can use parametric equations $x=u$, $y=3\cos v$, $z=3\sin v$ with the parameter
domain $0\le u\le 5$, $\frac{\pi}{2}\le v\le\frac{3\pi}{2}$. Alternatively, we can regard x and z as parameters. Then parametric equations are $x=x$, $z=z$, $y=-\sqrt{9-z^2}$, where $0\le x\le 5$ and $-3\le z\le 3$.

26. The surface appears to be a portion of a sphere of radius 1 centered at the origin. In spherical coordinates, the sphere has equation $\rho=1$, and imposing the restrictions $\frac{\pi}{2}\le\theta\le 2\pi$, $\frac{\pi}{4}\le\phi\le\pi$ will give only the portion of the sphere shown. Thus, to graph the surface on a CAS we can either use spherical coordinates with the stated restrictions, or we can use parametric equations: $x=\sin\phi\cos\theta$, $y=\sin\phi\sin\theta$, $z=\cos\phi$, $\frac{\pi}{2}\le\theta\le 2\pi$, $\frac{\pi}{4}\le\phi\le\pi$.

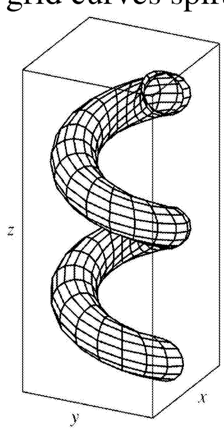
27. Using Equations 3, we have the parametrization $x=x$, $y=e^{-x}\cos\theta$, $z=e^{-x}\sin\theta$, $0\le x\le 3$, $0\le\theta\le 2\pi$.



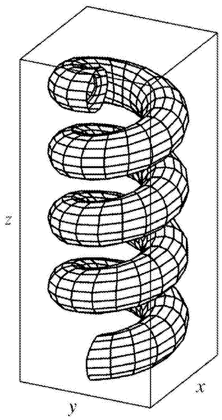
28. Letting θ be the angle of rotation about the y –axis, we have the parametrization $x=(4y^2-y^4)\cos\theta$, $y=y$, $z=(4y^2-y^4)\sin\theta$, $-2\leq y\leq 2$, $0\leq\theta\leq 2\pi$.



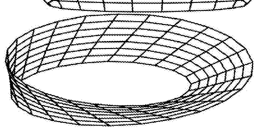
29. **(a)** Replacing cos u by sin u and sin u by cos u gives parametric equations $x=(2+\sin v)\sin u$, $y=(2+\sin v)\cos u$, $z=u+\cos v$. From the graph, it appears that the direction of the spiral is reversed. We can verify this observation by noting that the projection of the spiral grid curves onto the xy –plane, given by $x=(2+\sin v)\sin u$, $y=(2+\sin v)\cos u$, $z=0$, draws a circle in the clockwise direction for each value of v . The original equations, on the other hand, give circular projections drawn in the counterclockwise direction. The equation for z is identical in both surfaces, so as z increases, these grid curves spiral up in opposite directions for the two surfaces.

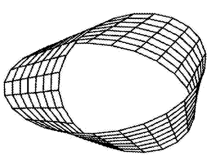


(b) Replacing cos u by cos $2u$ and sin u by sin $2u$ gives parametric equations $x=(2+\sin v)\cos 2u$, $y=(2+\sin v)\sin 2u$, $z=u+\cos v$. From the graph, it appears that the number of coils in the surface doubles within the same parametric domain. We can verify this observation by noting that the projection of the spiral grid curves onto the xy –plane, given by $x=(2+\sin v)\cos 2u$, $y=(2+\sin v)\sin 2u$, $z=0$ (where v is constant), complete circular revolutions for $0\leq u\leq\pi$ while the original surface requires $0\leq u\leq 2\pi$ for a complete revolution. Thus, the new surface winds around twice as fast as the original surface, and since the equation for z is identical in both surfaces, we observe twice as many circular coils in the same z –interval.



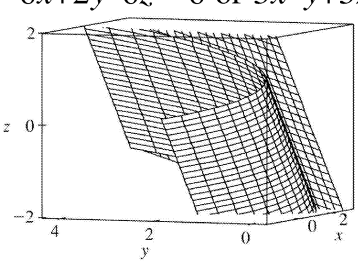
30. First we graph the surface as viewed from the front, then from two additional viewpoints.





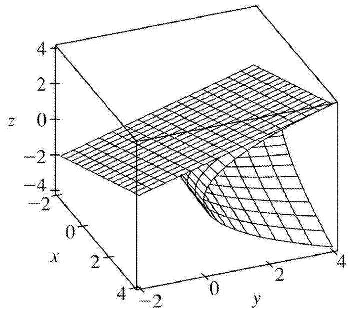
The surface appears as a twisted sheet, and is unusual because it has only one side. (The Mobius strip is discussed in more detail in Section 17.7).

31. $\boldsymbol{r}(u, v)=(u+v)\boldsymbol{i}+3u^2\boldsymbol{j}+(u-v)\boldsymbol{k}$.
$\boldsymbol{r}_u=\boldsymbol{i}+6u\boldsymbol{j}+\boldsymbol{k}$ and $\boldsymbol{r}_v=\boldsymbol{i}-\boldsymbol{k}$, so $\boldsymbol{r}_u\times\boldsymbol{r}_v=-6u\boldsymbol{i}+2\boldsymbol{j}-6u\boldsymbol{k}$. Since the point (2, 3, 0) corresponds to $u=1$, $v=1$, a normal vector to the surface at (2, 3, 0) is $-6\boldsymbol{i}+2\boldsymbol{j}-6\boldsymbol{k}$, and an equation of the tangent plane is $-6x+2y-6z=-6$ or $3x-y+3z=3$.

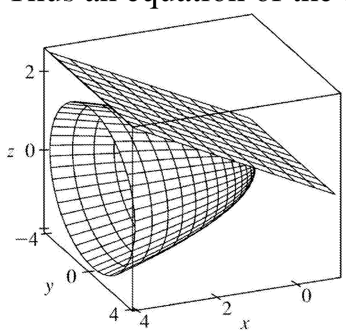


32. $\boldsymbol{r}(u, v)=u^2\boldsymbol{i}+v^2\boldsymbol{j}+uv\boldsymbol{k}\Rightarrow\boldsymbol{r}(1, 1)=(1, 1, 1)$.
$\boldsymbol{r}_u=2u\boldsymbol{i}+v\boldsymbol{k}$ and $\boldsymbol{r}_v=2v\boldsymbol{j}+u\boldsymbol{k}$, so a normal vector to the surface at the point (1, 1, 1) is

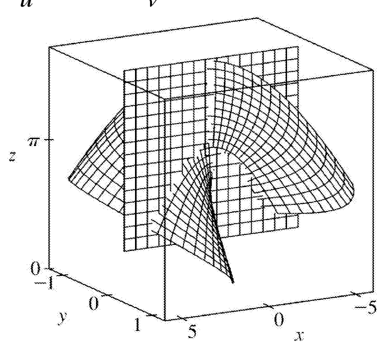
$\boldsymbol{r}_u(1, 1)\times\boldsymbol{r}_v(1, 1)=(2\boldsymbol{i}+\boldsymbol{k})\times(2\boldsymbol{j}+\boldsymbol{k})=-2\boldsymbol{i}-2\boldsymbol{j}+4\boldsymbol{k}$. Thus an equation of the tangent plane at the point (1, 1, 1) is $-2(x-1)-2(y-1)+4(z-1)=0$ or $x+y-2z=0$.



33. $\boldsymbol{r}(u, v)=u^2\boldsymbol{i}+2u\sin v\,\boldsymbol{j}+u\cos v\,\boldsymbol{k}\Rightarrow\boldsymbol{r}(1, 0)=(1, 0, 1)$.
$\boldsymbol{r}_u=2u\boldsymbol{i}+2\sin v\,\boldsymbol{j}+\cos v\,\boldsymbol{k}$ and $\boldsymbol{r}_v=2u\cos v\,\boldsymbol{j}-u\sin v\,\boldsymbol{k}$, so a normal vector to the surface at the point (1, 0, 1) is
$\boldsymbol{r}_u(1, 0)\times\boldsymbol{r}_v(1, 0)=(2\boldsymbol{i}+\boldsymbol{k})\times(2\boldsymbol{j})=-2\boldsymbol{i}+4\boldsymbol{k}$.
Thus an equation of the tangent plane at (1, 0, 1) is $-2(x-1)+0(y-0)+4(z-1)=0$ or $-x+2z=1$.



34. $\boldsymbol{r}(u, v)=uv\boldsymbol{i}+u\sin v\,\boldsymbol{j}+v\cos u\,\boldsymbol{k}\Rightarrow\boldsymbol{r}(0, \pi)=(0, 0, \pi)$
$\boldsymbol{r}_u=v\boldsymbol{i}+\sin v\,\boldsymbol{j}-v\sin u\,\boldsymbol{k}$ and $\boldsymbol{r}_v=u\boldsymbol{i}+u\cos v\,\boldsymbol{j}+\cos u\,\boldsymbol{k}$, so a normal vector to the surface at the point $(0, 0, \pi)$ is
$\boldsymbol{r}_u(0, \pi)\times\boldsymbol{r}_v(0, \pi)=(\pi\boldsymbol{i})\times(\boldsymbol{k})=-\pi\boldsymbol{j}$. Thus an equation of the tangent plane is $-\pi(y-0)=0$ or $y=0$.



35. Here $z=f(x,y)=4-x-2y$ and D is the disk $x^2+y^2\leq 4$. Thus, by Formula 9,

$$A(S)=\iint_D \sqrt{1+(-1)^2+(-2)^2}\,dA=\sqrt{6}\iint_D dA=\sqrt{6}\,A(D)=4\sqrt{6}\,\pi$$

36. $\boldsymbol{r}_u=\langle 0, 1, -5\rangle$, $\boldsymbol{r}_v=\langle 1, -2, 1\rangle$, and $\boldsymbol{r}_u\times\boldsymbol{r}_v=\langle -9, -5, -1\rangle$. Then by Definition 6, $A(S)=\iint_D |\boldsymbol{r}_u\times\boldsymbol{r}_v|dA=$ $\int_0^1\int_0^1 |\langle -9, -5, -1\rangle|\,du\,dv=\sqrt{107}\int_0^1 du\int_0^1 dv=\sqrt{107}$

37. $z=f(x, y)=xy$ with $0\le x^2+y^2\le 1$, so $f_x=y$, $f_y=x\Rightarrow$

$$A(S) = \iint_D \sqrt{1+y^2+x^2}\,dA = \int_0^{2\pi}\int_0^1 \sqrt{r^2+1}\,r\,dr\,d\theta = \int_0^{2\pi}\left[\frac{1}{3}(r^2+1)^{3/2}\right]_{r=0}^{r=1} d\theta$$

$$= \int_0^{2\pi}\frac{1}{3}(2\sqrt{2}-1)d\theta=\frac{2\pi}{3}(2\sqrt{2}-1)$$

38. $z=f(x,y)=1+3x+2y^2$ with $0\le x\le 2y$, $0\le y\le 1$. Thus, by Formula 9,

$$A(S)=\iint_D \sqrt{1+3^2+(4y)^2}\,dA=\int_0^1\int_0^{2y}\sqrt{10+16y^2}\,dx\,dy=\int_0^1 2y\sqrt{10+16y^2}\,dy$$

$$=\frac{1}{16}\cdot\frac{2}{3}(10+16y^2)^{3/2}\Big]_0^1=\frac{1}{24}(26^{3/2}-10^{3/2})$$

39. $z=f(x, y)=y^2-x^2$ with $1\le x^2+y^2\le 4$. Then

$$A(S) = \iint_D \sqrt{1+4x^2+4y^2}\,dA = \int_0^{2\pi}\int_1^2 \sqrt{1+4r^2}\,r\,dr\,d\theta = \int_0^{2\pi} d\theta\int_1^2 \sqrt{1+4r^2}\,r\,dr$$

$$= [\theta]_0^{2\pi}\left[\frac{1}{12}(1+4r^2)^{3/2}\right]_1^2=\frac{\pi}{6}(17\sqrt{17}-5\sqrt{5})$$

40. A parametric representation of the surface is $x=y^2+z^2$, $y=y$, $z=z$ with $0\le y^2+z^2\le 9$. Hence $\boldsymbol{r}_y\times\boldsymbol{r}_z=(2y\boldsymbol{i}+\boldsymbol{j})\times(2z\boldsymbol{i}+\boldsymbol{k})=\boldsymbol{i}-2y\,\boldsymbol{j}-2z\boldsymbol{k}$.

Note: In general, if $x=f(y, z)$ then $\boldsymbol{r}_y\times\boldsymbol{r}_z=\boldsymbol{i}-\frac{\partial f}{\partial y}\boldsymbol{j}-\frac{\partial f}{\partial z}\boldsymbol{k}$, and $A(S)=\iint_D \sqrt{1+\left(\frac{\partial f}{\partial y}\right)^2+\left(\frac{\partial f}{\partial z}\right)^2}\,dA.$

Then

$$A(S) = \iint_{0\le y^2+z^2\le 9} \sqrt{1+4y^2+4z^2}\,dA = \int_0^{2\pi}\int_0^3 \sqrt{1+4r^2}\,r\,dr\,d\theta$$
$$= \int_0^{2\pi} d\theta \int_0^3 r\sqrt{1+4r^2}\,dr = 2\pi\left[\frac{1}{12}(1+4r^2)^{3/2}\right]_0^3 = \frac{\pi}{6}(37\sqrt{37}-1)$$

41. A parametric representation of the surface is $x=x$, $y=4x+z^2$, $z=z$ with $0\le x\le 1$, $0\le z\le 1$. Hence $\mathbf{r}_x\times\mathbf{r}_z=(\mathbf{i}+4\mathbf{j})\times(2z\mathbf{j}+\mathbf{k})=4\mathbf{i}-\mathbf{j}+2z\mathbf{k}$.

Note: In general, if $y=f(x,z)$ then $\mathbf{r}_x\times\mathbf{r}_z=\frac{\partial f}{\partial x}\mathbf{i}-\mathbf{j}+\frac{\partial f}{\partial z}\mathbf{k}$ and $A(S)=\iint_D\sqrt{1+\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial z}\right)^2}\,dA$. Then

$$A(S) = \int_0^1\int_0^1\sqrt{17+4z^2}\,dx\,dz=\int_0^1\sqrt{17+4z^2}\,dz$$
$$= \frac{1}{2}\left(z\sqrt{17+4z^2}+\frac{17}{2}\ln\left|2z+\sqrt{4z^2+17}\right|\right)\Bigg]_0^1=\frac{\sqrt{21}}{2}+\frac{17}{4}\left[\ln(2+\sqrt{21})-\ln\sqrt{17}\right]$$

42. Let S_1 be that portion of the surface which lies above the plane $z=0$. Then $A(S)=2A(S_1)$ by symmetry. On S_1 , $z=\sqrt{a^2-x^2}$ so $|\mathbf{r}_x\times\mathbf{r}_y|=\sqrt{1+\frac{x^2}{a^2-x^2}}=\frac{a}{\sqrt{a^2-x^2}}$. Hence

$$A(S_1)=\iint_{0\le x^2+y^2\le a^2}\frac{a}{\sqrt{a^2-x^2}}\,dA=\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\frac{a}{\sqrt{a^2-x^2}}\,dy\,dx=\int_{-a}^{a}2a\,dx=4a^2.$$

Thus $A(S)=8a^2$.

Alternate solution: If $A(S_2)$ is the surface area in the first octant, then $A(S)=8A(S_2)$. A parametric representation of the surface in the first octant is $x=a\sin\theta$, $y=y$, $z=a\cos\theta$ (θ being the angle in the xz –plane measured from the positive z –axis), where $0\le\theta\le\frac{\pi}{2}$ and $0\le y\le a\cos\theta$. The restrictions on y follow from: $x^2+y^2\le a^2$ or $a^2\sin^2\theta+y^2\le a^2$ so $y^2\le a^2(1-\sin^2\theta)$; thus in the first octant $0\le y\le a\cos\theta$. Then $\mathbf{r}_y\times\mathbf{r}_\theta=\langle -a\sin\theta, 0, -a\cos\theta\rangle$ and $A(S_2)=\int_0^{\pi/2}\int_0^{a\cos\theta} a\,dy\,d\theta=\int_0^{\pi/2}a^2\cos\theta\,d\theta=a^2$.

Hence $A(S)=8a^2$.

43. Let $A(S_1)$ be the surface area of that portion of the surface which lies above the plane $z=0$. Then $A(S)=2A(S_1)$. Following Example 10, a parametric representation of S_1 is $x=a\sin\phi\cos\theta$, $y=a\sin\phi\sin\theta$,

$z=a\cos\phi$ and $|\mathbf{r}_\phi\times\mathbf{r}_\theta|=a^2\sin\phi$. For D , $0\le\phi\le\frac{\pi}{2}$ and for each fixed ϕ ,

$\left(x-\frac{1}{2}a\right)^2+y^2\le\left(\frac{1}{2}a\right)^2$ or $\left[a\sin\phi\cos\theta-\frac{1}{2}a\right]^2+a^2\sin^2\phi\sin^2\theta\le(a/2)^2$ implies

$a^2\sin^2\phi-a^2\sin\phi\cos\theta\le 0$ or $\sin\phi(\sin\phi-\cos\theta)\le 0$. But $0\le\phi\le\frac{\pi}{2}$, so $\cos\theta\ge\sin\phi$ or

$\sin\left(\frac{\pi}{2}+\theta\right)\ge\sin\phi$ or $\phi-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}-\phi$. Hence $D=\left\{(\phi,\theta)\mid 0\le\phi\le\frac{\pi}{2},\ \phi-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}-\phi\right\}$.

Then

$$A(S_1)=\int_0^{\pi/2}\int_{\phi-(\pi/2)}^{(\pi/2)-\phi}a^2\sin\phi\,d\theta\,d\phi=a^2\int_0^{\pi/2}(\pi-2\phi)\sin\phi\,d\phi$$

$$=a^2\left[(-\pi\cos\phi)-2(-\phi\cos\phi+\sin\phi)\right]_0^{\pi/2}=a^2(\pi-2)$$

Thus $A(S)=2a^2(\pi-2)$.

Alternate solution: Working on S_1 we could parametrize the portion of the sphere by $x=x$, $y=y$,

$z=\sqrt{a^2-x^2-y^2}$. Then $|\mathbf{r}_x\times\mathbf{r}_y|=\sqrt{1+\frac{x^2}{a^2-x^2-y^2}+\frac{y^2}{a^2-x^2-y^2}}=\frac{a}{\sqrt{a^2-x^2-y^2}}$ and

$$A(S_1)=\iint\limits_{0\le(x-(a/2))^2+y^2\le(a/2)^2}\frac{a}{\sqrt{a^2-x^2-y^2}}\,dA=\int_{-\pi/2}^{\pi/2}\int_0^{a\cos\theta}\frac{a}{\sqrt{a^2-r^2}}\,r\,dr\,d\theta$$

$$\int_{-\pi/2}^{\pi/2}-a(a^2-r^2)^{1/2}\Big]_{r=0}^{r=a\cos\theta}d\theta=\int_{-\pi/2}^{\pi/2}a^2\left[1-(1-\cos^2\theta)^{1/2}\right]d\theta$$

$$\int_{-\pi/2}^{\pi/2}a^2(1-|\sin\theta|)d\theta=2a^2\int_0^{\pi/2}(1-\sin\theta)d\theta=2a^2\left(\frac{\pi}{2}-1\right)$$

Thus $A(S)=4a^2\left(\frac{\pi}{2}-1\right)=2a^2(\pi-2)$.

Notes:

(1) Perhaps working in spherical coordinates is the most obvious approach here. However, you must be careful in setting up D.

(2) In the alternate solution, you can avoid having to use $|\sin\theta|$ by working in the first octant and

then multiplying by 4 . However, if you set up S_1 as above and arrived at $A(S_1)=a^2\pi$, you now see your error.

44. $\mathbf{r}_u=\langle \cos v, \sin v, 0\rangle, \mathbf{r}_v=\langle -u\sin v, u\cos v, 1\rangle$, and $\mathbf{r}_u\times\mathbf{r}_v=\langle \sin v, -\cos v, u\rangle$. Then

$$A(S) = \int_0^{\pi}\int_0^1 \sqrt{1+u^2}\,du\,dv=\int_0^{\pi} dv\int_0^1 \sqrt{1+u^2}\,du$$

$$= \pi\left[\frac{u}{2}\sqrt{u^2+1}+\frac{1}{2}\ln\left|u+\sqrt{u^2+1}\right|\right]_0^1=\frac{\pi}{2}\left[\sqrt{2}+\ln(1+\sqrt{2})\right]$$

45. $\mathbf{r}_u=\langle v, 1, 1\rangle$, $\mathbf{r}_v=\langle u, 1, -1\rangle$ and $\mathbf{r}_u\times\mathbf{r}_v=\langle -2, u+v, v-u\rangle$. Then

$$A(S) = \iint_{u^2+v^2\le 1}\sqrt{4+2u^2+2v^2}\,dA=\int_0^{2\pi}\int_0^1 r\sqrt{4+2r^2}\,dr\,d\theta=\int_0^{2\pi} d\theta\int_0^1 r\sqrt{4+2r^2}\,dr$$

$$= 2\pi\left[\frac{1}{6}(4+2r^2)^{3/2}\right]_0^1=\frac{\pi}{3}(6\sqrt{6}-8)=\pi\left(2\sqrt{6}-\frac{8}{3}\right)$$

46. $z=f(x, y)=\cos(x^2+y^2)$ with $x^2+y^2\le 1$.

$$A(S)=\iint_D\sqrt{1+(-2x\sin(x^2+y^2))^2+(-2y\sin(x^2+y^2))^2}\,dA$$

$$=\iint_D\sqrt{1+4x^2\sin^2(x^2+y^2)+4y^2\sin^2(x^2+y^2)}\,dA$$

$$=\iint_D\sqrt{1+4(x^2+y^2)\sin^2(x^2+y^2)}\,dA$$

$$=\int_0^{2\pi}\int_0^1\sqrt{1+4r^2\sin^2(r^2)}\,r\,dr\,d\theta=\int_0^{2\pi}d\theta\int_0^1 r\sqrt{1+4r^2\sin^2(r^2)}\,dr$$

$$=2\pi\int_0^1 r\sqrt{1+4r^2\sin^2(r^2)}\,dr\approx 4.1073$$

47. $z=f(x, y)=e^{-x^2-y^2}$ with $x^2+y^2\le 4$.

$$A(S)=\iint_D \sqrt{1+\left(-2xe^{-x^2-y^2}\right)^2+\left(-2ye^{-x^2-y^2}\right)^2}\,dA$$

$$=\iint_D \sqrt{1+4(x^2+y^2)e^{-2(x^2+y^2)}}\,dA$$

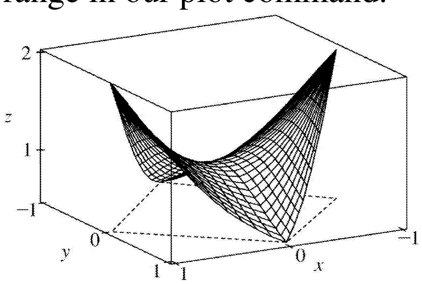
$$=\int_0^{2\pi}\int_0^2 \sqrt{1+4r^2e^{-2r^2}}\,r\,dr\,d\theta=\int_0^{2\pi} d\theta\int_0^2 r\sqrt{1+4r^2e^{-2r^2}}\,dr$$

$$=2\pi\int_0^2 r\sqrt{1+4r^2e^{-2r^2}}\,dr\approx 13.9783$$

48. Let $f(x, y)=\dfrac{1+x^2}{1+y^2}$. Then $f_x=\dfrac{2x}{1+y^2}$, $f_y=(1+x^2)\left[-\dfrac{2y}{(1+y^2)^2}\right]=-\dfrac{2y(1+x^2)}{(1+y^2)^2}$.

We use a CAS to estimate $\int_{-1}^{1}\int_{-(1-|x|)}^{1-|x|}\sqrt{1+f_x^2+f_y^2}\,dy\,dx\approx 2.6959$.

In order to graph only the part of the surface above the square, we use $-(1-|x|)\le y\le 1-|x|$ as the y – range in our plot command.



49. **(a)** The midpoints of the four squares are $\left(\frac{1}{4},\frac{1}{4}\right)$, $\left(\frac{1}{4},\frac{3}{4}\right)$, $\left(\frac{3}{4},\frac{1}{4}\right)$, and $\left(\frac{3}{4},\frac{3}{4}\right)$; the derivatives of the function $f(x, y)=x^2+y^2$ are $f_x(x, y)=2x$ and $f_y(x, y)=2y$, so the Midpoint Rule gives

$$A(S)=\int_0^1\int_0^1\sqrt{[f_x(x, y)]^2+[f_y(x, y)]^2+1}\,dy\,dx$$

$$\approx \frac{1}{4}\left(\sqrt{\left[2\left(\frac{1}{4}\right)\right]^2+\left[2\left(\frac{1}{4}\right)\right]^2+1}+\sqrt{\left[2\left(\frac{1}{4}\right)\right]^2+\left[2\left(\frac{3}{4}\right)\right]^2+1}\right.$$

$$+\sqrt{\left[2\left(\frac{3}{4}\right)\right]^2+\left[2\left(\frac{1}{4}\right)\right]^2+1}+\sqrt{\left[2\left(\frac{3}{4}\right)\right]^2+\left[2\left(\frac{3}{4}\right)\right]^2+1}\Bigg)$$

$$=\frac{1}{4}\left(\sqrt{\frac{3}{2}}+2\sqrt{\frac{7}{2}}+\sqrt{\frac{11}{2}}\right)\approx 1.8279$$

(b) A CAS estimates the integral to be $A(S)=\int_0^1\int_0^1\sqrt{f_x^2+f_y^2+1}\,dy\,dx=\int_0^1\int_0^1\sqrt{4x^2+4y^2+1}\,dy\,dx\approx 1.8616$.

This agrees with the Midpoint estimate only in the first decimal place.

50. $\mathbf{r}(u,v)=\left\langle \cos^3 u\cos^3 v, \sin^3 u\cos^3 v, \sin^3 v\right\rangle$, so $\mathbf{r}_u=\left\langle -3\cos^2 u\sin u\cos^3 v, 3\sin^2 u\cos u\cos^3 v, 0\right\rangle$,

$\mathbf{r}_v=\left\langle -3\cos^3 u\cos^2 v\sin v, -3\sin^3 u\cos^2 v\sin v, 3\sin^2 v\cos v\right\rangle$, and

$\mathbf{r}_u\times\mathbf{r}_v=\left\langle 9\cos u\sin^2 u\cos^4 v\sin^2 v, 9\cos^2 u\sin u\cos^4 v\sin^2 v, 9\cos^2 u\sin^2 u\cos^5 v\sin v\right\rangle$. Then

$$|\mathbf{r}_u\times\mathbf{r}_v|=9\sqrt{\cos^2 u\sin^4 u\cos^8 v\sin^4 v+\cos^4 u\sin^2 u\cos^8 v\sin^4 v+\cos^4 u\sin^4 u\cos^{10} v\sin^2 v}$$

$$=9\sqrt{\cos^2 u\sin^2 u\cos^8 v\sin^2 v(\sin^2 v+\cos^2 u\sin^2 u\cos^2 v)}$$

$$=9\cos^4 v|\cos u\sin u\sin v|\sqrt{\sin^2 v+\cos^2 u\sin^2 u\cos^2 v}$$

Using a CAS, we have $A(S)=\int_0^{\pi}\int_0^{2\pi}9\cos^4 v|\cos u\sin u\sin v|\sqrt{\sin^2 v+\cos^2 u\sin^2 u\cos^2 v}\,dv\,du\approx 4.4506$.

51. $z=1+2x+3y+4y^2$, so

$$A(S)=\iint_D\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA=\int_1^4\int_0^1\sqrt{1+4+(3+8y)^2}\,dy\,dx$$

$$=\int_1^4\int_0^1\sqrt{14+48y+64y^2}\,dy\,dx.$$

Using a CAS, we have

$$\int_1^4\int_0^1\sqrt{14+48y+64y^2}\,dy\,dx=\frac{45}{8}\sqrt{14}+\frac{15}{16}\ln\left(11\sqrt{5}+3\sqrt{14}\sqrt{5}\right)-\frac{15}{16}\ln\left(3\sqrt{5}+\sqrt{14}\sqrt{5}\right)\text{ or}$$

$$\frac{45}{8}\sqrt{14}+\frac{15}{16}\ln\frac{11\sqrt{5}+3\sqrt{70}}{3\sqrt{5}+\sqrt{70}}\ .$$

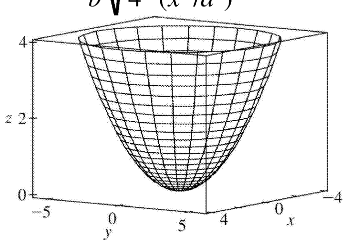
52. **(a)**

$\mathbf{r}_u = a\cos v\,\mathbf{i} + b\sin v\,\mathbf{j} + 2u\,\mathbf{k}$, $\mathbf{r}_v = -au\sin v\,\mathbf{i} + bu\cos v\,\mathbf{j} + 0\,\mathbf{k}$, and $\mathbf{r}_u \times \mathbf{r}_v = -2bu^2\cos v\,\mathbf{i} - 2au^2\sin v\,\mathbf{j} + abu\,\mathbf{k}$.

$$A(S) = \int_0^{2\pi}\int_0^2 \left|\mathbf{r}_u \times \mathbf{r}_v\right| du\,dv = \int_0^{2\pi}\int_0^2 \sqrt{4b^2u^4\cos^2 v + 4a^2u^4\sin^2 v + a^2b^2u^2}\, du\,dv$$

(b) $x^2 = a^2u^2\cos^2 v$, $y^2 = b^2u^2\sin^2 v$, $z = u^2 \Rightarrow x^2/a^2 + y^2/b^2 = u^2 = z$ which is an elliptic paraboloid. To find D , notice that $0 \le u \le 2 \Rightarrow 0 \le z \le 4 \Rightarrow 0 \le x^2/a^2 + y^2/b^2 \le 4$. Therefore, using Formula 9, we have

$$A(S) = \int_{-2a}^{2a}\int_{-b\sqrt{4-(x^2/a^2)}}^{b\sqrt{4-(x^2/a^2)}} \sqrt{1 + (2x/a^2)^2 + (2y/b^2)^2}\, dy\,dx.$$



(c)

(d) We substitute $a=2$, $b=3$ in the integral in part (a) to get

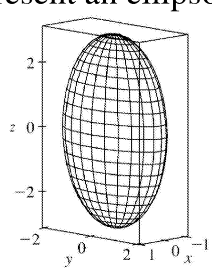
$A(S) = \int_0^{2\pi}\int_0^2 2u\sqrt{9u^2\cos^2 v + 4u^2\sin^2 v + 9}\, du\,dv$. We use a CAS to estimate the integral accurate to four decimal places. To speed up the calculation, we can set Digits:=7; (in Maple) or use the approximation command N (in Mathematica). We find that $A(S) \approx 115.6596$.

53. **(a)** $x = a\sin u\cos v$, $y = b\sin u\sin v$, $z = c\cos u \Rightarrow$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = (\sin u\cos v)^2 + (\sin u\sin v)^2 + (\cos u)^2$$

$$= \sin^2 u + \cos^2 u = 1$$

and since the ranges of u and v are sufficient to generate the entire graph, the parametric equations represent an ellipsoid.



(b)

(c) From the parametric equations (with $a=1$, $b=2$, and $c=3$),

we calculate $\mathbf{r}_u = \cos u\cos v\,\mathbf{i} + 2\cos u\sin v\,\mathbf{j} - 3\sin u\,\mathbf{k}$ and

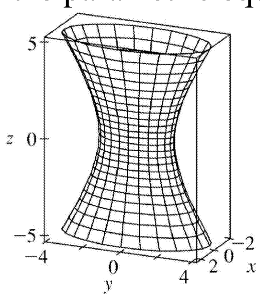
$\boldsymbol{r}_v=-\sin u \sin v\,\boldsymbol{i}+2\sin u\cos v\,\boldsymbol{j}$. So $\boldsymbol{r}_u\times\boldsymbol{r}_v=6\sin^2 u\cos v\,\boldsymbol{i}+3\sin^2 u\sin v\,\boldsymbol{j}+2\sin u\cos u\,\boldsymbol{k}$, and the surface area is given by

$$A(S)=\int_0^{2\pi}\int_0^{\pi}\left|\boldsymbol{r}_u\times\boldsymbol{r}_v\right|du\,dv$$
$$=\int_0^{2\pi}\int_0^{\pi}\sqrt{36\sin^4 u\cos^2 v+9\sin^4 u\sin^2 v+4\cos^{2u}\sin^2 u}\,du\,dv$$

54. **(a)** $x=a\cosh u\cos v$, $y=b\cosh u\sin v$, $z=c\sinh u\Rightarrow$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=\cosh^2 u\cos^2 v+\cosh^2 u\sin^2 v-\sinh^2 u$$
$$=\cosh^2 u-\sinh^2 u=1$$

and the parametric equations represent a hyperboloid of one sheet.



(b)

(c)

$\boldsymbol{r}_u=\sinh u\cos v\,\boldsymbol{i}+2\sinh u\sin v\,\boldsymbol{j}+3\cosh u\,\boldsymbol{k}$ and $\boldsymbol{r}_v=-\cosh u\sin v\,\boldsymbol{i}+2\cosh u\cos v\,\boldsymbol{j}$, so

$\boldsymbol{r}_u\times\boldsymbol{r}_v=-6\cosh^2 u\cos v\,\boldsymbol{i}-3\cosh^2 u\sin v\,\boldsymbol{j}+2\cosh u\sinh u\,\boldsymbol{k}$. We integrate between

$u=\sinh^{-1}(-1)=-\ln(1+\sqrt{2})$ and $u=\sinh^{-1}1=\ln(1+\sqrt{2})$, since then z varies between -3 and 3, as desired. So the surface area is

$$A(S)=\int_0^{2\pi}\int_{-\ln(1+\sqrt{2})}^{\ln(1+\sqrt{2})}\left|\boldsymbol{r}_u\times\boldsymbol{r}_v\right|du\,dv$$
$$=\int_0^{2\pi}\int_{-\ln(1+\sqrt{2})}^{\ln(1+\sqrt{2})}\sqrt{36\cosh^4 u\cos^2 v+9\cosh^4 u\sin^2 v+4\cosh^2 u\sinh^2 u}\,du\,dv$$

55. $\boldsymbol{r}(u,v)=\left\langle\cos^3 u\cos^3 v,\ \sin^3 u\cos^3 v,\ \sin^3 v\right\rangle$, so $\boldsymbol{r}_u=\left\langle-3\cos^2 u\sin u\cos^3 v,\ 3\sin^2 u\cos u\cos^3 v,\ 0\right\rangle$,

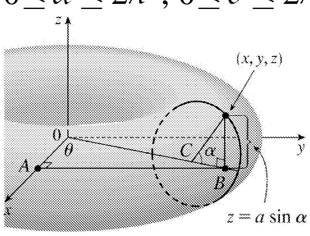
$\boldsymbol{r}_v = \left\langle -3\cos^3 u\cos^2 v\sin v,\ -3\sin^3 u\cos^2 v\sin v,\ 3\sin^2 v\cos v \right\rangle$, and

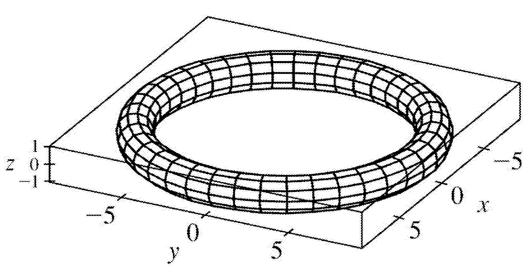
$\boldsymbol{r}_u \times \boldsymbol{r}_v = \left\langle 9\cos u\sin^2 u\cos^4 v\sin^2 v,\ 9\cos^2 u\sin u\cos^4 v\sin^2 v,\ 9\cos^2 u\sin^2 u\cos^5 v\sin v \right\rangle$. Then

$$\begin{aligned} \left|\boldsymbol{r}_u \times \boldsymbol{r}_v\right| &= 9\sqrt{\cos^2 u\sin^4 u\cos^8 v\sin^4 v+\cos^4 u\sin^2 u\cos^8 v\sin^4 v+\cos^4 u\sin^4 u\cos^{10} v\sin^2 v} \\ &= 9\sqrt{\cos^2 u\sin^2 u\cos^8 v\sin^2 v\,(\sin^2 v+\cos^2 u\sin^2 u\cos^2 v)} \\ &= 9\cos^4 v\,|\cos u\sin u\sin v|\sqrt{\sin^2 v+\cos^2 u\sin^2 u\cos^2 v} \end{aligned}$$

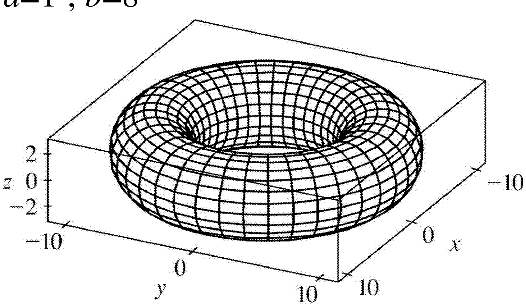
Using a CAS, we have $A(S)=\int_0^{\pi}\int_0^{2\pi} 9\cos^4 v\,|\cos u\sin u\sin v|\sqrt{\sin^2 v+\cos^2 u\sin^2 u\cos^2 v}\,dv\,du \approx 4.4506$.

56. **(a)** Here $z=a\sin\alpha$, $y=|AB|$, and $x=|OA|$. But $|OB|=|OC|+|CB|=b+a\cos\alpha$ and $\sin\theta=\dfrac{|AB|}{|OB|}$ so that $y=|OB|\sin\theta=(b+a\cos\alpha)\sin\theta$. Similarly $\cos\theta=\dfrac{|OA|}{|OB|}$ so $x=(b+a\cos\alpha)\cos\theta$. Hence a parametric representation for the torus is $x=b\cos\theta+a\cos\alpha\cos\theta$, $y=b\sin\theta+a\cos\alpha\sin\theta$, $z=a\sin\alpha$, where $0\le\alpha\le 2\pi$, $0\le\theta\le 2\pi$.

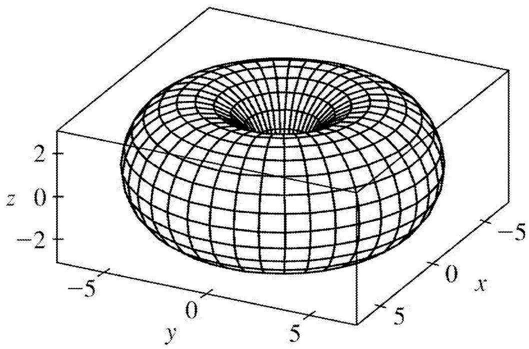




(b)
$a=1$, $b=8$



$a=3$, $b=8$



$a=3$, $b=4$

(c) $x=b\cos\theta+a\cos\alpha\cos\theta$, $y=b\sin\theta+a\cos\alpha\sin\theta$, $z=a\sin\alpha$, so $\mathbf{r}_\alpha=\langle -a\sin\alpha\cos\theta, -a\sin\alpha\sin\theta, a\cos\alpha\rangle$, $\mathbf{r}_\theta=\langle -(b+a\cos\alpha)\sin\theta, (b+a\cos\alpha)\cos\theta, 0\rangle$ and

$$\mathbf{r}_\alpha\times\mathbf{r}_\theta=(-ab\cos\alpha\cos\theta-a^2\cos\alpha\cos^2\theta)\mathbf{i}+(-ab\sin\alpha\cos\theta-a^2\sin\alpha\cos^2\theta)\mathbf{j}$$
$$+(-ab\cos^2\alpha\sin\theta-a^2\cos^2\alpha\sin\theta\cos\theta-ab\sin^2\alpha\sin\theta-a^2\sin^2\alpha\sin\theta\cos\theta)\mathbf{k}$$
$$=-a(b+a\cos\alpha)[(\cos\theta\cos\alpha)\mathbf{i}+(\sin\theta\cos\alpha)\mathbf{j}+(\sin\alpha)\mathbf{k}]$$

Then $|\mathbf{r}_\alpha\times\mathbf{r}_\theta|=a(b+a\cos\alpha)\sqrt{\cos^2\theta\cos^2\alpha+\sin^2\theta\cos^2\alpha+\sin^2\alpha}=a(b+a\cos\alpha)$.

Note: $b>a$, $-1\le\cos\alpha\le 1$ so $|b+a\cos\alpha|=b+a\cos\alpha$. Hence

$$A(S)=\int_0^{2\pi}\int_0^{2\pi}a(b+a\cos\alpha)\,d\alpha\,d\theta=2\pi\left[ab\alpha+a^2\sin\alpha\right]_0^{2\pi}=4\pi^2ab .$$

1. Each face of the cube has surface area $2^2=4$, and the points P_{ij}^* are the points where the cube intersects the coordinate axes. Here, $f(x, y, z)=\sqrt{x^2+2y^2+3z^2}$, so by Definition 1,

$$\iint_S f(x, y, z)dS \approx [f(1, 0, 0)](4)+[f(-1, 0, 0)](4)+[f(0, 1, 0)](4)+[f(0, -1, 0)](4)$$
$$+[f(0, 0, 1)](4)+[f(0, 0, -1)](4)$$
$$= 4(1+1+2\sqrt{2}+2\sqrt{3})=8(1+\sqrt{2}+\sqrt{3})\approx 33.170$$

2. Each quarter–cylinder has surface area $\frac{1}{4}[2\pi(1)(2)]=\pi$, and the top and bottom disks have surface area $\pi(1)^2=\pi$. We can take (0, 0, 1) as a sample point in the top disk, (0, 0, –1) in the bottom disk, and $(\pm 1, 0, 0)$, $(0, \pm 1, 0)$ in the four quarter–cylinders. Then $\iint_S f(x, y, z)dS$ can be approximated by the Riemann sum
$f(1, 0, 0)(\pi)+f(-1, 0, 0)(\pi)+f(0, 1, 0)(\pi)+f(0, -1, 0)(\pi)+f(0, 0, 1)(\pi)+f(0, 0, -1)(\pi)$
$=(2+2+3+3+4+4)\pi=18\pi\approx 56.5$.

3. We can use the xz – and yz –planes to divide H into four patches of equal size, each with surface area equal to $\frac{1}{8}$ the surface area of a sphere with radius $\sqrt{50}$, so $\Delta S=\frac{1}{8}(4)\pi(\sqrt{50})^2=25\pi$. Then $(\pm 3, \pm 4, 5)$ are sample points in the four patches, and using a Riemann sum as in Definition 1, we have

$$\iint_H f(x, y, z)dS \approx f(3, 4, 5)\Delta S+f(3, -4, 5)\Delta S+f(-3, 4, 5)\Delta S+f(-3,-4,5)\Delta S$$
$$= (7+8+9+12)(25\pi)=900\pi\approx 2827$$

4. On the surface, $f(x, y, z)=g\left(\sqrt{x^2+y^2+z^2}\right)=g(2)=-5$. So since the area of a sphere is $4\pi r^2$,

$$\iint_S f(x, y, z)dS=\iint_S g(2)dS=-5\iint_S dS=-5[4\pi(2)^2]=-80\pi .$$

5. $z=1+2x+3y$ so $\frac{\partial z}{\partial x}=2$ and $\frac{\partial z}{\partial y}=3$. Then by Formula 2,

$$\iint_S x^2yzdS = \iint_D x^2yz\sqrt{\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2+1}\,dA$$

$$= \int_0^3 \int_0^2 x^2 y(1+2x+3y)\sqrt{4+9+1}\, dy\, dx$$

$$= \sqrt{14}\int_0^3 \int_0^2 (x^2 y+2x^3 y+3x^2 y^2) dy\, dx$$

$$= \sqrt{14}\int_0^3 \left[\frac{1}{2} x^2 y^2+x^3 y^2+x^2 y^3 \right]_{y=0}^{y=2} dx$$

$$= \sqrt{14}\int_0^3 (10x^2+4x^3) dx=\sqrt{14}\left[\frac{10}{3} x^3+x^4 \right]_0^3=171\sqrt{14}$$

6. S is the region in the plane $2x+y+z=2$ or $z=2-2x-y$ over $D=\{ (x, y)|0\leq x\leq 1,0\leq y\leq 2-2x\}$. Thus

$$\iint_S xy dS = \iint_D xy\sqrt{(-2)^2+(-1)^2+1}\, dA$$

$$= \sqrt{6}\int_0^1 \int_0^{2-2x} xy dy dx=\sqrt{6}\int_0^1 \left[\frac{1}{2} xy^2 \right]_{y=0}^{y=2-2x} dx$$

$$= \frac{\sqrt{6}}{2}\int_0^1 (4x-8x^2+4x^3) dx= \frac{\sqrt{6}}{2}\left(2- \frac{8}{3}+1\right)= \frac{\sqrt{6}}{6}$$

7. S is the part of the plane $z=1-x-y$ over the region $D=\{ (x, y)|0\leq x\leq 1,\ 0\leq y\leq 1-x\}$. Thus

$$\iint_S yz dS = \iint_D y(1-x-y)\sqrt{(-1)^2+(-1)^2+1}\, dA$$

$$= \sqrt{3}\int_0^1 \int_0^{1-x} (y-xy-y^2) dy dx=\sqrt{3}\int_0^1 \left[\frac{1}{2} y^2 - \frac{1}{2} xy^2 - \frac{1}{3} y^3 \right]_{y=0}^{y=1-x} dx$$

$$= \sqrt{3}\int_0^1 \frac{1}{6} (1-x)^3 dx= \left. -\frac{\sqrt{3}}{24} (1-x)^4 \right]_0^1= \frac{\sqrt{3}}{24}$$

8. $z= \frac{2}{3} (x^{3/2}+y^{3/2})$ and

$$\iint_S y dS = \iint_D y\sqrt{(\sqrt{x})^2+(\sqrt{y})^2+1}\, dA=\int_0^1 \int_0^1 y\sqrt{x+y+1}\, dx dy$$

$$= \int_0^1 y\left[\frac{2}{3} (x+y+1)^{3/2} \right]_{x=0}^{x=1} dy=\int_0^1 \frac{2}{3} y\left[(y+2)^{3/2}-(y+1)^{3/2}\right] dy$$

Substituting u=y+2 in the first term and t=y+1 in the second, we have

$$\iint_S y\,dS = \frac{2}{3}\int_2^3 (u-2)u^{3/2}\,du - \frac{2}{3}\int_1^2 (t-1)t^{3/2}\,dt$$
$$= \frac{2}{3}\left[\frac{2}{7}u^{7/2} - \frac{4}{5}u^{5/2}\right]_2^3 - \frac{2}{3}\left[\frac{2}{7}t^{7/2} - \frac{2}{5}t^{5/2}\right]_1^2$$
$$= \frac{2}{3}\left[\frac{2}{7}(3^{7/2}-2^{7/2}) - \frac{4}{5}(3^{5/2}-2^{5/2}) - \frac{2}{7}(2^{7/2}-1) + \frac{2}{5}(2^{5/2}-1)\right]$$
$$= \frac{2}{3}\left(\frac{18}{35}\sqrt{3} + \frac{8}{35}\sqrt{2} - \frac{4}{35}\right) = \frac{4}{105}(9\sqrt{3}+4\sqrt{2}-2)$$

9. S is the portion of the cone $z^2=x^2+y^2$ for $1 \le z \le 3$, or equivalently, S is the part of the surface $z=\sqrt{x^2+y^2}$ over the region $D=\{(x, y) | 1 \le x^2+y^2 \le 9\}$. Thus

$$\iint_S x^2z^2\,dS = \iint_D x^2(x^2+y^2)\sqrt{\left(\frac{x}{\sqrt{x^2+y^2}}\right)^2 + \left(\frac{y}{\sqrt{x^2+y^2}}\right)^2 + 1}\,dA$$
$$= \iint_D x^2(x^2+y^2)\sqrt{\frac{x^2+y^2}{x^2+y^2}+1}\,dA = \iint_D \sqrt{2}\,x^2(x^2+y^2)\,dA$$
$$= \sqrt{2}\int_0^{2\pi}\int_1^3 (r\cos\theta)^2(r^2)\,r\,dr\,d\theta = \sqrt{2}\int_0^{2\pi}\cos^2\theta\,d\theta \int_1^3 r^5\,dr$$
$$= \sqrt{2}\left[\frac{1}{2}\theta + \frac{1}{4}\sin 2\theta\right]_0^{2\pi}\left[\frac{1}{6}r^6\right]_1^3 = \sqrt{2}(\pi)\cdot\frac{1}{6}(3^6-1) = \frac{364\sqrt{2}}{3}\pi$$

10. Using y and z as parameters, we have $\mathbf{r}(y, z)=(y+2z^2)\mathbf{i}+y\,\mathbf{j}+z\,\mathbf{k}$, $0 \le y \le 1$, $0 \le z \le 1$.
Then $\mathbf{r}_y \times \mathbf{r}_z = (\mathbf{i}+\mathbf{j}) \times (4z\mathbf{i}+\mathbf{k}) = \mathbf{i}-\mathbf{j}-4z\mathbf{k}$ and $|\mathbf{r}_y \times \mathbf{r}_z| = \sqrt{2+16z^2}$. Thus

$$\iint_S z\,dS = \int_0^1\int_0^1 z\sqrt{2+16z^2}\,dy\,dz = \int_0^1 z\sqrt{2+16z^2}\,dz = \left[\frac{1}{32}\cdot\frac{2}{3}(2+16z^2)^{3/2}\right]_0^1$$
$$= \frac{1}{48}(18^{3/2}-2^{3/2}) = \frac{13}{12}\sqrt{2}$$

11. Using x and z as parameters, we have $\mathbf{r}(x, z)=x\mathbf{i}+(x^2+z^2)\,\mathbf{j}+z\,\mathbf{k}$, $x^2+z^2 \le 4$. Then $\mathbf{r}_x \times \mathbf{r}_z = (\mathbf{i}+2x\,\mathbf{j}) \times (2z\,\mathbf{j}+\mathbf{k}) = 2x\mathbf{i}-\mathbf{j}+2z\mathbf{k}$ and $|\mathbf{r}_x \times \mathbf{r}_z| = \sqrt{4x^2+1+4z^2} = \sqrt{1+4(x^2+z^2)}$. Thus

$$\iint_S y\,dS = \iint_{x^2+z^2\le 4} (x^2+z^2)\sqrt{1+4(x^2+z^2)}\,dA = \int_0^{2\pi}\int_0^2 r^2\sqrt{1+4r^2}\,r\,dr\,d\theta$$

$$=\int_0^{2\pi} d\theta \int_0^2 r^2\sqrt{1+4r^2}\,r\,dr = 2\pi\int_0^2 r^2\sqrt{1+4r^2}\,r\,dr$$

$$\left[\text{ let } u=1+4r^2 \Rightarrow r^2=\frac{1}{4}(u-1) \text{ and } \frac{1}{8}du=r\,dr\ \right]$$

$$=2\pi\int_1^{17}\frac{1}{4}(u-1)\sqrt{u}\cdot\frac{1}{8}\,du=\frac{1}{16}\pi\int_1^{17}(u^{3/2}-u^{1/2})du$$

$$=\frac{1}{16}\pi\left[\frac{2}{5}u^{5/2}-\frac{2}{3}u^{3/2}\right]_1^{17}=\frac{1}{16}\pi\left[\frac{2}{5}(17)^{5/2}-\frac{2}{3}(17)^{3/2}-\frac{2}{5}+\frac{2}{3}\right]=\frac{\pi}{60}(391\sqrt{17}+1)$$

12. Here S consists of three surfaces: S_1, the lateral surface of the cylinder; S_2, the front formed by the plane $x+y=2$; and the back, S_3, in the plane $y=0$. On S_1: using cylindrical coordinates, $\boldsymbol{r}(\theta, y)=\sin\theta\,\boldsymbol{i}+y\,\boldsymbol{j}+\cos\theta\,\boldsymbol{k}$, $0\le\theta\le 2\pi$, $0\le y\le 2-\sin\theta$, $|\boldsymbol{r}_\theta\times\boldsymbol{r}_y|=1$ and

$$\iint_{S_1} xy\,dS=\int_0^{2\pi}\int_0^{2-\sin\theta}(\sin\theta)y\,dy\,d\theta=\int_0^{2\pi}\left[2\sin\theta-2\sin^2\theta+\frac{1}{2}\sin^3\theta\right]d\theta=-2\pi\ .$$

On S_2: $\boldsymbol{r}(x, z)=x\boldsymbol{i}+(2-x)\boldsymbol{j}+z\boldsymbol{k}$ and $|\boldsymbol{r}_x\times\boldsymbol{r}_z|=|-\boldsymbol{i}-\boldsymbol{j}|=\sqrt{2}$, where $x^2+z^2\le 1$ and

$$\iint_{S_2} xy\,dS = \iint_{x^2+z^2\le 1} x(2-x)\sqrt{2}\,dA=\int_0^{2\pi}\int_0^1\sqrt{2}\,(2r\sin\theta-r^2\sin^2\theta)\,r\,dr\,d\theta$$

$$=\sqrt{2}\int_0^{2\pi}\left[\frac{2}{3}\sin\theta-\frac{1}{4}\sin^2\theta\right]d\theta=-\frac{\sqrt{2}}{4}\pi$$

On S_3: $y=0$ so $\iint_{S_3} xy\,dS=0$. Hence $\int_S xy\,dS=-2\pi-\frac{\sqrt{2}}{4}\pi=-\frac{1}{4}(8+\sqrt{2})\pi$.

13. Using spherical coordinates and Example 17.6.10 we have $\boldsymbol{r}(\phi, \theta)=2\sin\phi\cos\theta\,\boldsymbol{i}+2\sin\phi\sin\theta\,\boldsymbol{j}+2\cos\phi\,\boldsymbol{k}$ and $|\boldsymbol{r}_\phi\times\boldsymbol{r}_\theta|=4\sin\phi$. Then

$$\iint_S (x^2z+y^2z)dS=\int_0^{2\pi}\int_0^{\pi/2}(4\sin^2\phi)(2\cos\phi)(4\sin\phi)\,d\phi\,d\theta=16\pi\sin^4\phi\Big]_0^{\pi/2}=16\pi.$$

14. Using spherical coordinates, $\boldsymbol{r}(\phi, \theta)=\sin\phi\cos\theta\,\boldsymbol{i}+\sin\phi\sin\theta\,\boldsymbol{j}+\cos\phi\,\boldsymbol{k}$,

$0 \le \phi \le \frac{\pi}{4}$, $0 \le \theta \le 2\pi$, and $|\boldsymbol{r}_\phi \times \boldsymbol{r}_\theta| = \sin\phi$ (see Example 17.6.10). Then

$$\iint_S xyz\,dS = \int_0^{2\pi}\int_0^{\pi/4} (\sin^3\phi\cos\phi\cos\theta\sin\theta)\,d\phi\,d\theta = 0 \text{ since } \int_0^{2\pi}\cos\theta\sin\theta\,d\theta = 0 .$$

15. Using cylindrical coordinates, we have $\boldsymbol{r}(\theta, z) = 3\cos\theta\,\boldsymbol{i} + 3\sin\theta\,\boldsymbol{j} + z\boldsymbol{k}$, $0 \le \theta \le 2\pi$, $0 \le z \le 2$, and $|\boldsymbol{r}_\theta \times \boldsymbol{r}_z| = 3$.

$$\iint_S (x^2y + z^2)\,dS = \int_0^{2\pi}\int_0^2 (27\cos^2\theta\sin\theta + z^2)\,3\,dz\,d\theta = \int_0^{2\pi}(162\cos^2\theta\sin\theta + 8)\,d\theta = 16\pi$$

16. Let S_1 be the lateral surface, S_2 the top disk, and S_3 the bottom disk.

On S_1 : $\boldsymbol{r}(\theta, z) = 3\cos\theta\,\boldsymbol{i} + 3\sin\theta\,\boldsymbol{j} + z\boldsymbol{k}$, $0 \le \theta \le 2\pi$, $0 \le z \le 2$, $|\boldsymbol{r}_\theta \times \boldsymbol{r}_z| = 3$,

$$\iint_{S_1} (x^2 + y^2 + z^2)\,dS = \int_0^{2\pi}\int_0^2 (9 + z^2)\,3\,dz\,d\theta = 2\pi(54 + 8) = 124\pi .$$

On S_2 : $\boldsymbol{r}(\theta, r) = r\cos\theta\,\boldsymbol{i} + r\sin\theta\,\boldsymbol{j} + 2\boldsymbol{k}$, $0 \le r \le 3$, $0 \le \theta \le 2\pi$, $|\boldsymbol{r}_\theta \times \boldsymbol{r}_r| = r$,

$$\iint_{S_2} (x^2 + y^2 + z^2)\,dS = \int_0^{2\pi}\int_0^3 (r^2 + 4)\,r\,dr\,d\theta = 2\pi\left(\frac{81}{4} + 18\right) = \frac{153}{2}\pi .$$

On S_3 : $\boldsymbol{r}(\theta, r) = r\cos\theta\,\boldsymbol{i} + r\sin\theta\,\boldsymbol{j}$, $0 \le r \le 3$, $0 \le \theta \le 2\pi$, $|\boldsymbol{r}_\theta \times \boldsymbol{r}_r| = r$,

$$\iint_{S_3} (x^2 + y^2 + z^2)\,dS = \int_0^{2\pi}\int_0^3 (r^2 + 0)\,r\,dr\,d\theta = 2\pi\left(\frac{81}{4}\right) = \frac{81}{2}\pi .$$

Hence $\iint_S (x^2 + y^2 + z^2)\,dS = 124\pi + \frac{153}{2}\pi + \frac{81}{2}\pi = 241\pi$.

17. $\boldsymbol{r}(u, v) = u^2\boldsymbol{i} + u\sin v\,\boldsymbol{j} + u\cos v\,\boldsymbol{k}$, $0 \le u \le 1$, $0 \le v \le \pi/2$ and

$\boldsymbol{r}_u \times \boldsymbol{r}_v = (2u\boldsymbol{i} + \sin v\,\boldsymbol{j} + \cos v\,\boldsymbol{k}) \times (u\cos v\,\boldsymbol{j} - u\sin v\,\boldsymbol{k}) = -u\boldsymbol{i} + 2u^2\sin v\,\boldsymbol{j} + 2u^2\cos v\,\boldsymbol{k}$ and

$|\boldsymbol{r}_u \times \boldsymbol{r}_v| = \sqrt{u^2 + 4u^4\sin^2 v + 4u^4\cos^2 v} = \sqrt{u^2 + 4u^4(\sin^2 v + \cos^2 v)} = u\sqrt{1 + 4u^2}$ (since $u \ge 0$). Then

$$\iint_S yz\,dS = \int_0^{\pi/2}\int_0^1 (u\sin v)(u\cos v)\cdot u\sqrt{1 + 4u^2}\,du\,dv = \int_0^1 u^3\sqrt{1 + 4u^2}\,du\int_0^{\pi/2}\sin v\cos v\,dv$$

$$\left[\text{ let } t = 1 + 4u^2 \Rightarrow u^2 = \frac{1}{4}(t - 1) \text{ and } \frac{1}{8}dt = u\,du\ \right]$$

$$=\int_1^5 \frac{1}{8}\cdot\frac{1}{4}(t-1)\sqrt{t}\,dt\int_0^{\pi/2}\sin v\cos v\,dv=\frac{1}{32}\int_1^5\left(t^{3/2}-\sqrt{t}\right)dt\int_0^{\pi/2}\sin v\cos v\,dv$$

$$=\frac{1}{32}\left[\frac{2}{5}t^{5/2}-\frac{2}{3}t^{3/2}\right]_1^5\left[\frac{1}{2}\sin^2 v\right]_0^{\pi/2}=\frac{1}{32}\left(\frac{2}{5}(5)^{5/2}-\frac{2}{3}(5)^{3/2}-\frac{2}{5}+\frac{2}{3}\right)\cdot\frac{1}{2}(1-0)$$

$$=\frac{5}{48}\sqrt{5}+\frac{1}{240}$$

18. $\boldsymbol{r}_u=\cos v\boldsymbol{i}+\sin v\boldsymbol{j}$, $\boldsymbol{r}_v=-u\sin v\boldsymbol{i}+u\cos v\boldsymbol{j}+\boldsymbol{k}\Rightarrow\boldsymbol{r}_u\times\boldsymbol{r}_v=\sin v\boldsymbol{i}-\cos v\boldsymbol{j}+u\boldsymbol{k}\Rightarrow|\boldsymbol{r}_u\times\boldsymbol{r}_v|=\sqrt{1+u^2}$, so

$$\iint_S\sqrt{1+x^2+y^2}\,dS=\int_0^{\pi}\int_0^1\sqrt{1+u^2}\sqrt{1+u^2}\,du\,dv=\frac{4}{3}\pi\ .$$

19. $\boldsymbol{F}(x,y,z)=xy\boldsymbol{i}+yz\boldsymbol{j}+zx\boldsymbol{k}$, $z=g(x,y)=4-x^2-y^2$, and D is the square $[0,1]\times[0,1]$, so by Equation 8

$$\iint_S\boldsymbol{F}\cdot d\boldsymbol{S}=\iint_D[-xy(-2x)-yz(-2y)+zx]dA$$

$$=\int_0^1\int_0^1[2x^2y+2y^2(4-x^2-y^2)+x(4-x^2-y^2)]dy\,dx$$

$$=\int_0^1\left(\frac{1}{3}x^2+\frac{11}{3}x-x^3+\frac{34}{15}\right)dx=\frac{713}{180}$$

20. $\boldsymbol{F}(x,y,z)=xy\boldsymbol{i}+4x^2\boldsymbol{j}+yz\boldsymbol{k}$, $z=g(x,y)=xe^y$, and D is the square $[0,1]\times[0,1]$, so by Equation 8

$$\iint_S\boldsymbol{F}\cdot d\boldsymbol{S}=\iint_D[-xy(e^y)-4x^2(xe^y)+yz]dA=\int_0^1\int_0^1(-xye^y-4x^3e^y+xye^y)dy\,dx$$

$$=\int_0^1\left[-4x^3e^y\right]_{y=0}^{y=1}dx=(e-1)\int_0^1(-4x^3)dx=1-e$$

21. $\boldsymbol{F}(x,y,z)=xze^y\boldsymbol{i}-xze^y\boldsymbol{j}+z\boldsymbol{k}$, $z=g(x,y)=1-x-y$, and $D=\{(x,y)|0\le x\le 1, 0\le y\le 1-x\}$. Since S has downward orientation, we have

$$\iint_S\boldsymbol{F}\cdot d\boldsymbol{S}=-\iint_D\left[-xze^y(-1)-(-xze^y)(-1)+z\right]dA=-\int_0^1\int_0^{1-x}(1-x-y)dy\,dx$$

$$=-\int_0^1\left(\frac{1}{2}x^2-x+\frac{1}{2}\right)dx=-\frac{1}{6}$$

22. $\boldsymbol{F}(x, y, z)=x\boldsymbol{i}+y\boldsymbol{j}+z^4\boldsymbol{k}$, $z=g(x, y)=\sqrt{x^2+y^2}$, and D is the disk $\{(x, y)|x^2+y^2\leq 1\}$. Since S has downward orientation, we have

$$\iint_S \boldsymbol{F}\cdot d\boldsymbol{S}=-\int_D\left[-x\left(\frac{x}{\sqrt{x^2+y^2}}\right)-y\left(\frac{y}{\sqrt{x^2+y^2}}\right)+z^4\right]dA$$

$$=-\iint_D\left[\frac{-x^2-y^2}{\sqrt{x^2+y^2}}+\left(\sqrt{x^2+y^2}\right)^4\right]dA=-\int_0^{2\pi}\int_0^1\left(\frac{-r^2}{r}+r^4\right)r\,dr\,d\theta$$

$$=-\int_0^{2\pi}d\theta\int_0^1(r^5-r^2)dr=-2\pi\left(\frac{1}{6}-\frac{1}{3}\right)=\frac{\pi}{3}$$

23. $\boldsymbol{F}(x, y, z)=x\boldsymbol{i}-z\boldsymbol{j}+y\boldsymbol{k}$, $z=g(x, y)=\sqrt{4-x^2-y^2}$ and D is the quarter disk $\{(x, y)|0\leq x\leq 2, 0\leq y\leq\sqrt{4-x^2}\}$. S has downward orientation, so by Formula 8,

$$\iint_S \boldsymbol{F}\cdot d\boldsymbol{S}=-\iint_D\left[-x\cdot\frac{1}{2}(4-x^2-y^2)^{-1/2}(-2x)-(-z)\cdot\frac{1}{2}(4-x^2-y^2)^{-1/2}(-2y)+y\right]dA$$

$$=-\iint_D\left(\frac{x^2}{\sqrt{4-x^2-y^2}}-\sqrt{4-x^2-y^2}\cdot\frac{y}{\sqrt{4-x^2-y^2}}+y\right)dA$$

$$=-\iint_D x^2(4-(x^2+y^2))^{-1/2}dA=-\int_0^{\pi/2}\int_0^2(r\cos\theta)^2(4-r^2)^{-1/2}r\,dr\,d\theta$$

$$=-\int_0^{\pi/2}\cos^2\theta\,d\theta\int_0^2 r^3(4-r^2)^{-1/2}dr$$

[let $u=4-r^2\Rightarrow r^2=4-u$ and $-\frac{1}{2}du=r\,dr$]

$$=-\int_0^{\pi/2}\left(\frac{1}{2}+\frac{1}{2}\cos 2\theta\right)d\theta\int_4^0-\frac{1}{2}(4-u)(u)^{-1/2}du$$

$$=-\left[\frac{1}{2}\theta+\frac{1}{4}\sin 2\theta\right]_0^{\pi/2}\left(-\frac{1}{2}\right)\left[8\sqrt{u}-\frac{2}{3}u^{3/2}\right]_4^0$$

$$=-\frac{\pi}{4}\left(-\frac{1}{2}\right)\left(-16+\frac{16}{3}\right)=-\frac{4}{3}\pi$$

24. $\boldsymbol{F}(x, y, z)=xz\boldsymbol{i}+x\boldsymbol{j}+y\boldsymbol{k}$

Using spherical coordinates, S is given by $x=5\sin\phi\cos\theta$, $y=5\sin\phi\sin\theta$, $z=5\cos\phi$, $0\leq\theta\leq\pi$, $0\leq\phi\leq\pi$. $\boldsymbol{F}(\boldsymbol{r}(\phi,\theta))=(5\sin\phi\cos\theta)(5\cos\phi)\boldsymbol{i}+(5\sin\phi\cos\theta)\boldsymbol{j}+(5\sin\phi\sin\theta)\boldsymbol{k}$ and

$\mathbf{r}_\phi \times \mathbf{r}_\theta = 25\sin^2\phi\cos\theta\,\mathbf{i} + 25\sin^2\phi\sin\theta\,\mathbf{j} + 25\cos\phi\sin\phi\,\mathbf{k}$, so

$\mathbf{F}(\mathbf{r}(\phi,\theta))\cdot(\mathbf{r}_\phi\times\mathbf{r}_\theta) = 625\sin^3\phi\cos\phi\cos^2\theta + 125\sin^3\phi\cos\theta\sin\theta + 125\sin^2\phi\cos\phi\sin\theta$

Then

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_D \left[\mathbf{F}(\mathbf{r}(\phi,\theta))\cdot(\mathbf{r}_\phi\times\mathbf{r}_\theta)\right]dA$$

$$= \int_0^\pi\int_0^\pi (625\sin^3\phi\cos\phi\cos^2\theta + 125\sin^3\phi\cos\theta\sin\theta + 125\sin^2\phi\cos\phi\sin\theta)\,d\theta\,d\phi$$

$$= 125\int_0^\pi \left[5\sin^3\phi\cos\phi\left(\frac{1}{2}\theta + \frac{1}{4}\sin 2\theta\right) + \sin^3\phi\left(\frac{1}{2}\sin^2\theta\right) + \sin^2\phi\cos\phi\,(-\cos\theta)\right]_{\theta=0}^{\theta=\pi} d\phi$$

$$= 125\int_0^\pi \left(\frac{5}{2}\pi\sin^3\phi\cos\phi + 2\sin^2\phi\cos\phi\right)d\phi$$

$$= 125\left[\frac{5}{2}\pi\cdot\frac{1}{4}\sin^4\phi + 2\cdot\frac{1}{3}\sin^3\phi\right]_0^\pi = 0$$

25. Let S_1 be the paraboloid $y = x^2 + z^2$, $0 \le y \le 1$ and S_2 the disk $x^2 + z^2 \le 1$, $y = 1$. Since S is a closed surface, we use the outward orientation. On S_1: $\mathbf{F}(\mathbf{r}(x,z)) = (x^2+z^2)\,\mathbf{j} - z\mathbf{k}$ and $\mathbf{r}_x\times\mathbf{r}_z = 2x\mathbf{i} - \mathbf{j} + 2z\mathbf{k}$ (since the **j** –component must be negative on S_1). Then

$$\iint_{S_1} \mathbf{F}\cdot d\mathbf{S} = \iint_{x^2+z^2\le 1} [-(x^2+z^2) - 2z^2]\,dA = -\int_0^{2\pi}\int_0^1 (r^2 + 2r^2\cos^2\theta)\,r\,dr\,d\theta$$

$$= -\int_0^{2\pi} \frac{1}{4}(1+2\cos^2\theta)\,d\theta = -\left(\frac{\pi}{2} + \frac{\pi}{2}\right) = -\pi$$

On S_2: $\mathbf{F}(\mathbf{r}(x,z)) = \mathbf{j} - z\mathbf{k}$ and $\mathbf{r}_z\times\mathbf{r}_x = \mathbf{j}$. Then $\iint_{S_2} \mathbf{F}\cdot d\mathbf{S} = \iint_{x^2+z^2\le 1} (1)\,dA = \pi$. Hence $\iint_S \mathbf{F}\cdot d\mathbf{S} = -\pi + \pi = 0$.

26. Here S consists of three surfaces: S_1, the lateral surface of the cylinder; S_2, the front formed by the plane $x + y = 2$; and the back, S_3, in the plane $y = 0$.

On S_1: $\mathbf{F}(\mathbf{r}(\theta, y)) = \sin\theta\,\mathbf{i} + y\,\mathbf{j} + 5\mathbf{k}$ and $\mathbf{r}_\theta\times\mathbf{r}_y = \sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{k} \Rightarrow$

$$\iint_{S_1} \mathbf{F}\cdot d\mathbf{S} = \int_0^{2\pi}\int_0^{2-\sin\theta} (\sin^2\theta + 5\cos\theta)\,dy\,d\theta$$

$$= \int_0^{2\pi} (2\sin^2\theta + 10\cos\theta - \sin^3\theta - 5\sin\theta\cos\theta)\, d\theta = 2\pi$$

On S_2 : $\boldsymbol{F}(\boldsymbol{r}(x, z)) = x\boldsymbol{i} + (2-x)\boldsymbol{j} + 5\boldsymbol{k}$ and $\boldsymbol{r}_z \times \boldsymbol{r}_x = \boldsymbol{i} + \boldsymbol{j}$.

$$\iint_{S_2} \boldsymbol{F} \cdot d\boldsymbol{S} = \iint_{x^2+z^2 \le 1} [x + (2-x)]\, dA = 2\pi$$

On S_3 : $\boldsymbol{F}(\boldsymbol{r}(x, z)) = x\boldsymbol{i} + 5\boldsymbol{k}$ and $\boldsymbol{r}_x \times \boldsymbol{r}_z = -\boldsymbol{j}$ so $\iint_{S_3} \boldsymbol{F} \cdot d\boldsymbol{S} = 0$. Hence $\iint_S \boldsymbol{F} \cdot d\boldsymbol{S} = 4\pi$.

27. Here S consists of the six faces of the cube as labeled in the figure. On S_1 : $\boldsymbol{F} = \boldsymbol{i} + 2y\boldsymbol{j} + 3z\boldsymbol{k}$,

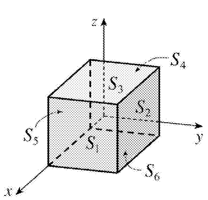
$\boldsymbol{r}_y \times \boldsymbol{r}_z = \boldsymbol{i}$ and $\iint_{S_1} \boldsymbol{F} \cdot d\boldsymbol{S} = \int_{-1}^{1}\int_{-1}^{1} dy\, dz = 4$;

S_2 : $\boldsymbol{F} = x\boldsymbol{i} + 2\boldsymbol{j} + 3z\boldsymbol{k}$, $\boldsymbol{r}_z \times \boldsymbol{r}_x = \boldsymbol{j}$ and $\iint_{S_2} \boldsymbol{F} \cdot d\boldsymbol{S} = \int_{-1}^{1}\int_{-1}^{1} 2\, dx\, dz = 8$;

S_3 : $\boldsymbol{F} = x\boldsymbol{i} + 2y\boldsymbol{j} + 3\boldsymbol{k}$, $\boldsymbol{r}_x \times \boldsymbol{r}_y = \boldsymbol{k}$ and $\iint_{S_3} \boldsymbol{F} \cdot d\boldsymbol{S} = \int_{-1}^{1}\int_{-1}^{1} 3\, dx\, dy = 12$;

S_4 : $\boldsymbol{F} = -\boldsymbol{i} + 2y\boldsymbol{j} + 3z\boldsymbol{k}$, $\boldsymbol{r}_z \times \boldsymbol{r}_y = -\boldsymbol{i}$ and $\iint_{S_4} \boldsymbol{F} \cdot d\boldsymbol{S} = 4$;

S_5 : $\boldsymbol{F} = x\boldsymbol{i} - 2\boldsymbol{j} + 3z\boldsymbol{k}$, $\boldsymbol{r}_x \times \boldsymbol{r}_z = -\boldsymbol{j}$ and $\iint_{S_5} \boldsymbol{F} \cdot d\boldsymbol{S} = 8$;



S_6 : $\boldsymbol{F} = x\boldsymbol{i} + 2y\boldsymbol{j} - 3\boldsymbol{k}$, $\boldsymbol{r}_y \times \boldsymbol{r}_x = -\boldsymbol{k}$ and $\iint_{S_6} \boldsymbol{F} \cdot d\boldsymbol{S} = \int_{-1}^{1}\int_{-1}^{1} 3\, dx\, dy = 12$. Hence $\iint_S \boldsymbol{F} \cdot d\boldsymbol{S} = \sum_{i=1}^{6} \iint_{S_i} \boldsymbol{F} \cdot d\boldsymbol{S} = 48.$

28. $\boldsymbol{r}_u = \cos v\boldsymbol{i} + \sin v\boldsymbol{j}$, $\boldsymbol{r}_v = -u\sin v\boldsymbol{i} + u\cos v\boldsymbol{j} + \boldsymbol{k} \Rightarrow \boldsymbol{r}_u \times \boldsymbol{r}_v = \sin v\boldsymbol{i} - \cos v\boldsymbol{j} + u\boldsymbol{k}$ and

$\boldsymbol{F}(\boldsymbol{r}(u, v)) = u\sin v\boldsymbol{i} + u\cos v\boldsymbol{j} + v^2\boldsymbol{k}$. Then

$$\iint_S \boldsymbol{F}\cdot d\boldsymbol{S} = \int_0^{\pi}\int_0^1 (u\sin^2 v - u\cos^2 v + uv^2)\,du\,dv = \int_0^{\pi}\int_0^1 (-u\cos 2v + uv^2)\,du\,dv$$

$$= \int_0^{\pi}\left[-\frac{1}{2}\cos 2v + \frac{1}{2}v^2\right]dv = \frac{1}{6}\pi^3.$$

29. $z=xy \Rightarrow \partial z/\partial x = y$, $\partial z/\partial y = x$, so by Formula 2, a CAS gives

$$\iint_S xyz\,dS = \int_0^1\int_0^1 xy(xy)\sqrt{y^2+x^2+1}\,dx\,dy \approx 0.1642 .$$

30. As in Exercise 29, we use a CAS to calculate

$$\iint_S x^2yz\,dS = \int_0^1\int_0^1 x^2y(xy)\sqrt{y^2+x^2+1}\,dx\,dy$$

$$= \frac{1}{60}\sqrt{3} - \frac{1}{12}\ln(1+\sqrt{3}) - \frac{1}{192}\ln(\sqrt{2}+1) + \frac{317}{2880}\sqrt{2} + \frac{1}{24}\ln 2$$

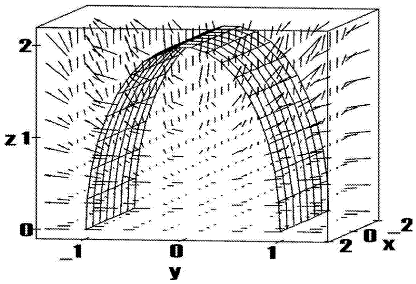
31. We use Formula 2 with $z=3-2x^2-y^2 \Rightarrow \partial z/\partial x = -4x$, $\partial z/\partial y = -2y$. The boundaries of the region $3-2x^2-y^2 \geq 0$ are $-\sqrt{\frac{3}{2}} \leq x \leq \sqrt{\frac{3}{2}}$ and $-\sqrt{3-2x^2} \leq y \leq \sqrt{3-2x^2}$, so we use a CAS (with precision reduced to seven or fewer digits; otherwise the calculation takes a very long time) to calculate

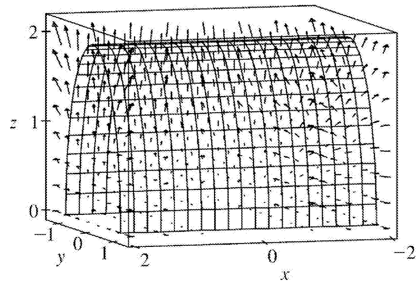
$$\iint_S x^2y^2z^2\,dS = \int_{-\sqrt{3/2}}^{\sqrt{3/2}}\int_{-\sqrt{3-2x^2}}^{\sqrt{3-2x^2}} x^2y^2(3-2x^2-y^2)^2\sqrt{16x^2+4y^2+1}\,dy\,dx \approx 3.4895$$

32. The flux of $\boldsymbol{F}$ across S is given by $\iint_S \boldsymbol{F}\cdot d\boldsymbol{S} = \iint_S \boldsymbol{F}\cdot\boldsymbol{n}\,dS$. Now on S , $z=g(x, y)=2\sqrt{1-y^2}$, so $\partial g/\partial x = 0$ and $\partial g/\partial y = -2y(1-y^2)^{-1/2}$. Therefore, by (8),

$$\iint_S \boldsymbol{F}\cdot d\boldsymbol{S} = \int_{-2}^{2}\int_{-1}^{1}\left(-x^2y\left[-2y(1-y^2)^{-1/2}\right] + \left[2\sqrt{1-y^2}\right]^2 e^{x/5}\right)dy\,dx$$

$$= \frac{1}{3}(16\pi + 80e^{2/5} - 80e^{-2/5})$$





33. If S is given by $y=h(x, z)$, then S is also the level surface $f(x, y, z)=y-h(x, z)=0$.

$\boldsymbol{n}=\dfrac{\nabla f(x, y, z)}{|\nabla f(x, y, z)|}=\dfrac{-h_x\boldsymbol{i}+\boldsymbol{j}-h_z\boldsymbol{k}}{\sqrt{h_x^2+1+h_z^2}}$, and $-\boldsymbol{n}$ is the unit normal that points to the left. Now we proceed as in the derivation of (8), using Formula 2 to evaluate

$$\iint_S \boldsymbol{F}\cdot d\boldsymbol{S}=\iint_S \boldsymbol{F}\cdot \boldsymbol{n}dS$$

$$=\iint_D (P\boldsymbol{i}+Q\boldsymbol{j}+R\boldsymbol{k})\,\frac{\dfrac{\partial h}{\partial x}\boldsymbol{i}-\boldsymbol{j}+\dfrac{\partial h}{\partial z}\boldsymbol{k}}{\sqrt{\left(\dfrac{\partial h}{\partial x}\right)^2+1+\left(\dfrac{\partial h}{\partial z}\right)^2}}\sqrt{\left(\frac{\partial h}{\partial x}\right)^2+1+\left(\frac{\partial h}{\partial z}\right)^2}\,dA$$

where D is the projection of $f(x, y, z)$ onto the xz –plane. Therefore

$$\iint_S \boldsymbol{F}\cdot d\boldsymbol{S}=\iint_D \left(P\frac{\partial h}{\partial x}-Q+R\frac{\partial h}{\partial z}\right)dA$$

34. If S is given by $x=k(y, z)$, then S is also the level surface $f(x, y, z)=x-k(y, z)=0$.

$\boldsymbol{n}=\dfrac{\nabla f(x, y, z)}{|\nabla f(x, y, z)|}=\dfrac{\boldsymbol{i}-k_y\boldsymbol{j}-k_z\boldsymbol{k}}{\sqrt{1+k_y^2+k_z^2}}$, and since the x –component is positive this is the unit normal that points forward. Now we proceed as in the derivation of (8), using Formula 2 for

$$\iint_S \boldsymbol{F}\cdot d\boldsymbol{S}=\iint_S \boldsymbol{F}\cdot \boldsymbol{n}dS$$

$$=\iint_D (P\boldsymbol{i}+Q\boldsymbol{j}+R\boldsymbol{k})\,\frac{\boldsymbol{i}-\dfrac{\partial k}{\partial y}\boldsymbol{j}-\dfrac{\partial k}{\partial z}\boldsymbol{k}}{\sqrt{1+\left(\dfrac{\partial k}{\partial y}\right)^2+\left(\dfrac{\partial k}{\partial z}\right)^2}}\sqrt{1+\left(\frac{\partial k}{\partial y}\right)^2+\left(\frac{\partial k}{\partial z}\right)^2}\,dA$$

where D is the projection of $f(x, y, z)$ onto the yz –plane. Therefore

$$\iint_S \boldsymbol{F}\cdot d\boldsymbol{S}=\iint_D \left(P-Q\frac{\partial k}{\partial y}-R\frac{\partial k}{\partial z}\right)dA$$

35. $m=\iint_S KdS=K\cdot 4\pi\left(\frac{1}{2}a^2\right)=2\pi a^2K$; by symmetry $M_{xz}=M_{yz}=0$, and

$M_{xy}=\iint_S zKdS=K\int_0^{2\pi}\int_0^{\pi/2}(a\cos\phi)(a^2\sin\phi)d\phi\, d\theta=2\pi Ka^3\left[-\frac{1}{4}\cos 2\phi\right]_0^{\pi/2}=\pi Ka^3$. Hence $(\bar{x}, \bar{y}, \bar{z})=\left(0, 0, \frac{1}{2}a\right)$.

36. S is given by $\boldsymbol{r}(x, y)=x\boldsymbol{i}+y\boldsymbol{j}+\sqrt{x^2+y^2}\,\boldsymbol{k}$, $|\boldsymbol{r}_x\times\boldsymbol{r}_y|=\sqrt{1+\frac{x^2+y^2}{x^2+y^2}}=\sqrt{2}$ so

$$m=\iint_S\left(10-\sqrt{x^2+y^2}\right)dS=\iint_{1\le x^2+y^2\le 16}\left(10-\sqrt{x^2+y^2}\right)\sqrt{2}\,dA$$
$$=\int_0^{2\pi}\int_1^4\sqrt{2}\,(10-r)r\,dr\,d\theta=2\pi\sqrt{2}\left[5r^2-\frac{1}{3}r^3\right]_1^4=108\sqrt{2}\,\pi$$

37. **(a)** $I_z=\iint_S(x^2+y^2)\rho(x, y, z)dS$

(b)

$$I_z=\iint_S(x^2+y^2)\left(10-\sqrt{x^2+y^2}\right)dS=\iint_{1\le x^2+y^2\le 16}(x^2+y^2)\left(10-\sqrt{x^2+y^2}\right)\sqrt{2}\,dA$$
$$=\int_0^{2\pi}\int_1^4\sqrt{2}\,(10r^3-r^4)dr\,d\theta=2\sqrt{2}\,\pi\left(\frac{4329}{10}\right)=\frac{4329}{5}\sqrt{2}\,\pi$$

38. S is given by $\boldsymbol{r}(x, y)=x\boldsymbol{i}+y\boldsymbol{j}+\sqrt{x^2+y^2}\,\boldsymbol{k}$ and $|\boldsymbol{r}_x\times\boldsymbol{r}_y|=\sqrt{2}$.

(a) $m=\iint_S kdS=k\int_{0\le x^2+y^2\le a^2}\sqrt{2}\,dS=\sqrt{2}\,a^2k\pi$; by symmetry $M_{xz}=M_{yz}=0$, and

$M_{xy}=\iint_S zk\,dS=k\int_0^{2\pi}\int_0^a \sqrt{2}\,r^2\,dr\,d\theta=\frac{2}{3}\sqrt{2}\,a^3k\pi$. Hence $(\bar{x},\bar{y},\bar{z})=(0,0,\frac{2}{3}a)$.

(b) $I_z=\iint_S (x^2+y^2)k\,dS=\int_0^{2\pi}\int_0^a \sqrt{2}\,kr^3\,dr\,d\theta=2\pi\sqrt{2}\,k\left(\frac{1}{4}a^4\right)=\frac{\sqrt{2}}{2}\pi ka^4$.

39. $\rho(x,y,z)=1200$, $\boldsymbol{V}=y\boldsymbol{i}+\boldsymbol{j}+z\boldsymbol{k}$, $\boldsymbol{F}=\rho\boldsymbol{V}=(1200)(y\boldsymbol{i}+\boldsymbol{j}+z\boldsymbol{k})$. S is given by $\boldsymbol{r}(x,y)=x\boldsymbol{i}+y\boldsymbol{j}+\left[9-\frac{1}{4}(x^2+y^2)\right]\boldsymbol{k}$, $0\le x^2+y^2\le 36$ and $\boldsymbol{r}_x\times\boldsymbol{r}_y=\frac{1}{2}x\boldsymbol{i}+\frac{1}{2}y\boldsymbol{j}+\boldsymbol{k}$.
Thus the rate of flow is given by

$$\iint_S \boldsymbol{F}\cdot d\boldsymbol{S}=\iint_{0\le x^2+y^2\le 36}(1200)\left(\frac{1}{2}xy+\frac{1}{2}y+\left[9-\frac{1}{4}(x^2+y^2)\right]\right)dA$$

$$=1200\int_0^6\int_0^{2\pi}\left[\frac{1}{2}r^2\sin\theta\cos\theta+\frac{1}{2}r\sin\theta+9-\frac{1}{4}r^2\right]r\,d\theta\,dr$$

$$=1200\int_0^6 2\pi\left(9r-\frac{1}{4}r^3\right)dr=(1200)(2\pi)(81)=194{,}400\pi$$

40. $\rho(x,y,z)=1500$, $\boldsymbol{F}=\rho\boldsymbol{V}=(1500)(-y\boldsymbol{i}+x\boldsymbol{j}+2z\boldsymbol{k})$. S is given by
$\boldsymbol{r}(\phi,\theta)=5\sin\phi\cos\theta\,\boldsymbol{i}+5\sin\phi\sin\theta\,\boldsymbol{j}+5\cos\phi\,\boldsymbol{k}$, $0\le\phi\le\pi$, $0\le\theta\le 2\pi$, and
$\boldsymbol{r}_\phi\times\boldsymbol{r}_\theta=25\sin^2\phi\cos\theta\,\boldsymbol{i}+25\sin^2\phi\sin\theta\,\boldsymbol{j}+25\sin\phi\cos\phi\,\boldsymbol{k}$. Thus the rate of outward flow is

$$\iint_S \boldsymbol{F}\cdot d\boldsymbol{S}=1500\int_0^{2\pi}\int_0^{\pi}(-125\sin^3\phi\sin\theta\cos\theta+125\sin^3\phi\sin\theta\cos\theta+250\sin\phi\cos^2\phi)\,d\phi\,d\theta$$

$$=(3000\pi)(250)\left(-\frac{1}{3}\cos^3\phi\right)\Big]_0^{\pi}=500{,}000\pi.$$

41. S consists of the hemisphere S_1 given by $z=\sqrt{a^2-x^2-y^2}$ and the disk S_2 given by $0\le x^2+y^2\le a^2$, $z=0$. On S_1 : $\boldsymbol{E}=a\sin\phi\cos\theta\,\boldsymbol{i}+a\sin\phi\sin\theta\,\boldsymbol{j}+2a\cos\phi\,\boldsymbol{k}$,
$\boldsymbol{T}_\phi\times\boldsymbol{T}_\theta=a^2\sin^2\phi\cos\theta\,\boldsymbol{i}+a^2\sin^2\phi\sin\theta\,\boldsymbol{j}+a^2\sin\phi\cos\phi\,\boldsymbol{k}$. Thus

$$\iint_{S_1}\boldsymbol{E}\cdot d\boldsymbol{S}=\int_0^{2\pi}\int_0^{\pi/2}(a^3\sin^3\phi+2a^3\sin\phi\cos^2\phi)\,d\phi\,d\theta$$

$$=\int_0^{2\pi}\int_0^{\pi/2}(a^3\sin\phi+a^3\sin\phi\cos^2\phi)\,d\phi\,d\theta=(2\pi)a^3\left(1+\frac{1}{3}\right)=\frac{8}{3}\pi a^3$$

On S_2: $\boldsymbol{E}=x\boldsymbol{i}+y\boldsymbol{j}$, and $\boldsymbol{r}_y\times\boldsymbol{r}_x=-\boldsymbol{k}$ so $\iint_{S_2}\boldsymbol{E}\cdot d\boldsymbol{S}=0$.

Hence the total charge is $q=\varepsilon_0\iint_S\boldsymbol{E}\cdot d\boldsymbol{S}=\frac{8}{3}\pi a^3\varepsilon_0$.

42. Referring to the figure in Exercise 27, on

S_1 : $\boldsymbol{E}=\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k}$, $\boldsymbol{r}_y\times\boldsymbol{r}_z=\boldsymbol{i}$ and $\iint_{S_1}\boldsymbol{E}\cdot d\boldsymbol{S}=\int_{-1}^{1}\int_{-1}^{1}dy\,dz=4$;

S_2 : $\boldsymbol{E}=x\boldsymbol{i}+\boldsymbol{j}+z\boldsymbol{k}$, $\boldsymbol{r}_z\times\boldsymbol{r}_x=\boldsymbol{j}$ and $\iint_{S_2}\boldsymbol{E}\cdot d\boldsymbol{S}=\int_{-1}^{1}\int_{-1}^{1}dx\,dz=4$;

S_3 : $\boldsymbol{E}=x\boldsymbol{i}+y\boldsymbol{j}+\boldsymbol{k}$, $\boldsymbol{r}_x\times\boldsymbol{r}_y=\boldsymbol{k}$ and $\iint_{S_3}\boldsymbol{E}\cdot d\boldsymbol{S}=\int_{-1}^{1}\int_{-1}^{1}dx\,dy=4$;

S_4 : $\boldsymbol{E}=-\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k}$, $\boldsymbol{r}_z\times\boldsymbol{r}_y=-\boldsymbol{i}$ and $\iint_{S_4}\boldsymbol{E}\cdot d\boldsymbol{S}=4$.

Similarly $\iint_{S_5}\boldsymbol{E}\cdot d\boldsymbol{S}=\iint_{S_6}\boldsymbol{E}\cdot d\boldsymbol{S}=4$. Hence $q=\varepsilon_0\iint_S\boldsymbol{E}\cdot d\boldsymbol{S}=\varepsilon_0\sum_{i=1}^{6}\iint_{S_i}\boldsymbol{E}\cdot d\boldsymbol{S}=24\varepsilon_0$.

43. $K\nabla u=6.5(4y\boldsymbol{j}+4z\boldsymbol{k})$. S is given by $\boldsymbol{r}(x,\theta)=x\boldsymbol{i}+\sqrt{6}\cos\theta\,\boldsymbol{j}+\sqrt{6}\sin\theta\,\boldsymbol{k}$ and since we want the inward heat flow, we use $\boldsymbol{r}_x\times\boldsymbol{r}_\theta=-\sqrt{6}\cos\theta\,\boldsymbol{j}-\sqrt{6}\sin\theta\,\boldsymbol{k}$. Then the rate of heat flow inward is given by

$$\iint_S(-K\nabla u)\cdot d\boldsymbol{S}=\int_0^{2\pi}\int_0^4-(6.5)(-24)dx\,d\theta=(2\pi)(156)(4)=1248\pi.$$

44. $u(x, y, z)=\dfrac{c}{\sqrt{x^2+y^2+z^2}}$,

$$\boldsymbol{F}=-K\nabla u=-K\left[-\frac{cx}{(x^2+y^2+z^2)^{3/2}}\,\boldsymbol{i}-\frac{cy}{(x^2+y^2+z^2)^{3/2}}\,\boldsymbol{j}-\frac{cz}{(x^2+y^2+z^2)^{3/2}}\,\boldsymbol{k}\right]$$

$$=\frac{cK}{(x^2+y^2+z^2)^{3/2}}(x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k})$$

and the outward unit normal is $\boldsymbol{n}=\frac{1}{a}(x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k})$.

Thus

$\boldsymbol{F}\cdot\boldsymbol{n}=\dfrac{cK}{a(x^2+y^2+z^2)^{3/2}}(x^2+y^2+z^2)$, but on S, $x^2+y^2+z^2=a^2$ so $\boldsymbol{F}\cdot\boldsymbol{n}=\dfrac{cK}{a^2}$. Hence the rate of heat flow across S is $\iint_S \boldsymbol{F}\cdot d\boldsymbol{S}=\dfrac{cK}{a^2}\iint_S dS=\dfrac{cK}{a^2}(4\pi a^2)=4\pi Kc$.

1. Both H and P are oriented piecewise–smooth surfaces that are bounded by the simple, closed, smooth curve $x^2+y^2=4$, $z=0$ (which we can take to be oriented positively for both surfaces). Then H and P satisfy the hypotheses of Stokes' Theorem, so by (3) we know

$\iint_H \operatorname{curl} \boldsymbol{F}\cdot d\boldsymbol{S}=\int_C \boldsymbol{F}\cdot d\boldsymbol{r}=\iint_P \operatorname{curl} \boldsymbol{F}\cdot d\boldsymbol{S}$ (where C is the boundary curve).

2. The plane $z=5$ intersects the paraboloid $z=9-x^2-y^2$ in the circle $x^2+y^2=4$, $z=5$. This boundary curve C is oriented in the counterclockwise direction, so the vector equation is $\boldsymbol{r}(t)=2\cos t\,\boldsymbol{i}+2\sin t\,\boldsymbol{j}+5\boldsymbol{k}$, $0\le t\le 2\pi$. Then $\boldsymbol{r}'(t)=-2\sin t\,\boldsymbol{i}+2\cos t\,\boldsymbol{j}$, $\boldsymbol{F}(\boldsymbol{r}(t))=10\sin t\,\boldsymbol{i}+10\cos t\,\boldsymbol{j}+4\cos t\sin t\,\boldsymbol{k}$, and by Stokes' Theorem,

$$\iint_S \operatorname{curl} \boldsymbol{F}\cdot d\boldsymbol{S}= \int_C \boldsymbol{F}\cdot d\boldsymbol{r}=\int_0^{2\pi} \boldsymbol{F}(\boldsymbol{r}(t))\cdot \boldsymbol{r}'(t)dt=\int_0^{2\pi} (-20\sin^2 t+20\cos^2 t)dt$$
$$= 20\int_0^{2\pi} \cos 2t\,dt=0$$

3. The boundary curve C is the circle $x^2+y^2=4$, $z=0$ oriented in the counterclockwise direction. The vector equation is $\boldsymbol{r}(t)=2\cos t\,\boldsymbol{i}+2\sin t\,\boldsymbol{j}$, $0\le t\le 2\pi$, so $\boldsymbol{r}'(t)=-2\sin t\,\boldsymbol{i}+2\cos t\,\boldsymbol{j}$ and $\boldsymbol{F}(\boldsymbol{r}(t))=(2\cos t)^2 e^{(2\sin t)(0)}\boldsymbol{i}+(2\sin t)^2 e^{(2\cos t)(0)}\boldsymbol{j}+(0)^2 e^{(2\cos t)(2\sin t)}\boldsymbol{k}=4\cos^2 t\,\boldsymbol{i}+4\sin^2 t\,\boldsymbol{j}$. Then, by Stokes' Theorem,

$$\iint_S \operatorname{curl} \boldsymbol{F}\cdot d\boldsymbol{S}= \int_C \boldsymbol{F}\cdot d\boldsymbol{r}=\int_0^{2\pi} \boldsymbol{F}(\boldsymbol{r}(t))\cdot \boldsymbol{r}'(t)\,dt=\int_0^{2\pi} (-8\cos^2 t\sin t+8\sin^2 t\cos t)dt$$
$$= 8\left[\frac{1}{3}\cos^3 t+\frac{1}{3}\sin^3 t\right]_0^{2\pi}=0$$

4. The boundary curve C is the circle $x^2+z^2=9$, $y=3$ with vector equation $\boldsymbol{r}(t)=3\sin t\,\boldsymbol{i}+3\boldsymbol{j}+3\cos t\,\boldsymbol{k}$, $0\le t\le 2\pi$ which gives the positive orientation. Then

$\boldsymbol{F}(\boldsymbol{r}(t))=729\sin^2 t\cos t\,\boldsymbol{i}+\sin(27\sin t\cos t)\,\boldsymbol{j}+27\sin t\cos t\,\boldsymbol{k}$ and

$\boldsymbol{F}(\boldsymbol{r}(t))\cdot \boldsymbol{r}'(t)=2187\sin^2 t\cos^2 t-81\sin^2 t\cos t$. Thus

$$\iint_S \operatorname{curl} \boldsymbol{F}\cdot d\boldsymbol{S}=\oint_C \boldsymbol{F}\cdot d\boldsymbol{r}=\int_0^{2\pi} \boldsymbol{F}(\boldsymbol{r}(t))\cdot \boldsymbol{r}'(t)dt$$
$$=\int_0^{2\pi} (2187\sin^2 t\cos^2 t-81\sin^2 t\cos t)dt=\int_0^{2\pi} \left(2187\left(\frac{1}{2}\sin 2t\right)^2-81\sin^2 t\cos t\right)dt$$

$$=\left[\frac{2187}{4}\left(\frac{1}{2}t-\frac{1}{8}\sin 4t\right)-81\cdot\frac{1}{3}\sin^3 t\right]_0^{2\pi}=\frac{2187}{4}(\pi)-0=\frac{2187}{4}\pi$$

5. C is the square in the plane $z=-1$. By (3), $\iint_{S_1}\operatorname{curl}\boldsymbol{F}\cdot d\boldsymbol{S}=\oint_C \boldsymbol{F}\cdot d\boldsymbol{r}=\iint_{S_2}\operatorname{curl}\boldsymbol{F}\cdot d\boldsymbol{S}$ where S_1 is the original cube without the bottom and S_2 is the bottom face of the cube. $\operatorname{curl}\boldsymbol{F}=x^2z\boldsymbol{i}+(xy-2xyz)\boldsymbol{j}+(y-xz)\boldsymbol{k}$. For S_2, we choose $\boldsymbol{n}=\boldsymbol{k}$ so that C has the same orientation for both surfaces. Then $\operatorname{curl}\boldsymbol{F}\cdot\boldsymbol{n}=y-xz=x+y$ on S_2 , where $z=-1$. Thus $\iint_{S_2}\operatorname{curl}\boldsymbol{F}\cdot d\boldsymbol{S}=\int_{-1}^{1}\int_{-1}^{1}(x+y)dx\,dy=0$ so

$\iint_{S_1}\operatorname{curl}\boldsymbol{F}\cdot d\boldsymbol{S}=0$.

6. The boundary curve C is the unit circle in the yz –plane. By Equation 3,

$\iint_{S_1}\operatorname{curl}\boldsymbol{F}\cdot d\boldsymbol{S}=\oint_C \boldsymbol{F}\cdot d\boldsymbol{r}=\iint_{S_2}\operatorname{curl}\boldsymbol{F}\cdot d\boldsymbol{S}$ where S_1 is the original hemisphere and S_2 is the disk $y^2+z^2\le 1$, $x=0$. $\operatorname{curl}\boldsymbol{F}=(x-x^2)\boldsymbol{i}-(y+e^{xy}\sin z)\boldsymbol{j}+(2xz-xe^{xy}\cos z)\boldsymbol{k}$, and for S_2 we choose $\boldsymbol{n}=\boldsymbol{i}$ so that C has the same orientation for both surfaces. Then curl $\boldsymbol{F}\cdot\boldsymbol{n}=x-x^2$ on S_2, where $x=0$. Thus

$$\iint_{S_2}\operatorname{curl}\boldsymbol{F}\cdot d\boldsymbol{S}=\iint_{y^2+z^2\le 1}(x-x^2)dA=\iint_{y^2+z^2\le 1}0dA=0.$$

Alternatively, we can evaluate $\oint_C \boldsymbol{F}\cdot d\boldsymbol{r}$. C with positive orientation is given by $\boldsymbol{r}(t)=\langle 0,\cos t,\sin t\rangle$, $0\le t\le 2\pi$, and

$$\begin{aligned}\iint_S\operatorname{curl}\boldsymbol{F}\cdot d\boldsymbol{S}&=\oint_C \boldsymbol{F}\cdot d\boldsymbol{r}\\&=\int_0^{2\pi}\left\langle e^{0(\cos t)}\cos(\sin t),(0)^2(\sin t),(0)(\cos t)\right\rangle\cdot\langle 0,-\sin t,\cos t\rangle dt\\&=\int_0^{2\pi}0dt=0\end{aligned}$$

7. curl $\boldsymbol{F}=-2z\boldsymbol{i}-2x\boldsymbol{j}-2y\boldsymbol{k}$ and we take the surface S to be the planar region enclosed by C , so S is the portion of the plane $x+y+z=1$ over $D=\{(x, y)|0\leq x\leq 1, 0\leq y\leq 1-x\}$. Since C is oriented counterclockwise, we orient S upward. Using Equation 17.7.8, we have $z=g(x, y)=1-x-y$, $P=-2z$, $Q=-2x$, $R=-2y$, and

$$\int_C \boldsymbol{F}\cdot d\boldsymbol{r}=\iint_S \text{curl}\,\boldsymbol{F}\cdot d\boldsymbol{S}=\iint_D \left[-(-2z)(-1)-(-2x)(-1)+(-2y)\right]dA$$

$$=\int_0^1\int_0^{1-x}(-2)\,dy\,dx=-2\int_0^1(1-x)\,dx=-1$$

8. curl $\boldsymbol{F}=e^x\boldsymbol{k}$ and S is the portion of the plane $2x+y+2z=2$ over $D=\{(x, y)|0\leq x\leq 1, 0\leq y\leq 2-2x\}$. We orient S upward and use Equation 17.7.8 with $z=g(x, y)=1-x-\frac{1}{2}y$:

$$\int_C \boldsymbol{F}\cdot d\boldsymbol{r}=\iint_S \text{curl}\,\boldsymbol{F}\cdot d\boldsymbol{S}=\iint_D (0+0+e^x)\,dA=\int_0^1\int_0^{2-2x} e^x\,dy\,dx$$

$$=\int_0^1(2-2x)e^x\,dx=\left[(2-2x)e^x+2e^x\right]_0^1 \quad \text{[by integrating by parts]}$$

$$=2e-4$$

9. curl $\boldsymbol{F}=(xe^{xy}-2x)\boldsymbol{i}-(ye^{xy}-y)\boldsymbol{j}+(2z-z)\boldsymbol{k}$ and we take S to be the disk $x^2+y^2\leq 16$, $z=5$. Since C is oriented counterclockwise (from above), we orient S upward. Then $\boldsymbol{n}=\boldsymbol{k}$ and curl $\boldsymbol{F}\cdot\boldsymbol{n}=2z-z$ on S, where $z=5$. Thus

$$\oint \boldsymbol{F}\cdot d\boldsymbol{r}=\iint_S \text{curl}\,\boldsymbol{F}\cdot\boldsymbol{n}\,dS=\iint_S (2z-z)\,dS=\iint_S (10-5)\,dS=5\,(\text{area of } S)=5(\pi\cdot 4^2)=80\pi$$

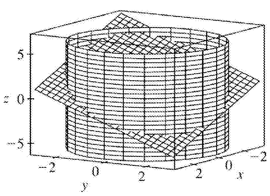
10. S is the part of the surface $z=1-x^2-y^2$ in the first octant. curl$\boldsymbol{F}=2y\boldsymbol{i}-2x\boldsymbol{j}$.
Using Equation 17.7.8 with $g(x, y)=1-x^2-y^2$, $P=2y$, $Q=-2x$, we have

$$\int_C \boldsymbol{F}\cdot d\boldsymbol{r}=\iint_S \text{curl}\,\boldsymbol{F}\cdot d\boldsymbol{S}=\iint_D \left[-2y(-2x)+(2x)(-2y)\right]dA=\iint_D 0\,dA=0.$$

11. **(a)** The curve of intersection is an ellipse in the plane $x+y+z=1$ with unit normal $\boldsymbol{n}=\frac{1}{\sqrt{3}}(\boldsymbol{i}+\boldsymbol{j}+\boldsymbol{k})$, curl$\boldsymbol{F}=x^2\boldsymbol{j}+y^2\boldsymbol{k}$ and curl$\boldsymbol{F}\cdot\boldsymbol{n}=\frac{1}{\sqrt{3}}(x^2+y^2)$. Then

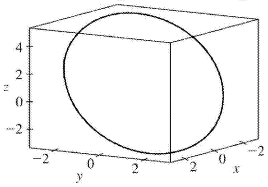
$$\oint_C \boldsymbol{F}\cdot d\boldsymbol{r} = \int_S \frac{1}{\sqrt{3}}\,(x^2+y^2)dS = \iint_{x^2+y^2\le 9} (x^2+y^2)dx\,dy$$

$$= \int_0^{2\pi}\int_0^3 r^3 dr\,d\theta = 2\pi\left(\frac{81}{4}\right) = \frac{81\pi}{2}$$



(b)

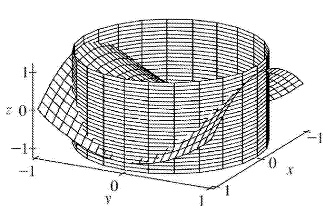
(c) One possible parametrization is $x=3\cos t$, $y=3\sin t$, $z=1-3\cos t-3\sin t$, $0\le t\le 2\pi$.



12. **(a)** S is the part of the surface $z=y^2-x^2$ that lies above the unit disk D .
$\text{curl}\boldsymbol{F}=x\boldsymbol{i}-y\boldsymbol{j}+(x^2-x^2)\boldsymbol{k}=x\boldsymbol{i}-y\boldsymbol{j}$. Using Equation 17.7.8 with $g(x, y)=y^2-x^2$, $P=x$, $Q=-y$, we have

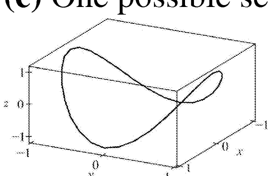
$$\int_C \boldsymbol{F}\cdot d\boldsymbol{r} = \iint_S \text{curl}\boldsymbol{F}\cdot d\boldsymbol{S} = \iint_D [-x(-2x)-(-y)(2y)]dA = 2\iint_D (x^2+y^2)dA$$

$$= 2\int_0^{2\pi}\int_0^1 r^2\, r\,dr\,d\theta = 2(2\pi)\left[\frac{1}{4}r^4\right]_0^1 = \pi$$



(b)

(c) One possible set of parametric equations is $x=\cos t$, $y=\sin t$, $z=\sin^2 t-\cos^2 t$, $0\le t\le 2\pi$.



13. The boundary curve C is the circle $x^2+y^2=1$, $z=1$ oriented in the counterclockwise direction as viewed from above. We can parametrize C by $\boldsymbol{r}(t)=\cos t\,\boldsymbol{i}+\sin t\,\boldsymbol{j}+\boldsymbol{k}$, $0\le t\le 2\pi$, and then

$\boldsymbol{r}'(t)=-\sin t\,\boldsymbol{i}+\cos t\,\boldsymbol{j}$. Thus $\boldsymbol{F}(\boldsymbol{r}(t))=\sin^2 t\,\boldsymbol{i}+\cos t\,\boldsymbol{j}+\boldsymbol{k}$, $\boldsymbol{F}(\boldsymbol{r}(t))\cdot\boldsymbol{r}'(t)=\cos^2 t-\sin^3 t$, and

$$\oint_C \boldsymbol{F}\cdot d\boldsymbol{r} = \int_0^{2\pi}(\cos^2 t-\sin^3 t)dt=\int_0^{2\pi}\frac{1}{2}(1+\cos 2t)dt-\int_0^{2\pi}(1-\cos^2 t)\sin t\,dt$$

$$=\frac{1}{2}\left[t+\frac{1}{2}\sin 2t\right]_0^{2\pi}-\left[-\cos t+\frac{1}{3}\cos^3 t\right]_0^{2\pi}=\pi$$

Now curl $\boldsymbol{F}=(1-2y)\boldsymbol{k}$, and the projection D of S on the xy –plane is the disk $x^2+y^2\leq 1$, so by Equation 17.7.8 [ET 16.7.8] with $z=g(x, y)=x^2+y^2$ we have

$$\iint_S \text{curl}\boldsymbol{F}\cdot d\boldsymbol{S}=\iint_D(1-2y)dA=\int_0^{2\pi}\int_0^1(1-2r\sin\theta)r\,dr\,d\theta=\int_0^{2\pi}\left(\frac{1}{2}-\frac{2}{3}\sin\theta\right)d\theta=\pi$$

14. The plane intersects the coordinate axes at $x=1$, $y=z=2$ so the boundary curve C consists of the three line segments C_1: $\boldsymbol{r}_1(t)=(1-t)\boldsymbol{i}+2t\,\boldsymbol{j}$, $0\leq t\leq 1$, C_2: $\boldsymbol{r}_2(t)=(2-2t)\boldsymbol{j}+2t\,\boldsymbol{k}$, $0\leq t\leq 1$, C_3: $\boldsymbol{r}_3(t)=t\,\boldsymbol{i}+(2-2t)\boldsymbol{k}$, $0\leq t\leq 1$. Then

$$\oint_C \boldsymbol{F}\cdot d\boldsymbol{r}=\int_0^1[(1-t)\boldsymbol{i}+2t\,\boldsymbol{j}]\cdot(-\boldsymbol{i}+2\boldsymbol{j})dt+\int_0^1[(2-2t)\boldsymbol{j}]\cdot(-2\boldsymbol{j}+2\boldsymbol{k})dt+\int_0^1(t\,\boldsymbol{i})\cdot(\boldsymbol{i}-2\boldsymbol{k})dt$$

$$=\int_0^1(5t-1)dt+\int_0^1(4t-4)dt+\int_0^1 t\,dt=\frac{3}{2}-2+\frac{1}{2}=0$$

Now curl $\boldsymbol{F}=xz\boldsymbol{i}-yz\boldsymbol{j}$, so by Equation 17.7.8 [ET 16.7.8] with $z=g(x, y)=2-2x-y$ we have

$$\iint_S \text{curl}\boldsymbol{F}\cdot d\boldsymbol{S}=\iint_D[-x(2-2x-y)(-2)+y(2-2x-y)(-1)]dA$$

$$=\int_0^1\int_0^{2-2x}(4x-4x^2-2y+y^2)dy\,dx$$

$$=\int_0^1\left[4x(2-2x)-4x^2(2-2x)-(2-2x)^2+\frac{1}{3}(2-2x)^3\right]dx$$

$$=\int_0^1\left(\frac{16}{3}x^3-12x^2+8x-\frac{4}{3}\right)dx=\left[\frac{4}{3}x^4-4x^3+4x^2-\frac{4}{3}x\right]_0^1=0$$

15. The boundary curve C is the circle $x^2+z^2=1$, $y=0$ oriented in the counterclockwise direction as viewed from the positive y –axis. Then C can be described by $\boldsymbol{r}(t)=\cos t\,\boldsymbol{i}-\sin t\,\boldsymbol{k}$, $0\leq t\leq 2\pi$, and $\boldsymbol{r}'(t)=-\sin t\,\boldsymbol{i}-\cos t\,\boldsymbol{k}$. Thus $\boldsymbol{F}(\boldsymbol{r}(t))=-\sin t\,\boldsymbol{j}+\cos t\,\boldsymbol{k}$, $\boldsymbol{F}(\boldsymbol{r}(t))\cdot\boldsymbol{r}'(t)=-\cos^2 t$, and

$$\oint_C \boldsymbol{F}\cdot d\boldsymbol{r} = \int_0^{2\pi} -\cos^2 t\,dt = \left. -\frac{1}{2}t - \frac{1}{4}\sin 2t \right]_0^{2\pi} = -\pi$$

Now curl $\boldsymbol{F} = -\boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k}$, and S can be parametrized (see Example 17.6.10 [ET 17.6.10]) by $\boldsymbol{r}(\phi, \theta) = \sin\phi\cos\theta\,\boldsymbol{i} + \sin\phi\sin\theta\,\boldsymbol{j} + \cos\phi\,\boldsymbol{k}$, $0 \le \theta \le \pi$, $0 \le \phi \le \pi$. Then $\boldsymbol{r}_\phi \times \boldsymbol{r}_\theta = \sin^2\phi\cos\theta\,\boldsymbol{i} + \sin^2\phi\sin\theta\,\boldsymbol{j} + \sin\phi\cos\phi\,\boldsymbol{k}$ and

$$\iint_S \operatorname{curl}\boldsymbol{F}\cdot d\boldsymbol{S} = \iint_{x^2+z^2\le 1} \operatorname{curl}\boldsymbol{F}\cdot(\boldsymbol{r}_\phi \times \boldsymbol{r}_\theta)\,dA$$

$$= \int_0^{\pi}\int_0^{\pi} (-\sin^2\phi\cos\theta - \sin^2\phi\sin\theta - \sin\phi\cos\phi)\,d\theta\,d\phi$$

$$= \int_0^{\pi} (-2\sin^2\phi - \pi\sin\phi\cos\phi)\,d\phi = \left[\frac{1}{2}\sin 2\phi - \phi - \frac{\pi}{2}\sin^2\phi\right]_0^{\pi} = -\pi$$

16. The components of $\boldsymbol{F}$ are polynomials, which have continuous partial derivatives throughout R^3 , and both the curve C and the surface S meet the requirements of Stokes' Theorem. If there is a vector field $\boldsymbol{G}$ where $\boldsymbol{F} = \operatorname{curl}\boldsymbol{G}$, then Stokes' Theorem says $\iint_S \boldsymbol{F}\cdot d\boldsymbol{S} = \iint_S$ curl, $\boldsymbol{G}\cdot d\boldsymbol{S}$ depends only on the values of $\boldsymbol{G}$ on C , and hence is independent of the choice of S . By Theorem 17.5.11 [ET 16.5.11], div curl$\boldsymbol{G} = 0$, so div$\boldsymbol{F} = 0 \Leftrightarrow (3ax^2 - 3z^2) + (x^2 + 3by^2) + (3cz^2) = 0 \Leftrightarrow (3a+1)x^2 + 3by^2 + (3c-3)z^2 = 0 \Leftrightarrow$ $a = -\frac{1}{3}$, $b = 0$, $c = 1$.

17. $\operatorname{curl}\boldsymbol{F} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^x + z^2 & y^y + x^2 & z^z + y^2 \end{vmatrix} = 2y\boldsymbol{i} + 2z\boldsymbol{j} + 2x\boldsymbol{k}$ and $W = \int_C \boldsymbol{F}\cdot d\boldsymbol{r} = \iint_S \operatorname{curl}\boldsymbol{F}\cdot d\boldsymbol{S}$.

To parametrize the surface, let $x = 2\cos\theta\sin\phi$, $y = 2\sin\theta\sin\phi$, $z = 2\cos\phi$, so that $\boldsymbol{r}(\phi, \theta) = 2\sin\phi\cos\theta\,\boldsymbol{i} + 2\sin\phi\sin\theta\,\boldsymbol{j} + 2\cos\phi\,\boldsymbol{k}$, $0 \le \phi \le \frac{\pi}{2}$, $0 \le \theta \le \frac{\pi}{2}$, and $\boldsymbol{r}_\phi \times \boldsymbol{r}_\theta = 4\sin^2\phi\cos\theta\,\boldsymbol{i} + 4\sin^2\phi\sin\theta\,\boldsymbol{j} + 4\sin\phi\cos\phi\,\boldsymbol{k}$. Then $\operatorname{curl}\boldsymbol{F}(\boldsymbol{r}(\phi, \theta)) = 4\sin\phi\sin\theta\,\boldsymbol{i} + 4\cos\phi\,\boldsymbol{j} + 4\sin\phi\cos\theta\,\boldsymbol{k}$, and $\operatorname{curl}\boldsymbol{F}\cdot(\boldsymbol{r}_\phi \times \boldsymbol{r}_\theta) = 16\sin^3\phi\sin\theta\cos\theta + 16\cos\phi\sin^2\phi\sin\theta + 16\sin^2\phi\cos\phi\cos\theta$. Therefore

$$\iint_S \operatorname{curl}\boldsymbol{F}\cdot d\boldsymbol{S} = \iint_D \operatorname{curl}\boldsymbol{F}\cdot(\boldsymbol{r}_\phi\times\boldsymbol{r}_\theta)\,dA$$

$$= 16\left[\int_0^{\pi/2}\sin\theta\cos\theta\,d\theta\right]\left[\int_0^{\pi/2}\sin^3\phi\,d\phi\right]+16\left[\int_0^{\pi/2}\sin\theta\,d\theta\right]\left[\int_0^{\pi/2}\sin^2\phi\cos\phi\,d\phi\right]$$

$$+16\left[\int_0^{\pi/2}\cos\theta\,d\theta\right]\left[\int_0^{\pi/2}\sin^2\phi\cos\phi\,d\phi\right]$$

$$= 8\left[-\cos\phi+\frac{1}{3}\cos^3\phi\right]_0^{\pi/2}+16(1)\left[\frac{1}{3}\sin^3\phi\right]_0^{\pi/2}+16(1)\left[\frac{1}{3}\sin^3\phi\right]_0^{\pi/2}$$

$$= 8\left[0+1+0-\frac{1}{3}\right]+16\left(\frac{1}{3}\right)+16\left(\frac{1}{3}\right)=\frac{16}{3}+\frac{16}{3}+\frac{16}{3}=16$$

18. $\int_C (y+\sin x)dx+(z^2+\cos y)dy+x^3dz=\int_C \boldsymbol{F}\cdot d\boldsymbol{r}$, where $\boldsymbol{F}(x, y, z)=(y+\sin x)\boldsymbol{i}+(z^2+\cos y)\,\boldsymbol{j}+x^3\boldsymbol{k}\Rightarrow$ $\operatorname{curl}\boldsymbol{F}=-2z\boldsymbol{i}-3x^2\boldsymbol{j}-\boldsymbol{k}$. Since $\sin 2t=2\sin t\cos t$, C lies on the surface $z=2xy$. Let S be the part of this surface that is bounded by C . Then the projection of S onto the xy –plane is the unit disk $D(x^2+y^2\leq 1)$. C is traversed clockwise (when viewed from above) so S is oriented downward. Using Equation 17.7.8 [ET 16.7.8] with $g(x, y)=2xy$, $P=-2(2xy)=-4xy$, $Q=-3x^2$, $R=-1$, we have

$$\int_C \boldsymbol{F}\cdot d\boldsymbol{r}= -\iint_S \operatorname{curl}\boldsymbol{F}\cdot d\boldsymbol{S}=-\iint_D\Big[-(-4xy)(2y)-(-3x^2)(2x)-1\Big]dA$$

$$= -\iint_D(8xy^2+6x^3-1)dA=-\int_0^{2\pi}\int_0^1(8r^3\cos\theta\sin^2\theta+6r^3\cos^3\theta-1)\,r\,dr\,d\theta$$

$$= -\int_0^{2\pi}\left(\frac{8}{5}\cos\theta\sin^2\theta+\frac{6}{5}\cos^3\theta-\frac{1}{2}\right)rdr\,d\theta$$

$$= -\left[\frac{8}{15}\sin^3\theta+\frac{6}{5}\left(\sin\theta-\frac{1}{3}\sin^3\theta\right)-\frac{1}{2}\theta\right]_0^{2\pi}=\pi$$

19. Assume S is centered at the origin with radius a and let H_1 and H_2 be the upper and lower hemispheres, respectively, of S . Then $\iint_S \operatorname{curl}\boldsymbol{F}\cdot d\boldsymbol{S}=\iint_{H_1}\operatorname{curl},\ \boldsymbol{F}\cdot d\boldsymbol{S}+\iint_{H_2}\operatorname{curl}\boldsymbol{F}\cdot d\boldsymbol{S}=\oint_{C_1}\boldsymbol{F}\cdot d\boldsymbol{r}+\oint_{C_2}\boldsymbol{F}\cdot d\boldsymbol{r}$ by Stokes' Theorem. But C_1 is the circle $x^2+y^2=a^2$ oriented in the counterclockwise direction while C_2 is the same circle oriented in the clockwise direction. Hence

$\oint_{C_2} \boldsymbol{F} \cdot d\boldsymbol{r} = -\oint_{C_1} \boldsymbol{F} \cdot d\boldsymbol{r}$ so $\int\int_S \text{curl}\,\boldsymbol{F} \cdot d\boldsymbol{S} = 0$ as desired.

20. **(a)** By Exercise 17.5.26 [ET 16.5.26], $\text{curl}(f \nabla g) = f \,\text{curl}(\nabla g) + \nabla f \times \nabla g = \nabla f \times \nabla g$ since $\text{curl}(\nabla g) = \mathbf{0}$. Hence by Stokes' Theorem $\int_C (f \nabla g) \cdot d\boldsymbol{r} = \int\int_S (\nabla f \times \nabla g) \cdot d\boldsymbol{S}$.

(b) As in (a), $\text{curl}(f \nabla f) = \nabla f \times \nabla f = \mathbf{0}$, so by Stokes' Theorem, $\int_C (f \nabla f) \cdot d\boldsymbol{r} = \int\int_S [\text{curl}(f \nabla f)] \cdot d\boldsymbol{S} = 0$.

(c) As in part (a),

$$\begin{aligned} \text{curl}(f \nabla g + g \nabla f) &= \text{curl}(f \nabla g) + \text{curl}(g \nabla f) \quad \text{(by Exercise 17.5.24)} \\ &= (\nabla f \times \nabla g) + (\nabla g \times \nabla f) = \mathbf{0} [\text{since}\, \boldsymbol{u} \times \boldsymbol{v} = -(\boldsymbol{v} \times \boldsymbol{u})] \end{aligned}$$

Hence by Stokes' Theorem, $\int_C (f \nabla g + g \nabla f) \cdot d\boldsymbol{r} = \int\int_S \text{curl}(f \nabla g + g \nabla f) \cdot d\boldsymbol{S} = 0$.

1. The vectors that end near P_1 are longer than the vectors that start near P_1 , so the net flow is inward near P_1 and div$\boldsymbol{F}(P_1)$ is negative. The vectors that end near P_2 are shorter than the vectors that start near P_2 , so the net flow is outward near P_2 and div$\boldsymbol{F}(P_2)$ is positive.

2. **(a)** The vectors that end near P_1 are shorter than the vectors that start near P_1 , so the net flow is outward and P_1 is a source. The vectors that end near P_2 are longer than the vectors that start near P_2 , so the net flow is inward and P_2 is a sink.

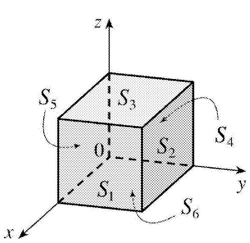
(b) $\boldsymbol{F}(x, y)=\left\langle x, y^2\right\rangle \Rightarrow \text{div}\boldsymbol{F}=\nabla\cdot\boldsymbol{F}=1+2y$. The y –value at P_1 is positive, so div$\boldsymbol{F}=1+2y$ is positive, thus P_1 is a source. At P_2 , $y<-1$, so div$\boldsymbol{F}=1+2y$ is negative, and P_2 is a sink.

3. div$\boldsymbol{F}=3+x+2x=3+3x$, so $\iiint_E \text{div}\boldsymbol{F}dV=\int_0^1\int_0^1\int_0^1(3x+3)dxdydz=\frac{9}{2}$ (notice the triple integral is three times the volume of the cube plus three times $\overline{x}$).

To compute $\iint_S \boldsymbol{F}\cdot d\boldsymbol{S}$ on S_1: $\boldsymbol{n}=\boldsymbol{i}$, $\boldsymbol{F}=3\boldsymbol{i}+y\boldsymbol{j}+2z\boldsymbol{k}$, and $\iint_{S_1}\boldsymbol{F}\cdot d\boldsymbol{S}=\iint_{S_1}3dS=3$;

S_2: $\boldsymbol{F}=3x\boldsymbol{i}+x\boldsymbol{j}+2xz\boldsymbol{k}$, $\boldsymbol{n}=\boldsymbol{j}$ and $\iint_{S_2}\boldsymbol{F}\cdot d\boldsymbol{S}=\iint_{S_2}xdS=\frac{1}{2}$;

S_3: $\boldsymbol{F}=3x\boldsymbol{i}+xy\boldsymbol{j}+2x\boldsymbol{k}$, $\boldsymbol{n}=\boldsymbol{k}$ and $\iint_{S_3}\boldsymbol{F}\cdot d\boldsymbol{S}=\iint_{S_3}2xdS=1$;



S_4: $\boldsymbol{F}=\boldsymbol{0}$, $\iint_{S_4}\boldsymbol{F}\cdot d\boldsymbol{S}=0$; S_5: $\boldsymbol{F}=3x\boldsymbol{i}+2x\boldsymbol{k}$, $\boldsymbol{n}=-\boldsymbol{j}$ and $\iint_{S_5}\boldsymbol{F}\cdot d\boldsymbol{S}=\iint_{S_5}0dS=0$;

S_6: $\boldsymbol{F}=3x\boldsymbol{i}+xy\boldsymbol{j}$, $\boldsymbol{n}=-\boldsymbol{k}$ and $\iint_{S_6}\boldsymbol{F}\cdot d\boldsymbol{S}=\iint_{S_6}0dS=0$. Thus $\iint_S\boldsymbol{F}\cdot d\boldsymbol{S}=\frac{9}{2}$.

4. div$\boldsymbol{F}=2x+x+1=3x+1$ so

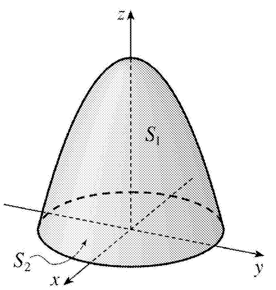
$$\iiint_E \operatorname{div} \boldsymbol{F}\, dV = \iiint_E (3x+1)\,dV = \int_0^{2\pi}\int_0^{2}\int_0^{4-r^2} (3r\cos\theta+1)\, r\,dz\,dr\,d\theta$$

$$= \int_0^{2}\int_0^{2\pi} r(3r\cos\theta+1)(4-r^2)\,d\theta\,dr$$

$$= \int_0^{2\pi} r(4-r^2)\left[3r\sin\theta+\theta\right]_{\theta=0}^{\theta=2\pi} dr$$

$$= 2\pi\int_0^{2}(4r-r^3)\,dr = 2\pi\left[2r^2-\frac{1}{4}r^4\right]_0^2$$

$$= 2\pi(8-4) = 8\pi$$



On S_1 : The surface is $z=4-x^2-y^2, x^2+y^2\le 4$, with upward orientation, and $\boldsymbol{F}=x^2\boldsymbol{i}+xy\,\boldsymbol{j}+(4-x^2-y^2)\boldsymbol{k}$. Then

$$\iint_{S_1} \boldsymbol{F}\cdot d\boldsymbol{S} = \iint_D [-(x^2)(-2x)-(xy)(-2y)+(4-x^2-y^2)]\,dA$$

$$= \iint_D \left[2x(x^2+y^2)+4-(x^2+y^2)\right]dA = \int_0^{2\pi}\int_0^{2}(2r\cos\theta\cdot r^2+4-r^2)\,r\,dr\,d\theta$$

$$= \int_0^{2\pi}\left[\frac{2}{5}r^5\cos\theta+2r^2-\frac{1}{4}r^4\right]_{r=0}^{r=2} d\theta = \int_0^{2\pi}\left(\frac{64}{5}\cos\theta+4\right)d\theta$$

$$= \left[\frac{64}{5}\sin\theta+4\theta\right]_0^{2\pi} = 8\pi$$

On S_2: The surface is $z=0$ with downward orientation, so $\boldsymbol{F}=x^2\boldsymbol{i}+xy\,\boldsymbol{j}$, $\boldsymbol{n}=-\boldsymbol{k}$ and

$$\iint_{S_2} \boldsymbol{F}\cdot \boldsymbol{n}\,dS = \iint_{S_2} 0\,dS = 0.$$

Thus

$\iint_S \boldsymbol{F}\cdot d\boldsymbol{S}=\iint_{S_1} \boldsymbol{F}\cdot d\boldsymbol{S}+\iint_{S_2} \boldsymbol{F}\cdot d\boldsymbol{S}=8\pi$.

5. div $\boldsymbol{F}=x+y+z$, so

$$\iiint_E \operatorname{div}\boldsymbol{F}dV = \int_0^{2\pi}\int_0^1\int_0^1 (r\cos\theta + r\sin\theta + z)r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^1 \left(r^2\cos\theta + r^2\sin\theta + \frac{1}{2}r\right)dr\,d\theta$$

$$= \int_0^{2\pi}\left(\frac{1}{3}\cos\theta + \frac{1}{3}\sin\theta + \frac{1}{4}\right)d\theta = \frac{1}{4}(2\pi) = \frac{\pi}{2}$$

Let S_1 be the top of the cylinder, S_2 the bottom, and S_3 the vertical edge. On S_1, $z=1$, $\boldsymbol{n}=\boldsymbol{k}$, and

$\boldsymbol{F}=xy\boldsymbol{i}+y\boldsymbol{j}+x\boldsymbol{k}$, so $\iint_{S_1} \boldsymbol{F}\cdot d\boldsymbol{S}=\iint_{S_1} \boldsymbol{F}\cdot \boldsymbol{n}dS=\iint_{S_1} x dS=\int_0^{2\pi}\int_0^1 (r\cos\theta)r\,dr\,d\theta=[\sin\theta]_0^{2\pi}\left[\frac{1}{3}r^3\right]_0^1=0$. On S_2,

$z=0$, $\boldsymbol{n}=-\boldsymbol{k}$, and $\boldsymbol{F}=xy\boldsymbol{i}$ so $\iint_{S_2} \boldsymbol{F}\cdot d\boldsymbol{S}=\iint_{S_2} 0 dS=0$. S_3 is given by $\boldsymbol{r}(\theta, z)=\cos\theta\,\boldsymbol{i}+\sin\theta\,\boldsymbol{j}+z\boldsymbol{k}$,

$0\le\theta\le 2\pi$, $0\le z\le 1$. Then $\boldsymbol{r}_\theta\times\boldsymbol{r}_z=\cos\theta\,\boldsymbol{i}+\sin\theta\,\boldsymbol{j}$ and

$$\iint_{S_3} \boldsymbol{F}\cdot d\boldsymbol{S} = \iint_D \boldsymbol{F}\cdot(\boldsymbol{r}_\theta\times\boldsymbol{r}_z)dA=\int_0^{2\pi}\int_0^1 (\cos^2\theta\sin\theta + z\sin^2\theta)dz\,d\theta$$

$$= \int_0^{2\pi}\left(\cos^2\theta\sin\theta + \frac{1}{2}\sin^2\theta\right)d\theta = \left[-\frac{1}{3}\cos^3\theta + \frac{1}{4}\left(\theta - \frac{1}{2}\sin 2\theta\right)\right]_0^{2\pi} = \frac{\pi}{2}$$

Thus $\iint_S \boldsymbol{F}\cdot d\boldsymbol{S}=0+0+\frac{\pi}{2}=\frac{\pi}{2}$.

6. div $\boldsymbol{F}=1+1+1=3$, so $\iiint_E \operatorname{div}\boldsymbol{F}dV=\iiint_E 3dV=3$ (volume of ball) $=3\left(\frac{4}{3}\pi\right)=4\pi$. To find $\iint_S \boldsymbol{F}\cdot d\boldsymbol{S}$ we use spherical coordinates. S is the unit sphere, represented by $\boldsymbol{r}(\phi,\theta)=\sin\phi\cos\theta\,\boldsymbol{i}+\sin\phi\sin\theta\,\boldsymbol{j}+\cos\phi\,\boldsymbol{k}$, $0\le\phi\le\pi$, $0\le\theta\le 2\pi$. Then $\boldsymbol{r}_\phi\times\boldsymbol{r}_\theta=\sin^2\phi\cos\theta\,\boldsymbol{i}+\sin^2\phi\sin\theta\,\boldsymbol{j}+\sin\phi\cos\phi\,\boldsymbol{k}$ (see Example 17.6.10 [ET 16.6.10]) and $\boldsymbol{F}(\boldsymbol{r}(\phi,\theta))=\sin\phi\cos\theta\,\boldsymbol{i}+\sin\phi\sin\theta\,\boldsymbol{j}+\cos\phi\,\boldsymbol{k}$. Thus

$$\iint_S \boldsymbol{F}\cdot d\boldsymbol{S} = \iint_D \boldsymbol{F}\cdot(\boldsymbol{r}_\phi\times\boldsymbol{r}_\theta)dA=\int_0^{2\pi}\int_0^{\pi} (\sin^3\phi\cos^2\theta + \sin^3\phi\sin^2\theta + \sin\phi\cos^2\phi)d\phi\,d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^{\pi} \sin\phi \, d\phi = (2\pi)(2) = 4\pi$$

7. div $\boldsymbol{F} = \frac{\partial}{\partial x}(e^x \sin y) + \frac{\partial}{\partial y}(e^x \cos y) + \frac{\partial}{\partial z}(yz^2) = e^x \sin y - e^x \sin y + 2yz = 2yz$, so by the Divergence Theorem,

$$\iint_S \boldsymbol{F} \cdot d\boldsymbol{S} = \iiint_E \text{div}\,\boldsymbol{F} dV = \int_0^1 \int_0^1 \int_0^2 2yz\,dz\,dy\,dx = 2\int_0^1 dx \int_0^1 y\,dy \int_0^1 z\,dz$$

$$= 2[x]_0^1 \left[\frac{1}{2}y^2\right]_0^1 \left[\frac{1}{2}z^2\right]_0^2 = 2$$

8. div $\boldsymbol{F} = \frac{\partial}{\partial x}(x^2 z^3) + \frac{\partial}{\partial y}(2xyz^3) + \frac{\partial}{\partial z}(xz^4) = 2xz^3 + 2xz^3 + 4xz^3 = 8xz^3$, so by the Divergence Theorem,

$$\iint_S \boldsymbol{F} \cdot d\boldsymbol{S} = \iiint_E \text{div}\,\boldsymbol{F} dV = \int_{-1}^1 \int_{-2}^2 \int_{-3}^3 8xz^3\,dz\,dy\,dx = 8\int_{-1}^1 x\,dx \int_{-2}^2 dy \int_{-3}^3 z^3\,dz$$

$$= 8\left[\frac{1}{2}x^2\right]_{-1}^1 [y]_{-2}^2 \left[\frac{1}{4}z^4\right]_{-3}^3 = 0$$

9. div $\boldsymbol{F} = 3y^2 + 0 + 3z^2$, so using cylindrical coordinates with $y = r\cos\theta$, $z = r\sin\theta$, $x = x$ we have

$$\iint_S \boldsymbol{F} \cdot d\boldsymbol{S} = \iiint_E (3y^2 + 3z^2)\,dV = \int_0^{2\pi} \int_0^1 \int_{-1}^2 (3r^2\cos^2\theta + 3r^2\sin^2\theta)\,r\,dx\,dr\,d\theta$$

$$= 3\int_0^{2\pi} d\theta \int_0^1 r^3\,dr \int_{-1}^2 dx = 3(2\pi)\left(\frac{1}{4}\right)(3) = \frac{9\pi}{2}$$

10. div $\boldsymbol{F} = 3x^2y - 2x^2y - x^2y = 0$, so $\iint_S \boldsymbol{F} \cdot d\boldsymbol{S} = \iiint_E 0\,dV = 0$.

11. div $\boldsymbol{F} = y\sin z + 0 - y\sin z = 0$, so by the Divergence Theorem, $\iint_S \boldsymbol{F} \cdot d\boldsymbol{S} = \iiint_E 0\,dV = 0$.

12. div$\boldsymbol{F} = 2xy + 2xy + 2xy = 6xy$, so

$$\iint_S \boldsymbol{F} \cdot d\boldsymbol{S} = \iiint_E 6xy\,dV$$

$$=\int_0^1\int_0^{2-2y}\int_0^{2-x-2y} 6xy\,dz\,dx\,dy=\int_0^1\int_0^{2-2y} 6xy(2-x-2y)\,dx\,dy$$

$$=\int_0^1\int_0^{2-2y}(12xy-6x^2y-12xy^2)\,dx\,dy=\int_0^1\Big[6x^2y-2x^3y-6x^2y^2\Big]_{x=0}^{x=2-2y}dy$$

$$=\int_0^1 y(2-2y)^3dy=\Big[-\frac{8}{5}y^5+6y^4-8y^3+4y^2\Big]_0^1=\frac{2}{5}$$

13. div $\boldsymbol{F}=y^2+0+x^2=x^2+y^2$ so

$$\iint_S \boldsymbol{F}\cdot d\boldsymbol{S}=\iiint_E(x^2+y^2)dV=\int_0^{2\pi}\int_0^2\int_{r^2}^4 r^2\cdot r\,dz\,dr\,d\theta=\int_0^{2\pi}\int_0^2 r^3(4-r^2)dr\,d\theta$$

$$=\int_0^{2\pi}d\theta\int_0^2(4r^3-r^5)dr=2\pi\Big[r^4-\frac{1}{6}r^6\Big]_0^2=\frac{32}{3}\pi$$

14. div $\boldsymbol{F}=4x^3+4xy^2$ so

$$\iint_S \boldsymbol{F}\cdot d\boldsymbol{S}=\iiint_E 4x(x^2+y^2)dV=\int_0^{2\pi}\int_0^1\int_0^{r\cos\theta+2}(4r^3\cos\theta)r\,dz\,dr\,d\theta$$

$$=\int_0^{2\pi}\int_0^1(4r^5\cos^2\theta+8r^4\cos\theta)dr\,d\theta=\int_0^{2\pi}\Big(\frac{2}{3}\cos^2\theta+\frac{8}{5}\cos\theta\Big)d\theta=\frac{2}{3}\pi$$

15. div $\boldsymbol{F}=12x^2z+12y^2z+12z^3$ so

$$\iint_S \boldsymbol{F}\cdot d\boldsymbol{S}=\iiint_E 12z(x^2+y^2+z^2)dV=\int_0^{2\pi}\int_0^{\pi}\int_0^{R}12(\rho\cos\phi)(\rho^2)\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

$$=12\int_0^{2\pi}d\theta\int_0^{\pi}\sin\phi\cos\phi\,d\phi\int_0^R\rho^5d\rho=12(2\pi)\Big[\frac{1}{2}\sin^2\phi\Big]_0^{\pi}\Big[\frac{1}{6}\rho^6\Big]_0^R=0$$

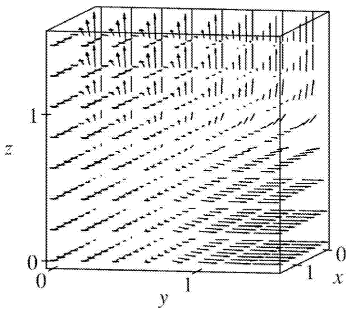
16.

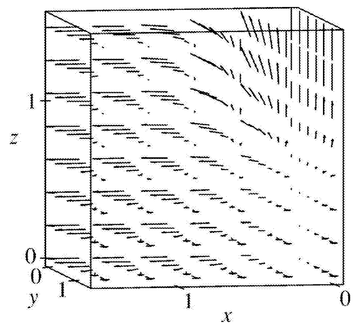
$$\iint_S \boldsymbol{F}\cdot d\boldsymbol{S}=\iiint_E 3(x^2+y^2+1)dV=\int_0^{2\pi}\int_0^{\pi/2}\int_1^2 3(\rho^2\sin^2\phi+1)\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

$$=2\pi\int_0^{\pi/2}\Big[\frac{93}{5}\sin^3\phi+7\sin\phi\Big]d\phi=2\pi\Big[\frac{93}{5}\Big(-\cos\phi+\frac{1}{3}\cos^3\phi\Big)-7\cos\phi\Big]_0^{\pi/2}=\frac{194}{5}\pi$$

17. $\iint_S \boldsymbol{F}\cdot d\boldsymbol{S}=\iiint_E \sqrt{3-x^2}\,dV=\int_{-1}^{1}\int_{-1}^{1}\int_0^{2-x^4-y^4}\sqrt{3-x^2}\,dz\,dy\,dx=\frac{341}{60}\sqrt{2}+\frac{81}{20}\sin^{-1}\left(\frac{\sqrt{3}}{3}\right)$

18.





By the Divergence Theorem, the flux of $\boldsymbol{F}$ across the surface of the cube is

$\iint_S \boldsymbol{F}\cdot d\boldsymbol{S}=\int_0^{\pi/2}\int_0^{\pi/2}\int_0^{\pi/2}\left[\cos x\cos^2 y+3\sin^2 y\cos y\cos^4 z+5\sin^4 z\cos z\cos^6 x\right]dz\,dy\,dx=\frac{19}{64}\pi^2.$

19. For S_1 we have $\boldsymbol{n}=-\boldsymbol{k}$, so $\boldsymbol{F}\cdot\boldsymbol{n}=\boldsymbol{F}\cdot(-\boldsymbol{k})=-x^2z-y^2=-y^2$ (since $z=0$ on S_1). So if D is the unit disk, we get $\iint_{S_1}\boldsymbol{F}\cdot d\boldsymbol{S}=\iint_{S_1}\boldsymbol{F}\cdot\boldsymbol{n}dS=\iint_D(-y^2)dA=-\int_0^{2\pi}\int_0^1 r^2(\sin^2\theta)r\,dr\,d\theta=-\frac{1}{4}\pi$. Now since S_2 is closed, we can use the Divergence Theorem. Since $\text{div}\boldsymbol{F}=\frac{\partial}{\partial x}(z^2x)+\frac{\partial}{\partial y}\left(\frac{1}{3}y^3+\tan z\right)+\frac{\partial}{\partial z}(x^2z+y^2)=z^2+y^2+x^2$, we use spherical coordinates to get $\iint_{S_2}\boldsymbol{F}\cdot d\boldsymbol{S}=\iiint_E \text{div}\boldsymbol{F}dV=\int_0^{2\pi}\int_0^{\pi/2}\int_0^1\rho^2\cdot\rho^2\sin\phi\,d\rho\,d\phi\,d\theta=\frac{2}{5}\pi.$ Finally

$\iint_S\boldsymbol{F}\cdot d\boldsymbol{S}=\iint_{S_2}\boldsymbol{F}\cdot d\boldsymbol{S}-\iint_{S_1}\boldsymbol{F}\cdot d\boldsymbol{S}=\frac{2}{5}\pi-\left(-\frac{1}{4}\pi\right)=\frac{13}{20}\pi.$

20. As in the hint to Exercise 19, we create a closed surface $S_2=S\cup S_1$, where S is the part of the paraboloid $x^2+y^2+z=2$ that lies above the plane $z=1$, and S_1 is the disk $x^2+y^2=1$ on the plane $z=1$ oriented downward, and we then apply the Divergence Theorem. Since the disk S_1 is oriented downward, its unit normal vector is $\boldsymbol{n}=-\boldsymbol{k}$ and $\boldsymbol{F}\cdot(-\boldsymbol{k})=-z=-1$ on S_1. So

$\iint_{S_1}\boldsymbol{F}\cdot d\boldsymbol{S}=\iint_{S_1}\boldsymbol{F}\cdot\boldsymbol{n}dS=\iint_{S_1}(-1)dS=-A(S_1)=-\pi.$ Let E be the region bounded by S_2. Then

$$\iint_{S_2} \boldsymbol{F}\cdot d\boldsymbol{S} = \iiint_E \operatorname{div} \boldsymbol{F} dV = \iiint_E 1 dV = \int_0^1 \int_0^{2\pi} \int_1^{2-r^2} r dz d\theta\, dr = \int_0^1 \int_0^{2\pi} (r-r^3) d\theta\, dr$$

$$=(2\pi)\frac{1}{4}=\frac{\pi}{2}$$

Thus the flux of $\boldsymbol{F}$ across S is $\iint_S \boldsymbol{F}\cdot d\boldsymbol{S} = \iint_{S_2} \boldsymbol{F}\cdot d\boldsymbol{S} - \iint_{S_1} \boldsymbol{F}\cdot d\boldsymbol{S} = \frac{\pi}{2} - (-\pi) = \frac{3\pi}{2}$.

21. Since $\dfrac{\boldsymbol{x}}{|\boldsymbol{x}|^3} = \dfrac{x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k}}{(x^2+y^2+z^2)^{3/2}}$ and $\dfrac{\partial}{\partial x}\left(\dfrac{x}{(x^2+y^2+z^2)^{3/2}}\right) = \dfrac{(x^2+y^2+z^2)-3x^2}{(x^2+y^2+z^2)^{5/2}}$ with similar expressions for $\dfrac{\partial}{\partial y}\left(\dfrac{y}{(x^2+y^2+z^2)^{3/2}}\right)$ and $\dfrac{\partial}{\partial z}\left(\dfrac{z}{(x^2+y^2+z^2)^{3/2}}\right)$, we have

$\operatorname{div}\left(\dfrac{\boldsymbol{x}}{|\boldsymbol{x}|^3}\right) = \dfrac{3(x^2+y^2+z^2)-3(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{5/2}} = 0$, except at (0, 0, 0) where it is undefined.

22. We first need to find $\boldsymbol{F}$ so that $\iint_S \boldsymbol{F}\cdot \boldsymbol{n} dS = \iint_S (2x+2y+z^2) dS$, so $\boldsymbol{F}\cdot\boldsymbol{n} = 2x+2y+z^2$.

But for S, $\boldsymbol{n} = \dfrac{x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k}}{\sqrt{x^2+y^2+z^2}} = x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k}$. Thus $\boldsymbol{F} = 2\boldsymbol{i}+2\boldsymbol{j}+z\boldsymbol{k}$ and $\operatorname{div}\boldsymbol{F}=1$. If

$B = \{(x, y, z) | x^2+y^2+z^2 \le 1\}$, then $\iint_S (2x+2y+z^2) dS = \iiint_B dV = V(B) = \frac{4}{3}\pi(1)^3 = \frac{4}{3}\pi$.

23. $\iint_S \boldsymbol{a}\cdot\boldsymbol{n} dS = \iiint_E \operatorname{div} \boldsymbol{a} dV = 0$ since $\operatorname{div} \boldsymbol{a} = 0$.

24. $\frac{1}{3}\iint_S \boldsymbol{F}\cdot d\boldsymbol{S} = \frac{1}{3}\iiint_E \operatorname{div}\boldsymbol{F} dV = \frac{1}{3}\iiint_E 3 dV = V(E)$

25. $\iint_S \operatorname{curl}\boldsymbol{F}\cdot d\boldsymbol{S} = \iiint_E \operatorname{div}(\operatorname{curl}\boldsymbol{F}) dV = 0$ by Theorem 17.5.11 [ET 16.5.11].

26.

$$\iint_S D_{\boldsymbol{n}} f\,dS=\iint_S (\nabla f\cdot \boldsymbol{n})dS=\iiint_E \operatorname{div}(\nabla f)dV=\iiint_E \nabla^2 f\,dV$$

27. $\iint_S (f\nabla g)\cdot \boldsymbol{n}dS=\iiint_E \operatorname{div}(f\nabla g)dV=\iiint_E (f\nabla^2 g+\nabla g\cdot \nabla f)dV$ by Exercise 17.5.25 [ET 16.5.25].

28. $\iint_S (f\nabla g-g\nabla f)\cdot \boldsymbol{n}dS=\iiint_E \left[(f\nabla^2 g+\nabla g\cdot \nabla f)-(g\nabla^2 f+\nabla g\cdot \nabla f)\right]dV$ [by Exercise 27].

But $\nabla g\cdot \nabla f=\nabla f\cdot \nabla g$, so that $\iint_S (f\nabla g-g\nabla f)\cdot \boldsymbol{n}dS=\iiint_E (f\nabla^2 g-g\nabla^2 f)dV$.

29. If $\boldsymbol{c}=c_1\boldsymbol{i}+c_2\boldsymbol{j}+c_3\boldsymbol{k}$ is an arbitrary constant vector, we define $\boldsymbol{F}=f\boldsymbol{c}=fc_1\boldsymbol{i}+fc_2\boldsymbol{j}+fc_3\boldsymbol{k}$. Then

$\operatorname{div}\boldsymbol{F}=\operatorname{div} f\boldsymbol{c}=\frac{\partial f}{\partial x}c_1+\frac{\partial f}{\partial y}c_2+\frac{\partial f}{\partial z}c_3=\nabla f\cdot \boldsymbol{c}$ and the Divergence Theorem says $\iint_S \boldsymbol{F}\cdot d\boldsymbol{S}=\iiint_E \operatorname{div}\boldsymbol{F}dV$

$\Rightarrow \iint_S \boldsymbol{F}\cdot \boldsymbol{n}dS=\iiint_E \nabla f\cdot \boldsymbol{c}dV$. In particular, if $\boldsymbol{c}=\boldsymbol{i}$ then $\iint_S f\boldsymbol{i}\cdot \boldsymbol{n}dS=\iiint_E \nabla f\cdot \boldsymbol{i}dV\Rightarrow$

$\iint_S fn_1dS=\iiint_E \frac{\partial f}{\partial x}dV$ (where $\boldsymbol{n}=n_1\boldsymbol{i}+n_2\boldsymbol{j}+n_3\boldsymbol{k}$). Similarly, if $\boldsymbol{c}=\boldsymbol{j}$ we have $\iint_S fn_2dS=\iiint_E \frac{\partial f}{\partial y}dV$,

and $\boldsymbol{c}=\boldsymbol{k}$ gives $\iint_S fn_3dS=\iiint_E \frac{\partial f}{\partial z}dV$. Then

$$\begin{aligned}\iint_S f\boldsymbol{n}dS &= \left(\iint_S fn_1dS\right)\boldsymbol{i}+\left(\iint_S fn_2dS\right)\boldsymbol{j}+\left(\iint_S fn_3dS\right)\boldsymbol{k}\\ &= \left(\iiint_E \frac{\partial f}{\partial x}dV\right)\boldsymbol{i}+\left(\iiint_E \frac{\partial f}{\partial y}dV\right)\boldsymbol{j}+\left(\iiint_E \frac{\partial f}{\partial z}dV\right)\boldsymbol{k}\\ &= \iiint_E \left(\frac{\partial f}{\partial x}\boldsymbol{i}+\frac{\partial f}{\partial y}\boldsymbol{j}+\frac{\partial f}{\partial z}\boldsymbol{k}\right)dV=\iiint_E \nabla f\,dV\end{aligned}$$

as desired.

30. By Exercise 29, $\iint_S p\boldsymbol{n}dS=\iiint_E \nabla p\,dV$, so

$$\boldsymbol{F} = -\iint_S p\boldsymbol{n}dS = -\iiint_E \nabla p dV = -\iiint_E \nabla(\rho g z)dV = -\iiint_E (\rho g \boldsymbol{k})dV$$
$$= -\rho g\left(\iiint_E dV\right)\boldsymbol{k} = -\rho g V(E)\boldsymbol{k}$$

But the weight of the displaced liquid is volume × density × $g = \rho g V(E)$, thus $\boldsymbol{F} = -W\boldsymbol{k}$ as desired.

1. The auxiliary equation is $r^2-6r+8=0 \Rightarrow (r-4)(r-2)=0 \Rightarrow r=4$, $r=2$. Then by (8) the general solution is $y=c_1e^{4x}+c_2e^{2x}$.

2. The auxiliary equation is $r^2-4r+8=0 \Rightarrow r=2\pm 2i$. Then by (11) the general solution is $y=e^{2x}(c_1\cos 2x+c_2\sin 2x)$.

3. The auxiliary equation is $r^2+8r+41=0 \Rightarrow r=-4\pm 5i$. Then by (11) the general solution is $y=e^{-4x}(c_1\cos 5x+c_2\sin 5x)$.

4. The auxiliary equation is $2r^2-r-1=(2r+1)(r-1)=0 \Rightarrow r=1$, $r=-\frac{1}{2}$. Then the general solution is $y=c_1e^{x}+c_2e^{-x/2}$.

5. The auxiliary equation is $r^2-2r+1=(r-1)^2=0 \Rightarrow r=1$. Then by (10), the general solution is $y=c_1e^{x}+c_2xe^{x}$.

6. The auxiliary equation is $3r^2-5r=r(3r-5)=0 \Rightarrow r=0$, $r=\frac{5}{3}$, so $y=c_1+c_2e^{5x/3}$.

7. The auxiliary equation is $4r^2+1=0 \Rightarrow r=\pm\frac{1}{2}i$, so $y=c_1\cos\left(\frac{1}{2}x\right)+c_2\sin\left(\frac{1}{2}x\right)$.

8. The auxiliary equation is $16r^2+24r+9=(4r+3)^2=0 \Rightarrow r=-\frac{3}{4}$, so $y=c_1e^{-3x/4}+c_2xe^{-3x/4}$.

9. The auxiliary equation is $4r^2+r=r(4r+1)=0 \Rightarrow r=0$, $r=-\frac{1}{4}$, so $y=c_1+c_2e^{-x/4}$.

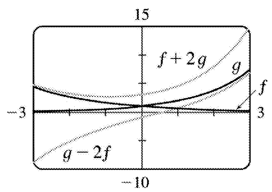
10. The auxiliary equation is $9r^2+4=0 \Rightarrow r=\pm\frac{2}{3}i$, so $y=c_1\cos\left(\frac{2}{3}x\right)+c_2\sin\left(\frac{2}{3}x\right)$.

11. The auxiliary equation is $r^2-2r-1=0 \Rightarrow r=1\pm\sqrt{2}$, so $y=c_1e^{(1+\sqrt{2})t}+c_2e^{(1-\sqrt{2})t}$.

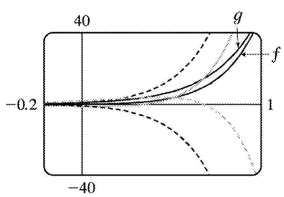
12. The auxiliary equation is $r^2-6r+4=0 \Rightarrow r=3\pm\sqrt{5}$, so $y=c_1e^{(3+\sqrt{5})t}+c_2e^{(3-\sqrt{5})t}$.

13. The auxiliary equation is $r^2+r+1=0 \Rightarrow r=-\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$, so
$y=e^{-t/2}\left[c_1\cos\left(\frac{\sqrt{3}}{2}t\right)+c_2\sin\left(\frac{\sqrt{3}}{2}t\right)\right]$.

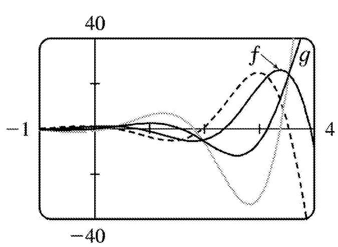
14. $6r^2-r-2=(2r+1)(3r-2)=0$ so $y=c_1e^{-x/2}+c_2e^{2x/3}$. The solutions $(c_1,c_2)=(0,1)$, $(1,0)$, $(1,2)$, $(-2,1)$ are shown. Each solution consists of a single continuous curve that approaches either 0 or $\pm\infty$ as $x\rightarrow\pm\infty$.



15. $r^2-8r+16=(r-4)^2=0$ so $y=c_1e^{4x}+c_2xe^{4x}$. The graphs are all asymptotic to the x –axis as $x\rightarrow-\infty$, and as $x\rightarrow\infty$ the solutions tend to $\pm\infty$.



16. $r^2-2r+5=0 \Rightarrow r=1\pm 2i$ and the solution is $y=e^x(c_1\cos 2x+c_2\sin 2x)$. Graphs for $(c_1,c_2)=(1,0)$, $(0,1)$, $(1,-1)$, $(-1,2)$ are shown. The solutions are all asymptotic to the x –axis as $x\rightarrow-\infty$ and they all oscillate. The amplitudes of the oscillations become arbitrarily large as $x\rightarrow\infty$ and arbitrarily small as $x\rightarrow-\infty$.



17. $2r^2+5r+3=(2r+3)(r+1)=0$, so $r=-\frac{3}{2}$, $r=-1$ and the general solution is $y=c_1e^{-3x/2}+c_2e^{-x}$. Then $y(0)=3\Rightarrow c_1+c_2=3$ and $y'(0)=-4\Rightarrow -\frac{3}{2}c_1-c_2=-4$, so $c_1=2$ and $c_2=1$. Thus the solution to the initial–

value problem is $y=2e^{-3x/2}+e^{-x}$.

18. $r^2+3=0 \Rightarrow r=\pm\sqrt{3}\,i$ and the general solution is
$y=e^{0x}\left(c_1\cos\left(\sqrt{3}\,x\right)+c_2\sin\left(\sqrt{3}\,x\right)\right)=c_1\cos\left(\sqrt{3}\,x\right)+c_2\sin\left(\sqrt{3}\,x\right)$. Then $y(0)=1 \Rightarrow c_1=1$ and $y'(0)=3$ $\Rightarrow c_2=\sqrt{3}$, so the solution to the initial–value problem is $y=\cos\left(\sqrt{3}\,x\right)+\sqrt{3}\sin\left(\sqrt{3}\,x\right)$.

19. $4r^2-4r+1=(2r-1)^2=0 \Rightarrow r=\frac{1}{2}$ and the general solution is $y=c_1e^{x/2}+c_2xe^{x/2}$. Then $y(0)=1 \Rightarrow c_1=1$ and $y'(0)=-1.5 \Rightarrow \frac{1}{2}c_1+c_2=-1.5$, so $c_2=-2$ and the solution to the initial–value problem is $y=e^{x/2}-2xe^{x/2}$.

20. $2r^2+5r-3=(2r-1)(r+3)=0 \Rightarrow r=\frac{1}{2}$, $r=-3$ and the general solution is $y=c_1e^{x/2}+c_2e^{-3x}$. Then $1=y(0)=c_1+c_2$ and $4=y'(0)=\frac{1}{2}c_1-3c_2$ so $c_1=2$, $c_2=-1$ and the solution to the initial–value problem is $y=2e^{x/2}-e^{-3x}$.

21. $r^2+16=0 \Rightarrow r=\pm 4i$ and the general solution is $y=e^{0x}(c_1\cos 4x+c_2\sin 4x)=c_1\cos 4x+c_2\sin 4x$. Then $y\left(\frac{\pi}{4}\right)=-3 \Rightarrow -c_1=-3 \Rightarrow c_1=3$ and $y'\left(\frac{\pi}{4}\right)=4 \Rightarrow -4c_2=4 \Rightarrow c_2=-1$, so the solution to the initial–value problem is $y=3\cos 4x-\sin 4x$.

22. $r^2-2r+5=0 \Rightarrow r=1\pm 2i$ and the general solution is $y=e^{x}(c_1\cos 2x+c_2\sin 2x)$. Then $0=y(\pi)=e^{\pi}(c_1+0)$ $\Rightarrow c_1=0$ and $2=y'(\pi)=(c_1+2c_2)e^{\pi} \Rightarrow c_2=1/e^{\pi}$ and the solution to the initial–value problem is $y=\frac{e^x}{e^{\pi}}\sin 2x=e^{x-\pi}\sin 2x$.

23. $r^2+2r+2=0 \Rightarrow r=-1\pm i$ and the general solution is $y=e^{-x}(c_1\cos x+c_2\sin x)$. Then $2=y(0)=c_1$ and $1=y'(0)=c_2-c_1 \Rightarrow c_2=3$ and the solution to the initial–value problem is $y=e^{-x}(2\cos x+3\sin x)$.

24. $r^2+12r+36=(r+6)^2=0 \Rightarrow r=-6$ and the general solution is $y=c_1e^{-6x}+c_2xe^{-6x}$. Then

$0=y(1)=c_1e^{-6}+c_2e^{-6}\Rightarrow c_1+c_2=0$ and $1=y'(1)=-6c_1e^{-6}-5c_2e^{-6}\Rightarrow 6c_1+5c_2=-e^{6}$, so $c_1=-e^{6}$ and $c_2=e^{6}$. The solution to the initial–value problem is $y=-e^{6}e^{-6x}+e^{6}xe^{-6x}=(x-1)e^{6-6x}$.

25. $4r^2+1=0\Rightarrow r=\pm\frac{1}{2}i$ and the general solution is $y=c_1\cos\left(\frac{1}{2}x\right)+c_2\sin\left(\frac{1}{2}x\right)$. Then $3=y(0)=c_1$ and $-4=y(\pi)=c_2$, so the solution of the boundary–value problem is $y=3\cos\left(\frac{1}{2}x\right)-4\sin\left(\frac{1}{2}x\right)$.

26. $r^2+2r=r(2+r)=0\Rightarrow r=0$, $r=-2$ and the general solution is $y=c_1+c_2e^{-2x}$. Then $1=y(0)=c_1+c_2$ and $2=y(1)=c_1+c_2e^{-2}$ so $c_2=\dfrac{e^2}{1-e^2}$, $c_1=\dfrac{1-2e^2}{1-e^2}$. The solution of the boundary–value problem is

$$y=\frac{1-2e^2}{1-e^2}+\frac{e^2}{1-e^2}\cdot e^{-2x} .$$

27. $r^2-3r+2=(r-2)(r-1)=0\Rightarrow r=1$, $r=2$ and the general solution is $y=c_1e^{x}+c_2e^{2x}$. Then $1=y(0)=c_1+c_2$ and $0=y(3)=c_1e^{3}+c_2e^{6}$ so $c_2=1/(1-e^{3})$ and $c_1=e^{3}/(e^{3}-1)$. The solution of the boundary–value problem is $y=\dfrac{e^{x+3}}{e^{3}-1}+\dfrac{e^{2x}}{1-e^{3}}$.

28. $r^2+100=0\Rightarrow r=\pm 10i$ and the general solution is $y=c_1\cos 10x+c_2\sin 10x$. But $2=y(0)=c_1$ and $5=y(\pi)=c_1$, so there is no solution.

29. $r^2-6r+25=0\Rightarrow r=3\pm 4i$ and the general solution is $y=e^{3x}(c_1\cos 4x+c_2\sin 4x)$. But $1=y(0)=c_1$ and $2=y(\pi)=c_1e^{3\pi}\Rightarrow c_1=2/e^{3\pi}$, so there is no solution.

30. $r^2-6r+9=(r-3)^2=0\Rightarrow r=3$ and the general solution is $y=c_1e^{3x}+c_2xe^{3x}$. Then $1=y(0)=c_1$ and $0=y(1)=c_1e^{3}+c_2e^{3}\Rightarrow c_2=-1$. The solution of the boundary–value problem is $y=e^{3x}-xe^{3x}$.

31. $r^2+4r+13=0\Rightarrow r=-2\pm 3i$ and the general solution is $y=e^{-2x}(c_1\cos 3x+c_2\sin 3x)$. But $2=y(0)=c_1$ and $1=y\left(\frac{\pi}{2}\right)=e^{-\pi}(-c_2)$, so the solution to the boundary–value problem is $y=e^{-2x}(2\cos 3x-e^{\pi}\sin 3x)$.

32. $9r^2-18r+10=0 \Rightarrow r=1\pm\frac{1}{3}i$ and the general solution is $y=e^x\left(c_1\cos\frac{x}{3}+c_2\sin\frac{x}{3}\right)$. Then $0=y(0)=c_1$ and $1=y(\pi)=e^{\pi}\left(\frac{1}{2}c_1+\frac{\sqrt{3}}{2}c_2\right)\Rightarrow c_2=\frac{2}{\sqrt{3}e^{\pi}}$. The solution of the boundary–value problem is $y=\frac{2e^x}{\sqrt{3}e^{\pi}}\sin\left(\frac{x}{3}\right)=\frac{2}{\sqrt{3}}e^{x-\pi}\sin\left(\frac{x}{3}\right)$.

33. **(a)** *Case 1* ($\lambda=0$) *:* $y''+\lambda y=0\Rightarrow y''=0$ which has an auxiliary equation $r^2=0\Rightarrow r=0\Rightarrow y=c_1+c_2x$ where $y(0)=0$ and $y(L)=0$. Thus, $0=y(0)=c_1$ and $0=y(L)=c_2L\Rightarrow c_1=c_2=0$. Thus, $y=0$.
Case 2 ($\lambda<0$) *:* $y''+\lambda y=0$ has auxiliary equation $r^2=-\lambda\Rightarrow r=\pm\sqrt{-\lambda}$ (distinct and real since $\lambda<0$) $\Rightarrow$ $y=c_1e^{\sqrt{-\lambda}\,x}+c_2e^{-\sqrt{-\lambda}\,x}$ where $y(0)=0$ and $y(L)=0$. Thus, $0=y(0)=c_1+c_2$ (*) and $0=y(L)=c_1e^{\sqrt{-\lambda}\,L}+c_2e^{-\sqrt{-\lambda}\,L}$ (**).
Multiplying (*) by $e^{\sqrt{-\lambda}\,L}$ and subtracting (**) gives $c_2\left(e^{\sqrt{-\lambda}\,L}-e^{-\sqrt{-\lambda}\,L}\right)=0\Rightarrow c_2=0$ and thus $c_1=0$ from (*). Thus, $y=0$ for the cases $\lambda=0$ and $\lambda<0$.
(b) $y''+\lambda y=0$ has an auxiliary equation $r^2+\lambda=0\Rightarrow r=\pm i\sqrt{\lambda}\Rightarrow y=c_1\cos\sqrt{\lambda}\,x+c_2\sin\sqrt{\lambda}\,x$ where $y(0)=0$ and $y(L)=0$. Thus, $0=y(0)=c_1$ and $0=y(L)=c_2\sin\sqrt{\lambda}\,L$ since $c_1=0$. Since we cannot have a trivial solution, $c_2\neq 0$ and thus $\sin\sqrt{\lambda}\,L=0\Rightarrow\sqrt{\lambda}\,L=n\pi$ where n is an integer $\Rightarrow\lambda=n^2\pi^2/L^2$ and $y=c_2\sin(n\pi x/L)$ where n is an integer.

34. The auxiliary equation is $ar^2+br+c=0$. If $b^2-4ac>0$, then any solution is of the form $y(x)=c_1e^{r_1x}+c_2e^{r_2x}$ where $r_1=\frac{-b+\sqrt{b^2-4ac}}{2a}$ and $r_2=\frac{-b-\sqrt{b^2-4ac}}{2a}$. But a, b, and c are all positive so both r_1 and r_2 are negative and $y(x)=0$. If $b^2-4ac=0$, then any solution is of the form $y(x)=c_1e^{rx}+c_2xe^{rx}$ where $r=-b/(2a)<0$ since a, b are positive. Hence $y(x)=0$. Finally if $b^2-4ac<0$, then any solution is of the form $y(x)=e^{\alpha x}(c_1\cos\beta x+c_2\sin\beta x)$ where $\alpha=-b/(2a)<0$ since a and b are positive. Thus $y(x)=0$.

1. The auxiliary equation is $r^2+3r+2=(r+2)(r+1)=0$, so the complementary solution is $y_c(x)=c_1e^{-2x}+c_2e^{-x}$. We try the particular solution $y_p(x)=Ax^2+Bx+C$, so $y_p'=2Ax+B$ and $y_p''=2A$. Substituting into the differential equation, we have $(2A)+3(2Ax+B)+2(Ax^2+Bx+C)=x^2$ or $2Ax^2+(6A+2B)x+(2A+3B+2C)=x^2$. Comparing coefficients gives $2A=1$, $6A+2B=0$, and $2A+3B+2C=0$, so $A=\frac{1}{2}$, $B=-\frac{3}{2}$, and $C=\frac{7}{4}$. Thus the general solution is $y(x)=y_c(x)+y_p(x)=c_1e^{-2x}+c_2e^{-x}+\frac{1}{2}x^2-\frac{3}{2}x+\frac{7}{4}$.

2. The auxiliary equation is $r^2+9=0$ with roots $r=\pm 3i$, so the complementary solution is $y_c(x)=c_1\cos(3x)+c_2\sin(3x)$. Try the particular solution $y_p(x)=Ae^{3x}$, so $y_p'=3Ae^{3x}$ and $y_p''=9Ae^{3x}$. Substitution into the differential equation gives $9Ae^{3x}+9(Ae^{3x})=e^{3x}$ or $18Ae^{3x}=e^{3x}$. Thus $A=\frac{1}{18}$ and the general solution is $y(x)=y_c(x)+y_p(x)=c_1\cos(3x)+c_2\sin(3x)+\frac{1}{18}e^{3x}$.

3. The auxiliary equation is $r^2-2r=r(r-2)=0$, so the complementary solution is $y_c(x)=c_1+c_2e^{2x}$. Try the particular solution $y_p(x)=A\cos 4x+B\sin 4x$, so $y_p'=-4A\sin 4x+4B\cos 4x$ and $y_p''=-16A\cos 4x-16B\sin 4x$. Substitution into the differential equation gives $(-16A\cos 4x-16B\sin 4x)-2(-4A\sin 4x+4B\cos 4x)=\sin 4x\Rightarrow(-16A-8B)\cos 4x+(8A-16B)\sin 4x=\sin 4x$. Then $-16A-8B=0$ and $8A-16B=1\Rightarrow A=\frac{1}{40}$ and $B=-\frac{1}{20}$. Thus the general solution is $y(x)=y_c(x)+y_p(x)=c_1+c_2e^{2x}+\frac{1}{40}\cos 4x-\frac{1}{20}\sin 4x$.

4. The auxiliary equation is $r^2+6r+9=(r+3)^2=0$, so the complementary solution is $y_c(x)=c_1e^{-3x}+c_2xe^{-3x}$. Try the particular solution $y_p(x)=Ax+B$, so $y_p'=A$ and $y_p''=0$. Substitution into the differential equation gives $0+6A+9(Ax+B)=1+x$ or $(9A)x+(6A+9B)=1+x$. Comparing coefficients, we have $9A=1$ and $6A+9B=1$, so $A=\frac{1}{9}$ and $B=\frac{1}{27}$. Thus the general solution is $y(x)=c_1e^{-3x}+c_2xe^{-3x}+\frac{1}{9}x+\frac{1}{27}$.

5. The auxiliary equation is $r^2-4r+5=0$ with roots $r=2\pm i$, so the complementary solution is

$y_c(x)=e^{2x}(c_1\cos x+c_2\sin x)$. Try $y_p(x)=Ae^{-x}$, so $y_p'=-Ae^{-x}$ and $y_p''=Ae^{-x}$. Substitution gives $Ae^{-x}-4(-Ae^{-x})+5(Ae^{-x})=e^{-x}\Rightarrow 10Ae^{-x}=e^{-x}\Rightarrow A=\frac{1}{10}$. Thus the general solution is $y(x)=e^{2x}(c_1\cos x+c_2\sin x)+\frac{1}{10}e^{-x}$.

6. $y_c(x)=e^{-x}(c_1x+c_2)$. Try $y_p(x)=x^2(Ax+B)e^{-x}$ so that no term in y_p is a solution of the complementary equation. Then $y_p'=[-Ax^3+(3A-B)x^2+2Bx]e^{-x}$, $y_p''=[Ax^3+(B-6A)x^2+(6A-4B)x+2B]e^{-x}$ and substitution gives $[Ax^3+(B-6A)x^2+(6A-4B)x+2B]+2[-Ax^3+(3A-B)x^2+2Bx]+(Ax^3+Bx^2)=x\Rightarrow$ $6Ax+2B=x$. So $y_p(x)=x^2\left(\frac{1}{6}x\right)e^{-x}$ and the general solution is $y(x)=e^{-x}\left(c_1x+c_2\right)+\frac{1}{6}x^3e^{-x}$.

7. The auxiliary equation is $r^2+1=0$ with roots $r=\pm i$, so the complementary solution is $y_c(x)=c_1\cos x+c_2\sin x$. For $y''+y=e^x$ try $y_{p_1}(x)=Ae^x$. Then $y_{p_1}'=y_{p_1}''=Ae^x$ and substitution gives $Ae^x+Ae^x=e^x\Rightarrow A=\frac{1}{2}$, so $y_{p_1}(x)=\frac{1}{2}e^x$. For $y''+y=x^3$ try $y_{p_2}(x)=Ax^3+Bx^2+Cx+D$.
Then $y_{p_2}'=3Ax^2+2Bx+C$ and $y_{p_2}''=6Ax+2B$. Substituting, we have
$6Ax+2B+Ax^3+Bx^2+Cx+D=x^3$, so $A=1$, $B=0$, $6A+C=0\Rightarrow C=-6$, and $2B+D=0\Rightarrow D=0$. Thus $y_{p_2}(x)=x^3-6x$ and the general solution is
$y(x)=y_c(x)+y_{p_1}(x)+y_{p_2}(x)=c_1\cos x+c_2\sin x+\frac{1}{2}e^x+x^3-6x$. But $2=y(0)=c_1+\frac{1}{2}\Rightarrow c_1=\frac{3}{2}$ and $0=y'(0)=c_2+\frac{1}{2}-6\Rightarrow c_2=\frac{11}{2}$. Thus the solution to the initial–value problem is
$y(x)=\frac{3}{2}\cos x+\frac{11}{2}\sin x+\frac{1}{2}e^x+x^3-6x$.

8. The auxiliary equation is $r^2-4=0$ with roots $r=\pm 2$, so the complementary solution is $y_c(x)=c_1e^{2x}+c_2e^{-2x}$. Try $y_p(x)=e^x(A\cos x+B\sin x)$, so $y_p'=e^x(A\cos x+B\sin x+B\cos x-A\sin x)$ and $y_p''=e^x(2B\cos x-2A\sin x)$. Substitution gives
$e^x(2B\cos x-2A\sin x)-4e^x(A\cos x+B\sin x)=e^x\cos x\Rightarrow(2B-4A)e^x\cos x+(-2A-4B)e^x\sin x=e^x\cos x\Rightarrow$

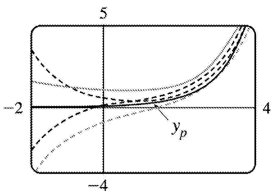
$A=-\frac{1}{5}$, $B=\frac{1}{10}$. Thus the general solution is $y(x)=c_1e^{2x}+c_2e^{-2x}+e^{x}\left(-\frac{1}{5}\cos x+\frac{1}{10}\sin x\right)$. But $1=y(0)=c_1+c_2-\frac{1}{5}$ and $2=y'(0)=2c_1-2c_2-\frac{1}{10}$. Then $c_1=\frac{9}{8}$, $c_2=\frac{3}{40}$, and the solution to the initial–value problem is $y(x)=\frac{9}{8}e^{2x}+\frac{3}{40}e^{-2x}+e^{x}\left(-\frac{1}{5}\cos x+\frac{1}{10}\sin x\right)$.

9. The auxiliary equation is $r^2-r=0$ with roots $r=0$, $r=1$ so the complementary solution is $y_c(x)=c_1+c_2e^{x}$. Try $y_p(x)=x(Ax+B)e^{x}$ so that no term in y_p is a solution of the complementary equation. Then $y_p'=(Ax^2+(2A+B)x+B)e^{x}$ and $y_p''=(Ax^2+(4A+B)x+(2A+2B))e^{x}$. Substitution into the differential equation gives $(Ax^2+(4A+B)x+(2A+2B))e^{x}-(Ax^2+(2A+B)x+B)e^{x}=xe^{x}\Rightarrow$ $(2Ax+(2A+B))e^{x}=xe^{x}\Rightarrow A=\frac{1}{2}$, $B=-1$. Thus $y_p(x)=\left(\frac{1}{2}x^2-x\right)e^{x}$ and the general solution is $y(x)=c_1+c_2e^{x}+\left(\frac{1}{2}x^2-x\right)e^{x}$. But $2=y(0)=c_1+c_2$ and $1=y'(0)=c_2-1$, so $c_2=2$ and $c_1=0$. The solution to the initial–value problem is $y(x)=2e^{x}+\left(\frac{1}{2}x^2-x\right)e^{x}=e^{x}\left(\frac{1}{2}x^2-x+2\right)$.

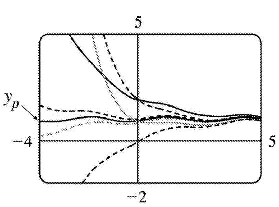
10. $y_c(x)=c_1e^{x}+c_2e^{-2x}$. For $y''+y'-2y=x$ try $y_{p_1}(x)=Ax+B$. Then $y_{p_1}'=A$, $y_{p_1}''=0$, and substitution gives $0+A-2(Ax+B)=x\Rightarrow A=-\frac{1}{2}$, $B=-\frac{1}{4}$, so $y_{p_1}(x)=-\frac{1}{2}x-\frac{1}{4}$. For $y''+y'-2y=\sin 2x$ try $y_{p_2}(x)=A\cos 2x+B\sin 2x$. Then $y_{p_2}'=-2A\sin 2x+2B\cos 2x$, $y_{p_2}''=-4A\cos 2x-4B\sin 2x$, and substitution gives $(-4A\cos 2x-4B\sin 2x)+(-2A\sin 2x+2B\cos 2x)-2(A\cos 2x+B\sin 2x)=\sin 2x\Rightarrow A=-\frac{1}{20}$, $B=-\frac{3}{20}$. Thus $y_{p_2}(x)=-\frac{1}{20}\cos 2x+-\frac{3}{20}\sin 2x$ and the general solution is $y(x)=c_1e^{x}+c_2e^{-2x}-\frac{1}{2}x-\frac{1}{4}-\frac{1}{20}\cos 2x-\frac{3}{20}\sin 2x$. But $1=y(0)=c_1+c_2-\frac{1}{4}-\frac{1}{20}$ and $0=y'(0)=c_1-2c_2-\frac{1}{2}-\frac{3}{10}\Rightarrow c_1=\frac{17}{15}$ and $c_2=\frac{1}{6}$. Thus the solution to the initial–value problem is $y(x)=\frac{17}{15}e^{x}+\frac{1}{6}e^{-2x}-\frac{1}{2}x-\frac{1}{4}-\frac{1}{20}\cos 2x-\frac{3}{20}\sin 2x$.

11. $y_c(x)=c_1e^{-x/4}+c_2e^{-x}$. Try $y_p(x)=Ae^{x}$. Then $10Ae^{x}=e^{x}$, so $A=\frac{1}{10}$ and the general solution is

$y(x)=c_1e^{-x/4}+c_2e^{-x}+\frac{1}{10}e^{x}$. The solutions are all composed of exponential curves and with the exception of the particular solution (which approaches 0 as $x\to-\infty$), they all approach either ∞ or $-\infty$ as $x\to-\infty$. As $x\to\infty$, all solutions are asymptotic to $y_p=\frac{1}{10}e^{x}$.



12. The auxiliary equation is $(2r+1)(r+1)=0$, so $r=-1$, $-\frac{1}{2}$ and $y_c(x)=c_1e^{-x}+c_2e^{-x/2}$. For $2y''+3y'+y=1$, try $y_{p_1}(x)=A$; substituting gives $y_{p_1}(x)=1$. For $2y''+3y'+y=\cos 2x$ try $y_{p_2}=A\cos 2x+B\sin 2x\Rightarrow y_{p_2}'=-2A\sin 2x+2B\cos 2x$, $y_{p_2}''=-4A\cos 2x-4B\sin 2x$.
Substituting into the differential equation gives $\cos 2x=(6B-7A)\cos 2x+(-7B-6A)\sin 2x$.
Then solving the equations $6B-7A=1$ and $-7B-6A=0$ gives $A=-\frac{7}{85}$, $B=\frac{6}{85}$. Thus, $y_{p_2}(x)=-\frac{7}{85}\cos 2x+\frac{6}{85}\sin 2x$ and the general solution is $y(x)=c_1e^{-x}+c_2e^{-x/2}+1-\frac{7}{85}\cos 2x+\frac{6}{85}\sin 2x$.
The graph shows $y_p=y_{p_1}+y_{p_2}$ and several other solutions. Notice that all solutions are asymptotic to y_p as $x\to\infty$.



13. Here $y_c(x)=c_1\cos 3x+c_2\sin 3x$. For $y''+9y=e^{2x}$ try $y_{p_1}(x)=Ae^{2x}$ and for $y''+9y=x^2\sin x$ try $y_{p_2}(x)=(Bx^2+Cx+D)\cos x+(Ex^2+Fx+G)\sin x$. Thus a trial solution is
$y_p(x)=y_{p_1}(x)+y_{p_2}(x)=Ae^{2x}+(Bx^2+Cx+D)\cos x+(Ex^2+Fx+G)\sin x$.

14. Since $y_c(x)=c_1+c_2e^{-9x}$, try $y_p(x)=(Ax+B)e^{-x}\cos\pi x+(Cx+D)e^{-x}\sin\pi x$.

15. Here $y_c(x)=c_1+c_2e^{-9x}$. For $y''+9y'=1$ try $y_{p_1}(x)=Ax$ (since $y=A$ is a solution to the complementary equation) and for $y''+9y'=xe^{9x}$ try $y_{p_2}(x)=(Bx+C)e^{9x}$.

16. Since $y_c(x)=c_1e^x+c_2e^{-4x}$ try $y_p(x)=x(Ax^3+Bx^2+Cx+D)e^x$ so that no term of $y_p(x)$ satisfies the complementary equation.

17. Since $y_c(x)=e^{-x}(c_1\cos 3x+c_2\sin 3x)$ we try $y_p(x)=x(Ax^2+Bx+C)e^{-x}\cos 3x+x(Dx^2+Ex+F)e^{-x}\sin 3x$ (so that no term of y_p is a solution of the complementary equation).

18. Here $y_c(x)=c_1\cos 2x+c_2\sin 2x$. For $y''+4y=e^{3x}$ try $y_{p_1}(x)=Ae^{3x}$ and for $y''+4y=x\sin 2x$ try $y_{p_2}(x)=x(Bx+C)\cos 2x+x(Dx+E)\sin 2x$ (so that no term of y_{p_2} is a solution of the complementary equation).

19. **(a)** The complementary solution is $y_c(x)=c_1\cos 2x+c_2\sin 2x$. A particular solution is of the form $y_p(x)=Ax+B$. Thus, $4Ax+4B=x\Rightarrow A=\frac{1}{4}$ and $B=0\Rightarrow y_p(x)=\frac{1}{4}x$. Thus, the general solution is $y=y_c+y_p=c_1\cos 2x+c_2\sin 2x+\frac{1}{4}x$.

(b) In (a), $y_c(x)=c_1\cos 2x+c_2\sin 2x$, so set $y_1=\cos 2x$, $y_2=\sin 2x$. Then $y_1y_2'-y_2y_1'=2\cos^2 2x+2\sin^2 2x=2$ so $u_1'=-\frac{1}{2}x\sin 2x\Rightarrow$
$u_1(x)=-\frac{1}{2}\int x\sin 2x\,dx=-\frac{1}{4}\left(-x\cos 2x+\frac{1}{2}\sin 2x\right)$ and $u_2'=\frac{1}{2}x\cos 2x\Rightarrow$
$u_2(x)=\frac{1}{2}\int x\cos 2x\,dx=\frac{1}{4}\left(x\sin 2x+\frac{1}{2}\cos 2x\right)$. Hence
$y_p(x)=-\frac{1}{4}\left(-x\cos 2x+\frac{1}{2}\sin 2x\right)\cos 2x+\frac{1}{4}\left(x\sin 2x+\frac{1}{2}\cos 2x\right)\sin 2x=\frac{1}{4}x$. Thus
$y(x)=y_c(x)+y_p(x)=c_1\cos 2x+c_2\sin 2x+\frac{1}{4}x$.

20. **(a)** Here $r^2-3r+2=0\Rightarrow r=1$ or 2 and

$y_c(x)=c_1e^{2x}+c_2e^{x}$. We try a particular solution of the form $y_p(x)=A\cos x+B\sin x \Rightarrow y_p'=-A\sin x+B\cos x$ and $y_p''=-A\cos x-B\sin x$. Then the equation $y''-3y'+2y=\sin x$ becomes $(A-3B)\cos x+(B+3A)\sin x=\sin x \Rightarrow A-3B=0$ and $B+3A=1 \Rightarrow A=\frac{3}{10}$ and $B=\frac{1}{10}$. Thus, $y_p(x)=\frac{3}{10}\cos x+\frac{1}{10}\sin x$. Therefore, the general solution is $y(x)=y_c(x)+y_p(x)=c_1e^{2x}+c_2e^{x}+\frac{3}{10}\cos x+\frac{1}{10}\sin x$.

(b) From (a) we know that $y_c(x)=c_1e^{2x}+c_2e^{x}$. Setting $y_1=e^{2x}$, $y_2=e^{x}$, we have $y_1y_2'-y_2y_1'=e^{3x}-2e^{3x}=-e^{3x}$. Thus $u_1'=-\frac{\sin xe^{x}}{-e^{3x}}=\sin xe^{-2x}$ and $u_2'=\frac{\sin xe^{2x}}{-e^{3x}}=-\sin xe^{-x}$. Then $u_1(x)=\int e^{-2x}\sin x\,dx=\frac{1}{5}e^{-2x}(-2\sin x-\cos x)$ and $u_2(x)=-\int e^{-x}\sin x\,dx=-\frac{1}{2}e^{-x}(-\sin x-\cos x)$. Thus $y_p(x)=\frac{1}{5}(-2\sin x-\cos x)+\frac{1}{2}(\sin x+\cos x)=\frac{1}{10}\sin x+\frac{3}{10}\cos x$ and the general solution is $y(x)=y_c(x)+y_p(x)=c_1e^{2x}+c_2e^{x}+\frac{1}{10}\sin x+\frac{3}{10}\cos x$.

21. **(a)** $r^2-r=r(r-1)=0 \Rightarrow r=0$, 1 , so the complementary solution is $y_c(x)=c_1e^{x}+c_2xe^{x}$. A particular solution is of the form $y_p(x)=Ae^{2x}$. Thus $4Ae^{2x}-4Ae^{2x}+Ae^{2x}=e^{2x} \Rightarrow Ae^{2x}=e^{2x} \Rightarrow A=1 \Rightarrow y_p(x)=e^{2x}$. So a general solution is $y(x)=y_c(x)+y_p(x)=c_1e^{x}+c_2xe^{x}+e^{2x}$.

(b) From (a), $y_c(x)=c_1e^{x}+c_2xe^{x}$, so set $y_1=e^{x}$, $y_2=xe^{x}$. Then, $y_1y_2'-y_2y_1'=e^{2x}(1+x)-xe^{2x}=e^{2x}$ and so $u_1'=-xe^{x} \Rightarrow u_1(x)=-\int xe^{x}dx=-(x-1)e^{x}$ and $u_2'=e^{x} \Rightarrow u_2(x)=\int e^{x}dx=e^{x}$. Hence $y_p(x)=(1-x)e^{2x}+xe^{2x}=e^{2x}$ and the general solution is $y(x)=y_c(x)+y_p(x)=c_1e^{x}+c_2xe^{x}+e^{2x}$.

22. **(a)** Here $r^2-2r+1=(r+1)^2=0 \Rightarrow r=-1$ and $y_c(x)=c_1+c_2e^{x}$ and so we try a particular solution of the form $y_p(x)=Axe^{x}$. Thus, after calculating the necessary derivatives, we get $y''-y'=e^{x} \Rightarrow$ $Ae^{x}(2+x)-Ae^{x}(1+x)=e^{x} \Rightarrow A=1$. Thus $y_p(x)=xe^{x}$ and the general solution is $y(x)=c_1+c_2e^{x}+xe^{x}$.

(b) From (a) we know that $y_c(x)=c_1+c_2e^{x}$, so setting $y_1=1$, $y_2=e^{x}$, then $y_1y_2'-y_2y_1'=e^{x}-0=e^{x}$. Thus $u_1'=-e^{2x}/e^{x}=-e^{x}$ and $u_2'=e^{x}/e^{x}=1$. Then $u_1(x)=-\int e^{x}dx=-e^{x}$ and $u_2(x)=x$. Thus $y_p(x)=-e^{x}+xe^{x}$ and the general solution is $y(x)=c_1+c_2e^x-e^x+xe^x=c_1+c_3e^x+xe^x$.

23. As in Example 6, $y_c(x)=c_1\sin x+c_2\cos x$, so set $y_1=\sin x$, $y_2=\cos x$. Then $y_1y_2'-y_2y_1'=-\sin^2 x-\cos^2 x=-1$, so $u_1'=-\dfrac{\sec x\cos x}{-1}=1\Rightarrow u_1(x)=x$ and $u_2'=\dfrac{\sec x\sin x}{-1}=-\tan x\Rightarrow$ $u_2(x)=-\int\tan x\,dx=\ln|\cos x|=\ln(\cos x)$ on $0<x<\dfrac{\pi}{2}$. Hence $y_p(x)=x\sin x+\cos x\ln(\cos x)$ and the general solution is $y(x)=(c_1+x)\sin x+\left[c_2+\ln(\cos x)\right]\cos x$.

24. Setting $y_1=\sin x$, $y_2=\cos x$, then $y_1y_2'-y_2y_1'=-\sin^2 x-\cos^2 x=-1$. Thus $u_1'=-\dfrac{\cot x\cos x}{-1}=\dfrac{\cos^2 x}{\sin x}$ and $u_2'=\dfrac{\cot x\sin x}{-1}=-\cos x$. Then $u_1(x)=\int\dfrac{\cos^2 x}{\sin x}dx=\int(\csc x-\sin x)dx=\ln(\csc x-\cot x)+\cos x$ and $u_2(x)=-\sin x$. Thus $y_p(x)=[\cos x+\ln(\csc x-\cot x)]\sin x+(-\sin x)(\cos x)$ and the general solution is $y(x)=c_1\sin x+c_2\cos x+\sin x\ln(\csc x-\cot x)$.

25. $y_1=e^x$, $y_2=e^{2x}$ and $y_1y_2'-y_2y_1'=e^{3x}$. So $u_1'=\dfrac{-e^{2x}}{(1+e^{-x})e^{3x}}=-\dfrac{e^{-x}}{1+e^{-x}}$ and $u_1(x)=\int-\dfrac{e^{-x}}{1+e^{-x}}dx=\ln(1+e^{-x})$. $u_2'=\dfrac{e^x}{(1+e^{-x})e^{3x}}=\dfrac{e^x}{e^{3x}+e^{2x}}$ so $u_2(x)=\int\dfrac{e^x}{e^{3x}+e^{2x}}dx=\ln\left(\dfrac{e^x+1}{e^x}\right)-e^{-x}=\ln(1+e^{-x})-e^{-x}$. Hence $y_p(x)=e^x\ln(1+e^{-x})+e^{2x}[\ln(1+e^{-x})-e^{-x}]$ and the general solution is $y(x)=[c_1+\ln(1+e^{-x})]e^x+[c_2-e^{-x}+\ln(1+e^{-x})]e^{2x}$.

26. $y_1=e^{-x}$, $y_2=e^{-2x}$ and $y_1y_2'-y_2y_1'=-e^{-3x}$. So $u_1'=-\dfrac{(\sin e^x)e^{-2x}}{-e^{-3x}}=e^x\sin e^x$ and $u_2'=\dfrac{(\sin e^x)e^{-x}}{-e^{-3x}}=-e^{2x}\sin e^x$. Hence $u_1(x)=\int e^x\sin e^x dx=-\cos e^x$ and $u_2(x)=\int-e^{2x}\sin e^x dx=e^x\cos e^x-\sin e^x$. Then $y_p(x)=-e^{-x}\cos e^x-e^{-2x}[\sin e^x-e^x\cos e^x]$ and the general solution is $y(x)=(c_1-\cos e^x)e^{-x}+[c_2-\sin e^x+e^x\cos e^x]e^{-2x}$.

27. $y_1=e^{-x}$, $y_2=e^x$ and $y_1y_2'-y_2y_1'=2$. So

$u_1' = -\frac{e^x}{2x}$, $u_2' = \frac{e^{-x}}{2x}$ and

$y_p(x) = -e^{-x}\int \frac{e^x}{2x}\,dx + e^x\int \frac{e^{-x}}{2x}\,dx$. Hence the general solution is

$y(x) = \left(c_1 - \int \frac{e^x}{2x}\,dx\right)e^{-x} + \left(c_2 + \int \frac{e^{-x}}{2x}\,dx\right)e^x$.

28. $y_1 = e^{-2x}$, $y_2 = xe^{-2x}$ and $y_1y_2' - y_2y_1' = e^{-4x}$. Then $u_1' = \frac{-e^{-2x}xe^{-2x}}{x^3e^{-4x}} = -\frac{1}{x^2}$ so $u_1(x) = x^{-1}$ and

$u_2' = \frac{e^{-2x}e^{-2x}}{x^3e^{-4x}} = \frac{1}{x^3}$ so $u_2(x) = -\frac{1}{2x^2}$. Thus $y_p(x) = \frac{e^{-2x}}{x} - \frac{xe^{-2x}}{2x^2} = \frac{e^{-2x}}{2x}$ and the general solution is

$y(x) = e^{-2x}[c_1 + c_2x + 1/(2x)]$.

1. By Hooke's Law $k(0.6)=20$ so $k=\frac{100}{3}$ is the spring constant and the differential equation is $3x''+\frac{100}{3}x=0$. The general solution is $x(t)=c_1\cos\left(\frac{10}{3}t\right)+c_2\sin\left(\frac{10}{3}t\right)$. But $0=x(0)=c_1$ and $1.2=x'(0)=\frac{10}{3}c_2$, so the position of the mass after t seconds is $x(t)=0.36\sin\left(\frac{10}{3}t\right)$.

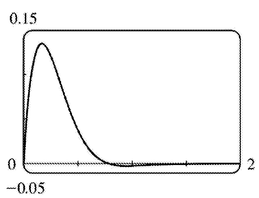
2. $k(0.3)=24.3$ or $k=81$ is the spring constant and the resulting initial–value problem is $4x''+81x=0$, $x(0)=-0.5$ (since compressed), $x'(0)=0$. The general solution is $x(t)=c_1\cos\left(\frac{9}{2}t\right)+c_2\sin\left(\frac{9}{2}t\right)$. But $-0.2=x(0)=c_1$ and $0=x'(0)=\frac{9}{2}c_2$. Thus the position is given by $x(t)=-0.2\cos(4.5t)$.

3. $k(0.5)=6$ or $k=12$ is the spring constant, so the initial–value problem is $2x''+14x'+12x=0$, $x(0)=1$, $x'(0)=0$. The general solution is $x(t)=c_1e^{-6t}+c_2e^{-t}$. But $1=x(0)=c_1+c_2$ and $0=x'(0)=-6c_1-c_2$. Thus the position is given by $x(t)=-\frac{1}{5}e^{-6t}+\frac{6}{5}e^{-t}$.

4.

(a) The differential equation is $3x''+30x'+123x=0$ with general solution $x(t)=e^{-5t}(c_1\cos 4t+c_2\sin 4t)$. Then $0=x(0)=c_1$ and $2=x'(0)=4c_2$, so the position is given by $x(t)=\frac{1}{2}e^{-5t}\sin 4t$.

(b)

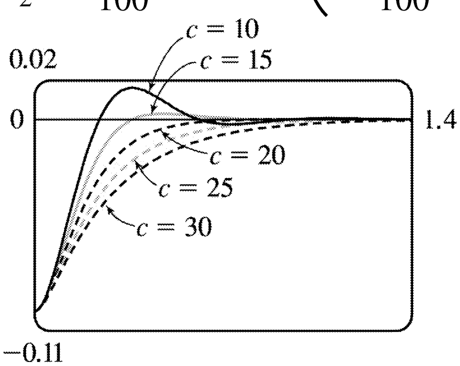


5. For critical damping we need $c^2-4mk=0$ or $m=c^2/(4k)=14^2/(4\cdot 12)=\frac{49}{12}$ kg.

6. For critical damping we need $c^2=4mk$ or $c=2\sqrt{mk}=2\sqrt{3\cdot 123}=6\sqrt{41}$.

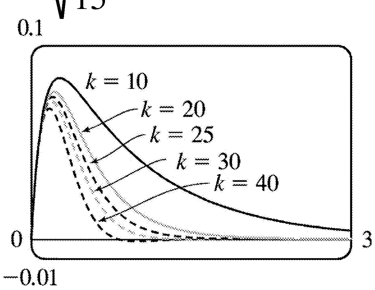
7. We are given $m=1$, $k=100$, $x(0)=-0.1$ and $x'(0)=0$. From (3), the differential equation is $\frac{d^2x}{dt^2}+c\frac{dx}{dt}+100x=0$ with auxiliary equation $r^2+cr+100=0$. If $c=10$, we have two complex roots

$r=-5\pm 5\sqrt{3}i$, so the motion is underdamped and the solution is $x=e^{-5t}\left[c_1\cos\left(5\sqrt{3}t\right)+c_2\sin\left(5\sqrt{3}t\right)\right]$. Then $-0.1=x(0)=c_1$ and $0=x'(0)=5\sqrt{3}c_2-5c_1 \Rightarrow c_2=-\frac{1}{10\sqrt{3}}$, so $x=e^{-5t}\left[-0.1\cos\left(5\sqrt{3}t\right)-\frac{1}{10\sqrt{3}}\sin\left(5\sqrt{3}t\right)\right]$. If $c=15$, we again have underdamping since the auxiliary equation has roots $r=-\frac{15}{2}\pm\frac{5\sqrt{7}}{2}i$. The general solution is $x=e^{-15t/2}\left[c_1\cos\left(\frac{5\sqrt{7}}{2}t\right)+c_2\sin\left(\frac{5\sqrt{7}}{2}t\right)\right]$, so $-0.1=x(0)=c_1$ and $0=x'(0)=\frac{5\sqrt{7}}{2}c_2-\frac{15}{2}c_1$ $\Rightarrow c_2=-\frac{3}{10\sqrt{7}}$. Thus $x=e^{-15t/2}\left[-0.1\cos\left(\frac{5\sqrt{7}}{2}t\right)-\frac{3}{10\sqrt{7}}\sin\left(\frac{5\sqrt{7}}{2}t\right)\right]$. For $c=20$, we have equal roots $r_1=r_2=-10$, so the oscillation is critically damped and the solution is $x=(c_1+c_2t)e^{-10t}$. Then $-0.1=x(0)=c_1$ and $0=x'(0)=-10c_1+c_2\Rightarrow c_2=-1$, so $x=(-0.1-t)e^{-10t}$. If $c=25$ the auxiliary equation has roots $r_1=-5$, $r_2=-20$, so we have overdamping and the solution is $x=c_1e^{-5t}+c_2e^{-20t}$. Then $-0.1=x(0)=c_1+c_2$ and $0=x'(0)=-5c_1-20c_2\Rightarrow c_1=-\frac{2}{15}$ and $c_2=\frac{1}{30}$, so $x=-\frac{2}{15}e^{-5t}+\frac{1}{30}e^{-20t}$. If $c=30$ we have roots $r=-15\pm 5\sqrt{5}$, so the motion is overdamped and the solution is $x=c_1e^{\left(-15+5\sqrt{5}\right)t}+c_2e^{\left(-15-5\sqrt{5}\right)t}$. Then $-0.1=x(0)=c_1+c_2$ and $0=x'(0)=\left(-15+5\sqrt{5}\right)c_1+\left(-15-5\sqrt{5}\right)c_2\Rightarrow c_1=\frac{-5-3\sqrt{5}}{100}$ and $c_2=\frac{-5+3\sqrt{5}}{100}$, so $x=\left(\frac{-5-3\sqrt{5}}{100}\right)e^{\left(-15+5\sqrt{5}\right)t}+\left(\frac{-5+3\sqrt{5}}{100}\right)e^{\left(-15-5\sqrt{5}\right)t}$.



8. We are given $m=1$, $c=10$, $x(0)=0$ and $x'(0)=1$. The differential equation is $\frac{d^2x}{dt^2}+10\frac{dx}{dt}+kx=0$ with auxiliary equation $r^2+10r+k=0$. $k=10$: the auxiliary equation has roots $r=-5\pm\sqrt{15}$ so we have

overdamping and the solution is $x=c_1e^{\left(-5+\sqrt{15}\right)t}+c_2e^{\left(-5-\sqrt{15}\right)t}$. Entering the initial conditions gives $c_1=\frac{1}{2\sqrt{15}}$ and $c_2=-\frac{1}{2\sqrt{15}}$, so $x=\frac{1}{2\sqrt{15}}e^{\left(-5+\sqrt{15}\right)t}-\frac{1}{2\sqrt{15}}e^{\left(-5-\sqrt{15}\right)t}$. $k=20$: $r=-5\pm\sqrt{5}$ and the solution is $x=c_1e^{\left(-5+\sqrt{5}\right)t}+c_2e^{\left(-5-\sqrt{5}\right)t}$ so again the motion is overdamped. The initial conditions give $c_1=\frac{1}{2\sqrt{5}}$ and $c_2=-\frac{1}{2\sqrt{5}}$, so $x=\frac{1}{2\sqrt{5}}e^{\left(-5+\sqrt{5}\right)t}-\frac{1}{2\sqrt{5}}e^{\left(-5-\sqrt{5}\right)t}$. $k=25$: we have equal roots $r_1=r_2=-5$, so the motion is critically damped and the solution is $x=(c_1+c_2t)e^{-5t}$. The initial conditions give $c_1=0$ and $c_2=1$, so $x=te^{-5t}$. $k=30$: $r=-5\pm\sqrt{5}\,i$ so the motion is underdamped and the solution is $x=e^{-5t}\left[c_1\cos\left(\sqrt{5}\,t\right)+c_2\sin\left(\sqrt{5}\,t\right)\right]$. The initial conditions give $c_1=0$ and $c_2=\frac{1}{\sqrt{5}}$, so $x=\frac{1}{\sqrt{5}}e^{-5t}\sin\left(\sqrt{5}\,t\right)$. $k=40$: $r=-5\pm\sqrt{15}\,i$ so we again have underdamping. The solution is $x=e^{-5t}\left[c_1\cos\left(\sqrt{15}\,t\right)+c_2\sin\left(\sqrt{15}\,t\right)\right]$, and the initial conditions give $c_1=0$ and $c_2=\frac{1}{\sqrt{15}}$. Thus $x=\frac{1}{\sqrt{15}}e^{-5t}\sin\left(\sqrt{15}\,t\right)$.



9. The differential equation is $mx''+kx=F_0\cos\omega_0t$ and $\omega_0\neq\omega=\sqrt{k/m}$. Here the auxiliary equation is $mr^2+k=0$ with roots $\pm\sqrt{k/m}\,i=\pm\omega i$ so $x_c(t)=c_1\cos\omega t+c_2\sin\omega t$. Since $\omega_0\neq\omega$, try $x_p(t)=A\cos\omega_0t+B\sin\omega_0t$. Then we need $(m)\left(-\omega_0^2\right)(A\cos\omega_0t+B\sin\omega_0t)+k(A\cos\omega_0t+B\sin\omega_0t)=F_0\cos\omega_0t$ or $A\left(k-m\omega_0^2\right)=F_0$ and $B\left(k-m\omega_0^2\right)=0$. Hence $B=0$ and $A=\frac{F_0}{k-m\omega_0^2}=\frac{F_0}{m\left(\omega^2-\omega_0^2\right)}$ since $\omega^2=\frac{k}{m}$. Thus the motion of the mass is given by

$x(t)=c_1\cos\omega t+c_2\sin\omega t+\dfrac{F_0}{m\left(\omega^2-\omega_0^2\right)}\cos\omega_0 t$.

10. As in Exercise 9, $x_c(t)=c_1\cos\omega t+c_2\sin\omega t$. But the natural frequency of the system equals the frequency of the external force, so try $x_p(t)=t(A\cos\omega t+B\sin\omega t)$. Then we need $m(2\omega B-\omega^2 At)\cos\omega t-m(2\omega A+\omega^2 Bt)\sin\omega t+kAt\cos\omega t+kBt\sin\omega t=F_0\cos\omega t$ or $2m\omega B=F_0$ and $-2m\omega A=0$ (noting $-m\omega^2 A+kA=0$ and $-m\omega^2 B+kB=0$ since $\omega^2=k/m$). Hence the general solution is $x(t)=c_1\cos\omega t+c_2\sin\omega t+\left[F_0 t/(2m\omega)\right]\sin\omega t$.

11. From Equation 6, $x(t)=f(t)+g(t)$ where $f(t)=c_1\cos\omega t+c_2\sin\omega t$ and $g(t)=\dfrac{F_0}{m\left(\omega^2-\omega_0^2\right)}\cos\omega_0 t$. Then f is periodic, with period $\dfrac{2\pi}{\omega}$, and if $\omega\neq\omega_0$, g is periodic with period $\dfrac{2\pi}{\omega_0}$. If $\dfrac{\omega}{\omega_0}$ is a rational number, then we can say $\dfrac{\omega}{\omega_0}=\dfrac{a}{b}\Rightarrow a=\dfrac{b\omega}{\omega_0}$ where a and b are non–zero integers. Then

$$x\left(t+a\cdot\frac{2\pi}{\omega}\right)=f\left(t+a\cdot\frac{2\pi}{\omega}\right)+g\left(t+a\cdot\frac{2\pi}{\omega}\right)=f(t)+g\left(t+\frac{b\omega}{\omega_0}\cdot\frac{2\pi}{\omega}\right)$$
$$=f(t)+g\left(t+b\cdot\frac{2\pi}{\omega_0}\right)=f(t)+g(t)=x(t)$$

so $x(t)$ is periodic.

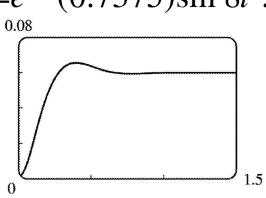
12. **(a)** The graph of $x=c_1e^{rt}+c_2te^{rt}$ has a t –intercept when $c_1e^{rt}+c_2te^{rt}=0\Leftrightarrow e^{rt}(c_1+c_2t)=0\Leftrightarrow c_1=-c_2t$. Since $t>0$, x has a t –intercept if and only if c_1 and c_2 have opposite signs.

(b) For $t>0$, the graph of x crosses the t –axis when $c_1e^{r_1t}+c_2e^{r_2t}=0\Leftrightarrow c_2e^{r_2t}=-c_1e^{r_1t}\Leftrightarrow$ $c_2=-c_1\dfrac{e^{r_1t}}{e^{r_2t}}=-c_1e^{(r_1-r_2)t}$. But $r_1>r_2\Rightarrow r_1-r_2>0$ and since $t>0$, $e^{(r_1-r_2)t}>1$. Thus

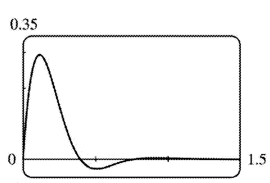
$\left|c_2\right| = \left|c_1\right| e^{(r_1-r_2)t} > \left|c_1\right|$, and the graph of x can cross the t –axis only if $\left|c_2\right| > \left|c_1\right|$.

13. Here the initial–value problem for the charge is $Q''+20Q'+500Q=12$, $Q(0)=Q'(0)=0$. Then $Q_c(t)=e^{-10t}(c_1\cos 20t+c_2\sin 20t)$ and try $Q_p(t)=A \Rightarrow 500A=12$ or $A=\frac{3}{125}$. The general solution is $Q(t)=e^{-10t}(c_1\cos 20t+c_2\sin 20t)+\frac{3}{125}$. But $0=Q(0)=c_1+\frac{3}{125}$ and $Q'(t)=I(t)=e^{-10t}\left[(-10c_1+20c_2)\cos 20t+(-10c_2-20c_1)\sin 20t\right]$ but $0=Q'(0)=-10c_1+20c_2$. Thus the charge is $Q(t)=-\frac{1}{250}e^{-10t}(6\cos 20t+3\sin 20t)+\frac{3}{125}$ and the current is $I(t)=e^{-10t}\left(\frac{3}{5}\right)\sin 20t$.

14. **(a)** Here the initial–value problem for the charge is $2Q''+24Q'+200Q=12$ with $Q(0)=0.001$ and $Q'(0)=0$. Then $Q_c(t)=e^{-6t}(c_1\cos 8t+c_2\sin 8t)$ and try $Q_p(t)=A \Rightarrow A=\frac{3}{50}$ and the general solution is $Q(t)=e^{-6t}(c_1\cos 8t+c_2\sin 8t)+\frac{3}{50}$. But $0.001=Q(0)=c+\frac{3}{50}$ so $c_1=-0.059$. Also $Q'(t)=I(t)=e^{-6t}\left[(-6c_1+8c_2)\cos 8t+(-6c_2-8c_1)\sin 8t\right]$ and $0=Q'(0)=-6c_1+8c_2$ so $c_2=-0.04425$. Hence the charge is $Q(t)=-e^{-6t}(0.059\cos 8t+0.04425\sin 8t)+\frac{3}{50}$ and the current is $I(t)=e^{-6t}(0.7375)\sin 8t$.



(b) charge, $Q(t)$



current, $I(t) = Q'(t)$

15. As in Exercise 13, $Q_c(t)=e^{-10t}(c_1\cos 20t+c_2\sin 20t)$ but $E(t)=12\sin 10t$ so try $Q_p(t)=A\cos 10t+B\sin 10t$. Substituting into the differential equation gives $(-100A+200B+500A)\cos 10t+(-100B-200A+500B)\sin 10t=12\sin 10t \Rightarrow 400A+200B=0$ and

$400B-200A=12$. Thus $A=-\frac{3}{250}$, $B=\frac{3}{125}$ and the general solution is

$Q(t)=e^{-10t}(c_1\cos 20t+c_2\sin 20t)-\frac{3}{250}\cos 10t+\frac{3}{125}\sin 10t$. But $0=Q(0)=c_1-\frac{3}{250}$ so $c_1=\frac{3}{250}$. Also

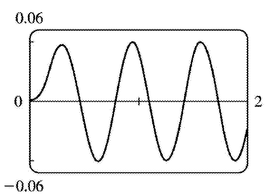
$Q'(t)=\frac{3}{25}\sin 10t+\frac{6}{25}\cos 10t+e^{-10t}\left[(-10c_1+20c_2)\cos 20t+(-10c_2-20c_1)\sin 20t\right]$ and

$0=Q'(0)=\frac{6}{25}-10c_1+20c_2$ so $c_2=-\frac{3}{500}$. Hence the charge is given by

$Q(t)=e^{-10t}\left[\frac{3}{250}\cos 20t-\frac{3}{500}\sin 20t\right]-\frac{3}{250}\cos 10t+\frac{3}{125}\sin 10t$.

16. **(a)** As in Exercise 14, $Q_c(t)=e^{-6t}(c_1\cos 8t+c_2\sin 8t)$ but try $Q_p(t)=A\cos 10t+B\sin 10t$. Substituting into the differential equation gives
$(-200A+240B+200A)\cos 10t+(-200B-240A+200B)\sin 10t=12\sin 10t$, so $B=0$ and
$A=-\frac{1}{20}$. Hence, the general solution is $Q(t)=e^{-6t}(c_1\cos 8t+c_2\sin 8t)-\frac{1}{20}\cos 10t$. But
$0.001=Q(0)=c_1-\frac{1}{20}$, $Q'(t)=e^{-6t}\left[(-6c_1+8c_2)\cos 8t+(-6c_2-8c_1)\sin 8t\right]-\frac{1}{2}\sin 10t$ and
$0=Q'(0)=-6c_1+8c_2$, so $c_1=0.051$ and $c_2=0.03825$. Thus the charge is given by

$Q(t)=e^{-6t}(0.051\cos 8t+0.03825\sin 8t)-\frac{1}{20}\cos 10t$.

(b) 

17. $x(t)=A\cos(\omega t+\delta)\Leftrightarrow x(t)=A[\cos\omega t\cos\delta-\sin\omega t\sin\delta]\Leftrightarrow x(t)=A\left(\frac{c_1}{A}\cos\omega t+\frac{c_2}{A}\sin\omega t\right)$ where $\cos\delta=c_1/A$ and $\sin\delta=-c_2/A\Leftrightarrow x(t)=c_1\cos\omega t+c_2\sin\omega t$. (Note that $\cos^2\delta+\sin^2\delta=1\Rightarrow c_1^2+c_2^2=A^2$.)

18. **(a)** We approximate $\sin\theta$ by θ and, with $L=1$ and $g=9.8$, the differential equation becomes
$\frac{d^2\theta}{dt^2}+9.8\theta=0$. The auxiliary equation is $r^2+9.8=0\Rightarrow r=\pm\sqrt{9.8}\,i$, so the general solution is
$\theta(t)=c_1\cos\left(\sqrt{9.8}\,t\right)+c_2\sin\left(\sqrt{9.8}\,t\right)$. Then $0.2=\theta(0)=c_1$ and $1=\theta'(0)=\sqrt{9.8}\,c_2\Rightarrow c_2=\frac{1}{\sqrt{9.8}}$, so the

equation is $\theta(t)=0.2\cos\left(\sqrt{9.8}\,t\right)+\frac{1}{\sqrt{9.8}}\sin\left(\sqrt{9.8}\,t\right)$.

(b) $\theta'(t)=-0.2\sqrt{9.8}\sin\left(\sqrt{9.8}\,t\right)+\cos\left(\sqrt{9.8}\,t\right)=0$ or $\tan\left(\sqrt{9.8}\,t\right)=\frac{5}{\sqrt{9.8}}$, so the critical numbers are $t=\frac{1}{\sqrt{9.8}}\tan^{-1}\left(\frac{5}{\sqrt{9.8}}\right)+\frac{n}{\sqrt{9.8}}\pi$ (n any integer). The maximum angle from the vertical is $\theta\left(\frac{1}{\sqrt{9.8}}\tan^{-1}\left(\frac{5}{\sqrt{9.8}}\right)\right)\approx 0.377$ radians (or about 21.7°).

(c) From part (b), the critical numbers of $\theta(t)$ are spaced $\frac{\pi}{\sqrt{9.8}}$ apart, and the time between successive maximum values is $2\left(\frac{\pi}{\sqrt{9.8}}\right)$. Thus the period of the pendulum is $\frac{2\pi}{\sqrt{9.8}}\approx 2.007$ seconds.

(d) $\theta(t)=0 \Rightarrow 0.2\cos\left(\sqrt{9.8}\,t\right)+\frac{1}{\sqrt{9.8}}\sin\left(\sqrt{9.8}\,t\right)=0 \Rightarrow \tan\left(\sqrt{9.8}\,t\right)=-0.2\sqrt{9.8} \Rightarrow$
$t=\frac{1}{\sqrt{9.8}}\left[\tan^{-1}\left(-0.2\sqrt{9.8}\right)+\pi\right]\approx 0.825$ seconds.

(e) $\theta'(0.825)\approx -1.180$ rad/s.

1. Let $y(x)=\Sigma_{n=0}^{\infty} c_n x^n$. Then $y'(x)=\Sigma_{n=1}^{\infty} nc_n x^{n-1}$ and the given equation, $y'-y=0$, becomes $\Sigma_{n=1}^{\infty} nc_n x^{n-1}-\Sigma_{n=0}^{\infty} c_n x^n=0$. Replacing n by $n+1$ in the first sum gives $\Sigma_{n=0}^{\infty}(n+1)c_{n+1}x^n-\Sigma_{n=0}^{\infty} c_n x^n=0$, so $\Sigma_{n=0}^{\infty}\left[(n+1)c_{n+1}-c_n\right]x^n=0$. Equating coefficients gives $(n+1)c_{n+1}-c_n=0$, so the recursion relation is $c_{n+1}=\dfrac{c_n}{n+1}$, $n=0,1,2,\ldots$. Then $c_1=c_0$, $c_2=\frac{1}{2}c_1=\frac{c_0}{2}$, $c_3=\frac{1}{3}c_2=\frac{1}{3}\cdot\frac{1}{2}c_0=\frac{c_0}{3!}$, $c_4=\frac{1}{4}c_3=\frac{c_0}{4!}$, and in general, $c_n=\dfrac{c_0}{n!}$. Thus, the solution is

$$y(x)=\Sigma_{n=0}^{\infty} c_n x^n=\Sigma_{n=0}^{\infty}\frac{c_0}{n!}x^n=c_0\Sigma_{n=0}^{\infty}\frac{x^n}{n!}=c_0e^x$$

2. Let $y(x)=\Sigma_{n=0}^{\infty} c_n x^n$. Then $y'=xy\Rightarrow y'-xy=0\Rightarrow\Sigma_{n=1}^{\infty} nc_n x^{n-1}-x\Sigma_{n=0}^{\infty} c_n x^n=0$ or $\Sigma_{n=1}^{\infty} nc_n x^{n-1}-\Sigma_{n=0}^{\infty} c_n x^{n+1}=0$. Replacing n with $n+1$ in the first sum and n with $n-1$ in the second gives $\Sigma_{n=0}^{\infty}(n+1)c_{n+1}x^n-\Sigma_{n=1}^{\infty} c_{n-1}x^n=0$ or $c_1+\Sigma_{n=1}^{\infty}(n+1)c_{n+1}x^n-\Sigma_{n=1}^{\infty} c_{n-1}x^n=0$. Thus, $c_1+\Sigma_{n=1}^{\infty}\left[(n+1)c_{n+1}-c_{n-1}\right]x^n=0$. Equating coefficients gives $c_1=0$ and $(n+1)c_{n+1}-c_{n-1}=0$. Thus, the recursion relation is $c_{n+1}=\dfrac{c_{n-1}}{n+1}$, $n=1,2$, $\ldots$. But $c_1=0$, so $c_3=0$ and $c_5=0$ and in general $c_{2n+1}=0$. Also, $c_2=\dfrac{c_0}{2}$, $c_4=\dfrac{c_2}{4}=\dfrac{c_0}{4\cdot 2}=\dfrac{c_0}{2^2\cdot 2!}$, $c_6=\dfrac{c_4}{6}=\dfrac{c_0}{6\cdot 4\cdot 2}=\dfrac{c_0}{2^3\cdot 3!}$ and in general $c_{2n}=\dfrac{c_0}{2^n\cdot n!}$. Thus, the solution is

$$y(x)=\Sigma_{n=0}^{\infty} c_n x^n=\Sigma_{n=0}^{\infty} c_{2n}x^{2n}=\Sigma_{n=0}^{\infty}\frac{c_0}{2^n\cdot n!}x^{2n}=c_0\Sigma_{n=0}^{\infty}\frac{\left(x^2/2\right)^n}{n!}=c_0e^{x^2/2}$$

3. Assuming $y(x)=\Sigma_{n=0}^{\infty} c_n x^n$, we have $y'(x)=\Sigma_{n=1}^{\infty} nc_n x^{n-1}=\Sigma_{n=0}^{\infty}(n+1)c_{n+1}x^n$ and $-x^2y=-\Sigma_{n=0}^{\infty} c_n x^{n+2}=-\Sigma_{n=2}^{\infty} c_{n-2}x^n$. Hence, the equation $y'=x^2y$ becomes $\Sigma_{n=0}^{\infty}(n+1)c_{n+1}x^n-\Sigma_{n=2}^{\infty} c_{n-2}x^n=0$ or $c_1+2c_2x+\Sigma_{n=2}^{\infty}\left[(n+1)c_{n+1}-c_{n-2}\right]x^n=0$. Equating coefficients gives $c_1=c_2=0$ and

$c_{n+1}=\frac{c_{n-2}}{n+1}$ for $n=2,3$, ... But $c_1=0$, so $c_4=0$ and $c_7=0$ and in general $c_{3n+1}=0$. Similarly $c_2=0$ so $c_{3n+2}=0$. Finally $c_3=\frac{c_0}{3}$, $c_6=\frac{c_3}{6}=\frac{c_0}{6\cdot 3}=\frac{c_0}{3^2\cdot 2!}$, $c_9=\frac{c_6}{9}=\frac{c_0}{9\cdot 6\cdot 3}=\frac{c_0}{3^3\cdot 3!}$, ... , and $c_{3n}=\frac{c_0}{3^n\cdot n!}$.
Thus, the solution is

$$y(x)=\sum_{n=0}^{\infty}c_n x^n=\sum_{n=0}^{\infty}c_{3n}x^{3n}=\sum_{n=0}^{\infty}\frac{c_0}{3^n\cdot n!}x^{3n}=c_0\sum_{n=0}^{\infty}\frac{x^{3n}}{3^n n!}=c_0\sum_{n=0}^{\infty}\frac{\left(x^3/3\right)^n}{n!}=c_0 e^{x^3/3}$$

4. Let $y(x)=\sum_{n=0}^{\infty}c_n x^n \Rightarrow y'(x)=\sum_{n=1}^{\infty}nc_n x^{n-1}=\sum_{n=0}^{\infty}(n+1)c_{n+1}x^n$. Then the differential equation becomes $(x-3)\sum_{n=0}^{\infty}(n+1)c_{n+1}x^n+2\sum_{n=0}^{\infty}c_n x^n=0\Rightarrow$
$\sum_{n=0}^{\infty}(n+1)c_{n+1}x^{n+1}-3\sum_{n=0}^{\infty}(n+1)c_{n+1}x^n+2\sum_{n=0}^{\infty}c_n x^n=0\Rightarrow \sum_{n=1}^{\infty}nc_n x^n-\sum_{n=0}^{\infty}3(n+1)c_{n+1}x^n+\sum_{n=0}^{\infty}2c_n x^n=0$
$\Rightarrow \sum_{n=0}^{\infty}\left[(n+2)c_n-3(n+1)c_{n+1}\right]x^n=0\left(\text{since}\sum_{n=1}^{\infty}nc_n x^n=\sum_{n=0}^{\infty}nc_n x^n\right)$. Equating coefficients gives $(n+2)c_n-3(n+1)c_{n+1}=0$, thus the recursion relation is $c_{n+1}=\frac{(n+2)c_n}{3(n+1)}$, $n=0,1,2,\ldots$. Then $c_1=\frac{2c_0}{3}$, $c_2=\frac{3c_1}{3(2)}=\frac{3c_0}{3^2}$, $c_3=\frac{4c_2}{3(3)}=\frac{4c_0}{3^3}$, $c_4=\frac{5c_3}{3(4)}=\frac{5c_0}{3^4}$, and in general, $c_n=\frac{(n+1)c_0}{3^n}$. Thus the solution is $y(x)=\sum_{n=0}^{\infty}c_n x^n=c_0\sum_{n=0}^{\infty}\frac{n+1}{3^n}x^n$. $\left[\text{Note that } c_0\sum_{n=0}^{\infty}\frac{n+1}{3^n}x^n=\frac{9c_0}{(3-x)^2}\text{ for } |x|<3.\right]$

5. Let $y(x)=\sum_{n=0}^{\infty}c_n x^n \Rightarrow y'(x)=\sum_{n=1}^{\infty}nc_n x^{n-1}$ and $y''(x)=\sum_{n=0}^{\infty}(n+2)(n+1)c_{n+2}x^n$. The differential equation becomes $\sum_{n=0}^{\infty}(n+2)(n+1)c_{n+2}x^n+x\sum_{n=1}^{\infty}nc_n x^{n-1}+\sum_{n=0}^{\infty}c_n x^n=0$ or
$\sum_{n=0}^{\infty}\left[(n+2)(n+1)c_{n+2}+nc_n+c_n\right]x^n\left(\text{since}\sum_{n=1}^{\infty}nc_n x^n=\sum_{n=0}^{\infty}nc_n x^n\right)$. Equating coefficients gives $(n+2)(n+1)c_{n+2}+(n+1)c_n=0$, thus the recursion relation is $c_{n+2}=\frac{-(n+1)c_n}{(n+2)(n+1)}=-\frac{c_n}{n+2}$, $n=0,1,2,\ldots$.
Then the even coefficients are given by $c_2=-\frac{c_0}{2}$, $c_4=-\frac{c_2}{4}=\frac{c_0}{2\cdot 4}$, $c_6=-\frac{c_4}{6}=-\frac{c_0}{2\cdot 4\cdot 6}$, and in general,

$c_{2n}=(-1)^n \dfrac{c_0}{2\cdot 4\cdot \cdot\cdot\cdot\cdot 2n} = \dfrac{(-1)^n c_0}{2^n \text{n!}}$. The odd coefficients are $c_3=-\dfrac{c_1}{3}$, $c_5=-\dfrac{c_3}{5}=\dfrac{c_1}{3\cdot 5}$,

$c_7=-\dfrac{c_5}{7}=-\dfrac{c_1}{3\cdot 5\cdot 7}$, and in general, $c_{2n+1}=(-1)^n \dfrac{c_1}{3\cdot 5\cdot 7\cdot \cdot\cdot\cdot\cdot (2n+1)} = \dfrac{(-2)^n \text{n!} c_1}{(2n+1)!}$. The solution is

$y(x)=c_0\Sigma_{n=0}^{\infty} \dfrac{(-1)^n}{2^n \text{n!}} x^{2n}+c_1\Sigma_{n=0}^{\infty} \dfrac{(-2)^n \text{n!}}{(2n+1)!} x^{2n+1}$.

6. Let $y(x)=\Sigma_{n=0}^{\infty} c_n x^n$. Then $y''(x)=\Sigma_{n=2}^{\infty} n(n-1)c_n x^{n-2}=\Sigma_{n=0}^{\infty}(n+2)(n+1)c_{n+2}x^n$. Hence, the equation $y''=y$ becomes $\Sigma_{n=0}^{\infty}(n+2)(n+1)c_{n+2}x^n-\Sigma_{n=0}^{\infty}c_n x^n=0$ or $\Sigma_{n=0}^{\infty}\left[(n+2)(n+1)c_{n+2}-c_n\right]x^n=0$.

So the recursion relation is $c_{n+2}=\dfrac{c_n}{(n+2)(n+1)}$, $n=0,1,\ldots$. Given c_0 and c_1 , $c_2=\dfrac{c_0}{2\cdot 1}$, $c_4=\dfrac{c_2}{4\cdot 3}=\dfrac{c_0}{4!}$,

$c_6=\dfrac{c_4}{6\cdot 5}=\dfrac{c_0}{6!}$, … , $c_{2n}=\dfrac{c_0}{(2n)!}$ and $c_3=\dfrac{c_1}{3\cdot 2}$, $c_5=\dfrac{c_3}{5\cdot 4}=\dfrac{c_1}{5\cdot 4\cdot 3\cdot 2}=\dfrac{c_1}{5!}$, $c_7=\dfrac{c_5}{7\cdot 6}=\dfrac{c_1}{7!}$, … ,

$c_{2n+1}=\dfrac{c_1}{(2n+1)!}$. Thus, the solution is

$$y(x)=\Sigma_{n=0}^{\infty}c_n x^n=\Sigma_{n=0}^{\infty}c_{2n}x^{2n}+\Sigma_{n=0}^{\infty}c_{2n+1}x^{2n+1}=c_0\Sigma_{n=0}^{\infty}\frac{x^{2n}}{(2n)!}+c_1\Sigma_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!}$$

The solution can be written as

$y(x)=c_0\cosh x+c_1\sinh x\left[\text{or}\,y(x)=c_0\dfrac{e^x+e^{-x}}{2}+c_1\dfrac{e^x-e^{-x}}{2}=\dfrac{c_0+c_1}{2}e^x+\dfrac{c_0-c_1}{2}e^{-x}\right]$.

7. Let $y(x)=\Sigma_{n=0}^{\infty}c_n x^n$. Then $y''=\Sigma_{n=0}^{\infty}n(n-1)c_n x^{n-2}$, $xy'=\Sigma_{n=0}^{\infty}nc_n x^n$ and $\left(x^2+1\right)y''=\Sigma_{n=0}^{\infty}n(n-1)c_n x^n+\Sigma_{n=0}^{\infty}(n+2)(n+1)c_{n+2}x^n$. The differential equation becomes

$\Sigma_{n=0}^{\infty}\left[(n+2)(n+1)c_{n+2}+[n(n-1)+n-1]c_n\right]x^n=0$. The recursion relation is $c_{n+2}=-\dfrac{(n-1)c_n}{n+2}$,

$n=0,1,2,\ldots$. Given c_0 and c_1 , $c_2=\dfrac{c_0}{2}$, $c_4=-\dfrac{c_2}{4}=-\dfrac{c_0}{2^2\cdot 2!}$, $c_6=-\dfrac{3c_4}{6}=(-1)^2\dfrac{3c_0}{2^3\cdot 3!}$, … ,

$c_{2n}=(-1)^{n-1}\dfrac{1\cdot 3\cdot\cdots\cdot(2n-3)\,c_0}{2^n n!}=(-1)^{n-1}\dfrac{(2n-3)!\,c_0}{2^n 2^{n-2} n!(n-2)!}=(-1)^{n-1}\dfrac{(2n-3)!\,c_0}{2^{2n-2} n!(n-2)!}$ for $n=2,3,\ldots$.

$c_3=\dfrac{0\cdot c_1}{3}=0 \Rightarrow c_{2n+1}=0$ for $n=1,2,\ldots$. Thus the solution is

$y(x)=c_0+c_1x+c_0\dfrac{x^2}{2}+c_0\sum_{n=2}^{\infty}\dfrac{(-1)^{n-1}(2n-3)!}{2^{2n-2}n!(n-2)!}x^{2n}$.

8. Assuming $y(x)=\sum_{n=0}^{\infty}c_nx^n$, $y''(x)=\sum_{n=2}^{\infty}n(n-1)c_nx^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)c_{n+2}x^n$ and $-xy(x)=-\sum_{n=0}^{\infty}c_nx^{n+1}=-\sum_{n=1}^{\infty}c_{n-1}x^n$. The equation $y''=xy$ becomes $\sum_{n=0}^{\infty}(n+2)(n+1)c_{n+2}x^n-\sum_{n=1}^{\infty}c_{n-1}x^n=0$ or $2c_2+\sum_{n=1}^{\infty}\left[(n+2)(n+1)c_{n+2}-c_{n-1}\right]x^n=0$. Equating coefficients gives $c_2=0$ and $c_{n+2}=\dfrac{c_{n-1}}{(n+2)(n+1)}$ for $n=1,2,\ldots$. Since $c_2=0$, $c_{3n+2}=0$ for $n=0,1,2,\ldots$. Given c_0 , $c_3=\dfrac{c_0}{3\cdot 2}$, $c_6=\dfrac{c_3}{6\cdot 5}=\dfrac{c_0}{6\cdot 5\cdot 3\cdot 2}$, … , $c_{3n}=\dfrac{c_0}{3n(3n-1)(3n-3)(3n-4)\cdot\cdots\cdot 6\cdot 5\cdot 3\cdot 2}$. Given c_1 , $c_4=\dfrac{c_1}{4\cdot 3}$, $c_7=\dfrac{c_4}{7\cdot 6}=\dfrac{c_1}{7\cdot 6\cdot 4\cdot 3}$, … , $c_{3n+1}=\dfrac{c_1}{(3n+1)3n(3n-2)(3n-3)\ldots 7\cdot 6\cdot 4\cdot 3}$. The solution can be written as

$$y(x)=c_0\sum_{n=0}^{\infty}\frac{(3n-2)(3n-5)\cdot\cdots\cdot 7\cdot 4\cdot 1}{(3n)!}x^{3n}+c_1\sum_{n=0}^{\infty}\frac{(3n-1)(3n-4)\cdot\cdots\cdot 8\cdot 5\cdot 2}{(3n+1)!}x^{3n+1}$$

9. Let $y(x)=\sum_{n=0}^{\infty}c_nx^n$. Then $-xy'(x)=-x\sum_{n=1}^{\infty}nc_nx^{n-1}=-\sum_{n=1}^{\infty}nc_nx^n=-\sum_{n=0}^{\infty}nc_nx^n$, $y''(x)=\sum_{n=0}^{\infty}(n+2)(n+1)c_{n+2}x^n$, and the equation $y''-xy'-y=0$ becomes $\sum_{n=0}^{\infty}\left[(n+2)(n+1)c_{n+2}-nc_n-c_n\right]x^n=0$. Thus, the recursion relation is

$c_{n+2}=\dfrac{nc_n+c_n}{(n+2)(n+1)}=\dfrac{c_n(n+1)}{(n+2)(n+1)}=\dfrac{c_n}{n+2}$ for $n=0,1,2,\ldots$. One of the given conditions is

$y(0)=1$. But $y(0)=\sum_{n=0}^{\infty}c_n(0)^n=c_0+0+0+\cdots=c_0$, so $c_0=1$. Hence, $c_2=\dfrac{c_0}{2}=\dfrac{1}{2}$, $c_4=\dfrac{c_2}{4}=\dfrac{1}{2\cdot 4}$,

$c_6=\dfrac{c_4}{6}=\dfrac{1}{2\cdot 4\cdot 6}$, … , $c_{2n}=\dfrac{1}{2^n n!}$. The other given condition is $y'(0)=0$. But

$y'(0)=\sum_{n=1}^{\infty} nc_n(0)^{n-1}=c_1+0+0+\cdots=c_1$, so $c_1=0$. By the recursion relation, $c_3=\frac{c_1}{3}=0$, $c_5=0$, … , $c_{2n+1}=0$ for $n=0$, 1 , 2 , … . Thus, the solution to the initial–value problem is

$$y(x)=\sum_{n=0}^{\infty} c_n x^n=\sum_{n=0}^{\infty} c_{2n} x^{2n}=\sum_{n=0}^{\infty} \frac{x^{2n}}{2^n n!}=\sum_{n=0}^{\infty} \frac{\left(x^2/2\right)^n}{n!}=e^{x^2/2}$$

10. Assuming that $y(x)=\sum_{n=0}^{\infty} c_n x^n$, we have $x^2 y=\sum_{n=0}^{\infty} c_n x^{n+2}$ and

$$y''(x)=\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}=\sum_{n=-2}^{\infty}(n+4)(n+3)c_{n+4}x^{n+2}=2c_2+6c_3x+\sum_{n=0}^{\infty}(n+4)(n+3)c_{n+4}x^{n+2}$$

Thus, the equation $y''+x^2y=0$ becomes $2c_2+6c_3x+\sum_{n=0}^{\infty}\left[(n+4)(n+3)c_{n+4}+c_n\right]x^{n+2}=0$. So

$c_2=c_3=0$ and the recursion relation is $c_{n+4}=-\frac{c_n}{(n+4)(n+3)}$, $n=0,1,2$, … .
But $c_1=y'(0)=0=c_2=c_3$ and by the recursion relation, $c_{4n+1}=c_{4n+2}=c_{4n+3}=0$ for $n=0,1,2$, … .
Also, $c_0=y(0)=1$, so
$c_4=-\frac{c_0}{4\cdot 3}=-\frac{1}{4\cdot 3}$, $c_8=-\frac{c_4}{8\cdot 7}=\frac{(-1)^2}{8\cdot 7\cdot 4\cdot 3}$, … , $c_{4n}=\frac{(-1)^n}{4n(4n-1)(4n-4)(4n-5)\cdot\cdots\cdot 4\cdot 3}$.
Thus, the solution to the initial–value problem is

$$y(x)=\sum_{n=0}^{\infty} c_n x^n=c_0+\sum_{n=0}^{\infty} c_{4n}x^{4n}=1+\sum_{n=1}^{\infty}(-1)^n \frac{x^{4n}}{4n(4n-1)(4n-4)(4n-5)\cdot\cdots\cdot 4\cdot 3}$$

11. Assuming that $y(x)=\sum_{n=0}^{\infty} c_n x^n$, we have $xy=x\sum_{n=0}^{\infty} c_n x^n=\sum_{n=0}^{\infty} c_n x^{n+1}$,
$x^2y'=x^2\sum_{n=1}^{\infty} nc_n x^{n-1}=\sum_{n=0}^{\infty} nc_n x^{n+1}$,

$y''(x)=\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}=\sum_{n=-1}^{\infty}(n+3)(n+2)c_{n+3}x^{n+1}$ [replace n with $n+3$] =
$2c_2+\sum_{n=0}^{\infty}(n+3)(n+2)c_{n+3}x^{n+1}$,
and the equation $y''+x^2y'+xy=0$ becomes $2c_2+\sum_{n=0}^{\infty}\left[(n+3)(n+2)c_{n+3}+nc_n+c_n\right]x^{n+1}=0$.
So

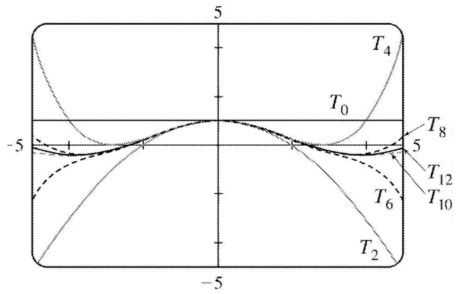
$c_2=0$ and the recursion relation is $c_{n+3}=\frac{-nc_n-c_n}{(n+3)(n+2)}=-\frac{(n+1)c_n}{(n+3)(n+2)}$, $n=0,1,2$, … .
But $c_0=y(0)=0=c_2$ and by the recursion relation, $c_{3n}=c_{3n+2}=0$ for $n=0$, 1 , 2 , … .
Also, $c_1=y'(0)=1$, so
$c_4=-\frac{2c_1}{4\cdot 3}=-\frac{2}{4\cdot 3}$, $c_7=-\frac{5c_4}{7\cdot 6}=(-1)^2\frac{2\cdot 5}{7\cdot 6\cdot 4\cdot 3}=(-1)^2\frac{2^2 5^2}{7!}$, … , $c_{3n+1}=(-1)^n\frac{2^2 5^2\cdot\cdots\cdot(3n-1)^2}{(3n+1)!}$.
Thus, the solution is

$$y(x)=\sum_{n=0}^{\infty}c_n x^n=x+\sum_{n=1}^{\infty}\left[(-1)^n\frac{2^2 5^2\cdot\cdots\cdot(3n-1)^2 x^{3n+1}}{(3n+1)!}\right]$$

12. **(a)** Let $y(x)=\sum_{n=0}^{\infty}c_n x^n$. Then $x^2 y''(x)=\sum_{n=2}^{\infty}n(n-1)c_n x^n=\sum_{n=0}^{\infty}(n+2)(n+1)c_{n+2}x^{n+2}$,
$xy'(x)=\sum_{n=1}^{\infty}nc_n x^n=\sum_{n=-1}^{\infty}(n+2)c_{n+2}x^{n+2}=c_1 x+\sum_{n=0}^{\infty}(n+2)c_{n+2}x^{n+2}$, and the equation
$x^2y''+xy'+x^2y=0$ becomes $c_1x+\sum_{n=0}^{\infty}\left\{[(n+2)(n+1)+(n+2)]c_{n+2}+c_n\right\}x^{n+2}=0$. So $c_1=0$ and the recursion relation is $c_{n+2}=-\frac{c_n}{(n+2)^2}$, $n=0,1,2$, … . But $c_1=y'(0)=0$ so $c_{2n+1}=0$ for $n=0,1,2,\ldots$. Also,
$c_0=y(0)=1$, so $c_2=-\frac{1}{2^2}$, $c_4=-\frac{c_2}{4^2}=(-1)^2\frac{1}{4^2 2^2}=(-1)^2\frac{1}{2^4(2!)^2}$, $c_6=-\frac{c_4}{6^2}=(-1)^3\frac{1}{2^6(3!)^2}$, … ,
$c_{2n}=(-1)^n\frac{1}{2^{2n}(n!)^2}$. The solution is

$$y(x)=\sum_{n=0}^{\infty}c_n x^n=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{2^{2n}(n!)^2}$$

(b) The Taylor polynomials T_0 to T_{12} are shown in the graph. Because T_{10} and T_{12} are close together throughout the interval $[-5,5]$, it is reasonable to assume that T_{12} is a good approximation to the Bessel function on that interval.


5
-5
5
-5
T_4
T_0
T_8
T_{12}
T_{10}
T_6
T_2